Surrogate Scoring Rules

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Strictly proper scoring rules (SPSR) are incentive compatible for eliciting information about random variables from strategic agents when the principal can reward agents after the realization of the random variables. They also quantify the quality of elicited information, with more accurate predictions receiving higher scores in expectation. In this paper, we extend such scoring rules to settings where a principal elicits private probabilistic beliefs but only has access to agents' reports. We name our solution *Surrogate Scoring Rules* (SSR). SSR is built on a bias correction step and an error rate estimation procedure for a reference answer defined using agents' reports. We show that, with a little information about the prior distribution of the random variables, SSR in a multi-task setting recover SPSR in expectation, as if having access to the ground truth. Therefore, a salient feature of SSR is that they quantify the quality of information despite the lack of ground truth, just as SPSR do for the setting *with* ground truth. As a by-product, SSR induce *dominant uniform strategy truthfulness* in reporting. Our method is verified both theoretically and empirically using data collected from real human forecasters.

CCS Concepts: • **Information systems** → **Incentive schemes**; • **Theory of computation** → **Quality of equilibria**.

Additional Key Words and Phrases: Strictly proper scoring rules, information elicitation without verification, peer prediction, dominant strategy incentive compatibility, information calibration

1 INTRODUCTION

Accurate assessment of random variables of interest (e.g. how likely the S&P 500 index will go up next week) plays a crucial role in a wide array of applications, including computational finance [7], geopolitical forecasting [10, 43], weather and climate forecasting [12], and the prediction of the replicability of social science studies [1, 15]. Since such assessments are often elicited from people, how to incentivize people to provide accurate assessments has been a topic of great scientific interests.

For settings where the principal will have access to the ground truth (e.g. after a week, knowing whether the S&P 500 index actually went up), strictly proper scoring rules (SPSR) [4, 13, 17, 37, 46] have been developed to elicit probabilistic assessments and evaluate them against the ground truth. SPSR have two desirable properties. First, they incentivize truthful information reporting: the SPSR score of an agent's reported prediction is strictly maximized in the agent's expectation if she truthfully reveals her prediction. Second, the SPSR score of a prediction measures the quality of the prediction in the sense that the closer the prediction is to the underlying, unknown true distribution of the random event, the higher the expected score.

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However, in many applications, the ground truth is not available in time or at all. For example, geopolitical events usually take months to resolve [43], and whether a study will be successfully replicated is not known if a replication test of it is not attempted. In this paper, we extend the literature of SPSR to the information elicitation *without* verification (IEWV) settings, where the principal has no access to the ground truth and still wants to elicit private probabilistic beliefs. We ask the following research question:

Can we extend SPSR to scoring mechanisms that can achieve truthful elicitation of probabilistic information and quantify the quality of the elicited information for IEWV?

Witkowski et al. [47] explored this question in a single-task setting (i.e., having a single random variable of interest to predict). When an unbiased proxy to the true probability distribution of the ground truth is available, they generalized SPSR to *proper proxy scoring rules*, which score a prediction against the unbiased proxy while maintaining the two properties of SPSR. However, when the principal only has access to agents' reports, it remains an open problem how such an unbiased proxy can be constructed without affecting the incentive properties.

In this paper, we study the research question in a multi-task setting, where a principal wants to predict multiple random variables that are similar a priori. We provide a positive answer to the question. In our solution, the principal only needs to know the order of the prior probability of each possible outcome (e.g., for binary random variables, the more likely outcome) and does not need to have an unbiased proxy for each task. Specifically, we develop a family of scoring mechanisms that utilize the similarity of tasks and the conditional independence of agents' beliefs to construct a biased proxy of the ground truth, and then, score a prediction against this proxy by removing the bias w.r.t. the underlying SPSR that one wants to recover. Our proxy is explicitly constructed only from agents' reported predictions. As a result, we achieve the dominant uniform strategy truthfulness [14] in eliciting probabilistic predictions, where truthful reporting is the strict best strategy when each agent adopts the same strategy across all tasks. Furthermore, the scores of our mechanisms recover the scores of SPSR in expectation. To the best of our knowledge, our work provides the first meta solution that enables applications of any SPSR to the IEWV setting without relying on access to unbiased proxies of the ground truth. We name our solution *Surrogate Scoring Rules (SSR)*.

As a building block, we first introduce SSR for a stylized setting where the principal has access to a noisy estimate of the ground truth, as well as the estimate's error rates, to evaluate the elicited information. We show that SSR preserve the same information quantification and truthful elicitation properties just as SPSR, despite the lack of access to the ground truth. These surrogate scoring rules are inspired by the use of surrogate loss functions in machine learning [2, 5, 29, 40, 41]. They remove the bias from the noisy estimate of the ground truth such that in expectation a report is as if evaluated against the ground truth.

Building on the above bias correction step, when the principal only has access to agents' reports and the order of the prior probabilities of each outcome, we develop the *SSR mechanisms* for the multi-task setting to achieve information quantification and the dominant uniform strategy truthfulness when the principal has sufficiently many tasks and agents. Our mechanisms rely on an estimation procedure to accurately estimate the average bias in the peer agents' reports. With the estimation, a random peer agent's report can serve as a noisy estimate of the ground truth, and SSR can then be applied to achieve the two desired properties. We evaluate the empirical performance of the SSR mechanisms using 14 real-world human forecast datasets. The results show that SSR effectively recover SPSR scores but using only agents' reports.

We summarize our contributions as follows:

- We extend SPSR to a family of scoring mechanisms, the SSR mechanisms, that operate in the IEWV setting. The SSR mechanisms only require access to peer reports and the order of the prior probabilities of the ground truth being each outcome, and they can truthfully elicit probabilistic beliefs. An SSR mechanism can build upon any SPSR and quantifies in expectation the value of the elicited information just as the corresponding SPSR does as if it had access to the ground truth. Therefore, our work complements the proper scoring rule literature and expands the application of SPSR in challenging elicitation settings where the ground truth is unavailable.
- For the IEWV setting, most existing mechanisms focus on incentivizing truthful reporting of categorical signals via rewarding the correlation between two agents' reports. Our SSR mechanisms complement this literature from two perspectives. First, SSR mechanisms induce dominant uniform strategy truthfulness in eliciting probabilistic predictions instead of categorical signals. Second, instead of scoring a prediction by assessing the correlation between two agents' reports, SSR mechanisms score predictions according to their prediction accuracy against the unknown ground truth. This property encourages agents to search for more accurate forecasts.
- We evaluate the empirical performance of SSR mechanisms on 14 real-world human prediction datasets. The results show that SSR mechanisms can better reflect the true accuracy of agents in terms of SPSR scores than other existing mechanisms designed for IEWV.

Organization. The rest of the paper is organized as follows. Section 2 provides a survey of related work. Section 3 introduces strictly proper scoring rules and their two main desirable properties in the information elicitation with verification setting. In Section 4, we introduce our model of IEWV and our main assumptions for eliciting predictions. In Section 5, we study the information elicitation problem in the stylized setting, where the principal has access to a noisy estimate of the ground truth with a known bias. We introduce surrogate scoring rules as a powerful solution in this section. In Section 6, we propose the dominant uniform strategy truthful mechanisms, SSR mechanisms, to address the IEWV in the multi-task binary-outcome task setting. We generalize our mechanisms and results to the multi-outcome task setting in Section 7. We present our experimental study of our mechanisms in Section 8. We discuss several restrictions of our mechanisms in Section 9. Omitted proofs can be found in the Appendix.

2 RELATED WORK

The most relevant literature to our paper is on *strictly proper scoring rules* (SPSR) and *peer prediction*. SPSR are designed to elicit subjective beliefs about random variables when the principal can evaluate agents' predictions after the random variables are realized. Brier [4] proposed the widely used Brier score to quantify the quality of forecasts. Subsequent work studied other SPSR and developed several characterizations of SPSR [13, 17, 37, 46].

Peer prediction refers to a collection of mechanisms developed for incentivizing truthful reporting in IEWV. Our SSR mechanisms are additions to this collection. The core idea of peer prediction is to leverage peer reports as references to score an agent's report. The pioneer work [28] considered a single-task elicitation setting where each agent observes a private signal associated with a single task of interest, and a principal who knows the joint distribution of these signals wants to elicit the exact realizations of the signals. It proposed the first mechanism where truthful reporting is a Bayesian Nash Equilibrium (BNE). Following this work, Jurca and Faltings [18, 19] proposed mechanisms where truthful reporting is a BNE with a strictly higher payment than any other pure-strategy equilibrium. Kong et al. [21] proposed a mechanism, in which truthful reporting is the BNE with the highest payoff for agents among all equilibria on a binary-outcome task. Frongillo

and Witkowski [11] characterized all mechanisms that admit a truthful reporting equilibrium in this setting. Another research thread for single-task elicitation asks agents to answer additional questions in addition to providing their signal. The Bayesian Truth Serum [31] additionally asks the agents to report their beliefs about other agents' reports and then uses this additional information to score the answer of each agent. The advantage of this approach is that the principal needs not to know the joint distribution of agents' signals and that the additional information can be used to identify the correct answer to the question [32]. However, this approach introduces extra work for the agents. For interested readers, this line of research has been further developed by other studies [33, 36, 39, 49].

To relax the requirement on the principal's knowledge of the signal distribution, many recent peer prediction studies have focused on a multi-task setting, where there exists a set of i.i.d. tasks, allowing the principal to leverage the statistical patterns in agents' reports to incentivize truthful reporting. Our work falls into this category. The multi-task setting was simultaneously developed by Dasgupta and Ghosh [6] and Witkowski and Parkes [50]. The latter was the first to explicitly estimate relevant aspects of agents' belief models from agents' reports (which our paper also uses), while the former achieves provably stronger equilibrium properties. In the mechanism of Dasgupta and Ghosh [6], the truthful reporting equilibrium has the highest expected payoff for agents among all equilibria when eliciting binary signals. Radanovic et al. [35] and Shnayder et al. [42] extended the mechanism of Dasgupta and Ghosh [6] to elicit categorical signals while maintaining the same incentive property. More recent studies have achieved the dominant uniform strategy truthfulness in the multi-task setting. Parallel to our work, Kong and Schoenebeck [23] developed a framework to design mechanisms to elicit general signals as long as certain notions of mutual information can be estimated from agents' reports. Their mechanisms, which includes the mechanism of Shnayder et al. [42] as a special case, are dominant uniform strategy truthful when there is an infinite number of tasks. Kong [20] further achieved this truthfulness property with a finite number of tasks for eliciting categorical signals. Kong et al. [24] and Schoenebeck and Yu [38] proposed dominant uniform strategy truthful mechanisms to elicit continuous signals with normal distributions and with general full-support marginal distributions, respectively. When there is a noisy estimate of the ground truth with a known confusion matrix, Goel and Faltings [14] proposed a mechanism that also achieves the dominant uniform strategy truthfulness; the reward of an agent in the mechanism is an affine transformation of the the agent's correctness rate over all classes. In comparison, our dominant uniform strategy truthfulness mechanisms focus on eliciting posterior beliefs of the ground truth and the rewards in our mechanisms recover in expectation the accuracy of agents in terms of the SPSR. Instead of assuming availability of an estimate of the confusion matrix, we construct an estimate from the agents' reports, assuming the principal knows the order of the prior probabilities of each possible outcome of the ground truth.

There are a few studies also focusing on eliciting probabilistic predictions like our paper. Among these studies, Witkowski and Parkes [48] and Radanovic and Faltings [34] consider single-task elicitation and ask agents to report additional information as required by the Bayesian Truth Serum [31]. The two mechanisms proposed make truthful reporting an ex-post equilibrium and a BNE, respectively. Kong and Schoenebeck [22] provided a mechanism to elicit probabilistic predictions for the multi-task setting. Although truthful reporting is an equilibrium strategy under their mechanism, the mechanism is not dominant uniform strategy truthful. When the principal has access to an unbiased proxy of the ground truth, the proxy scoring rules developed by Witkowski et al. [47] can be used to elicit probabilistic predictions for the single-task setting as what SPSR offer with access to the ground truth. In this case, proxy scoring rules score a prediction against the unbiased proxy using a SPSR, and the expected score is equal to the expected score given by the SPSR using the ground truth up to a positive affine transformation [8]. In comparison, our

mechanisms also offer a meta approach to recover the score for any SPSR. Our mechanisms do not require access to an unbiased proxy but a set of i.i.d. tasks.

Finally, our work borrows ideas from the machine learning literature on learning with noisy data [e.g. 9, 29, 40, 44]. At a high level, our goal in this paper aligns with the goal in learning from noisy labels – both aim to evaluate a prediction when the ground truth is missing, but instead a noisy signal of the ground truth is available. Our work addresses the additional challenge that the error rate of the noisy signal remains unknown a priori.

3 PRELIMINARIES

Before we introduce our model of information elicitation without verification, we first briefly introduce strictly proper scoring rules (SPSR), which are designed for the well-studied information elicitation with verification settings. We highlight two nice properties of SPSR: (1) SPSR quantify the value of information and (2) SPSR is incentive compatible for elicitation. Our goal of this paper is to develop scoring rules that match these properties for the more challenging without verification settings. Our solutions build upon the understanding of SPSR.

SPSR are designed for eliciting subjective distributions of random variables when the principal can reward agents after the realization of the random variables. SPSR apply to eliciting predictions for any random variables, but we introduce them for binary random variables in this section because the rest of our paper focuses on the binary case. Let $Y \in \{0, 1\}$ represent a binary event. An agent has a subjective belief $p \in [0, 1]$ for the likelihood of $Y = 1$. When the agent reports a probabilistic prediction $q \in [0, 1]$ of Y being 1, the principal rewards the agent using a scoring function $S(q, y)$ that depends on both the agent's report q and the realized outcome of Y. Strict properness of $S(\cdot, \cdot)$ is defined as follows.

Definition 3.1. A function $S : [0, 1] \times \{0, 1\} \to \mathbb{R}$ that maps a reported belief q and the ground truth Y into a score is a *strictly proper scoring rule* if it satisfies $\mathbb{E}[S(p, Y)] > \mathbb{E}[S(q, Y)]$, for all $p, q \in [0, 1]$ and $p \neq q$. Both expectations are taken with respect to $Y \sim \text{Bernoulli}(p)$.

There is a rich family of strictly proper scoring rules, including the Brier score ($S(q, Y) = 1 - (q - Y)^2$), the log scoring rule ($S(q, Y) = \log(q)$ if $Y = 1$ and $S(q, Y) = \log(1 - q)$ if $Y = 0$) and the spherical scoring rules [13].

Incentive compatibility of SPSR. The definition of SPSR immediately gives incentive compatibility. If an agent's belief is p, reporting p truthfully uniquely maximizes her expected score.

SPSR quantify the value of information. Another nice property of SPSR is that they quantify the value/accuracy of reported predictions. To give a rigorous argument, we use an indicator vector $\mathbf{y}$ of length 2 to represent the realization of Y, with 1 at the Y-th position and 0 otherwise. That is, $\mathbf{y} = (0, 1)$ if $Y = 1$ and $\mathbf{y} = (1, 0)$ if $Y = 0$. We use a probability vector $\mathbf{q} = (1 - q, q)$ to represent probability q. By the representation theorem [13, 26, 37], any strictly proper scoring rule can be characterized using a corresponding strictly convex function G as follows: $S(\mathbf{q}, \mathbf{y}) = G(\mathbf{y}) - D_G(\mathbf{y}, \mathbf{q})$, where D_G is the Bregman divergence function of G. Now consider the unknown true distribution of Y, denoted by $\mathbf{p}^* = (1 - p^*, p^*)$. The expected score for an agent predicting $\mathbf{q}$ is

$$\mathbb{E}[S(\mathbf{q}, \mathbf{y})] = \mathbb{E}[G(\mathbf{y})] - \mathbb{E}[D_G(\mathbf{y}, \mathbf{q})],$$

where all three expectations are taken over $Y \sim \text{Bernoulli}(p^*)$. This means that the maximum score an agent can receive in expectation is $\mathbb{E}_{Y \sim \text{Bernoulli}(p^*)}[G(\mathbf{y})]$, which happens when the agent's report $\mathbf{q} = \mathbf{p}^*$. Moreover, a prediction $\mathbf{q}$ with a smaller divergence $\mathbb{E}_{y \sim \mathbf{p}^*}[D_G(\mathbf{y}, \mathbf{q})]$ receives a higher score in expectation. Intuitively, $\mathbb{E}_{Y \sim \text{Bernoulli}(p^*)}[D_G(\mathbf{y}, \mathbf{q})]$ characterizes how "far away" $\mathbf{q}$ is from the true distribution of Y under divergence function D_G. This implies that a strictly proper

scoring rule S qualifies the the accuracy of a prediction $\mathbf{q}$ based on the corresponding divergence function. When S is taken as the Brier scoring rule, the corresponding Bregman divergence is the quadratic function, and $\mathbb{E}_{Y\sim\text{Bernoulli}(p^*)}[D_G(\mathbf{y},\mathbf{q})] = ||\mathbf{p}^* - \mathbf{q}||^2$, implying that a prediction $\mathbf{q}$ closer to $\mathbf{p}^*$ according to ℓ_2 norm receives a higher score in expectation. When S is taken as the log scoring rule, the corresponding Bregman divergence is the KL-divergence, D_{KL}, which is also called the relative entropy, and $\mathbb{E}_{Y\sim\text{Bernoulli}(p^*)}[D_G(\mathbf{y},\mathbf{q})] = D_{KL}(\mathbf{p}^*||\mathbf{q}) + H(\mathbf{p}^*)$, where H is the entropy function. A prediction with a smaller KL-divergence from $\mathbf{p}^*$ receives a higher score in expectation. This property of SPSR allows the principal to take an expert's average score over a set of prediction tasks as a proxy of his average accuracy and rank experts accordingly.

4 MODEL AND MECHANISM DESIGN PROBLEM

We consider a multi-task setting for the information elicitation without verification (IEWV) problem Under this setting, we aim to develop scoring mechanisms that are incentive compatible and are able to quantify the value of elicited information, recovering the two desirable properties that SPSR achieve in the presence of the ground truth. In this section, we formally introduce the information structure of our setting and the exact mechanism design problem we consider.

4.1 Model of Information Structure

A principal has a set of tasks $[m] = \{0, ..., m-1\}$. Each task asks for a prediction for an independent random variable of interest, denoted by $Y_k, k \in [m]$. For now, we assume that these random variables to predict are binary variables, i.e., $Y_k \in \{0, 1\}, \forall k \in [m]$. We will generalize our results to (non-binary) categorical random variables in Section 7. There is a set of informed agents $[n] = \{0, ..., n-1\}$. Each agent $i \in [n]$ privately observes a random signal $O_{i,k}$ generated by Y_k for each task $k \in [m]$, and thus holds a posterior belief about Y_k, represented by $P_{i,k} := \Pr[Y_k = 1|O_{i,k}]$. The posterior $P_{i,k}$ is a random variable as the signal $O_{i,k}$ is a random variable. Furthermore, we make the following main assumptions on the information structure among the signals and ground truth.

Assumption 1. *Tasks are independent and similar a priori, that is, the joint distribution of* $(O_{1,k}, ..., O_{n,k}, Y_k)$ *is i.i.d. for all tasks* $k \in [m]$.

This assumption is natural when the set of tasks are of similar nature, for example, tasks to predict the replicability of multiple studies published in the same journal and the same year. In this example, readers may a priori hold the same journal-wide belief about the features and the replicability of each study. After reading the journal, each agent receives a private signal for each individual study, which allows her to provide a more informed prediction for that study. This assumption is common for multi-task IEWV.[1]

Based on Assumption 1, each Y_k has the same prior, denoted by $p := \Pr[Y_k = 1]$. Also, for a fixed agent i, the distribution of signal $O_{i,k}$ conditioned on Y_k on each task $k \in [m]$ is the same. We use $\mathcal{D}_i^+$ and $\mathcal{D}_i^-$ to denote this conditional distribution for agent i for conditions $Y_k = 1$ and $Y_k = 0$, respectively. We assume that $\mathcal{D}_i^+ \neq \mathcal{D}_i^-$, otherwise, observation $O_{i,k}$ is independent and uninformative to Y_k. Each agent forms her posterior belief $P_{i,k}$ using the prior p and the conditional distributions $\mathcal{D}_i^+$ and $\mathcal{D}_i^-$. We require no knowledge of $\mathcal{D}_i^+$ and $\mathcal{D}_i^-$ for the principal and the agents other than agent i. Furthermore, we assume that agents' signals are independent conditioned on the ground truth.

[1]Kong [20], Kong and Schoenebeck [22] consider information elicitation for objective questions (i.e., questions where an objective ground truth exists). They make the same assumption as Assumption 1. Other studies (e.g., [6, 20, 23, 35, 42]) consider information elicitation for subjective questions (i.e., questions with no objective ground truth, e.g., how do you rate the movie?). These studies also assume that the joint distributions of agents' signals are the same across all tasks.

Assumption 2. *For each task, agents' signals are mutually independent conditioned on the ground truth, i.e.,* $\forall k \in [m], \Pr\left[O_{1,k}, ..., O_{n,k} | Y_k\right] = \prod_{i\in[n]} \Pr[O_{i,k}|Y_k]$.

This assumption excludes the scenarios where agents have some form of "side information" to coordinate their reports. With "side information", it is impossible to have any mechanism that can truthfully elicit agents' predictions without access to ground truth. This issue has been noted by Kong and Schoenebeck [22] and Kong [20] for tasks with ground truth and the same assumption has been adopted. Finally, we make a technical assumption about the prior p and the principal's knowledge.

Assumption 3. *It holds that* $p \neq 0.5$ *and the principal knows whether* $p > 0.5$ *or not.*

We do not assume that the principal knows the exact prior p of tasks but assume that she knows whether $p > 0.5$ or $p < 0.5$. This one binary-bit of information helps the principal distinguish between the set of truthful predictions and the set of inverted predictions (i.e. everyone reporting $1 - p_{i,k}$ instead of $p_{i,k}$), which otherwise is impossible to distinguish. In practice, this information is usually easy to obtain. In the example of predicting the replicability of studies, this assumption only requires that the principal knows whether the majority of the studies can be replicated or not. The assumption $p \neq 0.5$ is a technical condition we need in order to distinguish the truthful reporting scenario from the inverted reporting scenario.

We also assume that the posterior $P_{i,k}$ for any agent i on any task k is different under different realizations of private signal $O_{i,k}$. This assumption is without loss of generality, because different realizations of $O_{i,k}$ which lead to the same posterior $P_{i,k}$ for agent i on task k also lead to the same posterior about any other agent's signal $O_{j,k}$ for agent i due to Assumption 2. Therefore, we can merge multiple realizations of $O_{i,k}$ that lead to the same posterior $P_{i,k}$ into one realization without influencing agent i's belief about other agents' signals and the ground truth. Consequently, it is without loss of generality to assume that there exists a one-to-one correspondence between the realization of an agent's signal $O_{i,k}$ and her posterior $P_{i,k}$. According to this one-to-one correspondence and Assumptions 1 and 2, the following two conditions hold for $P_{i,k}$ for $i \in [n], k \in [m]$.

Proposition 4.1. *Under Assumptions 1 and 2, the following two conditions hold for agents' beliefs* $P_{i,k}, i \in [n], k \in [m]$.

(1) $P_{1,k}, ..., P_{n,k}$ *and* Y_k *are independent of their own counterparts across tasks* $k \in [m]$ *but have the same joint distribution, i.e.,* $(P_{1,k}, ..., P_{n,k}, Y_k)$ *are i.i.d. across tasks* $k \in [m]$.

(2) For each task $k \in [m]$, $P_{1,k}, ..., P_{n,k}$ *are independent conditioned on* Y_k*, i.e.,* $\Pr[P_{1,k}, ..., P_{n,k}|Y_k] = \prod_{i\in[n]} \Pr[P_{i,k}|Y_k], \forall k \in [m]$.

The first condition in Proposition 4.1 implies that an agent has the same expertise level across different tasks, as the joint distribution of her posterior belief and the ground truth is the same across tasks. The second condition implies that given the ground truth, each agent's probabilistic prediction is independent. The two conditions in Proposition 4.1 in fact characterize a broader space of information structure than the space captured by Assumptions 1 and 2. The former space includes the information structure where each task has a different prior but the distribution of the posterior beliefs of each agent are still the same across tasks. Our theoretical results in this paper hold for the model with this more broader information structure space characterized by the two conditions in Proposition 4.1 and with Assumption 3, where p refers to the mean prior over all tasks.

4.2 Mechanism design problem

We consider the multi-task peer prediction mechanisms where the principal assigns each task k to a subset $[n_k] \subseteq [n]$ of agents, collects a single probabilistic prediction $q_{i,k} \in [0, 1]$ from

each agent i assigned with task k, and pays each agent based on all predictions collected from all agents. We use $[m_i] \subseteq m$ to denote the set of tasks assigned to agent i. We use $q_{i,k} = \emptyset$ to denote that agent i has not been assigned to task k. Such a multi-task peer prediction mechanism can be formally expressed as a function $R : \{\emptyset \cup [0,1]\}^{n\times m} \to \mathbb{R}^n$, which maps a prediction profile on all tasks and all agents to a vector of total payments of all agents. In this paper, we restrict our attention to anonymous mechanisms that give each prediction from an agent an independent payment like SPSR. Thus, a mechanism that we consider can be fully expressed by a score function $R : \{\emptyset \cup [0,1]\} \times \{\emptyset \cup [0,1]\}^{n-1\times m} \to \mathbb{R}$, which maps a single prediction $q_{i,k}$ of agent i on task k and a profile of predictions from all other agents into a single reward score for that prediction, and agent i's total reward is the sum of the scores she obtains across the tasks she is assigned with.

Agents have no obligation to report their true beliefs. Instead, given a mechanism, an agent can report strategically to maximize her expected payment. As there exists a one-to-one correspondence between an agent's signal and her posterior on a single task in our model, we can define an agent's reporting strategy on a single task without loss of generality as a function that maps her posterior to a distribution where her reported prediction is drawn from.

Definition 4.2. Let $\Delta_{[0,1]}$ be the space of all probability distributions over $[0,1]$. The strategy of an agent i on task k is a mapping $\sigma_i : [0,1] \to \Delta_{[0,1]}$, which maps her posterior belief $P_{i,k}$ into a distribution $\sigma_i(P_{i,k})$ over [0,1], from which the agent draws the reported prediction $Q_{i,k}$.

We use the upper case $Q_{i,k}$ to denote the reported prediction when we want to emphasize that the reported prediction is a random variable determined by an agent's posterior belief and her reporting strategy jointly, otherwise, $q_{i,k}$ is used. We further assume that each agent adopts the same mixed strategy across all asigned tasks.

Assumption 4. *(Uniform Strategy) For any agent $i \in [n]$, she adopts the same strategy $\sigma_i(\cdot)$ over all assigned tasks $k \in [m_i]$.*

This assumption is reasonable as we assume that tasks are a priori similar to each other. We use $\sigma_i(\cdot)$ to denote the reporting strategy adopted by agent i on all tasks she answers and use σ_{-i} to denote the strategy profile used by all agents except agent i. Furthermore, we use $\mathbb{E}[R(q_{i,k}; \sigma_{-i})]$ to denote the expected score that agent i receives for reporting $q_{i,k}$ when other agents use strategy profile σ_{-i}, where the expectation is taken over the randomness in ground truth, other agents' signals and strategies and in the mechanism itself. We use $\mathbb{E}[R(\sigma_i; \sigma_{-i})]$ to denote the expected reward of agent i when her report is also a random variable generated by her belief $P_{i,k}$ and reporting strategy σ_i.

In this paper, our goal is to design a mechanism $R(\cdot)$ in the IEWV setting with similar properties that SPSR have for the information elicitation with verification setting: *quantification of the value of information* and *incentive compatibility*.

Quantifying value of information. The score of each prediction should reflect the true accuracy of the prediction, similar to what SPSR achieve. That is, for all i, k and $q_{i,k}$ and for any true distribution of the ground truth Y_k, $\underline{\mathbb{E}[R(q_{i,k}; \sigma_{-i})] = f\left(E_{Y_k}[S(q_{i,k}, Y_k)]\right)}$ holds for a SPSR $S(\cdot)$ and a strictly increasing function f, where the two expectations are taken over the true distributions of the random variables in the two expressions at each side of the equality. This design goal pursues that the score that an agent receives for a prediction in IEWV recovers what the agent would receive with a SPSR (with access to the ground truth) in expectation.

Incentive Compatibility. A mechanism satisfies incentive compatibility to some extent if truthful reporting is a strategy that maximizes an agent's expected utility under certain conditions. In this paper, we pursue the dominant uniform strategy truthfulness [14], where truthful reporting

is a dominant strategy if we restrict the strategy space with the uniform strategy assumption (Assumption 4).

Formally, in IEVW, a dominant uniform strategy truthful mechanism is a mechanism where truthful reporting on each task maximizes an agent's expected reward no matter what uniform strategies the other agents play and strictly maximize the agent's expected reward if other agents' reports are also informative.[2] Let σ_i^* be the truthful reporting strategy for agent i, i.e., σ_i^* is the function that maps a belief p_i to a distribution where all probability mass is put on p_i. Let $\bar{Q}_{-i,k} := \frac{1}{n-1}\sum_{j\neq i} Q_{j,k}$ be the mean of all agents' reported predictions on task k except agent i's. Note that $\bar{Q}_{-i,k}$ is a random variable, because of the randomness in reporting strategy σ_j and the randomness in signal $O_{j,k}$ for all $j \neq i$. We say that $\bar{Q}_{-i,k}$ is informative about the ground truth if $\mathbb{E}[\bar{Q}_{-i,k}|y_k = 1] \neq \mathbb{E}[\bar{Q}_{-i,k}|y_k = 0]$. We formally define the dominant uniform strategy truthful mechanisms as follows.

Definition 4.3. (Dominant uniform strategy truthfulness). A mechanism $R(\cdot)$ is *dominant uniform strategy truthful* if $\forall i \in [n], \forall k \in [m_i], \forall\{\mathcal{D}_j^+, \mathcal{D}_j^-\}_{j\in[n]}$ and for any realization $o_{i,k}$ of signal $O_{i,k}$: $\mathbb{E}[R(\sigma_i^*;\sigma_{-i})|O_{i,k} = o_{i,k}] \geq \mathbb{E}[R(\sigma_i;\sigma_{-i})|O_{i,k} = o_{i,k}]$ for any uniform strategy $\sigma_i \neq \sigma_i^*$ and any uniform strategy profile of other agents σ_{-i}, and the inequality holds strictly for any uniform strategy profile σ_{-i} under which $\bar{Q}_{-i,k}$ is informative about Y_k.

In Definition 4.3, we characterize the condition that peers' reports are informative by that the expectation of the mean of peers' reports on a task differs when conditioned on different realizations of the ground truth.

5 ELICITATION WITH A NOISY ESTIMATE OF GROUND TRUTH

Before we develop mechanisms with the two desirable properties we pursue, in this section we first obtain these two properties under a very stylized setting: *elicitation with a noisy estimate of ground truth*. In this stylized setting, we introduce surrogate scoring rules as an effective solution. These scoring rules will be the building blocks of our mechanisms designed for the general setting.

This stylized setting has only one event Y and one agent i, who observes a signal O_i generated from distribution $\mathcal{D}_i(Y)$ and forms a posterior $P_i = \Pr[Y = 1|O_i]$. The principal in this setting has access to a noisy estimate $Z \in \{0, 1\}$ of the ground truth Y, although she has no access to the exact realization of Y. The noisy estimate Z is characterized by two *error rates*, e_z^+ and e_z^-, defined as $e_z^+ := \Pr[Z = 0|Y = 1]$, $e_z^- := \Pr[Z = 1|Y = 0]$, which are the probabilities that Z mismatches Y under the two realizations of Y. The principal knows the realization Z and the exact error rates e_z^+, e_z^-. The principal cannot expect to do much if Z is independent of Y. Therefore, we assume that Z and Y are stochastically relevant, an assumption commonly adopted on the relation between a signal and the ground truth in the information elicitation literature [28].

Definition 5.1. A random variable Z is *stochastically relevant* to a random variable Y if the distribution of Y conditioned on Z differs for different realizations of Z.

[2] In a standard dominant truthful mechanism, truthful reporting strictly maximizes the agent's expected reward no matter what strategies other agents play. In IEWV, however, if all peer agents report predictions independently w.r.t. the ground truth on each task, then there will be no information available for the mechanism to incentivize truthful reporting. Therefore, it is inevitable to allow a dominant truthful mechanism in IEWV to pay truthful reporting strictly more only when the peer reports are informative about the ground truth. For example, in studies [20, 23], the dominant uniform strategy truthful mechanism is defined to be a mechanism that pays truthful reporting strictly more only when for each agent, there exists at least one peer agent reporting truthfully. We will see later that in our definition, we do not require that there is at least one peer agent reporting truthfully. We allow all peer agents to play non-truthfully, but require the mean of peer agents' reports to be informative with respective to the ground truth.

The following lemma shows that the stochastic relevance condition directly translates to a constraint on the error rates, that is, $e_z^+ + e_z^- \neq 1$. This lemma can be proved immediately by writing out the distribution of Y conditioned on Z in terms of the two error rates e_z^+, e_z^- and the prior of Z.

Lemma 5.2. *The noisy estimate Z is stochastically relevant to the ground truth Y if and only if $e_z^+ + e_z^- \neq 1$.*

The goal of the principal in this setting is to design a scoring rule to elicit the posterior P_i truthfully based on this noisy estimate Z and the error rates e_z^+, e_z^-. We define the design space of the scoring rules with the noisy estimate as follows.

Definition 5.3. Given a noisy estimate Z of ground truth Y with error rates $(e_z^+, e_z^-) \in [0,1]^2$, *a scoring rule against the noisy estimate of the ground truth* is a function $R : [0,1] \times \{0,1\} \to \mathbb{R}$ that maps a prediction $q_i \in [0,1]$ and a realized noisy estimate $z \in \{0,1\}$ to a score. The function R can depend on the two error rates (e_z^+, e_z^-).

Adopting the terminology from the scoring rule literature, we refer to strict properness of a scoring rule against a noisy estimate of ground truth as the property that the rule assigns a strictly better expected score to a truthful prediction of the ground truth than to a non-truthful prediction.

Definition 5.4. A scoring rule $R(q_i, Z)$ against a noisy estimate Z of ground truth is *strictly proper* for eliciting an agent's posterior belief generated by signal O_i if it holds for all realizations o_i of O_i and the posterior $p_i = \Pr[Y = 1 | O_i = o_i]$ that

$$\mathbb{E}_{Z|O_i=o_i}[R(p_i, Z)] > \mathbb{E}_{Z|O_i=o_i}[R(q_i, Z)], \forall q_i \in [0,1](q_i \neq p_i).$$

5.1 Surrogate scoring rules (SSR)

In this section, we present our solution, the *surrogate scoring rules* (SSR), for this stylized setting. SSR are a family of scoring rules that evaluate a prediction against a noisy estimate of ground truth. For any distribution of the ground truth and any stochastically relevant noisy estimate of the ground truth, the expected score that SSR give to the prediction, with expectation taken over the randomness of the noisy estimate, is equal to (up to a monotonic increasing transformation) the expected score that a SPSR gives to the same prediction, with expectation taken over the randomness of the ground truth. We will see that SSR are strictly proper under mild conditions.

Definition 5.5 (Surrogate Scoring Rules). $R : [0,1] \times \{0,1\} \to \mathbb{R}$ is a surrogate scoring rule if for some strictly proper scoring rule $S : [0,1] \times \{0,1\} \to \mathbb{R}$ and a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, it holds that $\forall p_i, q_i, e_z^+, e_z^- \in [0,1]$ and $e_z^+ + e_z^- \neq 1$, $\mathbb{E}_Z[R(q_i, Z)] = f(\mathbb{E}_Y[S(q_i, Y)])$, where Y is the ground truth drawn from Bernoulli(p_i) and Z is a noisy estimate of Y with error rates e_z^+, e_z^-.

Definition 5.5 defines the SSR $R(\cdot)$ as scoring rules that help us remove the bias in Z and return us the same score given by a SPSR in expectation. The idea of SSR is borrowed from the machine learning literature on learning with noisy data [5, 27, 29, 40, 44]. SSR can be viewed as a particular class of the proxy scoring rules proposed by Witkowski et al. [47]. Witkowski et al. [47] achieve properness of proxy scoring rules by plugging in an *unbiased* proxy of the ground truth to a SPSR. With SSR, we directly work with biased proxy and design scoring functions to de-bias the noise in the proxy. We have the following strict properness result for SSR straightforwardly:

Theorem 5.6. *Given the prior p of the ground truth Y and a private signal O_i, SSR $R(q_i, Z)$ against a noisy estimate Z is strictly proper for eliciting the posterior $P_i = \Pr[Y = 1 | O_i]$ if Z and O_i are independent conditioned on Y, and Z is stochastically relevant to Y.*

We provide an implementation of SSR, which we call SSR_α:

$$R(q_i, Z=1) = \frac{(1-e_z^-)\cdot S(q_i,1) - e_z^+ \cdot S(q_i,0)}{1-e_z^+-e_z^-}, \tag{1}$$

$$R(q_i, Z=0) = \frac{(1-e_z^+)\cdot S(q_i,0) - e_z^- \cdot S(q_i,1)}{1-e_z^+-e_z^-}, \tag{2}$$

where S can be any strictly proper scoring rule. This SSR implementation is inspired by Natarajan et al.[29]. As can been seen from Eqs. 1 and 2, the knowledge of the error rates e_z^+, e_z^- is crucial for defining SSR_α. Moreover, SSR_α has the property that the expected score $\mathbb{E}_{Z|Y}[R(q_i, Z)]$ conditioned on the realization of the ground truth Y is exactly the same as the score $S(q_i, Y)$ given by the SPSR. More formally, we have the following lemma.

Lemma 5.7 (Lemma 1, [29]). *For SSR_α, ground truth Y and noisy estimate Z, $\forall q_i, e_z^+, e_z^- \in [0,1]$ and $e_z^+ + e_z^- \neq 1, \forall y \in \{0,1\} : \mathbb{E}_{Z|Y=y}[R(q_i, Z)] = S(q_i, Y=y)$.*

Proof. Lemma 1 in [29] proves the statement for $e_z^+ + e_z^- < 1$. For completeness, we provide the proof for $e_z^+ + e_z^- \neq 1$ here. Let $q_i \in [0,1]$ be an arbitrary prediction. When $y = 1$, we have

$$\begin{aligned}
\mathbb{E}_{Z|Y=1}[R(q_i,Z)] &= (1-e_z^+)R(q_i,1) + e_z^+ R(q_i,0) \\
&= (1-e_z^+)\frac{(1-e_z^-)S(q_i,1) - e_z^+ S(q_i,0)}{1-e_z^+-e_z^-} + e_z^+ \frac{(1-e_z^+)S(q_i,0) - e_z^- S(q_i,1)}{1-e_z^+-e_z^-} \\
&= \frac{\left((1-e_z^+)(1-e_z^-) - e_z^+ e_z^-\right) S(q_i,1)}{1-e_z^+-e_z^-} \\
&= S(q_i,1)
\end{aligned}$$

When $y = 0$, we have

$$\begin{aligned}
\mathbb{E}_{Z|Y=0}[R(q_i,Z)] &= e_z^- R(q_i,1) + (1-e_z^-)R(q_i,0) \\
&= e_z^- \frac{(1-e_z^-)S(q_i,1) - e_z^+ S(q_i,0)}{1-e_z^+-e_z^-} + (1-e_z^-)\frac{(1-e_z^+)S(q_i,0) - e_z^- S(q_i,1)}{1-e_z^+-e_z^-} \\
&= S(q_i,0)
\end{aligned}$$

□

Intuitively, the linear transformation in SSR_α ensures that, in expectation, the prediction q_i is scored as if it was scored against the ground truth Y under the underlying SPSR. We would like to note that other surrogate loss functions designed for learning with noisy labels can also be leveraged to design SSR. With the conditional unbiasedness property of SSR_α, we can formally claim that SSR_α is a surrogate scoring rule, as stated in Theorem 5.8 below.

Theorem 5.8. *SSR_α is a surrogate scoring rule and $\forall p_i, q_i, e_z^+, e_z^- \in [0,1] (e_z^+ + e_z^- \neq 1), \mathbb{E}_Z[R(q_i, Z)] = \mathbb{E}_Y[S(q_i, Y)]$, where Y is the ground truth drawn from Bernoulli(p_i) and Z is the noisy estimate of ground truth Y with error rate e_z^+, e_z^-.*

Proof. As shown by Lemma 5.7, for SSR_α, we have $\forall p_i, q_i, e_z^+, e_z^-$ $(e_z^+ + e_z^- \neq 1)$ and $\forall y \in \{0,1\}$, $\mathbb{E}_{Z|Y=y}[R(q_i, z)] = S(q_i, Y=y)$, we have immediately

$$\mathbb{E}_Z[R(q_i,z)] = \mathbb{E}_Y\Big[\mathbb{E}_{Z|Y}[R(q_i,Z)]\Big] = \mathbb{E}_Y[S(q_i,Y)].$$

□

With Theorem 5.8 we know that SSR_α quantifies the quality of information of a prediction just as the underlying strictly proper scoring rule S does. Furthermore, SSR_α has the following variance:

Mechanism 1 SSR mechanisms (Sketch)

1: For each task k, we uniformly randomly pick at least 3 agents, assign task k to them and collect their predictions.
2: For each agent i and each task k the agent answers, we construct a reference report $Z_{i,k}$ using the agent's peer agents' reports, and estimate the error rates $e^+_{z_{i,k}}$ and $e^-_{z_{i,k}}$ for $Z_{i,k}$.
3: Pay each agent i for her prediction $q_{i,k}$ on task k by SSR $R(q_{i,k}, Z_{i,k})$ if $e^+_{z_{i,k}} + e^+_{z_{i,k}} \neq 1$, and pay 0, otherwise.

Theorem 5.9. *Let $p_z := \Pr[Z = 1]$. For a fixed prediction $q_i \in [0, 1]$, SSR_α suffers the following variance:*

$$\mathbb{E}_Z\Big[R(q_i, Z) - \mathbb{E}_z[R(q_i, Z)]\Big]^2 = \frac{p_z \cdot (1 - p_z)}{(1 - e^+_z - e^-_z)^2} \cdot (S(q_i, 1) - S(q_i, 0))^2 . \tag{3}$$

Proof.

$$\begin{aligned}
&\mathbb{E}_Z\Big[R(q_i, Z) - \mathbb{E}_Z[R(q_i, Z)]\Big]^2 \\
=&p_z\bigg(R(q_i, 1) - \big(p_z R(q_i, 1) + (1 - q_z)R(q_i, 0)\big)\bigg)^2 \\
&+ (1 - p_z)\bigg(R(q_i, 0) - \big(p_z R(q_i, 1) + (1 - q_z)R(q_i, 0)\big)\bigg)^2 \\
=&p_z(1 - p_z)^2\big(R(q_i, 1) - R(q_i, 0)\big)^2 + (1 - p_z)p_z^2\big(R(q_i, 0) - R(q_i, 1)\big)^2 \\
=&p_z(1 - p_z)\big(R(q_i, 0) - R(q_i, 1)\big)^2 \\
=&\frac{p_z(1 - p_z)}{(1 - e^+_z - e^-_z)^2}\bigg((1 - e^-_z)S(q_i, 1) - e^+_z S(q_i, 0) - \big((1 - e^+_z)S(q_i, 0) - e^-_z S(q_i, 1)\big)\bigg)^2 \\
=&\frac{p_z(1 - p_z)}{(1 - e^+_z - e^-_z)^2}\big(S(q_i, 1) - S(q_i, 0)\big)^2
\end{aligned}$$

□

6 ELICITATION WITHOUT VERIFICATION

The results in the previous section are built upon the fact that there exists a noisy estimate of ground truth with known error rates. In this section, we apply the idea of SSR to the IEWV setting. A reasonable way to do so is to use agents' reports as the source of the noisy estimate. Although the principal does not know the exact bias in agents' reports, we find a way to construct such a noisy proxy of ground truth and estimate its error rates. We refer to this noisy proxy as the *reference report*. Applying SSR with this reference report, we can finally get a family of mechanisms which are dominant uniform strategy truthful and which also quantify the value of information in agents' reports as what SPSR do. Within this family, we can choose different underlying SPSR for SSR to get different mechanisms. We call this family of mechanisms *SSR mechanisms*. We present a sketch of our SSR mechanisms in Mechanism 1.

The challenge of designing such mechanisms is to construct the reference report $Z_{i,k}$ in Mechanism 1 and successfully estimate its error rates $e^+_{z_{i,k}}, e^-_{z_{i,k}}$. In the following sections, we show how to construct this reference report and estimate its error rates.

6.1 Reference report and its property

Recall that we use $Q_{j,k}$ to denote the reported prediction of agent j on task k, which is generated by agent j's posterior belief $P_{j,k}$ and reporting strategy σ_j. Let $S_{j,k} \in \{0,1\}$ be a binary signal independently drawn from Bernoulli$(Q_{j,k})$. We refer to $S_{j,k}$ as the *prediction signal* of agent j on task k. We construct the reference report $Z_{i,k}$ for agent i as follows: *We uniformly randomly pick an agent j from agent i's peer agent set $[n]\backslash\{i\}$, collect agent j's prediction $Q_{j,k}$, and draw a prediction signal $S_{j,k} \sim$ Bernoulli$(Q_{j,k})$. We use this $S_{j,k}$ as the reference report $Z_{i,k}$ for agent i on task k.*

Conditioned on all peer agents' reports $Q_{j,k}$, $j \in [n]\backslash\{i\}$, the distribution of $Z_{i,k}$ is Bernoulli$\big(\bar{Q}_{-i,k}\big)$, because we pick a prediction signal from all peer agents uniformly randomly. Recall that in our model, $Q_{i,k} \sim \sigma_i(P_{i,k}), i \in [n], k \in [m]$. Due to Proposition 4.1 and Assumption 4, $\bar{Q}_{-i,k}$ is i.i.d. across tasks $k \in [m]$ and is independent to agent i's posterior $P_{i,k}$ conditioned on the ground truth Y_k for any task k. Therefore, $Z_{i,k}, k \in [m]$ that we construct have the following two properties.

Lemma 6.1. *$\forall i \in [n], k \in [m]$, $Z_{i,k}$ is independent to agent i's posterior $P_{i,k}$ conditioned on Y_k.*

This property ensures that $Z_{i,k}$ can be used as the conditionally independent noisy estimate of the ground truth in Theorem 5.6 and thus, SSR against $Z_{i,k}$ is strictly proper for eliciting the posterior belief $P_{i,k}$.

Lemma 6.2. *For any strategy profile agents play, the reference reports of a single agent i for any $i \in [n]$ are i.i.d. across tasks and have the same error rates w.r.t. their corresponding ground truth Y_k, i.e., $\forall \sigma_1, ..., \sigma_n, \forall i \in [n], \exists e_i^+, e_i^- \in [0,1], \forall k \in [m] : \Pr[Z_{i,k} = 0 | Y_k = 1] = e_i^+, \Pr[Z_{i,k} = 1 | Y_k = 0] = e_i^-$.*

This lemma shows that the error rates of the reference reports of an agent i are the same across all tasks. This property allows the estimation of the error rates using the multi-task prediction data. In the following sections, we introduce the estimation of the error rates and complete our mechanisms. We prove Lemma 6.1 and 6.2 below.

Proof. Proposition 4.1 and Assumption 4 directly imply that 1) for each task, $Q_{1,k}, ..., Q_{n,k}$ are mutually independent conditioned on the ground truth Y_k, and 2) $(Q_{1,k}, ..., Q_{n,k}, y_k)$ are i.i.d across tasks $k \in [M]$. As $Z_{i,k}$ is independently drawn from Bernoulli$(\bar{Q}_{-i,k})$, we immediately have that 1') for each task $k \in [m]$, $Z_{i,k}$ is independent to $O_{i,k}$ and thus to $P_{i,k} := \Pr[Y_k = 1 | O_{i,k}]$, and 2') $(Z_{i,k}, Y_k)$ have the same joint distribution for $k \in [m]$. As a result of 2'), $Z_{i,k}, k \in [m]$ have the same error rates w.r.t. the corresponding Y_k. □

6.2 Asymptotic setting

To better deliver our idea for error rates estimation, we start with an asymptotic setting with infinite amounts of tasks and agents, i.e., $m, n \to \infty$. We will later provide a finite sample justification for our mechanism. Based on Lemma 6.2, the reference reports of an agent on different tasks have the same distribution and error rates. Therefore, we focus on estimating the error rates of the reference report of agent i on a generic task k, while we use Z to denote this reference report, omitting the subscripts i and k, and use e_z^+, e_z^- to denote its error rates.

Our estimation algorithm resembles the "method of moments." We establish three equations on the first- to the third-order statistics, of which the parameters can be expressed by the unknown error rates e_z^+, e_z^-. We show that the three equations, with knowing the true parameters (which is true in the asymptotic setting), together uniquely determine e_z^+, e_z^-. Thus, we can solve the three equations to obtain e_z^+, e_z^-. In the next section, we argue that in the finite sample setting, with imperfect estimates of the parameters of the three questions, the solution from these three perturbed equations still approximate the true values of e_z^+, e_z^- with guaranteed accuracy.

To construct these three equations, we make the following preparation. Let $s_{j,k}$ be the realization of the prediction signal $S_{j,k}$ of agent j on task k, and let $\mathcal{S}_{-i} := \{s_{j,k}\}_{j\neq i,k\in[M]}$ be the realization profile of all prediction signals from all peer agents of agent i. On a generic task k, we draw three random variables Z_1, Z_2, Z_3. Z_1 represents the realization of a prediction signal uniformly randomly drawn from the set of all prediction signals $\{s_{j,k}\}_{j\neq i}$ on task k except agent i's. Z_2 represents the realization of another uniformly randomly picked prediction signal from set $\{s_{j,k}\}_{j\neq i}$ but excluding Z_1. Similarly, Z_3 represents the realization of another uniformly randomly picked prediction signal from set $\{s_{j,k}\}_{j\neq i}$ but excluding Z_1 and Z_2. Because agents' reports are conditionally independent, Z_1, Z_2, Z_3 are also independent conditioned on the ground truth. Moreover, Z_1 and the reference report Z have the same error rates, as they are generated by the same random process. With infinite number of agents, Z_2 and Z_3 also have the same error rates as Z. Furthermore, (Z_1, Z_2, Z_3) is i.i.d. across different tasks, according to Proposition 4.1 and Assumption 4. Therefore, with infinite number of tasks (and thus infinite number of samples from the joint distribution Z_1, Z_2, Z_3), we can know the exact distribution parameters of any statistics about Z_1, Z_2 and Z_3. We can then establish the following three equations.

---

1. First-order equation: The first equation is based on the distribution of Z. Let $\alpha_{-i} := \Pr[Z = 1]$. α_{-i} can be expressed as a function of e_z^+, e_z^- via spelling out the conditional expectation:

$$\alpha_{-i} = p \cdot \Pr[Z = 1|Y = 1] + (1-p) \cdot \Pr[Z = 1|Y = 0] = p \cdot (1 - e_Z^+) + (1-p) \cdot e_Z^-. \tag{4}$$

2. Matching between two prediction signals: The second equation is based on a second-order statistic called the matching probability. We consider the matching-on-1 probability of Z_1, Z_2, i.e., the matching-on-1 probability of the prediction signals from two uniformly randomly picked peer agents of agent i). Let $\beta_{-i} := \Pr[Z_1 = 1, Z_2 = 1]$. It can be written as a function of e_z^-, e_z^+ as follows:

$$\begin{aligned}\beta_{-i} &= p \cdot \Pr\left[Z_1 = 1, Z_2 = 1|Y = 1\right] + (1-p) \cdot \Pr\left[Z_1 = 1, Z_2 = 1|Y = 0\right] \\ &= p \cdot \Pr\left[Z_1 = 1|Y = 1\right] \cdot \Pr\left[Z_2 = 1|Y = 1\right] + (1-p) \cdot \Pr\left[Z_1 = 1|Y = 0\right] \Pr\left[Z_2 = 1|Y = 0\right] \\ &= p \cdot (1 - e_z^+)^2 + (1-p) \cdot (e_z^-)^2. \end{aligned} \tag{5}$$

3. Matching among three prediction signals: The third equation is obtained by going one order higher. We check the matching-on-1 probability over three prediction signals Z_1, Z_2, Z_3 uniformly randomly drawn from three different peer agents on the same task. Let $\gamma_{-i} := \Pr[Z_1 = Z_2 = Z_3 = 1]$. Similar to Eq. 5, we have:

$$\gamma_{-i} = p \cdot (1 - e_z^+)^3 + (1-p) \cdot (e_z^-)^3. \tag{6}$$

Notice that all three parameters $\alpha_{-i}, \beta_{-i}, \gamma_{-i}$ can be perfectly estimated using $\mathcal{S}_{-i}$ with infinite number of tasks and agents, yet without accessing any of the ground truth. With the knowledge of these three parameters, we prove the following:

Theorem 6.3. *p, e_z^-, e_z^+ are uniquely identified by Eqs. 4-6 under Assumption 3 ($p \neq 0.5$ and the principal knows whether $p > 0.5$ or not). The solution is in the closed form shown in Algorithm 1.*

Proof. Let $x^- := e_z^-,\ x^+ := 1 - e_z^+$. Recall the three equations we have

$$\alpha_{-i} = (1-p) \cdot x^- + p \cdot x^+ \tag{7}$$

$$\beta_{-i} = (1-p) \cdot (x^-)^2 + p \cdot (x^+)^2 \tag{8}$$

$$\gamma_{-i} = (1-p) \cdot (x^-)^3 + p \cdot (x^+)^3 \tag{9}$$

Algorithm 1 e_z^+, e_z^- solver

Input: $\alpha_{-i}, \beta_{-i}, \gamma_{-i}, \mathbb{1}(p > 0.5)$
Output: e_z^+, e_z^-

1: Compute the following quantities:

$$a := \frac{\gamma_{-i} - \alpha_{-i}\beta_{-i}}{\beta_{-i} - (\alpha_{-i})^2},\ b := \frac{\alpha_{-i}\gamma_{-i} - (\beta_{-i})^2}{\beta_{-i} - (\alpha_{-i})^2}.$$

2: Let

$$\underline{x} := \frac{a - \sqrt{a^2 - 4b}}{2},\ \overline{x} := \frac{a + \sqrt{a^2 - 4b}}{2}, p' := \frac{\alpha_{-i} - \underline{x}}{\overline{x} - \underline{x}}$$

3: If $\mathbb{1}(p' > 0.5) = \mathbb{1}(p > 0.5)$, then $e_z^+ = 1 - \overline{x},\ e_z^- = \underline{x}$, else $e_z^+ = 1 - \underline{x},\ e_z^- = \overline{x}$.

We can rewrite the three equations as:

$$\alpha_{-i} - x^+ = (1-p)(x^- - x^+) \tag{10}$$

$$\beta_{-i} = (1-p)(x^- - x^+)(x^- + x^+) + (x^+)^2 \tag{11}$$

$$\gamma_{-i} = (1-p)(x^- - x^+)\left((x^-)^2 + x^- \cdot x^+ + (x^+)^2\right) + (x^+)^3 \tag{12}$$

Plugging Eq. 10 into Eqs. 11 and 12 and re-organizing the two equations, we have respectively:

$$\beta_{-i} = \alpha_{-i}(x^- + x^+) - x^- \cdot x^+ \tag{13}$$

$$\gamma_{-i} = \alpha_{-i}\left((x^- + x^+)^2 - x^- \cdot x^+\right) - x^- \cdot x^+(x^- + x^+) \tag{14}$$

Let

$$x^- + x^+ = a,\ x^- \cdot x^+ = b,$$

then we have $a = \frac{b+\beta_{-i}}{\alpha_{-i}}$ from Eq 13. Note that a is well defined, as o.w. if $\alpha_{-i} = 0$, we have to have $x^- = x^+ = 0$ which leads to $e_z^- + e_z^+ = 1$, a contradiction.

Substituting $x^- + x^+$ and $x^- \cdot x^+$ with $\frac{b+\beta_{-i}}{\alpha_{-i}}$ and b correspondingly in Eq. 14, we have

$$\alpha_{-i} \cdot \left(\frac{(b+\beta_{-i})^2}{(\alpha_{-i})^2} - b\right) - b \cdot \frac{b+\beta_{-i}}{\alpha_{-i}} = \gamma_{-i} \tag{15}$$

$$\Rightarrow \frac{(b+\beta_{-i})^2}{\alpha_{-i}} - b \cdot \alpha_{-i} - \frac{b^2}{\alpha_{-i}} - \frac{b \cdot \beta_{-i}}{\alpha_{-i}} = \gamma_{-i} \tag{16}$$

$$\Rightarrow \left(\frac{\beta_{-i}}{\alpha_{-i}} - \alpha_{-i}\right) b = \gamma_{-i} - \frac{(\beta_{-i})^2}{\alpha_{-i}} \Rightarrow b = \frac{\alpha_{-i}\gamma_{-i} - (\beta_{-i})^2}{\beta_{-i} - (\alpha_{-i})^2} \tag{17}$$

Thus, $a = \frac{b+\beta_{-i}}{\alpha_{-i}} = \frac{\gamma_{-i} - \alpha_{-i}\beta_{-i}}{\beta_{-i} - (\alpha_{-i})^2}, b = \frac{\alpha_{-i}\gamma_{-i} - (\beta_{-i})^2}{\beta_{-i} - (\alpha_{-i})^2}$. Then from $x^- + x^+ = a,\ x^- \cdot x^+ = b$, we have

$$x^+ = \frac{a \pm \sqrt{a^2 - 4b}}{2}, x^- = \frac{a \mp \sqrt{a^2 - 4b}}{2}, p = \frac{\alpha_{-i} - x^-}{x^+ - x^-}$$

Thus, we have two pairs of solutions for the error rates and the prior:

$$e^+_{z,(1)} = 1 - \frac{a + \sqrt{a^2 - 4b}}{2}, e^-_{z,(1)} = \frac{a - \sqrt{a^2 - 4b}}{2}, p_{(1)} = \frac{\alpha_{-i} - e^-_{z,(1)}}{1 - e^+_{z,(1)} - e^-_{z,(1)}}$$

$$e^-_{z,(2)} = 1 - e^+_{z,(1)}, e^+_{z,(2)} = 1 - e^-_{z,(1)}, p_{(2)} = 1 - p_{(1)}$$

Mechanism 2 SSR mechanisms

1: For each task k, uniformly randomly pick at least 3 agents, assign task k to them, collect their reported predictions $q_{i,k}$ and generate the prediction signal $S_{i,k}$ for each prediction.
2: For each agent i and each task k the agent answers, uniformly randomly select one prediction signal $S_{j,k}$ from her peers' prediction signals on the same task and let the reference report $Z_{i,k} := S_{j,k}$.
3: Establish Eqs. 4-6 and solve out the error rates $e^-_{z_i}, e^+_{z_i}$ for $Z_{i,k}$ for any k using Algorithm 3.
4: Pay each agent i's prediction $q_{i,k}$ on each task k she answers by applying SSR_α with $q_{i,k}$ and the noisy estimate $Z_{i,k}$ with error rates $e^+_{z_i}, e^-_{z_i}$ if $e^+_{z_i} + e^+_{z_i} \neq 1$, and pay 0, otherwise.

As in these two solutions, the values for the prior is symmetric w.r.t. 0.5. Thus, by Assumption 3, the principal can identify the unique correct solution from the two.

□

We can continue to establish higher-order equations. However, we show that they do not provide additional information about the three unknown variables, p, e^+_z, and e^-_z.

Theorem 6.4. *Any higher order (≥ 4) matching equations can be expressed by the first- to the third-order equations, Eqs. 4-6.*

Proof. We follow the shorthand notations as in the proof of Theorem 6.3. The n-th equation is

$$\Pr[Z_1 = ... = Z_n = 1] = (1-p)(x^-)^n + p(x^+)^n.$$

For $n \geq 4$, the right-hand of the equation can be expressed as

$$\begin{aligned}(1-p)(x^-)^n + p(x^+)^n =& \left((1-p)(x^-)^{n-1} + p(x^+)^{n-1}\right)(x^- + x^+) \\ & - x^- \cdot x^+ \left((1-p)(x^-)^{n-2} + p(x^+)^{n-2}\right) \\ =& \Pr[Z_1 = ... = Z_{n-1}](x^- + x^+) \\ & - \Pr[Z_1 = ... = Z_{n-2}] x^- \cdot x^+\end{aligned}$$

As we know from the proof of Theorem 6.3, $x^- + x^+$ and $x^- \cdot x^+$ are uniquely determined by the first three equations, i.e., Eqs. 4-6 (no matter whether Assumption 3 is made or not). Therefore, by induction starting from $n = 4$, the n-th equation can be expressed by the first three equations. □

Now we have completed our SSR mechanisms. The full version of the mechanisms is presented in Mechanism 2. Intuitively speaking, Theorem 6.3 shows that without ground truth data, knowing how frequently agents' predictions reach consensus with each other will help us characterize the (average) subjective biases in their reports. Furthermore, it implies that SSR mechanisms are asymptotically (in m, n) preserving the information quantification property that strictly proper scoring rules have, i.e., $\mathbb{E}_Z[R(q_{i,k}, Z)] = \mathbb{E}_Y[S(q_{i,k}, Y)]$, and that SSR mechanisms induce truthful reporting as the unique best uniform strategy for an agent, when Z is informative (i.e., $1 - e^+_z - e^-_z \neq 0$), and as a best strategy otherwise. Formally, we have the following theorem.

Theorem 6.5. *Under Assumptions 1-4, SSR mechanisms are dominant uniform strategy truthful with infinite number of tasks and agents. Furthermore, for any agent i and task k, if the average prediction of all other agents are informative, i.e., $e^+_z + e^-_z \neq 1$ for the noisy estimate of the ground truth $Z_{i,k}$ constructed for agent i, then the expected score of SSR mechanisms for agent i's prediction on a task is equal to the expected score given by the corresponding strictly proper scoring rule S: $\forall q_{i,k} \in [0,1], \mathbb{E}_{Z_{i,k}}[R(q_{i,k}, Z_{i,k})] = \mathbb{E}_{Y_k}[S(q_{i,k}, Y_k)]$.*

Proof. Recall that in Assumption 4, we assume that each agent adopts the same reporting strategy across tasks. As long as this assumption is satisfied, for an agent i, no matter what exact strategy the other agents play, we can always correctly estimate the error rates e_z^+ and e_z^- of the reference report Z constructed for agent i, according to Theorem 6.3. Furthermore, by Lemma 6.1, Z is independent to agent i's belief conditioned on the ground truth. Therefore, according to Theorem 5.6, when $e_z^+ + e_z^- \neq 1$, i.e., the other agents' average prediction is informative about the ground truth Y, SSR give agent i's prediction $q_{i,k}$ a reward unbiased to the expected reward given by the corresponding SPSR, i.e., $\forall q_{i,k}, \mathbb{E}_{Z_{i,k}}[R(q_{i,k}, Z_{i,k})] = \mathbb{E}_{Y_k}[S(q_{i,k}, Y_k)]$. Consequently, truthful reporting strictly maximizes the expected reward of agent i. When $e_z^+ + e_z^- = 1$, i.e., the other agents' average prediction is uninformative about Y_k for task k, SSR mechanisms always reward agent i zero, where truthful reporting also maximizes the expected reward of agent i. Thus, SSR mechanisms are dominant uniform strategy truthful. □

Remark 1. *Theorems 6.3 and 6.5 rely on Proposition 4.1 and Assumptions 3 and 4. Proposition 4.1 and Assumption 4 guarantee that there exists a similar information pattern across the predictions of different tasks that we can learn to infer the ground truth. Therefore, they can be hardly relaxed in IEVW settings. For Assumption 3, we'd like to argue that at least one bit of information is needed in order to distinguish the case where agents are truthfully reporting from the case where agents are misreporting by reverting their observations. This is because for any distribution of the observed reports of agents resulted by a world with parameters* (p, e_z^+, e_z^-) *and with agents reporting truthfully, there always exists the following counterfactual world achieving the same distribution of the observed reports of agents: a world with parameters* $(1-p, 1-e_z^-, 1-e_z^+)$ *and with agents misreporting predictions via relabelling* $0 \to 1$ *and* $1 \to 0$*. Thus, the mechanism designer cannot tell the two worlds apart from only the observed reports. Some studies [20, 23] relax Assumption 3 by allowing the truthful reporting strategy to weakly dominate this "relabeling equilibrium".*

We will show in the next section, SSR mechanisms are also dominant uniform strategy truthful with finite number of tasks and agents under mild conditions. Several remarks follow. (1) We would like to emphasize again that for an agent i, both Z and $R(\cdot)$ come from the prediction signals of her peer agents' reports $\mathcal{S}_{-i}$: Z is directly picked from $\mathcal{S}_{-i}$; $R(\cdot)$ depends on the error rates e_z^+ and e_z^- of Z, which are also learnt from $\mathcal{S}_{-i}$. (2) When making reporting decisions under SSR mechanisms, agents can choose to be oblivious of how much error presents in others' reports, because truthful reporting is the dominant strategy, i.e., no matter what uniform reporting strategy other agents play, truthful reporting always maximizes the expected reward. This removes the practical concern of implementing truthful reporting as a particular Nash Equilibrium when there exists a non-truthful reporting equilibrium. (3) Another salient feature of SSR mechanisms is that they transfer the cognitive load of having prior knowledge from the agent side to the mechanism designer side. Yet we do not assume the designer has exact knowledge of the prior either (but the knowledge of whether the prior is greater than 0.5 or not); instead we will leverage the power of estimation from reported data to achieve our goals.

6.3 Finite sample analysis

With finite m, n, we use the same procedure as shown in Algorithm 1 to estimate the error rates e_z^+, e_z^- for each agent, except that we cannot have the exact value for $\alpha_{-i}, \beta_{-i}, \gamma_{-i}$ but only with finite-sample estimates for them. Specifically, for agent i, letting k_1, k_2, k_3 (which could be different on different tasks) be the three agents whose prediction signals are selected as Z_1, Z_2, Z_3 on each

task $k \in [M]$,[3] we estimate:

$$\widetilde{\alpha_{-i}} = \frac{\sum_{k=1}^{m} \mathbb{1}(S_{k_1,k} = 1)}{m},\ \widetilde{\beta_{-i}} = \frac{\sum_{k=1}^{m} \mathbb{1}(S_{k_1,k} = S_{k_2,k} = 1)}{m},\ \widetilde{\gamma_{-i}} = \frac{\sum_{k=1}^{m} \mathbb{1}(S_{k_1,k} = S_{k_2,k} = S_{k_3,k} = 1)}{m}.$$

We then use these three values to replace $\alpha_{-i}, \beta_{-i}, \gamma_{-i}$, respectively, in Algorithm 1 to solve Eqs. 4-6. We denote the resulted error rates as $\widetilde{e_z^+}$ and $\widetilde{e_z^-}$, and the corresponding SSR_α using these error rates as $\widetilde{R}(\cdot)$.

There are two reasons that these finite-sample estimates $\widetilde{e_z^+}$ and $\widetilde{e_z^-}$ are not equal to the exact true error rates e_z^+ and e_z^- for Z. First, in constructing Eqs. 4-6, the error rates of two randomly picked prediction signals Z_2, Z_3 will not have the exactly same error rates with Z, as these signals come from a slightly different agent pool. Second, $\widetilde{\alpha_{-i}}, \widetilde{\beta_{-i}}, \widetilde{\gamma_{-i}}$ are not exactly equal to $\alpha_{-i}, \beta_{-i}, \gamma_{-i}$ with finite samples. However, we will show that the errors induced by these two factors in estimating the error rates diminish with m and n. Consequently, the SSR computed using $\widetilde{e_z^+}, \widetilde{e_z^-}$ also have a small and diminishing error towards the SSR computed with the exact error rates e_z^+, e_z^-.

Lemma 6.6. *$\widetilde{e_z^+}, \widetilde{e_z^-}$ given by Algorithm 1 using $\widetilde{\alpha_{-i}}, \widetilde{\beta_{-i}}, \widetilde{\gamma_{-i}}$ satisfy that for an arbitrary $\delta \in (0,1)$, with probability at least $1-\delta$, $|\widetilde{e_z^+} - e_z^+| \le \epsilon$, $|\widetilde{e_z^-} - e_z^-| \le \epsilon$ for some $\epsilon = O\big(\frac{1}{n} + \sqrt{\frac{\ln \frac{1}{\delta}}{m}}\big)$, which can be made arbitrarily small with increasing m and n.*

Proof Sketch. We present the high-level idea of our proof here and defer the complete proof to the appendix. We consider the two aforementioned errors separately. Both of them can be transformed to a diminishing error attaching to the evaluation of α_{-i}, β_{-i}, and γ_{-i}. This diminishing noise in α_{-i}, β_{-i}, and γ_{-i} can then be transformed into a diminishing error in the final solution of e_z^+, e_z^-. □

Next, we show that the deviations of the rewards of SSR mechanisms due to the imperfect estimation of the error rates in the finite sample case can also be bounded to be arbitrarily small. We first deal with a special case: even if $1 - e_z^+ - e_z^-$ is far from zero, the estimated $1 - \widetilde{e_z^+} - \widetilde{e_z^-}$ in the denominator of SSR_α can be arbitrary close to zero by coincidence. In this case, agents can have unbounded scores, which may be far from the exact scores agents should obtain when the estimation is perfect. To address this special case, the principal can select a threshold κ greater but close to zero, and pay agents zero when $|1 - \widetilde{e_z^+} - \widetilde{e_z^-}| < \kappa$ instead of just when $1 - \widetilde{e_z^+} - \widetilde{e_z^-} = 0$. As a result, the final reward of each agent is always bounded. Next, we introduce a lemma we will use in our proof.

Lemma 6.7. $\forall l_1, l_2, t_1, t_2 \in [-1, 1], t_1, t_2 \neq 0, \left|\frac{l_1}{t_1} - \frac{l_2}{t_2}\right| \le \frac{|l_1 - l_2| + |t_1 - t_2|}{|t_1 t_2|}$

Proof. $\left|\frac{l_1}{t_1} - \frac{l_2}{t_2}\right| = \left|\frac{l_1 t_2 - l_2 t_1}{t_1 t_2}\right| = \left|\frac{l_1 t_2 - l_2 t_2 + l_2 t_2 - l_2 t_1}{t_1 t_2}\right| \le \frac{|t_2||l_1 - l_2| + |l_2||t_2 - t_1|}{|t_1 t_2|} \le \frac{|l_1 - l_2| + |t_1 - t_2|}{|t_1 t_2|}$ □

This lemma is an extension to Lemma 7 of [25], which considers the case where all variables are non-negative. Now we present our main theorem about the diminishing error in estimating the SSR scores.

Theorem 6.8. *For a bounded SPSR $S(\cdot)$ with supremum $\max S$, for an arbitrary $\delta \in (0,1)$, and some $\epsilon = O\big(\frac{1}{n} + \sqrt{\frac{\ln \frac{1}{\delta}}{m}}\big)$ such that with probability at least $1-\delta$, $|\widetilde{e_z^+} - e_z^+| \le \epsilon$, $|\widetilde{e_z^-} - e_z^-| \le \epsilon$, let m*

[3] In practice, we only need to assign task k to these three randomly selected agents.

and n be sufficiently large such that $\epsilon \le |1 - e_z^- - e_z^+|/4$, the SSR mechanism built upon $S(\cdot)$ satisfies, with probability at least $1-\delta$, that

$$|\widetilde{R}(q_{i,k}, Z) - R(q_{i,k}, Z)| \le \frac{12 \max S}{\Delta^2} \cdot \epsilon, \ \forall i \in [n], k \in [m], q_{i,k} \in [0,1], Z \in \{0,1\},$$

where $\Delta = |1 - e_z^- - e_z^+|$. Furthermore, taking over all the randomness in the score, we have

$$\left|\mathbb{E}[\widetilde{R}(q_{i,k}, Z)] - \mathbb{E}[S(q_{i,k}, Z)]\right| = O\left(\frac{1}{N} + \sqrt{\frac{\ln m}{m}}\right), \forall i, k.$$

Proof. This proof is straight-forward following the error rate bounding result (Lemma 6.6). We use $sgn(Z), Z \in \{0,1\}$ as the superscript, where $sgn(0)$ refers to super script "$-$" and $sgn(1)$ refers to super script "+".

Consider an arbitrary agent i and a task k, we have

$$\begin{aligned}
|\widetilde{R}(q_{i,k}, Z) - R(q_{i,k}, Z)| =& \left|\left(\frac{1 - \widetilde{e_z^{sgn(1-Z)}}}{1 - \widetilde{e_z^+} - \widetilde{e_z^-}} - \frac{1 - e_z^{sgn(1-Z)}}{1 - e_z^+ - e_z^-}\right) S(q_{i,k}, Z)\right. \\
&\left. - \left(\frac{\widetilde{e_z^{sgn(Z)}}}{1 - \widetilde{e_z^+} - \widetilde{e_z^-}} - \frac{e_z^{sgn(Z)}}{1 - e_z^+ - e_z^-}\right) S(q_{i,k}, 1-Z)\right| \\
\le& \left|\frac{1 - \widetilde{e_z^{sgn(1-Z)}}}{1 - \widetilde{e_z^+} - \widetilde{e_z^-}} - \frac{1 - e_z^{sgn(1-Z)}}{1 - e_z^+ - e_z^-}\right| \max S \\
&+ \left|\frac{\widetilde{e_z^{sgn(Z)}}}{1 - \widetilde{e_z^+} - \widetilde{e_z^-}} - \frac{e_z^{sgn(Z)}}{1 - e_z^+ - e_z^-}\right| \max S
\end{aligned}$$

Since $\epsilon \le (1 - e_z^- - e_z^+)/4$, we know that

$$|1 - \widetilde{e_z^+} - \widetilde{e_z^-}| \ge |1 - e_z^- - e_z^+|/2$$

Thus, with probability at least $1-\delta$,

$$\begin{aligned}
\left|\frac{1 - \widetilde{e_z^{sgn(1-Z)}}}{1 - \widetilde{e_z^+} - \widetilde{e_z^-}} - \frac{1 - e_z^{sgn(1-Z)}}{1 - e_z^+ - e_z^-}\right| \le& \frac{\left|\widetilde{e_z^{sgn(1-Z)}} - e_z^{sgn(1-Z)}\right| + \left|\widetilde{e_z^+} + \widetilde{e_z^-} - e_z^+ - e_z^-\right|}{|(1 - \widetilde{e_z^+} - \widetilde{e_z^-})(1 - e_z^+ - e_z^-)|} \\
\le& \frac{3\epsilon}{|(1 - \widetilde{e_z^+} - \widetilde{e_z^-})(1 - e_z^+ - e_z^-)|} \le \frac{6\epsilon}{\Delta^2}
\end{aligned}$$

In above inequalities, the first "$\le$" follows Lemma 6.7, the second follows Lemma 6.6, and the third follows $|1 - \widetilde{e_z^+} - \widetilde{e_z^-}| \ge |1 - e_z^- - e_z^+|/2$. Similarly, we have

$$\left|\frac{\widetilde{e_z^{sgn(Z)}}}{1 - \widetilde{e_z^+} - \widetilde{e_z^-}} - \frac{e_z^{sgn(Z)}}{1 - e_z^+ - e_z^-}\right| = \frac{6\epsilon}{\Delta^2}$$

Plugging back, we have proved the claim that with probability at least $1-\delta$,

$$|\widetilde{R}(q_{i,k}, Z) - R(q_{i,k}, Z)| \le \frac{12\epsilon \cdot \max S}{\Delta^2}, \ \forall q_{i,k} \in [0,1], Z \in \{0,1\}.$$

As $\mathbb{E}[S(q_{i,k}, Z)] = \mathbb{E}[R(q_{i,k}, Z)]$, letting $\delta = \frac{1}{m}$, we have the expected error $\left|\mathbb{E}[\widetilde{R}(q_{i,k}, Z)] - \mathbb{E}[S(q_{i,k}, Z)]\right|$ bounded by $O\left((1 - \frac{1}{m})\left(\frac{1}{n} + \sqrt{\frac{\ln m}{m}}\right) + \frac{1}{m}\right) = O\left(\frac{1}{m} + \sqrt{\frac{\ln m}{m}}\right)$.

□

Theorem 6.8 indicates that the errors of the expected scores given by SSR mechanisms w.r.t. the expected score given by the underlying SPSR can be made arbitrary small with sufficiently large m and n. As a result, for arbitrarily discretized report space of a prediction, SSR mechanisms are still dominant uniform strategy truthful with finite but sufficiently large m and n. To see this, we can make the error smaller than the minimum absolute difference of the SPSRs of any two allowed probability reports. In such way, there will be no beneficial deviation for agents to report non-truthfully. This result considers the reality that in real surveys, agents are often allowed to specify at most two decimal digits for probabilistic predictions.

Corollary 6.9. *For discretized report space of probabilistic predictions, SSR mechanisms which are built upon bounded SPSR are dominant uniform strategy truthful for finite but sufficiently large m and n.*

7 GENERALIZATIONS TO MULTI-OUTCOME TASKS

In this section, we discuss how to extend SSR and SSR mechanims to the multi-outcome multi-task setting. A multi-outcome task asks agents to provide predictions about a multi-outcome random variable Y, which takes value from a finite support set $[c] = \{0, ..., c-1\}$ with $c > 2$. A noisy estimate $Z \in [c]$ of the ground truth Y is characterized by a confusing matrix:

$$E_z = \begin{bmatrix} e_{0,0} & e_{0,1} & \dots & e_{0,c-1} \\ e_{1,0} & e_{1,1} & \dots & e_{1,c-1} \\ \dots & \dots & \dots & \dots \\ e_{c-1,0} & e_{c-1,1} & \dots & e_{c-1,c-1} \end{bmatrix},$$

where $e_{u,v}$ represents the flipping probability of Z w.r.t. Y, i.e., $e_{u,v} = \Pr[Z = v | Y = u], \forall u, v \in [c]$.

7.1 Generalization of SSR

The surrogate scoring rules for a task with c outcomes are defined as follows. Let Δ^{c-1} be the $(c-1)$-dimension probability simplex, i.e., $\Delta^{c-1} := \left\{(x_0, ..., x_{c-1}) | \sum_{i=0}^{c-1} x_i = 1, x_0, ..., x_{c-1} \geq 0\right\}$.

Definition 7.1 (Surrogate Scoring Rules). $R : \Delta^{c-1} \times [c] \to \mathbb{R}$ is a surrogate scoring rule for a c-outcome task if for some strictly proper scoring rule $S : \Delta^{c-1} \times [c] \to \mathbb{R}$ and a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, the following equation holds:

$$\forall \mathbf{p}, \mathbf{q} \in \Delta^{c-1}, \forall E_z \in [0,1]^{c \times c} (E_z \text{ is invertible}) : \underline{\mathbb{E}_Z[R(\mathbf{q}, Z)] = f\left(\mathbb{E}_Y[S(\mathbf{q}, Y)]\right)},$$

where the ground truth Y is drawn from Categorical($\mathbf{p}$) and Z is a noisy estimate of Y with confusing matrix E_z.

We have the following theorem immediately.

Theorem 7.2. *Given the prior* $\mathbf{p}$ *of the ground truth* Y *and a private signal* O_i*, SSR* $R(\mathbf{q}, z)$ *with a noisy estimate* Z *of the ground truth is strictly proper for eliciting an agent's posterior* $\mathbf{p}_i := \Pr[Y|O_i]$ *if* Z *and* O_i *are independent conditioned on* Y *and* E_z *is invertible.*

Now we give an implementation of SSR, SSR$_\alpha$, for a c-outcome task. Let $S(\mathbf{q}_i)$ be the vector of SPSR scores for a prediction $\mathbf{q}_i \in \Delta^{c-1}$ under each realization of Y, i.e., $S(\mathbf{q}_i) := (S(\mathbf{q}_i, Y = 0), ..., S(\mathbf{q}_i, Y = c-1))$. Similarly, let $R(\mathbf{q}_i) := (R(\mathbf{q}_i, Z = 0), ..., R(\mathbf{q}_i, Z = c-1))$. Our implementation SSR$_\alpha$ goes as follows:

$$R(\mathbf{q}_i) := (E_z)^{-1} S(\mathbf{q}_i)$$

Clearly, for SSR_α we have $S(\mathbf{q}_i) = E_z \cdot R(\mathbf{q}_i)$, which gives

$$\forall v \in [c], S(\mathbf{q}_i, Y = v) = \sum_{k=0}^{c-1} e_{v,k} R(\mathbf{q}_i, Z = k) = \mathbb{E}_{Z|Y=v}[R(\mathbf{q}_i, Z)].$$

Lemma 7.3. *For SSR_α: $\forall v \in [c], \mathbb{E}_{Z|Y=v}[R(p_i, Z)] = S(p_i, Y = v)$*

The following theorem follows immediately.

Theorem 7.4. *SSR_α is a surrogate scoring rule for a multi-outcome task, and for any distribution $\mathbf{p} \in \Delta^{c-1}$ of the ground truth Y and for any invertible confusing matrix E_z of a noisy estimate Z of the ground truth, we have $\forall \mathbf{q} \in \Delta^{c-1}, \mathbb{E}_Z[R(\mathbf{q}, Z)] = \mathbb{E}_Y[S(\mathbf{q}, Y)]$.*

We include a detailed example of SSR_α for a three-outcome task below.

Example 7.5. Let $c = 3$ and let the confusing matrix of a noisy signal Z being

$$E_z = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \Rightarrow (E_z)^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

We obtain a closed-form of SSR_α:

$$\begin{aligned} R(\mathbf{q}, Z = 0) &:= 3S(\mathbf{q}, 0) - S(\mathbf{q}, 1) - S(\mathbf{q}, 2) \\ R(\mathbf{q}, Z = 1) &:= -S(\mathbf{q}, 0) + 3S(\mathbf{q}, 1) - S(\mathbf{q}, 2) \\ R(\mathbf{q}, Z = 2) &:= -S(\mathbf{q}, 0) - S(\mathbf{q}, 1) + 3S(\mathbf{q}, 2) \end{aligned}$$

7.2 Generalization of SSR mechanisms

SSR mechanisms can also be extended to multi-outcome tasks and maintain the two properties we pursue: the dominant uniform strategy truthfulness and qualifying the value of information as what SPSR do.

We consider the same setting of information structures under Assumptions 1-3, except that $Y_k, k \in [m]$ in these assumptions are c-outcome categorical random variables, agents' beliefs are categorical distributions, and that in Assumption 3, the prior probabilities of Y_k being each outcome are different and the principal knows the order of these prior probabilities. As we have shown that SSR can be extended to multi-outcome events, to construct the corresponding SSR mechanism, we just need to construct the corresponding noisy estimate Z of the ground truth and estimate the confusion matrix E_z for multi-outcome tasks.

The noisy estimate Z for an agent i on task k can be constructed similarly as the counterpart in the binary case, i.e., we uniformly randomly pick an agent $j \neq i$ and draw $Z \sim \text{Categorical}(\mathbf{q}_{j,k})$, where $\mathbf{q}_{j,k}$ is the reported distribution of Y_k from agent j. Then, the confusion matrix can also be estimated using the method of moments. However, as there are $c^2 - 1$ unknown parameters in the confusion matrix E_z and the prior $\mathbf{p}$ of Y_k, we have to establish $c^2 - 1$ equations. These equations could be solved numerically. These $c^2 - 1$ equations will have $c!$ real-value symmetric solutions, each corresponds to a permutation of the labeling of the c outcomes. To identify the unique solution that yields the true confusion matrix and the prior of Y, i.e., to identify the correct labeling of the outcomes, the principal has to know the order of the prior probabilities of Y_k being each outcome, as what we assume in Assumption 3 for multi-outcome tasks. Thus, with the multi-task variant of Assumptions 1-3, we can still construct a noisy estimate Z of the ground truth, estimate its confusion matrix, and apply SSR to obtain unbiased estimates of agents' scores given by the underlying SPSR.

Despite the positive result in theory, there are some caveats of applying SSR mechanisms to multi-outcome tasks. First, Assumption 1 essentially assume that the confusion matrix of an agent

is homogeneous across different tasks. However, as there is no clear correspondence between the labels of the outcomes of different tasks, the confusion matrix of the noisy estimate Z for an agent is less likely to be homogeneous across different tasks. Therefore, the real data can deviate far from Assumption 1. Second, as there are more parameters in the confusion matrix to estimate in the multi-task case than in the binary case, we need a much larger number of agents and tasks and denser predictions to maintain decent estimation accuracy. Third, to apply a SSR mechanism to multi-outcome tasks, these tasks have to have the same number of outcomes. However, in most crowd forecasting projects, the number of multi-outcome tasks with the same number of outcomes is much smaller than the number of binary questions and may not be sufficient to make accurate estimation of the confusion matrix. These caveats leave a massive space for future research.

8 EMPIRICAL STUDIES

Using 14 real-world human forecasting datasets, we empirically examine the performance of SSR mechanisms in revealing agents' prediction accuracy in terms of SPSR. We focus on three aspects: the unbiasedness of SSR, the correlation of SSR scores to SPSR scores, and the accuracy of SSR in selecting true top forecasters in terms of SPSR. We also compare the performance of SSR mechanisms to several existing peer prediction mechanisms. The overall results show that our SSR mechanisms have an advantage in recovering SPSR.

8.1 Setting

8.1.1 Datasets. We conduct our experiments on 14 datasets from three human forecasting and crowdsourcing projects: the Good judgment Project (GJP), the Hybrid Forecasting Competition (HFC), and the human judgment datasets collected by MIT. These three projects differ in participant population, forecasting topics, and elicitation methods, offering a rich environment for empirical evaluation.

GJP datasets [3]. The GJP data consists of four datasets for geopolitical forecasting questions. The four datasets, denoted by G1~G4, were collected from 2011 to 2014, respectively. They contain different sets of forecasting questions and forecasters.

HFC datasets [16]. We use the forecast data of team participants in the Hybrid Forecasting Competition. The data consists of three datasets, denoted by H1~H3, referring to the forecasting data collected in the preseason competition, the first competition, and the second competition, respectively. The the preseason competition lasted half a year, and the two formal competitions lasted around one year. The three datasets have different forecasting questions and partially overlapped participating teams.

MIT datasets [32]. The MIT data consists of seven datasets, denoted by M1a, M1b, M1c, M2, M3, M4a, M4b, respectively. Each dataset uses one of four sets of questions and has a different participant pool. The questions range from guessing the capital of each state and predicting the price interval of artworks to some trivia questions. The forecasters were students in class and colleagues in labs. In datasets M1a, M1b, M4a, M4b, forecasters report only binary votes on forecasting questions. In datasets M1c, M2, M3, forecasters give probabilistic predictions.

Both GJP and HFC projects allow participants to make daily forecasts. For testing peer prediction mechanisms in our setting, we only need to use a single prediction for each participant on a forecasting question. In our experiments, we mainly focus on the final prediction of each participant made on each question (i.e., the last prediction made by each participant before the close date of the corresponding forecasting question) in these two projects. At the end of Section 8.2, we complement our analysis by verifying the robustness of SSR with respect to the choice of the time

Items	G1	G2	G3	G4	H1	H2	H3	M1a	M1b	M1c	M2	M3	M4a	M4b
# of questions (original)	94	111	122	94	44	86	203	50	50	50	80	80	90	90
# of agents (orginal)	1972	1238	1565	7019	79	317	222	51	32	33	39	25	20	20
After applying the filter														
# of questions	94	111	122	94	44	86	203	50	50	50	80	80	90	90
# of agents	1409	948	1033	3086	79	316	222	51	32	33	39	25	20	20
Avg. # of answers per question	851	533	369	1301	71	295	220	51	32	33	39	18	20	20
Avg. # of answers per agent	57	62	44	40	39	80	201	50	50	50	80	60	90	90
Majority vote correct ratio (%)	0.90	0.92	0.95	0.96	0.93	0.93	0.86	0.58	0.76	0.74	0.61	0.68	0.62	0.72

Table 1. Statistics about binary-outcome datasets from GJP, HFC and MIT datasets

the predictions are made. Also, we focus on the forecasting questions which have binary outcomes in these datasets. To have a relatively stable estimation over the accuracy of agents, we filter out participants who made predictions on less than 15 questions. The basic statistics of these datasets are presented in Table 1.

8.1.2 SPSR. We consider three SPSR, the Brier score, the log scoring rule, and the rank-sum scoring rule, because of their usage in practice and connections to machine learning concepts. The first two are the most widely adopted scoring rules. They are equivalent to two main loss functions, the squared error and the cross-entropy loss, respectively, used in the machine learning community. The rank-sum scoring rule can be written as an affine transformation of the area under the receiver operating characteristic curve (AUC-ROC),[4] which is also a widely adopted accuracy metric in the machine learning community.

In our experiments, we adopt the conventional formula of the Brier score used in the GJP and HFC projects. The Brier score ranges from 0 to 2, with a smaller score corresponds to higher accuracy. This is different from using SPSR as a payment method, where the higher the better. We can transfer between these two usages by applying a negative scalar. We orient the log scoring rule and the rank-sum score rule in the same direction as the Brier score, with a minimum (best) score of 0. The exact formula for each scoring rule is as follows: Recall that $q_{i,k}$ and Y_k are agent i's prediction and the ground truth for task k, respectively, and $[m_i]$ is the set of tasks answered by agent i.

- **Brier score**: $S^{\text{Brier}}(q_{i,k}, Y_k) = 2(q_{i,k} - Y_k)^2$. We use the mean Brier score, $\frac{1}{m_i} \cdot \sum_{k \in [m_i]} S^{\text{Brier}}(q_{i,k}, Y_k)$, to represent an agent's overall accuracy under the Brier score over the set of tasks she answered.
- **Log scoring rule**: $S^{\log}(q_{i,k}, Y_k) = Y_k \log(q_{i,k}) + (1 - Y_k) \log(1 - q_{i,k})$. We use the mean log score, $\frac{1}{m_i} \sum_{k \in [m_i]} S^{\log}(q_{i,k}, Y_k)$, to represent an agent's overall accuracy under the long scoring rule over the tasks she answered. As the log scoring rule is unbounded when the forecast predicts the opposite of the ground truth, we change all forecasts of 1 to 0.99 and forecasts of 0 to 0.01 to ensure that the score is always a real number.
- **Rank-sum scoring rule** is a multi-task scoring rule. For a single task k, it assigns a score

$$S^{\text{rank}}(q_{i,k}, y_k) = -y_k \cdot \psi\left(q_{i,k} | \{q_{i,k'}\}_{k' \in [m_i]}\right),$$

[4]The affine transformation coefficients are determined by the numbers of the tasks with ground truth 1 and with ground truth 0. (according to Eqs. 12 and 13, [30]). Thus, when evaluating agents' prediction accuracy on the same set of answered questions, the rank-sum scoring rule is equal to the AUC-ROC for each agent up to the same affine transformation determined by the ground truth of the questions. However, the AUC-ROC itself is not an SPSR, as when considering the incentive, the affine transformation coefficients may differ in different agents' beliefs.

where $\psi\left(q_{i,k}|\{q_{i,k'}\}_{k'\in[m_i]}\right) := \sum_{k'\in[m_i]}\mathbb{1}(q_{i,k'} < q_{i,k}) - \sum_{k'\in[m_i]}\mathbb{1}(q_{i,k'} > q_{i,k})$ is the rank of prediction $q_{i,k}$ among all predictions from agent i. Then, agent i's rank-sum score S_i^{rank} is defined as: $S_i^{\text{rank}} = \sum_{k\in[m_i]} S^{\text{rank}}(q_{i,k}, Y_k)$.[5] The range of the score increases with the number of answered tasks quadratically, thus we use the normalized score $1+\frac{4}{m_i^2}S_i^{\text{rank}}$ with range $[0, 2]$.

8.1.3 Treatments. Though existing peer prediction methods are not designed for recovery of SPSR, we add comparisons to them for completeness of our study.[6] In particular, we would like to understand whether in practice SSR mechanisms have the advantage of revealing the true scores given by SPSR while not accessing ground truth information.

In our experiments, we consider four popular existing peer prediction methods, serving as comparisons to SSR: proxy scoring rules (PSR) with extremized mean [47], peer truth serum (PTS) [35], correlated agreement (CA) [42], determinant mutual information (DMI) [20].

PSR is to directly apply the SPSR w.r.t. an unbiased proxy of the ground truth. When the principal knows no unbiased proxy, Witkowski et al. [47] recommend using the extremized mean of the reported predictions to serve as the proxy. In our experiments, we adopt the same formula for the extremized mean as in their experiments [47], i.e., $\frac{\bar{q}_k^2}{\bar{q}_k^2+(1-\bar{q}_k)^2}$, where $\bar{q}_k$ is the average reported prediction on task k. Using different SPSR as the underlying scoring rule, we can get different PSR and SSR.

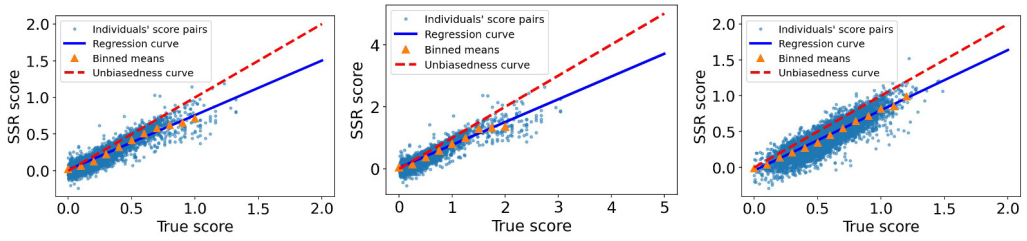
PTS, CA, and DMI do not depend on SPSR and are designed to elicit categorical labels instead of probabilistic predictions. So we make the following adaption for them to take probabilistic predictions as inputs. Our adaption is based on the fact that in essence, these mechanisms all appreciate the joint distribution of agents' reported labels to compute the scores: For a task k, an agent who reports probability $P_{i,k}$ believes that the true label of the task has probability $P_{i,k}$ to be 1. Therefore, on this task, the joint probability of agent i's believed true label and agent j's believed true label both being 1 is $P_{i,k}P_{j,k}$, assuming their believed true labels are independent conditioned on their predictions. By this way, we can compute the joint distribution of the believed true labels of two peer agents on each task and their joint distribution over the whole dataset is the mean of their joint distributions on each task. Using this joint distribution over the whole dataset, we can compute the scores for PTS, CA, and DMI directly. This adaption method for PTS, CA, and DMI turns out to give better correlations between the scores of these three mechanisms and the true SPSR scores than the alternative adaption method of using the mostly likely categorical labels indicated by the probabilistic predictions as inputs for these mechanisms (see how the correlations shown in Figs. 9 and 10 in the appendix (most likely labels as inputs) compare to the correlations shown in Figs. 2 and 3.).

8.2 Main results

Unbiasedness of SSR. Our theorem shows that under certain assumptions, the reward of an SSR mechanism is unbiased to the reward of the SPSR that the SSR mechanism is built upon. However, it is unclear to what extent this unbiasedness holds in real datasets where these assumptions are unlikely to hold strictly. Therefore, we empirically examine the concrete relationship between SSR scores and the corresponding SPSR scores.

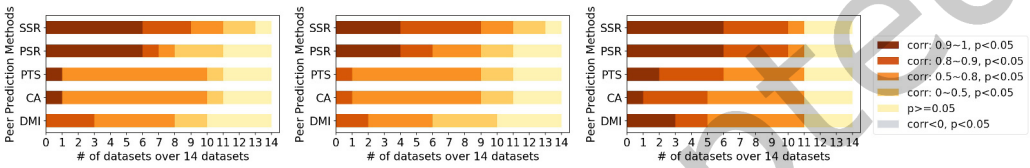
[5]The AUC-ROC of agent i is $\frac{1}{2}\left(1-\frac{1}{m_i^+(m_i-m_i^+)}S_i^{\text{rank}}\right)$, where $m_i+ := \sum_{k'\in[m_i]}\mathbb{1}(Y_{k'}=1)$ (given by Eqs. 12 and 13, [30]).

[6]We do not intend to claim our mechanism is better in any sense, as it would be an unfair comparison since the goals were different in each design of these mechanisms. For example, the mechanisms [20, 42] can characterize determinant mutual information between an agent's reports and the underlying ground truth.



(a) Brier ($y = 0.744 \cdot x + 0.014$) (b) Log ($y = 0.733 \cdot x - 0.041$) (c) Rank-sum ($y = 0.839 \cdot x - 0.041$)

Fig. 1. Regression of individuals' true accuracy and SSR score over 14 datasets under three different SPSR.



(a) Brier score (b) Log scoring rule (c) Rank-sum scoring rule

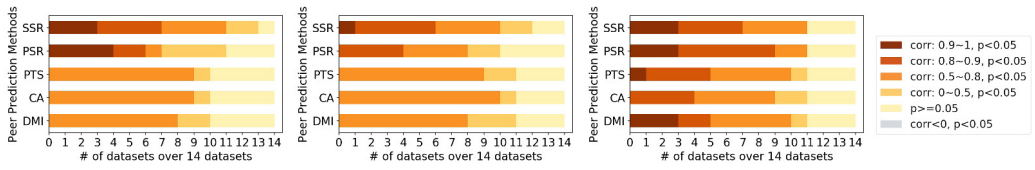
Fig. 2. The number of datasets in each level of correlation (measured by Pearson's correlation coefficient) between individuals' peer prediction scores and different SPSR.

Fig. 1 plots the score pairs received by forecasters in each of the 14 datasets. Each score pair represents the SPSR score and the SSR score that an individual forecaster receives in a single dataset. As can be seen, under each of the three SPSR we test, the SSR scores demonstrate a salient linear relationship to the true SPSR scores. We further draw a linear regression curve between the SSR scores and the true scores for each of the three SPSR of interest (the blue curves in Fig. 1). To draw this linear regression curve, we first cluster the score pairs into different groups based on the value of the SPSR scores and compute the center point (the mean score pair) for each group, represented by the orange triangles in Fig. 1. Then, we regress on these center points.[7] The three regression curves demonstrate a slope of 0.74, 0.73, and 0.84, respectively, all with an intercept near 0. This result indicates that though the SSR scores are not exactly unbiased in real data, they still follow an affine transformation of the true SPSR scores with decent approximate unbiasedness.

We also notice that under all three SPSR, the SSR scores tend to underestimate the true scores by around 20%. As the SSR scores follow an affine transformation of the SPSR scores empirically, this underestimation can possibly be mitigated by applying a constant scaling factor (e.g., 1.25 as suggested by our regression) without influencing the incentive properties of the SSR mechanisms.

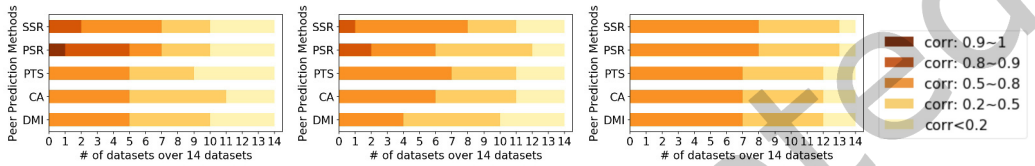
Correlation with SPSR. We compare the correlations between agents' SPSR scores and the scores given by the five peer prediction mechanisms we test. We first measure the correlations on each dataset independently using Pearson's correlation coefficient (corr) and then classify them into different levels based on the value of the coefficient. Finally, we count the number of datasets at different correlation levels for each peer prediction mechanism and present the results in Fig. 2.

[7]The reason for clustering score pairs before regression is that the SPSR scores of forecasters are not distributed evenly within the range of the SPSR score, with most forecasters' SPSR scores falling in the low range of the SPSR score. Consequently, drawing the regression curve directly on all score pairs will mainly reflect the regression pattern in the low range of the SPSR score instead of the whole range. In fact, for each of the three SPSR tested, the corresponding SSR mechanism obtains a regression slope closer to 1 at the low range of the SPSR score.



(a) Brier score (b) Log scoring rule (c) Rank sum scoring rule

Fig. 3. The number of datasets in each level of correlation (measured by Spearman's correlation coefficient) between individuals' peer prediction scores and different SPSR.



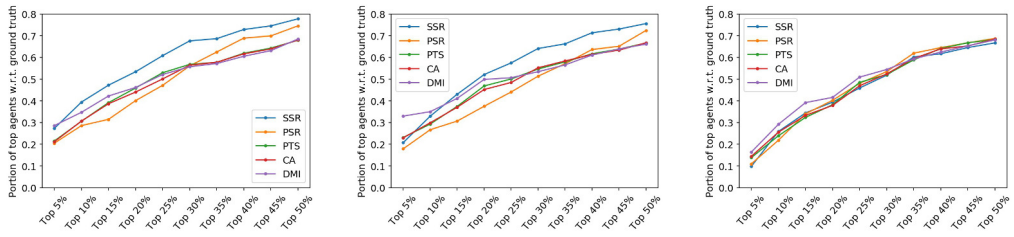
(a) Brier score (b) Log scoring rule (c) Rank-sum scoring rule

Fig. 4. The number of datasets in each level of correlation (measured by Pearson's correlation coefficient) between individuals' peer prediction scores and different SPSR on sampled datasets (the correlation is averaged over 100 runs of random sampling).

As can be seen, all five peer prediction mechanisms achieve a strong correlation (corr>0.5) to the SPSR on half of the 14 datasets, while the SSR mechanisms demonstrate an even stronger correlation pattern. In particular, the SSR mechanisms achieve a very strong correlation (corr>0.8) on 9 out of the 14 datasets under all three SPSR, and achieve correlations in more datasets than other mechanisms for each of the following levels: corr>0.9, corr>0.8, and corr>0.5. The advantage of SSR in the correlation to the SPSR is most salient under the Brier score and is more salient when compared to the PTS, CA, DMI mechanisms than compared to the PSR mechanisms. We observe similar results using Spearman's rank correlation test (Fig. 3), which implies that SSR mechanisms also rank the agents similarly to SPSR.

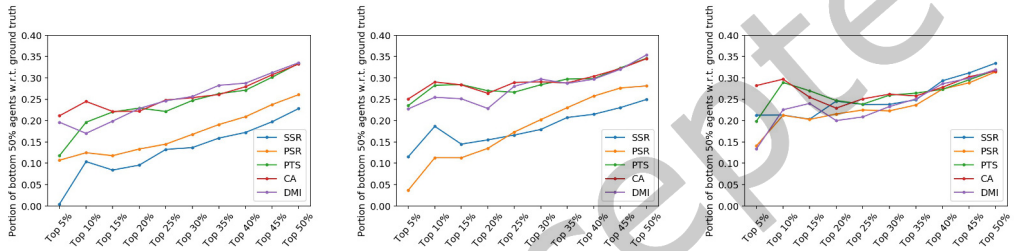
The performance of SSR mechanisms in reflecting the true SPSR scores depends on the accuracy of estimating the error rates of the constructed noisy estimate of ground truth in SSR mechanisms. This estimation accuracy depends on the number of prediction samples that SSR mechanisms have access to. In our previous experiments, each task receives a considerable number of predictions (no less than 20 on average), which may give an edge to the SSR mechanisms. However, a principal with a limited budget can often collect only a small number of predictions for each task. Therefore, we are also interested in comparing the performance of SSR mechanisms to other peer prediction mechanisms when each task receives only a limited number of predictions. To simulate this scenario, for each original dataset, we sample a subset of users to create a new dataset such that each new dataset has an average of 4~5 predictions per task with a minimum of 3 predictions, which is the minimum number of predictions per task required by our SSR mechanisms.[8] Fig. 4 shows the correlation results of each peer prediction mechanism based on the average Pearson's correlation

[8]To ensure a minimum of 3 predictions per task, we removed a small number of tasks that receive less than 3 predictions by this sampling method. Over the 100 runs of random sampling, around 20 tasks are removed on average from each GJP dataset, and no more than 2 tasks are removed from each of the other datasets in each run. This sampling operation keeps a decent number of predictions for each agent, which allows a stable computation for the scores of SSR, PTS, CA, and DMI mechanisms.



(a) Brier score (b) Log scoring rule (c) AUC-ROC

Fig. 5. The portion of top $t\%$ forecasters w.r.t. 3 different metrics (mean squared loss, cross-entropy loss, AUC-ROC loss) in the top $t\%$ forecasters selected by different methods (averaged over 14 datasets).



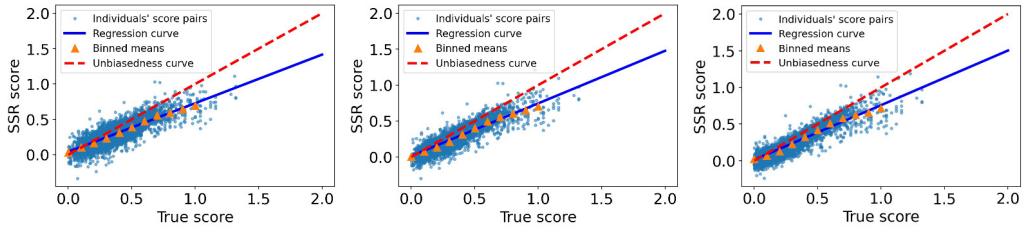
(a) Brier score (b) Log scoring rule (c) AUC-ROC

Fig. 6. The portion of bottom 50% forecasters w.r.t. 3 different metrics (mean squared loss, cross-entropy loss, AUC-ROC loss) in the top $t\%$ users selected by different methods (averaged over 14 datasets).

coefficient over 100 runs of random sampling. As can be seen, overall, the correlations between each peer prediction mechanism and the three SPSR in these sampled datasets decrease when compared to the corresponding correlations in the original datasets. SSR mechanisms still maintain a strong correlation (corr>0.5) over half of the 14 datasets, while the other mechanisms do not. However, the performance difference of SSR and other mechanisms shrinks. The PSR mechanisms outperform SSR mechanisms at two correlation levels, corr>0.8 and corr>0.9, under the Brier score and the log scoring rule. In fact, the single-task PSR mechanisms demonstrate smaller correlation decreases, indicating that they are more robust to the number of predictions than the other four multi-task mechanisms.

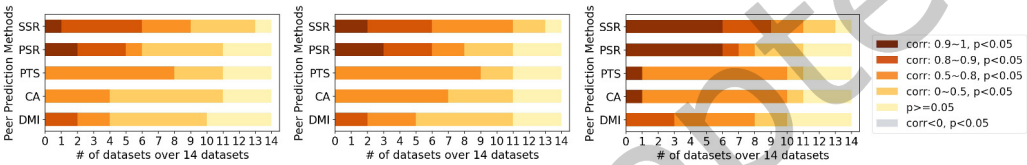
Expert identification. SPSR are sometimes used to identify top forecasters to assign prizes, e.g., in projects GJP and HFC. Moreover, accurate identification of true top forecasters without access to the ground truth can help improve the aggregation accuracy, when a principal wants to aggregate forecasters' predictions into a final prediction for each task [45]. Therefore, we examine to what extent different peer prediction scores can identify top-performing experts in terms of the true SPSR, without access to the ground truth.

In GJP and HFC projects, forecasters may answer different sets of forecasting questions. It is non-trivial to compare each forecaster's performance when they answer different sets of questions, whose difficulty levels vary. Here we use the mean peer prediction scores and mean SPSR scores to measure the forecaster's performance for simplicity, as our main purpose is to compare how close the evaluation results will be when we use true SPSR and when we use the SSR scores in the same way. Though we are demonstrating the application of SSR in expert identification, it is a by-product



(a) Initial ($y = 0.687x + 0.041$) (b) Average ($y = 0.731x + 0.014$) (c) Final ($y = 0.744 \cdot x + 0.014$)

Fig. 7. Regression and comparison of individuals' mean Brier scores and SSR scores over 14 datasets when the initial, average and final predictions are used for GJP and HFC datasets.



(a) Initial Prediction (b) Average Prediction (c) Final Prediction

Fig. 8. The number of datasets in each level of correlation (measured by Pearson's correlation coefficient) between individuals' peer prediction scores and the Brier score when the initial, average and final predictions are used for GJP and HFC datasets.

of its calibration property and we acknowledge that the identification might be affected by other factors, including how agents selected the forecasting questions. In projects like GJP, the organizers impute and standardize the scores for different questions and then prize the forecasters. Arguably, forecasters have no incentive to choose easy questions a prior, which alleviates the evaluation bias induced by heterogeneous difficulty levels of the forecasting questions.

We first rank the forecasters according to one of the three SPSR (when the rank-sum scoring rule is chosen, we use the AUC-ROC instead to evaluate agents' true accuracy, because as an accuracy metric instead of an incentive device, AUC-ROC is much more popular than the rank-sum scoring rule). We focus on two metrics about expert identification: (i) the percentage of top $t\%$ forecasters identified by the SPSR in the top $t\%$ forecasters selected by a peer prediction method, and (ii) the percentage of below-average forecasters (the bottom 50% forecasters) under the SPSR in the top $t\%$ forecasters selected by a peer prediction method. The results are shown in Figs. 5 and 6. For illustration, consider the left (Brier) pane of Figure 5. Of the top 10% forecasters according to SSR, 40% are indeed in the top 10% according to the Brier score. We find that for both the Brier score and the log score, there are more true top $t\%$ forecasters in the top $t\%$ forecasters selected by SSR than in the top $t\%$ forecasters selected by other peer prediction mechanisms, when $t\%$ ranges from 5% to 50%. Meanwhile, there are less true below-average forecasters in the top $t\%$ forecasters under SSR and PSR mechanisms than under the other peer prediction mechanisms. For AUC-ROC, while the SSR mechanism maintains a relatively smaller number of true below-average forecasters in its top 10% to 15% forecasters, all five peer prediction mechanisms perform similarly, which echos the correlation results under the rank-sum scoring rule, where the five peer prediction mechanisms all achieve strong correlation in most of the datasets (Fig. 2c).

Robustness of SSR in temporal forecasting. The above experiments use the final forecasts of each participant in the two temporal forecasting projects, the GJP and HFC projects. In this experiment, we test whether our evaluation is robust to the choice of the prediction time in temporal forecasting. In particular, we focus on the metric of the Brier score metric and re-do the experiments shown in Fig. 1a and Fig. 2a while using the initial prediction and the average prediction (from the open date to the close date of the forecasting question) of each participant instead of the final prediction (see Fig. 7 and 8).[9] Fig. 7 shows that the Pearson's correlation coefficient between SSR and the Brier score only decreases slightly when the initial and average predictions are used are used. Fig. 8 shows that when we use earlier predictions, the correlations between the five peer prediction scores and the Brier score slightly decrease, while SSR still maintains a relative advantage in correlation over other mechanisms.

9 DISCUSSION

In this paper, we propose the SSR mechanisms such that truthful reporting one's posterior belief is a dominant strategy in the multi-task IEWV setting, when each agent uses a consistent reporting strategy across all tasks. Moreover, the reward of a prediction given by an SSR mechanism quantifies the value of information in expectation as if the prediction is assessed by the corresponding SPSR with access to the ground truth. Because of these two properties, our mechanisms are particularly suitable for information elicitation scenarios where using SPSR to reward agents are favored but the ground truth is not available in time, such as forecasting long-term geopolitical events and predicting the replicability of social science studies.

There are also some limitations of applying our models and mechanisms. First, our Assumption 2 requires agents' signals on a task to be independent conditioned on the ground truth Y. This implies that our SSR mechanisms only apply to scenarios where there exists such an objective ground truth or where there is no objective ground truth but the agents' signals are correlated only through a single latent variable. An example of the latter is asking an agent how likely an essay is well-written or not. Although whether an essay is well-written or not may not have an objective answer, as long as the agents' signals are independent conditioned on a latent variable that captures the real quality of the essay, our mechanisms should incentivize truthful reporting as a dominant strategy when all agents adopt uniform strategies across tasks. In comparison, most existing multi-task peer prediction mechanisms [e.g. 20, 35, 42] that elicit categorical signals do not require agents' signals to be correlated only through a latent variable. Instead, they allow a broader correlation pattern (e.g., self-predicting [35]) or arbitrary correlations as long as signals are not completely independent [e.g. 20, 42].

Second, to estimate the error rates of the noisy estimate of the ground truth, our mechanisms require at least three reports for each task. In contrast, several multi-task mechanisms [e.g. 6, 20, 35, 42] only need one peer agent to achieve their incentive properties. Moreover, the variance of the rewards of SSR mechanisms depends on the number of tasks and reports that the mechanisms have access to. A relatively large number of tasks and reports is needed to obtain a low-variance reward for each agent. As can be seen from our empirical study, although SSR mechanisms still maintain better correlations to the true SPSR scores than the other mechanisms when there are only a few reports per task, SSR mechanisms have a more salient correlation decrease when compared to the case where each task receives a sufficient number of answers. SSR mechanisms are more sensitive to the size of the dataset. However, our analysis suggests that as long as agents adopt uniform strategies across tasks, it is possible to learn the statistical patterns of agents' reports without influencing the incentive. Therefore, a future direction to mitigate SSR mechanisms' sensitivity to

[9]Fig. 1a and Fig. 7c are the same. Fig. 2a and Fig. 8c are the same.

the amount of data is to develop or adopt more sophisticated estimation algorithms that require fewer tasks and reports to achieve stable performance.

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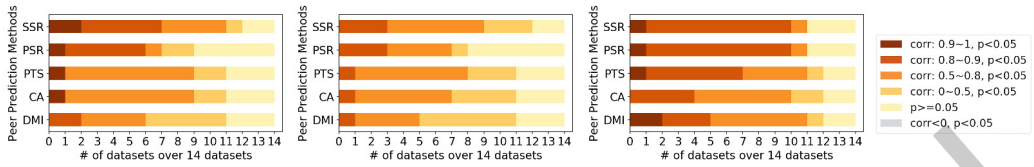
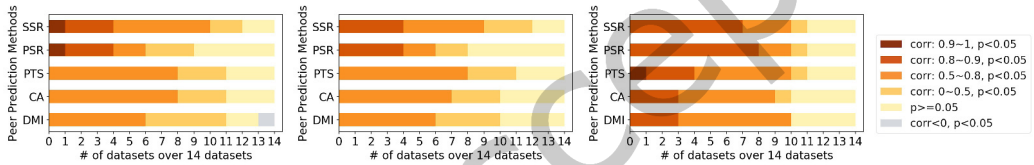
APPENDIX

A MISSING FIGURES



(a) Brier score (b) Log scoring rule (c) Rank-sum scoring rule

Fig. 9. The number of datasets in each level of correlation (measured by Pearson's correlation coefficient) between individuals' peer prediction scores and different SPSR when each probabilistic prediction is mapped to the most likely binary vote with uniform random tie breaking.



(a) Brier score (b) Log scoring rule (c) Rank sum scoring rule

Fig. 10. The number of datasets in each level of correlation (measured by Spearman's correlation coefficient) between individuals' peer prediction scores and different SPSR when each probabilistic prediction is mapped to the most likely binary vote with uniform random tie breaking.

B PROOF OF LEMMA 5.2

Proof. Suppose z and y are not stochastically relevant, we have

$$\Pr[y=0|z=0] = \Pr[y=0|z=1], \tag{18}$$

$$\Pr[y=1|z=0] = \Pr[y=1|z=1]. \tag{19}$$

From Eqn. (18) we know that

$$\frac{\Pr[y=0,z=1]}{\Pr[z=1]} = \frac{\Pr[y=0,z=0]}{\Pr[z=0]} \Leftrightarrow \frac{\Pr[y=0]e_z^-}{\Pr[z=1]} = \frac{\Pr[y=0](1-e_z^-)}{\Pr[z=0]},$$

When $\Pr[y=0] \neq 0$, we have $\frac{\Pr[z=1]}{\Pr[z=0]} = \frac{e_z^-}{1-e_z^-}$. Similarly from Eqn. (19), we have $\frac{\Pr[z=1]}{\Pr[z=0]} = \frac{1-e_z^+}{e_z^+}$., when $\Pr[y=1] \neq 0$. Therefore we, obtained

$$\frac{e_z^-}{1-e_z^-} = \frac{1-e_z^+}{e_z^+},$$

from which we have $e_z^- + e_z^+ = 1$. Contradiction. The other direction follows similarly. □

C PROOF OF LEMMA 6.6

Proof. We consider the estimation of the error rates e_z^+, e_z^- of an agent i, and we consider a generic task as tasks are a priori similar. Thus, in the proof, we drop the subscript k, which indexes the tasks. There are two layers of estimation error is solving the system of equations Eqn. (4, 5, 6):

- **1. Estimation error due to heterogeneous agents:** the higher order equations doesn't capture the true matching probability with heterogeneous agents. As we draw Z_2 and Z_3 in a task without replacement, with finite number of agents, Z_2 and Z_3 are dependent with Z_1, and the error rates of Z_2 and Z_3 are not exactly the same to the error rates of Z_1 (z).
- **2. Estimation errors due to finite estimation samples:** The last sources of errors come from the estimation errors of $\widetilde{\beta_{-i}}, \widetilde{\gamma_{-i}}$ and $\widetilde{\alpha_{-i}}$.

Next we bound the two errors separately.

1. Estimation error due to heterogeneous agents: The challenge lies in the fact that the higher order equations doesn't capture the true matching probability with heterogeneous agents.

We first consider Eqn. (5). (5) is not precise– randomly picking a prediction signal from all agents without replacement leads to a different error rates. This will complicate the solution for the system of equations. We show that our estimation, though being ignoring the above bias, will not affect our results by too much: Let k_1 be the agent whose prediction signal is picked to be Z_1. Conditioned on agent k_1 being picked and on reports $q_1, ..., q_N$, we have $\Pr[Z_1 = Z_2 = 1|q_1, ..., q_N, k_1] = q_{k_1} \cdot \left(\frac{\sum_{j\neq i,k_1} q_j}{N-2}\right)$. Recall that q_{k_1} is a random variable because of the private signal c_{k_1} received by agent k_1 and the randomness in σ_{k_1}, and that $e_z^+ = \mathbb{E}_{q_1,...,q_N|y=1}[\bar{q}_{-i}]$. We have that

$$
\begin{aligned}
\Pr[Z_1 = Z_2 = 1|y = 1] &= \mathbb{E}_{k_1}[\mathbb{E}_{q_1,...,q_N|y=1}[\Pr[Z_1 = Z_2 = 1|k_1, q_1, ..., q_N]]] \\
&= \mathbb{E}_{k_1}\left[\mathbb{E}_{q_1,...,q_N|y=1}\left[q_{k_1} \cdot \left(\frac{\sum_{j\neq i,k_1} q_j}{N-2}\right)\right]\right] \\
&= \mathbb{E}_{k_1}\left[\mathbb{E}_{q_1,...,q_N|y=1}[q_{k_1}] \cdot \mathbb{E}_{q_1,...,q_N|y=1}\left[\frac{\sum_{j\neq i,k_1} q_j}{N-2}\right]\right] \\
&= \mathbb{E}_{k_1}\left[\mathbb{E}_{q_1,...,q_N|y=1}[q_{k_1}] \cdot \mathbb{E}_{q_1,...,q_N|y=1}\left[\frac{(N-1)\bar{q}_{-i}}{N-2} - \frac{q_{k_1}}{N-2}\right]\right] \\
&= \mathbb{E}_{k_1}\left[\mathbb{E}_{q_1,...,q_N|y=1}[q_{k_1}] \cdot \left(\frac{N-1}{N-2}e_z^+ - \frac{1}{N-2}\mathbb{E}_{q_1,...,q_N|y=1}[q_{k_1}]\right)\right] \\
&= \frac{N-1}{N-2}e_z^+\mathbb{E}_{k_1}\left[\mathbb{E}_{q_1,...,q_N|y=1}[q_{k_1}]\right] - \frac{1}{N-2}\mathbb{E}_{k_1}\left[\mathbb{E}^2_{q_1,...,q_N|y=1}[q_{k_1}]\right] \\
&= \frac{N-1}{N-2}(e_z^+)^2 - \frac{1}{N-2}\omega,
\end{aligned}
$$

where $\omega := \mathbb{E}_{k_1}\left[\mathbb{E}^2_{q_1,...,q_N|y=1}[q_{k_1}]\right]$.

Note both e_z^+ and ω are no more than 1. Then ,

$$
\left|\frac{N-1}{N-2}(e_z^+)^2 - \frac{1}{N-2}\omega - (e_z^+)^2\right| \leq \frac{(e_z^+)^2}{N-2} + \frac{1}{N-2}\omega \leq \frac{2}{N-2}
$$

This adds $\frac{2}{N-2}$ error bias in the step where we replace $\Pr[z_1 = z_2 = 1|y = 1]$ with $(e_z^+)^2$ in the deduction of Eqn. (5). And, it finally adds $\frac{2}{N-2}$ error bias in estimating β_{-i} (through both $(e_z^-)^2$ and $(1 - e_z^+)^2$) in Eqn. (5).

Similarly for the matching among three agents (Eqn. (6)) we have

$$\left|\Pr[Z_1 = Z_2 = Z_3 = 1 | y = 1] - (e_z^+)^3\right| \leq \frac{3}{N-3}.$$

And this adds $\frac{3}{N-3}$ error bias in estimating γ_{-i}.

2. Estimation errors due to finite estimation samples: The last sources of errors come from the estimation errors of $\widetilde{\beta_{-i}}, \widetilde{\gamma_{-i}}$ and $\widetilde{\alpha_{-i}}$. Direct application of the Chernoff bound gives us the following lemma:

Lemma C.1. *When there are M samples for estimating $\widetilde{\beta_{-i}}, \widetilde{\gamma_{-i}}$ and $\widetilde{\alpha_{-i}}$ respectively (total budgeting $3M$), we have with probability at least $1-\delta$ taht*

$$|\widetilde{\beta_{-i}} - \beta_{-i}| \leq \sqrt{\frac{\ln\frac{6}{\delta}}{2M}},\ |\widetilde{\gamma_{-i}} - \gamma_{-i}| \leq \sqrt{\frac{\ln\frac{6}{\delta}}{2M}},\ |\widetilde{\alpha_{-i}} - \alpha_{-i}| \leq \sqrt{\frac{\ln\frac{6}{\delta}}{2M}}.$$

The error analysis in **1** and **2** jointly imply that with probability at least $1-\delta$

$$|\widetilde{\beta_{-i}} - \beta_{-i}| \leq \sqrt{\frac{\ln\frac{6}{\delta}}{2M}} + \frac{2}{N-2},\ |\widetilde{\gamma_{-i}} - \gamma_{-i}| \leq \sqrt{\frac{\ln\frac{6}{\delta}}{2M}} + \frac{3}{N-3},\ |\widetilde{\alpha_{-i}} - \alpha_{-i}| \leq \sqrt{\frac{\ln\frac{6}{\delta}}{2M}}.$$

Now we are ready to prove Lemma 6.6. First of all, from Algorithm 1, we can easily derive that

$$|\widetilde{e_z^-} - e_z^-| \leq \frac{|\tilde{a}-a|}{2} + \frac{|\sqrt{\tilde{a}^2 - 4\tilde{b}} - \sqrt{a^2-4b}|}{2} \tag{20}$$

$$|\widetilde{e_z^+} - e_z^+| \leq \frac{|\tilde{a}-a|}{2} + \frac{|\sqrt{\tilde{a}^2 - 4\tilde{b}} - \sqrt{a^2-4b}|}{2} \tag{21}$$

For the latter term in Eqn. (20) and (21), we have

$$\begin{aligned}
\frac{|\sqrt{\tilde{a}^2 - 4\tilde{b}} - \sqrt{a^2-4b}|}{2} =& \frac{|(\sqrt{\tilde{a}^2 - 4\tilde{b}} - \sqrt{a^2-4b}) \cdot (\sqrt{\tilde{a}^2 - 4\tilde{b}} + \sqrt{a^2-4b})|}{2(\sqrt{\tilde{a}^2 - 4\tilde{b}} + \sqrt{a^2-4b})} \\
\leq& \frac{|\tilde{a}^2 - a^2|}{2\sqrt{a^2-4b}} + \frac{4|\tilde{b}-b|}{2\sqrt{a^2-4b}} \\
\leq& \frac{|\tilde{a}-a|^2}{2\sqrt{a^2-4b}} + \frac{a\cdot|\tilde{a}-a|}{\sqrt{a^2-4b}} + \frac{2|\tilde{b}-b|}{\sqrt{a^2-4b}}
\end{aligned}$$

The first inequality is due to that we drop the positive $2\sqrt{\tilde{a}^2 - 4\tilde{b}}$ in the denominator. For the second inequality, note that a is non-negative as essentially, $a = 1 - e_z^+ + e_z^-$ shown in proof for Theorem 6.3.

To summarize, we have

$$|\widetilde{e_z^-} - e_z^-| \leq \left(\frac{1}{2} + \frac{a}{\sqrt{a^2-4b}}\right)|\tilde{a}-a| + \frac{2|\tilde{b}-b|}{\sqrt{a^2-4b}} + \frac{1}{2\sqrt{a^2-4b}}|\tilde{a}-a|^2 \tag{22}$$

$$|\widetilde{e_z^+} - e_z^+| \leq \left(\frac{1}{2} + \frac{a}{\sqrt{a^2-4b}}\right)|\tilde{a}-a| + \frac{2|\tilde{b}-b|}{\sqrt{a^2-4b}} + \frac{1}{2\sqrt{a^2-4b}}|\tilde{a}-a|^2 \tag{23}$$

The key tasks here reduce to bounding $|\tilde{a}-a|$ and $|\tilde{b}-b|$. Recall

$$a := \frac{\gamma_{-i}-\alpha_{-i}\beta_{-i}}{\beta_{-i}-(\alpha_{-i})^2}$$
$$b := \frac{\alpha_{-i}\gamma_{-i}-(\beta_{-i})^2}{\beta_{-i}-(\alpha_{-i})^2}$$

We know the following facts

$$\begin{aligned}|(\widetilde{\beta_{-i}}-(\widetilde{\alpha_{-i}})^2)-(\beta_{-i}-(\alpha_{-i})^2)| \le& |(\widetilde{\alpha_{-i}})^2-(\alpha_{-i})^2|+|\widetilde{\beta_{-i}}-\beta_{-i}| \\ \le& 2|\widetilde{\alpha_{-i}}-\alpha_{-i}|+|\widetilde{\beta_{-i}}-\beta_{-i}| \le 3\sqrt{\frac{\ln\frac{6}{\delta}}{2M}}+\frac{2}{N-2},\end{aligned}$$

$$\begin{aligned}|(\widetilde{\gamma_{-i}}-\widetilde{\beta_{-i}}\widetilde{\alpha_{-i}})-(\gamma_{-i}-\beta_{-i}\alpha_{-i})| \le& |\widetilde{\gamma_{-i}}-\gamma_{-i}|+|\widetilde{\beta_{-i}}\widetilde{\alpha_{-i}}-\beta_{-i}\alpha_{-i}| \\ \le& |\widetilde{\gamma_{-i}}-\gamma_{-i}|+|\widetilde{\beta_{-i}}-\beta_{-i}|+|\widetilde{\alpha_{-i}}-\alpha_{-i}| \\ \le& 3\sqrt{\frac{\ln\frac{6}{\delta}}{2M}}+\frac{2}{N-2}+\frac{3}{N-3},\end{aligned}$$

$$\begin{aligned}|(\widetilde{\alpha_{-i}}\widetilde{\gamma_{-i}}-(\widetilde{\beta_{-i}})^2)-(\alpha_{-i}\gamma_{-i}-(\beta_{-i})^2)| \le& |\widetilde{\alpha_{-i}}-\alpha_{-i}|+|\widetilde{\gamma_{-i}}-\gamma_{-i}|+2|\widetilde{\beta_{-i}}-\beta_{-i}| \\ \le& 4\sqrt{\frac{\ln\frac{6}{\delta}}{2M}}+\frac{2}{N-2}+\frac{3}{N-3},\end{aligned}$$

Next, denoting $\eta = p(1-p)(1-e_z^+-e_z^-)^2$ (which also means $\Delta = p(1-p)(x^- - x^+)^2$), we have

$$\begin{aligned}\beta_{-i}-(\alpha_{-i})^2 =& (1-p)\cdot(x^-)^2+p\cdot(x^+)^2-((1-p)\cdot x^-+p\cdot x^+)^2 \\ =& (1-p)\cdot p\cdot(x^- - x^+)^2 \\ =& \eta\end{aligned}$$

Let N be sufficiently large such that

$$3\sqrt{\frac{\ln\frac{6}{\delta}}{2M}}+\frac{2}{N-2} < \eta \tag{24}$$

then

$$\widetilde{\beta_{-i}}-(\widetilde{\alpha_{-i}})^2 \ge \frac{p(1-p)}{2}\cdot\frac{\eta}{2}$$

Therefore,

$$\begin{aligned}|\tilde{a}-a| =& \left|\frac{\widetilde{\gamma_{-i}}-\widetilde{\alpha_{-i}}\widetilde{\beta_{-i}}}{\widetilde{\beta_{-i}}-(\widetilde{\alpha_{-i}}^2)}-\frac{\gamma_{-i}-\alpha_{-i}\beta_{-i}}{\beta_{-i}-(\alpha_{-i})^2}\right| \\ \le& \frac{|(\widetilde{\beta_{-i}}-(\widetilde{\alpha_{-i}})^2)-(\beta_{-i}-(\alpha_{-i})^2)|+|(\widetilde{\gamma_{-i}}-\widetilde{\beta_{-i}}\widetilde{\alpha_{-i}})-(\gamma_{-i}-\beta_{-i}\alpha_{-i})|}{|\widetilde{\beta_{-i}}-(\widetilde{\alpha_{-i}})^2|\cdot|\beta_{-i}-(\alpha_{-i})^2|} \\ \le& \frac{2}{\eta^2}\left(6\sqrt{\frac{\ln\frac{6}{\delta}}{2M}}+\frac{4}{N-2}+\frac{3}{N-3}\right)\end{aligned}$$

Note that the first inequality uses Lemma 6.7.

Similarly for b, we have

$$|\tilde{b} - b| \leq \frac{2}{\eta^2}\left(7\sqrt{\frac{\ln\frac{6}{\delta}}{2M}} + \frac{4}{N-2} + \frac{3}{N-3}\right)$$

Together, we proved that when M and N are sufficiently large such that Eqn. (24) holds, i.e., $3\sqrt{\frac{\ln\frac{6}{\delta}}{2M}} + \frac{2}{N-2} < \frac{\eta}{2}$, we have

$$|\widetilde{e_z^-} - e_z^-| \leq O\left(\sqrt{\frac{\ln\frac{1}{\delta}}{M}} + \frac{1}{N}\right)$$

$$|\widetilde{e_z^+} - e_z^+| \leq O\left(\sqrt{\frac{\ln\frac{1}{\delta}}{M}} + \frac{1}{N}\right)$$

□