



INSTANTANEOUS SMOOTHING AND EXPONENTIAL DECAY OF SOLUTIONS FOR A DEGENERATE EVOLUTION EQUATION WITH APPLICATION TO BOLTZMANN'S EQUATION

FEDOR NAZAROV

Kent State University
Kent, OH 44240

KEVIN ZUMBRUN

Indiana University
Bloomington, IN 47405

(Communicated by Toan T. Nguyen)

In memory of Robert T. Glassey

Preface by Kevin Zumbrun: *It is a great privilege to submit a paper to this memorial issue in honor of long-time mentor and colleague Bob Glassey. Bob enlivened the department at Indiana University for many years with his gentle wit and incisive mathematical analysis. Bob is perhaps best known for his work on kinetic equations. What I know about Boltzmann's equation I know mainly from sitting in on his delightful graduate course, which later became his delightful text on the Cauchy problem for kinetic equations. So, this paper concerning Boltzmann's and related kinetic equations seems an appropriate submission, coming as a direct result of Bob's influence as a teacher and researcher. Of course, Bob is equally well known for his seminal work on blowup in nonlinear pde. So, it is perhaps also appropriate that our central estimate is a "blow-down" result consisting of the reverse-time version of finite-time blowup for the Riccati equation.*

ABSTRACT. We establish an instantaneous smoothing property for decaying solutions on the half-line $(0, +\infty)$ of certain degenerate Hilbert space-valued evolution equations arising in kinetic theory, including in particular the steady Boltzmann equation. Our results answer the two main open problems posed by Pogan and Zumbrun in their treatment of H^1 stable manifolds of such equations, showing that L^2_{loc} solutions that remain sufficiently small in L^∞ (i) decay exponentially, and (ii) are C^∞ for $t > 0$, hence lie eventually in the H^1 stable manifold constructed by Pogan and Zumbrun.

2020 *Mathematics Subject Classification.* Primary: 35Q35, 35Q20; Secondary: 76P05, 82C40.

Key words and phrases. Steady Boltzmann equation, evolution equation, boundary value problem, regularity, asymptotic behavior.

Research of F.N. was partially supported under NSF grant no. DMS-1600239; research of K.Z. was partially supported under NSF grant no. DMS-0300487.

*Corresponding author: Kevin Zumbrun.

1. Introduction. The goal of this paper is to prove instantaneous smoothing and decay properties for \mathbb{H} -valued solutions of the evolution equation

$$(d/dt)(Ax) = -x + \mathcal{G}(x), \quad (1)$$

where \mathbb{H} is a separable Hilbert space, $A : \mathbb{H} \rightarrow \mathbb{H}$ is a constant bounded self-adjoint operator, $\mathcal{G} : \mathbb{H} \rightarrow \mathbb{H}$ is an infinitely differentiable map with

$$\mathcal{G}(0) = 0, \sup_{x \in \mathbb{H}} \|D_x \mathcal{G}\| \leq 1/4, \text{ and } \sup_{x \in \mathbb{H}} \|D_x^k \mathcal{G}\| < +\infty \text{ for all } k. \quad (2)$$

Our a priori assumptions are just that $x : (0, +\infty) \rightarrow \mathbb{H}$ belongs to $(\mathbb{H}\text{-valued}) L_{loc}^2(0, +\infty)$ and $Ax \in H_{loc}^1(0, +\infty)$, with (1) holding as an equation in $L_{loc}^2(0, +\infty)$; we call such a function a “weak L_{loc}^2 solution” of (1). Note that these conditions do not imply any a priori smoothness for x itself because A may be quite degenerate; in particular, we have in mind the case considered for kinetic equations in [8, 9, 12] of A essentially singular, or *one-to-one but not boundedly invertible*.

These conditions seem quite natural to impose just to make sense of the differential equation (1). However it should be noted that our study was motivated by some questions that were left open in [8] where another notion of solution, the so-called “mild solution,” was introduced. We relate our results to [8] by showing that any mild solution is in fact a solution in our sense as well.

Remark 1.1. The studies of [8, 9, 12] concerned the situation $\mathcal{G}(x) = B(x, x)$ with B a bilinear map; we note that this satisfies the framework (1)–(2) for solutions with $\sup |x|$ sufficiently small. This is relevant to kinetic (in particular, Boltzmann’s) equations, as discussed in Section 4.

Remark 1.2. The condition $\sup_{x \in \mathbb{H}} \|D_x \mathcal{G}\| \leq 1/4$ can be weakened to

$$\sup_{x \in \mathbb{H}} \|D_x \mathcal{G}\| \leq \gamma$$

for $\gamma < 1$, with essentially no change in the arguments.¹ The choice $\gamma = 1/4$ is for expositional convenience.

Our main result is the following.

Theorem 1.3. *For A constant, bounded, and self-adjoint, and $\mathcal{G} \in C^\infty$ satisfying (2), every weak L_{loc}^2 solution x of (1) lying in $L^\infty(0, +\infty)$ is, in fact, C^∞ on $(0, +\infty)$, and decays exponentially with all derivatives as $t \rightarrow \infty$. Moreover, for all $t > 0$ and $k \geq 0$,*

$$|(d/dt)^k x(t)| \leq \begin{cases} Ct^{-k}, & 0 < t < 1, \\ Ce^{-ct}, & t \geq 1, \end{cases} \quad (3)$$

where c depends only on $\|A\|$, and C on $\|x\|_{L^\infty(0, +\infty)}$, $\|A\|$, k , and $\sup_{x \in \mathbb{H}} \|D_x^j \mathcal{G}\|$, $j = 1, 2, \dots, k+1$.

Remark 1.4. For $\mathcal{G} \in C^r$, we obtain instantaneous H_{loc}^r/C_{loc}^{r-1} smoothing by the same argument.

Remark 1.5. A similar approach gives instantaneous smoothing for $x \in L^2(0, +\infty)$, with bounds $|(d/dt)^k x(t)| \leq Ct^{-1/2-k}$, $t > 0$,²

¹Namely, in the first step of the proof of Theorem 1.3, we may use $|\mathcal{G}(x)| = |\mathcal{G}(x) - \mathcal{G}(0)| \leq \gamma|x|$ to obtain $-|x|^2 + \langle \mathcal{G}(x), x \rangle \leq -(1-\gamma)|x|^2 \lesssim -|x|^2$ in place of (6), and similarly in higher-derivative estimates throughout.

²Specifically, substituting $L^2(0, +\infty)$ for $L^\infty(0, +\infty)$ in item 1 of the proof in Section 2, and using the Sobolev estimate in item 2 to bound $|x(t)|$ for $t \leq 1$, gives $|x(\tau)| \leq Ct^{-1/2}$ for $\tau \geq t$, yielding the result by Theorem 1.3.

We believe that Theorem 1.3 may be of its own interest, but as far as the questions in [8] are concerned, it implies, in the slightly different setting of Remark 1.1, that $L^2_{loc}(0, +\infty)$ solutions remaining sufficiently small in sup norm must by exponential decay of $\|x\|_{H^1(t, +\infty)}$ belong eventually to the H^1 stable manifold constructed by Pogan-Zumbrun, which may be characterized [8] as the union of all trajectories x with $\|x\|_{H^1(0, +\infty)}$ sufficiently small. Moreover, it implies that this H^1 stable manifold, expressed as in [8] as the union of trajectories in $H^1(0, +\infty)$, lies in $C^\infty(t, +\infty)$ for any $t > 0$. We recall that the analysis of [8] was motivated by consideration of the steady Boltzmann equation, which, as shown in [6], may be put by an appropriate coordinate transformation into the framework (1) considered here, with $\mathcal{G}(x) = B(x, x)$, B a bounded bilinear map.

The paper is organized as follows. In Section 2, we prove the theorem itself. The proof is rather elementary and the reader interested in the key ideas will lose almost nothing assuming that we deal with a finite-dimensional Hilbert space \mathbb{H} , understand all derivatives in the classical sense, and just aim at quantitative bounds independent of dimension, etc. We will still clearly state all standard facts from the integration theory of Hilbert space-valued functions to make sure that everything works in the generality we need, but we refer the reader to, e.g., [3] for their proofs.

In Section 3, we revisit equation (1) from the standpoint of existence and uniqueness, in the process connecting with [8] and the notion of “mild solution”. Unlike Section 2, which can be read from scratch, Section 3 assumes of the reader at least some familiarity with standard Fourier transform, convolution, and Gagliardo-Nirenberg bounds, and, differently from the situation in Section 2, the issues of how exactly everything is defined and in what sense the equalities hold are central there. Though it is for the most part self-contained, the reader of Section 3 will benefit in Section 3.2 from familiarity with [8] and in Remark 3.12 will need it to make full sense of the remark.

In Section 4, we discuss implications of our results for the questions considered in [8], especially as they concern Boltzmann’s equation. Finally, in Section 5, we describe some perspectives and open problems. We delegate the proofs of one standard and one “semi-standard” Sobolev type embedding theorem (Lemmas A.1–A.2) to be used in Section 2 to an appendix so as not to interrupt the flow of the main argument. This appendix can be read completely independently of the main text. In a second appendix, we discuss integrability of Hilbert space-valued functions as it relates to the variation of constants formula in Section 3. This appendix refers to but is not needed in the main text. It may be skipped by the reader if desired.

2. Proof of the main theorem. We start with a technical lemma.

Lemma 2.1. *Suppose that $F : (0, +\infty) \rightarrow \mathbb{R}$ is absolutely continuous, f, g are nonnegative with $f \in L^1_{loc}(0, +\infty)$, g measurable, and $(d/dt)F \leq -F^2g + f$ a.e. on $(0, +\infty)$. Then, for every $t > t' > 0$,*

$$F(t) \leq \int_{t'}^t f + \left(\int_{t'}^t g \right)^{-1}.$$

Proof. Let $G(s) := F(s) - \int_{t'}^s f$, $s \geq t'$. If there exists $s \in (t', t)$ such that $G(s) \leq 0$, then

$$F(t) = F(s) + \int_s^t F' \leq \int_{t'}^s f + \int_s^t f = \int_{t'}^t f$$

and we are done. Otherwise, $G > 0$ on (t', t) , so $0 < G \leq F$ and

$$(d/dt)G \leq -F^2g \leq -G^2g,$$

or, equivalently, $(d/dt)(1/G) \geq g$, hence $(1/G)(t) \geq \int_{t'}^t g$ and $G(t) \leq \left(\int_{t'}^t g\right)^{-1}$. \square

Corollary 2.2. *Suppose that $h, y : (0, +\infty) \rightarrow \mathbb{H}$ are in $L^2(0, +\infty)$, $\langle h, y \rangle$ is absolutely continuous, $f : (0, +\infty) \rightarrow \mathbb{R}$ is in $L^1(0, +\infty)$ and nonnegative, and $(d/dt)\langle h, y \rangle \leq -|y|^2 + f$ a.e. on $(0, +\infty)$. Then, for every $t > t' > 0$, there holds $\langle h, y \rangle(t) \leq \int_{t'}^t f + \frac{\int_{t'}^t |h|^2}{(t-t')^2}$ and*

$$\int_t^{+\infty} |y|^2 \leq \int_{t'}^{+\infty} f + \frac{\int_{t'}^t |h|^2}{(t-t')^2}.$$

Proof. Note that $|y|^2 \geq \frac{\langle h, y \rangle^2}{|h|^2 + \varepsilon}$ for all $\varepsilon > 0$ and $\left(\int_{t'}^t (1/(|h|^2 + \varepsilon))\right)^{-1} \leq \frac{\int_{t'}^t (|h|^2 + \varepsilon)}{(t-t')^2}$, so Lemma 2.1 with $F = \langle h, y \rangle$, $g = 1/(|h|^2 + \varepsilon)$ implies $\langle h, y \rangle(t) \leq \int_{t'}^t f + \frac{\int_{t'}^t (|h|^2 + \varepsilon)}{(t-t')^2}$ for all $\varepsilon > 0$, yielding the first inequality in the limit as $\varepsilon \rightarrow 0^+$. Since there exists a sequence $t_k \rightarrow +\infty$ such that $\langle h, y \rangle(t_k) \rightarrow 0$, we get $\int_t^{t_k} |y|^2 \leq -\langle h, y \rangle|_t^{t_k} + \int_t^{t_k} f$. Taking the limit as $t_k \rightarrow +\infty$, we get the result. \square

In order to be able to use this corollary, we need the following key observation.

Lemma 2.3. *If A is a bounded self-adjoint operator, $x : (0, +\infty) \rightarrow \mathbb{H}$ is in $L_{loc}^2(0, +\infty)$, and $Ax \in H_{loc}^1(0, +\infty)$, then $\langle Ax, x \rangle$ is absolutely continuous, with*

$$(d/dt)\langle Ax, x \rangle = 2\langle (d/dt)(Ax), x \rangle. \quad (4)$$

Proof. Fix $\tau > 0$ and consider the difference quotient

$$\begin{aligned} \frac{\Delta_\tau \langle Ax, x \rangle}{\tau}(t) &= \frac{\langle \Delta_\tau(Ax)(t), x(t+\tau) \rangle}{\tau} + \frac{\langle Ax(t), (\Delta_\tau x)(t) \rangle}{\tau} \\ &= \frac{\langle \Delta_\tau(Ax)(t), x(t) + x(t+\tau) \rangle}{\tau}, \end{aligned}$$

where $\Delta_\tau v(t) := v(t+\tau) - v(t)$. Here, we have used in a crucial way self-adjointness of A . Since A is bounded, $\frac{\Delta_\tau \langle Ax, x \rangle}{\tau} \in L_{loc}^1(0, +\infty)$, and we have the integral identity

$$\int_{t'}^t \frac{\Delta_\tau \langle Ax, x \rangle}{\tau} = (S_\tau \langle Ax, x \rangle) \Big|_{t'}^t,$$

where

$$(S_\tau v)(t) := (1/\tau) \int_t^{t+\tau} v. \quad (5)$$

Note now that, as $\tau \rightarrow 0^+$, $S_\tau(\langle Ax, x \rangle) \rightarrow \langle Ax, x \rangle$ a.e. while

$$x(t) + x(t+\tau) \rightarrow 2x(t)$$

in L_{loc}^2 and, by Lemma A.1(ii), $\frac{\Delta_\tau(Ax)}{\tau} \rightarrow (d/dt)(Ax)$ in L_{loc}^2 . Hence, passing to the limit as $\tau \rightarrow 0^+$, we get

$$\langle Ax, x \rangle|_{t'}^t = \int_{t'}^t 2\langle (d/dt)(Ax), x \rangle$$

for a.e. $t', t \in (0, +\infty)$, verifying (4). \square

Proof of Theorem 1.3. We proceed by a series of steps bounding x in successively higher norms. The main idea of the argument may be seen in steps 1 and 2 showing instantaneous smoothing from L^∞ to H^1 . Steps 3–5, showing higher regularity, proceed in similar but more complicated fashion.

1. (*Proof that $x \in L^2(0, +\infty)$*). By time and space rescaling, we can always assume without loss of generality that $\|A\| = 1$ and $\|x\|_{L^\infty(0, +\infty)} \leq 1$. Now, we have

$$(d/dt)\langle Ax, x \rangle = 2\langle d/dt(Ax), x \rangle = -2|x|^2 + 2\langle \mathcal{G}(x), x \rangle \leq -|x|^2, \quad (6)$$

so $\langle Ax, x \rangle$ is decreasing. Moreover, since $|x|^2 \geq |\langle Ax, x \rangle|$, either $\langle Ax, x \rangle \rightarrow -\infty$ as $t \rightarrow +\infty$, which is impossible, since $x \in L^\infty(0, +\infty)$ and A is bounded, or else $\langle Ax, x \rangle \geq 0$ for all $t > 0$, and

$$\langle Ax, x \rangle(t) \leq e^{-t} \langle Ax, x \rangle(0) \leq e^{-t}.$$

For, otherwise, $(d/dt)\langle Ax, x \rangle \leq -|x|^2 \leq \langle Ax, x \rangle$ would be eventually uniformly negative.

Thus, for any $T > t > 0$, $\int_t^T |x|^2 \leq -\langle Ax, x \rangle|_t^T \leq \langle Ax, x \rangle(t) \leq e^{-t}$. Letting $T \rightarrow +\infty$, we get $\int_t^{+\infty} |x|^2 \leq e^{-t}$ for all $t > 0$. In particular, $x \in L^2(0, +\infty)$ and $\|x\|_{L^2(0, +\infty)} \leq 1$.

2. (*Proof that $x \in H_{loc}^1(0, +\infty)$*). It is enough to show that the difference ratios $\frac{\Delta_\tau x}{\tau}$ are uniformly bounded in $L_{loc}^2(0, +\infty)$ as $\tau \rightarrow 0^+$. To this end, write

$$(d/dt)\left(A \frac{\Delta_\tau x}{\tau}\right) = -\frac{\Delta_\tau x}{\tau} + \frac{\Delta_\tau \mathcal{G}(x)}{\tau}. \quad (7)$$

Note that $A \frac{\Delta_\tau x}{\tau} \in H_{loc}^1(0, +\infty)$ for any fixed $\tau > 0$. So, we get

$$(d/dt)\left\langle A \frac{\Delta_\tau x}{\tau}, \frac{\Delta_\tau x}{\tau} \right\rangle = -2\left|\frac{\Delta_\tau x}{\tau}\right|^2 + 2\left\langle \frac{\Delta_\tau \mathcal{G}(x)}{\tau}, \frac{\Delta_\tau x}{\tau} \right\rangle \leq -\left|\frac{\Delta_\tau x}{\tau}\right|^2, \quad (8)$$

because $\sup_{x \in \mathbb{H}} \|D_x \mathcal{G}\| \leq 1/4$, so $|\frac{\Delta_\tau \mathcal{G}(x)}{\tau}| \leq \frac{1}{4} |\frac{\Delta_\tau x}{\tau}|$. Note also that

$$\left|A \frac{\Delta_\tau x}{\tau}\right| = \left|\frac{\Delta_\tau(Ax)}{\tau}\right| = \left|S_\tau(d/dt(Ax))\right| = \left|S_\tau(-x + \mathcal{G}(x))\right| \leq 2S_\tau|x|$$

for all τ , where S_τ is the averaging operator of (5).

Thus, applying Corollary 2.2 with $h = A \frac{\Delta_\tau x}{\tau}$, $y = \frac{\Delta_\tau x}{\tau}$, $f = 0$, we get

$$\int_t^{+\infty} \left|\frac{\Delta_\tau x}{\tau}\right|^2 \leq \frac{4 \int_{t'}^t (S_\tau|x|)^2}{(t-t')^2}$$

for any $t > t' > 0$. Letting $t' \rightarrow 0$ when $0 < t < 1$ and putting $t' = t - 1$ when $t \geq 1$, then using the bounds $(S_\tau|x|)^2 \leq 1$ and $\int_{t-1}^\infty (S_\tau|x|)^2 \leq \int_{t-1}^{+\infty} |x|^2 \leq ee^{-t}$ respectively, we get

$$\int_t^{+\infty} \left|\frac{\Delta_\tau x}{\tau}\right|^2 \leq \begin{cases} \frac{4}{t}, & 0 < t < 1, \\ 4ee^{-t}, & t \geq 1. \end{cases}$$

Thus, $x \in H^1(t, +\infty)$ for any $t > 0$ and the same bounds hold for x' (Lemma A.1(i), Appendix A).

Moreover, we can now estimate $x(t)$ for $t \geq 1$ by writing

$$|x(t)|^2 = 2 \left| \int_t^{+\infty} \langle x, x' \rangle \right| \leq 2 \left(\int_t^{+\infty} |x|^2 \right)^{1/2} \left(\int_t^{+\infty} |x'|^2 \right)^{1/2} \leq Ce^{-t}.$$

Finally, applying Lemma A.1(i)–(ii), and passing to the L^2 -limit as $\tau \rightarrow 0^+$ in (7), we see that $Ax' \in H_{loc}^1(0, +\infty)$ and

$$(d/dt)(Ax') = -x' + D_x \mathcal{G}(x').$$

Specifically, we first apply Lemma A.1(ii) to the $H_{loc}^1(0, +\infty)$ functions x and $\mathcal{G}(x)$ to see that $\tau^{-1}\Delta_\tau x \rightarrow x'$ and $\tau^{-1}\Delta_\tau \mathcal{G}(x) \rightarrow (d/dt)\mathcal{G}(x) = D_x \mathcal{G}(x')$ on the right-hand side of (7). But, this implies that the limit

$$\lim_{\tau \rightarrow 0^+} \|(d/dt)A\tau^{-1}\Delta_\tau x\|_{L^2(t, +\infty)} = \lim_{\tau \rightarrow 0^+} \|\tau^{-1}\Delta_\tau Ax'\|_{L^2(t, +\infty)}$$

of the $L^2(t, +\infty)$ norm of the left-hand side of (7) exists and is bounded for all $t > 0$, hence $Ax' \in H_{loc}^1(0, +\infty)$ by Lemma A.1(i). Here, we have used the fact just established that $x \in H_{loc}^1(0, +\infty)$ to rewrite $(d/dt)A\tau^{-1}\Delta_\tau x = \tau^{-1}\Delta_\tau Ax'$. Applying Lemma A.1(ii), we find, finally, that

$$(d/dt)Ax' = \lim_{\tau \rightarrow 0^+} \tau^{-1}\Delta_\tau Ax' = -x' + D_x \mathcal{G}(x'). \quad (9)$$

3. (*Proof that $x \in W_{loc}^{1,4}(0, +\infty)$*). We shall start by fixing $\tau > 0$ and considering the difference $\Delta_\tau x'$. We have $A\Delta_\tau x' \in H^1(t, +\infty)$ for any $t > 0$ and, applying the linear difference operator Δ_τ to (9) and using the product rule $\Delta_\tau yz = y(\Delta_\tau z) + \Delta_\tau y(z(\cdot + \tau))$:

$$\begin{aligned} (d/dt)(A\Delta_\tau x') &= -\Delta_\tau x' + \Delta_\tau D_x \mathcal{G}(x') \\ &= -\Delta_\tau x' + D_x \mathcal{G}(\Delta_\tau x') + (\Delta_\tau(D_x \mathcal{G}))(x'(\cdot + \tau)). \end{aligned} \quad (10)$$

Passing to the corresponding differential equation for the quadratic form $\langle A\Delta_\tau x', \Delta_\tau x' \rangle$, we can combine the first two terms on the right-hand side using the bound

$$|D_x \mathcal{G}(\Delta_\tau x')| \leq \frac{1}{4} |\Delta_\tau x'|$$

to get

$$\begin{aligned} (d/dt)\langle A\Delta_\tau x', \Delta_\tau x' \rangle &\leq -\frac{3}{2} |\Delta_\tau x'|^2 + 2\langle \Delta_\tau x', \Delta_\tau(D_x \mathcal{G})(x'(\cdot + \tau)) \rangle \\ &\leq -|\Delta_\tau x'|^2 + 2\|\Delta_\tau D_x \mathcal{G}\|^2 |x'(\cdot + \tau)|^2 \\ &\leq -|\Delta_\tau x'|^2 + C|\Delta_\tau x'|^2 |x'(\cdot + \tau)|^2, \end{aligned}$$

where C is controlled by $\sup_{x \in \mathbb{H}} \|D_x^2 \mathcal{G}\|$. Here, we have used Young's inequality to bound the term $2\langle \Delta_\tau x', \Delta_\tau(D_x \mathcal{G})(x'(\cdot + \tau)) \rangle$ by an absorbable term $(1/2)|\Delta_\tau x'|^2$ plus $2\|\Delta_\tau D_x \mathcal{G}\|^2 |x'(\cdot + \tau)|^2$. Using the equation, and the condition $\|D_x \mathcal{G}\| \leq \frac{1}{4}$, we also have

$$|A(\Delta_\tau x')| = |\Delta_\tau(d/dt)(Ax)| = |\Delta_\tau(-x + \mathcal{G}(x))| \leq 2|\Delta_\tau x|.$$

Finally, we have by Jensen's inequality

$$|\Delta_\tau x|^2(t) = |\tau S_\tau x|^2(t) \leq \tau^2 S_\tau(|x'|^2)(t) \leq \tau \int_t^{+\infty} |x'|^2$$

for S_τ as in (5), which is at most $\frac{C\tau}{t}$ for $0 < t < 1$ and $C\tau e^{-t}$ for $t \geq 1$.

Applying Corollary 2.2 with $y = \Delta_\tau x'$, $h = A\Delta_\tau x'$, and $f = |\Delta_\tau x|^2 |x'(\cdot + \tau)|^2$, we get

$$\int_t^{+\infty} |\Delta_\tau x'|^2 \leq \int_{t'}^{+\infty} |\Delta_\tau x|^2 |x'(\cdot + \tau)|^2 + \frac{4 \int_{t'}^t |\Delta_\tau x|^2}{(t - t')^2}. \quad (11)$$

Plugging in the estimates obtained just above for $|\Delta_\tau x|^2(t)$ and in step 2 for $\int_t^{+\infty} |x'|^2$, and letting $t' = t/2$ for $0 < t < 1$ and $t' = t - 1/2$ for $t \geq 1$, we get, noting that $\tau > 0$ is arbitrary,

$$\sup_{\tau > 0} \int_t^{+\infty} \frac{|\Delta_\tau x'|^2}{\tau} \leq \begin{cases} Ct^{-2}, & 0 < t < 1, \\ Ce^{-t}, & t \geq 1. \end{cases}$$

By Lemma A.2, Appendix A, this together with our previous estimate

$$\int_t^{+\infty} |x'|^2 \leq C/t \quad \text{for } 0 < t < 1$$

is enough to conclude that $\int_t^{+\infty} |x'|^4 \leq (1/2)(C/t^2(C/t)) = (1/2)(C^2/t^3) \lesssim t^{-3}$ for $0 < t < 1$. Similarly, using our previous estimate $\int_t^{+\infty} |x'|^2 \leq Ce^{-t}$ for $t \geq 1$, we obtain $\int_t^{+\infty} |x'|^4 \leq C^2 e^{-2t}$ for $t \geq 1$. Combining, we have for some (possibly larger) $C > 0$

$$\int_t^{+\infty} |x'|^4 \leq \begin{cases} Ct^{-3}, & 0 < t < 1, \\ Ce^{-2t}, & t \geq 1. \end{cases}$$

4. (*Proof that $x \in H_{loc}^2(0, +\infty)$*). We shall use Corollary 2.2 once more with the same y, f, h , but this time we shall estimate the right-hand side of the inequality (11) in a different way. First note that

$$\begin{aligned} |A(\Delta_\tau x')|^2 &= |\Delta_\tau(Ax')|^2 \leq \tau^2 S_\tau(|(d/dt)(Ax')|)^2 \\ &= \tau^2 S_\tau(|-x' + D_x \mathcal{G}(x')|)^2 \leq 4\tau^2 S_\tau(|x'|)^2 \end{aligned}$$

with S_τ as in (5), so $\int_t^{+\infty} |A(\Delta_\tau x')|^2 \leq 4\tau^2 \int_t^{+\infty} |x'|^2$. Then observe that

$$\int_{t'}^{+\infty} |\Delta_\tau x|^2 |x'(\cdot + \tau)|^2 \leq \tau^2 \int_t^{+\infty} |S_\tau x'|^2 |x'(\cdot + \tau)|^2 \leq \tau^2 \|x'\|_{L^4(t, +\infty)}^4.$$

Thus, with the same choice $t' = t/2$ for $0 < t < 1$ and $t' = t - 1/2$ for $t \geq 1$, we arrive at the bound

$$\int_t^{+\infty} \frac{|\Delta_\tau x'|^2}{\tau^2} \leq \begin{cases} Ct^{-3}, & 0 < t < 1, \\ Ce^{-t}, & t \geq 1. \end{cases}$$

The difference ratios $\frac{\Delta_\tau x'}{\tau}$ are thus uniformly bounded in $L^2(t, +\infty)$ for any $t > 0$, hence, applying again Lemma A.1(i), Appendix A, we have $x \in H_{loc}^2(0, +\infty)$ and

$$\int_t^{+\infty} |x''|^2 \leq \begin{cases} Ct^{-3}, & 0 < t < 1, \\ Ce^{-t}, & t \geq 1. \end{cases}$$

Also,

$$\begin{aligned} |x'(t)|^2 &\leq 2 \left| \int_t^{+\infty} \langle x', x'' \rangle \right| \leq 2 \left(\int_t^{+\infty} |x'|^2 \right)^{1/2} \left(\int_t^{+\infty} |x''|^2 \right)^{1/2} \\ &\leq \begin{cases} Ct^{-2}, & 0 < t < 1, \\ Ce^{-t}, & t \geq 1, \end{cases} \end{aligned}$$

verifying (3) for $k = 1$. Finally, applying again Lemma A.1(i)–(ii) and passing to the L^2 -limit as $\tau \rightarrow 0^+$ on both sides of τ^{-1} times equation (10), we get $Ax'' \in H_{loc}^1(0, +\infty)$ and

$$(d/dt)(Ax'') = -x'' + D_x^2 \mathcal{G}(x', x') + D_x \mathcal{G}(x''). \quad (12)$$

Namely, we first observe that x' and $(d/dt)\mathcal{G}(x) = D_x\mathcal{G}(x')$ are both contained in $H_{loc}^1(0, +\infty)$, by the fact that $x' \in H_{loc}^1 \cap L_{loc}^\infty(0, +\infty)$ together with the uniform derivative bounds (2) on \mathcal{G} . Thus, we may argue as in the end of step 2 to obtain first convergence in $L^2(t, +\infty)$ of τ^{-1} times the right-hand side of (10) by Lemma A.1(ii), to $-x'' + (d/dt)D_x\mathcal{G}(x') = -x'' + D_x^2\mathcal{G}(x', x') + D_x\mathcal{G}(x'')$, for any $t > 0$. This implies convergence in $L^2(t, +\infty)$ of τ^{-1} times the left-hand side of (10), or

$$(d/dt)(\tau^{-1}\Delta_\tau x') = \tau^{-1}\Delta_\tau Ax'',$$

whence $Ax'' \in H_{loc}^1(0, +\infty)$ by Lemma A.1(i). Applying Lemma A.1(ii), we obtain finally that $\lim_{\tau \rightarrow 0^+} \tau^{-1}\Delta_\tau Ax'' = (d/dt)Ax''$, yielding (12) by equality of left- and right-hand limits.

5. (*Proof that $x \in H_{loc}^J(0, +\infty)$, $J \geq 3$*). The rest of the argument we carry out by induction. Specifically, at each level J , starting with $J = 2$, we make the following induction hypotheses.

- (I1) For $0 \leq k < J$, $x \in W_{loc}^{\infty, k}(0, +\infty)$, with $(d/dt)^k x$ satisfying (3) for $t > 0$.
- (I2) For $1 \leq k \leq J$, $x \in H_{loc}^k(0, +\infty)$, with

$$\int_t^{+\infty} |(d/dt)^k x|^2 \leq \begin{cases} Ct^{-2k+1}, & 0 < t < 1, \\ Ce^{-t}, & t \geq 1. \end{cases} \quad (13)$$

- (I3) For $1 \leq k \leq J$, $A(d/dt)^k x \in H_{loc}^1(0, +\infty)$ and $x, \mathcal{G}(x) \in H_{loc}^k(0, +\infty)$, with
- $$(d/dt)A(d/dt)^k x = -(d/dt)^k x + (d/dt)^k \mathcal{G}(x) \quad \text{in } L_{loc}^2(0, +\infty). \quad (14)$$

We have shown in Step 4 that (I1)–(I3) are satisfied for $J = 2$, i.e., $x \in W_{loc}^{\infty, 1}(0, +\infty)$ and (3) holds for $k = 0, 1$; $x \in H_{loc}^2(0, +\infty)$, and (13) holds for $k = 1, 2$; and $x, \mathcal{G}(x)$ are in $H_{loc}^2(0, +\infty)$, and $A(d/dt)^2 x$ is in $H_{loc}^1(0, +\infty)$, satisfying (14) in the $L_{loc}^2(0, +\infty)$ sense for $k = 0, 1, 2$. We now show that satisfaction of (I1)–(I3) at level $J = j \geq 2$ implies satisfaction of (I1)–(I3) at level $J = j + 1$, whence, by induction, (I1)–(I3) hold for all $J \geq 2$. This implies that x is C^∞ on $(0, +\infty)$ and satisfies (3) for all $k \geq 0$, completing the proof.

By (I3), we have that $(d/dt)A\Delta_\tau(d/dt)^j x$, $\Delta_\tau(d/dt)^j x$, and $\Delta_\tau(d/dt)^j \mathcal{G}(x)$ are in $L_{loc}^2(0, +\infty)$, with

$$(d/dt)A\Delta_\tau(d/dt)^j x = -\Delta_\tau(d/dt)^j x + \Delta_\tau(d/dt)^j \mathcal{G}(x). \quad (15)$$

Repeated application of the chain rule gives the expansion

$$(d/dt)^j \mathcal{G}(x) = \sum_{l=1}^j \sum_{s_1 + \dots + s_l = j-l, s_j \geq 0} C_{l,s}^j D_x^l \mathcal{G}((d/dt)^{s_1+1} x, \dots, (d/dt)^{s_l+1} x), \quad (16)$$

with $C_{l,s}^j$ denoting the number of occurrences of the derivative distribution $s = (s_1, \dots, s_l)$. In particular, there is only one term for $l = 1$, namely $C_{1,(j)}^j = 1$. Thus, we have

$$\begin{aligned} \Delta_\tau(d/dt)^j \mathcal{G}(x) &= \Delta_\tau \sum_{l=2}^j \sum_{s_1 + \dots + s_l = j-l, s_j \geq 0} C_{l,s}^j D_x^l \mathcal{G}((d/dt)^{s_1+1} x, \dots, (d/dt)^{s_l+1} x) \\ &\quad + (\Delta_\tau D_x \mathcal{G})((d/dt)^j x) + D_{x(\cdot+\tau)} \mathcal{G}(\Delta_\tau(d/dt)^j x). \end{aligned} \quad (17)$$

Noting that

$$|D_{x(\cdot+\tau)}\mathcal{G}(\Delta_\tau(d/dt)^j x)| \leq (1/4)|\Delta_\tau(d/dt)^j x| \quad (18)$$

by (2), and arguing as in step 3, we find therefore for $y = \Delta_\tau(d/dt)^j x$ and $h = A\Delta_\tau(d/dt)^j x$ that

$$\langle y, h \rangle' \leq -|y|^2 + f,$$

where $f = 2|g|^2$ with $g := \Delta_\tau(d/dt)^j \mathcal{G}(x) - D_{x(\cdot+\tau)}\mathcal{G}(\Delta_\tau(d/dt)^j x)$.

With an eye toward applying Corollary 2.2, we first bound $\int_t^{+\infty} |f| = 2 \int_t^{+\infty} |g|^2$. Evidently, g is given by the sum of the first two terms on the right-hand side of (17). The first, $\sum_{\ell=2}^j$ term, involves only derivatives $(d/dt)^{s_i+1}x$ of order $< j$, since there are at least two summands $s_i + 1$, and the sum of all such is j . Thus, we may use the relation $\Delta_\tau = \tau S_\tau(d/dt)$ to rewrite this term as

$$\tau S_\tau \sum_{l=2}^{j+1} \sum_{s_1+\dots+s_l=j+1-l, s_j \geq 0} B_{l,s}^j D_x^l \mathcal{G}((d/dt)^{s_1+1}x, \dots, (d/dt)^{s_l+1}x) \quad (19)$$

with $B_{l,s}^j$ integer valued, where, in each summand, at most one of the derivatives $(d/dt)^{s_i+1}x$ is of order j , and the rest are of order $< j$. Here, we are using in an important way the fact that $j \geq 2$. For $j = 1$, the multi-index $s = (1, 1)$ gives two derivatives of highest order j , a fact that cost some extra effort in steps 3–4.

By (I2)–(I3), therefore, the highest-order derivative appearing in each summand is bounded in $L^2(t, +\infty)$ by $Ct^{-(s_i+1)+1/2}$ for $0 < t < 1$ and $Ce^{-t/2}$ for $t \geq 1$, and the remaining derivatives are bounded in $L^\infty(t, +\infty)$ by $Ct^{-(s_i+1)}$ for $0 < t < 1$ and $Ce^{-t/2}$ for $t \geq 1$. Combining these bounds with the uniform derivative bounds (2) on \mathcal{G} and the fact that S_τ is bounded from $L^2(t, +\infty)$ to itself, we thus find that the $L^2(t, +\infty)$ norm of each summand in (19) is bounded by

$$\tau t^{-(\sum_{i=1}^l (s_i+1))+1/2} = \tau t^{-(j+1)+1/2}$$

for $0 < t < 1$ and $\tau e^{-t/2}$ for $t \geq 1$.

Likewise, in the remaining term $(\Delta_\tau D_x \mathcal{G})((d/dt)^j x)$ of g , operator

$$\Delta_\tau D_x \mathcal{G} = \tau S_\tau(d/dt) D_x \mathcal{G} = \tau S_\tau D_x^2 \mathcal{G}(x', \cdot)$$

involves only derivatives of x of order $1 < j$, hence its operator norm may be estimated using the $L^\infty(t, +\infty)$ bound (3) of induction hypothesis (I1), the bounds (2), and boundedness in operator norm of the averaging operator S_τ as $\|\Delta_\tau D_x \mathcal{G}\| \lesssim \tau t^{-1}$ for $0 < t < 1$ and $\lesssim \tau e^{-t/2}$ for $t \geq 1$. Together with bounds

$$\|(d/dt)^j x\|_{L^2(t, +\infty)} \lesssim t^{-j+1/2} \quad \text{for } 0 < t < 1$$

and $\|(d/dt)^j x\|_{L^2(t, +\infty)} \lesssim e^{-t}$ for $t \geq 1$ of induction hypothesis (I2), this gives

$$\|(\Delta_\tau D_x \mathcal{G})((d/dt)^j x)\|_{L^2(t, +\infty)} \lesssim \begin{cases} \tau t^{-1} t^{-j+1/2} = \tau t^{-(j+1)+1/2} & \text{for } 0 < t < 1, \\ \tau e^{-t/2} & \text{for } t \geq 1. \end{cases} \quad (20)$$

Combining the above estimates, we obtain finally

$$\|g\|_{L^2(t, +\infty)} \lesssim \begin{cases} \tau t^{-1} t^{-j+1/2} = \tau t^{-(j+1)+1/2} & \text{for } 0 < t < 1 \text{ and} \\ \tau e^{-t/2} & \text{for } t \geq 1, \end{cases} \quad (21)$$

and thus

$$\int_t^{+\infty} |f| = 2 \int_t^{+\infty} |g|^2 \lesssim \begin{cases} \tau^2 t^{-2(j+1)+1}, & 0 < t < 1, \\ \tau^2 e^{-t}, & t \geq 1. \end{cases}$$

We next bound $\int_t^{+\infty} |h|^2$. Arguing as in previous steps, we have

$$h = A\Delta_\tau(d/dt)^j x = \tau S_\tau(d/dt)A(d/dt)^j x = \tau S_\tau(-(d/dt)^j x + (d/dt)^j \mathcal{G}(x)).$$

Expanding $(d/dt)^j \mathcal{G}(x)$ as in (16) and noting that there appears in each term at most one derivative of x of order j and the rest of order $< j$, we find that the $L^2(t, +\infty)$ norm of $(d/dt)^j \mathcal{G}(x)$ can be bounded using induction hypotheses (I1)–(I2) by $Ct^{-j+1/2}$ for $0 < t < 1$ and $Ce^{-t/2}$ for $t \geq 1$. Likewise, the $L^2(t, +\infty)$ norm of $(d/dt)^j x$ is bounded, by the induction hypothesis, by $Ct^{-j+1/2}$ for $0 < t < 1$ and $Ce^{-t/2}$ for $t \geq 1$. Thus, noting again the harmless effect of averaging operator S_τ , we obtain

$$\int_t^{+\infty} |h|^2 \lesssim \begin{cases} \tau^2 t^{-2j+1}, & 0 < t < 1, \\ \tau^2 e^{-t}, & t \geq 1. \end{cases}$$

Applying Corollary 2.2 with $t' = t/2$ for $0 < t < 1$ and $t' = t - 1/2$ for $t \geq 1$, we obtain therefore $\int_t^{+\infty} |\Delta_\tau(d/dt)^j x|^2 \lesssim \tau^2 t^{-2(j+1)+1}$ for $0 < t < 1$ and $\lesssim \tau^2 e^{-t}$ for $t \geq 1$, hence, by Lemma A.1(i), $\int_t^{+\infty} |(d/dt)^{j+1} x|^2 \lesssim t^{-2(j+1)+1}$ for $0 < t < 1$ and $\lesssim e^{-t}$ for $t \geq 1$, giving $x \in H_{loc}^{j+1}(0, +\infty)$ and verifying the H^k bound of induction hypothesis (I2) for $k = j + 1$. This verifies (I2) for $J = j + 1$.

Moreover, applying Lemma A.1(i)–(ii) as in the last parts of Steps 2 and 4, we obtain in the $L^2(t, +\infty)$ limit as $\tau \rightarrow 0^+$ of

$$(d/dt)A\tau^{-1}\Delta_\tau(d/dt)^j x = -\tau^{-1}\Delta_\tau(d/dt)^j x + \tau^{-1}\Delta_\tau(d/dt)^j \mathcal{G}(x) \quad (22)$$

that $A(d/dt)^{j+1}x \in H_{loc}^1(0, +\infty)$ and $x, \mathcal{G}(x) \in H_{loc}^{j+1}(0, +\infty)$, with

$$(d/dt)A(d/dt)^{j+1}x = -(d/dt)^{j+1}x + (d/dt)^{j+1}\mathcal{G}(x), \quad (23)$$

verifying induction hypothesis (I3) at level $J = j + 1$.

Specifically, recalling that $\Delta_\tau(d/dt)^j \mathcal{G}(x) = g + D_{x(\cdot+\tau)}\mathcal{G}(\Delta_\tau(d/dt)^j x)$, we have from (18), (21), and the previously-obtained bounds on $y = \Delta_\tau(d/dt)^j x$ together with boundedness of \mathcal{G}_x that

$$\|\Delta_\tau(d/dt)^j \mathcal{G}(x)\|_{L_{loc}^2(t, +\infty)} \lesssim \begin{cases} \tau t^{-1} t^{-j+1/2} = \tau t^{-(j+1)+1/2} & \text{for } 0 < t < 1 \text{ and} \\ \tau e^{-t/2} & \text{for } t \geq 1, \end{cases}$$

giving $\mathcal{G}(x) \in H_{loc}^{j+1}$, by Lemma A.1(i), and therefore

$$\lim_{\tau \rightarrow 0^+} \tau^{-1}\Delta_\tau(d/dt)^j \mathcal{G}(x) = (d/dt)^{j+1}\mathcal{G}$$

by Lemma A.1(ii). By the fact $x \in H_{loc}^{j+1}$ verified in the previous step, we have by Lemma A.1(ii) also

$$\lim_{\tau \rightarrow 0^+} \tau^{-1}\Delta_\tau(d/dt)^j x = -(d/dt)^{j+1}x.$$

Thus, the right-hand side of (22) converges to $-(d/dt)^{j+1}x + (d/dt)^{j+1}\mathcal{G}(x)$ as $\tau \rightarrow 0^+$.

Using $x \in H_{loc}^{j+1}$ to re-express the left-hand side of (22) as $\tau^{-1}\Delta_\tau(A(d/dt)^{j+1}x)$, we obtain by Lemma A.1(i) that $A(d/dt)^{j+1}x \in H_{loc}^1$, and thus by Lemma A.1(ii), the left-hand side of (22) converges to $(d/dt)A(d/dt)^{j+1}x$. Comparing the limits

of left- and right-hand sides of (22) then yields (23), completing the verification of (I3) for $J = j + 1$.

Finally, from our H^{j+1} and H^j bounds, we obtain the $L^\infty(t, +\infty)$ estimate (3) for $k = j$ by the one-dimensional Sobolev embedding estimate

$$\begin{aligned} \|(d/dt)^j x\|_{L^\infty(t, +\infty)}^4 &\leq \|(d/dt)^j x\|_{L^2(t, +\infty)}^2 \|(d/dt)^{j+1} x\|_{L^2(t, +\infty)}^2 \\ &\lesssim t^{-2j+1} t^{-2(j+1)+1} = t^{-4j} \end{aligned}$$

for $0 < t < 1$ and $\lesssim e^{-2t}$ for $t \geq 1$. This verifies induction hypothesis (II) for $J = j + 1$, completing the induction and the proof. \square

Remark 2.4. The instantaneous smoothing result (3) of Theorem 1.3 can be motivated by the canonical example of linear diagonal flow $Ax' = -x$ on $\mathbb{H} = \ell^2$. Writing $x = \{x_j\}$, with $\sum_j |x_j|^2 < \infty$, take $Ax = \{a_j x_j\}$, with $a_j > 0$ and $a_j \rightarrow 0$ as $j \rightarrow \infty$. Then, for $|x(0)| = 1$,

$$|(d/dt)^k x(t)| \leq \sup_j a_j^{-k} e^{-a_j^{-1}t} |x(0)| \lesssim t^{-k}, \quad (24)$$

in agreement with (3). Noting that $\max_{z \in \mathbb{R}^+} z^{-k} e^{-z^{-1}t} = e^{-1} t^{-k}$ is attained at $z = t$, we find by considering initial data of form $e_n = (0, \dots, 0, 1, 0, \dots)$ with $n \rightarrow \infty$, and evaluating at $t = a_n$, that (24), hence also (3), is sharp in the sense that there is no better uniform bound as $t \rightarrow 0^+$.

3. Linear existence theory: Weak vs. mild solutions. We next compare our notion of “weak L_{loc}^2 ” solution with that of “mild” solution defined in [8], in the process establishing linear existence and uniqueness of weak L_{loc}^2 solutions. We show, first, that mild solutions are solutions in our sense as well, hence subject to the smoothing results of section 2, and, second, that solutions $x \in L^2(t_0, t_1)$ in our sense are mild solutions in the sense of [8] precisely if $\lim_{t \rightarrow \tau_0} |A|^{1/2} x(t)$ and $\lim_{t \rightarrow \tau_1} |A|^{1/2} x(t)$ lie in $\text{Range } |A|^{1/2}$. Here, $|A|^{1/2}$ as we now describe is defined via the spectral decomposition formula for bounded self-adjoint operators.

Recall [11] that a bounded self-adjoint operator A on \mathbb{H} admits a spectral decomposition

$$A = \int_{\mathbb{R}} \alpha dE_\alpha, \quad \text{Id} = \int_{\mathbb{R}} dE_\alpha, \quad \langle x, y \rangle = \int_{\mathbb{R}} \langle x, dE_\alpha y \rangle, \quad (25)$$

where dE_α is a projection-valued measure. Following the standard operator calculus, we define $\text{sgn}(A) := \int_{\mathbb{R}} \text{sgn}(\alpha) dE_\alpha$, $|A| = \text{sgn}(A)A := \int_{\mathbb{R}} |\alpha| dE_\alpha$, and

$$|A|^r := \int_{\mathbb{R}} |\alpha|^r dE_\alpha \text{ for real } r > 0.$$

3.1. The linear boundary-value problem and prescription of data for weak L_{loc}^2 solutions. The comparison of weak and mild solutions hinges ultimately on the question, of interest in its own right, of how or in what sense boundary data is attained for weak solutions that are merely L_{loc}^2 .

Formalizing the discussion of the introduction, we make the following definition.

Definition 3.1. For $f \in L_{loc}^2$ on a given domain D , a *weak L_{loc}^2 solution* of the linear inhomogeneous equation

$$(d/dt)(Ax) + x = f \quad (26)$$

is a function $x(t)$ such that $x \in L_{loc}^2(D)$, $Ax \in H_{loc}^1(D)$, and (26) holds on $L_{loc}^2(D)$.

We begin by observing that a weak L^2_{loc} solution, or indeed any function with $x \in L^2_{loc}$ and $Ax \in H^1_{loc}$, admits a representative for which $|A|^{1/2}x$ is continuous. Moreover, if $x, (d/dt)(Ax) \in L^2(\tau_0, \tau_1)$, then this representative extends continuously to $[\tau_0, \tau_1]$. Thus, for solutions on (t_0, t_1) , we may speak of the *boundary values* of $|A|^{1/2}x$ at $t = \tau_0, \tau_1$.

Lemma 3.2 (Extension to \mathbb{R}). *Let $x, (d/dt)(Ax) \in L^2(\tau_0, \tau_1)$. Then, there exists an extension \tilde{x} of x to the whole line such that*

$$\|\tilde{x}\|_{L^2(\mathbb{R})} + \|(d/dt)(A\tilde{x})\|_{L^2(\mathbb{R})} \lesssim \|x\|_{L^2(\tau_0, \tau_1)} + \|(d/dt)(Ax)\|_{L^2(\tau_0, \tau_1)}.$$

Proof. We define \tilde{x} as the even reflection of x across boundaries $t = \tau_0$ and $t = \tau_1$, supported on $[\tau_0 - d, \tau_1 + d]$, with $d = \tau_1 - \tau_0$, that is:

$$\tilde{x}(t) = \begin{cases} x(\tau_0 + (\tau_0 - t)), & t \in [\tau_0 - d, \tau_0] \\ x(t), & t \in [\tau_0, \tau_1] \\ x(\tau_1 - (t - \tau_1)), & t \in [\tau_1, \tau_1 + d]. \end{cases}$$

Clearly \tilde{x} is in L^2 , with L^2 norm bounded by three times that of x .

We show next that $A\tilde{x}$ has a weak derivative $\tilde{y} = (d/dt)(A\tilde{x})$ equal to the odd reflection of $y := (d/dt)(Ax)$ across boundaries $t = \tau_0$ and $t = \tau_1$, likewise with L^2 norm bounded by three times that of y on (τ_0, τ_1) :

$$\tilde{y}(t) = \begin{cases} -y(\tau_0 + (\tau_0 - t)), & t \in [\tau_0 - d, \tau_0] \\ y(t), & t \in [\tau_0, \tau_1] \\ -y(\tau_1 - (t - \tau_1)), & t \in [\tau_1, \tau_1 + d], \end{cases}$$

That is, we claim [4] that

$$\int \langle (d/dt)\phi, A\tilde{x} \rangle = - \int \langle \phi, \tilde{y} \rangle \quad (27)$$

for all test functions $\phi \in C^\infty_0(\tau_0 - d, \tau_1 + d)$.

For test functions ϕ that are even reflections about $t = \tau_1$ and supported in $(\tau_0, \tau_1 + d)$, this follows because $\langle (d/dt)\phi, A\tilde{x} \rangle$ and $\langle \phi, y \rangle$ are both odd about $t = \tau_1$ as inner products of even and odd functions, hence have integral zero. For test functions $\phi \in C^\infty_0(\tau_0, \tau_1)$, it follows because the restrictions of \tilde{y} and \tilde{x} to (τ_0, τ_1) are $y = (d/dt)(Ax)$ and x . Because test functions $\phi \in H^1_0(\tau_0, \tau_1)$ may be uniformly approximated in $H^1(\mathbb{R})$ by test functions in $C^\infty_0(\tau_0, \tau_1)$, (27) follows also for test functions $\phi \in H^1_0(\tau_0, \tau_1)$. By reflection, we find that (27) holds also for test functions ϕ in $H^1_0(\tau_0 - d, \tau_0)$ or $H^1_0(\tau_1, \tau_1 + d)$.

But, any test function in $C^\infty_0(\tau_0 - d, \tau_1 + d)$ may be decomposed into the sum of test functions that are even around τ_0 and supported in $(\tau_0 - d, \tau_1)$, even around τ_1 and supported in $(\tau_0, \tau_1 + d)$, plus test functions in $H^1_0(\tau_0 - d, \tau_0)$, $H^1_0(\tau_0, \tau_1)$, and $H^1_0(\tau_1, \tau_1 + d)$, whence (27) holds by linear superposition for arbitrary $\phi \in C^\infty_0(\mathbb{R})$ supported on $(\tau_0 - d, \tau_1 + d)$. Finally, we may multiply \tilde{x} by a smooth bump function that is identically equal to one on $[\tau_0, \tau_1]$ and identically equal to zero outside $(\tau_0 - d, \tau_1 + d)$ to obtain an extension of x defined on all of \mathbb{R} and satisfying the same bounds. \square

Corollary 3.3. *If $x, (d/dt)(Ax) \in L^2(\tau_0, \tau_1)$, then $|A|^{1/2}x$ may be taken to be in $C^0[\tau_0, \tau_1]$, with*

$$\||A|^{1/2}x\|_{C^0[\tau_0, \tau_1]} \lesssim \|x\|_{L^2(\tau_0, \tau_1)} + \|(d/dt)(Ax)\|_{L^2(\tau_0, \tau_1)}.$$

In particular, $|A|^{1/2}x$ may be taken to be continuous wherever $x, (d/dt)(Ax) \in L^2_{loc}$.

Proof. Observing that the bounded linear operator $\text{sgn}(A)$ commutes with weak differentiation, and applying Lemma 3.2, we may assume without loss of generality that $x, (d/dt)(|A|x) \in L^2(\mathbb{R})$, with $\|x\|_{L^2(\mathbb{R})} \lesssim \|x\|_{L^2(\tau_0, \tau_1)}$ and

$$\|(d/dt)(|A|x)\|_{L^2(\mathbb{R})} \lesssim \|x\|_{L^2(\tau_0, \tau_1)} + \|(d/dt)(Ax)\|_{L^2(\tau_0, \tau_1)}.$$

Arguing as in Section 2, we have that $\langle |A|^{1/2}x, |A|^{1/2}x \rangle = \langle |A|x, x \rangle$ is absolutely continuous and in $L^2(\mathbb{R})$, with derivative $2\langle (d/dt)(|A|x), x \rangle$, whence

$$\langle |A|^{1/2}x, |A|^{1/2}x \rangle(t) = 2 \int_{-\infty}^t \langle (d/dt)(|A|x), x \rangle d\tau.$$

Bounding $\int_{-\infty}^t \langle (d/dt)(|A|x), x \rangle d\tau \leq \|(d/dt)(|A|x)\|_{L^2(\mathbb{R})} \|x\|_{L^2(\mathbb{R})}$ by Cauchy-Schwarz, and applying Young's inequality, we obtain

$$\begin{aligned} \| |A|^{1/2}x \|_{L^\infty[\tau_0, \tau_1]} &\lesssim \|x\|_{L^2(\mathbb{R})} + \|(d/dt)(Ax)\|_{L^2(\mathbb{R})} \\ &\lesssim \|x\|_{L^2(\tau_0, \tau_1)} + \|(d/dt)(Ax)\|_{L^2(\tau_0, \tau_1)}. \end{aligned} \quad (28)$$

Continuity of $|A|^{1/2}x$ follows, finally, by a standard mollification argument, approximating x by $x^\varepsilon := x * \eta^\varepsilon$, where $\eta^\varepsilon(t) = \varepsilon^{-1}\eta(t/\varepsilon)$ is a smooth mollification kernel, $\eta \geq 0$ a C^∞ bump function equal to 1 for $|t| \leq 1/4$ and 0 for $|t| \geq 1$, with $\int_{\mathbb{R}} \eta(t)dt = 1$. Noting that $x^\varepsilon \rightarrow x$ and $(d/dt)(|A|x^\varepsilon) \rightarrow (d/dt)(|A|x)$ in L^2 [4], we find by (28) applied to $|A|^{1/2}x^\varepsilon$ that

$$\begin{aligned} \| |A|^{1/2}x^{\varepsilon_1} - |A|^{1/2}x^{\varepsilon_2} \|_{L^\infty[\tau_0, \tau_1]} &\leq \|x^{\varepsilon_1} - x^{\varepsilon_2}\|_{L^2(\mathbb{R})} \\ &\quad + \|(d/dt)(Ax^{\varepsilon_1}) - (d/dt)(Ax^{\varepsilon_2})\|_{L^2(\mathbb{R})} \rightarrow 0 \end{aligned}$$

as $\varepsilon_1, \varepsilon_2 \rightarrow 0$. Thus, the sequence $\{|A|^{1/2}x^\varepsilon\}$ is Cauchy in $L^\infty[\tau_0, \tau_1]$, whence $|A|^{1/2}x$ is a uniform limit of the continuous (indeed C^∞) functions $|A|^{1/2}x^\varepsilon$ as $\varepsilon \rightarrow 0$, and thus continuous. \square

We next recall two fundamental resolvent estimates from [8]. In the remainder of this section, in order to use Fourier transform techniques, we complexify the real Hilbert space \mathbb{H} in the standard way [2, Ch. 1, Ex. 1.7]. as $\mathbb{H} + i\mathbb{H}$ with inner product

$$\langle g_1 + ig_2, f_1 + if_2 \rangle := (\langle g_1, f_1 \rangle + \langle g_2, f_2 \rangle) + i(\langle g_1, f_2 \rangle - \langle g_2, f_1 \rangle).$$

Lemma 3.4 (cf. Lemma 3.4, [8]). *The Fourier symbol $(i\omega A + \text{Id})$, $\omega \in \mathbb{R}$ of $(A(d/dt) + \text{Id})$ satisfies*

$$\sup_{\omega \in \mathbb{R}} \|(i\omega A + \text{Id})^{-1}\| \leq 1, \quad \sup_{\omega \in \mathbb{R}} \|i\omega A(i\omega A + \text{Id})^{-1}\| \leq 2. \quad (29)$$

Proof. The first inequality follows by symmetry of A , which implies that the symmetric part of $(i\omega A + \text{Id})$ is just Id , hence bounded below by 1. The second one then follows by resolvent identity

$$i\omega A(i\omega A + \text{Id})^{-1} = \text{Id} - (i\omega A + \text{Id})^{-1}.$$

\square

From (29) and Parseval's identity, we find for $f \in L^2(\mathbb{R})$ that x defined by

$$\widehat{x}(\omega) := (i\omega A + \text{Id})^{-1} \widehat{f}(\omega) \quad (30)$$

\hat{g} denoting Fourier transform of g , gives a unique solution $x, (d/dt)(Ax) \in L^2(\mathbb{R})$ of (26) for $f \in L^2(\mathbb{R})$, that is, a weak L^2_{loc} solution in the sense of Definition 3.1 for the linear inhomogeneous problem on the line.

Now, define stable, center, and unstable projections

$$\begin{aligned}\Pi_s &:= \chi_{(0,+\infty)}(A) = \int_{\mathbb{R}} \chi_{\alpha>0} dE_\alpha, \\ \Pi_c &:= \chi_0(A) = \int_{\mathbb{R}} \chi_{\alpha=0} dE_\alpha, \\ \Pi_u &:= \chi_{(-\infty,0)}(A) = \int_{\mathbb{R}} \chi_{\alpha<0} dE_\alpha\end{aligned}$$

of $-A$, where χ_I denotes indicator function associated with set I , and stable, center, and unstable subspaces $\Sigma_s = \text{Range } \Pi_s$, $\Sigma_c = \text{Range } \Pi_c$, and $\Sigma_u = \text{Range } \Pi_u$.

It is straightforward to see that the operator-valued functions

$$\begin{aligned}T_s(t) &:= \int_{\mathbb{R}} \chi_{\alpha>0} e^{-t/\alpha} dE_\alpha, \quad t \geq 0 \\ T_u(t) &:= \int_{\mathbb{R}} \chi_{\alpha<0} e^{-t/\alpha} dE_\alpha, \quad t \leq 0,\end{aligned}\tag{31}$$

corresponding formally with $e^{-tA^{-1}}\Pi_s$ and $e^{-tA^{-1}}\Pi_u$, are strongly continuous with $T_s(0) = \text{Id}_{\Sigma_s}$ and $T_u(0) = \text{Id}_{\Sigma_u}$,³ and for $h_s \in \Sigma_s$ and $h_u \in \Sigma_u$ generate solutions $x_s(t) := T_s(t)h_s$ and $x_u(t) := T_u(t)h_u$ of the homogeneous equation $((d/dt)A + \text{Id})x = 0$ in forward and backward time, respectively, via $x_s(t) := T_s(t)h_s$ and $x_u(t) := T_s(t)h_u$. For,

$$\begin{aligned}|(T_s(t+\delta) - T_s(t))h|^2 &= \int_{\mathbb{R}} \chi_{\alpha>0} (1 - e^{-\delta/\alpha})^2 e^{-2(t+\delta)/\alpha} (h, dE_\alpha h) \\ &\leq \int_{\mathbb{R}} \chi_{\alpha>0} (1 - e^{-\delta/\alpha})^2 (h, dE_\alpha h) \rightarrow 0\end{aligned}$$

as $\delta \rightarrow 0^+$, for each fixed h , by Lebesgue Dominated Convergence, and similarly for $-t < \delta \rightarrow 0^-$. A symmetric argument yields the result for T_u .

These are exactly the “bi-stable semigroups” constructed by quite different, Fourier transform means in [8] (cf. [8, §2]). Note that in general, T_s is not bounded in the backward time direction, nor T_u in forward time direction, as $|\alpha|$ may be arbitrarily small, yielding arbitrarily large exponential growth $e^{|t/\alpha|}$. In particular, the Cauchy problem $((d/dt)A + \text{Id})x = 0$ for $t \geq 0$, $x(0) = x^0 \in \mathbb{H}$ is ill-posed in both forward and backward time directions. Note also that the only homogeneous solutions on center subspace Σ_c are, by inspection, the trivial ones $x(t) \equiv 0$.

For real $r > 0$, define the unbounded operator $|A|^{-r}$ as the inverse of $|A|^r$ from $\text{Range } |A|^r$ to $\Sigma_s \oplus \Sigma_u$, that is, $|A|^{-r}x = \int_{\mathbb{R} \setminus \{0\}} |\alpha|^{-r} dE_\alpha x$ for $x \in \text{Range } |A|^r$. The next result shows that $|A|^{-1/2}T_s$ and $|A|^{-1/2}T_u$, give solution operators for boundary data $|A|^{1/2}x(0)$ in Σ_s and Σ_u , respectively.

Lemma 3.5. *For $t > 0$ and $t < 0$, respectively, $T_s(t)$ and $T_u(t)$ take \mathbb{H} to $\text{Range } |A|^r$ for any $r > 0$, with sharp bounds*

$$||A|^{-r}T_s(t)h| \leq Ct^{-r}|h| \text{ for } t > 0 \text{ and } ||A|^{-r}T_u(t)h| \leq Ct^{-r}|h| \text{ for } t < 0.\tag{32}$$

³Recall [7] that strong continuity of an operator $T(t)$ on \mathbb{H} is defined as continuity of $T(t)x$ for each fixed $x \in \mathbb{H}$.

In particular, $x_s(t) := |A|^{-1/2}T_s(t)g_s$ and $x_u(t) := |A|^{-1/2}T_u(t)g_u$ are well-defined for any $g_s \in \Sigma_s$ and $g_u \in \Sigma_u$. Moreover,

$$x_s, (d/dt)(Ax_s) \in L^2(\mathbb{R}^+) \text{ and } x_u, (d/dt)(Ax_u) \in L^2(\mathbb{R}^-), \quad (33)$$

so that x_s and x_u solve $(d/dt)(Ax) + x = 0$ in forward and backward time, respectively, with boundary values

$$|A|^{1/2}x_s(0) = g_s \text{ and } |A|^{1/2}x_u(0) = g_u, \quad (34)$$

and these solutions are unique in the class $x, (d/dt)(Ax) \in L^2(0, \tau)$, for all $\tau > 0$ and $\tau < 0$, respectively.

Proof. Noting that $\text{Range } |A|^r$ consists of x such that $\int_{\mathbb{R} \setminus \{0\}} \langle x, |\alpha|^{-2r} dE_\alpha x \rangle < +\infty$, we obtain the first result from

$$\langle T_s(t)h, |\alpha|^{-2r} dE_\alpha T_s(t)h \rangle = \langle h, \chi_{\alpha>0} e^{-2t/\alpha} |\alpha|^{-2r} dE_\alpha h \rangle$$

together with $|\alpha|^{-2r} e^{-2t/\alpha} \lesssim t^{-2r}$ for $t, \alpha > 0$, as follows from $z^{2r} \lesssim e^{2z}$ for $z \in \mathbb{R}^+$. This gives at the same time (32), which, by taking data h with measure $(h, dE_\alpha h)$ supported near $\alpha = t$, is easily seen to be sharp.

The second assertion, (33), follows similarly by the observation that

$$\int_0^{+\infty} (|\alpha|^{-1/2} e^{-t/\alpha})^2 dt = \int_0^{+\infty} |\alpha|^{-1} e^{-2t/\alpha} dt = \int_0^{+\infty} e^{-2z} dz = \text{constant},$$

by substitution $z = t/\alpha$, $dz = \alpha^{-1} dt$. This gives by Fubini's Theorem

$$\begin{aligned} \|x_s\|_{L^2(\mathbb{R}^+)}^2 &= \int_{\mathbb{R}^+} \langle |A|^{-1/2} T_s g_0, |A|^{-1/2} T_s g_0 \rangle \\ &= \int_{\mathbb{R}^+} \langle g_0, |A|^{-1} T_s^2 g_0 \rangle \\ &= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |\alpha|^{-1} e^{-2t/\alpha} \langle g_0, dE_\alpha g_0 \rangle dt \\ &= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |\alpha|^{-1} e^{-2t/\alpha} dt \langle g_0, dE_\alpha g_0 \rangle \\ &= C |g_0|^2, \end{aligned}$$

hence $x_s \in L^2(\mathbb{R}^+)$ and, by $(d/dt)(Ax_s) = -x_s$, also $(d/dt)(Ax_s) \in L^2(\mathbb{R}^+)$. A similar computation gives (33) for T_u . Meanwhile, (34) follows by $|A|^{1/2}x_s = T_s(t)g_s$, $|A|^{1/2}x_u = T_u(t)g_u$ and strong continuity of T_s, T_u at $t = 0^\pm$.

Finally, uniqueness can be seen by an argument like that of Section 2, after first projecting by Π_s, Π_u , and Π_c onto Σ_s, Σ_u , and Σ_c . For, $|A|^{1/2}x(0) = 0$ implies

$$\langle x, Ax \rangle(0) = 0.$$

But, for solutions $x, (d/dt)(Ax) \in L^2$ of homogeneous equation $(d/dt)(Ax) = -x$, the quadratic form $\langle x, Ax \rangle$ is absolutely continuous, with derivative $-\langle x, x \rangle \leq 0$. Restricted to Σ_s , where $A \geq 0$, we thus have by

$$\langle x, Ax \rangle(t) = \langle x, Ax \rangle(0) - \int_0^t |x|(s)^2 ds \leq 0$$

that $\langle x, Ax \rangle(t) \equiv 0$ for $t \geq 0$. This gives forward uniqueness, or uniqueness for $t \geq 0$, of the projection onto Σ_s . A similar argument yields backward uniqueness, or uniqueness for $t \leq 0$, of the projection onto Σ_u . Finally, on Σ_c , the homogeneous equation reduces to $x = 0$, giving uniqueness for all t of the projection onto Σ_c .

Putting this information together, suppose we have a nontrivial solution $x \in (0, \tau)$ with $|A|^{1/2}x(0) = 0$, $\tau > 0$. Then, the projection $x_s = \Pi_s x$ vanishes on $(0, \tau)$, as does the projection $x_c = \Pi_c x$. It remains to verify that $x_u = \Pi_u x$ vanishes on $(0, \tau)$. If $|A|^{1/2}x_u(\tau) = 0$, then this follows by backward uniqueness on Σ_u . If $|A|^{1/2}x_u(\tau) = g \neq 0$, on the other hand, then $x_u(t) \equiv |A|^{-1/2}T_u(t - \tau)g$, by backward uniqueness of functions valued in Σ_u . In particular, we would have

$$0 = x_u(0) = |A|^{-1/2}T_u(-\tau)g = \int_{\mathbb{R}} \chi_{\alpha < 0} |\alpha|^{-1/2} e^{\tau/\alpha} dE_{\alpha} g,$$

which is evidently false unless $dE_{\alpha}g \equiv 0$ and thus $g = 0$. By contradiction, therefore, the result is proved. \square

Remark 3.6. Note that we have obtained not only forward (backward) existence for Cauchy data in Σ_s (Σ_u), but uniqueness in both forward and backward directions. For our purposes here, we only require forward (backward) uniqueness of solutions in Σ_s (Σ_u); however, the more general result seems interesting to note.

We have also the following more familiar reinterpretation of the Fourier-transform solution (30) via variation of constants. According to our earlier convention, define the unbounded operator A^{-r} for integer $r > 0$ as the inverse of A^r from $\text{Range } A^r = \text{Range } |A|^r$ to $\Sigma_s \oplus \Sigma_u$, that is,

$$A^{-r}x = \int_{\mathbb{R} \setminus \{0\}} \alpha^{-r} dE_{\alpha} x \text{ for } x \in \text{Range } |A|^r.$$

By Lemma 3.5, A^{-r} is well-defined on $T_s(t)x$ for $t > 0$ and $T_u(t)x$ for $t < 0$.

To state things most simply, define the spectral cutoffs $g^a(t) := \int_{|\alpha| \geq a} dE_{\alpha} g(t)$ for $a > 0$ of a function $g \in L^2(\mathbb{R})$. Evidently, $g^a \rightarrow g_s + g_u := \Pi_s g + \Pi_u g$ both pointwise and in $L^2(\mathbb{R})$ as $a \rightarrow 0^+$. Then, we have the following variation of constants type formula, expressed in terms of improper integrals with respect to the spectral parameter α .

Lemma 3.7. *The unique solution $y, (d/dt)(Ay) \in L^2(\mathbb{R})$ of $(d/dt)(Ay) + y = f$ defined by (30) may be expressed alternatively as*

$$\begin{aligned} y(t) = & \lim_{a \rightarrow 0^+} \left(\int_{-\infty}^t A^{-1} T_s(t - \tau) \Pi_s f^a(\tau) d\tau \right. \\ & \left. - \int_t^{+\infty} A^{-1} T_u(t - \tau) \Pi_u f^a(\tau) d\tau \right) + \Pi_c f(t), \end{aligned} \quad (35)$$

where the limit is taken in $L^2(\mathbb{R})$. In particular, for f supported on (τ_0, τ_1) ,

$$y_s(t) \equiv 0 \text{ for } t \leq \tau_0 \text{ and } y_u(t) \equiv 0 \text{ for } t \geq \tau_1. \quad (36)$$

Proof. Since Π_s , Π_c , and Π_u commute with A , it is equivalent to show that $y_s := \Pi_s y$, $y_u := \Pi_u y$, and $y_c := \Pi_c y$ are given by

$$\begin{aligned} y_s(t) &= \lim_{a \rightarrow 0^+} \int_{-\infty}^t A^{-1} T_s(t - \tau) \Pi_s f^a(\tau) d\tau, \\ y_u(t) &= - \lim_{a \rightarrow 0^+} \int_{-\infty}^t A^{-1} T_u(t - \tau) \Pi_u f^a(\tau) d\tau, \end{aligned}$$

and $y_c = \Pi_c f$.

The third relation is nothing other than the projection of the evolution equation onto Σ_c , since $\Pi_c A = 0$. For operators A with $|A| \geq a \text{Id} > 0$ in the sense of quadratic forms, the first two follow from the standard variation of constants formula for the

solution of $(d/dt)x = -A^{-1}x$. For, example, restricting for definiteness to the stable subspace Σ_s , we may first project the equation by Π_s onto Σ_s , then, noting that A in this case has bounded inverse, apply A^{-1} to obtain the bounded-coefficient ODE

$$(d/dt)y_s + A^{-1}y_s = (A^+)^{-1}\Pi_s f \quad (37)$$

Observing that $\|e^{-A^{-1}t}\Pi_s\| \leq Ce^{-t/\|A\|}$, we obtain by variation of constants that $y_s(t) = \int_{-\infty}^t A^{-1}T_s(t-\tau)\Pi_s f(\tau)d\tau$ is the unique solution of (37) in L^2 . A symmetric argument yields the result for $y_u(t)$.

Now, introduce the spectral cutoffs $f^a := \int_{|\alpha| \geq a} dE_\alpha f$ of f defined above, for $a > 0$, and denote the corresponding solutions y as y^a . By $f^a \rightarrow \Pi_s f + \Pi_u f$ in $L^2(\mathbb{R})$, together with the previously-shown boundedness of the solution operator, we have $y^a \rightarrow y_s + y_u$ in L^2 . But, also, $|A| \geq a\text{Id} > 0$ with respect to functions supported on spectra $|\alpha| \geq a$, hence, by the discussion of the previous paragraph,

$$y^a(t) = \int_{-\infty}^t A^{-1}T_s(t-\tau)\Pi_s f^a(\tau)d\tau - \int_t^{+\infty} A^{-1}T_u(t-\tau)\Pi_u f^a(\tau)d\tau$$

yielding (35). Property (36) is an immediate consequence. \square

Remark 3.8. The expression of (35) in terms of improper integrals as $|\alpha| \rightarrow 0^+$ highlights again the difference from the usual, nondegenerate case [5], for which the right-hand side of (35) may be expressed in terms of a standard Bochner integral on Hilbert space-valued functions [3]. We show by explicit counterexample in Appendix B that this is not necessarily the case in the present, degenerate context.

Remark 3.9. Equation (35) differs from the standard variation of constants formula in the final term $\Pi_u f(t)$, which appears to be of different form. But, note that kernels $\alpha^{-1}e^{-t/\alpha}$ for the first two terms converge as $\alpha \rightarrow 0$ to a δ -function, formally yielding the third upon convolution with f .

Combining the above facts, we obtain the following solution formula for the *boundary-value problem on an interval*, that is, for solutions $x(t)$ in our L_{loc}^2 sense of the linear problem (26) on (t_0, t_1) with boundary conditions imposed on the continuous image $|A|^{1/2}x(t)$ at $t = t_0, t_1$.

Proposition 3.10. *For each $f \in L^2(t_0, t_1)$, $g_0 \in \Sigma_s$, and $g_1 \in \Sigma_u$, there is a unique weak L_{loc}^2 solution x , $(d/dt)(Ax) \in L^2(t_0, t_1)$, of (26) satisfying boundary conditions*

$$(|A|^{1/2}\Pi_s x)(t_0) = g_0, \quad (|A|^{1/2}\Pi_u x)(t_1) = g_1 \quad (38)$$

on $|A|^{1/2}x$, given by

$$\begin{aligned} x(t) &= y(t) + |A|^{-1/2}T_s(t-t_0)g_0 + |A|^{-1/2}T_u(t-t_1)g_1, \\ \widehat{y}(\omega) &= (i\omega A + \text{Id})^{-1} \widehat{f_{|[t_0, t_1]}}(\omega), \end{aligned} \quad (39)$$

or alternatively, with y defined by (35), with f extended as 0 outside (τ_0, τ_1) .

Proof. Evidently, $z = x - y$ is a solution of the homogeneous equation

$$(d/dt)(Az) + z = 0$$

satisfying the prescribed boundary conditions, while y is a solution of

$$(d/dt)(Ay) + y = f_{[t_0, t_1]},$$

satisfying by (36) homogeneous boundary conditions $|A|^{1/2}\Pi_s y(t_0) = 0$ and

$$|A|^{1/2}\Pi_u y(t_1) = 0.$$

Thus, by linear superposition, $x = y + z$ satisfies (26) with boundary conditions (38). Uniqueness follows from uniqueness of solution z_s , z_u , and $z_c \equiv 0$ of the homogeneous equation for z on $[t_0, t_1]$ under the boundary conditions at t_0, t_1 , a consequence of Lemma 3.5. \square

3.2. Relation to mild solutions. With these preparations, we are now ready to compare our notion of weak L^2_{loc} solution with that of “mild” L^2_{loc} solution defined as follows in [8] (cf. [5]).

Definition 3.11 ([8, 5]). For $f \in L^2_{loc}$ on a given domain D , a *mild* L^2_{loc} solution of (26) is a function $x \in L^2_{loc}(D)$ with Fourier transform satisfying

$$\widehat{x}(\omega) = A(i\omega A + \text{Id})^{-1} [e^{-2\pi i \omega t_0} h_0 - e^{-2\pi i \omega t_1} h_1] + (i\omega A + \text{Id})^{-1} \widehat{f_{|[t_0, t_1]}}(\omega) \quad (40)$$

with $h_0 \in \Sigma_s$ and $h_1 \in \Sigma_u$,⁴ or, equivalently (see [8, Remark 3.2]), the variation of constants type formula

$$x(t) = y(t) + T_s(t - t_0)h_0 + T_u(t - t_1)h_1, \quad \widehat{y}(\omega) = (i\omega A + \text{Id})^{-1} \widehat{f_{|[t_0, t_1]}}(\omega). \quad (41)$$

Definition 3.11 gives a notion of a mild L^2_{loc} solution of (26); a mild $L^2(\mathbb{R}^+)$ solution of (1) may then be defined as in [8, 9] as a mild L^2_{loc} solution $x \in L^2(\mathbb{R}^+)$ of (26) with $f = \mathcal{G}(x)$ (cf. [8, Def. 3.1(ii)] and [8, Lemma 3.3]).

In (40)–(41), h_0 and h_1 are in general not connected with boundary values of x at t_0 and t_1 , which may not even be defined. However,

$$T_s(t - t_0)h_0 + T_u(t - t_1)h_1$$

is continuous, while $y(t)$ (since decaying at $\pm\infty$) vanishes on $(-\infty, t_0)$ in stable modes and on (t_1, ∞) in unstable modes. Thus, for H^1 solutions, admitting continuous representatives (the class ultimately considered in the invariant manifold constructions of [8, 9, 12]), we have $\Pi_s x(t_0) = h_0$ and $\Pi_u x(t_1) = h_1$; in particular, for $H^1(\mathbb{R}^+)$ solutions of (1), there is a well-defined boundary value $\Pi_s x(0)$ at $t = 0$.

Comparing (39) and (41), and noting that

$$|A|^{-1/2}T_s|A|^{1/2} = T_s, \quad |A|^{-1/2}T_u|A|^{1/2} = T_u,$$

we see that mild solutions are L^2_{loc} solutions as defined here, but not conversely. Indeed, they are precisely the subclass of L^2_{loc} solutions for which the boundary values g_0 and g_1 of $|A|^{1/2}x(t)$ at t_0 and t_1 are given by $|A|^{1/2}\Pi_s h_0$ and $|A|^{1/2}\Pi_u h_1$ with $h_j \in \mathbb{H}$, i.e., for which g_0 and g_1 lie in $\text{Range } |A|^{1/2}$.

Remark 3.12. The representations (40)–(41) were derived in [8] under the assumption that A be one-to-one; however, this assumption is not necessary, as shown by our analysis above. Indeed, one may check that the entire H^1 stable manifold construction of [8] goes through for general A . We note that the key relation $A(i\omega A + \text{Id})^{-1}e^{-2\pi i \omega t_0}x(t_0) = (T_s(t - t_0)x(t_0))$ linking (40) and (41) (see [8, Remark 3.2(ii)]) follows in the general case from the result in the invertible case, by the observation that

$$A(i\omega A + \text{Id})^{-1}e^{-2\pi i \omega t_0} = \Pi_{su}A(i\omega A + \text{Id})^{-1}e^{-2\pi i \omega t_0},$$

⁴In [8, 5], for which there was assumed no center subspace, this definition was given for $h_j \in \mathbb{H}$; however, without loss of generality it may be stated as above, thus allowing also the case of a nontrivial center subspace.

where $\Pi_{su} = \Pi_s + \Pi_u$ denotes projection onto $\Sigma_s \oplus \Sigma_u$. Likewise, (39) has the equivalent frequency-domain formulation

$$\widehat{x}(\omega) = A(i\omega A + \text{Id})^{-1} |A|^{-1/2} [e^{-2\pi i \omega t_0} g_0 - e^{-2\pi i \omega t_1} g_1] + (i\omega A + \text{Id})^{-1} \widehat{f_{|[t_0, t_1]}}(\omega).$$

4. Applications to Boltzmann's equation. The main example considered in [8] was the *steady Boltzmann equation* (plus cousins and discrete approximations),

$$\xi_1 f_z = Q(f, f), \quad z \in \mathbb{R}^1, \xi \in \mathbb{R}^3, \quad (42)$$

with hard-sphere collision operator Q , $f = f(z, \xi)$ denoting density at spatial point z of particles with velocity ξ , which, after the coordinate change $f \rightarrow \langle \xi \rangle^{1/2} f$, $Q \rightarrow \langle \xi \rangle^{1/2} Q$, $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$, can be put in form $A w_z = Q(w, w)$, with \mathbb{H} the standard square-root Maxwellian-weighted L^2 space in variable ξ , $A = \xi_1 / \langle \xi \rangle$ a multiplication operator, and Q a bounded bilinear map [6]. Note that A has no kernel on L^2 . However, 0 is in the essential range of the function $\xi_1 / \langle \xi \rangle$, hence in the essential spectrum of the operator of multiplication by $\xi_1 / \langle \xi \rangle$. That is, A is “essentially singular”.

Introducing the perturbation variable $x := w - w_0$, where w_0 is an equilibrium, $Q(w_0, w_0) = 0$, and performing some straightforward further reductions [8, 12] converts the equations to form (1), with $t = z$ and $\mathcal{G}(x) = B(x, x)$, B a bounded bilinear map. In this context, the problem considered here, of decay and smoothness of small solutions $x(t)$, amounts to the study of convergence and smoothness of solutions toward an equilibrium w_0 . The particular motivation described in [8] was the desire to study existence and temporal stability of large-amplitude heteroclinic connections, or planar *Boltzmann shock or boundary layers*, for which the study of stable manifolds and decay to equilibria is an important first step. The main result of [8] was construction of an H^1 stable manifold at w_0 containing all orbits sufficiently close to w_0 in $H^1(\mathbb{R}^+)$, exhibiting uniform exponential decay.

The global problem of existence and structure of large-amplitude Boltzmann shocks, as discussed by Truesdell, Ruggeri, Boillat, and others [1], is one of the fundamental open problems in Boltzmann theory. For this larger problem, it is important to know that the H^1 stable manifolds of [8] in fact contain all candidates for heteroclinic connections, i.e., that the $H^1(0, +\infty)$ regularity imposed on solutions in [8] is not too strong, eliminating potential connections. Thus, the questions of regularity considered in the present paper are not just technical, but central to the physical discussion.

In particular, we have answered here in the affirmative the two main open questions posed in [8]:

1. Do asymptotically decaying (or just sufficiently small) $L^2_{loc}(\mathbb{R}^+)$ solutions of (1) decay exponentially in t ?
2. Are small (in sup norm) $L^2_{loc}(\mathbb{R}^+)$ solutions necessarily in H^1 or higher Sobolev spaces?

These results, together with those of [8], imply that *the tail of any* (small- or large-amplitude) *Boltzmann shock or boundary layer is C^∞ and lies in the H^1 stable manifold constructed in [8].*

5. Discussion and open problems. The results of instantaneous smoothing obtained here are somewhat analogous to interior regularity results for more standard boundary-value problems, e.g., elliptic and other boundary-value ODE. However, here, it should be noted, due to allowed degeneracy of A , there is in general no gain

in regularity in solutions of the linear inhomogeneous problem $(d/dt)Ax + x = f$ of Proposition 3.10, with L^2 forcing f leading to L^2 regularity and not higher of solutions x . More, as discussed in [8, 12], the Fourier multiplier $(i\omega A + \text{Id})^{-1}$ is bounded on L^p for $1 < p < \infty$, but not $p = 1$ or ∞ ; thus, the solution operator is not associated with an integrable kernel as in more standard cases.

An interesting remaining open problem is to construct L^∞ decaying solutions of (1) that are not small in H^1 , i.e., backward extensions of the manifold of H^1 solutions constructed in [8]. (Here, we showed that solutions that are eventually small in L^∞ in fact decay to this manifold, but did not produce any such.) A related very interesting open problem, moving toward construction of full heteroclinic orbits, is the question of backward uniqueness of solutions of (1), i.e., whether $L^\infty(0, +\infty)$ solutions agreeing on $t \geq t' > 0$ must agree on $t > 0$. Existence of large Boltzmann shocks- the “structure problem of Ruggeri et al [1]- is a major open problem, involving in addition separate, and presumably more problem-dependent, issues of global analysis.

We note that forward uniqueness of small $L^\infty(0, +\infty)$ solutions holds for \mathcal{G} Lipschitz with small Lipschitz norm, by essentially the same argument as in step 1 of the proof of Theorem 1.3 applied to the error equation governing $e := x_1 - x_2$, where x_1 and x_2 are two solutions with $x_1(0) = x_2(0)$. This extends the result of forward uniqueness of $H^1(0, +\infty)$ solutions following from the H^1 stable manifold results of [8]. Thus, an interesting preliminary question is whether backward uniqueness of small $L^\infty(0, +\infty)$ solutions can fail for the same class of Lipschitz \mathcal{G} with small Lipschitz norm.

Appendix A. Sobolev estimates for difference operators.

Lemma A.1. (i) For $f : (0, +\infty) \rightarrow \mathbb{H}$ in $L^2(0, +\infty)$, there holds

$$\sqrt{\int_0^\infty |f'|^2} \lesssim \liminf_{\tau \rightarrow 0^+} \tau^{-1} \|f(\cdot + \tau) - f(\cdot)\|_{L^2(\mathbb{R}^+, \mathbb{H})}.$$

(ii) For $f : (0, +\infty) \rightarrow \mathbb{H}$ in $H_{loc}^1(0, +\infty)$, there holds $\tau^{-1}(f(\cdot + \tau) - f(\cdot)) \rightarrow f'$ in L_{loc}^2 as $\tau \rightarrow 0^+$.

Proof. (i). By Fatou’s Lemma,

$$\begin{aligned} \|(d/dt)g\|_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \omega^2 |\widehat{g}|^2 d\omega \\ &= \int_{\mathbb{R}} \lim_{\tau \rightarrow 0^+} |\tau^{-1}(e^{-i\tau\omega} - 1)\widehat{g}(\omega)|^2 d\omega \\ &\leq \liminf_{\tau \rightarrow 0^+} \int_{\mathbb{R}} |\tau^{-1}(e^{-i\tau\omega} - 1)\widehat{g}(\omega)|^2 d\omega \\ &= \liminf_{\tau \rightarrow 0^+} \left\| \tau^{-1}(g(\cdot + \tau) - g(\cdot)) \right\|_{L^2(\mathbb{R})} \end{aligned}$$

for functions $g \in L^2(\mathbb{R})$. Setting $g = \chi^\varepsilon f$ with $\chi^\varepsilon(z) := \chi(z/\varepsilon)$ a smooth cut-off function, $\chi(z)$ equal to zero for $z = 0$ and 1 for $z \geq 1$, and observing that $\sup |(\chi^\varepsilon)'| \lesssim \varepsilon^{-1}$, we thus have

$$\tau^{-1} \|\chi^\varepsilon f(\cdot + \tau) - \chi^\varepsilon f(\cdot)\|_{L^2(\mathbb{R})} \lesssim \tau^{-1} \|f(\cdot + \tau) - f(\cdot)\|_{L^2(0, +\infty)} + \varepsilon^{-1} \|f\|_{L^2(0, +\infty)},$$

whence $f \in H^1(\varepsilon, +\infty)$, and $f \in C^0[\varepsilon, +\infty)$, for each $\varepsilon > 0$ (albeit with bound $\sim \varepsilon^{-1}$ blowing up as $\varepsilon \rightarrow 0^+$).

In particular, f has a well-defined value $f(\varepsilon)$ at $x = \varepsilon$. Extending $f|_{[\varepsilon, +\infty)}$ by

$$\tilde{f}(t) := \begin{cases} f(t) & t \geq \varepsilon, \\ f(\varepsilon) & t \leq \varepsilon, \end{cases}$$

and defining a different smooth cutoff function $\tilde{\chi}^L(z) := \tilde{\chi}(z/L)$, with $\tilde{\chi}(z)$ equal to zero for $z \leq -1$ and 1 for $z \geq 0$, set $\tilde{g} = \tilde{\chi}^L \tilde{f}$. Using $\tilde{g}' = (\tilde{\chi}^L)'f(\varepsilon) + \tilde{\chi}^L f'|_{(\varepsilon, +\infty)}$, we have

$$\|\tilde{f}'\|_{L^2(\varepsilon, +\infty)} \leq \|\tilde{g}'\|_{L^2(\mathbb{R})} \leq \liminf_{\tau \rightarrow 0^+} \tau^{-1} \|\tilde{g}(\cdot + \tau) - \tilde{g}\|_{L^2(\mathbb{R})}. \quad (43)$$

Computing, by Jensen's inequality that

$$\begin{aligned} \tau^{-2} \int_0^\tau |f(\varepsilon + t) - f(\varepsilon)|^2 dt &= \int_0^\tau \left| \int_0^t f'(\varepsilon + s) ds \right|^2 dt \\ &\leq \tau^{-2} \int_0^\tau t dt \|f'\|_{L^2(\varepsilon, \varepsilon + \tau)}^2 \leq \|f'\|_{L^2(\varepsilon, \varepsilon + \tau)}^2 \rightarrow 0 \end{aligned}$$

as $\tau \rightarrow 0^+$, and using $\|(\tilde{\chi}^L)'\|_{L^2(\mathbb{R})} = L^{-1/2} \|\tilde{\chi}'\|_{L^2(\mathbb{R})}$ together with the reverse inequality $\tau^{-1} \|\tilde{\chi}^L(\cdot + \tau) - \tilde{\chi}^L\|_{L^2(-\infty, \varepsilon - \tau)} \leq \|(\tilde{\chi}^L)'\|_{L^2(-\infty, \varepsilon)}$, we have also, for $\tau < \varepsilon$,

$$\begin{aligned} \tau^{-1} \|\tilde{g}(\cdot + \tau) - \tilde{g}\|_{L^2(\mathbb{R})} - \tau^{-1} \|f(\cdot + \tau) - f\|_{L^2(\varepsilon, \infty)} &\leq \\ \tau^{-1} |f(\varepsilon)| \|\tilde{\chi}^L(\cdot + \tau) - \tilde{\chi}^L\|_{L^2(-\infty, \varepsilon - \tau)} & \\ + \tau^{-1} \left(\int_0^\tau |f(\varepsilon + t) - f(\varepsilon)|^2 dt \right)^{1/2} &\rightarrow 0 \end{aligned} \quad (44)$$

as $L \rightarrow \infty$ and $\tau \rightarrow 0^+$.

Taking $\liminf_{\tau \rightarrow 0^+} \lim_{L \rightarrow \infty}$ of (44), and combining with (43), we thus obtain

$$\|f'\|_{L^2(\varepsilon, +\infty)} \leq \liminf_{\tau \rightarrow 0^+} \tau^{-1} \|f(\cdot + \tau) - f(\cdot)\|_{L^2(\varepsilon, +\infty)}$$

for each $\varepsilon > 0$, whence the first assertion follows in the limit as $\varepsilon \rightarrow 0^+$.

(ii) Similarly, for functions $g \in H^1(\mathbb{R})$, we have $\tau^{-1}(g(\cdot + \tau) - g(\cdot)) \rightarrow g'$ in $L^2(\mathbb{R})$, by $\lim_{\tau \rightarrow 0^+} \tau^{-1}(e^{-i\tau\omega} - 1)\hat{g} = i\omega\hat{g}(\omega)$ a.e. and the Lebesgue dominated convergence theorem, noting that $|\tau^{-1}(e^{-i\tau\omega} - 1)\hat{g}| \lesssim |\omega\hat{g}| \in L^2(\mathbb{R})$ by $\sup_{z \in \mathbb{R}} |z^{-1}(e^{iz} - 1)| \lesssim 1$. Thus, for $f \in H^1(t_0 - \delta, t_0 + \delta)$, $\delta > 0$, we have, defining $g := \chi(t)f(t)$, where χ is a smooth cutoff equal to 1 for $t \in (t_0 - \delta/2, t_0 + \delta/2)$ and zero outside $(t_0 - \delta, t_0 + \delta)$, that $\tau^{-1}(g(\cdot + \tau) - g(\cdot)) \rightarrow g'$ in $L^2(\mathbb{R})$, and therefore $\tau^{-1}(f(\cdot + \tau) - f(\cdot)) \rightarrow f'$ in $L^2(t_0 - \delta/4, t_0 + \delta/4)$. As t_0 and $\delta > 0$ were arbitrary, the result follows. \square

Lemma A.2. *For any $t \in \mathbb{R}$, there holds*

$$\int_t^{+\infty} |x|^4 \leq \left(\frac{2}{2^{1/4} - 1} \right)^2 \left(\int_t^{+\infty} |x|^2 \right) \left(\sup_{\tau > 0} \int_t^{+\infty} \tau^{-1} |x(s + \tau) - x(s)|^2 ds \right).$$

Proof. Denoting $C_1 = \int_t^{+\infty} |x|^2$ and $C_2 = \sup_{\tau > 0} \int_t^{+\infty} \tau^{-1} |x(s + \tau) - x(s)|^2 ds$, we have

$$\int_t^{+\infty} |x|^2 \leq C_1$$

and $\int_t^{+\infty} |x(s + \tau) - x(s)|^2 ds \leq C_2 \tau$ for all $\tau > 0$.

Let

$$\phi_h(t) := \begin{cases} 1/h & \text{for } 0 \leq t \leq h, \\ 0 & \text{otherwise.} \end{cases}$$

Then, on $(t, +\infty)$, we have for any choice of scale $\sigma > 0$ the Haar decomposition

$$\begin{aligned} x(s) &= \lim_{k \rightarrow \infty} \int_0^{+\infty} x(s+u) \phi_{2^{-k}\sigma}(u) du \\ &= \int_0^{+\infty} x(s+u) \phi_\sigma(u) du + \sum_{k \geq 1, h=2^{-k}\sigma} \int_0^{+\infty} x(s+u) [\phi_h(u) - \phi_{2h}(u)] du \\ &=: x_\sigma + \sum_{k \geq 1, h=2^{-k}\sigma} \tilde{x}_h. \end{aligned}$$

Note by Jensen's inequality that

$$\begin{aligned} \int_t^{+\infty} |x_\sigma|^2 &\leq \int_t^{+\infty} \int_0^{+\infty} \phi_\sigma(u) |x(s+u)|^2 du ds \\ &= \int_0^{+\infty} \phi_\sigma(u) \int_t^{+\infty} |x(s+u)|^2 ds du \\ &\leq \left(\int_0^{+\infty} \phi_\sigma \right) \left(\int_t^{+\infty} |x|^2 \right) \leq C_1, \end{aligned}$$

and by Cauchy-Schwarz' inequality that

$$\|x_\sigma\|_{L^\infty(t, +\infty)} \leq \left(\int_t^{+\infty} |x|^2 \right)^{1/2} \left(\int_0^{+\infty} |\phi_\sigma|^2 \right)^{1/2} \leq \sqrt{C_1} (1/\sqrt{\sigma}) = \sqrt{C_1/\sigma},$$

whence $\int_t^{+\infty} |x_\sigma|^4 \leq \|x_\sigma\|_{L^\infty(t, +\infty)}^2 \int_t^{+\infty} |x_\sigma|^2 \leq C_1^2/\sigma$.

Note also that $\phi_h(u) - \phi_{2h}(u) = (1/2)[\phi_h(u) - \phi_h(u-h)]$, so that

$$\begin{aligned} \tilde{x}_h(s) &= (1/2) \int_0^{+\infty} x(s+u) [\phi_h(u) - \phi_h(u-h)] du \\ &= (1/2) \int_0^{+\infty} [x(s+u) - x(s+u-h)] \phi_h(u) du, \end{aligned}$$

and therefore, similarly to the previous computation,

$$\int_t^{+\infty} |\tilde{x}_h|^2 \leq (1/4) \int_t^{+\infty} |x(s+h) - x(s)|^2 ds \leq (1/4) C_2 h,$$

while

$$\begin{aligned} \|\tilde{x}_h\|_{L^\infty(t, +\infty)} &\leq (1/2) \sqrt{\int_t^{+\infty} |x(s+h) - x(s)|^2 ds} \sqrt{\int_0^{+\infty} \phi_h^2} \\ &\leq (1/2) \sqrt{C_2 h} (1/\sqrt{h}) = (1/2) \sqrt{C_2}. \end{aligned}$$

Thus, $\int_t^{+\infty} |\tilde{x}_h|^4 \leq \|\tilde{x}_h\|_{L^\infty(t, +\infty)}^2 \int_t^{+\infty} |\tilde{x}_h|^2 \leq (1/16) C_2^2 h \leq (1/16) C_2^2 \sigma 2^{-k}$ for $h = \sigma 2^{-k}$, whence

$$\begin{aligned} \|x\|_{L^4(t, +\infty)} &\leq \|x_\sigma\|_{L^4(t, +\infty)} + \sum_{k \geq 1, h=2^{-k}\sigma} \|\tilde{x}_h\|_{L^4(t, +\infty)} \\ &\leq (C_1^2/\sigma)^{1/4} + (C_2^2 \sigma/16)^{1/4} \sum_{k \geq 1} 2^{-k/4} \\ &= C_1^{1/2} \sigma^{1/4} + C_2^{1/2} \sigma^{1/4} / (2(2^{1/4} - 1)). \end{aligned}$$

Taking $\sigma^{1/4} = (C_1/C_2)^{1/4} \sqrt{2(2^{1/4} - 1)}$ to minimize the right-hand side gives

$$\|x\|_{L^4(t, +\infty)} \leq \left((2/(2^{1/4} - 1))^2 C_1 C_2 \right)^{1/4}$$

as required. \square

Appendix B. Integrals and counterexample. We conclude with an explicit counterexample showing that our expression (35) of the variation of constants formula in terms of improper integrals as $a \rightarrow 0^+$ cannot in general be reformulated in terms of standard Bochner integrals for Hilbert space-valued functions [3].

Let $\mathbb{H} = L^2(0, 1)$ and A the multiplication operator $Ah(\alpha) = \alpha h(\alpha)$. Then, $T_s(t)$ is also a multiplication operator, with $T_s(t)h(\alpha) = e^{-t/\alpha}h(\alpha)$ and $A^{-1}T_s(t)h(\alpha) = \alpha^{-1}e^{-t/\alpha}h(\alpha)$. Meanwhile $\Pi_s = \text{Id}$, Π_u , T_u , and Π_c are identically zero, hence (35) reduces to

$$y(t) = y_s(t) = \lim_{a \rightarrow 0^+} \int_{-\infty}^t A^{-1}T_s(t - \tau)f^a(\tau)d\tau. \quad (45)$$

The question is whether the righthand side of the above, or, equivalently, after the change of variables $\sigma := t - \tau$, expression $\lim_{a \rightarrow 0^+} \int_0^{+\infty} A^{-1}T_s(\sigma)f^a(t - \sigma)d\sigma$, can be interpreted as a Bochner integral $\int_0^{+\infty} A^{-1}T_s(\sigma)f(\cdot - \sigma)d\sigma$ in $L^2(\mathbb{R}, \mathbb{H})$, that is, as an integral over $\sigma \in (0, +\infty)$ of an integrand valued not in \mathbb{H} but in the large Hilbert space $L^2(\mathbb{R}, \mathbb{H})$.

Recall [3] that Bochner integrability requires Lebesgue integrability with respect to σ of the norm of the integrand, in this case, integrability on $(0, +\infty)$ of

$$\|A^{-1}T_s(\sigma)f(\cdot - \sigma)\|_{L^2(\mathbb{R}, \mathbb{H})} = \left(\int_{\mathbb{R}} \int_0^1 (\alpha^{-1}e^{-\sigma/\alpha}f(t - \sigma, \alpha))^2 d\alpha dt \right)^{1/2}. \quad (46)$$

Take now $f(t, \alpha) := \alpha^{-1/2}(|\log \alpha| + 1)^{-r/2}\phi(t)$, where $\phi \in L^2$ and $r > 1$. Then, $\|f\|_{L^2(\mathbb{R}, \mathbb{H})}^2 = \int_{\mathbb{R}} \int_0^1 (\alpha^{-1/2}(|\log \alpha| + 1)^{-r/2}\phi(t))^2 d\alpha dt = \|\phi\|_{L^2}^2 \int_0^1 \alpha^{-1}(|\log \alpha| + 1)^{-r} d\alpha$

is finite, hence $f \in L^2(\mathbb{R}, \mathbb{H})$. But,

$$\begin{aligned} & \left(\int_{\mathbb{R}} \int_0^1 (\alpha^{-1}e^{-\sigma/\alpha}f(t - \sigma, \alpha))^2 d\alpha dt \right)^{1/2} = \\ & \left(\int_{\mathbb{R}} \int_0^1 (\alpha^{-1}e^{-\sigma/\alpha}\alpha^{-1/2}(|\log \alpha| + 1)^{-r/2}\phi(t))^2 d\alpha dt \right)^{1/2} \\ & = \|\phi\|_{L^2(\mathbb{R}, \mathbb{R})} \left(\int_0^1 \alpha^{-3}e^{-2\sigma/\alpha}(|\log \alpha| + 1)^{-r} d\alpha \right)^{1/2}, \end{aligned}$$

after the change of coordinates $z := \alpha/\sigma$ becomes

$$\begin{aligned} \|\phi\|_{L^2(\mathbb{R}, \mathbb{R})} \sigma^{-1} \left(\int_0^{1/\sigma} \frac{z^{-3}e^{-2/z}}{(|\log z + \log \sigma| + 1)^r} dz \right)^{1/2} & \gtrsim \sigma^{-1} \left(\int_{1/2}^1 \frac{z^{-3}e^{-2/z}}{(|\log z + \log \sigma| + 1)^r} dz \right)^{1/2} \\ & \gtrsim \sigma^{-1}(|\log \sigma| + 1)^{-r/2} \end{aligned}$$

as $\sigma \rightarrow 0$, which, for $1 < r \leq 2$ is not integrable on $(0, +\infty)$. This example shows that (35) cannot in general be expressed in terms of standard, Bochner integrals.

Acknowledgments. Thanks to Benjamin Jaye for helpful discussions, and to Alin Pogan for several readings of the manuscript and helpful suggestions for the exposition. Thanks also to the anonymous referees for their careful reading and helpful suggestions.

REFERENCES

- [1] G. Boillat and T. Ruggeri, [On the shock structure problem for hyperbolic system of balance laws and convex entropy](#), *Contin. Mech. Thermodyn.*, **10** (1998), 285–292.
- [2] J. B. Conway, *A Course in Functional Analysis*, 2nd edition, Graduate Texts in Mathematics, 96. Springer-Verlag, New York, 1990.
- [3] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.
- [4] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.
- [5] Y. Latushkin and A. Pogan, [The dichotomy theorem for evolution bi-families](#), *J. Diff. Eq.*, **245** (2008), 2267–2306.
- [6] G. Métivier and K. Zumbrun, [Existence and sharp localization in velocity of small-amplitude Boltzmann shocks](#), *Kinet. Relat. Models*, **2** (2009), 667–705.
- [7] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.
- [8] A. Pogan and K. Zumbrun, [Stable manifolds for a class of singular evolution equations and exponential decay of kinetic shocks](#), *Kinet. Relat. Models*, **12** (2019), 1–36.
- [9] A. Pogan and K. Zumbrun, [Center manifolds for a class of degenerate evolution equations and existence of small-amplitude kinetic shocks](#), *J. Diff. Eq.*, **264** (2018), 6752–6808.
- [10] M. Reed and B. Simon, *Methods of Mathematical Physics*, 2nd edition, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980.
- [11] W. Rudin, *Functional Analysis*, 2nd edition, International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.
- [12] K. Zumbrun, [Invariant manifolds for a class of degenerate evolution equations and structure of kinetic shock layers](#), *Springer Proc. Math. Stat.*, **237** (2018), 691–714.

Received September 2021; revised December 2021; early access April 2022.

E-mail address: nazarov@math.kent.edu

E-mail address: kzumbrun@indiana.edu