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The matroid stratification of the Hilbert scheme of points on $\mathbb{P}^1$

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Abstract. Given a homogeneous ideal I in a polynomial ring over a field, one may record, for each degree d and for each polynomial $f \in I_d$, the set of monomials in f with nonzero coefficients. These data collectively form the *tropicalization* of I. Tropicalizing ideals induces a "matroid stratification" on any (multigraded) Hilbert scheme. Very little is known about the structure of these stratifications. In this paper, we explore many examples of matroid strata, including some with interesting combinatorial structure, and give a convenient way of visualizing them. We show that the matroid stratification in the Hilbert scheme of points $(\mathbb{P}^1)^{[k]}$ is generated by all Schur polynomials in k variables. We end with an application to the T-graph problem of $(\mathbb{A}^2)^{[n]}$; classifying this graph is a longstanding open problem, and we establish the existence of an infinite class of edges.

1. Introduction

Let $\Bbbk$ be a field. The *support* of a homogeneous polynomial $f \in \Bbbk[x_1, \ldots, x_r]$ is the set of monomials with nonzero coefficient in f. Let $I \subseteq R = \Bbbk[x_1, \ldots, x_r]$ be a homogeneous ideal. For each degree d, the data of all supports of polynomials in I_d comprise a combinatorial portrait called the *tropicalization* of I_d, denoted $\mathrm{Trop}(I_d)$. A *matroid* (see [13] or Definition 2.5) is the data of a finite set E, together with a subset of $M \subseteq 2^E$ satisfying certain combinatorial conditions. $\mathrm{Trop}(I_d)$ is an example of a matroid, where $E = \mathrm{Mon}_d$ is the set of degree d monomials in $x_1, \ldots, x_r$.

In this paper, we study I via the infinite sequence of matroids $\mathrm{Trop}(I) = (\mathrm{Trop}(I_d))_{d \geq 0}$; this sequence is the *tropicalization* of I. The matroids satisfy a certain combinatorial compatibility condition, namely the defining condition of a *tropical ideal* (Definition 2.8).

A (multigraded) Hilbert scheme is a moduli space parametrizing homogeneous ideals. The fibers of the function $I \mapsto \mathrm{Trop}(I)$ define a "matroid stratification" on any Hilbert scheme, possibly with countably many strata, analogous to, and generalizing, the more well-known matroid stratification on $\mathrm{Gr}(m, \Bbbk^n)$.

We identify the matroid stratification in the case of principal homogeneous ideals in $\Bbbk[x, y]$, i.e. in the Hilbert scheme of points $(\mathbb{P}^1)^{[k]}$. Note that a symmetric

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polynomial in k variables defines a divisor on $(\mathbb{P}^1)^{[k]}$ via the identification $(\mathbb{P}^1)^{[k]} \cong \mathrm{Sym}^k(\mathbb{P}^1)$. Then:

Theorem 3.7. *The matroid stratification on* $(\mathbb{P}^1)^{[k]}$ *is the stratification generated by all Schur polynomials* s_λ *in k variables.*

We end with an application to the *T-graph problem* for $(\mathbb{A}^2)^{[n]}$, which was our original motivation for the project. Let X be a variety with the action of an algebraic torus T such that the fixed point set X^T is finite. The T-graph of X is a graph with vertex set X^T, and an edge between two fixed points if they are the two limit points of a 1-dimensional T-orbit. A Hilbert scheme has a $T = (\mathbb{C}^*)^r$-action by scaling the variables $x_1, \ldots, x_r$. Determining the T-graphs of Hilbert schemes is a difficult problem that has been studied by Iarrobino, Evain, Altmann and Sturmfels, Hering and Maclagan, and others [1,4,6,8]. We show:

Theorem 5.11. *Let* $\Bbbk = \mathbb{C}$. *Let* $k \geq 1$ *and* $d > k$. *Let S be the set of 1-dimensional* $(\mathbb{C}^*)^2$*-orbits in* $(\mathbb{A}^2)^{[dk]}$ *whose limit points are the two fixed points* (x^k, y^d) *and* (x^d, y^k). *Then S is a finite set, in natural bijection with the set of binary necklaces with k black and* $d-k$ *white beads. (In particular,* (x^k, y^d) *and* (x^d, y^k) *are connected by an edge in the T-graph of* $(\mathbb{A}^2)^{[dk]}$*.)*

In Sect, 3.2, we pose some easily-stated questions from combinatorial linear algebra that we cannot answer. The answers would elucidate the relationship between Theorems 5.11 and 3.7.

Relation to other work. The forthcoming paper [5] of Fink–Giansiracusa–Giansiracusa is closely related to this one. Motivated by understanding "tropical Hilbert schemes," which are moduli spaces of tropical ideals over arbitrary valued fields, they also investigate the tropicalizations of ideals of points in $\mathbb{P}^1$. Our results complement each other: this paper considers trivially valued fields, and Hilbert schemes of arbitrarily many points on $\mathbb{P}^1$, while they consider arbitrary valued fields, but have results mainly for ≤ 2 points in $\mathbb{P}^1$. We hope that these perspectives can be merged to describe tropical Hilbert schemes of arbitrarily many points in $\mathbb{P}^1$.

Zajaczkowska's Ph.D thesis [14] studied the tropical Hilbert schemes of hypersurfaces of degrees 1 and 2 in $\mathbb{P}^1$ and $\mathbb{P}^2$. Among other things, the thesis contains the case $k = 2$ of Corollary 5.11.

2. Multigraded Hilbert schemes and their matroid stratifications

2.1. Multigraded Hilbert schemes

Multigraded Hilbert schemes are the natural moduli spaces of homogeneous ideals in a polynomial ring. Let $\Bbbk$ be a field, and consider the polynomial ring $R = \Bbbk[x_1, \ldots, x_r]$.

Definition 2.1. For $b \in \mathbb{Z}_{>0}$, a (positive) $\mathbb{Z}^b$*-multigrading*[1] $\mathbf{a} = (\vec{a}_1, \ldots, \vec{a}_r)$ on R is an assignment of a *multidegree* $\vec{a}_i \in \mathbb{Z}^b_{\geq 0} \setminus \{(0, \ldots, 0)\}$ to each variable x_i. A multigrading is *nondegenerate* if the rowspan of $\mathbf{a}$ is a rank-b lattice in $\mathbb{Z}^r$.

[1] There is a more general notion of multigrading that we will not need, see [7].

All multigradings from now on are assumed to be nondegenerate. A $\mathbb{Z}^b$-multigrading defines a decomposition $R = \bigoplus_{d \in \mathbb{Z}^b_{\geq 0}} R_d$. Any **a**-homogeneous ideal $I \subseteq R$ has a *multigraded Hilbert function*[2] $h : \mathbb{Z}^b_{\geq 0} \to \mathbb{N}$, defined by $h(d) = \dim_{\Bbbk}(R_d/(I \cap R_d))$.

Haiman and Sturmfels [7] define a *multigraded Hilbert scheme* $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ that is a projective fine moduli space for **a**-homogeneous ideals with multigraded Hilbert function h. For each $d \in \mathbb{Z}^b_{\geq 0}$, there is a short exact sequence of vector bundles on $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$:

$$0 \to \mathcal{I}_d \to \mathcal{R}_d \to \mathcal{Q}_d \to 0, \tag{1}$$

where $\mathcal{I}_d$ is the universal ideal sheaf, $\mathcal{R}_d$ denotes the trivial sheaf with fiber R_d, and $\mathcal{Q}_d$ is the rank-$h(d)$ universal quotient sheaf.

Example 2.2. An important special case is when I has finite colength, i.e. $\dim_{\Bbbk}(R/I) = \sum_{d \in \mathbb{Z}^b_{\geq 0}} h(d) < \infty$. In this case $V(I)$ has finite length, and there is a natural embedding $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r) \hookrightarrow (\mathbb{A}^r)^{[\sum_d h(d)]}$ into the Hilbert scheme of points in $\mathbb{A}^r$.

Example 2.3. When $r = 2$, $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^2)$ is smooth, irreducible, and rational [11], see also [4,8].

Example 2.4. If $b = 1$ and $\sum_d h(d)$ is *not* finite, then $\mathrm{Hilb}^h_{(a_1,\ldots,a_r)}(\mathbb{A}^r)$ has a natural map to a Hilbert scheme of subschemes of the weighted projective space $\mathbb{P}(a_1, \ldots, a_r)$, cut out by the same ideal.

This map need not be an embedding, essentially due to the fact that $I \in \mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ could have $(x_1, \ldots, x_r)$ as an embedded prime.

2.2. Tropicalizing ideals

Tropical geometry usually takes place over a valued field, but in this paper we will always assume $\Bbbk$ is trivially valued. We present the definitions we need only in this simpler context; see [10] for the general definitions.

First we briefly recall the basics of matroid theory. See [13] for details, including how to reconcile the following definition with the allusion in the introduction.

Definition 2.5. A *matroid* $M = (E, r)$ is the data of a finite set E, called the *groundset*, together with a function $r : 2^E \to \mathbb{Z}_{\geq 0}$ (where 2^E is the power set of E) called the *rank function*, such that:

(1) $r(\emptyset) = 0$,
(2) For all subsets $S, S' \subseteq E$, $r(S \cup S') + r(S \cap S') \leq r(S) + r(S')$, and
(3) For every subset $S \subseteq E$ and every element $x \in E \setminus S$, $r(S) \leq r(S \cup \{x\}) \leq r(S) + 1$.

[2] Positivity of **a** is necessary here; otherwise $R_d/(I \cap R_d)$ need not be finite-dimensional.

The *rank* of M is $r(E)$. A subset $S \subseteq E$ is called *dependent* if $r(S) < |S|$, and *independent* otherwise. A maximal independent subset is called a *basis*, and all bases have cardinality $r(E)$. A minimal dependent subset is called a *circuit*, and a union of circuits is called a *cycle*. A 1-element circuit is called a *loop*, and an element of E not contained in any dependent sets is a *coloop*. The *corank function* is $r^*(S) = |S| - r(S)$. A subspace $V \subseteq \Bbbk^E$ gives rise to a matroid $\mathrm{Trop}(V)$ with groundset E called its *tropicalization*, with rank function $r(S) = \dim(\Bbbk^S / V \cap \Bbbk^S)$ for $S \subseteq E$. (Note that this is dual to some definitions in the literature.)

Example 2.6. If $\Bbbk$ is algebraically closed, the tropicalization of a generic dimension-k subspace $V \in \mathrm{Gr}(k, \Bbbk^E)$ is the *uniform matroid* $U_{k,E}$, defined by the rank function

$$r(S) = \begin{cases} |S| & |S| \leq k \\ k & |S| \geq k. \end{cases}$$

We will use the following two standard facts.

Lemma 2.7. *Let $V \subseteq \Bbbk^E$ be a subspace.*

- *If $S \subseteq E$ is a circuit in* $\mathrm{Trop}(V)$*, then there exists $v = (v_e)_{e \in E} \in V$ such that $S = \{e \in E : v_e \neq 0\}$.*
- *For any $v = (v_e)_{e \in E} \in V$, the set $S = \{e \in E : v_e \neq 0\}$ is a cycle in* $\mathrm{Trop}(V)$.

(Over an infinite field, the converse of the second statement holds.)

Now we introduce our main objects of study.

Definition 2.8. Let $\mathbf{a} = (\vec{a}_1, \dots, \vec{a}_r)$ be a positive multigrading on $k[x_1, \dots, x_r]$. Let $\mathrm{Mon}_d(\mathbf{a})$ denote the set of monomials of degree d with respect to the grading $\mathbf{a}$. A *tropical (homogeneous) ideal $\mathscr{M} = (\mathscr{M}_d)_{d \in \mathbb{Z}^b_{\geq 0}}$ with respect to the grading* $\mathbf{a}$ (over the Boolean semifield) is the data of, for each $d \in \mathbb{Z}^b_{\geq 0}$, a matroid $\mathscr{M}_d = (\mathrm{Mon}_d(\mathbf{a}), r_d)$, such that

for any circuit S of $\mathscr{M}_d$, and any monomial $m' \in \mathrm{Mon}_{d'}(\mathbf{a})$, $m'S$ is a cycle in $\mathscr{M}_{d+d'}(\mathbf{a})$.

The *multigraded Hilbert function* of a tropical homogeneous ideal $\mathscr{M}$ is the function $d \mapsto r_d(\mathrm{Mon}_d(\mathbf{a}))$.

Just as a subspace of $\Bbbk^n$ gives rise to a matroid (a “tropical linear space over the Boolean semifield”), a homogeneous ideal with respect to the grading $\mathbf{a}$ gives rise to a tropical homogeneous ideal with grading $\mathbf{a}$:

Definition 2.9. Let $I \subseteq \Bbbk[x_1, \dots, x_r]$ be $\mathbf{a}$-homogeneous. The *tropicalization* of I is $\mathrm{Trop}(I) = (\mathrm{Trop}(I)_d)_{d \in \mathbb{Z}^b_{\geq 0}}$, where $\mathrm{Trop}(I)_d = \mathrm{Trop}(I_d)$.

Observe that Lemma 2.7 implies that $\mathrm{Trop}(I)$ satisfies the condition in Definition 2.8, and that the multigraded Hilbert functions of I and $\mathrm{Trop}(I)$ agree by definition.

2.3. *Pictures of tropical ideals*

When $r = 2$, we visualize a tropical ideal $\mathscr{M}$ as follows. We draw a grid whose boxes representing monomials in two variables x and y, where the bottom-leftmost square represents the monomial 1. We draw each circuit of $\mathscr{M}$ as a line segment connecting a collection of dots in the grid; these dots correspond to monomials in the circuit. We also label each degree d by the Hilbert function of $\mathscr{M}$, evaluated at d. (For simplicity, all examples shown have the standard grading $\mathbf{a} = (1, 1)$.)

To avoid clutter, we may omit a circuit S of $\mathscr{M}_d$ if we deem it "uninformative," i.e. if S is "forced" to be dependent by the existence of a circuit in lower degree. Precisely, from now on we omit a circuit S in degree d if there exists a circuit S' in degree $d' < d$ and a collection T of degree-$(d - d')$ monomials such that $S \subseteq \bigcup_{m \in T} mS'$ and $|S| > \left|\bigcup_{m \in T} mS'\right| - |T|$. In this case, S must be dependent, as follows.

Consider the ordering $\preceq$ on $\mathrm{Mon}_d(\mathbf{a})$ by increasing y-exponent. By Definition 2.8, each set mS' is a cycle. For each $m \in T$, select a circuit of mS' that contains the $\preceq$-minimal element of mS'. Then a circuit elimination argument, exactly analogous to matrix row-reduction, shows that the set $\bigcup_{m \in T} mS'$ has corank at least $|T|$.

Example 2.10. The ideal $I = (x^3 + x^2y + 2xy^2 + 3y^3, x^5, xy^4)$ has tropicalization pictured in Fig. 1, where in the left image all circuits are drawn, and in the right image uninformative circuits are omitted.

2.4. *Dependence loci and the matroid stratification*

The operation of tropicalization defines a stratification of any multigraded Hilbert scheme $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$, as follows. Fix $d \in \mathbb{Z}^b_{\geq 0}$ and $U \subseteq \mathrm{Mon}_d(\mathbf{a})$, with $\ell := |U|$. We give a scheme-theoretic restatement of the condition on $I \in \mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ that U be dependent in $\mathrm{Trop}(I)_d$. Consider the tautological sequence (1) on $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$. The

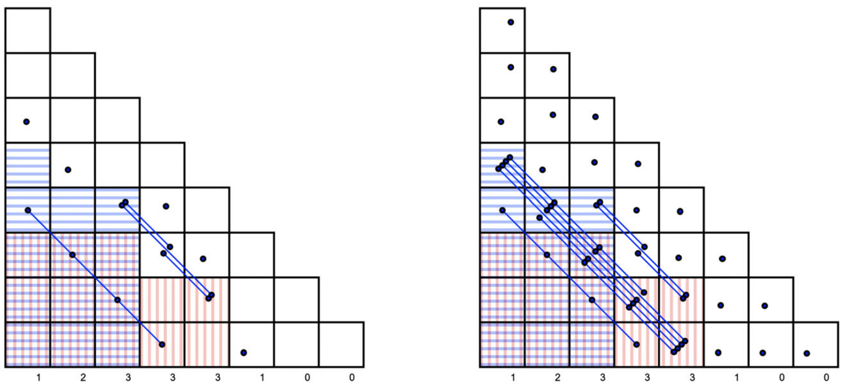


Fig. 1. Two pictures of $\mathrm{Trop}(x^3 + x^2y + 2xy^2 + 3y^3, x^5, xy^4)$. See Notation 4.4 for an explanation of the colors/shading

collection U defines, up to sign, an element of $\bigwedge^\ell R_d$. The wedge power of the map $R_d \to \mathcal{Q}_d$ gives a global section σ_U of $\bigwedge^\ell \mathcal{Q}_d$. This section vanishes at I if and only if the monomials in U are linearly dependent modulo I_d, i.e. if and only if U is a dependent set in $\mathrm{Trop}(I_d)$. Thus we define:

Definition 2.11. The *dependence scheme* of U is

$$\mathcal{D}(U) := V(\sigma_U) \subseteq \mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r).$$

It is immediate that dependence schemes are closed subschemes. Since matroids are uniquely defined by their dependent sets, we define:

Definition 2.12. Let $\mathscr{M}$ be a tropical ideal. The *matroid stratum* $\mathcal{S}(\mathscr{M}) \subseteq \mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ of $\mathscr{M}$ is the locally closed subscheme

$$\bigcap_{d \in \mathbb{Z}^b_{\geq 0}} \left(\bigcap_{U \text{ dependent in } \mathscr{M}_d} \mathcal{D}(U) \cap \bigcap_{U \text{ independent in } \mathscr{M}_d} \mathcal{D}(U)^C \right).$$

Note that each stratum involves an infinite intersection of Zariski-open sets, and therefore $\mathcal{S}(\mathscr{M})$ may not be Zariski-locally closed. Indeed we will see in Sect. 3 that this does occur! However, if $\sum_{d \in \mathbb{Z}^b_{\geq 0}} h(d) < \infty$, then there *are* no independent sets in sufficiently large degree—this implies there are finitely many strata in the matroid stratification of $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$, and they are Zariski-locally closed.

Remark 2.13. The number of strata in the matroid stratification of $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ is countable, as follows. A stratum $\mathcal{S}(\mathscr{M})$ is determined by the collection of sets U such that $\mathcal{D}(U) \supseteq \mathcal{S}(\mathscr{M})$; in particular, $\mathcal{S}(\mathscr{M})$ is the unique stratum whose Zariski closure is $\bigcap_d \bigcap_{\substack{U \subseteq \mathrm{Mon}_d(\mathbf{a}) \\ \mathcal{D}(U) \supseteq \mathcal{S}(\mathscr{M})}} \mathcal{D}(U)$. As $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ is Noetherian, any such intersection is actually finite; this defines an injective function from the set of matroid strata in $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ into the set of finite intersections of the countable collection $\{\mathcal{D}(U)\}_U$ of subsets.

Note, however, that this argument does not imply that the number of tropical *ideals* with fixed grading and Hilbert function is countable; indeed, we do not know whether this is the case.

Example 2.14. We here introduce a simple, but surprising, example of a dependence locus, which we will return to repeatedly. Assume $\Bbbk = \mathbb{C}$. Let $r = 2$, $b = 1$, and $\mathbf{a} = (1, 1)$, and suppose there exists $k \geq 1$ such that

$$h(d) = \begin{cases} d+1 & d < k \\ k & d \geq k. \end{cases} \tag{2}$$

The corresponding Hilbert scheme is the moduli space of **principal** homogeneous ideals in $\Bbbk[x, y]$ generated in degree k, i.e. the Hilbert scheme $(\mathbb{P}^1)^{[k]}$ of length-k subschemes of $\mathbb{P}^1$. Fix $d \geq k$, and let $U = \{x^d, y^d\}$. We classify $\mathcal{D}(U) \subseteq (\mathbb{P}^1)^{[k]}$.

Suppose $I = (f) \in \mathcal{D}(U) \subseteq (\mathbb{P}^1)^{[k]}$. Then (f) contains a polynomial of the form $c_1 x^d + c_2 y^d$, i.e. there exists a degree-$(d-k)$ polynomial p such that $pf = c_1 x^d + c_2 y^d$. Note that $V(pf)$ consists of the d points $\{[\zeta : 1] : \zeta^d = c_2/c_1\}$

(as long as $c_1 \neq 0$). These are the vertices of a regular d-gon in $\mathbb{C}$ centered at the origin. Since $V(f)$ is a length-k subscheme of $V(pf)$, $V(f)$ consists of k of the d vertices.

Conversely, given a collection of k points $z_1, \ldots, z_k \in \mathbb{C}$ that are distinct vertices of some regular d-gon centered at 0, the defining polynomial f of $\{z_1, \ldots, z_k\}$ satisfies $(f) \in \mathcal{D}(U)$ (simply by letting p be the defining polynomial of the other $d-k$ vertices).

To visualize $\mathcal{D}(U)$ further, consider the $\mathbb{C}^*$-action on regular d-gons centered at 0. This defines an action on $\mathcal{D}(U)$, and the collection of ratios $z_2/z_1, \ldots, z_k/z_1$ defines an orbit; this collection is equivalent to the data of a binary necklace with k black and $d-k$ white beads. Let $N_{d,k}$ denote the set of such necklaces. Then $\mathcal{D}(U)$ is a union of rational curves indexed by $N_{d,k}$, all of which intersect at the two points (x^k) (where the d-gon is scaled down to 0) and (y^k) (where the d-gon is scaled out to ∞).

In a rank-k matroid (E, r), a set $S \subseteq E$ with $|S| \leq k$ is dependent if and only if S' is dependent for every $S' \supseteq S$ with $\left|S'\right| = k$. (This follows from the fact that all bases have cardinality k.) We have the following scheme-theoretic version of this fact, which we will use in Sect. 3:

Proposition 2.15. *Fix a graded Hilbert function h. Let $U \subseteq \mathrm{Mon}_d(\mathbf{a})$ with $|U| \leq h(d)$. The dependence scheme $\mathcal{D}(U) \subseteq \mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ satisfies*

$$\mathcal{D}(U) = \bigcap_{\substack{W \supseteq U \\ |W| = h(d)}} \mathcal{D}(W).$$

(Note: This also holds if $|U| > h(d)$, in which case both sides are equal to $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$.)

Proof. Consider the sequence of maps

$$\bigwedge^{|U|} \mathrm{Span}(U) \xrightarrow{\iota} \bigwedge^{|U|} \mathcal{Q}_d \xrightarrow{w} \bigoplus_{U' \in \binom{\mathrm{Mon}_d}{h(d)-|U|}} \bigwedge^{h(d)} \mathcal{Q}_d \xrightarrow{p} \bigoplus_{\substack{U' \in \binom{\mathrm{Mon}_d}{h(d)-|U|} \\ U' \cap U \neq \emptyset}} \bigwedge^{h(d)} \mathcal{Q}_d,$$

where ι is the inclusion, p is the projection, and $w(\alpha) = \alpha \wedge \bigwedge_{u \in U'} u$. Note that w is injective, as it is induced by the nondegenerate pairing $\bigwedge^{|U|} \mathcal{Q}_d \otimes \bigwedge^{h(d)-|U|} \mathcal{Q}_d \to \Bbbk$. Thus $V(w(\sigma_U)) = V(\sigma_U)$.

Also, $p \circ w \circ \iota$ is zero, so $w \circ \iota$ factors through $\ker(p) = \bigoplus_{\substack{U' \in \binom{\mathrm{Mon}_d}{h(d)-|U|} \\ U' \cap U = \emptyset}} \bigwedge^{h(d)} \mathcal{Q}_d$.

Thus

$$\mathcal{D}(U) = V(\sigma_U) = V(w(\sigma_U)) = \bigcap_{\substack{U' \in \binom{\mathrm{Mon}_d}{h(d)-|U|} \\ U' \cap U = \emptyset}} V(\sigma_{U \cup U'}) = \bigcap_{\substack{U' \in \binom{\mathrm{Mon}_d}{h(d)-|U|} \\ U' \cap U = \emptyset}} \mathcal{D}(U' \cup U).$$

□

The matroid stratification of $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ satisfies the following straightforward recursivity relation, which implies that when studying strata, we may ignore ideals of the form $I = mI'$, where m is a monomial. There are natural inclusions $\iota_1, \ldots, \iota_r$ between $\mathbf{a}$-multigraded Hilbert schemes, defined by $\iota_i(I) = x_i I$.

Proposition 2.16. *Let* $U \subseteq \mathrm{Mon}_d(\mathbf{a})$. *Then*

$$\iota_i^{-1}(\mathcal{D}(U)) = \mathcal{D}\left(\frac{1}{x_i}(U \setminus \{m \in U : x_i \nmid m\})\right).$$

We omit the proof, as it is straightforward, and we will use the Proposition only to reduce the number of strata that are of interest.

3. The matroid stratification of $(\mathbb{P}^1)^{[k]}$

Let $k > 0$. In this section, we will describe (Theorem 3.7) the matroid stratification of $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ in the case $r = 2$, $b = 1$, $\mathbf{a} = (1, 1)$, and let h be as in (2). We write $R = \Bbbk[x, y]$. Note that $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ is simply the familiar Hilbert scheme of points $(\mathbb{P}^1)^{[k]}$. Recall that

$$(\mathbb{P}^1)^{[k]} \cong \mathrm{Sym}^k(\mathbb{P}^1) \cong \mathbb{P}^k,$$

where $[a_0 : a_1 : \cdots : a_k] \in \mathbb{P}^k$ corresponds the principal ideal $I = (a_0x^k + a_1x^{k-1}y + \cdots + a_ky^k) \in (\mathbb{P}^1)^{[k]}$, and to the set of roots (with multiplicity) $V(I) \in \mathrm{Sym}^k(\mathbb{P}^1)$.

We will describe the matroid stratification on $(\mathbb{P}^1)^{[k]}$ via vanishing loci of sections of line bundles. For convenience, we note that the sheaf $\mathcal{O}(n)$ on $\mathbb{P}^k$ is identified with the sheaf $\mathrm{Sym}^k(\mathbb{P}^1)$ of S_k-symmetric functions in k pairs of variables $x_1, y_1, \ldots, x_k, y_k$, bihomogeneous in each pair of variables of degree n. Also, the tautological line bundle $\mathcal{I}_k$ on $(\mathbb{P}^1)^{[k]}$ (see (1)) is identified with the bundle $\mathcal{O}(-1)$ on $\mathbb{P}^k$.

3.1. The correspondence between Schur polynomials and subsets of monomials

We introduce the following version of Schur polynomials, bihomogenized in each variable.

Definition 3.1. Let $k \geq 1$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, in nonincreasing order, with $c \leq k$ parts. We write $\lambda_i = 0$ for $c < i \leq k$. The *bihomogeneous Schur polynomial* s_λ *in* k *variables* is defined by

$$s_\lambda(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) = \frac{a_{(\lambda_1+k-1, \lambda_2+k-2, \ldots, \lambda_k+0)}(x_1, y_1, \ldots, x_k, y_k)}{a_{(k-1, k-2, \ldots, 0)}(x_1, y_1, \ldots, x_k, y_k)},$$

where

$$a_{(l_1, l_2, \ldots, l_k)}(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) = \det(x_j^{l_i} y_j^{l_1 - l_i})$$

is the Vandermonde determinant.

Similarly, the *bihomogeneous elementary symmetric polynomials* e_j are defined by

$$e_j(x_1, y_1, \dots, x_k, y_k) = \sum_{\substack{A \subseteq [k] \\ |A|=j}} \prod_{i \in A} x_i \prod_{i \notin A} y_i$$

Note that s_λ is bihomogeneous of degree λ_1 and e_j is bihomogeneous of degree 1 in each pair of variables x_i, y_i. To avoid confusion, we point out that $e_0 = y_1 \cdots y_k$.

Notation 3.2. Schur polynomials in k variables are indexed by partitions with at most k parts, or alternatively by Young diagrams that fit inside a $k \times \infty$ rectangle. Since Young diagrams also appear in this paper in relation to monomial ideals, we distinguish them as follows. We draw Young diagrams related to Schur polynomials with the longest row on top (English notation), as opposed the way we have been drawing monomial ideals (French notation).

We now give a correspondence between Young diagrams and sets of monomials.

Definition 3.3. Fix $h, k \geq 1$. Let λ be a partition whose Young diagram fits inside the $k \times h$ rectangle in $\mathbb{Z}^2$. (That is, λ has at most k parts, and $\lambda_1 \leq h$.) The *width-h, height-k rim path* of λ is the lattice path $P_\lambda^{h,k}$ in $\mathbb{Z}^2$ that begins at $(h, 0)$, and follows the edge of the Young diagram down and to the left until it reaches $(0, -k)$. We index the steps of $P_\lambda^{h,k}$ by $i = 0, \dots, h + k - 1$.

The *width-h, height-k monomial set* of λ is the set

$$U_\lambda^{h,k} = \{x^{h+k-1-i} y^i \in \mathrm{Mon}_{h+k-1} : i \in \{0, \dots, h + k - 1\} \text{ such that the } i\text{th step of } P_\lambda^{h,k} \text{ is vertical}\}.$$

The definition is illustrated on the left in Fig. 2. Note that $\left| U_\lambda^{h,k} \right| = k$.

Remark 3.4. The operation of taking the width-h, height-k monomial set has a clear inverse, hence gives a bijection between partitions with at most k parts and $\lambda_1 \leq h$, and k-element subsets of Mon_{h+k-1}. Thus Proposition 2.15 implies:

Proposition 3.5. *For any subset $U \subseteq \mathrm{Mon}_{h+k-1}$,*

$$\mathcal{D}(U) = \bigcap_{\lambda : U_\lambda^{h,k} \supseteq U} \mathcal{D}(U_\lambda^{h,k}).$$

We show how to visualize Proposition 3.5 in an example.

Example 3.6. Let $h = 7$ and $k = 5$. Let $U = \{x^{11}, x^6 y^5, x^2 y^9\}$. If λ is such that $U \subseteq U_\lambda^{7,5}$, then the 0th, 5th, and 9th steps of $P_\lambda^{7,5}$ are vertical. Concretely, this says precisely that the dashed red segments in Fig. 2 are not in $P_\lambda^{7,5}$. (This also disallows certain other segments from being in $P_\lambda^{7,5}$; we have shown these as dotted lines.) Then a partitions λ satisfies $U \subseteq U_\lambda^{7,5}$ if and only if $P_\lambda^{7,5}$ consists of solid segments. For example, $\lambda = (7, 4, 4, 1)$ satisfies $U \subseteq U_\lambda^{7,5}$; $P_\lambda^{7,5}$ is drawn in bold in Fig. 2.

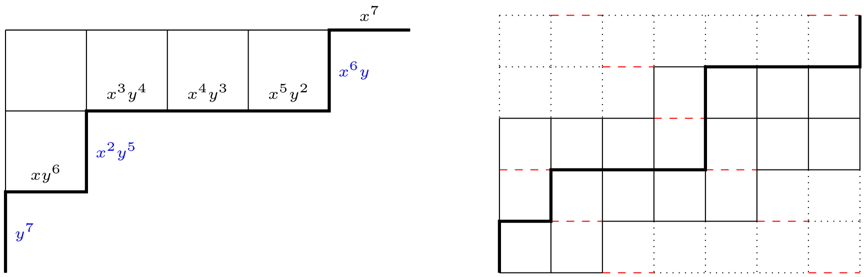


Fig. 2. Left: The width-5, height-3 rim path of $\lambda = (4, 1)$ is drawn in bold, and the monomial set is $U^{5,3}_{(4,1)} = \{x^6y, x^2y^5, y^7\}$. Right: Allowed rim path segments from Example 3.6, with the example $\lambda = (7, 4, 4, 1)$ in bold

By Proposition 3.5, the matroid stratification of $(\mathbb{P}^1)^{[k]}$ is "generated" (via taking intersections and complements) by the loci $\mathcal{D}(U^{h,k}_\lambda)$. Thus the stratification is entirely determined by the following:

Theorem 3.7. *The dependence subscheme $\mathcal{D}(U^{h,k}_\lambda) \subseteq (\mathbb{P}^1)^{[k]}$ is the vanishing locus of*

$$e_0(x_1, \dots, y_k)^{h-\lambda_1} s_\lambda(x_1, \dots, y_k)$$

(via the isomorphism $(\mathbb{P}^1)^{[k]} \cong \operatorname{Sym}^k \mathbb{P}^1$).

Remark 3.8. A. Fink independently observed a connection between matroid strata in $(\mathbb{P}^1)^{[k]}$ and Schur polynomials.

Proof. For notational convenience, in this proof we will write e_i for $e_i(x_1, \dots, y_k)$ and s_λ for $s_\lambda(x_1, y_1, \dots, x_k, y_k)$.

Recall the tautological sequences (1). Let f be a nonvanishing local section of the line bundle $\mathcal{I}_k$. Then in terms of the roots $[x_1 : y_1], \dots, [x_k : y_k]$, we have

$$f = \prod_{i=1}^{k} (y_i x - x_i y) = e_0 x^k - e_1 x^{k-1} y + \cdots + (-1)^k e_k y^k.$$

Step 0. Since $e_0^{h-\lambda_1} s_\lambda$ is homogeneous in $y_1, \dots, y_k$, we may instead work with the negative roots, and write

$$f = \prod_{i=1}^{k} (y_i x + x_i y) = e_0 x^k + e_1 x^{k-1} y + \cdots + e_k y^k.$$

The lack of signs will simplify Step 2.

Step 1.

By definition, $\mathcal{D}(U^{h,k}_\lambda)$ is the vanishing locus of the section

$$\sigma_{U^{h,k}_\lambda} \in \bigwedge^k \mathcal{Q}_{h+k-1} = \operatorname{Hom}\left(\bigwedge^k \operatorname{Span}(U^{h,k}_\lambda), \bigwedge^k \mathcal{Q}_{h+k-1}\right)$$

defined as the kth wedge of the chain of maps

$$\mathrm{Span}(U_\lambda^{h,k}) \hookrightarrow \mathcal{R}_{h+k-1} \to \mathcal{Q}_{h+k-1}.$$

By duality, there is a natural isomorphism

$$\begin{aligned}&\mathrm{Hom}\left(\bigwedge^k \mathrm{Span}(U_\lambda^{h,k}), \bigwedge^k \mathcal{Q}_{h+k-1}\right)\\ &\to \mathrm{Hom}\left(\left(\bigwedge^k \mathcal{Q}_{h+k-1}\right)^\vee, \left(\bigwedge^k \mathrm{Span}(U_\lambda^{h,k})\right)^\vee\right).\end{aligned}$$

The two exact sequences

$$0 \to \mathcal{I}_{h+k-1} \to \mathcal{R}_{h+k-1} \to \mathcal{Q}_{h+k-1} \to 0$$

and

$$0 \to \mathrm{Span}(U_\lambda^{h,k}) \to \mathcal{R}_{h+k-1} \to \mathcal{Q}_{h+k-1} \to 0$$

give identifications

$$\begin{aligned}\left(\bigwedge^k \mathcal{Q}_{h+k-1}\right)^\vee &\cong \bigwedge^h \mathcal{I}_{h+k-1}\\ \left(\bigwedge^k \mathrm{Span}(U_\lambda^{h,k})\right)^\vee &\cong \bigwedge^h \mathcal{R}_{h+k-1}/\mathrm{Span}(U_\lambda^{h,k}).\end{aligned}$$

Thus $\sigma_{U_\lambda^{h,k}}$ is identified with the section of $\mathrm{Hom}\left(\bigwedge^h \mathcal{I}_{h+k-1}, \bigwedge^h \mathcal{R}_{h+k-1}/\mathrm{Span}(U_\lambda^{h,k})\right)$ defined as the h-th (top) wedge of the chain of maps

$$\mathcal{I}_{h+k-1} \xrightarrow{A} \mathcal{R}_{h+k-1} \xrightarrow{B} \mathcal{R}_{h+k-1}/\mathrm{Span}(U_\lambda^{h,k}), \tag{3}$$

i.e. $\det(B \circ A)$. Note that

$$\begin{aligned}&\mathrm{Hom}\left(\bigwedge^h \mathcal{I}_{h+k-1}, \bigwedge^h \mathcal{R}_{h+k-1}/\mathrm{Span}(U_\lambda^{h,k})\right)\\ &\cong \mathrm{Hom}\left(\bigwedge^h (\mathcal{R}_{h-1} \otimes \mathcal{I}_k), \bigwedge^h \mathcal{R}_{h+k-1}/\mathrm{Span}(U_\lambda^{h,k})\right)\\ &\cong \mathrm{Hom}\left(\bigwedge^h \mathcal{R}_{h-1} \otimes \mathcal{I}_k^{\otimes h}, \bigwedge^h \mathcal{R}_{h+k-1}/\mathrm{Span}(U_\lambda^{h,k})\right)\\ &\cong (\mathcal{I}_k^*)^{\otimes h} \cong \mathcal{O}(h),\end{aligned}$$

Step 2. By principality,

there is a natural multiplication isomorphism

$$R_{h-1} \otimes \mathcal{I}_k \to \mathcal{I}_{h+k-1}.$$

The inclusion A from (3) has the following matrix X_A with respect to the basis $\{m \otimes f : \text{monomials } m \in R_{h-1}\}$ for $\mathcal{I}_{h+k-1}$ and the monomial basis for R_{h+k-1} (ordered such that larger powers of x appear first):

$$X_A = \left(e_{b-j}\right)_{0 \le b \le h+k-1}^{0 \le j \le h-1} = \begin{pmatrix} e_0 & 0 & \cdots & 0 & 0 \\ e_1 & e_0 & \cdots & 0 & 0 \\ e_2 & e_1 & \ddots & e_0 & 0 \\ \vdots & \vdots & \ddots & e_1 & e_0 \\ e_k & e_{k-1} & \ddots & e_2 & e_1 \\ 0 & e_k & \ddots & \vdots & e_2 \\ \vdots & \vdots & \ddots & e_k & \vdots \\ 0 & 0 & \cdots & 0 & e_k \end{pmatrix} \quad (4)$$

The (square) matrix $X_{B\circ A}$ of $B \circ A$ is obtained by deleting the rows corresponding to elements of $U_\lambda^{h,k}$. Let $X'_{B\circ A}$ be the matrix obtained by reversing the order of the rows and the order of the columns in $X_{B\circ A}$. Define $b_0, \ldots, b_{h-i}$ so that the i-th row of $X'_{B\circ A}$ is the b_i-th row of X_A.

Note that rows of $X'_{B\circ A}$ correspond to rightward steps in the *reversed* width-h, height-k rim path of λ—that is, to columns in the Young diagram of λ. The i-th row of $X'_{B\circ A}$ (starting with $i = 0$) has entries $e_{\ell_i}, e_{\ell_i+1}, \ldots, e_{\ell_i+h-1}$, where

$$\ell_i = k - i - \#\{b \le b_i : x^b y^{h+k-1-b} \in U_\lambda^{h,k}\}.$$

Since elements of $\{b \le b_i : x^b y^{h+k-1-b} \in U_\lambda^{h,k}\}$ correspond to upward steps in the reversed rim path, we see that $\ell_i + i = k - \#\{b \le b_i : x^b y^{h+k-1-b}\}$ is the i-th entry of the conjugate partition λ'. (Here λ' is taken to have exactly h entries, some of which may be zero.) Thus $X'_{B\circ A} = (e_{\lambda'_i+j-i})_{i,j=0}^{h-1}$. Note that $\ell_i + i = 0$ for $\lambda_1 < i \le h$. Expanding the determinant along the last $h - \lambda_1$ rows gives

$$\det(X_{B\circ A}) = e_0^{h-\lambda_1} \det((e_{\lambda'_i+j-i})_{i,j=0}^{\lambda_1}).$$

By the second Jacobi-Trudi formula, $\det(X_{B\circ A}) = \pm \det(X'_{B\circ A}) = \pm e_0^{h-\lambda_1} s_\lambda$.

□

Remark 3.9. If $\lambda_k > 0$, then expanding the Jacobi-Trudi formula gives

$$s_\lambda = e_k^{\lambda_k} s_{(\lambda_1-\lambda_k, \lambda_2-\lambda_k, \ldots, \lambda_{k-1}-\lambda_k)}(x_1, \ldots, y_k).$$

In particular, Theorem 3.7 now implies that

$$\mathcal{D}(U_\lambda^{h,k}) = V(e_0^{h-\lambda_1} e_k^{\lambda_k} s_{(\lambda_1-\lambda_k, \lambda_2-\lambda_k, \ldots, \lambda_{k-1}-\lambda_k)}(x_1, \ldots, y_k)).$$

This is a manifestation of Proposition 2.16.

Remark 3.10. Theorem 3.7 reduces all questions about the matroid stratification to questions about the intersection theory of Schur polynomials – however, it appears that intersection theory of Schur polynomials has not been actively studied.

Remark 3.11. If $\Bbbk = \mathbb{C}$ (or more generally if $\Bbbk$ is uncountable and algebraically closed), it follows from Theorem 3.7 that for a very general point $I \in (\mathbb{P}^1)^{[k]}$, $\mathrm{Trop}(I_d)$ is the uniform matroid of the appropriate rank, for all d. This is a special case of a forthcoming result by Maclagan and the author, which states that under the same assumptions on $\Bbbk$, a very general point $I \in \mathrm{Hilb}^h_{(a,b)}(\mathbb{A}^2)$ satisfies $\mathrm{Trop}(I_d) = U_{h(d),\mathrm{Mon}_d(a,b)}$ for all d, where (a, b) is any positive grading and h is any Hilbert function. (Recall Example 2.6.)

Example 3.12. We continue Example 2.14. Fix $k \geq 1$ and $d \geq k$. By Proposition 3.5 and Theorem 3.7, there is a certain set of Schur polynomials in k variables whose common vanishing locus is a collection of rational curves indexed by the set $N_{d,k}$ of binary necklaces with k black and $d - k$ white beads. One may also show this directly, as follows.

By the analysis in Example 3.6, the Schur polynomials in question are precisely those s_λ such that λ has at most $k - 1$ parts, and $\lambda_1 = d - k + 1$. The vanishing of these polynomials is highly non-transverse; however, calculations using the first Jacobi-Trudi formula show that the ideal they generate is in fact equal to the ideal $J = (h_{d-k+1}, h_{d-k+2}, \ldots, h_{d-1})$, where h_i is the i-th (bihomogeneous) *complete* symmetric polynomial, i.e.

$$h_i = \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=i}} \prod_{j=1}^{k} x_j^{i_j} y_j^{i-i_j}.$$

By [2], these polynomials form a regular sequence, so $V(J) \subseteq (\mathbb{P}^1)^{[k]}$ is 1-dimensional, as desired.

One may show directly that if $z_1, \ldots, z_k \in \mathbb{CP}^1$ are distinct vertices of a regular d-gon centered at 0, then the polynomials $h_{d-k+1}, \ldots, h_{d-1}$ vanish at $(z_1, \ldots, z_k)$. This shows that $\mathcal{D}(U)$ contains the collection of rational curves in Example 2.14. One may then show by a degree calculation that the $\mathcal{D}(U)$ does not contain any other points.

3.2. *An open problem interlude: the tropical ideal associated to a necklace*

Following Examples 2.14 and 3.12, we now pose a natural combinatorial question, to which we do not know the answer. Let $\gamma \in N_{d,k}$ be a necklace with k black beads and $d - k$ white beads. There is a corresponding curve $C_\gamma \cong \mathbb{C}^*$ in $\mathcal{D}(\{x^d, y^d\}) \subseteq (\mathbb{P}^1)^{[k]}$. In fact, as C_γ is a torus orbit (see Sect. 4), it has the property that any $I \in C_\gamma$ has the same tropicalization $\mathrm{Trop}(\gamma) := \mathrm{Trop}(I)$. (In other words, C_γ is in a single matroid stratum; we will see that it may not be an entire matroid stratum.) For example, see Figs. 3 and 4.

Question 3.13. Is there a combinatorial algorithm to compute the function $\gamma \mapsto \mathrm{Trop}(\gamma)$?

We do not have a full answer to this question, but we now discuss it further. Let $\gamma \in N_{d,k}$. We know that $\{x^d, y^d\}$ is dependent in $\mathrm{Trop}(\gamma)$. Note that for any

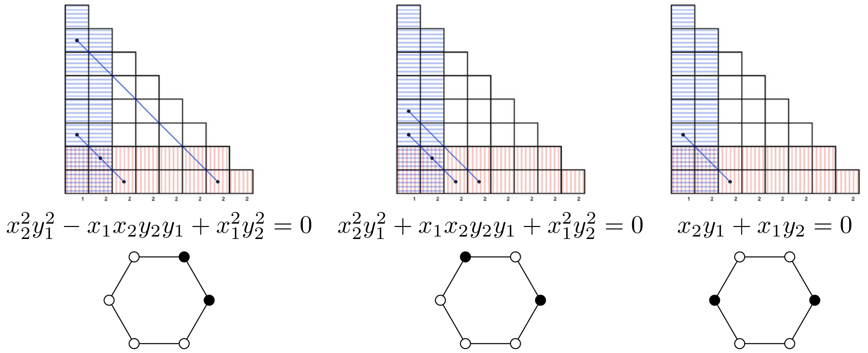


Fig. 3. The three elements of $N_{6,2}$ and their tropicalizations. Each equation defines (the closure of) the corresponding stratum in $(\mathbb{P}^1)^{[2]} \cong \mathrm{Sym}^2\,\mathbb{P}^1$

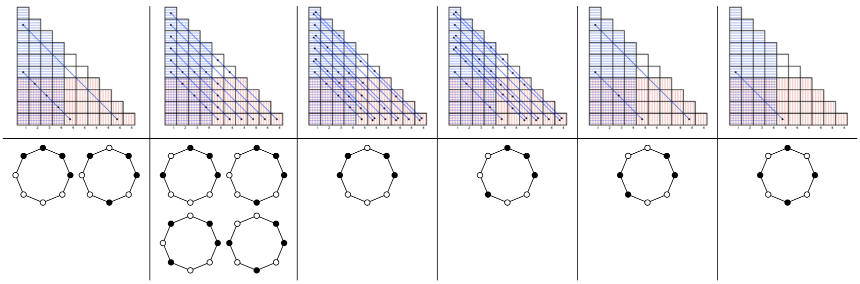


Fig. 4. The ten elements of $N_{8,4}$ and their tropicalizations

$d' \geq k$, $\{x^{d'}, y^{d'}\}$ is dependent in $\mathrm{Trop}(\gamma)$ if and only if the black beads of γ are a subset of the vertices of a regular d'-gon. Rephrasing this:

Proposition 3.14. *Let $\gamma \in N_{d,k}$, and let α be the* gcd *of the k distances between consecutive beads in γ. (Since the sum of these distances is d, d is divisible by α.) Then $\{x^{d'}, y^{d'}\}$ is dependent in* $\mathrm{Trop}(\gamma)$ *if and only if d' is a multiple of d/α.*

Note that this explains all circuits in Fig. 3. We also note the following condition, which implies certain necklaces have the same tropicalization.

Proposition 3.15. *Let $\gamma \in N_{d,k}$, and let $a \in (\mathbb{Z}/d\mathbb{Z})^\times$. We define $a\gamma$ to be the necklace obtained by traversing γ by jumps of length a. (For example, if $\gamma =$ , then $3\gamma =$  $=$ ). Then* $\mathrm{Trop}(\gamma) = \mathrm{Trop}(a\gamma)$.

Proof. The independence of any k-element set $U_\lambda^{g,k}$ in $\mathrm{Trop}(\gamma)$ is determined by the nonvanishing of an element of $\mathbb{C}$ obtained by field operations applied to a primitive dth root of unity ζ (namely, the determinant of the associated Schur matrix). This nonvanishing is preserved by the field automorphism that sends $\zeta \mapsto \zeta^a$, which determines the independence of $U_\lambda^{g,k}$ in $\mathrm{Trop}(\gamma)$. $\square$

Question 3.16. Does the converse of Proposition 3.15 hold? That is, can there exist $\gamma_1, \gamma_2 \in N_{d,k}$ such that $\mathrm{Trop}(\gamma_1) = \mathrm{Trop}(\gamma_2)$, but $\gamma_2 \neq a\gamma_1$ for $a \in (\mathbb{Z}/d\mathbb{Z})^\times$? Observe that no counterexamples appear in Figs. 3 or 4.

In order to fully characterize $\mathrm{Trop}(\gamma)$, we need to know not only which sets $\{x^{d'}, y^{d'}\}$ are dependent, but which sets $U_\lambda^{g,k}$ are dependent.

Let $\gamma \in N_{d,k}$. Given λ a partition with at most k parts, let $\eta_{d,k}(\lambda) = \{\zeta_d^{\lambda_i+k-i-1} : i = 1, \ldots, k\}$. Let $\zeta_d^{a_1}, \ldots, \zeta_d^{a_k}$ be a set of points representing γ, and note that

$$s_\lambda(\zeta_d^{a_1}, \ldots, \zeta_d^{a_k}) = \det\left((\zeta_d^{a_j(\lambda_i-i-1+k)})_{i,j=1}^k\right)/V,$$

where V is a Vandermonde determinant (which is guaranteed to be nonzero at $\zeta_d^{a_1}, \ldots, \zeta_d^{a_k}$). If λ is such that two elements of $\eta_{d,k}(\lambda)$ coincide, then $s_\lambda(\zeta_d^{a_1}, \ldots, \zeta_d^{a_k}) = 0$ since two rows of the defining matrix are equal.

On the other hand, if $\eta_{d,k}(\lambda)$ contains k distinct elements, then $\eta_{d,k}(\lambda)$ naturally corresponds to a necklace with k black beads and $d-k$ white beads. In particular, reordering and scaling $\eta_{d,k}(\lambda)$ corresponds to reordering and scaling the rows of the matrix in the definition of $s_\lambda(\zeta_d^{a_1}, \ldots, \zeta_d^{a_k})$, which does not affect its rank – hence, the question of whether $\gamma \in \mathcal{D}(U_\lambda^{g,k})$ for some $\gamma \in N_{d,k}$ depends only on the necklace $\eta_{d,k}(\lambda)$, not λ itself. This dependence is, interestingly, commutative in the following sense.

Proposition 3.17. *Let $\gamma \in N_{d,k}$ such that $\gamma = \gamma(\lambda)$. Then $\gamma \in \mathcal{D}(U_{\lambda'}^{g,k})$ if and only if $\gamma(\lambda') \in \mathcal{D}(U_\lambda^{g,k})$.*

Proof. This follows immediately from $\det(A) = \det(A^T)$. □

Answering Question 3.13 now boils down to:

Question 3.18. Let γ_1 and γ_2 be necklaces. We choose identifications of γ_1 and γ_2 with k-element subsets $\{\gamma_{1,i}\}$ and $\{\gamma_{2,j}\}$ of $\mathbb{Z}/d\mathbb{Z}$. Is there a combinatorial algorithm to determine whether the determinant $D(\gamma_1, \gamma_2) := \det\left((\zeta_d^{\gamma_{1,i}\gamma_{2,j}})_{i,j=1}^k\right)$ vanishes?

Remark 3.19. Experimentally, one may find sufficient conditions for the vanishing of the above determinant. In particular, one may prove a statement of the following form: if a divides d, and the k black beads of γ_1 are distributed "sufficiently unequally" among the μ_a-orbits of the dth roots of unity, *and* the k black beads of γ_2 are distributed "sufficiently unequally" among the $\mu_{d/a}$-orbits of the dth roots of unity, then $D(\gamma_1, \gamma_2) = 0$. However, we do not know of any necessary conditions; an additional idea would be needed to prove that any determinants are *nonzero*.

4. Equivariant structure of $\mathrm{Hilb}_{\mathbf{a}}^h(\mathbb{A}^r)$ and the T-graph problem

In this section, we rephrase some of the preceding material in terms of torus actions. Let $\Bbbk$ be algebraically closed in this section. The condition of **a**-homogeneity for an ideal I is equivalent to the invariance of I under the action of a certain subtorus of $T := (\Bbbk^*)^r$—specifically, the image of the homomorphism $(\Bbbk^*)^b \to (\Bbbk^*)^r$ defined by the matrix of exponents **a**.

Example 4.1. The ideal $(x^2y+z^6) \subseteq \Bbbk[x, y, z]$ is homogeneous with respect to the grading $((3, 0), (0, 6), (1, 1))$. This ideal is invariant under elements of $T = (\Bbbk^*)^3$ of the form $(\lambda_1^3, \lambda_2^6, \lambda_1\lambda_2)$, which act on the polynomial x^2y+z^6 by multiplication by $\lambda_1^6\lambda_2^6$.

The torus T acts on $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ with stabilizers isomorphic to $T/(\Bbbk^*)^b$, so the dimension of any T-orbit $T \cdot I$ is at most $r - b$ (by nondegeneracy). There is a stratification of $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ by T-orbit dimension; it is easy to check that the (finite set of) monomial ideals with graded Hilbert function h are the T-fixed points.

If $b = r - 1$, then for every h and $\mathbf{a}$, every T-orbit in $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ is either a monomial ideal or is isomorphic to $T/(\Bbbk^*)^b \cong \Bbbk^*$. Each 1-dimensional orbit $T \cdot I$ has two "endpoints;" these are initial monomial ideals of I with respect to appropriate term orders. (When $r = 2$, the two term orders are $x > y$ and $y > x$. See [6].)

Observation 4.2. *Since* Trop(I) *is defined in terms of supports of polynomials, and T acts by multiplying coefficients by nonzero scalars, we always have* Trop$(T \cdot I)$ = Trop(I). *In particular, every dependence locus and stratum of the matroid stratification is T-invariant. As initial ideals* $\mathrm{in}_\prec(I)$ *are defined via supports of polynomials, they are recoverable from* Trop(I)*, as in the following definition.*

Definition 4.3. Let $M = (E, r)$ be a matroid, and let $\preceq$ be a total ordering on E. The *initial matroid* of M with respect to $\preceq$ is the matroid $\mathrm{in}_\preceq(M) = (E, r')$ whose circuits are $\{\min_\preceq(c) : c \text{ a circuit of } M\}$. (It is a straightforward exercise to check that these circuits define a matroid. In fact, $\mathrm{in}_\preceq(M)$ is a *discrete matroid*, i.e. every element of E is either a loop or a coloop of $\mathrm{in}_\preceq(M)$.)

If $\mathscr{M}$ is a tropical ideal, and $\preceq$ is a monomial order, then the *initial tropical ideal* of M is defined by $\mathrm{in}_\preceq(\mathscr{M})_d = (\mathrm{in}_\preceq(\mathscr{M}_d))$.

Notation 4.4. When $r = 2$, there is a natural term order $x \preceq y$. When drawing a tropical ideal $\mathscr{M}$, we color-code each monomial m as follows:

- Blue and horizontally striped if m is *not* a circuit of $\mathrm{in}_\preceq(\mathscr{M})$, and
- Red and vertically striped if m is *not* a circuit of $\mathrm{in}_\succeq(\mathscr{M})$.

More simply, a box is blue if it does not contain the bottom-right-most dot of any line segment, and red if it does not contain the top-left-most dot of any line segment.

The *T-graph problem* (see [1]) asks which pairs of T-fixed points in $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ are endpoints of a 1-dimensional T-orbit. The problem has been studied extensively [1,3,4,6,8]. An algebraic algorithm via Gröbner theory for generating the T-graph was given in [1] and later implemented as the `TEdges` Macaulay2 package [9]; given two monomial ideals M_1 and M_2, the algorithm produces equations that cut out the "edge scheme" $E(M_1, M_2) \subseteq \mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$ consisting of ideals I such that M_1 and M_2 are the endpoints of $T \cdot I$. By the observation above, $E(M_1, M_2)$ is a union of matroid strata.

Recall (Example 2.2) that if h has finitely many nonzero entries, there is an (equivariant) embedding $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r) \hookrightarrow (\mathbb{A}^r)^{[\sum_d h(d)]}$. In particular, $(\mathbb{A}^r)^{[n]}$ contains every multigraded Hilbert scheme $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$, where $\sum_d h(d) = n$ and $\mathbf{a}$ is

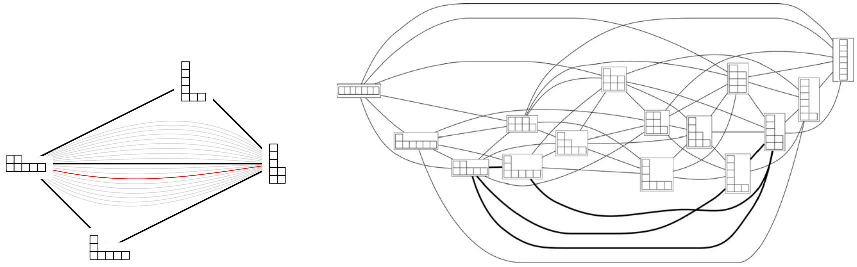


Fig. 5. The T-graphs of $\mathrm{Hilb}^{(1,2,2,1,1,0,\dots)}_{(1,1)}$ and $(\mathbb{A}^2)^{[7]}$

arbitrary, as a closed subscheme. If $r = 2$, then any non-monomial ideal is homogeneous with respect to at most one positive grading. Thus the T-graph problem for $(\mathbb{A}^2)^{[n]}$ is equivalent to the T-graph problem for every multigraded Hilbert scheme $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^2)$, where $\sum_d h(d) = n$.

Example 4.5. In Fig. 1, the colored boxes signify that the two endpoints of $T \cdot I$ are (x^5, x^3y^2, y^3) and (x^3, xy^4, y^5). Thus I is a point of the edge scheme $E\Big($  ,  $\Big) \subseteq \mathrm{Hilb}^{(1,2,3,3,3,1,0,\dots)}_{(1,1)}(\mathbb{A}^2) \subseteq (\mathbb{A}^2)^{[13]}$.

Example 4.6. One may compute using `TEdges` that $\mathrm{Hilb}^{(1,2,2,1,1,0,0,\dots)}_{(1,1)}(\mathbb{A}^2)$ is equivariantly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with the diagonal action of $\mathbb{k}^*$. On the left in Fig. 5 is its T-graph, drawn in thick black lines, and on the right is the T-graph of $(\mathbb{A}^2)^{[7]}$, with the corresponding edges thickened. The gray curves on the left are intended to depict the 1-dimensional family of T-orbits that correspond to the single diagonal edge in the T-graph. (The edges of the outer rectangle correspond to single 1-dimensional T-orbits.) There are ten matroid strata:

- The four monomial ideals,
- The four black edges of the outer rectangle,
- The red curve, representing ideals of the form $(x^2 - c^2y^2, xy^2 + cy^3, y^5)$, and
- The open stratum, representing ideals of the form $(x^2 + c_1xy + (c_1c_2 - c_2^2)y^2, xy^2 + c_2y^3, y^5)$ with $c_1 \neq 0$.

By viewing T-orbit-closures as rational curves in $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$, and using machinery of *unbroken stable maps* [12], one may associated to each multigraded Hilbert scheme (or edge scheme, or intersection of dependence loci) a moduli space $\overline{\mathcal{M}}^h_{\mathbf{a}}(\mathbb{A}^r)$, which roughly parametrizes T-orbit-closures and their degenerations. (More specifically, the moduli space parametrizes T-invariant maps $f : C \to \mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$, possibly ramified, from nodal rational curves to $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^r)$, such that f is locally T-equivariantly smoothable at every node of C.)

Example 4.7. In Example 4.6, $\overline{\mathcal{M}}^{(1,2,2,1,1,0,\dots)}_{(1,1)}(\mathbb{A}^2) \cong \mathbb{P}^1$, with two points corresponding to the two degenerations of orbits into nodal rational curves (unions of orbits). Another example is $\overline{\mathcal{M}}^{(1,2,1,0,\dots)}_{(1,1)}(\mathbb{A}^2) \cong \mathbb{P}(2,1)$, a weighted projective stack. The orbifold point corresponds to a family of 1-dimensional orbits whose limit is a doubled line. (In fact, $\mathrm{Hilb}^{(1,2,1,0,\ldots)}_{(1,1)}$ is equivariantly isomorphic to $\mathbb{P}^2$ with the $\mathbb{k}^*$-action $\lambda \cdot [x : y : z] = [\lambda x : \lambda^{-1} y : z]$. The orbits are conics $xy = cz^2$, and the doubled line above is the limit $c \to \infty$.)

Example 4.8. Consider again Example (2). and let $\gamma \in N_{d,k}$. Suppose γ has order d' rotational symmetry, where $d'|d$.. Then the element $\zeta_d^{d/d'} \in \mathbb{C}^* = T$ acts trivially on the T-orbit in $\mathcal{D}(\{x^d, y^d\}) \subseteq (\mathbb{P}^1)^{[k]}$ associated to γ, and T acts with weight d' on this orbit. It follows that the moduli space $\overline{\mathcal{M}}$ associated to $\mathcal{D}(\{x^d, y^d\})$ contains a single orbifold point with isotropy group $\mathbb{Z}/d'\mathbb{Z}$ corresponding to γ. Altogether, $\overline{\mathcal{M}}$ is isomorphic (as a stack) to the *moduli space of necklaces* $\mathcal{N}_{d,k} = \left[\binom{\{1,\ldots,d\}}{k}/(\mathbb{Z}/d\mathbb{Z})\right]$.

Example 4.9. Using TEdges, we compute that the moduli space associated to the $E\Big($  ,  $\Big)$ is a single point.

However, this moduli space has "empty interior," in the sense that $E\Big($  ,  $\Big)$ is actually empty, and the point in question corresponds to the nodal union of two T-orbits, with the node mapping to .

Question 4.10. The moduli spaces defined above have essentially not been studied. We ask, for example: Is $\overline{\mathcal{M}}^h_{\mathbf{a}}(\mathbb{A}^2)$ smooth (as a stack) for all $\mathbf{a}$ and h? Rational? What about the moduli spaces associated to edge schemes? (From Example 2.14, these may be disconnected.)

Note that in light of Mnëv's universality theory, the moduli spaces associated to arbitrary matroid strata-closures are expected to be arbitrarily badly-behaved.

5. Applications to finite-length Hilbert schemes

Finally, we give a way to apply Theorem 3.7 to the T-graph problem for $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{A}^2)$. First we need the following, which is quite useful for working with initial ideals.

Lemma 5.1. *Let $M = (E, r_M)$ be a matroid, and let $\preceq$ be a total order on E. Let $B_{\preceq}(M)$ be the set of coloops of the (discrete) initial matroid* $\mathrm{in}_{\preceq}(M)$ *(in other words, $B_{\preceq}(M)$ is the unique basis for* $\mathrm{in}_{\preceq}(M)$*), and let $B_{\succeq}(M)$ be the set of coloops of* $\mathrm{in}_{\succeq}(M)$.

Let $m \in E$, and suppose

$$\left|\{m' \in B_{\preceq}(M) : m' \preceq m\}\right| - \left|\{m' \in B_{\succeq}(M) : m' \preceq m\}\right| \leq \left|\{m' \in B_{\preceq}(M) : m' \succeq m\}\right| - \left|\{m' \in B_{\succeq}(M) : m' \succeq m\}\right|. \quad (5)$$

Then m is either a loop or a coloop of M.

Proof. The following are easy to check using matroid contraction and deletion operations:

$$\left|\{m' \in B_{\preceq}(M) : m' \preceq m\}\right| = r(\{m' \in E : m' \preceq m\})$$

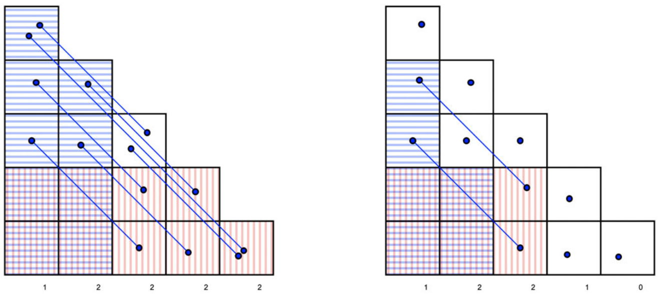


Fig. 6. Tropicalizations of $I = (x^2 - y^2)$ (left) and $I + (x^3)$ (right)

$$\begin{aligned}
\left|\{m' \in B_{\succeq}(M) : m' \preceq m\}\right| &= r(E) - r(\{m' \in E : m' \succ m\}) \\
\left|\{m' \in B_{\preceq}(M) : m' \succeq m\}\right| &= r(E) - r(\{m' \in E : m' \prec m\}) \\
\left|\{m' \in B_{\succeq}(M) : m' \succeq m\}\right| &= r(\{m' \in E : m' \succeq m\}).
\end{aligned}$$

Note that $r(\{m' \in E : m' \preceq m\}) + r(\{m' \in E : m' \succ m\}) \geq r(E)$, with equality if and only if M is the direct sum of matroids on the groundsets $\{m' \in E : m' \preceq m\}$ and $\{m' \in E : m' \succ m\}$. Similarly, $r(\{m' \in E : m' \prec m\}) + r(\{m' \in E : m' \succeq m\}) \geq r(E)$, with equality if and only if M is the direct sum of matroids on the groundsets $\{m' \in E : m' \prec m\}$ and $\{m' \in E : m' \succeq m\}$. Thus the left side of (5) is nonnegative, the right side is nonpositive, and both are zero if and only if M is a direct sum of matroids on the groundsets $\{m' \in E : m' \prec m\}$, $\{m\}$, and $\{m' \in E : m' \succ m\}$. Thus m is either a loop (if the summand on $\{m\}$ has rank zero), or a coloop (if that summand has rank 1). □

We apply Theorem 3.7 via the observation that one can obtain a finite-colength ideal from a principal ideal by adding an appropriate monomial ideal: if I is **a**-homogeneous, and N is a monomial ideal, then $I + N$ is also **a**-homogeneous. (Of course, not all finite-colength ideals can be obtained this way, e.g. $(x^2 - xy, xy - y^2, x^3)$ cannot.)

Example 5.2. Consider $I = (x^2 - y^2) \in (\mathbb{P}^1)^{[2]}$. If $N = (x^3)$, then $I + N = (x^2 - y^2, x^3)$ is an ideal of colength 6, with tropicalization shown in Fig. 6. Note that adding N does not commute with taking initial ideals; for example, $\mathrm{in}_{x<y}(I) + N = (x^2) + (x^3) = (x^2)$, while $\mathrm{in}_{x<y}(I + N) = (x^2, xy^2, y^4)$.

Definition 5.3. A homogeneous ideal $I \subseteq R$ is *PPM* (short for *principal plus monomial*) if $I = (f) + N$ for some $f \in R$ homogeneous, and some monomial ideal N.

The analogous operation of matroids is the "looped contraction." (We do not know of a standard term for this operation.)

Definition 5.4. Let $M = (E, r)$ be a matroid, and let $S \subseteq E$. The *contraction* M/S of M at S is the matroid with groundset $E \setminus S$ whose circuits are the minimal elements of $\{S' \cap (E \setminus S) : S' \text{ a circuit of } M\}$. In other words, for $T \subseteq E \setminus S$, $r_{M/S}(T) = r_M(S \cup T) - r_M(S)$.

The *looped contraction* $M \div S$ is the matroid $M \div S = M/S \oplus U_{0,S}$, where $U_{0,S}$ is the uniform matroid from Example 2.6. Note $M \div S$ has groundset E and rank $r(M) - r(S)$, and elements of S are loops in $M \div S$. The rank function is given, for $T \subseteq E$, $r_{M \div S}(T) = r_M(S \cup T) - r_M(S)$.

Let $\mathscr{M}$ be a tropical ideal, and let N be a monomial ideal. The *looped contraction* $\mathscr{M} \div N$ of $\mathscr{M}$ at N is the tropical ideal defined by $(\mathscr{M} \div N)_d = \mathscr{M}_d \div S_d$, where S_d is the set of monomials in N_d. It is straightforward to check that $\mathscr{M} \div N$ is a tropical ideal.

A tropical ideal $\mathscr{M}$ is *tropically principal* if it has the Hilbert function of a principal ideal, i.e. if there exists $c \in \mathbb{Z}^b_{\geq 0}$ such that

$$\mathrm{rk}(\mathscr{M}_d) = |\mathrm{Mon}_d(\mathbf{a})| - |\mathrm{Mon}_{d-c}(\mathbf{a})|$$

for all $d \in \mathbb{Z}^b_{\geq 0}$. (We say $\mathscr{M}$ is *generated in degree* c.) A tropical ideal $\mathscr{M}$ is *PPM* if there exists a tropically principal tropical ideal $\mathscr{M}'$ and a monomial ideal N such that $\mathscr{M} = \mathscr{M}' \div N$.

A straightforward calculation using the rank function in Definition 5.4 yields:

Proposition 5.5. *For any homogeneous ideal I and any monomial ideal N,* $\mathrm{Trop}(I + N) = \mathrm{Trop}(I) \div N$.

Corollary 5.6. *Let $\mathscr{M}$ be a tropically principal tropical ideal, and let N be a monomial ideal. Then $I \mapsto I + N$ defines a morphism $\mathcal{S}(\mathscr{M}) \to \mathcal{S}(\mathscr{M} \div N)$. (Note that $\mathcal{S}(\mathscr{M} \div N)$ lies in a single multigraded Hilbert scheme.)*

Corollary 5.7. *Let $J \subseteq R$ be an ideal. If J is PPM, then* $\mathrm{Trop}(J)$ *is PPM.*

The converse of Corollary 5.7 does not hold:

Example 5.8. Let $\Bbbk = \mathbb{C}$. The tropical ideal $\mathscr{M}$ in Fig. 7 is PPM, since $\mathscr{M} = \mathrm{Trop}((f) + N)$, where $f = x^3 + x^2y + 2xy^2 + y^3$ and $N = (x^4, x^3y^2, x^2y^3, y^4)$. (It is straightforward to check that the roots of f do not differ by 4th roots of unity, hence $f \notin \mathcal{D}(\{x^4, y^4\})$. This implies that $\mathrm{Trop}((f)+N)_4$ has rank 1, as shown.) On the other hand, we also have $\mathscr{M} = \mathrm{Trop}(((x-y)(x-iy)(x+y), x^3y+2x^2y^2)+N)$, as follows. Since $((x-y)(x-iy)(x+y)) \in \mathcal{D}(\{x^4, y^4\})$, $\mathrm{Trop}(((x-y)(x-iy)(x+y)) + N)_4$ has rank 2, and adding $x^3y + 2x^2y^2$ reduces the rank to 1. (It is again easy to check that neither ideal contains any extra monomials in degree 4.)

Lastly, we observe that $((x-y)(x-iy)(x+y), x^3y+2x^2y^2)+N$ is not PPM. If it were, it would necessarily be generated in degree 4 by $\{x(x-y)(x-iy)(x+y), y(x-y)(x-iy)(x+y), x^4, y^4\}$; these span too small a subspace.

The following is the key observation for applying Sect. 3 to Hilbert schemes of finite-length subschemes.

Lemma 5.9. *Let $I = (f) + N$ be a PPM ideal. Let $U \subseteq N_d$ be a set of monomials such that $|U| > r_{\mathrm{Trop}((f))}(\mathrm{Mon}_d(a_1, a_2)) - r_{\mathrm{Trop}(I)}(\mathrm{Mon}_d(a_1, a_2))$. Then $(f) \in \mathcal{D}(U)$.*

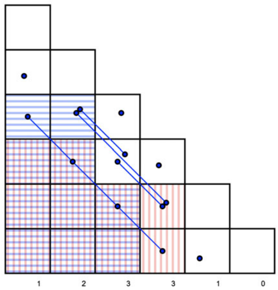


Fig. 7. A tropical ideal that is the tropicalization of both a PPM ideal and a non-PPM ideal

Proof. By Proposition 5.5,

$$r_{\mathrm{Trop}(I)}(\mathrm{Mon}_d(a_1, a_2)) = r_{\mathrm{Trop}((f))}(\mathrm{Mon}_d(a_1, a_2)) - r_{\mathrm{Trop}((f))}(N_d).$$

By assumption,

$$\begin{aligned} |U| &> r_{\mathrm{Trop}((f))}(\mathrm{Mon}_d(a_1, a_2)) - r_{\mathrm{Trop}(I)}(\mathrm{Mon}_d(a_1, a_2)) \\ &= r_{\mathrm{Trop}((f))}(N_d) \geq r_{\mathrm{Trop}((f))}(U), \end{aligned}$$

so U is dependent. □

Remark 5.10. One can apply Lemma 5.9 as follows. Often, it can be argued that a given matroid stratum (or edge scheme) $\mathcal{S}$ must consist *only* of PPM ideals. In this case, recording the nonmonomial generator (where monomials in N are given coefficient zero) defines a natural embedding from $\mathcal{S}$ to a Hilbert scheme $\mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{P}^{r-1}) \cong \mathbb{P}^N$ of *principal* ideals. Lemma 5.9 then says that the embedding factors through $\bigcap_U \mathcal{D}(U) \subseteq \mathrm{Hilb}^h_{\mathbf{a}}(\mathbb{P}^{r-1})$, where U runs over sets satisfying the condition in the hypothesis.

We conclude by illustrating this method in our running example, Example 2.14.

Theorem 5.11. *Let $k \geq 1$ and $d_0 > k$. Let M_1 (resp. M_2) be the partition whose Young diagram is an $d_0 \times k$ (resp. $k \times d_0$) rectangle. Then the edge scheme $E(M_1, M_2) \subseteq (\mathbb{A}^2)^{[d_0 \cdot k]}$ is isomorphic to $\mathcal{D}(\{x^{d_0}, y^{d_0}\}) \subseteq (\mathbb{P}^1)^{[k]}$, i.e. it consists of a collection of rational curves, indexed by necklaces with k black and $d_0 - k$ white beads, all of which meet at two points.*

Proof. First, we argue that any ideal I in the edge scheme $E(M_1, M_2)$ is PPM, with nonmonomial generator in degree k. Note that the Hilbert function of M_1 and M_2 with respect to the grading $(1, 1)$ is

$$(1, 2, \ldots, \underbrace{k, k, \ldots, k}_{d_0-k+1}, k-1, \ldots, 1, 0, 0, \ldots)$$
$$= \begin{cases} d+1 & 0 \leq d \leq k-1 \\ k & k \leq d \leq d_0 - 1 \\ d_0 + k - 1 - d & d_0 \leq d \leq d_0 + k - 1 \\ 0 & d > d_0 + k - 1. \end{cases}$$

For $d \leq k-1$, $\dim I_d = 0$. For $k \leq d \leq d_0 - 1$, $\dim I_d = d + 1 - k$, which implies I_d is spanned by the $d+1-k$ linearly independent monomial multiples of the generator of I_k.

For $d_0 \leq d \leq d_0 + k - 1$, I_d contains the $d+1-k$ monomial multiples of the generator of I_k, as well as the $d - d_0 + 1$ consecutive monomials $x^d, x^{d-1}y, \ldots, x^{d_0}y^{d-d_0}$, by Lemma 5.1. Since $d - d_0 < k$, by an upper-triangularity argument, these $(d+1-k)+(d-d_0+1)$ vectors are all linearly independent. On the other hand,

$$\dim I_d = 2d - k - d_0 + 2 = (d+1-k) + (d - d_0 + 1).$$

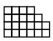
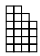
In particular, if f is a generator of I_k, we have shown that $I = (f) + (x^{d_0})$, hence is PPM. (This is from the case $d = d_0$.)

Next, we apply Lemma 5.9. Again by Lemma 5.1, $y^{d_0} \in I$, so we may as well write $I = (f) + (x^{d_0}, y^{d_0})$. Let $U = \{x^{d_0}, y^{d_0}\}$, and note that

$$2 = |U| > r_{\mathrm{Trop}((f))}(\mathrm{Mon}_{d_0}) - r_{\mathrm{Trop}(I)}(\mathrm{Mon}_{d_0}) = k - (k-1) = 1.$$

By Lemma 5.9, $f \in \mathcal{D}(U)$. This shows that $\mathcal{M}(M_1, M_2) \subseteq \mathcal{N}_{d_0,k}$, and the opposite inclusion follows immediately from counting ranks in each grade. □

Theorem 5.11 immediately generalizes, with the same proof, to the case where M_1 and M_2 are both "cut off" in some degree $d_1 > d_0$. For example, $\mathcal{M}(M_1, M_2) \cong \mathcal{N}_{6,4}$, where

$M_1 =$  $M_2 =$  .

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