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ELLIPSE SYNTHESIS OF A FIVE-BAR LINKAGE

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ABSTRACT

This paper formulates synthesis equations for a two degree-of-freedom planar five-bar linkage to reproduce specified velocity ellipses at specified points in its workspace. The synthesis procedure finds a mechanism that exactly reproduces two ellipses, and affords the designer the free choice of one ground pivot. This paper shows how four solutions to these synthesis equations can be obtained in closed form. Each solution describes a five-bar linkage. The solution procedure reveals shared structure between these five-bars. The new synthesis procedure is applied to a few examples.

INTRODUCTION

This paper presents a new way to think about the synthesis of multi-degree-of-freedom mechanisms. That is to use the ellipses produced by Jacobian transformations as constraints that form synthesis equations. These synthesis equations can be solved to find the dimensions of mechanisms that reproduce specified ellipses. This paper deals with exact ellipse reproduction. Such a procedure can be used to synthesize multi-degree-of-freedom mechanisms that exhibit tailored directional velocity and force characteristics at different parts of its workspace. The synthesis equations are formulated for a planar five-bar linkage to exactly reproduce two specified velocity ellipses. The designer is afforded the choice of one ground pivot. Four solutions of the synthesis equations are found in closed form, each describing a five-bar linkage.

The study of mechanism Jacobians as transformations from spheres/circles in an input space to ellipsoids/ellipses in an output space is an old topic in robotics [1–5]. Similar results exist on both velocity and force/torque production. The literature on this topic suggests many different performance indices derived from Jacobians to assess a mechanism's kinematic design [6–12], mostly geared toward optimization. This work poses Jacobian ellipses themselves as constraints that form synthesis equations. From this perspective, this work is more similar to four-bar motion generation problems of Burmester [13] rather than the performance index literature.

ELLIPSES and JACOBIANS

To specify ellipses, the theory of the singular value decomposition of matrices is leveraged. That is that any real, square matrix $[J]$ can be decomposed into the product,

$$\begin{aligned}
&[J] = [U][\Sigma][V]^T \\
&\text{where} \quad [U] = [\mathcal{R}(\theta_u)] \\
&\qquad\qquad [\Sigma] = diag(\sigma_x, \sigma_y) \\
&\qquad\qquad [V] = [\mathcal{R}(\theta_v)] \quad \text{or} \quad [\hat{\mathcal{R}}(\theta_v)].
\end{aligned} \tag{1}$$

Interpreting $[J]$ as a transformation, it can be seen that if it transforms the points belonging to a circle in an input space to an ellipse in an output space. The matrix $[U]$ sets the orientation of that ellipse, the matrix $[\Sigma]$ sets its semi-axis lengths, and the matrix $[V]$ sets the orientation of the coordinate transformation onto the ellipse. To explain the last one, note that the image of $\{1,0\}$ and $\{0,1\}$ generally do not lie on the semi-axes of the transformed ellipse. $[V]$ can either be a rotation $[\mathcal{R}]$ or a reflection $[\hat{\mathcal{R}}]$.

$$[\mathcal{R}(\theta)] := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$[\hat{\mathcal{R}}(\theta)] := \begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

If $[V]$ is a reflection, the small angle between the image of $\{1,0\}$ and $\{0,1\}$ reverses, i.e. the handedness of the transformation reverses. We will define variable $\eta = |V|$ to denote whether $[V]$ is a rotation ($\eta = 1$) or a reflection ($\eta = -1$).

In this way all real 2×2 matrices, and their underlying ellipse transformations, are parameterized by θ_u, σ_x, σ_y, θ_v, and η. If $[\,J\,]$ is a velocity Jacobian, then this parameterization has physical significance. It describes how the movements of input actuators multiply into movements at an output end-effector. A long semi-axis indicates a direction which may be moved at faster velocities (at the cost of force production), and a short semi-axis indicates the opposite. Note that force ellipses share the same orientation as velocity ellipses, but with reciprocal semi-axis lengths. Therefore, the results of this paper are as equally important to synthesizing multi-degree-of-freedom force production as they are to synthesizing velocity production.

In this work, the aim is to synthesize specified velocity ellipses at points throughout the workspace, $\mathbf{P}_j$, $j = 0,1,\ldots,N-1$. Geometric ellipse information is used to define Jacobian matrices $[J_j]$ at each $\mathbf{P}_j$ via Eqn. (1). Therefore, instantiating desired velocity ellipses is accomplished by finding mechanism dimensions that satisfy constraints imposed by Jacobian elements.

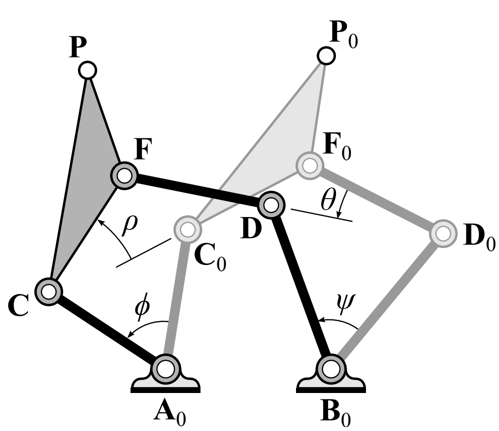

FIGURE 1. A FIVE-BAR LINKAGE DISPLACED FROM A REFERENCE CONFIGURATION.

SYNTHESIS SOLUTION

A five-bar linkage is shown in Fig. 1. Its dimensions are given by six points specified in a reference configuration: $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{D}_0$, $\mathbf{F}_0$, and $\mathbf{P}_0$. To begin the formulation, first the velocity Jacobian is represented. The loop equations are formed,

$$\begin{aligned} \mathbf{A}_0 + [\mathcal{R}(\phi)](\mathbf{C}_0 - \mathbf{A}_0) + [\mathcal{R}(\rho)](\mathbf{F}_0 - \mathbf{C}_0) = \\ \mathbf{B}_0 + [\mathcal{R}(\psi)](\mathbf{D}_0 - \mathbf{B}_0) + [\mathcal{R}(\theta)](\mathbf{F}_0 - \mathbf{D}_0) \quad (2) \\ \mathbf{A}_0 + [\mathcal{R}(\phi)](\mathbf{C}_0 - \mathbf{A}_0) + [\mathcal{R}(\rho)](\mathbf{P}_0 - \mathbf{C}_0) = \mathbf{P}, \quad (3) \end{aligned}$$

To write this more concisely, the following vector functions are introduced,

$$\begin{aligned} \mathbf{r}_{AC}(\phi) &:= [\mathcal{R}(\phi)](\mathbf{C}_0 - \mathbf{A}_0) & \mathbf{r}_{CF}(\rho) &:= [\mathcal{R}(\rho)](\mathbf{F}_0 - \mathbf{C}_0) \\ \mathbf{r}_{BD}(\psi) &:= [\mathcal{R}(\psi)](\mathbf{D}_0 - \mathbf{B}_0) & \mathbf{r}_{DF}(\theta) &:= [\mathcal{R}(\theta)](\mathbf{F}_0 - \mathbf{D}_0) \\ \mathbf{r}_{CP}(\rho) &:= [\mathcal{R}(\rho)](\mathbf{P}_0 - \mathbf{C}_0) & \mathbf{r}_{BA} &= \mathbf{A}_0 - \mathbf{B}_0 \end{aligned} \quad (4)$$

Then the loop equations take the form,

$$\mathbf{A}_0 + \mathbf{r}_{AC}(\phi) + \mathbf{r}_{CF}(\rho) = \mathbf{B}_0 + \mathbf{r}_{BD}(\psi) + \mathbf{r}_{DF}(\theta) \quad (5)$$
$$\mathbf{A}_0 + \mathbf{r}_{AC}(\phi) + \mathbf{r}_{CP}(\rho) = \mathbf{P}, \quad (6)$$

Taking the derivative of these equations yields,

$$\dot{\phi}[\,i\,]\mathbf{r}_{AC}(\phi) + \dot{\rho}[\,i\,]\mathbf{r}_{CF}(\rho) = \dot{\psi}[\,i\,]\mathbf{r}_{BD}(\psi) + \dot{\theta}[\,i\,]\mathbf{r}_{DF}(\theta) \quad (7)$$
$$\dot{\phi}[\,i\,]\mathbf{r}_{AC}(\phi) + \dot{\rho}[\,i\,]\mathbf{r}_{CP}(\rho) = \dot{\mathbf{P}} \quad (8)$$

where $[\,i\,]$ is the 90° rotation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with the property $[\,i\,]^2 = -[\,I\,]$. Eqn. (7) may be rewritten as

$$\begin{Bmatrix} \dot{\rho} \\ \dot{\theta} \end{Bmatrix} = \quad (9)$$
$$\frac{1}{\mathbf{r}_{DF}(\theta)\times\mathbf{r}_{CF}(\rho)} \begin{bmatrix} ([\,i\,]\mathbf{r}_{DF}(\theta))^T \\ ([\,i\,]\mathbf{r}_{CF}(\rho))^T \end{bmatrix} \begin{bmatrix} -\mathbf{r}_{AC}(\phi) & \mathbf{r}_{BD}(\psi) \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\psi} \end{Bmatrix}$$

where the "×" operator defines $\mathbf{v}_1 \times \mathbf{v}_2 = |\mathbf{v}_1\ \mathbf{v}_2|$, i.e. the determinant of a matrix formed by two vectors.

The expression for $\dot{\rho}$ from Eqn. (9) can be substituted into Eqn. (8) to obtain the Jacobian transformation,

$$\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\psi} \end{Bmatrix} = \dot{\mathbf{P}} \quad (10)$$

$$\text{where} \quad \mathbf{J}_1 = [\,i\,]\Big(\mathbf{r}_{AC}(\phi) - \frac{\mathbf{r}_{DF}(\theta)\times\mathbf{r}_{AC}(\phi)}{\mathbf{r}_{DF}(\theta)\times\mathbf{r}_{CF}(\rho)}\mathbf{r}_{CP}(\rho)\Big) \quad (11)$$

$$\mathbf{J}_2 = \frac{\mathbf{r}_{DF}(\theta)\times\mathbf{r}_{BD}(\psi)}{\mathbf{r}_{DF}(\theta)\times\mathbf{r}_{CF}(\rho)}[\,i\,]\mathbf{r}_{CP}(\rho) \quad (12)$$

The synthesis equations are formed by repeating Eqns. (5), (6), (11), and (12) for N task positions, indexed by $j = 0,1,\ldots,N-1$.

$$\mathbf{A}_0+\mathbf{r}_{AC}(\phi_j)+\mathbf{r}_{CF}(\rho_j)=\mathbf{B}_0+\mathbf{r}_{BD}(\psi_j)+\mathbf{r}_{DF}(\theta_j) \tag{13}$$

$$\mathbf{A}_0+\mathbf{r}_{AC}(\phi_j)+\mathbf{r}_{CP}(\rho_j)=\mathbf{P}_j, \tag{14}$$

$$[\,i\,]\Big(\mathbf{r}_{AC}(\phi_j)-\frac{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{AC}(\phi_j)}{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{CF}(\rho_j)}\mathbf{r}_{CP}(\rho_j)\Big)=\mathbf{J}_{1,j} \tag{15}$$

$$\frac{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BD}(\psi_j)}{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{CF}(\rho_j)}[\,i\,]\mathbf{r}_{CP}(\rho_j)=\mathbf{J}_{2,j}\quad j=0,1,\ldots,N-1 \tag{16}$$

The input data to this design problem is a list of point locations $\mathbf{P}_j$, $j=0,1,\ldots,N-1$, and the Jacobian information that should be reproduced at each point, that is $\mathbf{J}_{1,j}$ and $\mathbf{J}_{2,j}$. The design parameters are the unknown reference pivot positions $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{D}_0$, and $\mathbf{F}_0$. Along for the ride, are also several unknown intermediate angle values: ϕ_j, ρ_j, ψ_j, and θ_j. At the reference configuration, $\phi_0=\rho_0=\psi_0=\theta_0=0$, making Eqns. (13) and (14) identically equal. For exact two position synthesis, i.e. $j=0,1$, this makes for 12 equations and 14 unknowns. The system is made square by redesignating the components of $\mathbf{B}_0=\{B_{x0},B_{y0}\}$ as free choices rather than unknown variables.

To embark on the solution procedure for two position synthesis, first multiply both sides of Eqns. (15) and (16) by $-[\,i\,]$, then solve for $\mathbf{r}_{CP}(\rho_j)$ in (16) and substitute into (15). After simplification, (15) and (16) become

$$\mathbf{r}_{AC}(\phi_j)=-[\,i\,]\mathbf{J}_{1,j}-\frac{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{AC}(\phi_j)}{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BD}(\psi_j)}[\,i\,]\mathbf{J}_{2,j} \tag{17}$$

$$\mathbf{r}_{CP}(\rho_j)=-\frac{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{CF}(\rho_j)}{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BD}(\psi_j)}[\,i\,]\mathbf{J}_{2,j} \tag{18}$$

Eqns. (17) and (18) can be substituted in to (14) to obtain

$$\mathbf{A}_0-[\,i\,]\mathbf{J}_{1,j}-\alpha_j[\,i\,]\mathbf{J}_{2,j}=\mathbf{P}_j\qquad j=0,1 \tag{19}$$

$$\text{where}\quad \alpha_j=\beta_{1,j}+\beta_{2,j} \tag{20}$$

$$\beta_{1,j}=\frac{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{AC}(\phi_j)}{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BD}(\psi_j)} \tag{21}$$

$$\beta_{2,j}=\frac{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{CF}(\rho_j)}{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BD}(\psi_j)} \tag{22}$$

Eqn. (19) consists of four equations which are linear in the four unknowns A_{x0}, A_{y0}, α_0, and α_1. The linear system takes the form,

$$\begin{bmatrix} 1 & 0 & \multirow{2}{*}{$-[\,i\,]\mathbf{J}_{2,0}$} & 0 \\ 0 & 1 & & 0 \\ 1 & 0 & 0 & \multirow{2}{*}{$-[\,i\,]\mathbf{J}_{2,1}$} \\ 0 & 1 & 0 & \end{bmatrix}\begin{Bmatrix} A_{x0}\\ A_{y0}\\ \alpha_0\\ \alpha_1\end{Bmatrix}=\begin{bmatrix}\mathbf{P}_0+[\,i\,]\mathbf{J}_{1,0}\\ \mathbf{P}_1+[\,i\,]\mathbf{J}_{1,1}\end{bmatrix} \tag{23}$$

and can be solved outright, yielding numeric values for $\mathbf{A}_0$, α_0, and α_1.

Upon introducing the new β coefficients, Eqns. (17) and (18) can be rewritten as

$$\mathbf{r}_{AC}(\phi_j)=-[\,i\,]\mathbf{J}_{1,j}-\beta_{1,j}[\,i\,]\mathbf{J}_{2,j} \tag{24}$$

$$\mathbf{r}_{CP}(\rho_j)=-\beta_{2,j}[\,i\,]\mathbf{J}_{2,j} \tag{25}$$

Angles ϕ_j and ρ_j can be eliminated from (24) and (25), respectively, by taking the dot product of each side of each equation with itself. The result is

$$(\mathbf{C}_0-\mathbf{A}_0)\cdot(\mathbf{C}_0-\mathbf{A}_0)=(\mathbf{J}_{1,j}+\beta_{1,j}\mathbf{J}_{2,j})\cdot(\mathbf{J}_{1,j}+\beta_{1,j}\mathbf{J}_{2,j}) \tag{26}$$

$$(\mathbf{P}_0-\mathbf{C}_0)\cdot(\mathbf{P}_0-\mathbf{C}_0)=\beta_{2,j}^2\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j}\qquad j=0,1 \tag{27}$$

Eqns. (20), (26), and (27) form a subsystem of six equations in the six unknowns C_{x0}, C_{y0}, $\beta_{1,0}$, $\beta_{1,1}$, $\beta_{2,0}$, and $\beta_{2,1}$. To eliminate the β coefficients, solve for $\beta_{1,j}$ and $\beta_{2,j}$ in (26) and (27), respectively,

$$\beta_{1,j}=-\frac{\mathbf{J}_{1,j}\cdot\mathbf{J}_{2,j}}{\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j}} \pm\frac{\sqrt{(\mathbf{J}_{1,j}\cdot\mathbf{J}_{2,j})^2-(\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j})(\mathbf{J}_{1,j}\cdot\mathbf{J}_{1,j}-(\mathbf{C}_0-\mathbf{A}_0)\cdot(\mathbf{C}_0-\mathbf{A}_0))}}{\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j}} \tag{28}$$

$$\beta_{2,j}=\pm\sqrt{\frac{(\mathbf{P}_0-\mathbf{C}_0)\cdot(\mathbf{P}_0-\mathbf{C}_0)}{\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j}}}\qquad j=0,1 \tag{29}$$

Substitution into (20) yields an equation of the form,

$$a_j=\sqrt{b_j}+\sqrt{c_j}\,,\qquad j=0,1 \tag{30}$$

where a_j, b_j, and c_j are introduced to aid in equation manipulation,

$$a_j=\alpha_j+\frac{\mathbf{J}_{1,j}\cdot\mathbf{J}_{2,j}}{\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j}} \tag{31}$$

$$b_j=\frac{(\mathbf{J}_{1,j}\cdot\mathbf{J}_{2,j})^2-(\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j})(\mathbf{J}_{1,j}\cdot\mathbf{J}_{1,j}-(\mathbf{C}_0-\mathbf{A}_0)\cdot(\mathbf{C}_0-\mathbf{A}_0))}{(\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j})^2} \tag{32}$$

$$c_j=\frac{(\mathbf{P}_0-\mathbf{C}_0)\cdot(\mathbf{P}_0-\mathbf{C}_0)}{\mathbf{J}_{2,j}\cdot\mathbf{J}_{2,j}}\qquad j=0,1 \tag{33}$$

Eqn. (30) may be converted to a polynomial. To do so, square both sides. There will still be a leftover radical. Isolate it to one side of the equation and square both sides again. Then expand the result, witness some cancellations, and do some factoring. The result will yield,

$$a_j^2(a_j^2-2(b_j+c_j))+(b_j-c_j)^2=0\,, \qquad j=0,1 \tag{34}$$

Eqn. (34) looks to be quartic in terms of C_{x0} and C_{y0} in light of (31)–(33). But upon closer inspection of the last term, it can be found that expansion of b_j-c_j according to (32) and (33) leads to the term $\mathbf{C}_0\cdot\mathbf{C}_0$ cancelling out. Because of this, Eqn. (34) is quadratic, and the system, $j=0,1$, has four solutions for $\mathbf{C}_0$, which may be obtained numerically. Once values of $\mathbf{C}_0$ are obtained, then values of the β variables are obtained through back-substitution into Eqns. (28) and (29). However, note that it is unclear whether "+" or "−" solutions should be used in (28) and (29). Across these equations, and considering $j=0,1$, there are 16 cases of different sign combinations from which β coefficients can be calculated. All 16 cases were evaluated and tested for satisfaction of (20). During the numerical exercises of this work, it was always the case that only one of the 16 sign combinations lead to satisfaction of (20) for every $\mathbf{C}_0$ solution, and this combination was not the same every time.

Next, the numeric values of $\mathbf{C}_0$, $\beta_{1,j}$, $\beta_{2,j}$ are back-substituted into Eqns. (24) and (25) to obtain unique solutions of ϕ_j and ρ_j, $j=0,1$ (not considering 2π periodicity),

$$\begin{Bmatrix}\cos\phi_j\\ \sin\phi_j\end{Bmatrix}= \frac{1}{|\mathbf{C}_0-\mathbf{A}_0|^2}\Big[\mathbf{C}_0-\mathbf{A}_0\;\;[\,i\,](\mathbf{C}_0-\mathbf{A}_0)\Big]^T[\,i\,]\Big(-\mathbf{J}_{1,j}-\beta_{1,j}\mathbf{J}_{2,j}\Big) \tag{35}$$

$$\begin{Bmatrix}\cos\rho_j\\ \sin\rho_j\end{Bmatrix}=\frac{1}{|\mathbf{P}_0-\mathbf{C}_0|^2}\Big[\mathbf{P}_0-\mathbf{C}_0\;\;[\,i\,](\mathbf{P}_0-\mathbf{C}_0)\Big]^T[\,i\,]\Big(-\beta_{2,j}\mathbf{J}_{2,j}\Big) \tag{36}$$

where the arctan function is used to compute the angles.

At this point, four solutions of the *ACP* dyad have been computed and can be analyzed. In all numeric exercises, it was found that four unique values of $\mathbf{C}_0$ were computed (generally), but dyad link lengths occurred in identical pairs. Within each pair, only one solution (generally) had a zero value for ϕ_0 and ρ_0, which is expected given the design space parameterization. The other partner solution described the same dyad but measured from a different reference configuration and rotated onto the reference configuration of the first solution. Therefore, these partner solutions do not satisfy Eqn. (14) when $\phi_0=\rho_0=0$, leaving only two unique *ACP* dyad solutions going forward. To solve for the remaining pivots of the five-bar, there is left Eqns. (13), (21), and (22), which amounts to eight equations in the eight unknowns D_{x0}, D_{y0}, F_{x0}, F_{y0}, ψ_0, ψ_1, θ_0, θ_1,

Next, manipulate Eqns. (13), (21), and (22) to find θ_j. Apply the "$\times$" operation between $\mathbf{r}_{DF}(\theta_j)$ and the terms of Eqn. (13). This cancels out the last term, resulting in

$$\mathbf{r}_{DF}(\theta_j)\times(\mathbf{A}_0-\mathbf{B}_0)\;+\;\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{AC}(\phi_j)\;+\;\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{CF}(\rho_j)\;=\;\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BD}(\psi_j) \tag{37}$$

Divide both sides by the right hand side to obtain,

$$\frac{\mathbf{r}_{DF}(\theta_j)\times(\mathbf{A}_0-\mathbf{B}_0)}{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BD}(\psi_j)}+\beta_{1,j}+\beta_{2,j}=1 \tag{38}$$

considering the β definitions of (21), and (22). Solve for $\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BD}(\psi_j)$ in (22), and substitute in to (38) to obtain

$$\frac{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{BA}}{\mathbf{r}_{DF}(\theta_j)\times\mathbf{r}_{AC}(\phi_j)}\beta_{1,j}=1-\alpha_j \tag{39}$$

considering (20) and defining $\mathbf{r}_{BA}=\mathbf{A}_0-\mathbf{B}_0$. Clear the denominator of (39) and factor out $\mathbf{r}_{DF}(\theta_j)$ to obtain,

$$\mathbf{r}_{DF}(\theta_j)\times\boldsymbol{\delta}_j=0 \tag{40}$$

$$\text{where}\quad \boldsymbol{\delta}_j=\beta_{1,j}\,\mathbf{r}_{BA}-(1-\alpha_j)\mathbf{r}_{AC}(\phi_j) \tag{41}$$

This statement requires colinearity of link DF with the known vector $\boldsymbol{\delta}_j$ Define r_{DF} to be the length of link DF and $\hat{\theta}$ to be its angle measured from horizontal such that

$$[\mathcal{R}(\theta_j)](\mathbf{F}_0-\mathbf{D}_0)=r_{DF}\begin{Bmatrix}\cos\hat{\theta}_j\\ \sin\hat{\theta}_j\end{Bmatrix}\quad\text{and}\quad\hat{\theta}_j=\hat{\theta}_0+\theta_j \tag{42}$$

For each position, $j=0,1$, there are two possible values for $\hat{\theta}_j$,

$$\hat{\theta}_0=\arctan\frac{\delta_{y0}}{\delta_{x0}}+m\pi \qquad \hat{\theta}_1=\arctan\frac{\delta_{y1}}{\delta_{x1}}+n\pi \tag{43}$$

case	m	n
I	0	0
II	1	1
III	1	0
IV	0	1

leading to four sets of $\{\hat{\theta}_0,\hat{\theta}_1\}$ solutions per every *ACP* dyad, of which there are two. The proceeding steps of the solution process are performed for all eight sets of solution variables generated at this point. From Eqn. (42), transform back to reference configuration angles, $\theta_0=0$ and $\theta_1=\hat{\theta}_1-\hat{\theta}_0$.

Next, define new variables $\gamma_{1,j}$, $\gamma_{2,j}$, and $\gamma_{3,j}$,

$$\gamma_{1,j} = \begin{Bmatrix} \cos\hat{\theta}_j \\ \sin\hat{\theta}_j \end{Bmatrix} \times \big([\mathcal{R}(\phi_j)](\mathbf{C}_0 - \mathbf{A}_0)\big) \tag{44}$$

$$\gamma_{2,j} = \begin{Bmatrix} \cos\hat{\theta}_j \\ \sin\hat{\theta}_j \end{Bmatrix} \times \big([\mathcal{R}(\rho_j)](\mathbf{F}_0 - \mathbf{C}_0)\big) \tag{45}$$

$$\gamma_{3,j} = \begin{Bmatrix} \cos\hat{\theta}_j \\ \sin\hat{\theta}_j \end{Bmatrix} \times \big([\mathcal{R}(\psi_j)](\mathbf{D}_0 - \mathbf{B}_0)\big) \qquad j = 0,1 \tag{46}$$

which were chosen to set up

$$\beta_{1,j} = \frac{\gamma_{1,j}}{\gamma_{3,j}} \qquad \text{and} \qquad \beta_{2,j} = \frac{\gamma_{2,j}}{\gamma_{3,j}} \tag{47}$$

See that $\gamma_{1,j}$ is fully defined as the right hand side of Eqn. (44) is completely known at this point. Next, $\gamma_{3,j}$ and then $\gamma_{2,j}$ are found sequentially using (47). $\mathbf{F}_0$ remains as the sole unknown in (45) and can be solved for across $j = 0,1$. To do so, expand and rearrange (45), replacing $\{\cos\hat{\theta}_j, \sin\hat{\theta}_j\}^T$ with $[\mathcal{R}(\hat{\theta}_j)]\{1,0\}^T$,

$$[\mathcal{R}(\hat{\theta}_j)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \times [\mathcal{R}(\rho_j)]\mathbf{F}_0 = \gamma_{2,j} + [\mathcal{R}(\hat{\theta}_j)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \times [\mathcal{R}(\rho_j)]\mathbf{C}_0 \qquad j = 0,1 \tag{48}$$

Note that the "×" product may be rewritten as a dot product, i.e. $\mathbf{a} \times \mathbf{b} = [\,i\,]\mathbf{a} \cdot \mathbf{b}$, which is invariant to rotation of its operands. Therefore, the left hand side of (48) can be rewritten as

$$[\,i\,][\mathcal{R}(\hat{\theta}_j - \rho_j)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \cdot \mathbf{F}_0 \qquad j = 0,1 \tag{49}$$

Combining $j = 0$ and $j = 1$ equations, the first operand of (49) forms the rows of a matrix which may be inverted to yield numeric values of $\mathbf{F}_0$,

$$\mathbf{F}_0 = \frac{1}{\sin(\hat{\theta}_1 - \hat{\theta}_0 - \rho_1 + \rho_0)} \begin{bmatrix} \cos(\hat{\theta}_1 - \rho_1) & -\cos(\hat{\theta}_0 - \rho_0) \\ \sin(\hat{\theta}_1 - \rho_1) & -\sin(\hat{\theta}_0 - \rho_0) \end{bmatrix} \times \begin{Bmatrix} \gamma_{2,0} + [\mathcal{R}(\hat{\theta}_0)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \times [\mathcal{R}(\rho_0)]\mathbf{C}_0 \\ \gamma_{2,1} + [\mathcal{R}(\hat{\theta}_1)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \times [\mathcal{R}(\rho_1)]\mathbf{C}_0 \end{Bmatrix} \tag{50}$$

Note that $\rho_0 = 0$, but it is left in to illustrate the equations pattern. To find the remaining unknown pivot $\mathbf{D}_0$, combine the $j = 1$ equation of (13) and the $j = 0$ equation of (46), that is

$$\boldsymbol{\varepsilon}_1 - [\mathcal{R}(\theta_1)](\mathbf{F}_0 - \mathbf{D}_0) = [\mathcal{R}(\psi_1)](\mathbf{D}_0 - \mathbf{B}_0) \tag{51}$$

$$\gamma_{3,0} = \begin{Bmatrix} \cos\hat{\theta}_0 \\ \sin\hat{\theta}_0 \end{Bmatrix} \times (\mathbf{D}_0 - \mathbf{B}_0) \tag{52}$$

$$\text{where} \quad \boldsymbol{\varepsilon}_1 = \mathbf{r}_{BA} + \mathbf{r}_{AC}(\phi_1) + \mathbf{r}_{CF}(\rho_1) \tag{53}$$

and where $\boldsymbol{\varepsilon}_1$ is introduced to represent several known terms and shorten the length of equations. The angle ψ_1 is eliminated from (51) by taking the dot product of each side with itself. Expansion of the resulting equation cancels out $\mathbf{D}_0 \cdot \mathbf{D}_0$ and yields,

$$2\big([\mathcal{R}(\theta_1)]^T\boldsymbol{\varepsilon}_1 - \mathbf{F}_0 + \mathbf{B}_0\big) \cdot \mathbf{D}_0 = 2\boldsymbol{\varepsilon}_1 \cdot [\mathcal{R}(\theta_1)]\mathbf{F}_0 - \boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_1 - \mathbf{F}_0 \cdot \mathbf{F}_0 + \mathbf{B}_0 \cdot \mathbf{B}_0 \tag{54}$$

Expansion of (52) yields

$$\left([\,i\,][\mathcal{R}(\hat{\theta}_0)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}\right) \cdot \mathbf{D}_0 = \gamma_{3,0} + [\mathcal{R}(\hat{\theta}_0)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \times \mathbf{B}_0 \tag{55}$$

The first operands of the dot product on the left hand sides of Eqns. (54) and (55) can be stacked into a matrix that is inverted to solve for $\mathbf{D}_0$,

$$\mathbf{D}_0 = \frac{1}{2\left([\mathcal{R}(\theta_1)]^T\boldsymbol{\varepsilon}_1 - \mathbf{F}_0 + \mathbf{B}_0\right) \times [\,i\,][\mathcal{R}(\hat{\theta}_0)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}} [\,i\,] \times \left[-[\,i\,][\mathcal{R}(\hat{\theta}_0)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad 2\big([\mathcal{R}(\theta_1)]^T\boldsymbol{\varepsilon}_1 - \mathbf{F}_0 + \mathbf{B}_0\big) \right] \times \begin{Bmatrix} 2\,\boldsymbol{\varepsilon}_1 \cdot [\mathcal{R}(\theta_1)]\mathbf{F}_0 - \boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_1 - \mathbf{F}_0 \cdot \mathbf{F}_0 + \mathbf{B}_0 \cdot \mathbf{B}_0 \\ \gamma_{3,0} + [\mathcal{R}(\hat{\theta}_0)] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \times \mathbf{B}_0 \end{Bmatrix} \tag{56}$$

Finally, all pivot locations are now known. The remaining unknown angle ψ_j is determined from Eqn. (51),

$$\begin{Bmatrix} \cos\psi_j \\ \sin\psi_j \end{Bmatrix} = \frac{1}{|\mathbf{D}_0 - \mathbf{B}_0|^2} \Big[\mathbf{D}_0 - \mathbf{B}_0 \quad [\,i\,](\mathbf{D}_0 - \mathbf{B}_0)\Big]^T (\boldsymbol{\varepsilon}_j - \mathbf{r}_{DF}(\theta_j)) \qquad j = 0,1 \tag{57}$$

where the arctan function is called on the left hand side of (57). The number of solutions computed at this point is eight: that is four stemming from the combinations of arctan solutions for $\{\hat{\theta}_0, \hat{\theta}_1\}$, Eqn. (43), for each of the two *ACP* dyads. However, for each *ACP* dyad only two unique values of $\mathbf{F}_0$ were found for all examples. Therefore, for each of the two dyad *ACP* solutions, there are two dyad *BDF* solutions, yielding four five-bar solutions in total.

TABLE 1. ELLIPSE SPECIFICATION AND SYNTHESIS SOLUTIONS FOR THE FIRST EXAMPLE. ELLIPSES AND SYNTHESIS SOLUTIONS ARE DRAWN IN FIG. 2.

Ellipse specification

j	0	1
P_x	0.260000	−0.320000
P_y	0.256000	−0.040000
θ_u	−0.291457	−0.117109
σ_x	0.352477	0.122066
σ_y	0.104403	0.342345
θ_v	−1.395103	−1.234371
η	1	1

Synthesis solutions

	1	2	3	4
$\mathbf{A}_0$	0.355430	0.355430	0.355430	0.355430
	0.836371	0.836371	0.836371	0.836371
$\mathbf{B}_0$	0.260000	0.260000	0.260000	0.260000
	−0.400000	−0.400000	−0.400000	−0.400000
$\mathbf{C}_0$	0.557885	0.557885	0.170474	0.170474
	1.087540	1.087540	0.006091	0.006091
$\mathbf{D}_0$	0.609264	0.341047	0.379773	0.242951
	−0.405995	0.024940	−0.145966	0.255987
$\mathbf{F}_0$	0.451863	0.451863	0.247668	0.247668
	−0.153103	−0.153103	0.242130	0.242130

TABLE 2. ELLIPSE SPECIFICATION AND SYNTHESIS SOLUTIONS FOR THE SECOND EXAMPLE. ELLIPSES AND SYNTHESIS SOLUTIONS ARE DRAWN IN FIG. 3.

Ellipse specification

j	0	1
P_x	0.006000	0.012000
P_y	−0.006000	0.008000
θ_u	−1.561894	0.004843
σ_x	0.678955	0.822000
σ_y	0.074673	0.070114
θ_v	1.411372	−0.283472
η	−1	1

Synthesis solutions

	1	2	3	4
$\mathbf{A}_0$	−0.345764	−0.345764	−0.345764	−0.345764
	−0.365612	−0.365612	−0.365612	−0.365612
$\mathbf{B}_0$	−0.460000	−0.460000	−0.460000	−0.460000
	−0.860000	−0.860000	−0.860000	−0.860000
$\mathbf{C}_0$	−0.801636	−0.801636	0.013569	0.013569
	−0.283086	−0.283086	−0.003403	−0.003403
$\mathbf{D}_0$	−0.621189	−1.163860	−0.409607	0.013118
	−0.932387	−0.676899	−0.833723	−0.001692
$\mathbf{F}_0$	−1.288407	−1.288407	0.012703	0.012703
	−0.618262	−0.618262	−0.002509	−0.002509

EXAMPLES

To validate the solution procedure for the two position ellipse synthesis procedure for a five-bar linkage, it was applied to three example synthesis tasks. Ellipse specifications for each of the synthesis tasks are given in Tables 1, 2, and 3. The ellipse parameters, i.e. θ_u, σ_x, σ_y, θ_v, and η, define Jacobian matrices according to Eqn. (1). The four corresponding synthesis solutions also appear in Tables 1, 2, and 3. The target ellipses and five-bar solutions are drawn in Figs. 2, 3, and 4. Task ellipse input data was specified by mouse clicks through an interactive interface.

The first example demonstrates four five-bar linkages capable of reproducing two generically selected ellipses. Of the four solutions, a workspace check shows that three would be practically useful. The fourth solution, Figs. 2g-h, possesses a very small link length, such that it is practically a four-bar linkage. That being the case, it has a very narrow workspace.

The second example attempted to select ellipses that share the same center point, but have long semi-axis lengths nearly perpendicular to each. A mechanism with a high aspect ratio velocity ellipse has great velocity production (and poor force production) in one direction, and poor velocity production (and great force production) in the orthogonal direction. The task specification tests whether a five-bar can be found that can reconfigure to completely flip such anisotropic velocity/force characteristics at the same workspace point. Two five-bar solutions capable of this are shown in Fig. 3.

The third example aimed to find five-bar linkages that possess a “floor,” so to say, of high aspect ratio ellipses that favor force in the vertical direction and velocity in the horizontal direction. Such a five-bar might be useful for mobile robots that need to resist their own weight in the vertical direction, but otherwise favor fast motions. Of the four synthesis solutions, two of them (Fig. 4a-d) possessed pivots in favorable locations, and the

TABLE 3. ELLIPSE SPECIFICATION AND SYNTHESIS SOLUTIONS FOR THE THIRD EXAMPLE. ELLIPSES AND SYNTHESIS SOLUTIONS ARE DRAWN IN FIG. 4.

Ellipse specification

j	0	1
P_x	0.398000	−0.462000
P_y	−0.235000	−0.220000
θ_u	0.000000	−0.022862
σ_x	0.640078	0.656305
σ_y	0.070711	0.070114
θ_v	−2.984176	−1.087663
η	1	1

Synthesis solutions

	1	2	3	4
$\mathbf{A}_0$	−0.492586	−0.492586	−0.492586	−0.492586
	0.396535	0.396535	0.396535	0.396535
$\mathbf{B}_0$	0.260000	0.260000	0.260000	0.260000
	0.480000	0.480000	0.480000	0.480000
$\mathbf{C}_0$	−0.160275	−0.160275	0.256709	0.256709
	0.567150	0.567150	−0.031988	−0.031988
$\mathbf{D}_0$	0.332286	0.063780	0.349471	0.488824
	0.707879	0.215627	0.038313	−0.380754
$\mathbf{F}_0$	0.163433	0.163433	0.571720	0.571720
	0.398321	0.398321	−0.630043	−0.630043

other two (Fig. 4e-h) did not. A workspace check shows that the first two solutions are generally tenable. However, what Figs. 4a-d does not show is that in between the two specified ellipses, the velocity ellipses are not high aspect ratio, which thwarts the effort to find a so-called "floor."

CONCLUSION

In this paper, the formulation was presented for the synthesis of a five-bar linkage capable of exactly achieving two specified velocity ellipses at two points of its workspace. Four solutions to the synthesis equations were found, where each describes the dimensions of a five-bar linkage. The synthesis procedure was illustrated with a variety of examples. This work represents a new way of performing the synthesis of multi-degree-of-freedom linkages that consider velocity ellipses directly into constraint equations, rather than parsed for performance indices.

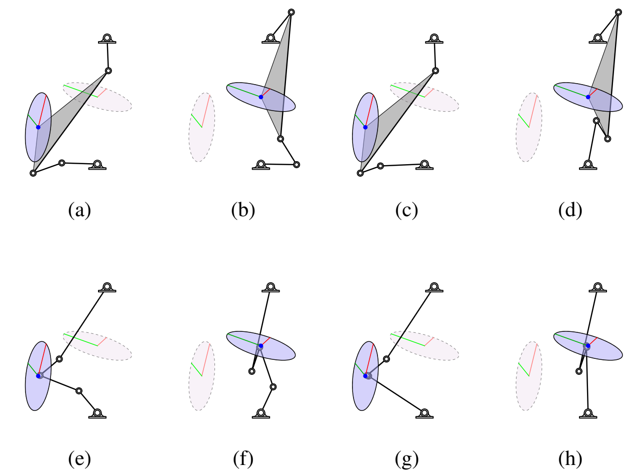

FIGURE 2. The five-bar linkages computed for the first example. Drawings correspond to ellipses and solutions of Table 1. (a), (b) depict Solution 1 achieving each ellipsoid. (c), (d) depict Solution 2 achieving each ellipsoid. (e), (f) depict Solution 3 achieving each ellipsoid. (g), (h) depict Solution 4 achieving each ellipsoid.

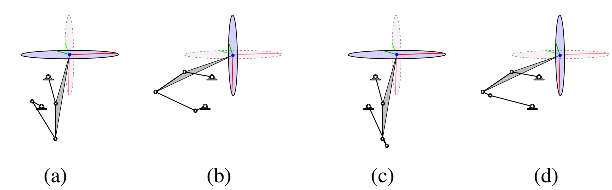

FIGURE 3. The five-bar linkages computed for the second example. Drawings correspond to ellipses and solutions of Table 2. (a), (b) depict Solution 1 achieving each ellipsoid. (c), (d) depict Solution 2 achieving each ellipsoid. The third and fourth solutions were omitted as they possessed small link lengths that made them practically useless.

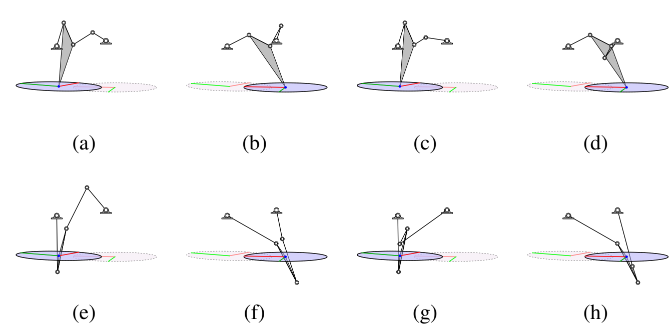

FIGURE 4. The five-bar linkages computed for the third example. Drawings correspond to ellipses and solutions of Table 3. (a), (b) depict Solution 1 achieving each ellipsoid. (c), (d) depict Solution 2 achieving each ellipsoid. (e), (f) depict Solution 3 achieving each ellipsoid. (g), (h) depict Solution 4 achieving each ellipsoid.

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