Distributed Time-Varying Quadratic Optimal Resource Allocation Subject to Nonidentical Time-Varying Hessians With Application to Multiquadrotor Hose Transportation

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***Abstract*—This article considers the distributed time-varying optimal resource allocation problem with time-varying quadratic cost functions and a time-varying coupled equality constraint for multiagent systems. The objective is to design a distributed algorithm for agents with single-integrator dynamics to cooperatively satisfy the coupled equality constraint and minimize the sum of all local cost functions. Here, both the coupled equality constraint and cost functions depend explicitly on time. The cost functions are in quadratic form and may have nonidentical time-varying Hessians. To solve the problem in a distributed manner, an estimator based on the distributed average tracking method is first developed for each agent to estimate certain global information. By leveraging the estimated global information and an adaptive gain scheme, a distributed continuous-time algorithm is proposed, which ensures the agents to find and track the time-varying optimal trajectories with vanishing errors. We illustrate the applicability of the proposed method in the optimal hose transportation problem using multiple quadrotors.**

***Index Terms*—Continuous-time algorithm, distributed control, optimal resource allocation, time-varying system.**

I. Introduction

Distributed network optimization methods have facilitated the development of multiagent systems in the past decade (see [1] and references therein). Based on the control techniques, many researchers have developed various distributed algorithms to dynamically solve the network optimization problems, such as unconstrained optimization problems (see [2]–[6]) and optimal resource allocation problems (see [7]–[14]) using only local information and interaction among agents.

The aforementioned works on optimal resource allocation problems in [7]–[14] focus on dealing with time-invariant local cost functions and constraints. However, it should be noted that the cost functions and constraints in practical optimization problems might depend explicitly on time. Recently, the discrete-time (see [15] and references therein) and continuous-time algorithms (see [16]–[23]) have been proposed to handle the time-varying optimization problems. Most of the existing discrete-time algorithms dealing with time-varying optimization problems require that each optimizer be computed in each discrete-time instance, which is infeasible when the optimal solution trajectory varies fast. The sampling period, step size, and computation time at each step in the discrete-time algorithms will all affect the upper bound of the tracking errors. On the other hand, the discrete-time algorithms can hardly be employed for multiple robots with continuous-time dynamics to swarm around a common time-varying optimal trajectory as in [16]. Recently, many scholars have concentrated on developing the continuous-time methods to track the variations of the optimal solution trajectories such that the general time-varying optimization problems can be solved with vanishing errors (see [16]–[23]). Time-varying unconstrained optimization algorithms are developed for the swarm tracking behavior of a multirobot system in [16], and the power output consensus problem in [17] and [18]. Contrary to the unconstrained optimization problems, time-varying inequality and equality constraints are considered in [19] and [20], respectively, while the cost functions are time invariant. Time-varying cost functions and constraints are considered simultaneously in [21] and [22] by proposing the centralized algorithms. The prediction-correction interior-point method, which requires the inverse Hessian matrices of the cost functions, is presented in [21] to solve the collision-free robot navigation problem. A novel zeroing neural network is applied in [22] to solve the time-varying nonlinear optimization problem. A distributed approach is developed in [23] to solve the optimization problem with time-varying cost functions and time-varying local inequality constraints. As discussed in [16] and [17], the time-varying optimization problems with quadratic objective functions exist in the multirobotic system where all robots aim to swarm around a common time-varying optimal trajectory and the grid-connected battery energy storage system that involves with the state-of-charge balancing among multiple battery packages. Distributed algorithms have been proved to be

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more appropriate for the above spatially distributed multiagent systems where one agent has only limited access to its neighbors. In addition, while the resource allocation optimization methods in [8] for a smart grid system consisting of multiple generators focus on time-invariant cost functions and coupled equality constraints, the local objective functions and constraints can be time varying when generating electricity from light, wind, water, and other energy sources that will consistently change due to the variability of the environment. There are only a few distributed algorithms, however, dealing with the time-varying optimization problems where both the objective functions and the coupled constraints are time varying (see [15], [24], and their references). When both the cost functions and equality constraints are time varying, distributed continuous-time algorithms are proposed in [24] to address the optimal resource allocation problem under, respectively, identical time-varying and nonidentical time-invariant Hessians. However, in some applications, such as the optimal estimation of distributed processes in [15], the cost functions have nonidentical time-varying Hessians. Unfortunately, distributed continuous-time algorithms to solve the quadratic optimal resource allocation problem subject to nonidentical time-varying Hessians are not addressed in all the above-mentioned articles. The main contributions are given as follows. We propose a distributed continuous-time algorithm to address the optimal resource allocation problem with time-varying cost functions and a time-varying coupled equality constraint. Here, each agent only knows its own cost function and weight coefficient that contributes to the coupled equality constraint. In addition, all local cost functions are in quadratic form with nonidentical time-varying Hessians. By developing estimators based on the distributed average tracking method, which is used for estimating certain global information, and an adaptive control idea, our continuous-time algorithm ensures that each agent will track its corresponding optimal trajectory with vanishing errors. Furthermore, we analyze the applicability to the optimal hose transportation using multiple quadrotors from a new point of view. In such a situation, each quadrotor is able to determine its optimal thrust output in a distributed manner while reducing the total energy consumption of the team.

Comparison With the Existing Literature: In contrast to the methods in [16]–[18] dealing with time-varying unconstrained optimization problems, our algorithm focuses on addressing the optimal resource allocation problem involving a time-varying coupled equality constraint. We emphasize that it is nontrivial to develop a dynamical system that can track the time-varying optimal solutions when the time-varying coupled constraint is taken into consideration, which leads to a completely different Karush–Kuhn–Tucker (KKT) condition. Although the methods in [19]–[24] can deal with time-varying constrained optimization problems, they are not applicable to the case in this article. While the time-varying inequality and equality constraints are dealt with in [19] and [20], respectively, the considered cost functions are time invariant. That is, the variation of the optimal solutions due to time-varying cost functions cannot be tracked by using the methods in [19] and [20]. The algorithms in [21] and [22] are able to address some constrained time-varying optimization problems with zero tracking error, but they are centralized and requires global information to compute the inverse of certain global matrix. In contrast, our proposed algorithm aims to use only the local information of each agent and its neighbors. The methods in [23] focus on the optimization problem with time-varying cost functions and local inequality constraints. Here, the inequality constraints are local and known to each agent. In contrast, our algorithm aims to address a global coupled equality constraint. All agents need to cooperatively satisfy the global constraint while having only partial information of the coupled time-varying equality constraint. The quadratic cost functions in [24] are assumed to have identical time-varying or nonidentical time-invariant Hessians. In contrast, the proposed algorithm in this article aims to deal with nonidentical time-varying Hessians for quadratic cost functions. That is, it is for the first time to design distributed continuous-time algorithms for the time-varying optimal resource allocation problem with a time-varying coupled equality constraint and cost functions that have nonidentical time-varying Hessians.

The remainder of this article is organized as follows. Section II introduces the notation and problem description. In Section III, an estimator is first developed for all agents to estimate certain global information. By adopting an adaptive gain scheme and leveraging the estimated global information, the distributed continuous-time algorithm is then proposed. It is shown that each agent is able to find and track the time-varying optimal trajectories. The applicability to the optimal hose transportation using multiple quadrotors is illustrated in Section IV, and a numerical simulation is then performed to corroborate the theoretical result. Finally, Section V offers the general conclusions.

II. Notation and Problem Formulation

A. Notation

We use $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ to represent, respectively, the set of real, nonnegative, and positive numbers. The set of n-dimensional real vectors is denoted by $\mathbb{R}^n$. Let $\mathbb{R}^{m\times n}$ represent the set of $m \times n$ real matrices. The column vector with all zeros is denoted by $\mathbf{0}_n$. Let $|x|$ represent the absolute value of any $x \in \mathbb{R}$. For a vector $x \in \mathbb{R}^n$, let $\|x\|_p$ and $\|x\|_\infty$ denote, respectively, the p-norm and ∞-norm of the vector x. The 2-norm and ∞-norm of a matrix A are represented by $\|A\|_2$ and $\|A\|_\infty$, respectively. Let $\mathrm{sgn}(x)$ denote the signum function for $x \in \mathbb{R}$, i.e., $\mathrm{sgn}(x) = 1$ when x is positive, $\mathrm{sgn}(x) = -1$ when x is negative, and $\mathrm{sgn}(x) = 0$ otherwise. In addition, for a vector $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, let $\mathrm{sgn}(x) = [\mathrm{sgn}(x_1), \dots, \mathrm{sgn}(x_n)]^T$. For a square matrix $A \in \mathbb{R}^{n\times n}$, let $\Lambda(A)$ denote the vector of the main diagonal elements of A. For a vector $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, let $\mathrm{diag}(x)$ denote the square matrix with the elements of the vector x on the main diagonal. Given a function $f(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, $\nabla_x f(x, t) \in \mathbb{R}^n$ and $H(x, t, f(\cdot)) \in \mathbb{R}^{n\times n}$ are used to represent the gradient and Hessian of the function $f(x, t)$. In addition, we use $\nabla_{xt} f(x, t) \in \mathbb{R}^n$ to denote the partial derivative of $\nabla_x f(x, t)$ with respect to t.

Lemma 1 [25, Th. 1]: Consider the following system:

$$\dot{z}_i(t) = \alpha \sum_{j\in\mathcal{N}_i} \operatorname{sgn}\big(x_j(t) - x_i(t)\big)$$
$$x_i(t) = z_i(t) + r_i(t) \tag{1}$$

where $x_i(t) \in \mathbb{R}^m$, $r_i(t) \in \mathbb{R}^m$, $\sum_{i=1}^n z_i(0) = \mathbf{0}_m$, $\alpha \in \mathbb{R}_{++}$, $\bar{r} \in \mathbb{R}_{++}$ satisfies $\bar{r} \geq \sup_{t\geq 0} \|\dot{r}_i(t)\|_\infty$. For the system (1), if the communication topology among the agents is undirected and connected and $\alpha > \bar{r}$, then the states of all agents will reach average consensus in finite time. That is, there exists a positive number $T \in \mathbb{R}_{++}$ satisfying $T \leq (1/2)(\alpha - \bar{r}) \sum_{i=1}^n \sum_{j\in\mathcal{N}_i} \|x_i(0) - x_j(0)\|_\infty$ such that $\|x_i(t) - (1/n)\sum_{i=1}^n r_i(t)\|_2 = 0$ for $t \geq T$.

B. Problem Description

In this section, we consider the time-varying optimal resource allocation problem with quadratic convex cost functions and a coupled equality constraint. Suppose that the networked multiagent system consists of n agents. The communication graph that is connected and undirected among agents is represented by $\mathcal{G}$. If there is an undirected path between any two distinct agents in the graph $\mathcal{G}$, then the undirected graph $\mathcal{G}$ is connected. The undirected path is denoted by a sequence of edges $(i, j), (j, k), \ldots$, where $i, j, k \in \mathcal{I}$, and $\mathcal{I} = \{1, 2, \ldots, n\}$ denotes the index set of agents. In addition, let $\mathcal{N}_i$ denote the set of neighbors of agent i.

In particular, we suppose that each agent has the continuous-time single-integrator dynamics

$$\dot{x}_i(t) = u_i(t) \tag{2}$$

where $x_i(t) \in \mathbb{R}^m$ is the state variable associated with the ith agent, and $u_i(t) \in \mathbb{R}^m$ is the control input for the ith agent. The objective is to design the distributed control input $u_i(t)$, $i \in \mathcal{I}$, for all agents to cooperatively satisfy a time-varying coupled equality constraint while minimizing the sum of all local time-varying cost functions, each of which is known to only one agent. In other words, by employing the proposed algorithm, we ensure that each agent is able to actively update its state variable using only local information and interaction such that the aggregate column vector composed of all local state variables will track the optimal trajectory

$$x^*(t) = \underset{x(t)\in\mathbb{R}^{mn}}{\operatorname{argmin}} \sum_{i=1}^n f_i(x_i(t), t), \tag{3a}$$
$$\text{subject to } \sum_{i=1}^n \eta_i(t) x_i(t) = \sum_{i=1}^n b_i(t) \tag{3b}$$

where $x(t) = [x_1^T(t), \ldots, x_n^T(t)]^T \in \mathbb{R}^{mn}$ is the aggregate column vector involving all local state variables, $\eta_i(t) \in \mathbb{R}_{++}$ is the weight coefficient that the ith agent contributes to the coupled equality constraint, $\eta_i(t)$ and $b_i(t) \in \mathbb{R}^m$ are local variables that are only known to agent i, and $f_i(x_i(t), t)$ is the time-varying convex cost function associated with the ith agent. For further studies, we define the Lagrange function associated with the time-varying optimal resource allocation problem (3) as

$$L(z(t), t) = \sum_{i=1}^n f_i(x_i(t), t) + \lambda^T(t)\left(\sum_{i=1}^n \eta_i(t) x_i(t) - \sum_{i=1}^n b_i(t)\right) \tag{4}$$

where $\lambda(t) \in \mathbb{R}^m$ is the Lagrange multiplier, and $z(t) = [x(t)^T, \lambda(t)^T]^T$. Then, we define the dual function of $L(z(t), t)$ as $\mathcal{F}(\lambda(t), t) = \min_{x(t)\in\mathbb{R}^{mn}} L(x(t), \lambda(t), t)$. The optimal dual variable is denoted by $\lambda^*(t) = \operatorname{argmax}_{\lambda(t)\in\mathbb{R}^m} \mathcal{F}(\lambda(t), t)$. The optimal primal–dual variable is denoted by $z^*(t) = [x^*(t)^T, \lambda^*(t)^T]^T$.

Note that the optimal resource allocation problem (3) appears in many applications (see [20] and references therein). Different from the case in [20] that $\eta_i(t) = 1$ for $i \in \mathcal{I}$, the weight coefficients in problem (3) can be time varying. In addition, problem (3) is a more general case than that in [20] because the cost functions $f_i(x_i(t), t)$ for $i \in \mathcal{I}$ are time varying.

III. Main Results

In this section, we consider the time-varying optimal resource allocation problem defined by (3) for a networked multiagent system, in which all agents have the continuous-time single-integrator dynamics (2). For simplicity of notation, we use x_i and u_i instead of $x_i(t)$ and $u_i(t)$ in the following to remove the time dependence. Before moving on, we need the following assumptions throughout this article.

Assumption 1: The time-varying optimal resource allocation problem (3) is feasible at all times. That is, there exists at least $\hat{x}(t) = [\hat{x}_1^T(t), \ldots, \hat{x}_n^T(t)]^T \in \mathbb{R}^{mn}$ such that $\sum_{i=1}^n \eta_i(t)\hat{x}_i(t) = \sum_{i=1}^n b_i(t)$ for any given time $t \geq 0$.

Assumption 2: The cost function $f_i(x_i, t)$ for the ith agent is in the form of

$$f_i(x_i, t) = \frac{1}{2} x_i^T H_i(t) x_i + C_i^T(t) x_i + d_i(t) \tag{5}$$

where $H_i(t) = \operatorname{diag}([h_{i1}(t), \ldots, h_{im}(t)]^T) \in \mathbb{R}^{m\times m}$, $h_{ij}(t) \geq m_f$ for some $m_f \in \mathbb{R}_{++}$, $|h_{ij}(t)|$ and $|\dot{h}_{ij}(t)|$ are upper bounded, $C_i(t) = [c_{i1}(t), \ldots, c_{im}(t)]^T \in \mathbb{R}^m$, $c_{ij}(t) \in \mathbb{R}$, $|c_{ij}(t)|$ and $|\dot{c}_{ij}(t)|$ are upper bounded, $j = 1, \ldots, m$, and $d_i(t) \in \mathbb{R}$.

Assumption 3: The local weight coefficient $\eta_i(t)$ and its time derivative $\dot{\eta}_i(t)$ are bounded for $i \in \mathcal{I}$. In addition, $\|b_i(t)\|_\infty$ and $\|\dot{b}_i(t)\|_\infty$ are bounded for $i \in \mathcal{I}$.

Remark 1: It follows from (5) that $\nabla_{x_i} f_i(x_i, t) = H_i(t) x_i + C_i(t)$, $\nabla_{x_i t} f_i(x_i, t) = \dot{H}_i(t) x_i + \dot{C}_i(t)$, and $H(x_i, t, f_i(\cdot)) = H_i(t)$ for $i \in \mathcal{I}$. The strong convexity of the cost function (5) ensures that for any given time $t \geq 0$, the optimal solution of (3) is unique. In addition, $d_i(t)$ in (5) is insignificant and does not affect the variation of the optimal solution.

Remark 2: The time-varying quadratic cost function (5) is suitable for a great class of applications, such as the swarm tracking behavior in [16] and moving targets tracking in [21] for multirobot systems. In particular, the class of the local time-varying cost functions addressed in [16]–[18] is assumed to have the gradient $\nabla_{x_i} f_i(x_i, t) = \rho x_i + \zeta_i(t)$, where $\rho \in \mathbb{R}_{++}$, and $\zeta_i(t) \in \mathbb{R}^m$. That is, these local cost functions are a special case of Assumption 2.

Under Assumptions 1–3, we propose the following distributed continuous-time algorithm:

$$\lambda_i = -(\mathrm{diag}(\theta_i(t)))^{-1}\omega_i(t) \tag{6a}$$

$$e_i = \frac{\nabla_{x_i} f_i(x_i, t)}{\eta_i(t)} + \lambda_i \tag{6b}$$

$$\dot{\hat{\beta}}_i = \alpha \|e_i\|_1 \tag{6c}$$

$$u_i = -H_i^{-1}(t)\Bigg(\nabla_{x_i t} f_i(x_i, t) + \alpha\hat{\beta}_i\eta_i(t)\mathrm{sgn}(e_i) + \eta_i(t)e_i - \nabla_{x_i} f_i(x_i, t)\frac{\dot{\eta}_i(t)}{\eta_i(t)}\Bigg) \tag{6d}$$

where $\lambda_i \in \mathbb{R}^m$, $e_i \in \mathbb{R}^m$, $\alpha \in \mathbb{R}_{++}$, the initial value of $\hat{\beta}_i \in \mathbb{R}$ is a positive constant, i.e., $\hat{\beta}_i(0) \in \mathbb{R}_{++}$, and $\theta_i(t), \omega_i(t) \in \mathbb{R}^m$ are auxiliary variables driven by the distributed estimators satisfying

$$\dot{\xi}_i(t) = \gamma \sum_{j\in\mathcal{N}_i} \mathrm{sgn}\big(\omega_j(t) - \omega_i(t)\big) \tag{7a}$$

$$\omega_i(t) = \xi_i(t) + g_i(t) + b_i(t) \tag{7b}$$

$$\dot{\psi}_i(t) = \beta \sum_{j\in\mathcal{N}_i} \mathrm{sgn}\big(\theta_j(t) - \theta_i(t)\big) \tag{7c}$$

$$\theta_i(t) = \psi_i(t) + \phi_i(t) \tag{7d}$$

where $g_i(t) = \eta_i(t)H_i^{-1}(t)C_i(t)$, $\gamma \in \mathbb{R}_{++}$ satisfies $\gamma > \sup_{t\ge 0}\|\dot{g}_i(t) + \dot{b}_i(t)\|_\infty$, $\phi_i(t) = \eta_i^2(t)\Lambda(H_i^{-1}(t))$, $\beta \in \mathbb{R}_{++}$ satisfies $\beta > \sup_{t\ge 0}\|\dot{\phi}_i(t)\|_\infty$, the variables $\xi_i(t) \in \mathbb{R}^m$ and $\psi_i(t) \in \mathbb{R}^m$ satisfy $\sum_{i=1}^n \xi_i(0) = \sum_{i=1}^n \psi_i(0) = \mathbf{0}_m$ for $i \in \mathcal{I}$, the initial value of $\theta_i(t)$ for $i \in \mathcal{I}$ is positive, and $\mathcal{N}_i$ denotes the set of neighbors of agent i.

The derivation of the algorithm in (6) is inspired by the KKT condition and the distributed finite-time average tracking method. By transforming the KKT condition of the convex optimization problem, we can estimate some information about the optimal primal–dual variable $z^*(t) = [x^*(t)^T, \lambda^*(t)]^T$. Note that the Lagrange function associated with the time-varying optimal resource allocation problem (3) is defined as (4). According to the convex optimization theory, the optimal pair $x^*(t)$ and $\lambda^*(t)$ must satisfy the following KKT conditions:

$$\nabla_x L\big(z^*(t), t\big) = \nabla_x\left(\sum_{i=1}^n f_i\big(x_i^*(t), t\big)\right) + \lambda^*(t)\nabla_x\left(\sum_{i=1}^n \eta_i(t)x_i^*(t)\right) = \mathbf{0}_{mn} \tag{8}$$

$$\nabla_\lambda L\big(z^*(t), t\big) = \sum_{i=1}^n \eta_i(t)x_i^*(t) - \sum_{i=1}^n b_i(t) = \mathbf{0}_m. \tag{9}$$

Equation (8) can be written as

$$\nabla_{x_i} f_i\big(x_i^*(t), t\big) + \lambda^*(t)\eta_i(t) = \mathbf{0}_m. \tag{10}$$

For the quadratic objective function (5), we have

$$\nabla_{x_i} f_i\big(x_i^*, t\big) = H_i(t)x_i^* + C_i(t) \tag{11}$$

where $C_i(t) = [c_{i1}(t), \ldots, c_{im}(t)]^T \in \mathbb{R}^m$. Substituting (11) to (10), we have

$$x_i^* = -\eta_i(t)H_i^{-1}(t)\lambda^*(t) - H_i^{-1}(t)C_i(t).$$

Then, we have

$$\sum_{i=1}^n \eta_i x_i^* = -\sum_{i=1}^n \eta_i^2(t)H_i^{-1}(t)\lambda^*(t) - \sum_{i=1}^n \eta_i H_i^{-1}(t)C_i(t). \tag{12}$$

Substituting (12) to (9), we have

$$\lambda^*(t)\sum_{i=1}^n \eta_i^2(t)H_i^{-1}(t) = -\sum_{i=1}^n \eta_i H_i^{-1}(t)C_i(t) - \sum_{i=1}^n b_i(t).$$

Then, we have

$$\lambda^*(t) = -\left(\sum_{i=1}^n \eta_i^2(t)H_i^{-1}(t)\right)^{-1} \times \left(\sum_{i=1}^n \eta_i H_i^{-1}(t)C_i(t) + \sum_{i=1}^n b_i(t)\right) \tag{13}$$

which can be estimated in a distributed way. It follows from (13) that the optimal Lagrange multiplier at any time is independent on the states of the agents. Hence, the right-hand side of (13) can be estimated according to Lemma 1. Actually, the aim of (6a) is to obtain the above common Lagrange multiplier $\lambda^*(t)$ by estimating the right-hand side of (13) in a distributed way. Specifically, θ_i and ω_i in (6a) are introduced to estimate $\sum_{i=1}^n \eta_i^2(t)H_i^{-1}(t)$ and $\sum_{i=1}^n \eta_i H_i^{-1}(t)C_i(t) + \sum_{i=1}^n b_i(t)$, respectively. After tracking the Lagrange multiplier, we design (6b)–(6d) to drive the states x_i to satisfy the remaining part of the KKT condition, which is actually an output tracking problem. The error term in (6b) means the difference between the gradient and the weighted Lagrange multiplier. The adaptive gains $\hat{\beta}_i$ in (6c) are designed to make the error term converge to zero by updating $x_i(t)$ using (6d) such that the KKT condition (10) can be satisfied. The convergence analysis of algorithm (6) with (7) is provided in the following.

Theorem 1: Suppose that the fixed undirected graph $\mathcal{G}$ is connected and Assumptions 1–3 hold. By using the controller (6), the state x_i and Lagrange multiplier λ_i of the ith agent with dynamics (2) will converge to the corresponding optimal state and Lagrange multiplier as $t \to \infty$ for the time-varying optimal resource allocation problem defined by (3), respectively. In addition, there exists a constant $x_{\max} \in \mathbb{R}_{++}$ such that $\|x_i\|_\infty \le x_{\max}$ for all time.

Proof: Noting that $g_i(t) = \eta_i(t)H_i^{-1}(t)C_i(t)$ and $\phi_i(t) = \eta_i^2(t)\Lambda(H_i^{-1}(t))$ for $i \in \mathcal{I}$, we first show that the global information $\sum_{i=1}^n(g_i(t) + b_i(t))$ and $\sum_{i=1}^n \phi_i(t)$ can be estimated for each agent in a finite time by leveraging (7). Note from Assumption 3 that $\eta_i(t)$, $\dot{\eta}_i(t)$, and $\|\dot{b}_i(t)\|_\infty$ are bounded. Also, note that $h_{ij}(t) \in \mathbb{R}_{++}$, $|h_{ij}(t)|$, $|\dot{h}_{ij}(t)|$, $|c_{ij}(t)|$, and $|\dot{c}_{ij}(t)|$ for $j = 1, \ldots, m$ are upper bounded due to Assumption 2. Then, we have that $\|\dot{g}_i(t)\|_\infty$ and $\|\dot{\phi}_i(t)\|_\infty$ are upper bounded. Note that the fixed undirected graph $\mathcal{G}$ is connected, and $\|\dot{g}_i(t)\|_\infty$, $\|\dot{b}_i(t)\|_\infty$, and $\|\dot{\phi}_i(t)\|_\infty$ are upper bounded for $i \in \mathcal{I}$. Then, there exists a positive time $T \in \mathbb{R}_{++}$ such that $\|\omega_i(t) - (1/n)\sum_{i=1}^n(g_i(t) + b_i(t))\|_2 = 0$ and $\|\theta_i(t) - (1/n)\sum_{i=1}^n \phi_i(t)\|_2 = 0$ for all $t \ge T$ according to Lemma 1. Hence, the Lagrange multiplier λ_i with dynamics (6a) for $i \in \mathcal{I}$ will reach consensus in finite time. That is, we have $\lambda_i = \lambda_j$ for any $t \geq T\ \forall i, j \in \mathcal{I}$.

We then show that the gradient of $f_i(x_i, t)$, i.e., $\nabla_{x_i} f_i(x_i, t)$, converges to $-\lambda_i \eta_i(t)$ as $t \to \infty$ for $i \in \mathcal{I}$ by employing (6b)–(6d). For the ith agent, we define the Lyapunov function candidate as $V_i = (1/2)e_i^T e_i + (1/2)(\hat{\beta}_i - \beta^*)^2$, where $\beta^* \in \mathbb{R}_{++}$ is a positive constant to be determined later. The time derivative of V_i along (6) is then obtained as

$$\dot{V}_i = e_i^T \dot{e}_i + \left(\hat{\beta}_i - \beta^*\right)\dot{\hat{\beta}}_i. \tag{14}$$

Note from (6b) that $\dot{e}_i = \dot{\nabla}_{x_i} f_i(x_i, t)[1/\eta_i(t)] + \nabla_{x_i} f_i(x_i, t)([-\dot{\eta}_i(t)]/[\eta_i^2(t)]) + \dot{\lambda}_i$, where the time derivative of $\nabla_{x_i} f_i(x_i, t)$ at (x_i, t) can be written as $H_i(t)\dot{x}_i + \nabla_{x_i t} f_i(x_i, t)$. That is, we have

$$\begin{aligned}\dot{e}_i = &\left(H_i(t)\dot{x}_i + \nabla_{x_i t} f_i(x_i, t)\right)\frac{1}{\eta_i(t)}\\ &+ \nabla_{x_i} f_i(x_i, t)\frac{-\dot{\eta}_i(t)}{\eta_i^2(t)} + \dot{\lambda}_i.\end{aligned} \tag{15}$$

Substituting (15) and (6c) to (14), we have

$$\begin{aligned}\dot{V}_i = e_i^T\Bigg(&\left(H_i(t)\dot{x}_i + \nabla_{x_i t} f_i(x_i, t)\right)\frac{1}{\eta_i(t)}\\ &+ \nabla_{x_i} f_i(x_i, t)\frac{-\dot{\eta}_i(t)}{\eta_i^2(t)} + \dot{\lambda}_i\Bigg) + (\hat{\beta}_i - \beta^*)\alpha\|e_i\|_1.\end{aligned}$$

It follows from (2) and (6d) that:

$$\begin{aligned}\dot{V}_i &= e_i^T\left(-\alpha\hat{\beta}_i \operatorname{sgn}(e_i) - e_i + \dot{\lambda}_i\right) + (\hat{\beta}_i - \beta^*)\alpha\|e_i\|_1\\ &= -e_i^T e_i + e_i^T\dot{\lambda}_i - \beta^*\alpha\|e_i\|_1\\ &\leq -e_i^T e_i + \|e_i\|_1\left\|\dot{\lambda}_i\right\|_\infty - \beta^*\alpha\|e_i\|_1.\end{aligned} \tag{16}$$

Note from (7b) that $\dot{\omega}_i(t) = \gamma\sum_{j\in\mathcal{N}_i} \operatorname{sgn}(\omega_j(t) - \omega_i(t)) + \dot{g}_i(t) + \dot{b}_i(t)$. Because $\|\dot{g}_i(t)\|_\infty$ and $\|\dot{b}_i(t)\|_\infty$ are upper bounded, we have that $\|\dot{\omega}_i(t)\|_\infty$ is bounded. Note from (7d) that $\dot{\theta}_i(t) = \beta\sum_{j\in\mathcal{N}_i} \operatorname{sgn}(\theta_j(t) - \theta_i(t)) + \dot{\phi}_i(t)$. Because $\|\dot{\phi}_i(t)\|_\infty$ is upper bounded, we have that $\|\dot{\theta}_i(t)\|_\infty$ is bounded. Then, we have that $\|\dot{\lambda}_i\|_\infty$ is upper bounded for $i \in \mathcal{I}$. Hence, by selecting β^* satisfying $\beta^*\alpha > \sup_{t\geq 0}\|\dot{\lambda}_i\|_\infty + 1$, we then obtain from (16) that

$$\dot{V}_i \leq -\|e_i\|_1. \tag{17}$$

Because $V_i \geq 0$ and $\dot{V}_i \leq 0$, we obtain that e_i is bounded, that is, $e_i \in \mathcal{L}_\infty$. Integrating both sides of (17), we have that $e_i \in \mathcal{L}_2$. It follows from (16) that $\dot{e}_i = -\alpha\hat{\beta}_i \operatorname{sgn}(e_i) - e_i + \dot{\lambda}_i$, we have that $\|\dot{e}_i\|_\infty$ is bounded. According to Barbălat's lemma in [26], we have that e_i with (6b) for $i \in \mathcal{I}$ will converge to zero, i.e., $\lim_{t\to\infty} e_i(t) = \mathbf{0}_m$. That is, we have

$$\nabla_{x_i} f_i(x_i, t) = -\lambda_i \eta_i(t) \tag{18}$$

as $t \to \infty$ for $i \in \mathcal{I}$.

It follows from (5) that $\nabla_{x_i} f_i(x_i, t) = H_i(t)x_i + C_i(t)$ for $i \in \mathcal{I}$. Then, we have $x_i = H_i^{-1}(t)\nabla_{x_i} f_i(x_i, t) - H_i^{-1}(t)C_i(t)$ for $i \in \mathcal{I}$. Because $\lambda_i = \lambda_j$ after a finite time $T\ \forall i, j \in \mathcal{I}$, let the variable $\lambda(t)$ denote the identical Lagrange multiplier such that $\lambda_i = \lambda(t)$ for any $t \geq T$. Because $\lim_{t\to\infty} \nabla_{x_i} f_i(x_i, t) = -\lambda_i \eta_i(t)$, we have $x_i \to -\eta_i(t)H_i^{-1}(t)\lambda(t) - H_i^{-1}(t)C_i(t)$ as $t \to \infty$ for $i \in \mathcal{I}$. Summing up all the weighted x_i as $t \to \infty$, we have

$$\begin{aligned}\sum_{i=1}^n \eta_i(t)x_i \to &-\sum_{i=1}^n \eta_i^2(t)H_i^{-1}(t)\lambda(t)\\ &-\sum_{i=1}^n \eta_i(t)H_i^{-1}(t)C_i(t).\end{aligned} \tag{19}$$

Note from (7) that $g_i(t) = \eta_i(t)H_i^{-1}(t)C_i(t)$ and $\phi_i(t) = \eta_i^2(t)\Lambda(H_i^{-1}(t))$. It follows from the estimated global information by using (7) and (6a) that:

$$\lambda(t) = -\left(\operatorname{diag}\left(\sum_{i=1}^n \phi_i(t)\right)\right)^{-1}\left(\sum_{i=1}^n g_i(t) + \sum_{i=1}^n b_i(t)\right) \tag{20}$$

for $t \geq T$. Substituting (20) to (19), we then have

$$\sum_{i=1}^n \eta_i(t)x_i \to \sum_{i=1}^n b_i(t) \tag{21}$$

as $t \to \infty$.

It follows from Assumption 1 that the optimal solution of the problem (3) can be characterized by the KKT conditions for all time. For $t \geq T$, we obtain the gradient of $L(z(t), t)$ with respect to $z(t)$ as $\nabla_z L(z(t), t) = [(\nabla_{x_1} f_1(x_1, t) + \eta_1(t)\lambda(t))^T, \ldots, (\nabla_{x_n} f_n(x_n, t) + \eta_n(t)\lambda(t))^T, (\sum_{i=1}^n \eta_i(t)x_i - \sum_{i=1}^n b_i(t))^T]^T$, where $z(t) = [x_1^T, \ldots, x_n^T, \lambda^T(t)]^T$. Because $\lambda_i = \lambda(t)$ for $t \geq T$ and $\lim_{t\to\infty} \nabla_{x_i} f_i(x_i, t) = -\lambda_i \eta_i(t)$, we have $\lim_{t\to\infty} \nabla_{x_i} f_i(x_i, t) + \eta_i(t)\lambda(t) = \mathbf{0}_m$. Because $\sum_{i=1}^n \eta_i(t)x_i \to \sum_{i=1}^n b_i(t)$ as $t \to \infty$, we have $\lim_{t\to\infty} \nabla_z L(z(t), t) = \mathbf{0}_{m(n+1)}$. Motivated by the mean-value theorem presented in the proof of [21, Proposition 2], we obtain that the expansion of $\nabla_z L(z(t), t)$ with respect to $z(t)$ around $z^*(t)$ where $\nabla_z L(z^*(t), t) = \mathbf{0}_{m(n+1)}$ can be written as $\nabla_z L(z(t), t) = \nabla_{zz} L(\eta(t), t)(z(t) - z^*(t))$, in which the argument $\eta(t)$ is formed by a convex combination of $z(t)$ and $z^*(t)$. Then, we have $\|z(t) - z^*(t)\|_2 = \|\nabla_{zz}^{-1} L(\eta(t), t)\nabla_z L(z(t), t)\|_2 \leq \|\nabla_{zz}^{-1} L(\eta(t), t)\|_2 \|\nabla_z L(z(t), t)\|_2$. Due to the strong convexity, we have $\|\nabla_{zz}^{-1} L(\eta(t), t)\|_2$ is upper bounded. We then obtain that the variable $z(t)$ will converge to the optimal solution $z^*(t)$ for the optimization problem defined by (3) because $\lim_{t\to\infty} \nabla_z L(z(t), t) = \mathbf{0}_{m(n+1)}$. That is, the states x_i and Lagrange multiplier λ_i of the ith agent using the algorithm (6) with (7) will track the corresponding optimal trajectories as $t \to \infty$, respectively.

It follows from (5) that $\nabla_{x_i} f_i(x_i, t) = H_i(t)x_i + C_i(t)$. Because $\lambda_i = \lambda(t)$ for $t \geq T$, it follows from (18) that $x_i \to -\eta_i(t)H_i^{-1}(t)\lambda(t) - H_i^{-1}(t)C_i(t)$ as $t \to \infty$ for $i \in \mathcal{I}$. Defining $\tilde{e}_i = x_i - (1/n)\sum_{j=1}^n x_j$, we have $\tilde{e}_i = (1/n)\sum_{j=1}^n (x_i - x_j)$. Then, we have

$$\begin{aligned}\tilde{e}_i \to -\frac{1}{n}\sum_{j=1}^n\Big(&\lambda(t)\left(\eta_i(t)H_i^{-1}(t) - \eta_j(t)H_j^{-1}(t)\right)\\ &+ H_i^{-1}(t)C_i(t) - H_j^{-1}(t)C_j(t)\Big)\end{aligned} \tag{22}$$

as $t \to \infty$ for $i \in \mathcal{I}$. According to Assumptions 2 and 3, we have that $|\eta_i(t)|$, $\|H_i(t)\|_\infty$, and $\|C_i(t)\|_\infty$ are bounded.

Then, we have $\tilde{e}_i$ is bounded as $t \to \infty$. Note from (21) that $\sum_{i=1}^{n} \eta_i(t)x_i \to \sum_{i=1}^{n} b_i(t)$ as $t \to \infty$. It follows from (3) that $\eta_i(t)$ is always positive. According to Assumption 3, $\|b_i(t)\|_\infty$ are bounded for $i \in \mathcal{I}$. Then, there is a positive constant $\eta_{\min}$ satisfying $\eta_{\min} < \sup_{t\geq 0}|\eta_i(t)|$ for $i \in \mathcal{I}$ such that $(1/n)\sum_{i=1}^{n}\|x_i\|_\infty \leq (1/n\eta_{\min})\sum_{i=1}^{n}\|b_i(t)\|_\infty$. By invoking the definition of $\tilde{e}_i$, we have $x_i = \tilde{e}_i + (1/n)\sum_{j=1}^{n} x_j$ and hence

$$\|x_i\|_\infty \leq \|\tilde{e}_i\|_\infty + \frac{1}{n\eta_{\min}}\sum_{i=1}^{n}\|b_i(t)\|_\infty. \tag{23}$$

Specifically, substituting $g_i(t) = \eta_i(t)H_i^{-1}(t)C_i(t)$ and $\phi_i(t) = \eta_i^2(t)\Lambda(H_i^{-1}(t))$ to (20), we have

$$\lambda(t) = -\left(\mathrm{diag}\left(\sum_{i=1}^{n}\eta_i^2(t)\Lambda\left(H_i^{-1}(t)\right)\right)\right)^{-1} \times \left(\sum_{i=1}^{n}\eta_i(t)H_i^{-1}(t)C_i(t) + \sum_{i=1}^{n} b_i(t)\right). \tag{24}$$

Substituting (24) to (22), we have

$$\begin{aligned}\|\tilde{e}_i\|_\infty \leq \frac{1}{n}&\left\|\left(\mathrm{diag}\left(\sum_{i=1}^{n}\eta_i^2(t)\Lambda\left(H_i^{-1}(t)\right)\right)\right)^{-1}\right\|_\infty\\ &\times\left(\sum_{i=1}^{n}\left\|\eta_i(t)H_i^{-1}(t)C_i(t)\right\|_\infty + \sum_{i=1}^{n}\|b_i(t)\|_\infty\right)\\ &\times\sum_{j=1}^{n}\left\|\left(\eta_i(t)H_i^{-1}(t) - \eta_j(t)H_j^{-1}(t)\right)\right\|_\infty\\ &+\frac{1}{n}\sum_{j=1}^{n}\left\|H_i^{-1}(t)C_i(t) - H_j^{-1}(t)C_j(t)\right\|_\infty.\end{aligned} \tag{25}$$

Note from (25) that $\|\tilde{e}_i\|_\infty$ is bounded. Also, note that $\sum_{i=1}^{n}\sup_{t\geq 0}\|b_i(t)\|_\infty$ is bounded. There exists a constant $x_{\max} \geq \sup_{t\geq 0}\|\tilde{e}_i\|_\infty + (1/n\eta_{\min})\sum_{i=1}^{n}\sup_{t\geq 0}\|b_i(t)\|_\infty$ such that $\|x_i\|_\infty \leq x_{\max}$ for all time. ∎

Remark 3: Both of the algorithms in [24] and (6) employ the distributed average tracking method (7) to estimate certain global information to guarantee the consensus of Lagrange multipliers. The algorithm (6), however, adopts a different method to update the states and Lagrange multipliers. The algorithms in [24] are able to address the problem (3) with nonidentical constant Hessians and identical time-varying Hessians. In contrast, the proposed algorithm (6) can address the general case of nonidentical time-varying Hessians.

Remark 4: Different from the existing Laplacian-gradient algorithm dealing with optimal resource allocation problems as in [27] and so on, where each agent requires exchanging gradient information, the algorithm (6) with (7) is able to address the time-varying optimal resource allocation problem (3) for privacy-sensitive situations where each individual agent is allowed to access only the estimated information. In addition, the methods in [27] focus on static optimization problems while the proposed algorithm (6) with (7) can address the optimization problem with time-varying cost functions.

Remark 5: Given that all states are upper bounded, the algorithm (6) will also be feasible when a linear inequality constraint containing (23) is imposed on the optimization problem (3).

Remark 6: To avoid the singularity in (6a), the minimum initial value of $\theta_i(t)$ for $i \in \mathcal{I}$ is supposed to be positive. Letting $\theta_1(0) > 0$, for example, be the minimum initial variable, it follows that $\beta\sum_{j\in\mathcal{N}_1}\mathrm{sgn}(\theta_j(0)-\theta_1(0)) > 0$ and hence, $\dot{\psi}_i(t) > 0$. Noting that $\phi_i(t) = \eta_i^2(t)\Lambda(H_i^{-1}(t))$ and $H_i(t)$ is a diagonal matrix with positive elements, we have $\dot{\phi}_i(t) > 0$. Then, it follows from (7d) that $\dot{\theta}_i(t) = \dot{\psi}_i(t) + \dot{\phi}_i(t) > 0$, which implies that θ_1 will increase. Because all θ_is will reach consensus such that $\|\theta_i(t) - (1/n)\sum_{i=1}^{n}\phi_i(t)\|_2 = 0$ for all $t \geq T$ and $\phi_i(t) > 0$, we have that θ_i of each agent will be positive all the time. Hence, there will be no singularity problem in (6a).

IV. Case Study for Hose Transportation Using Multiple Quadrotors

In this section, we aim at showing that the result of Theorem 1 for the time-varying optimal resource allocation problem (3) subject to nonidentical time-varying Hessians can be employed for multiple quadrotors to cooperatively transport a hose.

The collaboration of multiple quadrotors has received growing attention, and the multiquadrotor system is becoming a promising robotic platform for aerial transportation due to the simplicity, practicality, and agility (see [28]–[31]). In particular, multiple quadrotors are employed to transport hoses in [32] and [33] or fabrics in [34] for practical applications in recent years. As for the hose transportation using multiple quadrotors in [32] and [33], a catenary-based method is proposed in [32] to calculate the desired space configuration among all quadrotors, where each quadrotor towing the hose will undertake the equal load and hence, have the same energy consumption. Different from the case in [32] that all quadrotors have the same energy consumption, the aim of this section is to employ nonidentical quadrotors to transport a hose while minimizing the total energy consumption of the team. By taking into consideration the different energy cost function of each quadrotor and employing the result of Theorem 1, each quadrotor is able to determine the optimal thrust that it should undertake, which leads to reducing the total energy consumption of the team. Then, the calculated optimal thrust can be used by the method in [32] to determine the desired space configuration of the team.

It is common that all working quadrotors have different energy consumption functions due to the different configuration and payload for applications, such as the spraying system in Fig. 1, the recovery system for fixed-wing UAVs in [30], and the fabrics transportation system in [34]. Therefore, to reduce the total energy consumption of the team, it is to be hoped that quadrotors are able to find the optimal thrust according to their own energy consumption functions and hence, adjust the space configuration by the method in [32] with the help of tension sensors.

The hoses in Fig. 1 can be considered deformable linear objects as discussed in [32] and [33]. The quasistationary

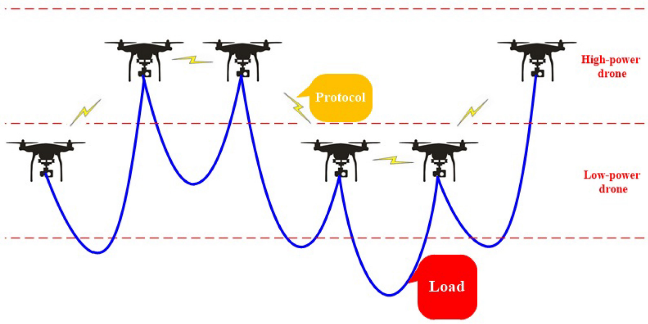

Fig. 1. Illustration of a spraying system using multiple quadrotors.

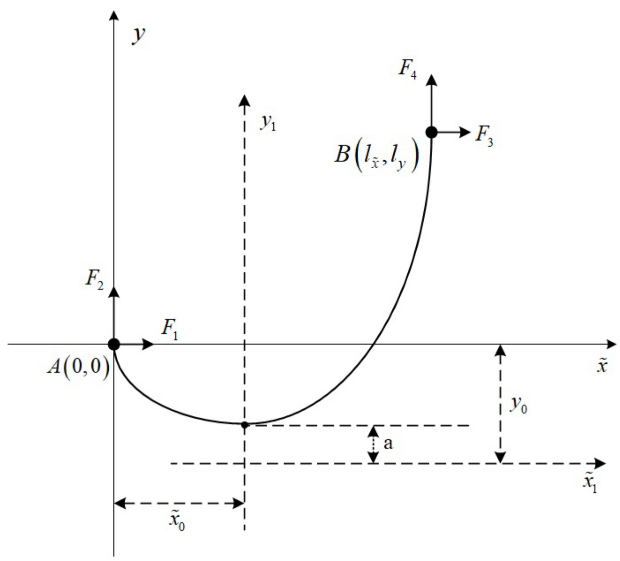

Fig. 2. Model of a catenary curve between two points A and B.

dynamics of hoses hanging from multiple quadrotors has been formulated in [32] and [35] by employing a catenary curve model. In particular, the catenary curve model between two distinct nodes A and B is illustrated in Fig. 2, where we have the geometric relationship $y - y_0 = a \cdot \cosh([\tilde{x} - \tilde{x}_0/a])$ with respect to the reference frame $(\tilde{x}, y)$. Here, the parameter $a \in \mathbb{R}$ is the ratio between T_0 and $\hat{\omega}$, i.e., $a = (T_0/\hat{\omega})$, where $\hat{\omega} \in \mathbb{R}_{++}$ is the weight of unit length of the catenary hoses and $T_0 \in \mathbb{R}$ is the horizontal tension of the catenary, $\tilde{x}_0 \in \mathbb{R}$ is the distance from the lowest point of the catenary to the reference frame $(\tilde{x}, y)$ in the horizon direction, $y_0 \in \mathbb{R}$ is the sum of a and the distance from the lowest point of the catenary to the reference frame $(\tilde{x}, y)$ in the vertical direction, and $\cosh(\cdot)$ is the hyperbolic cosine function. As discussed in [32], the tension forces on nodes A and B can be expressed by

$$\begin{cases} F_1 = -\frac{\hat{\omega} \cdot l_{\tilde{x}}}{2 \cdot \lambda_0} \\ F_2 = \frac{\hat{\omega}}{2}\left(-l_y \cdot \coth(\lambda_0) + L_0\right) \\ F_3 = -F_1 \\ F_4 = \hat{\omega} \cdot L_0 - F_2 \end{cases} \tag{26}$$

where $L_0 \in \mathbb{R}_{++}$ denotes the total length of the catenary in Fig. 2, $l_{\tilde{x}}, l_y \in \mathbb{R}$ denote the distances between A and B in the reference $(\tilde{x}, y)$, $\coth(\cdot)$ is the hyperbolic cotangent function, and $\lambda_0 \in \mathbb{R}_+$ satisfies

$$\lambda_0 = \begin{cases} 10^6, & \text{if } \left(l_{\tilde{x}}^2 + l_y^2\right) = 0 \\ \sqrt{3\left(\frac{L_0^2 - l_y^2}{l_{\tilde{x}}^2} - 1\right)}, & \text{if } \left(L_0^2 \geq l_{\tilde{x}}^2 + l_y^2\right). \end{cases}$$

Note from (26) that the tension forces $F_1, F_2, F_3, F_4 \in \mathbb{R}$ are functions related to $l_{\tilde{x}}$ and l_y. Based on the catenary theory, Estévez *et al.* [32] and Suzuki *et al.* [33] have verified that it is feasible to estimate the positions of quadrotors towing hoses by measuring the tension forces, which means that the tension forces that each quadrotor provides can be employed to determine its relative position. That is, once we obtain the optimal thrust that each quadrotor should undertake, the desired flying position can be calculated by using the aforementioned catenary theory-based method.

On the other hand, as discussed in [36], the minimum power $P \in \mathbb{R}_{++}$ that the quadrotors expend when moving forward in the air is given by

$$P = F\left(v\sin(\sigma) + \frac{2F}{\pi n_r D_r^2 \hat{\rho} \sqrt{[v\cos(\sigma)]^2 + [v\sin(\sigma) + v_{in}]^2}}\right) \tag{27}$$

where $F \in \mathbb{R}$ is the thrust force generated on the propellers, $\hat{\rho} \in \mathbb{R}$ is the density of the surrounding air, $\sigma \in \mathbb{R}$ is the angle of attack for steady flight, $\sin(\cdot)$ and $\cos(\cdot)$ are the sine and cosine functions, respectively, the positive integer n_r is the number of rotors with diameter $D_r \in \mathbb{R}$, $v \in \mathbb{R}$ is the total free stream speed, i.e., the translation speed in addition with wind velocity, and $v_{in} \in \mathbb{R}$ is the induced velocity. In addition, according to the dynamics about quadrotors presented in [36] and [37], we have that the required thrust F_t to sustain the flying height and forward velocity is formulated by $F_t = \sqrt{F_w^2 + F_h^2}$, where $F_t \in \mathbb{R}_+$ and $F_w \in \mathbb{R}$ satisfying $F_w = (m_q + m_p)g$ denote the total weight, including the quadrotor and payload, where $m_q \in \mathbb{R}_{++}$ and $m_p \in \mathbb{R}_+$ are the mass of the quadrotor and payload, respectively, and $g \in \mathbb{R}$ denotes the gravitational acceleration, and $F_h \in \mathbb{R}$ satisfying $F_h = (1/2)\hat{\rho} A_f C_d v_h^2$ denotes the drag force caused by the airflow in the horizontal direction, where $C_d \in \mathbb{R}$ is the drag coefficient, $v_h \in \mathbb{R}$ denotes the air speed, and $A_f \in \mathbb{R}$ is the projected area perpendicular to the air speed v_h. Generally speaking, the thrust generated by a quadrotor exactly balances the gravity and drag forces due to the translation motion and air flow. That is, we have

$$F = F_t = \sqrt{F_w^2 + F_h^2} \tag{28}$$

for a quadrotor when it is moving forward in the air. Accordingly, to optimize the total energy consumption for multiple quadrotors towing a hose, we need to solve the following time-varying optimal resource allocation problem:

$$\min \sum_{i=1}^{n} f_i(x_i, t) \quad \text{s.t.} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} F_{t,i}(t) \tag{29}$$

where $x_i \in \mathbb{R}$ denotes the thrust generated by the ith quadrotor, $f_i(x_i, t) = (1/2)a_i(t)x_i^2 + \hat{c}_i(t)x_i$ denotes the power consumption derived from (27) for the ith quadrotor, in which $a_i(t)$ and $\hat{c}_i(t)$ are the corresponding coefficients, $F_{t,i}(t)$ is the required thrust to counter the gravity and drag forces for the ith quadrotor with payload, and n is the number of quadrotors. Because the cost functions in (29) satisfy Assumption 2, $a_i(t)$, $\hat{c}_i(t)$, and their derivatives for $i \in \mathcal{I}$ in practice are bounded, and $\eta_i(t) = 1$, it follows from Theorem 1 that the time-varying optimal resource allocation problem (29) for the transportation of hoses can be solved by using the algorithm (6) with (7). Here, it should be noted that (23) becomes

$$\|x_i\|_\infty \le \|\tilde{e}_i\|_\infty + \frac{1}{n}\sum_{i=1}^{n}\left\|F_{t,i}(t)\right\|_\infty \tag{30}$$

where $\tilde{e}_i$ is given by (22). Specifically, as for problem (29), we have $\eta_i(t) = 1$ and $H_i(t) = a_i(t)$ for $i \in \mathcal{I}$. Then, (22) becomes

$$\begin{aligned}\tilde{e}_i \to & -\frac{1}{n}\lambda(t)\sum_{j=1}^{n}\left(a_i^{-1}(t) - a_j^{-1}(t)\right)\\ & -\frac{1}{n}\sum_{j=1}^{n}\left(a_i^{-1}(t)\hat{c}_i(t) - a_j^{-1}(t)\hat{c}_j(t)\right).\end{aligned} \tag{31}$$

Substituting $\phi_i(t) = a_i^{-1}(t)$ and $g_i(t) = a_i^{-1}(t)\hat{c}_i(t)$ to (20), we have

$$\lambda(t) = -\left(\sum_{i=1}^{n} a_i^{-1}(t)\right)^{-1}\left(\sum_{i=1}^{n} a_i^{-1}(t)\hat{c}_i(t) + \sum_{i=1}^{n} F_{t,i}(t)\right). \tag{32}$$

Substituting (32) to (31), we have

$$\begin{aligned}\tilde{e}_i \le & \frac{1}{n}\left(\sum_{i=1}^{n} a_i^{-1}(t)\right)^{-1}\left(\sum_{i=1}^{n} a_i^{-1}(t)|\hat{c}_i(t)| + \sum_{i=1}^{n} |F_{t,i}(t)|\right)\\ & \times \sum_{j=1}^{n}\left|a_i^{-1}(t) - a_j^{-1}(t)\right|\\ & + \frac{1}{n}\sum_{j=1}^{n}\left|a_i^{-1}(t)\hat{c}_i(t) - a_j^{-1}(t)\hat{c}_j(t)\right|.\end{aligned} \tag{33}$$

Hence, to employ the result of Theorem 1, we impose a stricter assumption that the thrust that each quadrotor can provide is always greater than the upper bound given by (30) where $\tilde{e}_i$ satisfies (33). Furthermore, once the optimal thrust that each quadrotor should undertake is determined, the team of these quadrotors will move to the corresponding desired position calculated by the method in [32] to reduce the total energy consumption.

In what follows, we consider the problem of hose transportation using multiple quadrotors as shown in Fig. 1. Because the key of the optimal hose transportation problem is to find and track the time-varying desired thrust that each quadrotor should undertake, we focus on dealing with the time-varying optimal resource allocation problem (29). The optimal strategy (6) with (7) ensures that each quadrotor actively determines its desired optimal thrust in a distributed way. A numerical simulation is performed to demonstrate the effectiveness of the proposed distributed continuous-time algorithm for the problem (29). Suppose that there are six quadrotors. Fig. 3 shows the connected undirected communication graph among all quadrotors.

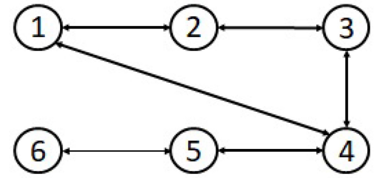

Fig. 3. Communication graph among $n = 6$ agents.

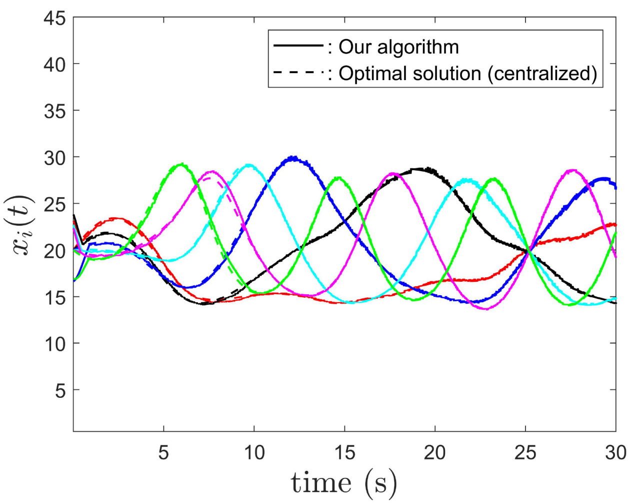

Fig. 4. State trajectories $x_i(t)$ using (6) and (7) under the graph shown in Fig. 3 for time-varying cost functions and constraints. $x_i(t)$ from $i = 1$ to $i = 6$ are indicated by, respectively, red, black, blue, cyan, magenta, and green lines.

In the numerical simulation, we suppose that the cost function of each agent is given by $f_i(x_i, t) = (1/2)a_i(t)x_i^2 + \hat{c}_i(t)x_i$, where $a_i(t) = 3 + \sin(0.25it)$ and $\hat{c}_i(t) = \cos(0.508it)$. All quadrotors should collectively satisfy the coupled time-varying constraint denoted by $\sum_{i=1}^{n} b_i(t)$, where $b_i(t) = 20 + \sin(0.15it)$. We set the parameters of estimator (7) as $\gamma = 0.76$ and $\beta = 0.12$. Because $2 \le a_i(t) \le 4$, $-1 \le \hat{c}_i(t) \le 1$, $19 \le b_i(t) \le 21$, and $n = 6$, we have $(1/3) \le (\sum_{i=1}^{n} a_i^{-1}(t))^{-1} \le (2/3)$, $\sum_{i=1}^{n} a_i^{-1}(t)|\hat{c}_i(t)| \le 3$, $114 \le \sum_{i=1}^{n} |b_i(t)| \le 126$, $\sum_{j=1}^{n} |a_i^{-1}(t) - a_j^{-1}(t)| \le (3/2)$, and $\sum_{j=1}^{n} |a_i^{-1}(t)\hat{c}_i(t) - a_j^{-1}(t)\hat{c}_j(t)| \le 6$. Then, it follows from (33) that $\tilde{e}_i \le (135/6)$ and hence, $x_i \le (135/6) + 21 = 43.5$. The solid lines in Fig. 4 show the trajectories of the states of all quadrotors leveraging algorithm (6) with (7). The dashed lines in Fig. 4 represent the optimal solutions obtained by the MATLAB tool, which is a centralized method. It can be seen from Fig. 4 that the state of each quadrotor converges to the corresponding optimal solution of the optimization problem (29). It can also be seen that the states of all agents are less than 43.5 in Fig. 4. For comparison, we also employ the algorithm in [20] to solve the optimal resource allocation problem (29). As shown in Fig. 5, the state of each quadrotor cannot converge to its corresponding time-varying optimal solution.

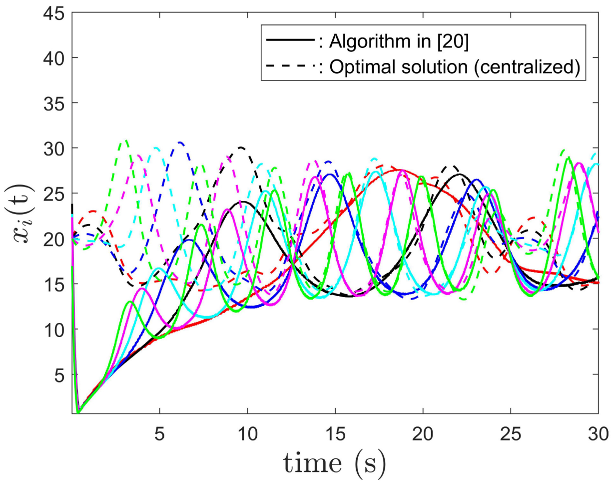

Fig. 5. State trajectories $x_i(t)$ using the algorithm in [20] under the graph shown in Fig. 3 for time-varying cost functions and constraints. $x_i(t)$ from $i = 1$ to $i = 6$ are indicated by, respectively, red, black, blue, cyan, magenta, and green lines.

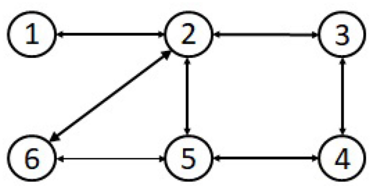

Fig. 6. New communication graph among $n = 6$ agents.

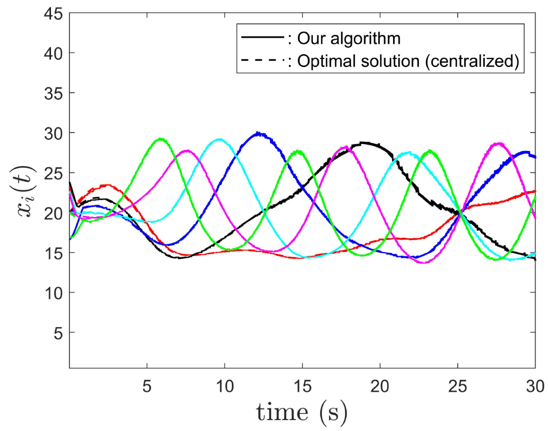

Fig. 7. States trajectories using (6) and (7) under the graph shown in Fig. 6. $x_i(t)$ from $i = 1$ to $i = 6$ are indicated by different colored lines as before.

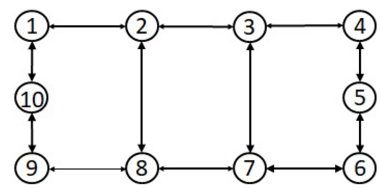

Fig. 8. Communication graph among $n = 10$ agents.

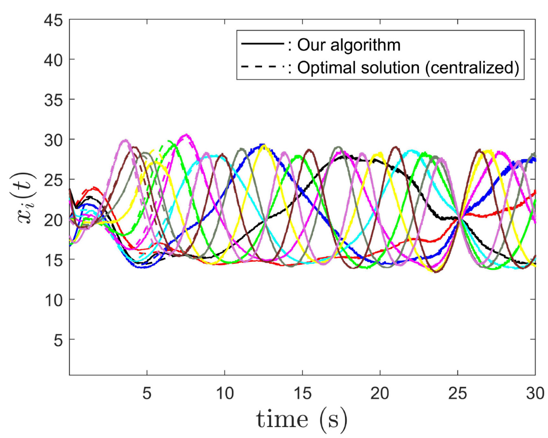

Fig. 9. State trajectories $x_i(t)$ using (6) and (7) under the graph shown in Fig. 8. $x_i(t)$ from $i = 1$ to $i = 10$ are indicated by different colored lines as before.

Because Lemma 1 always holds provided that the graph $\mathcal{G}$ is fixed, undirected, and connected, the number of agents and the connectivity of the graph (e.g., in terms of the diameter) do not affect the convergence of our algorithm. They only affect the convergence rate of the distributed average tracking method while all the agents track the optimal trajectories eventually. The reason is that the upper bound of the finite convergence time for the distributed average tracking method is given by $T \le (1/2)(\alpha - \bar{r}) \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} \|x_i(0) - x_j(0)\|_\infty$, and both the number of the agents and the connectivity of the graph affect the value of $\sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} \|x_i(0) - x_j(0)\|_\infty$. To illustrate, we have realized a simulation under a new communication graph as shown in Fig. 6 where the number of agents is still $n = 6$ but the connectivity of the graph is different from Fig. 3. The state trajectories of all the agents are shown in Fig. 7. Another simulation is realized under the graph as shown in Fig. 8 where the number of agents is $n = 10$. The state trajectories of all the agents are shown in Fig. 9. In addition, because the coefficient matrix $H_i(t)$ is required to be a diagonal matrix such that the elements of the optimization variable are not affected by each other, the behavior of our algorithm will not change with a different size of the optimization variable.

Remark 7: Because the position of each quadrotor can be derived by the tensions on the hose and the position of a neighboring quadrotor according to [32] and [33], the key of determining the optimal flying position is to calculate the optimal thrust that each quadrotor should undertake. Therefore, by assuming that each quadrotor is able to fly to the desired position steadily with external disturbance, which has already been achieved by some reliable flight control software, such as PX4 and Crazyflie, our algorithm focuses on how to calculate the desired optimal thrust in a distributed way. That is, $x_i(t)$ in Figs. 4 and 5 is the optimal desired thrust that each quadrotor should undertake rather than the desired position.

Remark 8: The key of carrying out a realistic and complete simulation is to build a model that can give the final states or shape of the catenary curve quickly for every quasi-stationary state of quadrotors with different tensions, which has been discussed in [32]. That is, once the desired thrust for each quadrotor is determined, the desired formation of quadrotors can be provided to all the quadrotors for transporting the hoses. Here, the quasistationary dynamics of the chain of vehicles are not neglected but modeled by a multicatenary system where the forces exerted on the quadrotors can be calculated based on the theory of catenary curve. In addition, to employ our algorithm, the detailed cost functions of all quadrotors over any arbitrary time horizon are necessary.

V. Conclusion

In this article, we have proposed a distributed continuous-time algorithm for the optimal resource allocation problem under quadratic time-varying cost functions subject to nonidentical time-varying Hessians and a time-varying coupled equality constraint. We have employed an estimator driven from the distributed average tracking idea to estimate certain global information involving all local cost functions and the equality constraint. By leveraging an adaptive gain scheme and the estimated global information, the proposed distributed continuous-time algorithm has been proved to converge to the time-varying optimal solutions under some mild assumptions. The proposed algorithm has been shown to deal with the problem of transporting hoses using multiple quadrotors. The performance of the proposed algorithm has been demonstrated in a numerical simulation. Noting that our proposed algorithms need to employ a distributed estimator driven from the average tracking idea to track the time-varying optimal Lagrange multiplier, how to design finite-time distributed average tracking algorithms is the primary difficulty for the distributed time-varying optimal resource allocation problem under a directed graph. While there are a few works dealing with the distributed average tracking problem under directed graphs, to the best of our knowledge, the existing distributed average tracking algorithms under directed graphs can just guarantee a bounded tracking error or have restrictive assumptions which make them not applicable to our problem. We will consider the time-varying optimization problem under directed graphs by means of developing the distributed average tracking algorithms that can guarantee zero tracking error in the future.

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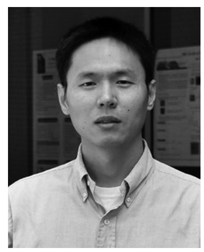

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