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Distributed Continuous-Time Optimization for Networked Lagrangian Systems with Time-Varying Cost Functions Under Fixed Graphs

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Abstract— **In this paper, the distributed time-varying optimization problem is addressed for networked Lagrangian systems with parametric uncertainties. Usually, in the literature, to address some distributed control problems for nonlinear systems, a networked virtual system is constructed, and a tracking algorithm is designed such that the agents' physical states tracks the virtual states. It is worth pointing out that such an idea requires the exchange of the virtual states and hence necessitates communication among the group. In addition, due to the complexities of the Lagrangian dynamics and the distributed time-varying optimization problem, there exist significant challenges. This paper proposes a distributed time-varying optimization algorithm achieving zero optimum-tracking error for the networked Lagrangian agents without the communication requirement. The main idea behind the proposed algorithm is to construct a dynamic system for each agent to generate a reference velocity using absolute and relative physical state measurements with no exchange of virtual states needed, and to design adaptive controllers for Lagrangian systems such that the physical states are able to track the reference velocities and hence the optimal trajectory. The algorithm introduces mutual feedback between reference systems and local controllers via physical states/measurements and is amenable to implementation via local onboard sensing in a communication unfriendly environment.**

I. Introduction

In distributed optimization of networked systems, each member has a local cost function, and the goal is to cooperatively minimize the sum of the all the local cost functions. A number of distributed optimization algorithms have been presented in the literature. See [1] and the references therein for instance. These results (e.g., [1] and the references therein) usually assume fixed local cost functions for the agents. However, the local cost functions are time varying in many practical applications, which reflects the fact that the optimal point might be changing over time and forms an optimal trajectory. Hence, it is meaningful to investigate the distributed time-varying optimization problem.

In the literature, there are extensive distributed discrete-time algorithms that solve the time-varying optimization problem. See [2]–[5] for examples. There usually exist bounded convergence errors to the optimal trajectory by using the discrete-time algorithms. There is another body of literature on distributed continuous-time optimization algorithms with time-varying cost functions. These distributed continuous-time optimization algorithms have various applications in practice. One application lies in the coordination of a team of robots, where each robot's dynamics are described by differential equations and the team objective is to track an optimal trajectory defined by all the team members' cost functions. For instance, by constructing a quadratic objective function for each agent, the distributed time-varying optimization algorithms can be applied to solve the distributed average tracking of multi-agent systems, where each agent aims to track the average of all the reference signals. A few distributed time-varying optimization algorithms are established for single-integrator agents [6], [7] and double-integrator agents [8]. In reality, a broad class of robots can be modeled by Lagrangian systems, for example, the planar elbow manipulator and autonomous vehicles [9]. The Lagrangian systems are nonlinear and more complicated than single- and double-integrator systems, which are the focus of this paper. The complexity of the nonlinear Lagrangian dynamics creates more challenges to solve the distributed time-varying optimization problem.

Some results addressing distributed time-invariant coordination problems (e.g., consensus) for agents with nonlinear dynamics introduce distributed observers or virtual systems at a higher level, where the agents communicate their observer states (virtual states independent of the agents' physical states/measurements) with neighbors to achieve consensus. Then control algorithms are designed for the agents to track the virtual states (serving as reference trajectories). However, due to the lack of physical states/feedback (e.g., agent positions) in the observers, the reference trajectories generated by such an approach do not explicitly take into account the physical agents' interaction with the environment and their capability. Also, such an approach cannot be implemented based on local measurements via onboard sensors without communication in a communication unfriendly environment.

In this paper, we propose a communication-free distributed time-varying optimization algorithm for networked Lagrangian agents with parametric uncertainties. The main idea of the proposed algorithm is constructing a dynamic system for each agent, which is driven by the physical states instead of virtual states between neighbors and generates a reference velocity, and then designing adaptive controllers such that the agents' physical states track their reference velocities, and hence the optimal trajectory. The algorithm introduces mutual influence/feedback between reference sys-

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tems and local controllers via physical states/measurements and is amenable to implementation via local onboard sensing in a communication unfriendly environment. Due to the coupling and mutual influence of the constructed dynamic systems and the agents' dynamics, there are significant new challenges in the convergence analysis. In particular, the constructed dynamic system is rewritten as a coupled and perturbed networked second-order system by taking the tracking errors between agents' velocity and their own reference states as disturbances. Due to the use of the signum function in the construction of the constructed dynamic systems, the coupled and perturbed networked system has disturbances inside and outside the signum function, and the general input-to-state stability analysis might not be directly applicable. This requires novel rigorous analysis on the impact of disturbance on the optimum-tracking performance of the perturbed system. To this end, this paper carefully examines the perturbed system, and obtains that the optimum-tracking error remains bounded if the disturbances are bounded in a certain sense and converges to zero if the disturbances converge to zero (See Proposition 2 for details). These intermediate results facilitate the convergence analysis of the proposed algorithm for the networked Lagrangian agents.

Comparison with Related Works. The works [10]–[12] focus on solving the distributed time-invariant optimization problem for networked Lagrangian agents. They follow the aforementioned distributed observer idea which rely on the exchange of virtual states between neighbors. The work [10] also considers the case of time-invariant cost functions with additive uncertainties modeled by time-dependent functions, and nonzero bounded optimum-tracking errors are achieved. In contrast, the proposed algorithm in this paper solves the optimization problem with time-varying cost functions, which is not addressed in [11], [12]. Compared with [10], the problem considered in this paper is more general and can be solved with zero optimum-tracking error. More importantly, the proposed algorithm in this paper relies purely on physical states without the need for exchange of virtual states and can be implemented in a communication unfriendly application. In contrast, the communication of virtual states between neighbors is necessary in [10]–[12]. The structure of the proposed algorithm is inspired by [13], where the consensus and leader-following tracking of networked Lagrangian systems are addressed. However, the problem considered in this paper is more complex and challenging, and includes the consensus and leader-following tracking of networked agents as special cases. While the construction of the dynamic system is partially inspired by [8], the results there cannot be directly applied to solve the problem considered in this paper due to the complexity of the Lagrangian dynamics.

II. Preliminaries

A. Notations

Throughout this paper, let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_+$ denote the sets of all real numbers, all nonnegative real numbers and all positive real numbers, respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$, and for a real number $x \in \mathbb{R}$, $|x|$ denotes the absolute value of x. The transpose of matrix A is denoted by A^T. For a given vector $x = [x_1, \dots, x_p]^T \in \mathbb{R}^p$, define $\|x\|_1 = \sum_{i=1}^{p} |x_i|$, $\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_p|^2}$, and $\|x\|_\infty = \max_{i=1,\dots,p} |x_i|$. For a symmetric matrix $A \in \mathbb{R}^{p\times p}$, let $\lambda_1(A) \leq \cdots \leq \lambda_p(A)$ denote its eigenvalues. The Kronecker product of matrices A and B is denoted by $A \otimes B$. For a vector $x \in \mathbb{R}^p$, define $\mathrm{sgn}(x) = [\mathrm{sgn}(x_1), \dots, \mathrm{sgn}(x_p)]^T$ where $\mathrm{sgn}(x_i) = 1$ if $x_i > 0$, $\mathrm{sgn}(x_i) = 0$ if $x_i = 0$, and $\mathrm{sgn}(x_i) = -1$ if $x_i < 0$. Let $\mathbf{0}_{m\times n} \in \mathbb{R}^{m\times n}$ and $\mathbf{1}_{m\times n} \in \mathbb{R}^{m\times n}$ denote the $m \times n$ dimensional zero and all-ones matrix, respectively, and for simplicity, let $\mathbf{0}_m = \mathbf{0}_{m\times 1}$ and let $\mathbf{1}_m = \mathbf{1}_{m\times 1}$. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. Define $\mathcal{L}^p_\infty = \left\{x : \mathbb{R}_{\geq 0} \to \mathbb{R}^p \mid \sup_{t\in\mathbb{R}_{\geq 0}} \|x(t)\|_\infty < \infty\right\}$ and $\mathcal{L}^p_2 = \left\{x : \mathbb{R}_{\geq 0} \to \mathbb{R}^p \mid \sqrt{\int_0^\infty u^T(t)u(t)\mathrm{d}t} < \infty\right\}$. For a time-varying function $f : \mathbb{R}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, its gradient, denoted by $\nabla f(q,t) \in \mathbb{R}^p$ with $q \in \mathbb{R}^p$ and $t \in \mathbb{R}_{\geq 0}$, is the partial derivative of $f(q,t)$ with respect to q, and its Hessian, denoted by $H(q,t) \in \mathbb{R}^{p\times p}$, is the partial derivative of the gradient $\nabla f(q,t)$ with respect to q.

B. Graph Theory

For a multi-agent system consisting of N agents, the interaction topology can be modeled by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \dots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the node set and edge set, respectively. An edge denoted by $(i,j) \in \mathcal{E}$, means that agent i and j can obtain information from each other. In an undirected graph, the edges (i,j) and (j,i) are equivalent. It is assumed that $(i,i) \notin \mathcal{E}$. The neighbor set of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ of the graph G is defined such that $a_{ij} = 1$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. For an undirected graph, $a_{ij} = a_{ji}$. The Laplacian matrix $L = [L_{ij}] \in \mathbb{R}^{N\times N}$ associated with the adjacency matrix A is defined as $L_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$ and $L_{ij} = -a_{ij}$ for $i \neq j$. By arbitrarily assigning an orientation for every edge in $\mathcal{G}$, let $B = [B_{ij}] \in \mathbb{R}^{N\times|\mathcal{E}|}$ denote the incidence matrix associated with graph $\mathcal{G}$, where $B_{ij} = -1$ if edge e_j leaves node i, $B_{ij} = 1$ if it enters node i, and $B_{ij} = 0$ otherwise. It holds that $L = BB^T$.

An undirected path between node i_1 and i_k is a sequence of edges of the form (i_1, i_2), (i_2, i_3), $\dots$, (i_{k-1}, i_k), where $i_k \in \mathcal{V}$. A connected graph means that there exists an undirected path between any pair of nodes in $\mathcal{V}$.

Assumption 1: The graph $\mathcal{G}$ is connected.

C. Agents' Dynamics

In this paper, we consider N Lagrangian systems, and the interaction topology among these agents is characterized as the graph $\mathcal{G}$. The equations of motion of the i-th Lagrangian system can be described by [9]

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i \quad (1)$$

where $q_i \in \mathbb{R}^p$ is the generalized position (or configuration), $M_i(q_i) \in \mathbb{R}^{p\times p}$ is the inertia matrix, $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{p\times p}$

is the Coriolis and centrifugal matrix, $g_i(q_i) \in \mathbb{R}^p$ is the gravitational torque, and $\tau_i \in \mathbb{R}^p$ is the exerted control torque. Three well-known properties associated with the dynamics (1) are listed as follows [9], [14].

Property 1: The inertial matrix $M_i(q_i)$ is symmetric and uniformly positive definite, and there exist positive constants $k_{\bar{C}}$ and $k_{\bar{g}}$ such that $\|C_i(q_i, \dot{q}_i)\|_2 \leq k_{\bar{C}} \|\dot{q}_i\|_2$ and $\|g_i(q_i)\|_2 \leq k_{\bar{g}}$, $\forall i \in \mathcal{V}$.

Property 2: The Coriolis and centrifugal matrix $C_i(q_i, \dot{q}_i)$ can be suitably chosen such that the matrix $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew-symmetric.

Property 3: The dynamics (1) depend linearly on an unknown constant parameter vector $\vartheta_i \in \mathbb{R}^m$, that is, for any $x, y \in \mathbb{R}^p$, it holds that

$$M_i(q_i)x + C_i(q_i, \dot{q}_i)y + g_i(q_i) = Y_i(q_i, \dot{q}_i, y, x)\vartheta_i, \quad (2)$$

where $Y_i(q_i, \dot{q}_i, y, x)$ is the regressor matrix.

III. Problem Statement

In the distributed time-varying optimization problem, each Lagrangian agent aims to cooperatively track the optimal trajectory determined by the group cost function. Let $q^*(t) \in \mathbb{R}^p$ denote the optimal trajectory, which is defined as

$$q^*(t) = \arg\min \sum_{i=1}^{N} f_i[q_i(t), t], \quad \text{s.t.} \quad q_i(t) = q_j(t)\ \forall i \neq j,$$

where $f_i[q_i(t), t] : \mathbb{R}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the local cost function associated with agent $i \in \mathcal{V}$. In the rest of the paper, it is assumed that $q^* \in \mathcal{L}_\infty^p$. This assumption is satisfied in most applications in practice. It is assumed that $f_i[q(t), t]$ is known only to agent i. The goal is to design τ_i for the system (1) such that all agents cooperatively optimize the group cost function $\sum_{i=1}^N f_i[q_i(t), t]$. That is, design τ_i for each agent i such that $q_i(t)$ is capable of tracking $q^*(t)$, i.e., $\lim_{t\to\infty}[q_i(t) - q^*(t)] = \mathbf{0}_p$, $\forall i \in \mathcal{V}$. We make the following assumptions on the objective functions.

Assumption 2: Each cost function $f_i(q_i, t)$, $i \in \mathcal{V}$, is twice continuously differentiable both in $q_i \in \mathbb{R}^p$ and t, and strongly convex in q_i and uniformly in t. That is, $H_i(q_i, t)$ is always positive definite and bounded below by $\underline{m}$ for all $q_i \in \mathbb{R}^p$ and uniformly in t, i.e., $\|H_i(q_i, t)\|_2 \geq \underline{m}$ $\forall i \in \mathcal{V}$. In addition, each $H_i(q_i, t)$ is upper-bounded, i.e., $\|H_i(q_i, t)\|_2 \leq \bar{m}$ $\forall i \in \mathcal{V}$.

Assumption 3: The Hessian matrices satisfy $H_i(q_i, t) = H_j(q_j, t)$ $\forall i, j \in \mathcal{V}$.

Assumption 4: For each agent $i \in \mathcal{V}$, $\frac{\partial^2}{\partial t^2}\nabla f_i(q_i, t)$, $\frac{\partial^2}{\partial q_i^2}\nabla f_i(q_i, t)$ and $\frac{\partial^2}{\partial t \partial q_i}\nabla f_i$ exist. In addition, if agent i's position q_i, $i \in \mathcal{V}$, is bounded, then $\frac{\partial}{\partial t}\nabla f_i(q_i, t)$, $\frac{\partial^2}{\partial t^2}\nabla f_i(q_i, t)$, $\frac{\partial^2}{\partial q_i^2}\nabla f_i(q_i, t)$ and $\frac{\partial^2}{\partial t \partial q_i}\nabla f_i$ are all bounded.

In Assumption 2, the uniform strong convexity of the objective functions guarantees that the optimal trajectory q^* is unique for all $t \geq 0$, and it also ensures that $H_i(q_i, t)$ $\forall i \in \mathcal{V}$ is invertible for all t. The upper-boundedness of the Hessian matrix is equivalent to the Lipschitz continuity of the gradient $\nabla f_i(q_i, t)$. In Assumption 4, one sufficient condition for the existence of $\frac{\partial}{\partial t}\nabla f_i(q_i, t)$, $\frac{\partial^2}{\partial t^2}\nabla f_i(q_i, t)$, $\frac{\partial^2}{\partial q_i^2}\nabla f_i(q_i, t)$ and $\frac{\partial^2}{\partial t \partial q_i}\nabla f_i$, can be that each cost function $f_i(q_i, t)$, $i \in \mathcal{V}$, is at least three times continuously differentiable in q_i and t. Assumptions 2-4 are some similar/same assumptions that are used in prior related works [5], [8], [15].

Lemma 1: [16] Let $f(x) : \mathbb{R}^p \to \mathbb{R}$ be a continuously differentiable convex function with respect to x. The function $f(x)$ is minimized at x^* if and only if $\nabla f(x^*) = \mathbf{0}_p$.

IV. Distributed Time-Varying Optimization for Networked Lagrangian Agents

For each agent $i \in \mathcal{V}$, define a differentiable vector $v_i \in \mathbb{R}^p$ by a dynamic system

$$\begin{aligned} \dot{v}_i = &- \sum_{j \in \mathcal{N}_i} \left[\alpha(q_i - q_j) + \beta(\dot{q}_i - \dot{q}_j)\right] \\ &- \gamma \sum_{j \in \mathcal{N}_i} \operatorname{sgn}\left[\alpha(q_i - q_j) + \beta(\dot{q}_i - \dot{q}_j)\right] + \varphi_i, \end{aligned} \quad (3)$$

where α and β are some positive constants to be determined, and φ_i is defined by

$$\varphi_i = -\dot{F}_i(q_i, t) - H_i(q_i, t)\nabla f_i(q_i, t), \quad (4)$$

with

$$F_i(q_i, t) = H_i^{-1}(q_i, t)\left[\frac{\partial}{\partial t}\nabla f_i(q_i, t) + \nabla f_i(q_i, t)\right]. \quad (5)$$

Note that Assumptions 2 and 4 guarrantee the existence of φ_i, $i \in \mathcal{V}$. Define

$$s_i = \dot{q}_i - v_i. \quad (6)$$

The adaptive controller for the Lagrangian system (1) is given by

$$\tau_i = -K_i s_i + Y_i(q_i, \dot{q}_i, v_i, \dot{v}_i)\hat{\vartheta}_i, \quad (7)$$

$$\dot{\hat{\vartheta}}_i = -\Gamma_i Y_i^T(q_i, \dot{q}_i, v_i, \dot{v}_i)s_i, \quad (8)$$

where K_i and Γ_i are symmetric positive definite matrices, and $\hat{\vartheta}_i$ is the estimate of ϑ_i. In the algorithm (7)-(8), the signal v_i can be regarded as the desired reference velocity for each agent i, and the adaptive controller (7)-(8) is used to drive each agent's velocity $\dot{q}_i$ to track its local v_i, and in the meantime, q_i to track the optimal trajectory.

Remark 1: It is worth emphasizing that the algorithm (3)-(8) does not rely on exchange of virtual variables between neighbors. Especially, the dynamic system (3) is driven by agents' physical state information, i.e., q_i, $\dot{q}_i$, $q_i - q_j$ and $\dot{q}_i - \dot{q}_j$. Such design excludes the usage of communication channels, and can be implemented by onboard sensors. This feature distinguishes this algorithm from existing results on distributed optimization of networked Lagrangian systems, e.g., [10]–[12], where inter-agent communication is required. In addition, the algorithm (7)-(8) with $\dot{v}_i$ defined in (3) addresses the distributed time-varying optimization problem with zero optimum-tracking error, while the works [11], [12] are limited to distributed time-invariant optimization, and the work [10] only addresses a special case of time-varying cost functions with nonzero bounded optimum-tracking errors.

Assumption 5: For any $i,j \in \mathcal{V}$, there exist positive constants c_1 and c_2 such that $\|\varphi_i - \varphi_j\|_1 \leq c_1(\|q_i - q_j\|_1 + \|\dot{q}_i - \dot{q}_j\|_1) + c_2$.

Remark 2: Assumptions 2-5 can be satisfied in many situations in practice. If the cost function are constructed as $f_i(q_i,t) = \|q_i(t) - r_i(t)\|_2^2$ where $q_i(t) \in \mathbb{R}^p$ and $r_i(t) \in \mathbb{R}^p$ are agent i's position and local reference signal, respectively, the distributed time-varying optimization algorithms can be applied to address the distributed average tracking of networked agents, which has found applications in region following formation control [17] and coordinated path planning [18]. Note that Assumption 2 holds trivially from the above construction of $f_i(q_i,t)$. Also, the boundedness assumptions of r_i, $\dot{r}_i$ and $\ddot{r}_i$ are commonly placed when dealing with the distributed average tracking of networked agents [19], and such boundedness assumptions implies that Assumptions 4 and 5 hold. In addition, when the cost functions have a slightly more general form as $f_i(q_i,t) = \|\rho q_i + g_i(t)\|_2^2$, where $\rho \in \mathbb{R}_+$ and $g_i(t)$ is a time-varying function, which is a commonly used cost function for energy minimization [15], [19], Assumptions 4 and 5 are satisfied under the boundedness assumption of $g_i(t)$, $\dot{g}_i(t)$ and $\ddot{g}_i(t)$. It is also worth pointing out that under Assumption 3, the value of the constants c_1 and c_2 in Assumption 5 depend mostly on the structure of the cost functions and their state-independent parts.

Using the definition of s_i in (6), the dynamic system (3) can be rewritten as

$$\dot{q}_i = v_i + s_i \tag{9}$$

$$\begin{aligned}\dot{v}_i = &-\sum_{j\in\mathcal{N}_i}\left[\alpha(q_i - q_j) + \beta(v_i - v_j + s_i - s_j)\right] + \varphi_i \\ &- \gamma\sum_{j\in\mathcal{N}_i}\mathrm{sgn}\left[\alpha(q_i - q_j) + \beta(v_i - v_j + s_i - s_j)\right].\end{aligned} \tag{10}$$

Proposition 1: Consider a group of N agents, and their interaction is described by the graph $\mathcal{G}$. Each agent's dynamics are given by (9)-(10). Suppose that Assumptions 1-5 hold. Let α and β be chosen such that $\alpha > \frac{2k}{\lambda_2(L)}$ and $\beta > \frac{3k+2\sqrt{k[\alpha\lambda_2(L)+2k]+4\alpha\lambda_2(L)-k}}{4\alpha\lambda_2(L)-k}\alpha$ with $k = c_1 p\lambda_N(L)(N-1)^2|\mathcal{E}|$, and γ be chosen such that $\gamma > c_2(N-1)^2|\mathcal{E}|$. Then, the following two statements hold.

1) If $s_i \in \mathcal{L}_\infty^p \cap \mathcal{L}_2^p$ $\forall i \in \mathcal{V}$, it holds that $q_i - q^* \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$.
2) If $s_i \in \mathcal{L}_\infty^p \cap \mathcal{L}_2^p$ and $s_i(t) \to \mathbf{0}_p$ $\forall i \in \mathcal{V}$ as $t \to \infty$, it holds that $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$.

Proof: The proof of statements is divided into two steps: the coordination step and the optimum-tracking step. In the coordination step, it is proved that the coordination errors, $q_i - \frac{1}{N}\sum_{j=1}^N q_j$ and $v_i - \frac{1}{N}\sum_{j=1}^N v_j$, are bounded and convergent to zero if s_i $\forall i \in \mathcal{V}$ are bounded and convergent to zero, respectively. In the optimum-tracking step, it is proved that $\sum_{j=1}^N \nabla f_j(q_j,t) \in \mathcal{L}_\infty^p$ if $s_j \in \mathcal{L}_2^p$ $\forall i \in \mathcal{V}$, and $\sum_{j=1}^N \nabla f_j(q_j,t) \to \mathbf{0}_p$ as $t \to 0$ if $s_j \in \mathcal{L}_2^p$ and $s_i \to \mathbf{0}_p$ $\forall i \in \mathcal{V}$. Hence, the statements follow by combining these two steps.

First, consider the coordination step. Let q, v, s and φ be the column stack vectors of q_i, v_i, s_i and φ_i, respectively. Define $x = (M \otimes I_p)q$ and $y = (M \otimes I_p)v$, where $M = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$. Then it holds that

$$\dot{x} = y + (M \otimes I_p)s \tag{11}$$

$$\begin{aligned}\dot{y} = &-(L \otimes I_p)(\alpha x + \beta y + \beta s) + (M \otimes I_p)\varphi \\ &- \gamma(B \otimes I_p)\mathrm{sgn}[(B^T \otimes I_p)(\alpha x + \beta y + \beta s)].\end{aligned} \tag{12}$$

Define the function $V = \frac{1}{2}[x^T\, y^T]P[x^T\, y^T]^T$ with $P = \begin{bmatrix} 2\alpha\beta L & \alpha I_N \\ \alpha I_N & \beta I_N \end{bmatrix} \otimes I_p$. Note that the function V is positive definite if $\frac{\alpha}{\beta^2} < 2\lambda_2(L)$. Taking the derivative along the solution of (11)-(12) yields $\dot{V} = U_1 + U_2$ where $U_1 = -\alpha^2 x^T(L \otimes I_p)x - y^T[(\beta^2 L - \alpha I_N) \otimes I_p]y + 2\alpha\beta x^T(L \otimes I_p)s - \beta^2 y^T(L \otimes I_p)s + \alpha y^T(M \otimes I_p)s$ and $U_2 = (\alpha x^T + \beta y^T)(M \otimes I_p)\varphi - \gamma(\alpha x^T + \beta y^T)(B \otimes I_p)\mathrm{sgn}\left[\left(B^T \otimes I_p\right)(\alpha x + \beta y + \beta s)\right]$.

Consider the term U_1. For notational simplicity, let $z = \alpha x + \beta y$ and $\xi = \left[x^T, y^T\right]^T$. It holds that

$$\begin{aligned}U_1 \leq &-\alpha^2\lambda_2(L)\|x\|_2^2 - \left[\beta^2\lambda_2(L) - \alpha\right]\|y\|_2^2 \\ &- \beta^2 y^T(L \otimes I_p)s + 2\alpha\beta x^T(L \otimes I_p)s + \alpha y^T(M \otimes I_p)s \\ \leq &-X^T Q_1 X + c_M\sqrt{2Np}\,\|\xi\|_2\|s\|_\infty,\end{aligned}$$

where $X = [\|x\|_2, \|y\|_2]^T$, $Q_1 = \mathrm{diag}\{\alpha^2\lambda_2(L), \beta^2\lambda_2(L) - \alpha\}$, and $c_M = \max\left\{2\alpha\beta\lambda_N(L)\sqrt{Np}, \beta^2\lambda_N(L)\sqrt{Np} + \alpha\right\}$.

Consider the term U_2. Let $z_i = \alpha x_i + \beta y_i$, $\mathcal{P} = \{1,\ldots,p\}$, and $z_{i,k}$ and $s_{i,k}$ be the k-th entry in vector z_i and s_i. It holds that $-\gamma z^T(B \otimes I_p)\mathrm{sgn}\left[\left(B^T \otimes I_p\right)(z + \beta s)\right] = -\gamma\sum_{k\in\mathcal{P}}\sum_{(i,j)\in\mathcal{E}}\Lambda_{i,j}^k$, where $\Lambda_{i,j}^k = (z_{i,k} - z_{j,k})\mathrm{sgn}[z_{i,k} - z_{j,k} + \beta(s_{i,k} - s_{j,k})]$. For any $k \in \mathcal{P}$, define $\mathcal{E}_0^k = \left\{(i,j) \in \mathcal{E} \mid z_{i,k} - z_{j,k} + \beta(s_{i,k} - s_{j,k}) = 0\right\}$. Note that $\Lambda_{i,j}^k = 0$ if $(i,j) \in \mathcal{E}_0^k$. Then, it holds that $-\gamma z^T(B \otimes I_p)\mathrm{sgn}\left[\left(B^T \otimes I_p\right)(z + \beta s)\right] = -\gamma\sum_{k\in\mathcal{P}}\sum_{(i,j)\in\mathcal{E}\setminus\mathcal{E}_0^k}\Lambda_{i,j}^k$. For any $(i,j) \in \mathcal{E}\setminus\mathcal{E}_0^k$, it holds that

$$\begin{aligned}-\gamma\Lambda_{i,j}^k &= -\gamma\frac{(z_{i,k} - z_{j,k})^2 + \beta(z_{i,k} - z_{j,k})(s_{i,k} - s_{j,k})}{|z_{i,k} - z_{j,k} + \beta(s_{i,k} - s_{j,k})|} \\ &\leq -\gamma\big||z_{i,k} - z_{j,k}| - \beta|s_{i,k} - s_{j,k}|\big| + \gamma\beta|s_{i,k} - s_{j,k}|.\end{aligned}$$

Define $\mathcal{E}_+^k = \left\{(i,j) \in \mathcal{E} \mid |z_{i,k} - z_{j,k}| \geq \beta|s_{i,k} - s_{j,k}|\right\}$ and $\mathcal{E}_-^k = \left\{(i,j) \in \mathcal{E} \mid |z_{i,k} - z_{j,k}| < \beta|s_{i,k} - s_{j,k}|\right\}$. Then, it holds that $-\gamma\sum_{(i,j)\in\mathcal{E}\setminus\mathcal{E}_0^k}\Lambda_{i,j}^k \leq -\gamma\sum_{(i,j)\in\mathcal{E}}|z_{i,k} - z_{j,k}| + 2\gamma\beta\sum_{(i,j)\in\mathcal{E}}|s_{i,k} - s_{j,k}|$. Hence,

$$\begin{aligned}&-\gamma z^T(B \otimes I_p)\mathrm{sgn}\left[\left(B^T \otimes I_p\right)(z + \beta s)\right] \\ &\leq -\gamma\left\|\left(B^T \otimes I_p\right)z\right\|_1 + 2\gamma\beta\left\|\left(B^T \otimes I_p\right)s\right\|_1.\end{aligned} \tag{13}$$

Note that $\|z\|_1 \leq \frac{(N-1)}{2}\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\|z_i - z_j\|_1 = (N-1)\left\|\left(B^T \otimes I_p\right)z\right\|_1$, and it follows from Assumption 5 that $\|(M \otimes I_p)\varphi\|_\infty \leq \|(M \otimes I_p)\varphi\|_1 \leq (N-1)\left[c_1\left\|\left(B^T \otimes I_p\right)x\right\|_1 + c_1\left\|\left(B^T \otimes I_p\right)y\right\|_1 + c_1\left\|\left(B^T \otimes I_p\right)s\right\|_1 + c_2|\mathcal{E}|\right]$. Then, $z^T(M \otimes I_p)\varphi \leq \|z\|_1\|(M \otimes I_p)\varphi\|_\infty + X^T Q_2 X + k_4\|\xi\|_2\|s\|_\infty + \pi\left\|\left(B^T \otimes I_p\right)z\right\|_1$, where $Q_2 = k[\alpha,\beta]^T\mathbf{1}_2^T$, $k_1 = \sqrt{|\mathcal{E}|p\lambda_N(L)}$, $k_2 = \sqrt{N|\mathcal{E}|p^2\lambda_N(L)}$,

$k_3 = \sqrt{|\mathcal{E}|p\lambda_N(L)(\alpha^2+\beta^2)}$, $k_4 = c_1(N-1)^2Npk_3\left\|B^T\otimes I_p\right\|_\infty$, $k = c_1k_1^2(N-1)^2$, and $\pi = c_2(N-1)^2|\mathcal{E}|$. From (13), it follows that

$$U_2 \leq -(\gamma-\pi)\sqrt{\lambda_2(L)}\left\|z\right\|_2 + 2\gamma\beta k_2\left\|s\right\|_\infty + X^TQ_2X + k_4\left\|\xi\right\|_2\left\|s\right\|_\infty.$$

Hence,

$$\dot{V} \leq -X^TQX + \left(c_M\sqrt{2Np}+k_4\right)\left\|\xi\right\|_2\left\|s\right\|_\infty - (\gamma-\pi)\sqrt{\lambda_2(L)}\left\|z\right\|_2 + 2\gamma\beta k_2\left\|s\right\|_\infty,$$

where $Q = Q_1 - Q_2$. Note that Q is positive definite if $\alpha > \frac{2k}{\lambda_2(L)}$ and $\beta > \frac{3k+2\sqrt{k[\alpha\lambda_2(L)+2k]+4\alpha\lambda_2(L)-k}}{4\alpha\lambda_2(L)-k}\alpha$. Then, $-X^TQX \leq -\lambda_m\left\|X\right\|_2^2$, where λ_m is the smallest eigenvalue of Q, i.e., $\lambda_m = \lambda_1(Q)$. It then holds that $\dot{V} \leq -\lambda_m(1-2\eta)\left\|\xi\right\|_2^2 - (\gamma-\pi)\sqrt{\lambda_2(L)}\left\|z\right\|_2 - 2\lambda_m\eta\left\|\xi\right\|_2^2 + \left(c_M\sqrt{2Np}+k_4\right)\left\|\xi\right\|_2\left\|s\right\|_\infty + 2\gamma\beta k_2\left\|s\right\|_\infty$, where $\eta \in \left(0,\frac{1}{2}\right)$. Note that the term $-2\lambda_m\eta\left\|\xi\right\|_2^2 + \left(c_M\sqrt{2Np}+k_4\right)\left\|\xi\right\|_2\left\|s\right\|_\infty + 2\gamma\beta k_2\left\|s\right\|_\infty$ is nonnegative if $\left\|\xi\right\|_2 \geq \max\left\{d_1\left\|s\right\|_\infty, d_2\sqrt{\left\|s\right\|_\infty}\right\}$, where $d_1 = \frac{c_M\sqrt{2Np}+k_4}{\lambda_m\eta}$ and $d_2 = \sqrt{\frac{2\gamma\beta k_2}{\lambda_m\eta}}$. Note that $\rho(r) = \max\left\{d_1r, d_2\sqrt{r}\right\}$ is a class $\mathcal{K}$ function [22, p. 144]. It holds that

$$\dot{V} \leq -\lambda_m(1-2\eta)\left\|\xi\right\|_2^2 - (\gamma-\pi)\sqrt{\lambda_2(L)}\left\|z\right\|_2 \quad \forall\left\|\xi\right\|_2 \geq \rho(\left\|s\right\|_\infty).$$

It then follows from [22, Theorem 4.19] and the property of the input-to-state stability [22, p. 175] that $x \in \mathcal{L}_\infty^{Np}$ and $y \in \mathcal{L}_\infty^{Np}$ if $s \in \mathcal{L}_\infty^{Np}$, and that $x(t) \to \mathbf{0}_{Np}$ and $y(t) \to \mathbf{0}_{Np}$ as $t \to \infty$ if $s(t) \to \mathbf{0}_{Np}$ as $t \to \infty$. Then, the coordination step is concluded from the definitions of x and y.

Consider the optimum-tracking step. Let $\chi = \sum_{j=1}^N \nabla f_j(q_j,t)$ and $\psi = \sum_{j=1}^N\left[v_j + F_j(q_j,t)\right]$. Define the Lyapunov function candidate $W = \frac{1}{2}\chi^T\chi + \frac{1}{2}\psi^T\psi$. Taking the derivative of W yields that $\dot{W} = -\chi^T\chi + \chi^T\left[\sum_{j=1}^N H_j(q_j,t)s_j\right]$. Note that $\chi^T\left[\sum_{j=1}^N H_j(q_j,t)s_j\right] \leq \frac{1}{2}\left\|\chi\right\|_2^2 + \frac{N\bar{m}^2}{2}\sum_{j=1}^N\left\|s_j\right\|_2^2$. It follows that $\dot{W} \leq -\frac{1}{2}\left\|\chi\right\|_2^2 + \frac{N\bar{m}^2}{2}\sum_{j=1}^N\left\|s_j\right\|_2^2$. Then, it holds that $2\dot{W} + \left\|\chi\right\|_2^2 \leq N\bar{m}^2\sum_{j=1}^N\left\|s_j\right\|_2^2$. Hence, $\int_0^t\left\|s_j\right\|_2^2\mathrm{d}\tau < \infty$ $\forall i \in \mathcal{V}$ $\forall t \geq 0$. Note that $W(t) \geq 0$ $\forall t \geq 0$. It then holds that $2W(t) + \int_0^t\left\|\chi\right\|_2^2\mathrm{d}\tau < \infty$ $\forall t \geq 0$, which implies that $W(t) \in \mathcal{L}_\infty^1$ and $\chi \in \mathcal{L}_2^p$. Hence, $\chi \in \mathcal{L}_\infty^p$ and $\psi \in \mathcal{L}_\infty^p$. Since $q_i - \frac{1}{N}\sum_{j=1}^N q_j \in \mathcal{L}_\infty^p$, then $q_i - q^* \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$.

Recall that $\chi \in \mathcal{L}_2^p$ and $\psi \in \mathcal{L}_\infty^p$. Since $s_j \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$, it holds that $\dot{\chi} \in \mathcal{L}_\infty^p$. It then follows from Barbalat's Lemma [23, p. 125] that $\sum_{j=1}^N\nabla f_j(q_j,t) \to \mathbf{0}_p$ as $t \to \infty$. It follows from the coordination step that $x_i \to x_j$ and $v_i \to v_j$ $\forall i,j \in \mathcal{V}$ as $t \to \infty$. Then, it follows from Lemma 1 that $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$. ■

Proposition 2: Suppose that Assumptions 1-5 hold. For the system (9)-(10), if $s_i \in \mathcal{L}_\infty^p \cap \mathcal{L}_2^p$ $\forall i \in \mathcal{V}$, then all $\varphi_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$.

Proof: The proof follows from Proposition 1 and Assumptions 2 and 4. ■

Theorem 1: Suppose that Assumptions 1-5 hold, α and β be chosen such that $\alpha > \frac{2k}{\lambda_2(L)}$ and $\beta > \frac{3k+2\sqrt{k[\alpha\lambda_2(L)+2k]+4\alpha\lambda_2(L)-k}}{4\alpha\lambda_2(L)-k}\alpha$ with $k = c_1p\lambda_N(L)(N-1)^2|\mathcal{E}|$, and γ be chosen such that $\gamma > c_2(N-1)^2|\mathcal{E}|$. Using the controller (7)-(8) with $\dot{v}_i$ defined in (3) for the networked Lagrangian system (1) yields that $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$.

Proof: For any $i \in \mathcal{V}$, define Lyapunov function candidate $W_i = \frac{1}{2}s_i^TM_i(q_i)s_i + \frac{1}{2}\Delta\vartheta_i^T\Gamma_i^{-1}\Delta\vartheta_i$ with $\Delta\vartheta_i = \hat{\vartheta}_i - \vartheta_i$. By using Property 2, it holds that $\dot{W}_i = -s_i^TK_is_i \leq 0$. Hence, it holds that $s_i \in \mathcal{L}_\infty^p \cap \mathcal{L}_2^p$ and $\hat{\vartheta}_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$.

Since $s_i \in \mathcal{L}_\infty^p \cap \mathcal{L}_2^p$ $\forall i \in \mathcal{V}$, it follows the analysis of Proposition 2 that $\varphi_i \in \mathcal{L}_\infty^p$, $q_i \in \mathcal{L}_\infty^p$ and $v_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$. From (9), it then holds that $\dot{q}_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$. From (10), it holds that $\dot{v}_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$.

Substituting (7) into (1) and using Property 3 yield that $M_i(q_i)\dot{s}_i + C_i(q_i,\dot{q}_i)s_i = -K_is_i + Y_i(q_i,\dot{q}_i,v_i,\dot{v}_i)\Delta\vartheta_i$. Then by using Property 1, it follows that $\dot{s}_i \in \mathcal{L}_\infty^p$ $\forall i \in \mathcal{V}$. It can thus be shown that s_i $\forall i \in \mathcal{V}$ are uniformly continuous. Using Barbalat's lemma [23, p. 125], we obtain that $s_i(t) \to \mathbf{0}_p$ as $t \to \infty$ for any i in $\mathcal{V}$. Then, it follows from Proposition 1 that $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$. ■

Remark 3: The construction of $\dot{v}_i$ in (3) is inspired by the work [8]. However, this work focuses on the networked Lagrangian systems, whose dynamics are more complex compared with single- and double-integrators considered in [8]. In the convergence analysis, it is proved that the optimum-tracking errors are input-to-state stable with respect to the disturbances s_i, $i \in \mathcal{V}$, for the networked system (9)-(10), where there are disturbances inside the system, and hence it is different from the disturbance-free double-integrator model considered in [8], and there are significant technical challenges.

Remark 4: The structure of the algorithm (7)-(8) with $\dot{v}_i$ defined in (3) for networked Lagrangian agents are partially inspired by [13], where the consensus and/or leader-following tracking of networked Lagrangian systems are investigated. In this paper, the distributed time-varying optimization problem is addressed, which is more complex and challenging and includes the consensus and leader-following tracking as special cases. Moreover, while dealing with the distributed time-varying optimization for networked Lagrangian agents, the analysis is quite different from the work [13]. The signum function is used to constructing $\dot{v}_i$, which forms a perturbed closed-loop networked double-integrator systems with s_i as disturbance in the model and inside the nonlinear function (see (9)-(10) for an example). This paper provide rigorous analysis on the performance of the perturbed systems under bounded and convergent disturbances. In addition, during the convergence analysis of the distributed time-varying optimization algorithm, additional analysis steps are required, see the optimum-tracking steps in the proof of Proposition 1 for instance.

Remark 5: As shown in Theorem 1, the lower bound of the design parameters α, β and γ depend on information of the cost functions and the graph. It is worth mentioning that these design parameters are constants, and they can be determined off-line. Once chosen, one can embed them into each agent and implement the proposed algorithm using relative and absolute physical state measurements, which implies that the proposed algorithm can be implemented in a distributed way. In addition, one can use some existing algorithms [24], [25] to estimate the relative values about the cost functions and the graph, and then choose appropriate parameters based on the estimated values. One can also be conservative and select large enough values for γ and α, and then large enough value for β compared with the chosen value of α.

V. Illustrative Examples

In this section, we provide an example to illustrate the results in this paper. We consider a group of ten two-link rovolute joint arms [9, pp. 259-262] ($N = 10$) modeled by (1), which are labeled as $1, \ldots, 10$, and let each agent i have a local cost function $f_i(q_i, t) = [q_{i_1} - i\sin(t)]^2 + [q_{i_2} - i\cos(t)]^2$, where $q_i = [q_{i_1}, q_{i_2}]^T$, $i \in \mathcal{V}$. The interaction among these ten agents are described by a ring topology. For the distributed optimization algorithm (7)-(8) with $\dot{v}_i$ defined in (3), we let $\Gamma_i = 30I_5$ and $K_i = 30I_2$ for any $i \in \mathcal{V}$, $\alpha = 35$, $\beta = 100$, and $\gamma = 1500$. Let $q^* = [q_1^*, q_2^*]^T$ denote the optimal trajectory that minimize the sum of all the local cost functions $f_i(q_i, t)$. The simulation results are presented in Fig. 1, and they show that all the agents track the optimal trajectory, i.e., $q_{i_1} \to q_1^*$ and $q_{i_2} \to q_2^*$ $\forall i \in \mathcal{V}$.

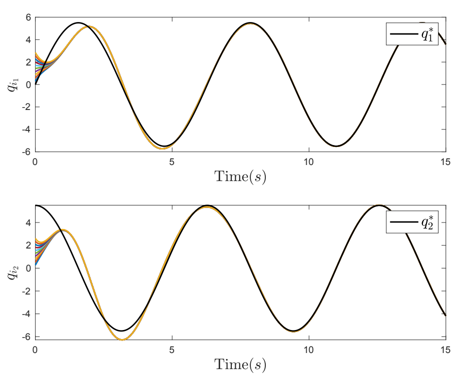

Fig. 1. The black lines are the optimal trajectories for each dimension, and the rest are the trajectories of q_{i_1} and q_{i_2}, $i = 1, \ldots, 10$.

VI. Conclusion

In this paper, a distributed algorithm has been proposed to solve the time-varying optimization problem for networked Lagrangian systems. The proposed algorithm does not need the exchange of any virtual variables and achieves zero-error tracking to the optimal trajectory, which show its advantages over the existing related works.

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