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Tropical flag varieties

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ABSTRACT

Flag matroids are combinatorial abstractions of flags of linear subspaces, just as matroids are of linear subspaces. We introduce the flag Dressian as a tropical analogue of the partial flag variety, and prove a correspondence between: (a) points on the flag Dressian, (b) valuated flag matroids, (c) flags of projective tropical linear spaces, and (d) coherent flag matroidal subdivisions. We introduce and characterize projective tropical linear spaces, which serve as a fundamental tool in our proof. We apply the correspondence to prove that all valuated flag matroids on ground set up to size 5 are realizable, and give an example where this fails for a flag matroid on 6 elements.

1. Introduction

The Grassmannian $Gr(r;n)$ over a field $\Bbbk$ parameterizes r-dimensional linear subspaces in $\Bbbk^{[n]}$, or equivalently, realizations of matroids of rank r on the ground set $[n] = \{1, \ldots, n\}$. It can be embedded in $\mathbb{P}\big(\Bbbk^{\binom{[n]}{r}}\big)$, where it is cut out by the quadratic

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Grassmann-Plücker relations. For a fixed matroid M, one can modify the Grassmann-Plücker relations to cut out only the points in the Grassmannian realizing M. The tropical prevariety of these equations is the **Dressian of** M, denoted $Dr(M)$, which was introduced in [20]. The Dressian of a loopless matroid M has multiple interpretations as

(a) the tropical prevariety of (modified) Grassmann-Plücker relations,
(b) the set of all valuated matroids with underlying matroid M [10],
(c) the weight vectors inducing a matroidal subdivision of the base polytope of M [33], or
(d) the parameter space of all tropical linear spaces given by M [33].

The (partial) flag variety $Fl(r_1, \dots, r_k; n)$ parameterizes flags of linear spaces $L_1 \subseteq \cdots \subseteq L_s$ in $\mathbb{k}^{[n]}$ where $\dim_{\mathbb{k}}(L_i) = r_i$. A point on $Fl(r_1, \dots, r_k; n)$ corresponds to a realization of a **flag matroid**, which is a sequence of matroids $\boldsymbol{M} = (M_1, \dots, M_k)$ of ranks $(r_1, \dots, r_k)$ on $[n]$ such that every circuit of M_j is a union of circuits of M_i for all $1 \le i < j \le k$. Flag matroids are the Coxeter matroids of type A [5]. The flag variety $Fl(r_1, \dots, r_k; n)$ can be embedded in $\mathbb{P}\left(\mathbb{k}^{\binom{[n]}{r_1}}\right) \times \cdots \times \mathbb{P}\left(\mathbb{k}^{\binom{[n]}{r_k}}\right)$, where it is cut out by the quadratic incidence-Plücker relations (see Equation (IP)) in addition to the Grassmann-Plücker relations. For a fixed flag matroid $\boldsymbol{M}$, one can modify these relations to cut out only the points in the flag variety which realize $\boldsymbol{M}$. We define the **flag Dressian of** $\boldsymbol{M}$, denoted $FlDr(\boldsymbol{M})$, as the tropical prevariety of these equations, and establish several characterizations.

Theorem A. *Let* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ *be a sequence of valuated matroids such that its sequence of underlying matroids* $\boldsymbol{M} = (M_1, \dots, M_k)$ *is a flag matroid. Then the following are equivalent:*

(a) $\boldsymbol{\mu}$ *is a point on* $FlDr(\boldsymbol{M})$*, i.e. it satisfies tropical incidence-Plücker relations,*
(b) $\boldsymbol{\mu}$ *is a valuated flag matroid with underlying flag matroid* $\boldsymbol{M}$*,*
(c) $\boldsymbol{\mu}$ *induces a subdivision of the base polytope of* $\boldsymbol{M}$ *into base polytopes of flag matroids, and*
(d) the projective tropical linear spaces $\overline{\operatorname{trop}}(\mu_i)$ *form a flag* $\overline{\operatorname{trop}}(\mu_1) \subseteq \cdots \subseteq \overline{\operatorname{trop}}(\mu_k)$

The concepts appearing here are introduced in Definition 4.2.1 for (a), Definition 4.2.2 for (b), Definition 4.1.4 for (c), and Theorem B.(i) for (d).

Example 1.0.1. Consider the flag matroid $\boldsymbol{U}_{1,3;4} = (U_{1,4}, U_{3,4})$ consisting of uniform matroids on 4 elements. Its flag Dressian of $\boldsymbol{U}_{1,3;4}$, denoted $FlDr(\boldsymbol{U}_{1,3;4})$, is

(a) the tropical prevariety of the flag variety $Fl(1, 3; 4)$ embedded in $\mathbb{P}\left(\mathbb{k}^{\binom{4}{1}}\right) \times \mathbb{P}\left(\mathbb{k}^{\binom{4}{3}}\right)$ by the single equation $p_1p_{234} - p_2p_{134} + p_3p_{124} - p_4p_{123}$,
(b) the valuations on $U_{1,4}$ and $U_{3,4}$ making $(U_{1,4}, U_{3,4})$ a valuated flag matroid,

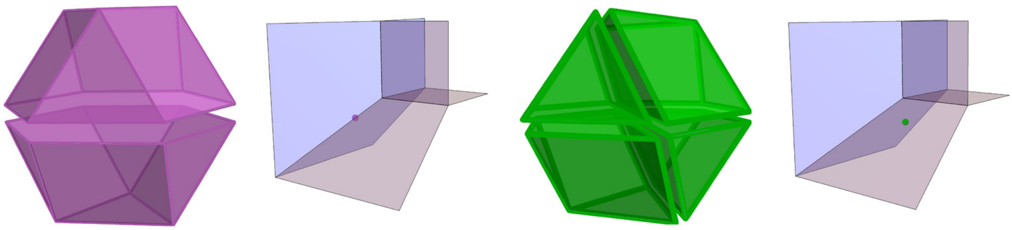

Fig. 1. Base polytope subdivisions and their tropical flags for $(U_{1,4}, U_{3,4})$.

(c) the space parameterizing weights that induce flag matroidal subdivisions of the base polytope of $(U_{1,4}, U_{3,4})$, which is the cuboctohedron $\text{Conv}\big(\sigma(1,1,2,0) \mid \sigma \in S_4\big) \subset \mathbb{R}^4$, and

(d) the space parameterizing the data of a (tropical) point on a tropical plane.

It is a pure simplicial fan in $\mathbb{R}^{\binom{4}{1}}/\mathbb{R}\mathbf{1} \times \mathbb{R}^{\binom{4}{3}}/\mathbb{R}\mathbf{1}$ of dimension 5 with a 3 dimensional lineality space. The 3 dimensional lineality space corresponds to the 3 dimensional freedom of selecting the location of the vertex of the tropical plane. Modulo the lineality space, it consists of 4 rays and 6 two-dimensional cones, as does a tropical plane in 3-space. Up to combinatorial equivalence, there are two types of nontrivial subdivisions of the cuboctohedron into smaller flag matroid polytopes, with the corresponding data of a point in a tropical plane as indicated in Fig. 1. For more examples, see Figs. 6, 7, and 8.

A fundamental tool in our proof of Theorem A is the notion of *projective tropical linear spaces*. The usual tropical linear spaces are defined only for matroids without loops, which is a harmless restriction in studying matroids, but not in studying flag matroids (Remark 4.1.6). In order to treat matroids with and without loops consistently, we introduce projective tropical linear spaces, which have previously appeared in the literature in various guises (Remark 3.2.6). We collect their characterizations, adding two new ones ((iii) and (v)) to this list (Theorem B). See §2.1 for terminology in projective tropical geometry, and §3.1 for terminology concerning valuated matroids.

Theorem B. *Let μ be a valuated matroid on a ground set $[n]$. Let $\ell \subseteq [n]$ be the set of loops of its underlying matroid. The following sets in the tropical projective space $\mathbb{P}(\mathbb{T}^{[n]})$ coincide:*

(i) The ***projective tropical linear space****, defined as*

$$\overline{\text{trop}}(\mu) := \bigcup_{\emptyset \subseteq S \subsetneq [n]} \Big(\text{trop}(\mu/_S) \times \{\infty\}^S\Big) \subset \mathbb{P}(\mathbb{T}^{[n]}),$$

(ii) The projective tropical prevariety of the valuated circuits of μ*, i.e.*

$$\bigcap_{\substack{\text{valuated}\\ \text{circuits } \mathbf{C}}} \left\{ \overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^{[n]}) \;\middle|\; \text{the minimum is achieved at least twice among } \{C_i + v_i\}_{i\in[n]} \right\},$$

(iii) The union of coloopless cells of the closure of the dual complex of μ^* *in* $\mathbb{P}(\mathbb{T}^{[n]})$*, i.e.*

$$\left\{ \overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^{[n]}) \mid \Delta_{\mu^*}^{\overline{\mathbf{u}}} \text{ is a base polytope of a coloopless matroid} \right\},$$

(iv) The tropical span of the valuated cocircuits of μ*, i.e.*

$$\left\{ \begin{array}{c} \text{the image in } \mathbb{P}(\mathbb{T}^{[n]}) \text{ of} \\ (a_1 \odot \mathbf{C}_{\mathbf{1}}^*) \oplus \cdots \oplus (a_l \odot \mathbf{C}_{\mathbf{m}}^*) \in \mathbb{T}^{[n]} \end{array} \;\middle|\; \begin{array}{c} \mathbf{C}_{\mathbf{i}}^* \in \mathbb{T}^{[n]} \text{ a valuated cocircuit of } \mu, \\ a_i \in \mathbb{R}, \quad \forall 1 \le i \le m \end{array} \right\},$$

(v) The closure of $\operatorname{trop}(\mu/_\ell) \times \{\infty\}^\ell$ *inside* $\mathbb{P}(\mathbb{T}^{[n]})$*.*

We apply Theorem A to establish a relation between Dressians and flag Dressians, and deduce a realizability result for valuated flag matroids. First, let us recall that the **Dressian** $Dr(r;n)$ is defined as the union of $Dr(M)$ over all matroids M of rank r on $[n]$. We define the **flag Dressian** $FlDr(r_1,\dots,r_k;n)$ as the union of $FlDr(\boldsymbol{M})$ over all flag matroids $\boldsymbol{M}$ of rank $(r_1,\dots,r_k)$ on $[n]$.

Theorem 5.1.2 & Theorem 5.2.1. *The natural isomorphism* $\mathbb{R}^{\binom{[n+1]}{r+1}} \xrightarrow{\sim} \mathbb{R}^{\binom{[n]}{r}} \times \mathbb{R}^{\binom{[n]}{r+1}}$ *induces a surjective map from a subset of* $Dr(r+1;n+1)$ *to* $FlDr(r,r+1;n)$*, whose fiber over each point is isomorphic to* $\mathbb{R}$*. As a consequence, every valuated flag matroid on a ground set of size* ≤ 5 *is realizable; the tropicalization of a flag variety* $Fl(r_1,\dots,r_k;n)$ *coincides with the flag Dressian* $FlDr(r_1,\dots,r_k;n)$ *whenever* $n \le 5$*.*

The tropicalization of a flag variety may differ from the flag Dressian when $n \ge 6$. See Example 5.2.4.

1.1. Previous works

In the unpublished manuscript [19], the author established (a) $\iff$ (d) in Theorem A for loopless matroids.[1] In [25, §4.3], the flag Dressian $FlDr(1,r;n)$ appeared implicitly as the universal family over $Dr(r;n)$. In [6], the authors computed the tropicalizations of the full flag varieties $Fl(1,2,3;4)$ and $Fl(1,2,3,4;5)$ in order to compute toric degenerations. In a related work [13], the authors identified some distinguished maximal cones in the tropicalizations of full flag varieties to study PBW-degenerations. In [14, §5] and [15, §6],

[1] We also note an error in the proof of Proposition 3 of [19]: there can be many more sets I satisfying $T \cap S \subseteq I \subseteq T \cup S$ than are considered. This nullifies his Lemma 2 and Theorem 2 on the convex hull of the base polytopes of a matroid quotient.

in order to describe the parameter space of matroids over valuation rings, the authors studied the space of valuated flag matroids (μ_1, μ_2) of ranks $(r, r+1)$ given a fixed valuated matroid μ_2. In [21], the authors studied the same space as tropicalized Fano schemes under the assumption that μ_2 is realizable.

1.2. Organization

In §2, we review projective tropical geometry, dual complexes, and M-convex functions. In §3, we review Dressians of matroids and prove Theorem B. In §4, after a review of flag matroids, we introduce flag Dressians and prove Theorem A. In §5, we apply Theorem A to relate Dressians and flag Dressians, and obtain a realizability result for valuated flag matroids.

1.3. Notation

For a finite set E, we write $\{\mathbf{e}_i \mid i \in E\}$ for the standard basis of $\mathbb{R}^E$, and denote $\mathbf{e}_S := \sum_{i\in S} \mathbf{e}_i$ for subsets $S \subseteq E$. All-one-vectors $(1, 1, \dots, 1)$ in appropriate coordinate spaces are denoted $\mathbf{1}$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product. We will follow the "min" convention for all polyhedral operations, such as taking faces and coherent subdivisions. Likewise, the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ is the min-plus algebra, with operations $a \odot b := a + b$ and $a \oplus b := \min\{a, b\}$. The topology on $\mathbb{T}$ is the standard one that makes $\mathbb{T}$ homeomorphic to $(-\infty, 0]$. The field $\Bbbk$ is algebraically closed, with a (possibly trivial) valuation $\mathrm{val} : \Bbbk \to \mathbb{T}$. Denote $[n] = \{1, \dots, n\}$. For $0 \le r \le n$, the set of r-subsets of $[n]$ is denoted $\binom{[n]}{r}$.

2. Preliminaries

In §2.1, we review tropical projective spaces and their products, since these are the ambient spaces of Dressians, flag Dressians, and projective tropical linear spaces. In §2.2, we review point configurations, dual complexes, and mixed subdivisions, since we will need these notions to study mixed subdivisions of base polytopes of flag matroids in §4.4. Our novel contribution here is Theorem 2.2.9 concerning coherence of mixed subdivisions. In §2.3, we review M-convex functions because the structure of their dual complexes will play a central role in the proof of Theorem B and Theorem 4.4.3. Theorem 2.3.8 explicitly describes the closures of their dual complexes inside tropical projective spaces. Let E be a finite set throughout.

2.1. Projective tropical geometry

We review projective tropical geometry, and explain the underlying algebraic geometry in the remarks. See [25, Chapter 6] for a detailed treatment.

Definition 2.1.1. Let $E = [n]$. The **tropical projective space** $\mathbb{P}(\mathbb{T}^E)$ is

$$\begin{aligned}\mathbb{P}(\mathbb{T}^E) :=& \big(\mathbb{T}^E \setminus \{(\infty,\ldots,\infty)\}\big)/\mathbb{R}\mathbf{1}\\ =& \{\mathbf{u} \in \mathbb{T}^E \mid \mathbf{u} \neq (\infty,\ldots,\infty)\}/\sim, \text{ where } \mathbf{u} \sim \mathbf{u}' \text{ if } \mathbf{u}' = \mathbf{u} + c\mathbf{1} \text{ for some } c \in \mathbb{R}.\end{aligned}$$

For $\mathbf{u} = (u_i)_{i\in E}$ in $\mathbb{R}^E$ or $\mathbb{T}^E$, write $\overline{\mathbf{u}}$ for its image in $\mathbb{R}^E/\mathbb{R}\mathbf{1}$ or $\mathbb{P}(\mathbb{T}^E)$. The **support** of $\mathbf{u}$ is $\operatorname{supp}(\mathbf{u}) := \{i \in E \mid u_i \neq \infty\}$. For a nonempty subset $S \subseteq E$, denote by

$$T_S := \{\overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^E) \mid \operatorname{supp}(\mathbf{u}) = S\},$$

the image of $\mathbb{R}^S \times \{\infty\}^{E\setminus S}$ in $\mathbb{P}(\mathbb{T}^E)$. The set $T_E = \mathbb{R}^E/\mathbb{R}\mathbf{1}$ is the *tropical projective torus*. By abuse of notation, we often identify $\mathbb{R}^S/\mathbb{R}\mathbf{1}$ with T_S, and $\mathbb{P}(\mathbb{T}^S)$ with the closure of T_S in $\mathbb{P}(\mathbb{T}^E)$. The subsets $\{T_S\}_{\emptyset\subsetneq S\subseteq E}$ partition $\mathbb{P}(\mathbb{T}^E)$.

Remark 2.1.2. The space $\mathbb{P}(\mathbb{T}^E)$ is the **tropicalization** of the projective space $\mathbb{P}(\Bbbk^E)$. The projective space $\mathbb{P}(\Bbbk^E)$ is a toric variety with the projective torus $(\Bbbk^*)^E/\Bbbk^*$. For each nonempty subset $S \subseteq E$, the torus orbit $O_S := \big((\Bbbk^*)^S \times \{0\}^{E\setminus S}\big)/\Bbbk^*$ in $\mathbb{P}(\Bbbk^E)$ tropicalizes to be the stratum T_S of $\mathbb{P}(\mathbb{T}^E)$. We often identify $(\Bbbk^*)^S/\Bbbk^*$ with O_S, and $\mathbb{P}(\Bbbk^S)$ with the closure $\overline{O_S} = \{\mathbf{y} \in \mathbb{P}(\Bbbk^E) \mid y_i = 0 \text{ if } i \notin S\}$. The orbits $\{O_S\}_{\emptyset\subsetneq S\subseteq E}$ partition the space $\mathbb{P}(\Bbbk^E)$. See [25, §6.2] or [24, §3.2] for tropicalizations of toric varieties in general.

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^E_{\geq 0}$. A **tropical polynomial** F with **support** $\operatorname{supp}(F) = \mathcal{A}$ is

$$F = \bigoplus_{\mathbf{v}\in\mathcal{A}} c_{\mathbf{v}} \odot \mathbf{x}^{\odot \mathbf{v}}.$$

It represents the function $\mathbb{T}^E \to \mathbb{T}$, $(x_i)_{i\in E} \mapsto \min_{\mathbf{v}\in\mathcal{A}}\{c_{\mathbf{v}} + \sum_{i\in E} v_i \cdot x_i\}$. Here, by convention $0 \odot \infty = 0$ and $a \odot \infty = \infty$ if $a \neq 0$. We always assume that a tropical polynomial F is homogeneous; that is, there exists $d \in \mathbb{Z}_{\geq 0}$ such that $d = \sum_{i\in E} v_i$ for all $\mathbf{v} \in \operatorname{supp}(F)$.

Definition 2.1.3. Let F be a tropical polynomial with support in $\mathbb{Z}^E_{\geq 0}$. We define the **projective tropical hypersurface of** F to be

$$\overline{\operatorname{trop}}(F) := \left\{\overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^E) \;\middle|\; \text{the minimum in } \Big\{c_{\mathbf{v}} + \sum_{i\in E} v_i \cdot u_i\Big\}_{\mathbf{v}\in\operatorname{supp}(F)} \text{ is achieved at least twice}\right\}.$$

When $\{c_{\mathbf{v}} + \sum_{i\in E} v_i \cdot u_i\}_{\mathbf{v}\in\operatorname{supp}(F)} = \{\infty\}$, by convention the minimum in $\{c_{\mathbf{v}} + \sum_{i\in E} v_i \cdot u_i\}_{\mathbf{v}\in\operatorname{supp}(F)}$ is said to be achieved at least twice even if $\operatorname{supp}(F)$ is a single element. The

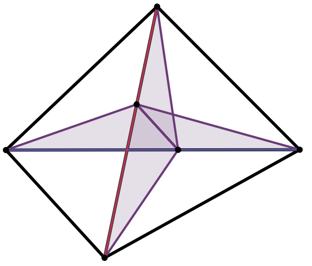

Fig. 2. The projective tropical hypersurface from Example 2.1.4. The red line is where $x_0 = x_3 = \infty$ and the blue line is where $x_1 = x_2 = \infty$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

set $\overline{\text{trop}}(F)$ is well-defined in $\mathbb{P}(\mathbb{T}^E)$ because one may pass from $\mathbb{T}^E$ to $\mathbb{P}(\mathbb{T}^E)$ by the homogeneity of F.

Example 2.1.4. Let $F = x_0 \odot x_1 \oplus x_0 \odot x_2 \oplus x_1 \odot x_3 \oplus x_2 \odot x_3 = \min(x_0 + x_1, x_0 + x_2, x_1 + x_3, x_2 + x_3)$. Then the projective tropical hypersurface $\overline{\text{trop}}(F) \subset \mathbb{P}(\mathbb{T}^{\{0,1,2,3\}})$ is as pictured in Fig. 2.

Suppose F is multi-homogeneous; that is, there is a partition $E = \bigsqcup_{j\in J} E_j$ and integers $\{d_j\}_{j\in J}$ such that $d_j = \sum_{i\in E_j} v_i$ for all $j \in J$ and $\mathbf{v} \in \text{supp}(F)$. Then the **multi-projective tropical hypersurface** of F is defined analogously as a subset of $\prod_{j\in J} \mathbb{P}(\mathbb{T}^{E_j})$.

Definition 2.1.5. If $F_1, \ldots, F_l$ are tropical polynomials with supports in $\mathbb{Z}^E_{\geq 0}$, we define their **projective tropical prevariety** to be

$$\overline{\text{trop}}(F_1, \ldots, F_l) := \bigcap_{i=1}^{l} \overline{\text{trop}}(F_i) \subset \mathbb{P}(\mathbb{T}^E).$$

If there is a common partition $S = \bigsqcup_{j\in J} E_j$ such that each F_i is multi-homogeneous in S, the **multi-projective tropical prevariety** is defined analogously as a subset of $\prod_{j\in J} \mathbb{P}(\mathbb{T}^{E_j})$. Multi-projective tropical prevarieties are closed.

In §3, Dressians and projective tropical linear spaces are defined as projective tropical prevarieties in $\mathbb{P}\big(\mathbb{T}^{\binom{[n]}{r}}\big)$ and $\mathbb{P}(\mathbb{T}^{[n]})$, respectively. In §4, flag Dressians will be defined as multi-projective tropical prevarieties in $\mathbb{P}\big(\mathbb{T}^{\binom{[n]}{r_1}}\big) \times \cdots \times \mathbb{P}\big(\mathbb{T}^{\binom{[n]}{r_k}}\big)$.

Remark 2.1.6. The intersection $\overline{\text{trop}}(F) \cap T_E \subset \mathbb{R}^E/\mathbb{R}\mathbf{1}$ is the usual **tropical hypersurface** of a tropical polynomial F, and is denoted $\text{trop}(F)$. More generally, for a nonempty subset $S \subseteq E$, consider the intersection $\overline{\text{trop}}(F) \cap T_S$ as a subset of $\mathbb{R}^S/\mathbb{R}\mathbf{1}$. Then it is equal to $\text{trop}(F_S)$, where F_S is the tropical polynomial obtained from F by keeping only the terms with exponent supports in $\mathbb{Z}^S_{\geq 0}$. The set $\overline{\text{trop}}(F)$ is the closure of $\text{trop}(F)$ in $\mathbb{P}(\mathbb{T}^E)$ when F has no nontrivial monomial factors, i.e. there is no $\mathbf{0} \neq \mathbf{v}' \in \mathbb{Z}^E_{\geq 0}$ such that $\mathbf{v} - \mathbf{v}' \in \mathbb{Z}^E_{\geq 0}$ for all $\mathbf{v} \in \text{supp}(F)$.

We now give the underlying algebraic geometry. See [25, §6.2] for proofs of statements.

Remark 2.1.7. Let val : $\Bbbk \to \mathbb{T}$ be a (possibly trivial) valuation on $\Bbbk$. Let $Y = V(f) \subset \mathbb{P}(\Bbbk^E)$ be a projective subvariety defined by a homogeneous polynomial

$$f = \sum_{\mathbf{v} \in \mathbb{Z}^E_{\geq 0}} c_{\mathbf{v}} \mathbf{x}^{\mathbf{v}} \in \Bbbk[x_i \mid i \in E] \qquad \text{(all but finitely many } c_{\mathbf{v}} \text{ are zero).}$$

The **projective tropicalization** of Y, denoted $\overline{\operatorname{trop}}(Y)$, is the set $\overline{\operatorname{trop}}(f^{\operatorname{trop}})$ where

$$f^{\operatorname{trop}} = \bigoplus_{\mathbf{v} \in \operatorname{supp}(f)} \operatorname{val}(c_{\mathbf{v}}) \odot \mathbf{x}^{\odot \mathbf{v}}.$$

If $f = 0$, then $\overline{\operatorname{trop}}(Y) = \mathbb{P}(\mathbb{T}^E)$. Recall the notation $O_S = \big((\Bbbk^*)^S \times \{0\}^S\big)/\Bbbk^*$ for a nonempty subset $S \subseteq E$. For $\mathring{Y} := Y \cap O_E$ a subvariety of the projective torus $(\Bbbk^*)^E/\Bbbk^*$, the usual tropical hypersurface $\operatorname{trop}(f^{\operatorname{trop}}) \subset \mathbb{R}^S/\mathbb{R}\mathbf{1}$ is the usual tropicalization of $\mathring{Y}$, denoted $\operatorname{trop}(\mathring{Y})$. More generally, consider $\mathring{Y}_S := Y \cap O_S$, regarded as a subvariety in $(\Bbbk^*)^S/\Bbbk^*$. Then $\operatorname{trop}(\mathring{Y}_S)$ is equal to $\overline{\operatorname{trop}}(Y) \cap T_S$, regarded as a subset of $\mathbb{R}^S/\mathbb{R}\mathbf{1}$. The set $\overline{\operatorname{trop}}(Y)$ is the closure of $\operatorname{trop}(\mathring{Y})$ in $\mathbb{P}(\mathbb{T}^E)$ when Y is the closure of $\mathring{Y}$ in $\mathbb{P}(\Bbbk^E)$.

Remark 2.1.8. Suppose now that $Y \subset \mathbb{P}(\Bbbk^E)$ is a projective subvariety defined by a homogeneous ideal $I \subset \Bbbk[x_i \mid i \in S \subset E]$. The projective tropicalization of Y is defined as

$$\overline{\operatorname{trop}}(Y) := \bigcap_{f \in I} \overline{\operatorname{trop}}(f^{\operatorname{trop}}),$$

which is a finite intersection for a suitable choice of generators of I, and hence $\overline{\operatorname{trop}}(Y)$ is a projective tropical prevariety. As in the hypersurface case (Remark 2.1.7), the usual tropicalization of $\mathring{Y} = Y \cap O_E$ is $\operatorname{trop}(\mathring{Y}) := \bigcap_{f \in I} \operatorname{trop}(f^{\operatorname{trop}})$. For a nonempty subset $S \subseteq E$, we have $\overline{\operatorname{trop}}(Y) \cap T_S = \operatorname{trop}(\mathring{Y}_S)$ where $\mathring{Y}_S := Y \cap O_S$. The set $\overline{\operatorname{trop}}(Y)$ is the closure of $\operatorname{trop}(\mathring{Y})$ when Y the closure of $\mathring{Y}$ in $\mathbb{P}(\Bbbk^S)$. If I is principal, generated by f, then $\overline{\operatorname{trop}}(Y) = \overline{\operatorname{trop}}(f^{\operatorname{trop}})$, but in general, the set $\overline{\operatorname{trop}}(Y)$ may not equal $\bigcap_{i=1}^{l} \overline{\operatorname{trop}}(f_i^{\operatorname{trop}})$ for an arbitrary generating set $\{f_1, \ldots, f_l\}$ of I.

2.2. Point configurations, dual complexes, and mixed subdivisions

We review point configurations, dual complexes of their coherent subdivisions, and mixed subdivisions. Point configurations, which generalize the notion of subsets of points, are necessary for discussing mixed subdivisions. See [9] for a detailed treatment of subdivisions of point configurations. Our novel contribution here is Theorem 2.2.9 concerning mixed coherent subdivisions.

Definition 2.2.1. Let $\mathcal{A}$ be a finite index set. A **point configuration** $(\mathcal{A}, \mathbf{a})$ **in** $\mathbb{R}^E$ is a map $\mathbf{a}_{(\cdot)} : \mathcal{A} \to \mathbb{R}^E$. In other words, it is a finite set of points $\{\mathbf{a}_i \in \mathbb{R}^E : i \in \mathcal{A}\}$ labeled by the set $\mathcal{A}$, where some points may have multiple labels.

We often abbreviate $(\mathcal{A}, \mathbf{a})$ to $\mathcal{A}$ when the map $\mathbf{a}$ is understood. For $A \subset \mathbb{R}^E$ a finite subset, we write A also for the point configuration $(A, s \mapsto s)$. For Q a lattice polytope in $\mathbb{R}^E$, we write Q for the point configuration of its lattice points. We write $\mathrm{Conv}(\mathcal{A})$ for the polytope $\mathrm{Conv}(\mathbf{a}_i \mid i \in \mathcal{A}) \subset \mathbb{R}^E$.

Assumption. The point configuration $\mathcal{A}$ is always integral, i.e. the image $\{\mathbf{a}_i\}_{i \in \mathcal{A}}$ lies in $\mathbb{Z}^E$, and it is homogeneous, i.e. there exists $d \in \mathbb{Z}$ such that $d = \langle \mathbf{e}_E, \mathbf{a}_i \rangle$ for all $i \in \mathcal{A}$, where $e_E = \sum_{i \in E} e_i$.

For a point configuration $(\mathcal{A}, \mathbf{a})$, a subset $\mathcal{A}' \subset \mathcal{A}$ defines a subconfiguration $(\mathcal{A}', \mathbf{a}|_{\mathcal{A}'})$. In particular, a vector $\overline{\mathbf{u}} \in \mathbb{R}^E / \mathbb{R}\mathbf{1}$ defines a subconfiguration $\mathcal{A}^{\overline{\mathbf{u}}}$ by

$$\mathcal{A}^{\overline{\mathbf{u}}} := \left\{ i \in \mathcal{A} \mid \langle \mathbf{u}, \mathbf{a}_i \rangle = \min_{j \in \mathcal{A}} \langle \mathbf{u}, \mathbf{a}_j \rangle \right\}.$$

This does not depend on the choice of the representative $\mathbf{u}$ of $\overline{\mathbf{u}}$ because $\mathcal{A}$ is homogeneous. A subconfiguration $\mathcal{F} \subset \mathcal{A}$ arising in this way is called a **face** of $\mathcal{A}$, denoted $\mathcal{F} \leq \mathcal{A}$.

Definition 2.2.2. A collection Δ of subconfigurations of $\mathcal{A}$ is a **subdivision of** $\mathcal{A}$ if

(1) for all $\mathcal{F} \in \Delta$ and $\mathcal{F}' \leq \mathcal{F}$, one has $\mathcal{F}' \in \Delta$, and
(2) the set of polytopes $\{\mathrm{Conv}(\mathcal{F})\}_{\mathcal{F} \in \Delta}$ forms a polyhedral subdivision of $\mathrm{Conv}(\mathcal{A})$. So, we have $\bigcup \{\mathrm{Conv}(\mathcal{F})\}_{\mathcal{F} \in \Delta} = \mathrm{Conv}(\mathcal{A})$, and for any $\mathcal{F}_1 \neq \mathcal{F}_2 \in \Delta$, the intersection $\mathrm{Conv}(\mathcal{F}_1) \cap \mathrm{Conv}(\mathcal{F}_2)$ of $\mathrm{Conv}(\mathcal{F}_1)$ and $\mathrm{Conv}(\mathcal{F}_2)$ is a proper face of each.

The elements $\mathcal{F} \in \Delta$ are called the **faces of** Δ. A subdivision of $\mathcal{A}$ is **tight** if every $i \in \mathcal{A}$ is in some face of the subdivision.

We will study subdivisions of $\mathcal{A}$ induced by weights. A **weight** on a point configuration $(\mathcal{A}, \mathbf{a})$ is a function $w : \mathcal{A} \to \mathbb{R}$. Like point configurations, we write $w^{\overline{\mathbf{u}}}$ for the restriction $w|_{\mathcal{A}^{\overline{\mathbf{u}}}}$ for $\overline{\mathbf{u}} \in \mathbb{R}^E / \mathbb{R}\mathbf{1}$. We set the following notations for the subdivision induced by a weighted point configuration w.

Notation 2.2.3.

- Δ_w is the **coherent subdivision** of $\mathcal{A}$, consisting of the lower faces of the point configuration $\Gamma_w(\mathcal{A}) := (\mathcal{A}, (\mathbf{a}, \nu))$ where $(\mathbf{a}, w) : i \mapsto (\mathbf{a}_i, w(i)) \in \mathbb{R}^E \times \mathbb{R}$ for $i \in \mathcal{A}$.

- $\Delta_w^{\overline{\mathbf{u}}}$ is the face of the coherent subdivision Δ_w corresponding to $\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1}$, defined by

$$\Delta_w^{\overline{\mathbf{u}}} := \Gamma_w(\mathcal{A})^{(\mathbf{u},1)} = \left\{ i \in \mathcal{A} \;\middle|\; \langle \mathbf{u}, \mathbf{a}_i \rangle + w(i) = \min_{j \in \mathcal{A}} \Big(\langle \mathbf{u}, \mathbf{a}_j \rangle + w(j) \Big) \right\}.$$

- Σ_w is the **dual complex** in $\mathbb{R}^E/\mathbb{R}\mathbf{1}$ of the coherent subdivision Δ_w. It is a polyhedral complex consisting of polyhedra corresponding to faces of Δ_w by

$$\{\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1} \mid \Delta_w^{\overline{\mathbf{u}}} \geq \mathcal{F}\} \longleftrightarrow \mathcal{F} \in \Delta_w. \tag{\dagger}$$

  The relative interiors $\{\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1} \mid \Delta_w^{\overline{\mathbf{u}}} = \mathcal{F}\}$ as $\mathcal{F}$ ranges over all faces of Δ_w partition $\mathbb{R}^E/\mathbb{R}\mathbf{1}$. We call the relative interiors the **cells** of the polyhedral complex Σ_w.

We note a useful observation.

Lemma 2.2.4. *Let w be a weight on a point configuration $(\mathcal{A}, \mathbf{a})$ in $\mathbb{R}^E$, and $\mathbf{u} \in \mathbb{R}^E$. Consider a new weight defined by $i \mapsto w(i) + \langle \mathbf{u}, \mathbf{a}_i \rangle$ for $i \in \mathcal{A}$. Then $\Delta_w^{\overline{\mathbf{u}}} = \Delta^{\mathbf{0}}_{w(\cdot)+\langle \mathbf{u}, \mathbf{a}_{(\cdot)} \rangle}$.*

In Corollary 2.3.10, we will extend the correspondence (†) to a correspondence between points of $\mathbb{P}(\mathbb{T}^E)$ and projections of faces of Δ_w for a particular family of weight configurations w. For now, we discuss mixed subdivisions of Minkowski sums, because Minkowski sums of base polytopes of matroids and their mixed subdivisions are the focus of §4.4.

Definition 2.2.5. Let $(\mathcal{A}_1, \mathbf{a_1}), \ldots, (\mathcal{A}_k, \mathbf{a_k})$ be point configurations in $\mathbb{R}^E$. Their **Minkowski sum**, denoted $\sum_{i=1}^k \mathcal{A}_i$, is a point configuration $(\mathcal{A}_1 \times \cdots \times \mathcal{A}_k, \sum_i \mathbf{a_i})$ defined by

$$\textstyle\sum_i \mathbf{a_i} : (j_1, \ldots, j_k) \mapsto \displaystyle\sum_{i=1}^k \mathbf{a_i}_{j_i} \qquad \text{for } (j_1, \ldots, j_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k.$$

If $w_1, \ldots, w_k$ are weights on $\mathcal{A}_1, \ldots, \mathcal{A}_k$ (respectively), then their Minkowski sum $\sum_i w_i$ is a weight on $\sum_i \mathcal{A}_i$ defined by $(j_1, \ldots, j_k) \mapsto \sum_{i=1}^k w_i(j_i)$.

We will repeatedly make use of the following observation.

Lemma 2.2.6. *Let $w = \sum_{i=1}^k w_i$ be a Minkowski sum of weight point configurations. Then for $\overline{\mathbf{u}} \in \mathbb{R}^S/\mathbb{R}\mathbf{1}$, we have $\Delta_w^{\overline{\mathbf{u}}} = \sum_{i=1}^k \Delta_{w_i}^{\overline{\mathbf{u}}}$.*

Definition 2.2.7. A subdivision Δ of a Minkowski sum $\sum_i \mathcal{A}_i$ is **mixed** if there exist subdivisions $\Delta_1, \ldots, \Delta_k$ of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ (respectively) such that each face $\mathcal{F} \in \Delta$ is a

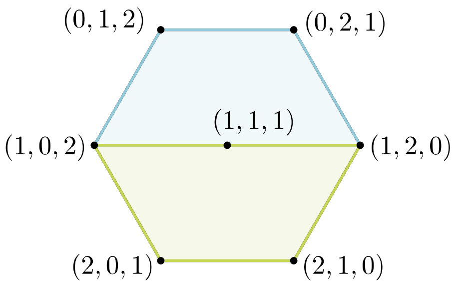

Fig. 3. The point configuration in Example 2.2.8.

Minkowski sum $\sum_{i=1}^{k} \mathcal{F}_i$ of faces $\mathcal{F}_i$ of Δ_i. If there exist weights $w_i : \mathcal{A}_i \to \mathbb{R}$ such that their Minkowski sum $w := \sum_i w_i$ satisfies $\Delta_w = \Delta$, we say that Δ is a **mixed coherent subdivision**, which is mixed by Lemma 2.2.6.

A priori, the terminology "mixed coherent subdivision" can be ambiguous: if a weight w on $\sum_i \mathcal{A}_i$ induces a coherent subdivision that is mixed, is w necessarily a Minkowski sum of weights? In general, the answer is no, as displayed in the following example.

Example 2.2.8. Let $A = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $B = \{(0,1,1),(1,0,1),(1,1,0)\}$ be two point configurations in $\mathbb{R}^3$. Their Minkowski sum $A+B$ is labeled by the nine elements of $A \times B$ and is the collection of points $\{(2,0,1),(2,1,0),(1,2,0),(0,2,1),(0,1,2),(1,0,2),(1,1,1)\}$. The first six points have unique labels, and the last point has three labels, because it arises in three ways: $(0,0,1)+(1,1,0) = (1,0,0)+(0,1,1) = (0,1,0)+(1,0,1)$. This is shown in Fig. 3.

Consider the following two weight vectors.

	201	210	120	021	012	102	001 + 110	100 + 011	010 + 101
w_1	0	0	0	1	1	0	0	0	17
w_2	0	0	0	1	1	0	0	1	0

Both w_1 and w_2 induce the subdivision indicated in Fig. 3, which is mixed. The first is not a Minkowski sum of weights on A and B, while the second is the Minkowski sum of weight vectors w_A and w_B where

$$w_A : \begin{cases} (1,0,0) \mapsto 0 \\ (0,1,0) \mapsto 0 \\ (0,0,1) \mapsto 0 \end{cases} \quad \text{and} \quad w_B : \begin{cases} (0,1,1) \mapsto 1 \\ (1,0,1) \mapsto 0 \\ (1,1,0) \mapsto 0. \end{cases}$$

This example shows that not every weight vector inducing a coherent subdivision that is mixed is a Minkowski sum of weights. However, there does exist a weight vector which is a Minkowski sum inducing the same subdivision.

We establish the following weaker statement about coherent mixed subdivisions. Together with Theorem 4.4.3, it will imply a strengthening of the equivalence (a) $\iff$ (c) in Theorem A (Corollary 4.4.5). We will only need Theorem 2.2.9 for the proof of Corollary 4.4.5.

Theorem 2.2.9. *Let $\mathcal{A} = \sum_{i=1}^{k} \mathcal{A}_i$ be a Minkowski sum of point configurations. For simplicity, let us assume that if $\dim \mathrm{Conv}(A_i) = 1$ then $|\mathcal{A}_i| = 2$. Suppose that a weight $w : \mathcal{A} \to \mathbb{R}$ induces a coherent subdivision Δ_w that is mixed. Then there exist weights $w_1, \dots, w_k$ on $\mathcal{A}_1, \dots, \mathcal{A}_k$ such that $\Delta_{\sum_{i=1}^{k} w_i} = \Delta_w$.*

We prepare with the following observation.

Lemma 2.2.10. *Let Q be a d-dimensional polytope, and let $\{Q_1, \dots, Q_m\}$ be the maximal (i.e. d-dimensional) faces of a polyhedral subdivision of Q. The graph on $[m]$ with edges (i, j) whenever $Q_i \cap Q_j$ has dimension $d - 1$ is connected. In particular, if $d \geq 2$, or if $d = 1$ and the subdivision is trivial, then the maximal cells $Q_1, \dots, Q_m$ are connected through dimension ≥ 1.*

Proof. For any two vertices $i, j \in [m]$, pick points p_i and p_j in the interior of Q_i and Q_j (respectively). Perturbing p_i and p_j if necessary, we have that the line segment $\overline{p_i p_j}$ meets faces of the polyhedral subdivision only of dimension $\geq d - 1$. □

Proof of Theorem 2.2.9. Let $\Delta_1, \dots, \Delta_k$ be subdivisions of $\mathcal{A}_1, \dots, \mathcal{A}_k$ (respectively) making up the mixed subdivision Δ_w. For each $\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1}$, the face $\Delta_w^{\overline{\mathbf{u}}}$ is a Minkowski sum $\sum_{i=1}^{s} \mathcal{F}_{i,\overline{\mathbf{u}}}$ where $\mathcal{F}_{i,\overline{\mathbf{u}}}$ is a face of Δ_i. For each $i = 1, \dots, k$, consider the partition of $\mathbb{R}^E$ by the equivalence relation $\overline{\mathbf{u}} \sim_i \overline{\mathbf{u}}' \iff \mathcal{F}_{i,\overline{\mathbf{u}}} = \mathcal{F}_{i,\overline{\mathbf{u}}'}$. This partition consists of components whose closures define a polyhedral complex Σ_i that coarsens the dual complex Σ_w. We claim that each Σ_i is a dual complex Σ_{w_i} for some weight $w_i : \mathcal{A}_i \to \mathbb{R}$ such that $\Delta_{w_i}^{\overline{\mathbf{u}}} = \mathcal{F}_{i,\overline{\mathbf{u}}}$ for all $\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1}$. We are then done by Lemma 2.2.6.

For the claim, fix $\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1}$ lying in a non-maximal cell of Σ_w. By [25, Lemma 3.3.6], the polyhedral complex $\mathrm{star}_{\Sigma_w}(\overline{\mathbf{u}})$ is the normal fan of the polytope $\mathrm{Conv}(\Delta_w^{\overline{\mathbf{u}}}) = \sum_{i=1}^{k} \mathrm{Conv}(\mathcal{F}_{i,\overline{\mathbf{u}}})$. Now fix any $1 \leq i \leq k$. By construction of Σ_i, the normal fan of $\mathrm{Conv}(\mathcal{F}_{i,\overline{\mathbf{u}}})$ is equal to $\mathrm{star}_{\Sigma_i}(\overline{\mathbf{u}})$. As $\mathrm{Conv}(\mathcal{F}_{i,\overline{\mathbf{u}}})$ is a lattice polytope by our running integrality assumption on point configurations, it follows that the union of non-maximal cells of Σ_i is a rational, pure, balanced, polyhedral complex of codimension 1. In other words, the complex Σ_i satisfies the condition of [25, Proposition 3.3.10], which states that there exists a weighted point configuration $\widetilde{w}_i : \widetilde{\mathcal{A}}_i \to \mathbb{R}$ with $\Sigma_i = \Sigma_{\widetilde{w}_i}$.

We now use $\widetilde{w}_i$ to define weights w_i' on $\mathscr{V}_i$, where $\mathscr{V}_i = \mathrm{Vert}(\Delta_i)$ is the set of elements of $\mathcal{A}_i$ that appear as vertices of the subdivision Δ_i. This will have the property that the induced coherent subdivision satisfies $\mathrm{Conv}(\Delta_{w_i'}^{\overline{\mathbf{u}}}) = \mathrm{Conv}(\mathcal{F}_{i,\overline{\mathbf{u}}})$ for all $\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1}$, so that w_i' naturally extends to a weight w_i on $\mathcal{A}_i$ satisfying $\Delta_{w_i}^{\overline{\mathbf{u}}} = \mathcal{F}_{i,\overline{\mathbf{u}}}$ for all $\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1}$.

By construction, the two polytopes $\mathrm{Conv}(\Delta_{\widetilde{w}_i}^{\overline{\mathbf{u}}})$ and $\mathrm{Conv}(\mathcal{F}_{i,\overline{\mathbf{u}}})$ are dilates of each other (up to translation) for every $\overline{\mathbf{u}} \in \mathbb{R}^E/\mathbb{R}\mathbf{1}$. Since we assumed that $|\mathcal{A}_i| = 1$ if

$\dim \mathrm{Conv}(\mathcal{A}_i) = 1$, by Lemma 2.2.10 the polyhedral subdivision from Δ_i is connected through dimension ≥ 1. Hence, the dilation factor is global; that is, (up to translation) the set $\mathscr{V}_i$ is a dilation of the set of vertices of $\Delta_{\widetilde{w}_i}$. Assign the weight w_i' on $\mathscr{V}_i$ via this dilation correspondence. □

Remark 2.2.11. Note that if w was already a Minkowski sum $w_1' + \cdots + w_k'$, then the constructed weights $\{w_i\}_{1\leq i\leq k}$ in the proof satisfy $\Sigma_{w_i} = \Sigma_{w_i'}$ for all $1 \leq i \leq k$.

2.3. M-convex functions and their dual complexes

We review M-convex functions, and establish Theorem 2.3.8 concerning the structure of their dual complexes.

Definition 2.3.1. A function $\mu : \mathbb{Z}^{[n]} \to \mathbb{T}$ is **M-convex** if for $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^{[n]}$ and $i \in [n]$ such that $a_i > b_i$, there exists $j \in [n]$ such that $a_j < b_j$ and

$$\mu(\mathbf{a}) + \mu(\mathbf{b}) \geq \mu(\mathbf{a} - \mathbf{e}_i + \mathbf{e}_j) + \mu(\mathbf{b} - \mathbf{e}_j + \mathbf{e}_i). \tag{M}$$

The set $\{\mathbf{v} \in \mathbb{Z}^{[n]} \mid \mu(\mathbf{v}) \neq \infty\}$ is the **effective domain** $\mathrm{dom}(\mu)$ of μ, and is assumed to be finite.

We view μ as a weighted point configuration $\mu : \mathrm{dom}(\mu) \to \mathbb{R}$. For M-convex functions μ_1 and μ_2, their Minkowski sum as weighted point configurations (not as functions) is denoted $\mu_1 + \mu_2$.

M-convex functions are studied in several contexts. For instance, they are foundational objects of discrete convex analysis [27]. We focus on their connection to generalized permutohedra.

Definition 2.3.2. A lattice polytope Q in $\mathbb{R}^{[n]}$ is a **generalized permutohedron** if every edge of Q is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in [n]$.

The definition implies that a generalized permutohedron is homogeneous as a point configuration.

Generalized permutohedra form a rich combinatorial class of lattice polytopes [11,30,1]. For example, base polytopes of matroids and flag matroids, which we discuss in §3.1 and §4.4, are examples of generalized permutohedra [17,5]. Generalized permutohedra are related to M-convex functions in the following way.

Theorem 2.3.3. *Let* $\mu : \mathbb{Z}^{[n]} \to \mathbb{T}$ *be a function with an effective domain* $\mathrm{dom}(\mu)$.

(1) If μ *takes only two values* $\{c, \infty\}$ *for some* $c \in \mathbb{R}$*, then* μ *is M-convex if and only if* $\mathrm{dom}(\mu)$ *is the set of lattice points of a generalized permutohedron.*

(2) More generally, μ is M-convex if and only if the subdivision Δ_μ of $\mathrm{dom}(\mu)$ *is tight and its faces are the sets of lattice points of generalized permutohedra.*

In particular, if μ is M-convex, the point configuration $\mathrm{dom}(\mu)$ *is a generalized permutohedron, and hence is homogeneous.*

Proof. The first statement (1) is [27, Theorem 4.15]. For the second statement (2), we note the following observations.

- Let $\mu : \mathbb{Z}^{[n]} \to \mathbb{T}$. For any $\mathbf{u} \in \mathbb{R}^{[n]}$, the function $\mu(\cdot) + \langle \mathbf{u}, \cdot \rangle : \mathbb{Z}^{[n]} \to \mathbb{T}$ defined by $\mathbf{v} \mapsto \mu(\mathbf{v}) + \langle \mathbf{u}, \mathbf{v} \rangle$ is M-convex if and only if μ is.
- Let $\mu : \mathbb{Z}^{[n]} \to \mathbb{T}$ be an M-convex function. Then the function defined by $\mathbf{v} \mapsto \min(\mu)$ if $\mu(\mathbf{v}) = \min(\mu)$ and $\mathbf{v} \mapsto \infty$ otherwise is also M-convex. In other words, by the first statement, the face $\Delta_\mu^{\mathbf{0}}$ is the set of lattice points of a generalized permutohedron.

The second statement now follows from the first by applying Lemma 2.2.4 to these observations. □

Let us now turn to the dual complex Σ_μ of μ. Its polyhedral cells are subsets of $\mathbb{R}^{[n]}/\mathbb{R}\mathbf{1}$. Consider the closures of these polyhedral cells inside $\mathbb{P}(\mathbb{T}^{[n]})$. For each nonempty proper subset $S \subsetneq [n]$, this defines a polyhedral complex structure on the boundary $T_S \subset \mathbb{P}(\mathbb{T}^{[n]})$. While these polyhedral complex structures can be difficult to describe for general weighted point configurations, for M-convex functions we give an explicit description in Theorem 2.3.8. This explicit description will be instrumental in our proof of Theorem B and Theorem 4.4.3. We first note the following general boundary behavior.

Lemma 2.3.4. *Let w be a weight on a point configuration $\mathcal{A}$ in $\mathbb{R}^{[n]}$. For a nonempty subset S, fix $\overline{\mathbf{u}'} \in T_S$. For a sufficiently small open neighborhood U of $\overline{\mathbf{u}'}$, one has $\Gamma_w(\mathcal{A})^{(\overline{\mathbf{u}},1)} = \Gamma_w(\mathcal{A}^{\mathbf{e}_{[n]\setminus S}})^{(\overline{\mathbf{u}},1)}$ for any $\overline{\mathbf{u}} \in U \cap T_{[n]}$. In other words, near T_S, the dual complex Σ_w is the same as the dual complex of the restriction of w to $\mathcal{A}^{\mathbf{e}_{[n]\setminus S}}$.*

Proof. Let $\mathbf{u} = (u_i)_{i\in S} \times (u_j)_{j\notin S}$. Shrinking U if necessary, we can make $\min\{u_i - u_j \mid i \in S,\ j \notin S\}$ arbitrarily large. Since $\mathcal{A}$ is finite and w is fixed, this means that for $i \in \mathcal{A}$ to minimize $\langle \mathbf{u}, \mathbf{a}_i\rangle + w(i)$, it must first minimize $\langle \mathbf{e}_{[n]\setminus S}, \mathbf{a}_i\rangle$. □

Next, we note that a property known as the *Hopf monoid structure* of generalized permutohedra extends to M-convex functions.

Notation 2.3.5. We need the following notations: For a lattice polytope $Q \subset \mathbb{R}^{[n]}$ and a nonempty subset $S \subseteq [n]$, the projection of the face $Q^{\mathbf{e}_{[n]\setminus S}}$ under $\mathbb{R}^{[n]} \to \mathbb{R}^S$ is denoted $Q|_S$, and the projection of $Q^{\mathbf{e}_{[n]\setminus S}}$ under $\mathbb{R}^{[n]} \to \mathbb{R}^{[n]\setminus S}$ is denoted $Q/_S$. Both are lattice polytopes, and we write $Q|_S \times Q/_S \subset \mathbb{R}^S \times \mathbb{R}^{[n]\setminus S} \simeq \mathbb{R}^{[n]}$ for their product, considered as a polytope in $\mathbb{R}^{[n]}$.

Our notation here differs from [1] by a complementation ($\mathbf{e}_{[n]\setminus S}$ instead of $\mathbf{e}_S$). Since $-\mathbf{e}_S$ and $\mathbf{e}_{[n]\setminus S}$ are equal as elements in $\mathbb{R}^{[n]}/\mathbb{R}\mathbf{1}$, the difference is due to our "min" convention for polyhedral operations instead of the "max" convention used in [1].

Theorem 2.3.6. *[1, Theorem 6.1] Let Q be a generalized permutohedron in $\mathbb{R}^{[n]}$, and $S \subseteq [n]$ be a nonempty subset. Then the polytopes $Q|_S$ and $Q/_S$ are generalized permutohedra in their respective spaces. Moreover, $Q^{\mathbf{e}_{[n]\setminus S}} = Q|_S \times Q/_S$ and in particular is a generalized permutohedron.*

This property of generalized permutohedra extends to M-convex functions. If w_1, w_2 are weights on $\mathcal{A}_1, \mathcal{A}_2$ in $\mathbb{R}^{S_1}, \mathbb{R}^{S_2}$ (respectively), let us write $w_1 \times w_2$ for the weight on $\mathcal{A}_1 \times \mathcal{A}_2$ in $\mathbb{R}^{S_1} \times \mathbb{R}^{S_2}$ defined by $w(i_1, i_2) := w(i_1) + w(i_2)$.

Lemma 2.3.7. *Let $\mu : \mathbb{Z}^{[n]} \to \mathbb{T}$ be M-convex, and write $Q = \operatorname{dom}(\mu)$. For a nonempty subset $S \subseteq [n]$, there exist weights $\mu|_S$ and $\mu/_S$ on $Q|_S$ and $Q/_S$ (respectively), each unique up to adding a constant globally, such that*

$$\mu^{\mathbf{e}_{[n]\setminus S}} = \mu|_S \times \mu/_S.$$

The weighted point configurations $\mu^{\mathbf{e}_{[n]\setminus S}}$, $\mu|_S$, and $\mu/_S$ are M-convex.

Proof. As Q is a generalized permutohedron, we have $Q^{\mathbf{e}_{[n]\setminus S}} = Q|_S \times Q/_S$. Thus, for the first statement, it suffices to show that for every choice of lattice points $p, p' \in Q|_S$ and $q, q' \in Q/_S$, one has $\mu(p,q) - \mu(p',q) = \mu(p,q') - \mu(p',q')$. Moreover, as $Q|_S$ and Q/S are both generalized permutohedra, it suffices to check in the case where $p - p' = \mathbf{e}_i - \mathbf{e}_{i'}$ and $q - q' = \mathbf{e}_j - \mathbf{e}_{j'}$ where $i, i' \in S$ and $j, j' \in [n] \setminus S$. Applying the defining property (M) of an M-convex function twice, once with $(\mathbf{a}, \mathbf{b}) = ((p,q),(p',q'))$ and again with $(\mathbf{a}, \mathbf{b}) = ((p,q'),(p',q))$, gives the desired equality.

For the second statement, applying the forward direction of Theorem 2.3.3.(2) to μ implies that the face $Q^{\mathbf{e}_{[n]\setminus S}}$ is subdivided into generalized permutohedra, which implies that both $Q|_S$ and $Q/_S$ are too. (If one of them has an edge not parallel to $\mathbf{e}_i - \mathbf{e}_j$, so does the product). The converse direction of Theorem 2.3.3.(2) then implies that $\mu^{\mathbf{e}_{[n]\setminus S}}$, $\mu|_S$, and $\mu/_S$ are M-convex. □

We are now ready to describe explicitly the closure of Σ_μ inside $\mathbb{P}(\mathbb{T}^{[n]})$.

Theorem 2.3.8. *Let μ be an M-convex function, considered as a weighted point configuration in $\mathbb{R}^{[n]}$. For a cell $\sigma \subset \mathbb{R}^{[n]}/\mathbb{R}\mathbf{1}$ of the dual complex Σ_μ, denote by $\overline{\sigma}$ its closure in $\mathbb{P}(\mathbb{T}^{[n]})$. For a nonempty subset $S \subseteq [n]$, we have $\{\overline{\sigma} \cap T_S \mid \sigma \in \Sigma_\mu\} = \Sigma_{\mu|_S}$, where T_S is identified with $\mathbb{R}^S/\mathbb{R}\mathbf{1}$.*

Proof. Lemma 2.3.4 implies that $\{\overline{\sigma} \cap T_S \mid \sigma \in \Sigma_\mu\} = \{\overline{\sigma} \cap T_S \mid \sigma \in \Sigma_{\mu^{e_{[n]\setminus S}}}\}$. Applying Lemma 2.3.7 then gives the desired equality. $\square$

Notation 2.3.9. Let μ be M-convex and $\overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^{[n]})$. We denote

$$\Delta_\mu^{\overline{\mathbf{u}}} := \Delta_{\mu|_S}^{\overline{\mathbf{u}'}},$$

where $S \subseteq [n]$ is the subset satisfying $\overline{\mathbf{u}} \in T_S$, so that $\mathbf{u} = \mathbf{u}' \times \infty^{[n]\setminus S}$ for some $\mathbf{u}' \in \mathbb{R}^S$.

Corollary 2.3.10. *Let μ be M-convex and $\emptyset \subsetneq S \subseteq [n]$. The correspondence (†) for $T_S = \mathbb{R}^S/\mathbb{R}\mathbf{1}$ gives*

$$\{\overline{\mathbf{u}} \in T_S \mid \Delta_\mu^{\overline{\mathbf{u}}} \geq \mathcal{F}\} \longleftrightarrow \mathcal{F} \in \Delta_{\mu|_S}.$$

This correspondence now extends to all of $\mathbb{P}(\mathbb{T}^{[n]})$: the set $\mathbb{P}(\mathbb{T}^{[n]})$ is partitioned by the relative interiors $\{\overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^{[n]}) \mid \Delta_\mu^{\overline{\mathbf{u}}} = \mathcal{F}\}$ as $\mathcal{F}$ ranges over all faces $\mathcal{F}$ of $\Delta_{\mu|_S}$ over all $\emptyset \subsetneq S \subseteq [n]$.

3. Dressians and projective tropical linear spaces

We review Dressians and valuated matroids in §3.1. Then, we introduce projective tropical linear spaces in §3.2, and prove Theorem B, which characterizes projective tropical linear spaces in many different ways. We assume familiarity with matroids. We point to [34,28] as references.

Notation 3.0.1. We adopt the following notations for a matroid M on a ground set $[n]$:

- $\mathcal{B}(M)$ is the set of bases, which we will often view as a point configuration $(\mathcal{B}(M), \mathbf{e})$, where $B \in \mathcal{B}(M) \subset \binom{[n]}{r}$ maps to $\mathbf{e}_B \in \mathbb{R}^{[n]}$,
- $\mathcal{C}(M)$ is the set of circuits.
- $\mathrm{rk}_M : 2^{[n]} \to \mathbb{Z}$ is the rank function.
- $Q(M) := \mathrm{Conv}(\mathbf{e}_B \mid B \in \mathcal{B}(M)) \subset \mathbb{R}^{[n]}$ the base polytope of M, which as a point configuration is identical to $(\mathcal{B}(M), \mathbf{e})$ because $Q(M)$ has no non-vertex lattice points.
- M^* is the dual matroid of M.
- $M|_S$ (resp. $M/_S$) is the restriction (resp. contraction) of M to (resp. by) a subset $S \subseteq [n]$.

As it is customary in matroid theory, we write $S \cup i$ to mean $S \cup \{i\}$ and $S \setminus i$ to mean $S \setminus \{i\}$ for a set S and an element i. We will often use the following.

Theorem 3.0.2. *[17] A lattice polytope contained in the cube* $\mathrm{Conv}(\mathbf{e}_S \mid \emptyset \subsetneq S \subseteq [n]) \subset \mathbb{R}^{[n]}$ *is a generalized permutohedron if and only if it is a base polytope of a matroid.*

3.1. Dressians and valuated matroids

We review Dressians and valuated matroids. As before, the underlying algebraic geometry is explained in the remarks.

Definition 3.1.1. For $0 \leq r \leq n$, the **tropical Grassmann-Plücker relations** are tropical polynomials in variables $\{P_I \mid I \in \binom{[n]}{r}\}$ defined as

$$\mathscr{P}^{\mathrm{trop}}_{r;n} := \left\{ (P_I \odot P_J) \oplus \bigoplus_{j \in J \setminus I} (P_{I \setminus i \cup j} \odot P_{J \setminus j \cup i}) \,\middle|\, I, J \in \binom{[n]}{r},\ |I \cap J| < r-1,\ i \in I \setminus J \right\}. \tag{GP}$$

The **Dressian (of rank** r **in** $[n]$**)** is the projective tropical prevariety of these tropical Grassmann-Plücker relations. That is, we define

$$Dr(r;n) := \overline{\mathrm{trop}}(\mathscr{P}^{\mathrm{trop}}_{r;n}) \subset \mathbb{P}(\mathbb{T}^{\binom{[n]}{r}}).$$

Points on Dressians were previously described in several ways [33,25,20]; we collect them together in Theorem 3.1.3. Let us first recall the definition of valuated matroids from [10].

Definition 3.1.2. Let M be a matroid of rank r on $[n]$. A **valuated matroid** with underlying matroid M is a function $\mu : \mathcal{B}(M) \to \mathbb{R}$ such that for every $B, B' \in \mathcal{B}(M)$ and $i \in B \setminus B'$ there exists $j \in B' \setminus B$ satisfying

$$\mu(B) + \mu(B') \geq \mu(B \setminus i \cup j) + \mu(B' \setminus j \cup i).$$

Theorem 3.1.3. *Let $\mu \in \mathbb{T}^{\binom{[n]}{r}}$. Then the following are equivalent:*

(a) The image $\overline{\mu} \in \mathbb{P}(\mathbb{T}^{\binom{[n]}{r}})$ is a point of $Dr(r;n)$.
(b) μ is a valuated matroid with an underlying matroid of rank r on $[n]$.
(c) When μ is regarded as a weight on $\{\mathbf{e}_I \in \mathbb{R}^{[n]} \mid \mu(I) \neq \infty\}$, the faces of Δ_μ are base polytopes of matroids.

Proof. Let us consider $\mu \in \mathbb{T}^{\binom{[n]}{r}}$ as a function $\mu : \mathbb{Z}^{[n]} \to \mathbb{T}$ where

$$\mathbf{v} \mapsto \begin{cases} \mu(I) & \text{if } \mathbf{v} = \mathbf{e}_I \text{ for some } I \in \binom{[n]}{r} \\ \infty & \text{otherwise.} \end{cases}$$

One can check from the definitions that μ is M-convex if and only if the image $\overline{\mu} \in \mathbb{P}(\mathbb{T}^{\binom{[n]}{r}})$ lies in $Dr(r;n)$. The equivalence of (a) and (b) now follows by comparing the definitions of M-convexity and valuated matroids. The equivalence of (b) and (c) follows from Theorem 2.3.3.(2) and Theorem 3.0.2. □

For a valuated matroid μ with underlying matroid M, we will freely switch between considering μ as a point on $\mathbb{T}^{\binom{[n]}{r}}$, as an M-convex function with effective domain $Q(M)$, and as a weight on the point configuration $(\mathcal{B}(M), \mathbf{e})$.

Definition 3.1.4. Recall the notation $T_S := \mathbb{R}^S/\mathbb{R}\mathbf{1} \times \{\infty\}^{E\setminus S} \subset \mathbb{P}(\mathbb{T}^E)$ for sets $S \subseteq E$. For M a matroid of rank r on $[n]$, the **Dressian of** M, denoted $Dr(M)$, is the intersection

$$Dr(M) = Dr(r;n) \cap T_{\mathcal{B}(M)} \subset \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r}}\right).$$

By Theorem 3.1.3, the set $Dr(M)$, which was introduced in [20], parametrizes valuated matroids with underlying matroid M, or equivalently, weights $\mathcal{B}(M) \to \mathbb{R}$ that induce coherent subdivisions of $Q(M)$ into base polytopes of matroids. By Remark 2.1.6, the set $Dr(M)$ is the usual tropical prevariety in $\mathbb{R}^{\mathcal{B}(M)}/\mathbb{R}\mathbf{1}$ of appropriately modified tropical Grassmann-Plücker relations.

Many aspects of matroids extend to valuated matroids. We will use the following notions.

Definition 3.1.5. Let μ be a valuated matroid of rank r on $[n]$ with underlying matroid M.

- For each $S \in \binom{[n]}{r+1}$, define an element $\mathbf{C}_\mu(S) \in \mathbb{T}^{[n]}$ by

$$C_\mu(S)_i := \begin{cases} \mu(S \setminus i) & i \in S \\ \infty & i \notin S. \end{cases}$$

  Then the set of **valuated circuits** of μ is defined as

$$\mathcal{C}(\mu) := \left\{ \mathbf{C}_\mu(S) \;\middle|\; S \in \binom{[n]}{r+1} \right\} \setminus \{(\infty, \dots, \infty)\}.$$

- The **dual** of μ is the valuated matroid μ^* defined by setting $\mu^*([n] \setminus I) := \mu(I)$ for $I \in \binom{[n]}{r}$.
- The **valuated cocircuits** of μ are defined as the circuits of μ^*. Explicitly, the set of valuated cocircuits is

$$\mathcal{C}^*(\mu) = \left\{ \mathbf{C}^*_\mu(S) \;\middle|\; S \in \binom{[n]}{r-1} \right\} \setminus \{(\infty, \dots, \infty)\},$$

  where

$$C^*_\mu(S)_i := \begin{cases} \mu(S \cup i) & i \notin S \\ \infty & i \in S. \end{cases}$$

- For a nonempty subset $S \subseteq [n]$, the **restriction to** S (resp. **contraction by** S) **of** μ is $\mu|_S$ (resp. $\mu/_S$), where $\mu|_S$ and $\mu/_S$ are as in Lemma 2.3.7. Lemma 2.3.7, combined with Theorem 3.0.2, implies that these are valuated matroids.

The following facts are easy to verify:

- The set $\{\mathrm{supp}(\mathbf{C}) \mid \mathbf{C} \in \mathcal{C}(\mu)\}$ is the set of circuits of M. (Recall the notation $\mathrm{supp}(\mathbf{C}) := \{i \in [n] \mid C_i \neq \infty\}$ for $\mathbf{C} \in \mathbb{T}^{[n]}$).
- The underlying matroid of μ^* is M^*, and $(\mu^*)^* = \mu$.
- The underlying matroid of $\mu|_S$ (resp. $\mu/_S$) is $M|_S$ (resp. $M/_S$).

Lastly, we will need the following description of the valuated circuits in terms of restrictions and contractions. It is a consequence of [4, Theorem 3.29 & Corollary 4.10.(1)].

Theorem 3.1.6. *Let μ be a valuated matroid of rank r on $[n]$, and $S \subseteq [n]$ a nonempty subset. Then we have*

$$\mathcal{C}(\mu|_S) = \{\mathbf{C} \in \mathcal{C}(\mu) \mid \mathrm{supp}(\mathbf{C}) \subseteq S\}, \text{ and}$$
$$\mathcal{C}(\mu/_S) = \{\text{points of } \mathbb{T}^{[n]\setminus S} \text{ of minimal support among } \{(C_i)_{i\in[n]\setminus S} \mid \mathbf{C} \in \mathcal{C}(M)\}\}.$$

Moreover, the two operations are dual to each other, in the sense that $(\mu/_S)^* = \mu^*|_{([n]\setminus S)}$.

The underlying geometry behind the definition of Dressians follows.

Remark 3.1.7. See [16, §9] for Plücker embeddings, and see Remarks 2.1.7 and 2.1.8 for tropicalizations of projective subvarieties. The Grassmannian $Gr(r;n)$, whose points are r-dimensional subspaces of $\Bbbk^{[n]}$, is embedded in $\mathbb{P}(\Bbbk^{\binom{[n]}{r}})$ by the Plücker embedding. When $\mathrm{char}\,\Bbbk = 0$, the defining ideal is generated by the Grassmann-Plücker relations:

$$\mathscr{P}_{r;n} := \left\{ -P_I P_J + \sum_{j\in J\setminus I} \mathrm{sign}(i,j,I,J) P_{I\setminus i\cup j} P_{J\setminus j\cup i} = 0 \;\middle|\; I,J \in \binom{[n]}{r},\; i \in I\setminus J \right\} \tag{1}$$

where $\mathrm{sign}(i,j,I,J) := (-1)^{\#\{a\in I \mid \min(i,j)<a<\max(i,j)\}+\#\{b\in J \mid \min(i,j)<b<\max(i,j)\}}$. When $\mathrm{char}\,\Bbbk > 0$ they generate the ideal up to radical. The tropical Grassmann-Plücker relations are tropicalizations of these polynomials. That is, we have $\mathscr{P}^{\mathrm{trop}}_{r;n} = \{f^{\mathrm{trop}} \mid f \in \mathscr{P}_{r;n}\}$. The projective tropicalization $\overline{\mathrm{trop}}(Gr(r;n))$ is thus a subset of $Dr(r;n)$.

The inclusion $\overline{\mathrm{trop}}(Gr(r;n)) \subseteq Dr(r;n)$ is often strict, precisely because not all valuated matroids are realizable in the following sense: For a linear subspace $L \in Gr(r;n)$, let $(P_I(L))_{I\in\binom{[n]}{r}} \in \mathbb{P}(\Bbbk^{\binom{[n]}{r}})$ be its coordinates in the Plücker embedding. Then the function $I \mapsto \mathrm{val}(P_I(L))\ \forall I \in \binom{[n]}{r}$ is a valuated matroid, denoted $\mu(L)$, whose underlying matroid is denoted $M(L)$. Valuated matroids arising in this way are said to be **realizable (over** $\Bbbk$**)**. The points of $\overline{\mathrm{trop}}(Gr(r;n))$ are exactly the valuated matroids realizable over $\Bbbk$. When $r \geq 3$ and $n \geq 7$, there are non-realizable valuated matroids, and hence, in these cases the inclusion $\overline{\text{trop}}(Gr(r;n)) \subseteq Dr(r;n)$ is strict. Realizability can fail in many ways. For example, there are valuated matroids where every cell of the induced subdivision is a realizable matroid, but the valuated matroid is not realizable [32].

3.2. The many faces of projective tropical linear spaces

We introduce projective tropical linear spaces, and prove Theorem B, which characterizes them in many different ways. We start by reviewing usual tropical linear spaces.

Proposition 3.2.1. *[25, Lemma 4.4.7] Let μ be a valuated matroid on $[n]$. The following two subsets of $\mathbb{R}^{[n]}/\mathbb{R}\mathbf{1}$ coincide:*

(1) The set

$$\bigcap_{\mathbf{C}\in\mathcal{C}(\mu)} \left\{ \overline{\mathbf{u}} \in \mathbb{R}^{[n]}/\mathbb{R}\mathbf{1} \;\middle|\; \text{the minimum in } \{C_i + u_i\}_{i\in\text{supp}(\mathbf{C})} \text{ is achieved at least twice} \right\},$$

which is the usual tropical prevariety of the valuated circuits of μ (Remark 2.1.6).

(2) With μ regarded as a weighted point configuration, the set

$$\{\overline{\mathbf{u}} \in \mathbb{R}^{[n]}/\mathbb{R}\mathbf{1} \mid \Delta_\mu^{-\overline{\mathbf{u}}} \text{ is a base polytope of a loopless matroid}\},$$

which is the union of "loopless cells" of the dual complex Σ_μ in $\mathbb{R}^{[n]}/\mathbb{R}\mathbf{1}$.

Definition 3.2.2. Let μ be a valuated matroid of rank r on $[n]$, and let M be its underlying matroid. The subset of $\mathbb{R}^{[n]}/\mathbb{R}\mathbf{1}$ in the previous proposition is defined as the **tropical linear space of** μ, denoted $\text{trop}(\mu)$. Note that if M has loops, then $\text{trop}(\mu) = \emptyset$.

We will extend Proposition 3.2.1 to projective tropical linear spaces. Since projective tropical linear spaces are subsets of $\mathbb{P}(\mathbb{T}^{[n]})$, the negative sign $-\overline{\mathbf{u}}$ in Proposition 3.2.1.(2) can be problematic because $-\infty$ is not an element of $\mathbb{T}$. We will thus use the following reformulation:

Lemma 3.2.3. *Let μ be a valuated matroid, and μ^* its dual. Then we have*

$$\text{trop}(\mu) = \{\overline{\mathbf{u}} \in \mathbb{R}^{[n]}/\mathbb{R}\mathbf{1} \mid \Delta_{\mu^*}^{\overline{\mathbf{u}}} \text{ is a base polytope of a coloopless matroid}\},$$

which is the union of "coloopless cells" of the dual complex Σ_{μ^*}.

Proof. For a weight w on a point configuration $(\mathcal{A}, \mathbf{a})$, let us write w^{op} for the weight on the point configuration $(\mathcal{A}, -\mathbf{a})$, defined by $w^{op}(i) := w(i)\ \forall i \in \mathcal{A}$. It is easy to verify that $\Delta_w^{-\overline{\mathbf{u}}} = \Delta_{w^{op}}^{\overline{\mathbf{u}}}$ as subsets of $\mathcal{A}$. Now, if M is the underlying matroid of μ, then $Q(M^*) = -Q(M) + \mathbf{1}$, so that $\mu^* = \mu^{op}$. The lemma now follows from the description of

$\mathrm{trop}(\mu)$ in Proposition 3.2.1.(2), since a matroid is loopless if and only if its dual matroid is coloopless. □

The following remark explains the geometry behind tropical linear spaces via tropicalizations of subvarieties (see Remark 2.1.8). It also motivates our definition of projective tropical linear spaces.

Remark 3.2.4. Recall from Remark 3.1.7 that a linear subspace $L \subset \Bbbk^{[n]}$ defines a valuated matroid $\mu(L)$. Let us consider L as a linear projective subvariety of $\mathbb{P}(\Bbbk^{[n]})$, and write $\mathring{L} := L \cap (\Bbbk^*)^{[n]}/\Bbbk^*$. Then the usual tropicalization $\mathrm{trop}(\mathring{L})$ of $\mathring{L}$ is the tropical linear space $\mathrm{trop}(\mu(L))$. When L is contained in a coordinate hyperplane, or equivalently, when the matroid $M(L)$ has a loop, the intersection $\mathring{L}$ is empty, and hence $\mathrm{trop}(\mathring{L})$ is empty, as is $\mathrm{trop}(\mu(L))$. See [25, §4.3] for a more details on tropicalizations of linear subvarieties in a torus $(\Bbbk^*)^{[n]}/\Bbbk^*$.

Now consider the projective tropicalization $\overline{\mathrm{trop}}(L)$. For each nonempty subset $S \subseteq [n]$, the torus orbit O_S intersects L to give another (possibly empty) linear subvariety of $(\Bbbk^*)^S/\Bbbk^*$, denoted $\mathring{L}_S$. Similarly, let $L_S := L \cap \overline{O_S}$, considered as a subvariety of $\mathbb{P}(\Bbbk^S)$. Then the valuated matroid $\mu(L_S)$ is the contraction $\mu(L)/_{([n]\setminus S)}$. We thus have $\overline{\mathrm{trop}}(L) \cap T_S = \mathrm{trop}(\mathring{L}_S) = \mathrm{trop}(\mu(L)/_{([n]\setminus S)})$. This motivates our definition of projective tropical linear spaces.

Definition 3.2.5. Let μ be a valuated matroid on $[n]$. The **projective tropical linear space** $\overline{\mathrm{trop}}(\mu)$ of μ is a subset of $\mathbb{P}(\mathbb{T}^{[n]})$ defined by setting

$$\overline{\mathrm{trop}}(\mu) \cap T_{[n]\setminus S} := \mathrm{trop}(\mu/_S) \times \{\infty\}^S$$

for each $\emptyset \subseteq S \subsetneq [n]$.

Projective tropical linear spaces have previously appeared in various forms (see Remark 3.2.6). Theorem B, reproduced below, unifies them and adds two new characterizations ((iii) and (v)).

Theorem B. *Let μ be a valuated matroid on a ground set $[n]$. Let $\ell \subseteq [n]$ be the set of loops of its underlying matroid M. The following sets in the tropical projective space $\mathbb{P}(\mathbb{T}^E)$ coincide:*

(i) The projective tropical linear space of μ, i.e.

$$\overline{\mathrm{trop}}(\mu) := \bigcup_{\emptyset \subseteq S \subsetneq [n]} \left(\mathrm{trop}(\mu/_S) \times \{\infty\}^S\right) \subset \mathbb{P}(\mathbb{T}^E),$$

(ii) The projective tropical prevariety of the valuated circuits of μ, i.e.

$$\bigcap_{\substack{\text{valuated}\\ \text{circuits } \mathbf{C}}} \left\{ \overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^{[n]}) \;\middle|\; \text{the minimum is achieved at least twice among } \{C_i + v_i\}_{i \in [n]} \right\},$$

(iii) The union of coloopless cells of the closure of the dual complex of μ^* in $\mathbb{P}(\mathbb{T}^{[n]})$, i.e.

$$\left\{ \overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^{[n]}) \mid \Delta_{\mu^*}^{\overline{\mathbf{u}}} \text{ is a base polytope of a coloopless matroid} \right\},$$

(iv) The tropical span of the valuated cocircuits of μ, i.e.

$$\left\{ \begin{array}{c} \text{the image in } \mathbb{P}(\mathbb{T}^E) \text{ of} \\ (a_1 \odot \mathbf{C}_\mathbf{1}^*) \oplus \cdots \oplus (a_l \odot \mathbf{C}_\mathbf{m}^*) \in \mathbb{T}^E \end{array} \;\middle|\; \begin{array}{c} \mathbf{C}_\mathbf{i}^* \in \mathbb{T}^E \text{ a valuated cocircuit of } \mu, \\ a_i \in \mathbb{R}, \quad \forall 1 \le i \le m \end{array} \right\},$$

(v) The closure of $\operatorname{trop}(\mu/_\ell) \times \{\infty\}^\ell$ *inside* $\mathbb{P}(\mathbb{T}^{[n]})$.

Remark 3.2.6. For ordinary matroids (not valuated), the description (i) appeared in [31, Definition 2.20]. The authors of [24] also considered the description (i), and characterized $\overline{\operatorname{trop}}(\mu)$ as a tropical cycle of projective degree 1 [24, Remark 7.4.15]. In the language of hyperfields (see [4]), the description (ii) says that a projective tropical linear space is the set of covectors of a matroid over the tropical hyperfield. This characterization appeared in [26], along with the proof of (ii)=(iv) [26, Theorem 3.8], and was generalized to perfect tracts in [2].

Proof of Theorem B. The equality (ii) = (iv) is [26, Theorem 3.8]. Recalling from Theorem 3.1.6 that the dual of $\mu/_S$ is $\mu^*|_{([n]\setminus S)}$, combining Theorem 2.3.8 with Lemma 3.2.3 then implies (i) = (iii). We now show (iii) $\subseteq$ (v) $\subseteq$ (ii) $\subseteq$ (i).

For all subsets $S \subseteq [n]$ such that $S \not\supset \ell$, the matroid M/S has loops, and so the intersection of the set (iii) with $T_{[n]\setminus S}$ is empty (since (iii) = (i)). The same is true for the set (v). Hence, for showing (iii) $\subseteq$ (v) we may assume that M is loopless. In this case, both sets are $\operatorname{trop}(\mu)$ on $T_{[n]}$ by Lemma 3.2.3. Now, suppose $\mathbf{u} \times \{\infty\}^{[n]\setminus S} \in T_S$ is in the set (iii). We need to show that it is in the closure of $\operatorname{trop}(\mu)$. Since M is loopless, so is $M|_{[n]\setminus S}$, and hence $\operatorname{trop}(\mu|_{[n]\setminus S})$ is nonempty. Let $\overline{\mathbf{u}'} \in \operatorname{trop}(\mu|_{[n]\setminus S})$ and pick its representative $\mathbf{u}' \in \mathbb{R}^{[n]\setminus S}$ to have all positive coordinates. For a point $\mathbf{u} \times c\mathbf{u}' \in \mathbb{R}^{[n]}$, if $c > 0$ is sufficiently high (equivalently, if $\mathbf{u} \times c\mathbf{u}'$ is in a small enough open neighborhood of $\mathbf{u} \times \{\infty\}^{[n]\setminus S}$), Lemma 2.3.4 implies that $\Delta_{\mu^*}^{\mathbf{u}\times c\mathbf{u}'} = \Delta_w^{\mathbf{u}\times c\mathbf{u}'}$, where $w = (\mu^*)^{\mathbf{e}_{[n]\setminus S}}$. Then by Lemma 2.3.7, we have $w = \mu^*|_S \times \mu^*/_S$, so that $\Delta_w^{\mathbf{u}\times c\mathbf{u}'} = \Delta_{\mu^*|_S}^{\mathbf{u}} \times \Delta_{\mu^*/_S}^{c\mathbf{u}'}$. By assumption the matroids of $\Delta_{\mu^*|_S}^{\mathbf{u}}$ and $\Delta_{\mu^*/_S}^{c\mathbf{u}'}$ are both coloopless. We thus conclude that $\mathbf{u} \times c\mathbf{u}'$ is in $\operatorname{trop}(\mu)$ for all sufficiently large $c > 0$, and hence the point $\mathbf{u} \times \{\infty\}^{[n]\setminus S}$ is in the closure of $\operatorname{trop}(\mu)$.

For (v) $\subseteq$ (ii), we may again assume M loopless, since the fact that a loop is a circuit implies that (ii) is contained in the closure of $T_{[n]\setminus \ell}$. In this case, both sets are $\operatorname{trop}(\mu)$

on $T_{[n]}$ by Proposition 3.2.1.(1). Since projective tropical prevarieties are closed, we thus have (v) $\subseteq$ (ii).

Lastly, for any proper subset $\emptyset \subseteq S \subsetneq [n]$, consider the intersection of the set (ii) with $T_{[n]\setminus S}$. In other words, for each valuated circuit $\mathbf{C}$ defining a tropical polynomial $\bigoplus_{i\in[n]} C_i \odot x_i$, we give ignore all C_i with $i \in S$ since $x_i = \infty$. Thus, the description of the valuated circuits of the contraction $\mu/_S$ in Theorem 3.1.6, combined with Proposition 3.2.1, imply (ii) $\subseteq$ (i). □

4. Valuated flag matroids and flag Dressians

We now introduce flag Dressians and valuated flag matroids, and prove Theorem A. We review flag matroids in §4.1. In §4.2 we define flag Dressians and valuated flag matroids, and show (a) $\iff$ (b), which is mostly definitional (Proposition 4.2.3). In §4.3, we prove (b) $\iff$ (d) (Theorem 4.3.1). In §4.4, we define flag matroidal subdivisions and prove (b) $\implies$ (c) (Theorem 4.4.2) and (c) $\implies$ (d) (Theorem 4.4.3). We give an extended illustration of Theorem A in Example 4.4.6.

4.1. Flag matroids

Flag matroids are defined through matroid quotients.

Definition 4.1.1. Let M and N be matroids on a common ground set $[n]$. We say that M is a **(matroid) quotient** of N, denoted $M \twoheadleftarrow N$, if any of the following equivalent conditions are satisfied [7, Proposition 7.4.7]:

(1) For all $A \subseteq B \subseteq [n]$, we have $\mathrm{rk}_M(B) - \mathrm{rk}_M(A) \leq \mathrm{rk}_N(B) - \mathrm{rk}_N(A)$,
(2) each circuit of N is a union of circuits of M,
(3) there exist a matroid $\widetilde{M}$ on $[n] \sqcup [n']$ such that $M = \widetilde{M}/_{[n']}$ and $N = \widetilde{M}\setminus_{[n']}$,
(4) N^* is a quotient of M^*.

A sequence $\boldsymbol{M} = (M_1, \ldots, M_k)$ of matroids on $[n]$ is a **flag matroid** if $M_i \twoheadleftarrow M_j$ for every $1 \leq i < j \leq k$. The **rank** of $\boldsymbol{M}$ is the sequence of its constituent matroids $(\mathrm{rk}(M_1), \ldots, \mathrm{rk}(M_k))$.

The following example gives the geometric origin of the terminology.

Example 4.1.2 *(Realizable quotients and flag matroids).* Let $L'^* \twoheadleftarrow L^* \twoheadleftarrow \Bbbk^{[n]}$ be quotients of linear spaces. Equivalently, we have an inclusion of linear subspaces $L' \subseteq L \subseteq \Bbbk^{[n]}$. Then, the matroids of L' and L, which we denote $M(L')$ and $M(L)$ (Remark 3.1.7), form a matroid quotient $M(L') \twoheadleftarrow M(L)$. Matroid quotients arising in this way are said to be **realizable (over** $\Bbbk$**)**. Similarly, a flag of linear subspaces $\boldsymbol{L} = L_1 \subseteq L_2 \subseteq \cdots \subseteq L_k \subseteq \Bbbk^{[n]}$ defines a flag matroid $\boldsymbol{M}(\boldsymbol{L}) = (M(L_1), \ldots, M(L_k))$. Such flag matroids are **realizable (over** $\Bbbk$**)**.

Remark 4.1.3. A quotient $M \twoheadleftarrow N$ can fail to be realizable even if M and N are realizable over the field. For a concrete example, see [5, §1.7.5. Example 7].

Definition 4.1.4. Given a flag matroid $\boldsymbol{M} = (M_1, \dots, M_k)$ on $[n]$, its **base configuration** $\mathcal{B}(\boldsymbol{M})$ is a point configuration obtained as the Minkowski sum of the bases of its constituent matroids. That is, $\mathcal{B}(\boldsymbol{M}) := \mathcal{B}(M_1) + \cdots + \mathcal{B}(M_k) = (\mathcal{B}(M_1) \times \cdots \times \mathcal{B}(M_k), \mathbf{e})$, where

$$(B_1, \dots, B_k) \in \mathcal{B}(M_1) \times \cdots \times \mathcal{B}(M_k) \mapsto \mathbf{e}_{B_1} + \cdots + \mathbf{e}_{B_k} \in \mathbb{R}^{[n]}.$$

The **base polytope** $Q(\boldsymbol{M})$ of $\boldsymbol{M}$ is the convex hull of the image of the base configuration, i.e.

$$Q(\boldsymbol{M}) := \operatorname{Conv}(\mathbf{e}_{B_1} + \cdots + \mathbf{e}_{B_k} \mid (B_1, \dots, B_s) \in \mathcal{B}(M_1) \times \cdots \times \mathcal{B}(M_k)) \subset \mathbb{R}^{[n]}.$$

Properties of matroid polytopes found in Theorem 3.0.2 extend to flag matroid base polytopes.

Theorem 4.1.5.

(1) [5, Theorem 1.11.1] A lattice polytope $Q \subset \mathbb{R}^{[n]}$ is a base polytope of a flag matroid of rank $(r_1, \dots, r_k)$ if and only if it is a generalized permutohedron and its vertices are a subset of the orbit of $\mathbf{e}_{\{1,2,\dots,r_1\}} + \cdots + \mathbf{e}_{\{1,2,\dots,r_k\}}$ under the permutation group S_n.

(2) For a flag matroid $\boldsymbol{M} = (M_1, \dots, M_k)$ on $[n]$, and a subset $S \subseteq [n]$, the sequences $\boldsymbol{M}|_S := (M_1|_S, \dots, M_k|_S)$ and $\boldsymbol{M}/_S := (M_1/_S, \dots, M_k/_S)$ are flag matroids, and the face $Q(\boldsymbol{M})^{\mathbf{e}_{[n]\setminus S}}$ is the product $Q(\boldsymbol{M}|_S) \times Q(\boldsymbol{M}/_S)$.

Proof. The first part of statement (2) is checked directly from the description of matroid quotients by rank functions. The second part of (2) follows by Lemma 2.2.6 and Theorem 2.3.6. □

Remark 4.1.6. Restricting to only loopless matroids is harmless in studying matroids because the only data lost by deleting the loops of a matroid is the number of loops: if ℓ is the set of loops of a matroid M, then $M = M\setminus_\ell \oplus U_{0,\ell}$, so one easily recovers M from $M \setminus \ell$ and $|\ell|$. However, for a flag matroid $\boldsymbol{M} = (M_1, \dots, M_k)$ on $[n]$, an element $e \in [n]$ can be a loop in some but not all of the matroids $M_1, \dots, M_k$, and in such cases one cannot always recover $\boldsymbol{M}$ from $\boldsymbol{M}\setminus_e = (M_1\setminus_e, \dots, M_k\setminus_e)$ and $\boldsymbol{M}|_e = (M_1|_e, \dots, M_k|_e)$. So, it is necessary for us to develop the theory for matroids with loops in the flag setting.

Remark 4.1.7. According to [18,5], flag matroids are exactly the Coxeter matroids of type A. Coxeter matroids in general are defined by modifying Theorem 4.1.5.(1) with the notion of *Coxeter generalized permutohedra*. See [3] for a modern treatment of Coxeter generalized permutohedra and their connection to combinatorics.

4.2. Definition of flag Dressians and valuated flag matroids

We now extend Dressians and valuated matroids, described in Section 3.1, to the setting of flag matroids.

Definition 4.2.1. Let $0 \leq r \leq s \leq n$. The **tropical incidence-Plücker relations** are tropical polynomials in variables $\{P_I \mid I \in \binom{[n]}{r}\} \cup \{P_J \mid J \in \binom{[n]}{s}\}$ defined as

$$\mathscr{P}^{\text{trop}}_{r,s;n} = \left\{ \bigoplus_{j' \in J' \setminus I'} P_{I' \cup j'} \odot P_{J' \setminus j'} \;\middle|\; I' \in \binom{[n]}{r-1},\ J' \in \binom{[n]}{s+1} \right\}. \tag{IP}$$

When $r = s$, the sets $\{P_I \mid I \in \binom{[n]}{r}\}$ and $\{P_J \mid J \in \binom{[n]}{s}\}$ coincide, and the relations $\mathscr{P}^{\text{trop}}_{r,s;n}$ in (IP) above degenerate to $\mathscr{P}^{\text{trop}}_{r;n}$ in (GP). These tropical polynomials are multi-homogeneous with respect to the partition $\binom{[n]}{r} \sqcup \binom{[n]}{s}$. For $0 \leq r_1 \leq \cdots \leq r_k \leq n$, the **flag Dressian** (of rank $(r_1, \ldots, r_k)$ on $[n]$) is the multi-projective tropical prevariety inside $\mathbb{P}\big(\mathbb{T}^{\binom{[n]}{r_1}}\big) \times \cdots \times \mathbb{P}\big(\mathbb{T}^{\binom{[n]}{r_k}}\big)$ defined by the tropical Grassmann-Plücker relations (GP) and the tropical incidence-Plücker relations (IP):

$$FlDr(r_1, \ldots, r_k) := \overline{\text{trop}}\left(F \in \{\mathscr{P}^{\text{trop}}_{r_i;n}\}_{1 \leq i \leq k} \cup \left\{\mathscr{P}^{\text{trop}}_{r_i, r_j; n}\right\}_{1 \leq i < j \leq n} \right).$$

We interpret the tropical incidence-Plücker relations as a condition for *valuated matroid quotients*, and points on the flag Dressian as *valuated flag matroids*, defined as follows.

Definition 4.2.2. Let μ and ν be valuated matroids on a common ground set $[n]$, whose underlying matroids are M and N of ranks r and s (respectively) with $r \leq s$. We say that μ is a **valuated (matroid) quotient of** ν, denoted $\mu \twoheadleftarrow \nu$, if for any $I \in \mathcal{B}(M)$, $J \in \mathcal{B}(N)$, and $i \in I \setminus J$, there exists $j \in J \setminus I$ such that

$$\mu(I) + \nu(J) \geq \mu(I \setminus i \cup j) + \nu(J \setminus j \cup i).$$

A sequence $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ of valuated matroids on $[n]$ is a **valuated flag matroid** if $\mu_i \twoheadleftarrow \mu_j$ for every $1 \leq i < j \leq k$. It follows from the definition that $\mu \twoheadleftarrow \nu$ if and only if $\nu^* \twoheadleftarrow \mu^*$.

We will show that the underlying matroids of a valuated matroid quotients form a matroid quotient (Corollary 4.3.2). Thus, for a valuated flag matroid $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$, its sequence of underlying matroids $(M_1, \ldots, M_k)$ is called the underlying flag matroid of $\boldsymbol{\mu}$.

We first note that points of the flag Dressian correspond to valuated flag matroids. The following is the equivalence (a) $\iff$ (b) in Theorem A.

Proposition 4.2.3. *Let $\mu \times \nu$ be a point on $\mathbb{T}^{\binom{[n]}{r}} \times \mathbb{T}^{\binom{[n]}{s}}$. Its image $\overline{\mu} \times \overline{\nu} \in \mathbb{P}(\mathbb{T}^{\binom{[n]}{r}}) \times \mathbb{P}(\mathbb{T}^{\binom{[n]}{s}})$ is a point on the flag Dressian $FlDr(r, s; n)$ if and only if μ and ν are valuated matroids that form a valuated matroid quotient $\mu \twoheadleftarrow \nu$. In other words, the points on the flag Dressian $FlDr(r_1, \ldots, r_k; n)$ correspond to valuated flag matroids of rank $(r_1, \ldots, r_k)$ on $[n]$.*

Proof. Each of μ and ν satisfies its respective tropical Grassmann-Plücker relations if and only if it is a valuated matroid by Theorem 3.1.3. Now, note that the tropical incidence-relation

$$\bigoplus_{j' \in J' \setminus I'} P_{I' \cup j'} \odot P_{J' \setminus j'}$$

for $I' \in \binom{[n]}{r-1}$, $J' \in \binom{[n]}{s+1}$ can be rewritten as follows: Fix any $i \in J' \setminus I'$, and set $I = I' \cup i$ and $J = J' \setminus i$. Then, the above tropical polynomial is the same as

$$P_I \odot P_J \oplus \bigoplus_{j \in J \setminus I} P_{I \setminus i \cup j} \odot P_{J \setminus j \cup i}.$$

The condition that the minimum (if achieved) is achieved by at least two terms of these tropical polynomials is equivalent to the condition imposed by the inequalities in the definition of valuated matroid quotients. □

Definition 4.2.4. Recall the notation $T_S := \mathbb{R}^S / \mathbb{R}\mathbf{1} \times \{\infty\}^{E \setminus S} \subset \mathbb{P}(\mathbb{T}^E)$ for sets $S \subseteq E$. Let $\boldsymbol{M} = (M_1, \ldots, M_k)$ be a flag matroid of rank $(r_1, \ldots, r_k)$ on $[n]$. The **flag Dressian of $\boldsymbol{M}$**, denoted $FlDr(\boldsymbol{M})$, is the intersection

$$FlDr(\boldsymbol{M}) := FlDr(r_1, \ldots, r_k; n) \cap \left(T_{\mathcal{B}(M_1)} \times \cdots \times T_{\mathcal{B}(M_k)}\right) \subset \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r_1}}\right) \times \cdots \times \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r_k}}\right).$$

In other words, by Proposition 4.2.3 the flag Dressian $FlDr(\boldsymbol{M})$ parametrizes all valuated flag matroids whose underlying flag matroid is $\boldsymbol{M}$.

Remark 4.2.5. For linear subspaces K and L of $\Bbbk^{[n]}$ of rank r and s, let $(p_I)_{I \in \binom{[n]}{r}}$ and $(p_J)_{I \in \binom{[n]}{s}}$ be their Plücker coordinates (respectively). Then $K \subseteq L$ if and only if the two Plücker coordinates satisfy the **incidence-Plücker relations** [16, §9, Lemma 2]:

$$\mathscr{P}_{r,s;[n]} = \left\{ \sum_{j' \in J' \setminus I'} \operatorname{sign}(j'; I', J') P_{I' \cup j'} P_{J' \setminus j'} \;\middle|\; I' \in \binom{[n]}{r-1},\ J' \in \binom{[n]}{s+1} \right\} \tag{2}$$

where $\operatorname{sign}(j'; I', J') = (-1)^{\#\{a \in I' | a < j'\} + \#\{b \in J' | b < j'\}}$. The tropical incidence-Plücker relations are tropicalizations of these polynomials. That is, we have $\mathscr{P}^{\text{trop}}_{r,s;n} = \{f^{\text{trop}} \mid f \in \mathscr{P}_{r,s;n}\}$. Thus, if $K \subseteq L$, then the corresponding valuated matroids $\mu(K)$ and $\mu(L)$ form a valuated matroid quotient $\mu(K) \twoheadleftarrow \mu(L)$. Valuated matroid quotients arising in this way are said to be **realizable (over $\Bbbk$)**.

Remark 4.2.6. The (partial) flag variety $Fl(r_1, \ldots, r_k; n)$ consists of flags of linear subspaces $L_1 \subseteq \cdots \subseteq L_k \subseteq \Bbbk^{[n]}$ with $\dim_{\Bbbk} L_i = r_i$. It has an embedding by $Fl(r_1, \ldots, r_k; n) \hookrightarrow Gr(r_1; n) \times \cdots \times Gr(r_k; n) \hookrightarrow \mathbb{P}(\Bbbk^{\binom{[n]}{r_1}}) \times \cdots \times \mathbb{P}(\Bbbk^{\binom{[n]}{r_1}})$. When $\operatorname{char} \Bbbk = 0$, the Grassmann-Plücker relations (1) combined with the incidence-Plücker relations (2) generate the multi-homogeneous ideal of this embedding. When $\operatorname{char} \Bbbk > 0$, they generate the ideal up to radical. The multi-projective tropicalization of the flag variety $\overline{\operatorname{trop}}(Fl(r_1, \ldots, r_k; n))$ is thus a subset of $FlDr(r_1, \ldots, r_k; n)$. The points of $\overline{\operatorname{trop}}(Fl(r_1, \ldots, r_k; n))$ correspond to valuated flag matroids that are **realizable (over $\Bbbk$)**. The inclusion of $\overline{\operatorname{trop}}(Fl(r_1, \ldots, r_k; n))$ in $FlDr(r_1, \ldots, r_k; n)$ is strict precisely when there are valuated flag matroids of rank $(r_1, \ldots, r_k)$ on $[n]$ that are not realizable (over $\Bbbk$).

Remark 4.2.7. A valuated matroid quotient $\mu \twoheadleftarrow \nu$, where μ and ν are realizable over $\Bbbk$ and the underlying matroid quotient is realizable over $\Bbbk$, can fail to be realizable over $\Bbbk$. See Example 5.2.4.

We address realizability of valuated flag matroids in more depth in §5.

4.3. Flags of projective tropical linear spaces

We show that a valuated matroid quotient is equivalent to the inclusion of the corresponding projective tropical linear spaces. This proves the equivalence (b) $\iff$ (d) in Theorem A.

Theorem 4.3.1. *Let μ and ν be valuated matroids of ranks r and s respectively on a common ground set $[n]$. Then $\mu \twoheadleftarrow \nu$ if and only if $\overline{\operatorname{trop}}(\mu) \subseteq \overline{\operatorname{trop}}(\nu)$. In other words, a sequence $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ of valuated matroids is a valuated flag matroid if and only if $\overline{\operatorname{trop}}(\mu_1) \subseteq \cdots \subseteq \overline{\operatorname{trop}}(\mu_k)$.*

Proof. By Theorem B.(iv), the projective tropical linear space $\overline{\operatorname{trop}}(\mu)$ is the tropical span of its valuated cocircuits $\mathcal{C}^*(\mu)$. Hence, we have $\overline{\operatorname{trop}}(\mu) \subset \overline{\operatorname{trop}}(\nu)$ if and only if $\mathcal{C}^*(\mu) \subset \overline{\operatorname{trop}}(\nu)$.

By Theorem B.(ii), the projective tropical linear space $\overline{\operatorname{trop}}(\nu)$ is cut out by its valuated circuits:

$$\overline{\operatorname{trop}}(\nu) := \bigcap_{\mathbf{C} \in \mathcal{C}(\nu)} \overline{\operatorname{trop}} \left(\bigoplus_{j' \in [n]} C_{j'} \odot x_{j'} \right).$$

The description of valuated circuits and cocircuits (Definition 3.1.5) implies that $\mathcal{C}^*(\mu) \subseteq \overline{\operatorname{trop}}(\nu)$ if and only if the minimum in

$$\left\{C_\nu(J')_{j'} + C^*_\mu(I')_{j'}\right\}_{j'\in[n]} = \{\nu(J' \setminus j') \odot \mu(I' \cup j')\}_{j'\in[n]}$$

is attained at least twice for each $I' \in \binom{[n]}{r-1}$ and $J' \in \binom{[n]}{s+1}$. Removing the terms that are ∞ on the right hand side, this is the same as saying that the minimum in

$$\{\mu(I' \cup j') \odot \nu(J' \setminus j')\}_{j'\in J'\setminus I'}$$

is attained at least twice for every I' and J', which is exactly the condition defined by the tropical incidence-Plücker relations (IP). □

This proof closely mirrors that of [19, Theorem 1], where Theorem 4.3.1 was proved for loopless matroids. We list some properties of valuated matroid quotients that follow from Theorem 4.3.1.

Corollary 4.3.2. *Let μ, ν, ξ be valuated matroids on $[n]$.*

(1) A composition of valuated matroid quotients is a valuated matroid quotient. That is, if $\mu \twoheadrightarrow \nu$ and $\nu \twoheadrightarrow \xi$ then $\mu \twoheadrightarrow \xi$.
(2) If $\mu \twoheadleftarrow \nu$, then $\mu/_S \twoheadleftarrow \nu/_S$ and $\mu\backslash_S \twoheadleftarrow \nu\backslash_S$ for any subset $S \subseteq [n]$.
(3) For any $S \subset [n]$, we have $\mu/_S \twoheadleftarrow \mu\backslash_S$.
(4) The underlying matroids M and N of μ and ν (respectively) satisfy $M \twoheadleftarrow N$ if $\mu \twoheadleftarrow \nu$.

Proof. (1) is immediate from Theorem 4.3.1. For (2), the two statements are duals of each other since $\mu \twoheadleftarrow \nu$ if and only if $\nu^* \twoheadleftarrow \mu^*$, so we only need show $\mu/S \twoheadrightarrow \nu/S$, which follows from Theorem 4.3.1 and Theorem B.(i). For (3), note that $\mathcal{C}^*(\mu/_S) = \mathcal{C}(\mu^*\backslash_S) \subseteq \mathcal{C}(\mu^*/_S) = \mathcal{C}^*(\mu\backslash_S)$ by Theorem 3.1.6, and then combine Theorem 4.3.1 with Theorem B.(iv). For (4), again from Theorem 4.3.1 and Theorem B.(iv), we have that the valuated cocircuits $\mathcal{C}^*(\mu)$ are in the tropical span of $\mathcal{C}^*(\nu)$. Considering the supports of these as elements in $\mathbb{T}^{[n]}$, we have that every cocircuit of M is a union of cocircuits of N. Hence, we have $N^* \twoheadleftarrow M^*$, or equivalently, $M \twoheadleftarrow N$. □

Remark 4.3.3. The results here about valuated matroid quotients generalize to **matroid morphisms** (see [12]) in the following way. For a map of finite sets $\varphi : [n] \to [m]$ and M a matroid of rank r on $[m]$, define a matroid $\varphi^{-1}M$ on $[n]$ by $\mathcal{B}(\varphi^{-1}M) := \{B \in \binom{[n]}{r} \mid \varphi(B) \in \mathcal{B}(M)\}$. If μ is a valuated matroid on M, then $\varphi^{-1}\mu : B \mapsto \mu(\varphi(B))$ is a valuated matroid on $\varphi^{-1}M$. A **valuated matroid morphism** $\nu \to \mu$ consists of valuated matroids ν, μ on $[n], [m]$ (respectively) and a map $\varphi : [n] \to [m]$ such that $\nu \twoheadrightarrow \varphi^{-1}(\mu)$. By Theorem 4.3.1, this is equivalent to saying that the following diagram commutes

$$\begin{array}{ccc} \overline{\mathrm{trop}}(\mu) & \longrightarrow & \overline{\mathrm{trop}}(\nu) \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}(\mathbb{T}^{[m]}) & \xrightarrow{\mathrm{trop}_\varphi} & \mathbb{P}(\mathbb{T}^{[n]}), \end{array} \tag{‡}$$

where $\mathrm{trop}_\varphi : \mathbb{P}(\mathbb{T}^{[m]}) \to \mathbb{P}(\mathbb{T}^{[n]})$ is given by $(u_j)_{j\in[m]} \mapsto (u_{\varphi(i)})_{i\in[n]}$. The diagram (‡) mirrors the diagram defining realizable matroid morphisms [12, Remark 2.1]. The map $\overline{\mathrm{trop}}(\mu) \to \overline{\mathrm{trop}}(\nu)$ in (‡) is a tropical morphism in the sense of [23].

4.4. Flag matroidal subdivisions of base polytopes

We now come to subdivisions of base configurations of flag matroids. Let $\boldsymbol{M}$ be a flag matroid.

Definition 4.4.1. A subdivision of the base configuration $\mathcal{B}(\boldsymbol{M})$ is **flag matroidal** if each face of the subdivision is a base configuration of a flag matroid. A subdivision of the base polytope $Q(\boldsymbol{M})$ is **flag matroidal** if each face of the subdivision is a base polytope of a flag matroid.

A flag matroidal subdivision of $\mathcal{B}(\boldsymbol{M})$ is necessarily mixed, and gives a flag matroidal subdivision of $Q(\boldsymbol{M})$. In general one cannot recover the subdivision of a point configuration $\mathcal{A}$ from the resulting polyhedral subdivision of $\mathrm{Conv}(\mathcal{A})$. But in this case, since one can recover the constituent matroids $M_1, \dots, M_k$ of a flag matroid $\boldsymbol{M}$ from the data of its base polytope $Q(\boldsymbol{M})$ alone, a flag matroidal subdivision of the base polytope $Q(\boldsymbol{M})$ determines a subdivision of the base configuration $\mathcal{B}(\boldsymbol{M})$. We now prove (b) $\Longrightarrow$ (c) in Theorem A.

Theorem 4.4.2. *Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ be a valuated flag matroid with underlying flag matroid $\boldsymbol{M} = (M_1, \dots, M_k)$. Regard each μ_i as a weighted point configuration on $\mathcal{B}(M_i)$. Then, their Minkowski sum $\sum_{i=1}^k \mu_i$, which is a weight on the base configuration $\mathcal{B}(\boldsymbol{M})$, induces a flag matroidal subdivision of $\mathcal{B}(\boldsymbol{M})$.*

Proof. We begin by making the following observations for valuated matroids μ and ν with underlying matroids M and N (respectively). Both observations are straightforward to verify.

- Fix $\mathbf{u} \in \mathbb{R}^{[n]}$, and consider a new weight μ' defined by

  $$\mu'(I) := \mu(I) + \langle \mathbf{u}, \mathbf{e}_I \rangle \quad \text{for } I \in \mathcal{B}(M),$$

  and similarly define ν'. Then $\mu \twoheadleftarrow \nu$ if and only if $\mu' \twoheadleftarrow \nu'$.
- Let r be the rank of M. Define a new weight $\mu_{min} : \binom{[n]}{r} \to \mathbb{R}$ by $\mu_{min}(I) = \min(\mu)$ if $\mu(I) = \min(\mu)$ and ∞ otherwise, which is a (valuated) matroid also known as the

initial matroid of μ (see [25, Definition 4.2.7] and [8, Definition 2.1]). Similarly define ν_{min}. Then $\mu \twoheadleftarrow \nu$ implies $\mu_{min} \twoheadleftarrow \nu_{min}$. In particular, by Corollary 4.3.2.(4), the underlying matroids of μ_{min} and ν_{min} form a matroid quotient if $\mu \twoheadleftarrow \nu$.

Now, fix an arbitrary $\mathbf{u} \in \mathbb{R}^{[n]}$ and consider the face $\Delta^{\overline{\mathbf{u}}}_{\sum_{i=1}^k \mu_i}$ of the subdivision of $\mathcal{B}(\boldsymbol{M})$. By Lemma 2.2.6, this face is the Minkowski sum $\sum_{i=1}^k \Delta^{\overline{\mathbf{u}}}_{\mu_i}$. By Lemma 2.2.4 and the first observation above, we may assume that $\mathbf{u} = \mathbf{0}$. In this case, each face $\Delta^{\mathbf{0}}_{\mu_i}$ is the underlying matroid of $(\mu_i)_{min}$, and the second observation thus implies that the faces form a flag matroid. $\square$

The following theorem proves (c) $\implies$ (d) in Theorem A. The characterization in Theorem B.(iii) plays a fundamental role here.

Theorem 4.4.3. *Let $\boldsymbol{M} = (M_1, \ldots, M_k)$ be a flag matroid on $[n]$, and let w be a weight on the base configuration $\mathcal{B}(\underline{\mathcal{M}})$. Suppose the coherent subdivision Δ_w is flag matroidal, and let $\mu_1, \ldots, \mu_k$ be any weights on $\mathcal{B}(M_1), \ldots, \mathcal{B}(M_k)$ satisfying $\Delta_w = \Delta_{\sum_{i=1}^k \mu_i}$. Then $\mu_1, \ldots, \mu_k$ are valuated matroids, and they satisfy $\overline{\operatorname{trop}}(\mu_1) \subseteq \cdots \subseteq \overline{\operatorname{trop}}(\mu_k)$.*

Lemma 4.4.4. *Let $\boldsymbol{M} = (M_1, \ldots, M_k)$ be a flag matroid on $[n]$. If M_i is loopless, then so are M_j for any $1 \leq i < j \leq k$. Equivalently, if M_i^* is coloopless, then so are M_j^* for any $1 \leq i < j \leq k$.*

Proof. The definition of matroid quotients by rank functions implies that if an element $l \in [n]$ satisfies $\operatorname{rk}_{M_j}(l) = 0$, then $\operatorname{rk}_{M_i}(l) = 0$ also for all $1 \leq i < j \leq k$. $\square$

Proof of Theorem 4.4.3. By Lemma 2.2.6, for every $\mathbf{u} \in \mathbb{R}^{[n]}/\mathbb{R}\mathbf{1}$ the face $\Delta^{\overline{\mathbf{u}}}_w = \Delta^{\overline{\mathbf{u}}}_{\sum_{i=1}^k \mu_i}$ is the Minkowski sum $\sum_{i=1}^s \Delta^{\overline{\mathbf{u}}}_{\mu_i}$. Since Δ_w is flag matroidal, in particular each face of Δ_w is a Minkowski sum of base polytopes of matroids. In other words, for each $1 \leq i \leq k$ the face $\Delta^{\overline{\mathbf{u}}}_{\mu_i}$ is a base polytope of a matroid. Thus, each Δ_{μ_i} is a subdivision of $\mathcal{B}(M_i)$ whose faces are all also matroids. Thus, by the equivalence of (a) and (c) in Theorem 3.1.3, each μ_i is a valuated matroid.

We will apply Theorem B.(iii) to prove the rest of the theorem. In preparation, we first note that for a matroid M, one has $Q(M^*) = -Q(M) + \mathbf{1}$. Hence, if μ is a valuated matroid, then the map $\Delta_\mu \to \Delta_{\mu^*}$ defined by $\mathcal{F} \mapsto -\mathcal{F} + \mathbf{1}$ is a bijection. Therefore, the duals $\mu_1^*, \ldots, \mu_k^*$ induce a flag matroidal subdivision $\Delta_{\sum_{i=1}^k \mu_i^*}$ of the flag matroid base polytope of $(M_k^*, \ldots, M_1^*)$, because its faces are in bijection with the faces of Δ_w by $\mathcal{F} \mapsto -\mathcal{F} + k\mathbf{1}$.

Now, let $\overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^{[n]})$, and let $S \subseteq [n]$ be the subset such that $\overline{\mathbf{u}} \in T_S$. Write $\mathbf{u} = \mathbf{u}' \times \infty^{[n]\setminus S}$. Combining Lemma 2.3.4 and Lemma 2.3.7 implies that for some $\mathbf{u}'' \in \mathbb{R}^{[n]\setminus S}$, we have $\Delta^{\mathbf{u}' \times \mathbf{u}''}_{\mu_i^*} = \Delta^{\mathbf{u}'}_{\mu_i^*|_S} \times \Delta^{\mathbf{u}''}_{\mu_i^*/_S}$, and thus Theorem 4.1.5.(2) implies that $\Delta_{\sum_{i=1}^k \mu_i^*|_S}$ is a flag matroidal subdivision. Theorem 2.3.8 therefore implies that the sequence of faces $(\Delta^{\overline{\mathbf{u}}}_{\mu_1^*}, \ldots, \Delta^{\overline{\mathbf{u}}}_{\mu_k^*})$ form the dual of a flag matroid, that is, $(\Delta^{\overline{\mathbf{u}}}_{\mu_k^*}, \ldots, \Delta^{\overline{\mathbf{u}}}_{\mu_1^*})$ is a flag

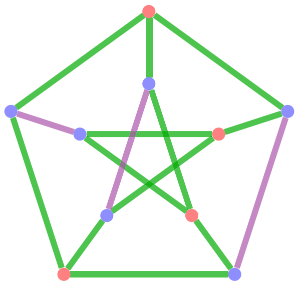

Fig. 4. The flag Dressian $FlDr(\boldsymbol{U}_{1,2;4})$.

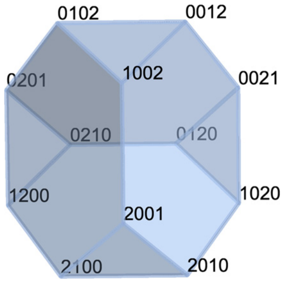

Fig. 5. The base polytope of the flag matroid $(\boldsymbol{U}_{1,2;4})$.

matroid. By Lemma 4.4.4, if the matroid of $\Delta^{\overline{\mathbf{u}}}_{\mu_i^*}$ is coloopless for some $1 \leq i \leq k$, then so are $\Delta^{\overline{\mathbf{u}}}_{\mu_j^*}$ for any $1 \leq i < j \leq k$. The desired inclusion $\overline{\text{trop}}(\mu_i) \subseteq \overline{\text{trop}}(\mu_j)$ for all $1 \leq i < j \leq k$ now follows from Theorem B.(iii), which states that $\overline{\text{trop}}(\mu_i) = \{\overline{\mathbf{u}} \in \mathbb{P}(\mathbb{T}^{[n]}) \mid \text{matroid of } \Delta^{\overline{\mathbf{u}}}_{\mu_i^*} \text{ is coloopless}\}$. □

We have now proven Theorem A. The equivalence (a) $\Longleftrightarrow$ (c) states that a mixed coherent subdivision of a base configuration of a flag matroid is flag matroidal if and only if the weights form a valuated flag matroid. One can further ask whether all coherent flag matroidal subdivisions arise in this way. Combining Theorems 2.2.9 and 4.4.3 implies the following.

Corollary 4.4.5. *Every coherent flag matroidal subdivision of a base polytope of a flag matroid arises from a valuated flag matroid.*

We now feature an extended illustration of Theorem A.

Example 4.4.6. Consider the tropical prevariety of $Fl(1, 2; 4)$, the (closure of) the flag Dressian of the flag matroid $\boldsymbol{U}_{1,2;4} := (U_{1,4}, U_{2,4})$. Compare this example to [25, Example 4.3.19].

We embed the variety $Fl(1, 2; 4)$ inside $\mathbb{P}^5 \times \mathbb{P}^3$, where the first factor has Plücker coordinates P_{ij} while the second factor has Plücker coordinates P_i for $i, j = 1, \ldots, 4$ and $i < j$. The equations defining $FlDr(1, 2; 4)$ in this embedding are given by

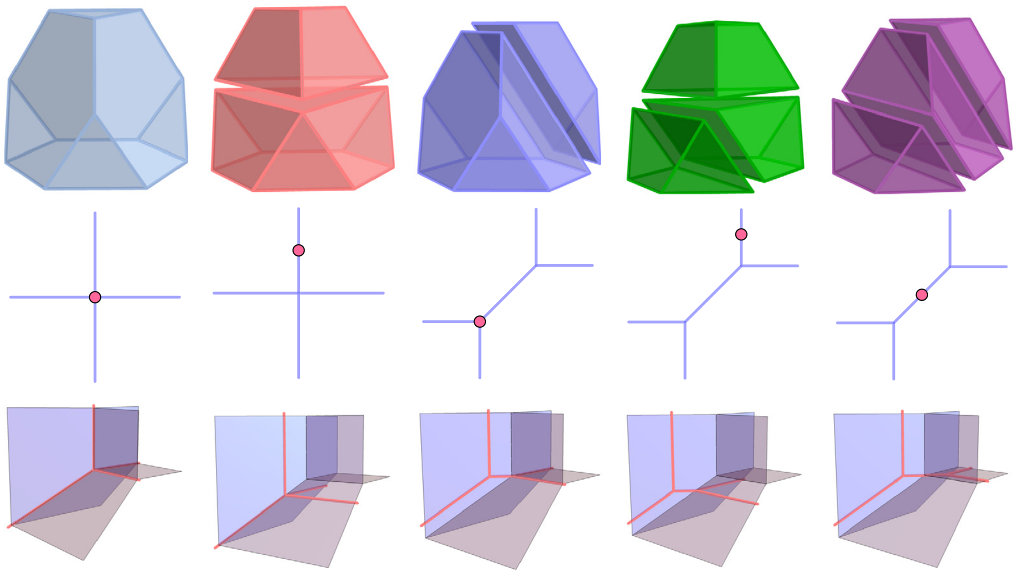

Fig. 6. The subdivisions of $Q(\boldsymbol{U}_{1,2;4})$ induced by points in the flag Dressian $FlDr(\boldsymbol{U}_{1,2;4})$. The colors correspond to which points in $FlDr(\boldsymbol{U}_{1,2;4})$ induce that subdivision, see Fig. 4. Each is displayed with the corresponding flag of a tropical point in a tropical line, and dually the tropical line in a tropical plane.

$$\langle P_{14}P_{23} - P_{13}P_{24} + P_{12}P_{34}, \quad P_4P_{23} - P_3P_{24} + P_2P_{34}, \quad P_4P_{13} - P_3P_{14} + P_1P_{34},$$
$$P_4P_{12} - P_2P_{14} + P_1P_{24}, \quad P_3P_{12} - P_2P_{13} + P_1P_{23}\rangle.$$

We compute the tropical prevariety defined by these equations to obtain (the affine cone of) the flag Dressian $FlDr(\boldsymbol{U}_{1,2;4})$ using the command `tropicalintersection` in the software `gfan` [22]. In its 10 dimensional ambient space, the affine cone of $FlDr(\boldsymbol{U}_{1,2;4})$ is a pure simplicial fan of dimension 7 with a 5 dimensional lineality space. Modulo its lineality space, it consists of 10 rays and 15 two-dimensional cones. Intersected with the sphere, we obtain Fig. 4. This is also $Dr(2;5)$, in agreement with $\widehat{FlDr}(1,2;4) = \widehat{Dr}(2;5)$ as we will see in Corollary 5.1.5.

The base polytope $Q(\boldsymbol{U}_{1,2;4})$ is the *truncated tetrahedron*, pictured in Fig. 5. This is the orbit polytope Conv $\left\{g \cdot (1,2,0,0) \subset \mathbb{R}^4 \ : \ g \in S_4\right\}$. The subdivisions induced on $Q(\boldsymbol{U}_{1,2;4})$ by points in $FlDr(\boldsymbol{U}_{1,2;4})$ come in five types, as indicated in Fig. 6. These correspond to the colored edges and vertices in Fig. 4. By Theorem A these are the subdivisions of $Q(\boldsymbol{U}_{1,2;4})$ into flag matroid polytopes. We display each subdivision with the corresponding flag of tropical linear spaces.

5. Realizability

We now give an application of Theorem A to realizability. In Example 4.4.6 we saw that the flag Dressian $FlDr(1,2;4)$ is the Petersen graph, which is the same as the Dressian $Dr(2;5)$. In Theorem 5.1.2, we explain this equality. In Theorem 5.2.1 we see that every valuated flag matroid on ground set of size at most 5 is realizable. We conclude with two examples. The first gives an interpretation of the tropicalization of the complete flag $Fl(1,2,3;4)$ as parameterizing two points in a tropical line. The second gives an example of non-realizability for a flag matroid on 6 elements.

5.1. Relating Dressians and flag Dressians

We begin by recalling a classical fact about matroid quotients with rank difference 1. See [28, §7.3] for a proof.

Proposition 5.1.1. *Let $M' \twoheadleftarrow M$ be a matroid quotient on $[n]$ where the ranks of M' and M differ by 1. Let $[\widetilde{n}] := \{0, 1, \dots, n\}$. Then the collection of subsets of $[\widetilde{n}]$*

$$\mathcal{B}(\widetilde{M}) := \{I' \cup 0 \mid I' \in \mathcal{B}(M')\} \cup \mathcal{B}(M)$$

is a set of bases of a matroid $\widetilde{M}$ on $[\widetilde{n}]$. The matroid $\widetilde{M}$ is the unique matroid on $[\widetilde{n}]$ satisfying $M' = \widetilde{M}/_0$ and $M = \widetilde{M}\backslash_0$.

We extend this fact to valuated matroid quotients with rank difference 1. Let us first consider the following partially defined map

$$\mathbb{P}(\mathbb{T}^{\binom{[\tilde{n}]}{r+1}}) \dashrightarrow \mathbb{P}(\mathbb{T}^{\binom{[n]}{r}}) \times \mathbb{P}(\mathbb{T}^{\binom{[n]}{r+1}}) \text{ defined by } (\mathbf{u}_I)_{I \in \binom{[\tilde{n}]}{r+1}} \mapsto (\mathbf{u}_{I\backslash 0})_{I \ni 0} \times (\mathbf{u}_I)_{I \not\ni 0}.$$

This map is well-defined on the set

$$\Omega(r+1;\widetilde{n}) := \left\{ \overline{\mathbf{u}} \in \mathbb{P}\left(\Bbbk^{\binom{[\tilde{n}]}{r+1}}\right) \,\middle|\, \begin{array}{c} \mathbf{u}_I \neq \infty \text{ for some } I \in \binom{[\tilde{n}]}{r+1} \text{ with } I \ni 0, \\ \text{and } \mathbf{u}_J \neq \infty \text{ for some } J \in \binom{[\tilde{n}]}{r+1} \text{ with } J \not\ni 0 \end{array} \right\}.$$

If $\widetilde{\mu}$ is a valuated matroid on $[\widetilde{n}]$ for which the map is well-defined, then its image under the map is the product of the two valuated matroids $\mu/_0$ and $\mu\backslash_0$ on $[n]$, which form a valuated matroid quotient by Corollary 4.3.2.(3). The following theorem generalizes Proposition 5.1.1 by showing that every valuated matroid quotient of rank difference 1 arises in this way.

Theorem 5.1.2. *Consider the Dressian $Dr(r+1;n+1)$ as a subset of $\mathbb{P}(\mathbb{T}^{\binom{[\tilde{n}]}{r+1}})$. The map*

$$\Omega(r+1;\widetilde{n}) \cap Dr(r+1;n+1) \to FlDr(r,r+1;n)$$

induced by the partially defined map $\mathbb{P}(\mathbb{T}^{\binom{[\tilde{n}]}{r+1}}) \dashrightarrow \mathbb{P}(\mathbb{T}^{\binom{[n]}{r}}) \times \mathbb{P}(\mathbb{T}^{\binom{[n]}{r+1}})$ is surjective, and the fiber over a point $(\mu',\mu) \in FlDr(r,r+1;n)$ is

$$\{(a \odot \mu' \oplus b \odot \mu) \in Dr(r+1;n+1) \mid a,b \in \mathbb{R}\} \simeq \mathbb{R}^2/\mathbb{R}(1,1),$$

where μ' *and* μ *are considered as elements of* $\mathbb{P}(\mathbb{T}^{\binom{[\tilde{n}]}{r+1}})$ *by*

$$\mu'(I) = \begin{cases} \mu'(I \setminus 0) & \text{if } I \ni 0 \\ \infty & \text{otherwise} \end{cases} \quad \text{and} \quad \mu(J) = \begin{cases} \mu(J) & \text{if } J \not\ni 0 \\ \infty & \text{otherwise.} \end{cases}$$

Proof. We have established that the map is well-defined: it sends a point of $Dr(r+1;n+1)$ to a point of $Fl(r,r+1;n)$ by Corollary 4.3.2.(3). Consider the fiber over a point $(\mu',\mu) \in FlDr(r,r+1;n)$, and write (M',M) for the underlying flag matroid. Let $\widetilde{M}$ be the matroid on $[\tilde{n}]$ given by Proposition 5.1.1. For any $a,b \in \mathbb{R}$, we need to show that $\tilde{\mu} : \binom{[\tilde{n}]}{r+1} \to \mathbb{T}$ defined by

$$\tilde{\mu}(I) := \begin{cases} a + \mu'(I \setminus 0) & \text{if } I \ni 0 \\ b + \mu(I) & \text{if } I \not\ni 0 \end{cases}$$

is a valuated matroid. By the equivalence of (b) and (c) in Theorem 3.1.3, it suffices to show that the induced subdivision $\Delta_{\tilde{\mu}}$ of the base polytope $Q(\widetilde{M})$ consists only of base polytopes of matroids.

The base polytope $Q(\widetilde{M}) \subset \mathbb{R}^{[\tilde{n}]} = \mathbb{R} \times \mathbb{R}^{[n]}$ is the convex hull of $\{\mathbf{e}_0\} \times Q(M')$ and $\{0\} \times Q(M)$, so it is equivalent to the Cayley polytope of $Q(M')$ and $Q(M)$. Thus, by the Cayley trick [9, Theorem 9.2.16], the faces of the subdivision $\Delta_{\tilde{\mu}}$ of $Q(\widetilde{M})$ are in bijection with the faces of the subdivision $\Delta_{(a+\mu')+(b+\mu)}$ of $\mathcal{B}(M') + \mathcal{B}(M)$. The subdivisions $\Delta_{(a+\mu')+(b+\mu)}$ and $\Delta_{\mu'+\mu}$ are the same, and by Theorem 4.4.2 each face of $\Delta_{\mu'+\mu}$ is a Minkowski sum of base polytopes of two matroids that form a matroid quotient. Hence, we conclude from Proposition 5.1.1 that each face of $\Delta_{\tilde{\mu}}$ is a base polytope of a matroid. $\square$

Theorem 5.1.2 is a tropical analogue of the following geometry.

Remark 5.1.3. Let $[\tilde{n}] := \{0,1,\dots,n\}$, and let $\{P_I \mid I \in \binom{[\tilde{n}]}{r+1}\}$ be the Plücker coordinates of the embedding $Gr(r+1;n+1) \hookrightarrow \mathbb{P}(\Bbbk^{\binom{[\tilde{n}]}{r+1}})$. Consider the rational map

$$\mathbb{P}(\Bbbk^{\binom{[\tilde{n}]}{r+1}}) \dashrightarrow \mathbb{P}(\Bbbk^{\binom{[n]}{r}}) \times \mathbb{P}(\Bbbk^{\binom{[n]}{r+1}}) \text{ where } (P_I)_{I \in \binom{[\tilde{n}]}{r+1}} \mapsto (P_{I\setminus 0})_{I \ni 0} \times (P_I)_{I \not\ni 0}.$$

With $Fl(r,r+1;n)$ embedded in $\mathbb{P}(\Bbbk^{\binom{[n]}{r}}) \times \mathbb{P}(\Bbbk^{\binom{[n]}{r+1}})$, this gives a rational map $Gr(r+1,n+1) \dashrightarrow Fl(r,r+1;n)$. The fiber over a point $(P_{I'}) \times (P_J) \in Fl(r,r+1;n)$ is

$$\{(aP_{I' \cup 0}, bP_J) \in Gr(r+1;n+1) \mid a,b \in \Bbbk^*\} \simeq (\Bbbk^*)^2/\Bbbk^*,$$

so that the map is a $\Bbbk^*$-fibration. Theorem 5.1.2 shows that a similar map in the tropical setting is an $\mathbb{R}$-fibration.

Theorem 5.1.2 relates Dressians and flag Dressians by their affine cones.

Definition 5.1.4. The **affine cone** of a projective tropical prevariety $X \subset \mathbb{P}(\mathbb{T}^E)$ is

$$\widehat{X} := \{\mathbf{u} \in \mathbb{T}^E \setminus \{\infty^E\} \mid \overline{\mathbf{u}} \in X\} \cup \{\infty^E\}.$$

Affine cones of multi-projective tropical prevarieties are similarly defined.

Corollary 5.1.5. *Under the identification $\mathbb{T}^{\binom{n+1}{r+1}} \simeq \mathbb{T}^{\binom{n}{r}} \times \mathbb{T}^{\binom{n}{r+1}}$, the affine cones $\widehat{Dr}(r+1;n+1)$ and $\widehat{FlDr}(r,r+1;n)$ of $Dr(r+1;n+1)$ and $FlDr(r,r+1;n)$ are identical.*

Proof. Let $(\mu',\mu) \in \mathbb{T}^{\binom{n}{r}} \times \mathbb{T}^{\binom{n}{r+1}}$. First consider the case where $\mu' = \infty^{\binom{n}{r}}$ or $\mu = \infty^{\binom{n}{r+1}}$. Then $(\mu',\mu) \in \widehat{FlDr}(r,r+1;n)$ if and only if μ is a valuated matroid of rank $r+1$ on $n+1$ elements where the element 0 is a loop (or respectively, μ' as a valuated matroid where 0 is a coloop). In other words $(\mu',\mu) \in \widehat{FlDr}(r,r+1;n)$ is equivalent to $(\mu',\mu) \in \widehat{Dr}(r+1;n+1)$ in this case.

If neither of μ' and μ is an all-∞ vector, then Theorem 5.1.2 implies that

$$(\mu',\mu) \in \widehat{FlDr}(r,r+1;n) \implies (\mu',\mu) \in \widehat{Dr}(r+1;n+1),$$

and Corollary 4.3.2.(3) implies that $(\mu',\mu) \in Dr(r+1;n+1) \implies (\mu',\mu) \in \widehat{FlDr}(r,r+1;n)$. $\square$

When $r = 1$, Corollary 5.1.5 follows from observing that the collections of tropical Plücker relations that define $Dr(2;n+1)$ and $Fl(1,2;n)$ are identical after simply renaming the variables $P_i \in \mathbb{P}(\mathbb{T}^{\binom{[n]}{1}})$ to $P_{i\cup 0}$. This observation however fails for $r > 1$.

5.2. Realizability for small ground sets

We compare the tropicalization of a partial flag variety and a flag Dressian in this subsection. Due to the nature of this subsection, we use the contents of the geometric Remarks 2.1.7, 2.1.8, 3.1.7, 4.2.5, and 4.2.6.

A non-realizable valuated flag matroid corresponds to a point on the flag Dressian that does not lie in the tropicalization of the partial flag variety over any valued field $\Bbbk$ (Remark 4.2.6). Realizability of a valuated flag matroid can be subtle. In Example 5.2.4, we give a valuated flag matroid (μ',μ) that is not realizable, but its underlying flag matroid is realizable, and both valuated matroids μ' and μ are realizable over a common field. For small ground sets realizability is guaranteed.

Theorem 5.2.1. *For $n \leq 5$, the tropicalization $\mathrm{trop}(Fl(r_1,\dots,r_s;n))$ of a flag variety $Fl(r_1,\dots,r_s;n)$ embedded in $\mathbb{P}(\Bbbk^{\binom{E}{r_1}}) \times \cdots \times \mathbb{P}(\Bbbk^{\binom{E}{r_s}})$ is equal to the flag Dressian $FlDr(r_1,\dots,r_s;n)$. Equivalently, for a valued field $\Bbbk$ satisfying $\mathrm{val}(\Bbbk) = \mathbb{T}$, every valuated flag matroid on a ground set of size at most 5 is realizable over $\Bbbk$.*

A valued field $\Bbbk$ satisfying $\mathrm{val}(\Bbbk) = \mathbb{T}$ exists in every characteristic; see [29, §3] for an example known as Mal'cev-Neumann rings. Theorem 5.2.1 fails for $n \geq 6$; see Example 5.2.4. We prepare the proof of the theorem with a lemma.

Lemma 5.2.2. *Let $\Bbbk$ be a valued field, and write $\Gamma := \mathrm{val}(\Bbbk) \subseteq \mathbb{T}$. Suppose $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$ is a valuated flag matroid on $[n]$ with $\mathrm{rk}(\mu_1) = 1$ such that $(\mu_2, \dots, \mu_k)$ is realizable over $\Bbbk$, and μ_1 as an element of $\mathbb{T}^{\binom{[n]}{1}}$ has coordinates in Γ. Then $\boldsymbol{\mu}$ is realizable over $\Bbbk$. By duality, if $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$ is a valuated flag matroid such that $(\mu_1, \dots, \mu_{k-1})$ is realizable over $\Bbbk$ and $\mathrm{rk}(\mu_k) = n-1$, then $\boldsymbol{\mu}$ is also realizable over $\Bbbk$ when $\mu_k \in \Gamma^{\binom{[n]}{n-1}}$.*

Proof. Let a flag $L_2 \subset \cdots \subset L_k \subset \Bbbk^E$ be a realization of $(\mu_2, \dots, \mu_k)$. We need to show that there exists a one-dimensional space L_1, that is, a point in $\mathbb{P}(\Bbbk^E)$, such that $L_1 \subset L_2$ and $\overline{\mathrm{trop}}(\mu_1) = \overline{\mathrm{trop}}(L_1)$. But since $\mathrm{rk}(\mu_1) = 1$, the space $\overline{\mathrm{trop}}(\mu_1)$ is a single point, which by Theorem 4.3.1 is on $\overline{\mathrm{trop}}(\mu_2) = \overline{\mathrm{trop}}(L_2)$. By the lifting property in the Fundamental Theorem of Tropical Geometry [25, Theorem 3.2.3, Theorem 6.2.15], there exists a point $p_1 \in \mathbb{P}(L_2) \subset \mathbb{P}(\Bbbk^E)$ with $\overline{\mathrm{trop}}(p_1) = \overline{\mathrm{trop}}(\mu_1)$. □

Proof of Theorem 5.2.1. We first note some previous results:

- One has $\mathrm{trop}(Gr(1;n)) = Dr(1;n) = \mathbb{P}(\mathbb{T}^E)$, and dually, $\mathrm{trop}(Gr(n-1;n)) = Dr(n-1;n)$.
- For any n, one has $\mathrm{trop}(Gr(2;n)) = Dr(2;n)$, and dually, $\mathrm{trop}(Gr(n-2;n)) = Dr(n-2;n)$ [25, Corollary 4.3.12].
- One has $\mathrm{trop}(Gr(3;6)) = Dr(3;6)$ [25, Example 4.4.10].

By Theorem 5.1.2, the desired statement thus holds for $Fl(1,2;n)$, its dual $Fl(n-2, n-1;n)$, and $Fl(2,3;5)$. The rest of the cases for $n \leq 5$ then follow from Lemma 5.2.2. □

Example 5.2.3. Let $Fl_4 := Fl(1,2,3;4)$, and denote by $\mathring{F}l_4$ the very affine variety obtained as the intersection of Fl_4 embedded in $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$ with the torus $(\Bbbk^*)^4/\Bbbk^* \times (\Bbbk^*)^6/\Bbbk^* \times (\Bbbk^*)^4/\Bbbk^*$. The f-vector of its tropicalization $\mathrm{trop}(\mathring{F}l_4)$, with the Gröbner complex for its polyhedral complex structure, was computed in [6] to be $(1, 20, 79, 78)$ with the aid of a computer. We now give an explicit description of the combinatorial structure of $\mathrm{trop}(\mathring{F}l_4)$.

By Theorem 5.2.1, we have that $\mathrm{trop}(\mathring{F}l_4) = FlDr(\boldsymbol{U}_{1,2,3;4})$ where $\boldsymbol{U}_{1,2,3;4} = (U_{1,4}, U_{2,4}, U_{3,4})$. If μ is a valuated matroid whose underlying matroid is $U_{2,4}$, then $\mathrm{trop}(\mu)$ is a translate of $\mathrm{trop}(\mu^*)$. Thus, by Theorem 4.3.1, one can identify the space $FlDr(\boldsymbol{U}_{1,2,3;4})$ as the parameter space of two labeled points on a tropical line. Using this, we completely describe the polyhedral complex structure of $\mathrm{trop}(\mathring{F}l_4) = FlDr(\boldsymbol{U}_{1,2,3;4})$ in Figs. 7 and 8. The pictorial representations of the maximal cells in [6, Fig. 2] are related to but different from ours.

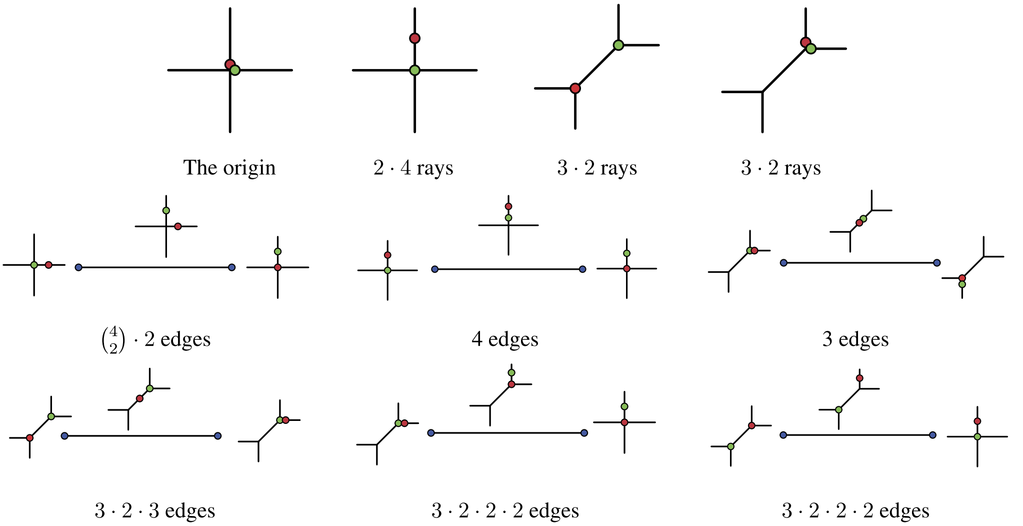

Fig. 7. The origin, 20 rays, and 79 edges of $FlDr(\boldsymbol{U}_{1,2,3;4})$.

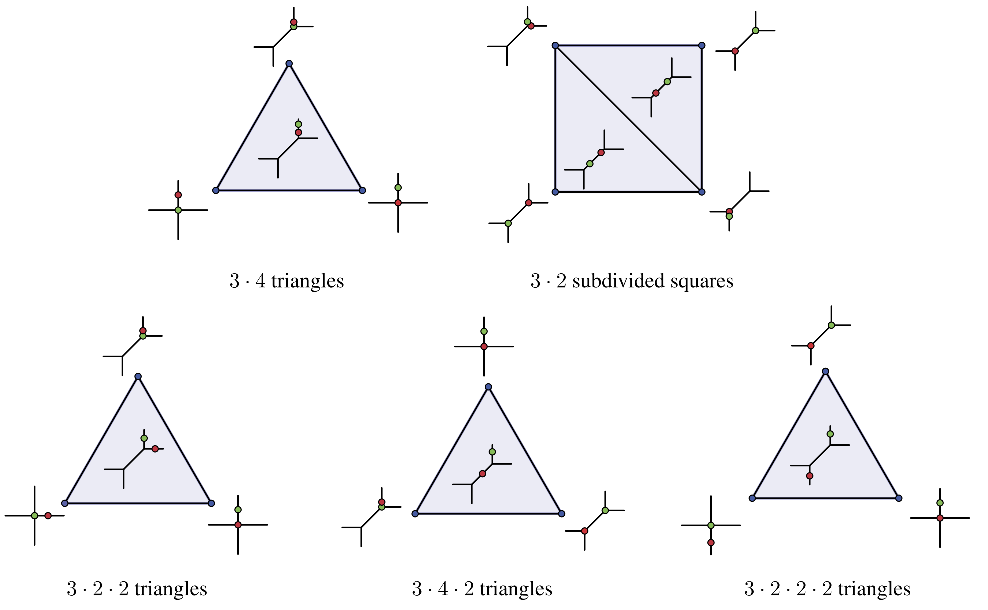

Fig. 8. The 78 2-cells of $FlDr(\boldsymbol{U}_{1,2,3;4})$. 78 is also $3 \cdot 5^2 + 3$ (three kinds of tropical lines, each with five 1-cells, plus 3 from the subdivided squares).

The next example highlights the subtleties of realizability. In light of Theorem 5.1.2, the example below is closely related to [25, Example 4.3.14].

Example 5.2.4. Consider the flag matroid $\boldsymbol{M} = (U_{2,6}, M_4)$ pictured in Fig. 9. The matroid M_4 is the rank 3 matroid on $E = \{1, \ldots, 6\}$ with circuit hyperplanes $\{124, 135, 236, 456\}$.

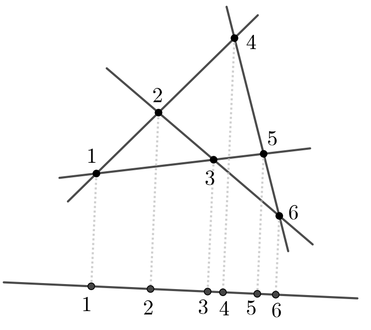

Fig. 9. The flag matroid considered in Example 5.2.4.

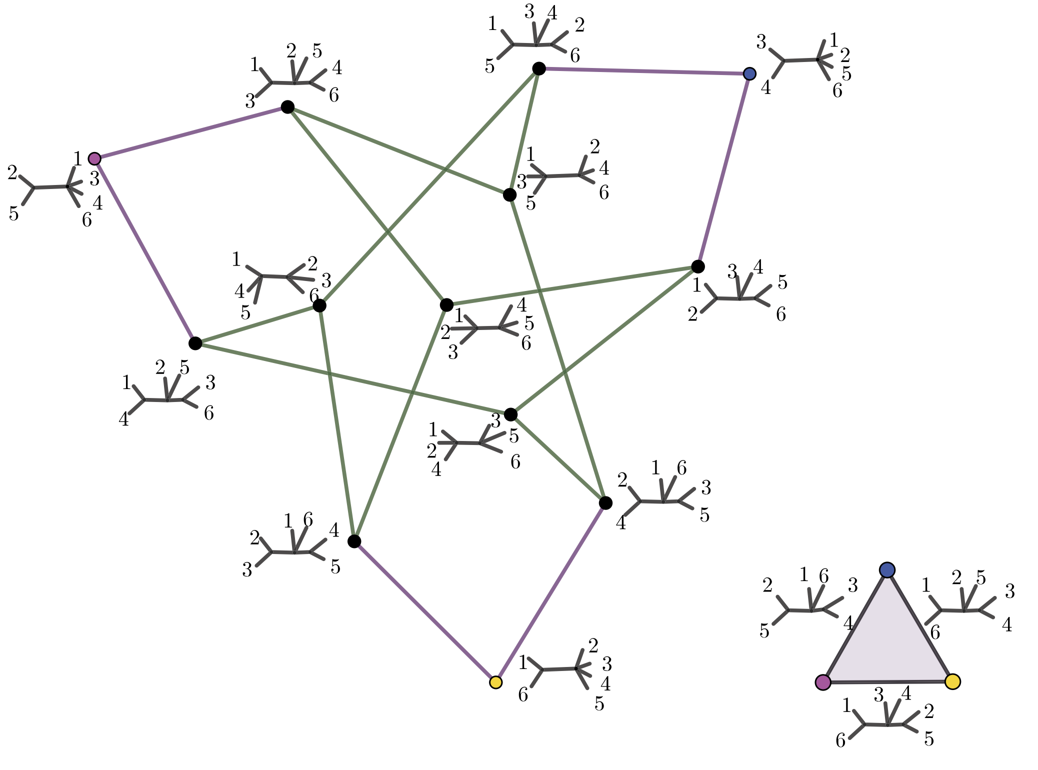

Fig. 10. The Dressian of the flag matroid in Example 5.2.4.

The affine cone of the flag Dressian $FlDr(\boldsymbol{M})$ is a 10 dimensional fan, with a 7 dimensional lineality space. Modulo lineality and intersecting with a sphere, it has 13 rays, 21 edges, and one triangle, depicted in Fig. 10. In Fig. 10, rays are labeled with the tree given by the $U_{2,6}$ coordinates. The green edges in the graph correspond to points where the corresponding tree is a caterpillar, and the purple points give snowflake trees. The triangle is glued to the pink, blue, and yellow vertices as indicated.

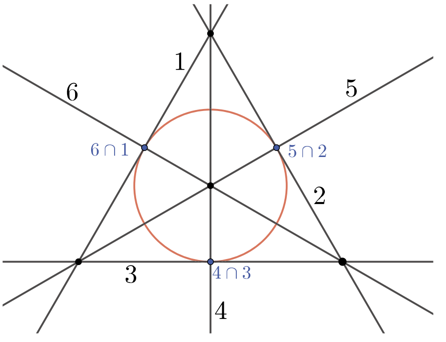

Fig. 11. The dual matroid to M_4 is pictured in black. The red line must be realizable to produce the trees appearing in the interior of the triangle in Fig. 10.

Let $V \subset Fl(2,3;6)$ be the space of realizations of $(M_4, U_{2,6})$. The affine cone of this subvariety V of $Fl(2,3;6)$ has dimension 9, but the affine cone of the flag Dressian $FlDr(\boldsymbol{M})$ has dimension 10. Since $\dim \operatorname{trop}(V) = \dim V$ [25, Theorem 3.3.8], the flag Dressian $FlDr(\boldsymbol{M})$ must strictly contain $\overline{\operatorname{trop}}(V)$. Indeed, the tropicalization $\operatorname{trop}(V)$ when $\Bbbk$ has characteristic 0 consists of all zero and one dimensional cells in Fig. 10; the interior of the single triangle is removed. Over characteristic 2, some points in the interior of the triangle may be on $\operatorname{trop}(V)$, but not the entire triangle.

Let us understand the non-realizable points in the interior of this triangle in more detail when $\Bbbk$ has characteristic 0. The Dressians for $U_{2,6}$ and M_4 are each tropical varieties, meaning that every point w in each of their Dressians can be realized as vectors over $\Bbbk$ whose Plücker coordinates valuate to w. So, the points in the interior of the triangle in $FlDr(\boldsymbol{M})$ correspond to two realizable valuated matroids that fail to form a realizable valuated matroid quotient.

We see why it is not possible to realize these points as follows. Points on the interior of the triangle correspond to snowflake trees with pairs $\{2,5\}$, $\{1,6\}$, and $\{3,4\}$. In order to realize this over $\Bbbk$, we would need to make a configuration as in Fig. 9 such that over the residue field, the projections of the points $\{2,5\}$ coincide, the projections of the points $\{1,6\}$ coincide, and the projections of the points $\{3,4\}$ coincide. The dual picture is shown in Fig. 11. In order to realize the desired snowflake, we need to find a line that intersects the six lines pictured at each of the points of intersection of the lines 2 and 5, 1 and 6, and 3 and 4. This is only possible over fields of characteristic 2, where the Fano plane is realizable.

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References

[1] M. Aguiar, F. Ardila, Hopf monoids and generalized permutahedra, arXiv:1709.07504, 2017.
[2] L. Anderson, Vectors of matroids over tracts, J. Comb. Theory, Ser. A 161 (2019) 236–270.
[3] F. Ardila, F. Castillo, C. Eur, A. Postnikov, Coxeter submodular functions and deformations of Coxeter permutahedra, Adv. Math. 365 (2020) 107039.
[4] M. Baker, N. Bowler, Matroids over partial hyperstructures, Adv. Math. 343 (2019) 821–863.
[5] A. Borovik, I. Gelfand, N. White, Coxeter Matroids, Progress in Mathematics, vol. 216, Birkhäuser Boston Inc., Boston, MA, 2003.
[6] L. Bossinger, S. Lamboglia, K. Mincheva, F. Mohammadi, Computing toric degenerations of flag varieties, in: Combinatorial Algebraic Geometry, Toronto, ON, in: Fields Inst. Commun. Fields Inst. Res. Math. Sci., vol. 80, 2017, pp. 247–281.
[7] T. Brylawski, Constructions, in: Theory of Matroids, in: Encyclopedia Math. Appl., vol. 26, Cambridge Univ. Press, Cambridge, 1986, pp. 127–223.
[8] M. Brandt, D. Speyer, Matroids and their Dressians, https://arxiv.org/pdf/1902.05592.pdf, 2020.
[9] J. De Loera, J. Rambau, F. Santos, Triangulations. Structures for Algorithms and Applications, Algorithms and Computation in Mathematics, vol. 25, Springer-Verlag, Berlin, 2010.
[10] A. Dress, W. Wenzel, Valuated matroids, Adv. Math. 93 (2) (1992) 214–250.
[11] J. Edmonds, Submodular functions, matroids, and certain polyhedra, in: Combinatorial Structures and Their Applications, Proc. Calgary Internat. Conf., Calgary, Alta., 1969, Gordon and Breach, New York, 1970, pp. 69–87.
[12] C. Eur, J. Huh, Logarithmic concavity for morphisms of matroids, Adv. Math. 367 (2020) 107094.
[13] X. Fang, E. Feigin, G. Fourier, I. Makhlin, Weighted PBW degenerations and tropical flag varieties, Commun. Contemp. Math. (ISSN 0219-1997) 21 (1) (2019) 1850016, https://doi.org/10.1142/S0219199718500165, https://doi-org.libproxy.berkeley.edu/10.1142/S0219199718500165.
[14] A. Fink, L. Moci, Matroids over a ring, J. Eur. Math. Soc. 18 (4) (2016) 681–731.
[15] A. Fink, L. Moci, Polyhedra and parameter spaces for matroids over valuation rings, Adv. Math. 343 (2019) 448–494.
[16] W. Fulton, Young Tableaux, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry.
[17] I. Gelfand, M. Goresky, R. MacPherson, V. Serganova, Combinatorial geometries, convex polyhedra, and Schubert cells, Adv. Math. 63 (3) (1987) 301–316.
[18] I. Gelfand, V. Serganova, Combinatorial geometries and the strata of a torus on homogeneous compact manifolds, Usp. Mat. Nauk 42.2 (254) (1987) 107–134, p. 287.
[19] M.M. Haque, Tropical incidence relations, polytopes, and concordant matroids, arXiv:1211.2841, 2012.
[20] S. Herrmann, A. Jensen, M. Joswig, B. Sturmfels, How to draw tropical planes, Electron. J. Comb. 16 (2) (2009) 6, Special volume in honor of Anders Björner.
[21] P. Jell, H. Markwig, F. Rincón, B. Schröter, Tropical lines in planes and beyond, arXiv:2003.02660, 2020.
[22] A. Jensen, Gfan, a software system for Gröbner fans and tropical varieties, Available at http://home.imf.au.dk/jensen/software/gfan/gfan.html.
[23] G. Mikhalkin, Tropical geometry and its applications, in: International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 827–852.
[24] G. Mikhalkin, J. Rau, Tropical geometry. Book in progress (Version Nov. 2018). Available at https://www.math.uni-tuebingen.de/user/jora/downloads/main.pdf.
[25] D. Maclagan, B. Sturmfels, Introduction to Tropical Geometry, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015.
[26] K. Murota, A. Tamura, On circuit valuation of matroids, Adv. Appl. Math. 26 (3) (2001) 192–225.

[27] K. Murota, Discrete Convex Analysis, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.
[28] J. Oxley, Matroid Theory, second edition, Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011.
[29] B. Poonen, Maximally complete fields, Enseign. Math. (2) (ISSN 0013-8584) 39 (1–2) (1993) 87–106.
[30] A. Postnikov, Permutohedra, associahedra, and beyond, Int. Math. Res. Not. 6 (2009) 1026–1106.
[31] K. Shaw, A tropical intersection product in matroidal fans, SIAM J. Discrete Math. 27 (1) (2013) 459–491.
[32] D. Speyer, Tropical geometry, Available at http://www-personal.umich.edu/~speyer/thesis.pdf.
[33] D. Speyer, Tropical linear spaces, SIAM J. Discrete Math. 22 (4) (2008) 1527–1558.
[34] D. Welsh, Matroid Theory, L. M. S. Monographs, vol. 8, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976, xi+433.