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The Number of Traveling Wave Families in a Running Water with Coriolis Force

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Abstract

In this paper, we study the number of traveling wave families near a shear flow under the influence of Coriolis force, where the traveling speeds lie outside the range of the flow u. Under the β-plane approximation, if the flow u has a critical point at which u attains its minimal (resp. maximal) value, then a unique transitional β value exists in the positive (resp. negative) half-line such that the number of traveling wave families near the shear flow changes suddenly from finite to infinite when β passes through it. On the other hand, if u has no such critical points, then the number is always finite for positive (resp. negative) β values. This is true for general shear flows under mildly technical assumptions, and for a large class of shear flows including a cosine jet $u(y) = \frac{1+\cos(\pi y)}{2}$ (i.e. the sinus profile) and analytic monotone flows unconditionally. The sudden change of the number of traveling wave families indicates that long time dynamics around the shear flow is much richer than the non-rotating case, where no such traveling wave families exist.

1. Introduction

The earth's rotation influences dynamics of large-scale flows significantly. Under the β-plane approximation, the motion for such a flow could be described by 2-D incompressible Euler equation with rotation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla P - \beta y J \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v} = 0, \tag{1.1}$$

where $\boldsymbol{v} = (v_1, v_2)$ is the fluid velocity, P is the pressure, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the rotation matrix, and β is the Rossby number. Here we study the fluid in a *periodic* finite channel $\Omega = D_T = \mathbb{T}_T \times [y_1, y_2]$, $\mathbb{T}_T = \mathbf{R}/(T\mathbf{Z})$ with the following non-permeable boundary condition on $\partial\Omega$:

$$v_2 = 0 \quad \text{on} \quad y = y_1, y_2. \tag{1.2}$$

The β-plane approximation is commonly used for large-scale motions in geophysical fluid dynamics [34,35]. The vorticity form of (1.1) is

$$\partial_t \omega + (\boldsymbol{v} \cdot \nabla)\omega + \beta v_2 = 0, \tag{1.3}$$

where $\omega = \partial_x v_2 - \partial_y v_1$. By the incompressible condition, we introduce the stream function ψ such that $\boldsymbol{v} = \nabla^{\perp}\psi = (\partial_y \psi, -\partial_x \psi)$. Consider the shear flow $(u(y), 0)$, which is a steady solution of (1.3). The linearized equation of (1.3) around $(u(y), 0)$ is

$$\partial_t \omega + u\partial_x \omega - (\beta - u'')\partial_x \psi = 0, \tag{1.4}$$

which was derived in [41].

In the study of long time dynamics near a shear flow, the most rigid case is the nonlinear inviscid damping (to a shear flow), a kind of asymptotic stability. This means that if the initial velocity is taken close enough to the given shear flow in some function space, then the velocity tends asymptotically to a nearby shear flow in this space. The existence of nearby non-shear steady states or traveling waves means that nonlinear inviscid damping (to a shear flow) is not true, and long time dynamics near the shear flow may be richer and fruitful. To understand the richer long time dynamics near the shear flow in this situation, an important step is to clarify whether the number of curves of nearby traveling waves with traveling speeds converging to different points is infinite. Indeed, if the number is finite, then the velocity might tend asymptotically to some nonlinear superpositions of finite such non-shear states when the initial data is taken close to the flow, and quasi-periodic nearby solutions are expected, which indicates new but not so complicated dynamics. If the number is infinite, then the evolutionary velocity might tend asymptotically to superpositions of infinite such non-shear states, and almost periodic nearby solutions potentially exist, which predicts complicated even chaotic long time behavior near the flow. Similar phenomena were observed numerically in the study of Vlasov-Poisson system, a model describing collisionless plasmas [4,5,9,24]. This model shares many similarities with the 2D incompressible Euler equation. By numerical simulations, it was found that for some initial perturbation near homogeneous states, the asymptotic state toward which the system evolves can be described by a superposition of BGK modes [9]. This offers a hint for further numerical study in the 2D Euler case. It is very challenging to study long time dynamics near a shear flow in a fully analytic way when such non-shear steady states or traveling waves exist. The first step towards this direction is to construct nonlinear superpositions of traveling waves as in the Vlasov-Poisson case.

When there is no Coriolis force, long time dynamics near monotone flows is relatively rigid in strong topology, while it is still highly non-trivial to give a mathematical confirmation. A first step is to understand the linearized equation. Orr

[37] observed the linear damping for Couette flow, and Case [6] predicted the decay of velocity for monotone shear flows. Recently, their predictions are confirmed in [14,19,20,28,43,47,48] and are extended to non-monotone flows in [44,45]. Meanwhile, great progress has been made in the study of nonlinear dynamics near shear flows. Bedrossian and Masmoudi [3] proved nonlinear inviscid damping near Couette flow for the initial perturbation in some Gevrey space on $\mathbb{T} \times \mathbf{R}$. Ionescu and Jia [17] extended the above asymptotic stability to a periodic finite channel $\mathbb{T} \times [-1, 1]$ under the assumption that the initial vorticity perturbation is compacted supported in the interior of the channel. Later, nonlinear inviscid damping was proved near a class of Gevrey smooth monotone shear flows in a periodic finite channel if the perturbation is taken in a suitable Gevrey space, where $u''(y)$ is compactly supported [18,33]. It is still challenging to study the long time behavior near general, rough, monotone or non-monotone shear flows. On the other hand, inviscid damping (to a shear flow) depends on the regularity of the perturbation, and the existence of non-shear stationary structures is shown near some specific flows. Lin and Zeng [28] found cats' eyes flows near Couette for $H^{<\frac{3}{2}}$ vorticity perturbation in a periodic finite channel, while no non-shear traveling waves near Couette exist if the regularity is $H^{>\frac{3}{2}}$, in contrast to the linear level, where damping is always true for any initial vorticity in L^2. For Kolmogorov flows, which is non-monotone, Coti Zelati, Elgindi and Widmayer [8] constructed non-shear stationary states near Kolmogorov at analytic regularity on the square torus, while there are no nearby non-shear steady states at regularity H^3 for velocity on a rectangular torus. They also proved that any traveling wave near Poiseuille must be shear for $H^{>5}$ vorticity perturbation in a periodic finite channel.

As indicated in [35], the study of the dynamics of large-scale oceanic or atmospheric motions must include the Coriolis force to be geophysically relevant, and once the Coriolis force is included a host of subtle and fascinating dynamical phenomena are possible. By numerical computation, Kuo [23] found the boundary of barotropic instability for the sinus profile, which is far from linear instability in no Coriolis case. Later, based on Hamiltonian index theory and spectral analysis, Lin, Yang and Zhu theoretically confirmed large parts of the boundary and corrected the rest. New traveling waves, which are purely due to the Coriolis effects, are found near the sinus profile [27]. Barotropic instability of other geophysical shear flows has also attracted much attention. For instance, by looking for the neutral solutions, most of the stability boundary, which is again different from no Coriolis case, of bounded and unbounded Bickley jet is found numerically and analytically in [2,12,16,29,31]. More fruitful geophysical fluid dynamics, such as Rossby wave and baroclinic instability, could be found in [10,23,30,34,35]. On the other hand, similar to no Coriolis case, linear inviscid damping is still true for a large class of flows and moreover, the same decay estimates of the velocity can be obtained for a class of monotone flows [46]. Elgindi and Widmayer [11] viewed Coriolis effect as one mechanism helping to stabilize the motion of an ideal fluid, and proved the almost global stability of the zero solution for the β-plane equation. Global stability of the zero solution is further to be confirmed in [36].

When Coriolis force is involved, long time dynamics near a shear flow becomes fruitful. One of the main reasons is that, compared with the no Coriolis case, there are new traveling waves with fluid trajectories moving in one direction. This paper is devoted to studying the number of such traveling wave families near a general shear flow u under the influence of Coriolis force. Here, a traveling wave family roughly includes the sets of nearby traveling waves with traveling speeds converging to a same number outside the range of the flow, see Definition 2.8 for details. Precisely, we prove that if the flow u has a critical point at which u attains its minimal (resp. maximal) value, then a unique transitional β value β_+ (resp. β_-) exists in the positive (resp. negative) half-line, through which the number of traveling wave families changes suddenly from finite to infinite. The transitional β values are defined in (1.12)–(1.12). If the flow u has no such critical points, then the number of traveling wave families is always finite for positive (resp. negative) β values. This is true for general shear flows under mildly technical assumptions. Based on Hamiltonian structure and index theory, we unconditionally prove the above results for a flow in class $\mathcal{K}^+$, which is defined as follows:

Definition 1.1. A flow u in class $\mathcal{K}^+$ means that $u \in H^4(y_1, y_2)$ is not a constant function, and for any $\beta \in \mathrm{Ran}(u'')$, there exists $u_\beta \in \mathrm{Ran}(u)$ such that $K_\beta = (\beta - u'')/(u - u_\beta)$ is positive and bounded on $[y_1, y_2]$.

A typical example of such a flow is a cosine jet $u(y) = \frac{1+\cos(\pi y)}{2}$, $y \in [-1, 1]$ (i.e. the sinus profile), which was studied in geophysical literature [22,23,35]. For $\beta = 0$ and a general shear flow $u \in C^2([y_1, y_2])$, Rayleigh [38] gave a necessary condition for spectral instability that $u''(y_0) = 0$ for some $y_0 \in (y_1, y_2)$, and even under this condition, Fjrtoft [13] provided a sufficient condition for spectral stability that $(u - u(y_0))u'' \geqq 0$ on (y_1, y_2). For $\beta \neq 0$ and $u \in C^2([y_1, y_2])$, the above two conditions can be extended as $\beta - u''(y_\beta) = 0$ for some $y_\beta \in (y_1, y_2)$ and $(\beta - u'')(u - u(y_\beta)) \leqq 0$ on (y_1, y_2), respectively; see, for example, (6.3)–(6.4) in [23]. For a flow in class $\mathcal{K}^+$, the extended Rayleigh's condition implies that $\beta \in \mathrm{Ran}(u'')$ is necessary for spectral instability, but the flow does not satisfy the extended Fjørtoft's sufficient condition for spectral stability. The sharp condition for spectral stability indeed depends on β and the wave number α, which was obtained in [26] for $\beta = 0$ and in [27] for $\beta \neq 0$.

Consider a class of general shear flows satisfying

$$\textbf{(H1)} \quad u \in H^4(y_1, y_2), \; u'' \neq 0 \text{ on } u\text{'s critical level } \{u' = 0\}.$$

A flow u in class $\mathcal{K}^+$ satisfies the assumption (**H1**). In fact, it is trivial for $0 \notin \mathrm{Ran}(u'')$; if $0 \in \mathrm{Ran}(u'')$ and $y_0 \in [y_1, y_2]$ satisfies $u'(y_0) = 0$ and $u''(y_0) = 0$, then $u(y_0) - u_0 = -\frac{1}{K_0(y_0)} u''(y_0) = 0$. Thus, $\varphi \equiv u - u_0$ solves $\varphi'' + K_0\varphi = 0$, $\varphi(y_0) = \varphi'(y_0) = 0$. Then $u \equiv u_0$, which is a contradiction.

To state our main results with few restrictions, we first consider flows in class $\mathcal{K}^+$, and left the extension to general shear flows satisfying (**H1**) in Section 2.

Theorem 1.2. *Let $\beta \neq 0$ and the flow u be in class $\mathcal{K}^+$.*

(1) If $\{u' = 0\} \cap \{u = u_{\min}\} \neq \emptyset$*, then there exists* $\beta_+ \in (0, \infty)$ *such that there exist at most finitely many traveling wave families near* $(u, 0)$ *for* $\beta \in (0, \beta_+]$*, and infinitely many traveling wave families near* $(u, 0)$ *for* $\beta \in (\beta_+, \infty)$*. Moreover,* β_+ *is specified in (1.12).*
(2) If $\{u' = 0\} \cap \{u = u_{\max}\} \neq \emptyset$*, then there exists* $\beta_- \in (-\infty, 0)$ *such that there exist at most finitely many traveling wave families near* $(u, 0)$ *for* $\beta \in [\beta_-, 0)$*, and infinitely many traveling wave families near* $(u, 0)$ *for* $\beta \in (-\infty, \beta_-)$*. Moreover,* β_- *is specified in (1.12).*
(3) If $\{u' = 0\} \cap \{u = u_{\min}\} = \emptyset$*, then there exist at most finitely many traveling wave families near* $(u, 0)$ *for* $\beta \in (0, \infty)$*.*
(4) If $\{u' = 0\} \cap \{u = u_{\max}\} = \emptyset$*, then there exist at most finitely many traveling wave families near* $(u, 0)$ *for* $\beta \in (-\infty, 0)$*.*

Here, the precise description of a traveling wave family near $(u, 0)$ *is given in Definition 2.8.*

Unless otherwise specified, "near $(u, 0)$" always means "in a (velocity) H^3 neighborhood of $(u, 0)$" in Theorem 1.2 and the rest of this paper, as indicated in Definition 2.4. These traveling wave families do not exist if there is no Coriolis force. By Theorem 1.2, Coriolis force and its magnitude indeed bring fascinating dynamics near the shear flow. On the one hand, for flows having no critical point which is meanwhile a minimal point, the number of traveling wave families is always finite no matter how much magnitude of Coriolis force, which is a mild Coriolis effect. On the other hand, for flows having such a critical point, there is a surprisingly sharp difference, namely, when the Coriolis parameter passes through the transitional point β_+, the number of traveling wave families changes suddenly from finite to infinite. In particular, quasi-periodic solutions to (1.1)–(1.2) can be expected near the shear flow for $\beta \in (0, \beta_+]$, while almost periodic solutions potentially exist for $\beta \in (\beta_+, \infty)$. This could be regarded as a strong Coriolis effect and predicts chaotic long time dynamics near these flows.

The same dynamical phenomena are true for general shear flows under some mildly technical assumptions. The explicit result is stated in Theorem 2.2. For $\beta > 0$, the technical assumption for flows having a critical and meanwhile minimal point is that $u_{\min}$ is not an embedding eigenvalue of the linearized Euler operator for small wave numbers. The assumption for flows having no such critical points is some regularized condition near the endpoints of u. Note that the first spectral assumption has only restriction for one point $u_{\min}$, no matter whether the interior of Ran (u) has embedding eigenvalues. The second assumption is more generic and quite easy to verify. Both the two technical assumptions are used for ruling out eigenvalues' oscillation for Rayleigh-Kuo boundary value problem (BVP) as the parameter c tends to $u_{\min}$, see Subsection 2.2 for details.

Let us give some remarks on properties of such traveling waves near the flow u.

- The traveling waves have fluid trajectories moving in one direction, see (5.6) in the proof of Lemma 2.5. Thus unlike the constructed steady flow near Couette flow in [28], the streamlines here have no cat's eyes structure.

- The traveling waves can be constructed near a smooth shear flow for $H^{\geqq 3}$ (including $H^{>6}$) velocity perturbation when the Coriolis parameter is large, see Corollary 2.6. In contrast, in the case of no Coriolis force, no traveling waves could be found near Couette flow for $H^{>\frac{5}{2}}$ velocity perturbation [28] and near Poiseuille flow for $H^{>6}$ velocity perturbation [8].
- Let $\{u' = 0\} \cap \{u = u_{\min}\} \neq \emptyset$ and $\beta > \frac{9}{8}\kappa_+$. The directions of vertical velocities of the nearby traveling waves might change frequently with small amplitude as the traveling speeds converge to $u_{\min}^-$, see Remark 5.2.

We apply the main results to analytic monotone flows (including Couette flow) and the sinus profile. For an analytic monotone flow, there exist at most finitely many nearby traveling wave families for $\beta \neq 0$, see Corollary 2.3. For the sinus profile, as mentioned above, it is in class $\mathcal{K}^+$, and so applying Theorem 1.2 (1)–(2) we get that $\{u' = 0\} \cap \{u = u_{\min}\} = \{\pm 1\}$, $\{u' = 0\} \cap \{u = u_{\max}\} = \{0\}$, $\beta_+ = \frac{9}{16}\pi^2$, $\beta_- = -\frac{1}{2}\pi^2$, there exist at most finitely many traveling wave families near the sinus profile for $\beta \in [-\frac{1}{2}\pi^2, \frac{9}{16}\pi^2]$, and infinitely many nearby traveling wave families for $\beta \notin [-\frac{1}{2}\pi^2, \frac{9}{16}\pi^2]$. Moreover, we will give a systematical study on the number of isolated real eigenvalues of the linearized Euler operator and traveling wave families near the sinus profile on the whole (α, β)'s region in Section 7 (here α is the wave number in the x-direction), which plays an important role in further study on its long time dynamics. We make a comparison with the previous work in [27]. By Theorem 2.1, the number of isolated real eigenvalues of the linearized Euler operator (i.e. non-resonant modes) determines that of traveling wave families. The explicit number of isolated real eigenvalues in the region $(\alpha, \beta) \in (0, \infty) \times [-\frac{\pi^2}{2}, \frac{\pi^2}{2}]$ can be obtained in [27], but no information can be concluded outside this region, see the discussion below Fig. 4 in [27]. Our new contribution for the sinus profile in this paper is that we calculate the explicit number of isolated real eigenvalues in the remaining area $(\alpha, \beta) \in (0, \infty) \times (-\infty, -\frac{\pi^2}{2}) \cup (\frac{\pi^2}{2}, \infty)$, and thus completely get the number of traveling wave families near the sinus profile on the whole (α, β)'s region. For the sinus profile, the novelty is that we give the asymptotic behavior of the n-th eigenvalue $\lambda_n(c)$ of the Rayleigh-Kuo BVP (2.6) as $c \to 0^-$ for $\beta \in (\frac{\pi^2}{2}, \infty)$ and as $c \to 1^+$ for $\beta \in (-\infty, -\frac{\pi^2}{2})$, from which we find the transitional β values such that the number of traveling wave families changes suddenly from finite to infinite. For general shear flows satisfying (**H1**), the key is to study whether $\lambda_n(c)$ is unbounded from below as c is close to $u_{\min}$ (or $u_{\max}$) in Theorem 2.9 and to rule out the oscillation of $\lambda_n(c)$ in Theorems 2.11-2.13. In this paper, we focus on the description of the eigenvalues of the Rayleigh-Kuo BVP (2.6), which in turn, by Theorem 2.1, yields information on traveling wave families.

The rest of this paper is organized as follows: in Section 2, we extend Theorem 1.2 to general shear flows and give the outline of the proof. In Sections 3-4, we study the asymptotic behavior of the n-th eigenvalue of Rayleigh-Kuo BVP, where we determine the transitional values for the n-th eigenvalue of Rayleigh-Kuo BVP in Section 3, and rule out oscillation of the n-th eigenvalue in Section 4. In Section 5, we establish the correspondence between a traveling wave family and an isolated real eigenvalue of the linearized Euler operator. In Section 6, we prove the main

Theorems 2.2 and 1.2. As a concrete application, we thoroughly study the number of traveling wave families near the sinus profile in the last section.

Notation

We provide the notations that we use in this paper. Let $u_{\min} = \min(u)$ and $u_{\max} = \max(u)$ for $u \in C([y_1, y_2])$. For a shear flow u satisfying (**H1**), we use the following characteristic quantities of the flow. If $\{u' = 0\} \cap \{u = u_{\min}\} \neq \emptyset$, we define

$$\kappa_+ := \min\{u''(y)|y \in [y_1, y_2] \text{ such that } u'(y) = 0 \text{ and } u(y) = u_{\min}\}. \tag{1.5}$$

If $\{u' = 0\} \cap \{u = u_{\max}\} \neq \emptyset$, we define

$$\kappa_- := \max\{u''(y)|y \in [y_1, y_2] \text{ such that } u'(y) = 0 \text{ and } u(y) = u_{\max}\}. \tag{1.6}$$

Note that $\kappa_+ \in (0, \infty)$ and $\kappa_- \in (-\infty, 0)$ in (1.6)–(1.6). In fact, (**H1**) implies $u''(y_0) > 0$ for $y_0 \in A := \{y \in [y_1, y_2]|u'(y) = 0 \text{ and } u(y) = u_{\min}\}$. Then y_0 is an isolated point of A. Thus, A is a finite set and $\kappa_+ \in (0, \infty)$ in (1.6). Similarly, $\kappa_- \in (-\infty, 0)$ in (1.6). Besides (1.6)–(1.6), we define

$$\kappa_+ := \infty, \quad \text{if } \{u' = 0\} \cap \{u = u_{\min}\} = \emptyset, \tag{1.7}$$

$$\kappa_- := -\infty, \quad \text{if } \{u' = 0\} \cap \{u = u_{\max}\} = \emptyset. \tag{1.8}$$

If $\{u = u_{\min}\} \cap (y_1, y_2) \neq \emptyset$, we define

$$\mu_+ := \min\{u''(y)|y \in (y_1, y_2) \text{ such that } u(y) = u_{\min}\}. \tag{1.9}$$

If $\{u = u_{\max}\} \cap (y_1, y_2) \neq \emptyset$, we define

$$\mu_- := \max\{u''(y)|y \in (y_1, y_2) \text{ such that } u(y) = u_{\max}\}. \tag{1.10}$$

Note that $\mu_+ \in [\kappa_+, \infty)$ and $\mu_- \in (-\infty, \kappa_-]$ in (1.10)–(1.10). Then we define

$$\beta_+ := \begin{cases} \min\{\frac{9}{8}\kappa_+, \mu_+\}, & \text{if } \{u = u_{\min}\} \cap (y_1, y_2) \neq \emptyset, \\ \frac{9}{8}\kappa_+, & \text{if } \{u = u_{\min}\} \cap (y_1, y_2) = \emptyset, \end{cases} \tag{1.11}$$

and

$$\beta_- := \begin{cases} \max\{\frac{9}{8}\kappa_-, \mu_-\}, & \text{if } \{u = u_{\max}\} \cap (y_1, y_2) \neq \emptyset, \\ \frac{9}{8}\kappa_-, & \text{if } \{u = u_{\max}\} \cap (y_1, y_2) = \emptyset. \end{cases} \tag{1.12}$$

We denote

$$(\mathbf{E}_+) \qquad u_{\min} \text{ is not an embedding eigenvalue of } \mathcal{R}_{\alpha,\beta}, \tag{1.13}$$

$$(\mathbf{E}_-) \qquad u_{\max} \text{ is not an embedding eigenvalue of } \mathcal{R}_{\alpha,\beta}, \tag{1.14}$$

where $\mathcal{R}_{\alpha,\beta}$ is defined in (2.5). Moreover, we define

$$m_\beta := \begin{cases} \sharp\{a \in (y_1, y_2)|u(a) = u_{\min}, u''(a) - \beta < 0\}, & \text{if } 0 < \beta \leqq \frac{9}{8}\kappa_+, \\ \sharp\{a \in (y_1, y_2)|u(a) = u_{\max}, u''(a) - \beta > 0\}, & \text{if } \frac{9}{8}\kappa_- \leqq \beta < 0, \end{cases} \tag{1.15}$$

and

$$M_\beta := \begin{cases} -\inf\limits_{c\in(-\infty,u_{\min})} \lambda_{m_\beta+1}(c), & \text{if } 0<\beta \leqq \frac{9}{8}\kappa_+, \\ -\inf\limits_{c\in(u_{\max},\infty)} \lambda_{m_\beta+1}(c), & \text{if } \frac{9}{8}\kappa_- \leqq \beta < 0, \end{cases} \tag{1.16}$$

where $\lambda_{m_\beta+1}(c)$ is the $(m_\beta+1)$-th eigenvalue of the Rayleigh-Kuo BVP (2.6).

$\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{Z}^+$ denote the set of all the real numbers, integers and positive integers, respectively. $\sharp(K)$ or $\sharp\, K$ is the cardinality of the set K. Let L be a linear operator from a Banach space X to X. X^* is the dual space of X. $\sigma(L)$, $\sigma_e(L)$ and $\sigma_d(L)$ are the spectrum, essential spectrum and discrete spectrum of the operator L, respectively. For $\psi \in L^2(D_T)$, the Fourier transform of ψ in x is denoted by $\widehat{\psi}$.

2. Extension to general shear flows and outline of the proof

In this section, we first extend the main Theorem 1.2 to general shear flows under mild assumptions, and then discuss our approach in its proof.

2.1. Main results for general shear flows

For a shear flow in $H^4(y_1, y_2)$, we give the exact number of traveling wave families near the flow.

Theorem 2.1. *Now $\alpha = 2\pi/T$, $\beta \neq 0$ and $u \in H^4(y_1, y_2)$. Then $\sharp(\bigcup_{k\geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}))$ is exactly the number of traveling wave families near $(u, 0)$, where $\mathcal{R}_{k\alpha,\beta}$ is defined in (2.5) and the precise description of a traveling wave family near $(u, 0)$ is given in Definition 2.8.*

Then we state our main theorem for a shear flow satisfying (**H1**).

Theorem 2.2. *Let $\beta \neq 0$ and u satisfy* (**H1**).

(1) If $\{u' = 0\} \cap \{u = u_{\min}\} \neq \emptyset$ and $(\mathbf{E}_+)$ holds for every $\alpha \in (0, \sqrt{M_\beta}] \cap \{\frac{2k\pi}{T} | k \in \mathbf{Z}^+\}$ and $\beta \in (0, \frac{9}{8}\kappa_+)$, then there exists $\beta_+ \in (0, \infty)$ such that there exist at most finitely many traveling wave families near $(u, 0)$ for $\beta \in (0, \beta_+)$, and infinitely many traveling wave families near $(u, 0)$ for $\beta \in (\beta_+, \infty)$, where κ_+, $(\mathbf{E}_+)$ and M_β are defined in (1.6), (1.14) and (1.16), respectively. Moreover, β_+ is specified in (1.12).

(2) If $\{u' = 0\} \cap \{u = u_{\max}\} \neq \emptyset$ and $(\mathbf{E}_-)$ holds for every $\alpha \in (0, \sqrt{M_\beta}] \cap \{\frac{2k\pi}{T} | k \in \mathbf{Z}^+\}$ and $\beta \in (\frac{9}{8}\kappa_-, 0)$, then there exists $\beta_- \in (-\infty, 0)$ such that there exist at most finitely many traveling wave families near $(u, 0)$ for $\beta \in (\beta_-, 0)$, and infinitely many traveling wave families near $(u, 0)$ for $\beta \in (-\infty, \beta_-)$, where κ_- and $(\mathbf{E}_-)$ are defined in (1.6) and (1.14), respectively. Moreover, β_- is specified in (1.12).

Assume that $u(y_1) \neq u(y_2)$ and for $i = 1, 2$, there exist $\delta > 0$, $C > 0$ and $m_i > 0$ such that (i) *$u''(y) = \beta_i$ for $y \in (y_i - \delta, y_i + \delta) \cap [y_1, y_2]$ or* (ii) *$C^{-1}|y - y_i|^{m_i} \leqq |u''(y) - \beta_i| \leqq C|y - y_i|^{m_i}$ for $y \in (y_i - \delta, y_i + \delta) \cap [y_1, y_2]$ or* (iii) *$\beta_i u'(y_i)(-1)^i \geqq 0$, where $\beta_i = u''(y_i)$.*

(3) If $\{u' = 0\} \cap \{u = u_{\min}\} = \emptyset$*, then there exist at most finitely many traveling wave families near* $(u, 0)$ *for* $\beta \in (0, \infty)$.

(4) If $\{u' = 0\} \cap \{u = u_{\max}\} = \emptyset$*, then there exist at most finitely many traveling wave families near* $(u, 0)$ *for* $\beta \in (-\infty, 0)$.

Here, the precise description of a traveling wave family near $(u, 0)$ *is given in Definition 2.8.*

As mentioned in the Introduction, ($\mathbf{E}_+$) or ($\mathbf{E}_-$) is "one spectral point" assumption for small wave numbers. Note that if $\frac{2\pi}{T} > \sqrt{M_\beta}$, then $(0, \sqrt{M_\beta}] \cap \{\frac{2k\pi}{T} | k \in \mathbf{Z}^+\} = \emptyset$, and ($\mathbf{E}_\pm$) is not needed in Theorem 2.2 (1)–(2). One of the conditions (i)–(iii) is the "good" endpoints assumption and rather generic. For example, if $u \in C^m([y_1, y_2])$, $m \geqq 3$ and $u^{(k_i)}(y_i) \neq 0$ for some $3 \leqq k_i \leqq m$, then (ii) is true for $m_i = k_i - 2$. Thus, for analytic flows, (ii) holds if $u^{(k_i)}(y_i) \neq 0$ for some $k_i \geqq 3$ and (i) holds otherwise. Applying Theorem 2.2 (3)–(4) to analytic monotone flows, we have the following result:

Corollary 2.3. *Let* u *be an analytic monotone flow:* $u'(y) \neq 0$ *for* $y \in [y_1, y_2]$. *Then there exist at most finitely many traveling wave families near* $(u, 0)$ *for* $\beta \neq 0$.

2.2. Outline and our approach in the proof

Non-parallel steady flows or traveling waves may be bifurcated from a shear flow if the linearized Euler operator has an embedding or isolated real eigenvalues [1,27,28]. Based on the existence of an embedding eigenvalue for a class of monotone shear flows near Couette flow, cat's eyes steady states are bifurcated from these flows [28]. When the Coriolis force is involved, non-parallel traveling waves are bifurcated from the sinus profile on account of the existence of an isolated real eigenvalue [27]. The traveling speeds lie outside the range of the sinus profile and are contiguous to the isolated real eigenvalue.

Now, we consider such bifurcation theorem for general shear flows, namely, using an isolated real eigenvalue of the linearized Euler operator, we prove that such traveling waves can be bifurcated from general shear flows. We use the following concept:

Definition 2.4. $\{\boldsymbol{u}_\varepsilon (x - c_\varepsilon t, y) = (u_\varepsilon (x - c_\varepsilon t, y), v_\varepsilon (x - c_\varepsilon t, y)) | \varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0\}$ is called a set of traveling wave solutions near $(u, 0)$ with traveling speeds converging to c_0, if for each $\varepsilon \in (0, \varepsilon_0)$, $\boldsymbol{u}_\varepsilon (x - c_\varepsilon t, y) = (u_\varepsilon (x - c_\varepsilon t, y), v_\varepsilon (x - c_\varepsilon t, y))$ is a traveling wave solution to (1.1)–(1.2) which has period T in x such that

$$\|(u_\varepsilon, v_\varepsilon) - (u, 0)\|_{H^3(D_T)} \leqq \varepsilon, \tag{2.1}$$

$\|v_\varepsilon\|_{L^2(D_T)} \neq 0$, $c_\varepsilon \notin \text{Ran}(u)$ and $c_\varepsilon \to c_0$.

Now we give the bifurcation result for general shear flows.

Lemma 2.5. *Let $\alpha = 2\pi/T$, $\beta \neq 0$ and $u \in H^4(y_1, y_2)$. Assume that $c_0 \in \bigcup_{k \geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$, where $\mathcal{R}_{k\alpha,\beta}$ is defined in (2.5). Then there exists a set of traveling wave solutions near $(u, 0)$ with traveling speeds converging to c_0. Moreover, we have $u_\varepsilon(x, y) - c_\varepsilon \neq 0$.*

Here, we mention some differences from the construction of traveling waves in the literature. First, the horizontal period of constructed traveling waves in Proposition 7 of [27] is not the given period T, and for the sinus profile, the period of traveling waves is modified to T by adjusting the traveling speed in Theorem 7 of [27]. But the price is an additional condition, namely, the isolated eigenvalue c_0 can not be an extreme point of λ_1 (i.e. $\alpha_0 \neq \sqrt{\Lambda_\beta}$ in Theorem 7 (ii) of [27]), where λ_n is the n-th eigenvalue of (2.6). In Lemma 2.5, we can construct traveling waves for general flows no matter whether c_0 is an extreme point of λ_n, and thus improve the result in Theorem 7 of [27] even for the sinus profile. Second, it is possible that $c_0 \in \sigma_d(\mathcal{R}_{0,\beta}) \cap \mathbf{R}$, which makes it subtle to guarantee that the bifurcated solutions near the flow u is not a shear flow. Thus, the extension of the bifurcation result for the sinus profile in [27] to general shear flows in Lemma 2.5 is still non-trivial, since we have to treat the unsolved case that c_0 is an extreme point of λ_{n_0} for some $n_0 \in \mathbf{Z}^+$ or $c_0 \in \sigma_d(\mathcal{R}_{0,\beta}) \cap \mathbf{R}$. To overcome the difficulty, we carefully modify the flow u to a suitable shear flow, which satisfies that λ_{n_0} is locally monotone near c_0 and $c_0 \notin \sigma_d(\mathcal{R}_{0,\beta}) \cap \mathbf{R}$, and then study the bifurcation at the suitable shear flow. Finally, the minimal horizontal periods of constructed traveling waves are possibly less than $2\pi/\alpha$ if $c_0 \in \sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}$. In fact, the Sturm-Liouville operator $\mathcal{L}_{c_0}$ could indeed have more than one negative eigenvalues (e.g., if $\kappa_+ < \infty$, $\beta > 9\kappa_+/8$ and c_0 is close to $u_{\min}$), where $\mathcal{L}_{c_0}$ is defined in (2.6). In this case, we give sufficient condition to guarantee that the minimal period is $2\pi/\alpha$ in Lemma 5.3. In contrast, the minimal period must be $2\pi/\alpha$ in Theorem 5.1 of [25] and Theorem 1 of [28], since the corresponding Sturm–Liouville operator has only one negative eigenvalue.

Since the isolated real eigenvalue c_0 lies outside the range of the flow u, by a similar proof of Lemma 2.5 we can improve the regularity of traveling waves as follows.

Corollary 2.6. *Let $\alpha = 2\pi/T$, $\beta \neq 0$, $u \in C^\infty([y_1, y_2])$ and $s \geqq 3$. Assume that $c_0 \in \bigcup_{k \geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$, where $\mathcal{R}_{k\alpha,\beta}$ is defined in (2.5). Then the conclusion in Lemma 2.5 holds true with (2.1) replaced by $\|(u_\varepsilon, v_\varepsilon) - (u, 0)\|_{H^s(D_T)} \leqq \varepsilon$.*

One naturally asks whether the assumption $c_0 \in \bigcup_{k \geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$ in Lemma 2.5 is necessary. By studying the asymptotic behavior of traveling speeds and L^2 normalized vertical velocities for nearby traveling waves, we confirm that it is true.

Lemma 2.7. *Let $\alpha = 2\pi/T$, $\beta \neq 0$ and $u \in H^4(y_1, y_2)$. Assume that $\{\boldsymbol{u}_\varepsilon(x - c_\varepsilon t, y) = (u_\varepsilon(x - c_\varepsilon t, y), v_\varepsilon(x - c_\varepsilon t, y)) | \varepsilon \in (0, \varepsilon_0)\}$ is a set of traveling wave solutions near $(u, 0)$ with traveling speeds converging to c_0.*

Then $c_0 \in \bigcup_{k \geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}) \cup \{u_{\min}, u_{\max}\}$, where $\mathcal{R}_{k\alpha,\beta}$ is defined in (2.5). Moreover, if $c_0 \in \bigcup_{k \geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$, then there exists $\varphi_{c_0} \in \ker(\mathcal{G}_{c_0})$

such that

$$\tilde{v}_\varepsilon \longrightarrow \varphi_{c_0} \ \ in \ \ H^2(D_T), \tag{2.2}$$

where the operator $\mathcal{G}_{c_0}$ *is defined by*

$$\mathcal{G}_{c_0} = -\Delta - \frac{\beta - u''(y)}{u(y) - c_0} \ : \ H^2(D_T) \to L^2(D_T) \tag{2.3}$$

with periodic boundary condition in x and Dirichlet boundary condition in y, and $\tilde{v}_\varepsilon = v_\varepsilon / \|v_\varepsilon\|_{L^2(D_T)}$.

The limit function φ_{c_0} in Lemma 2.7 is a superposition of finite normal modes, see Remark 5.1. If $c_0 \in \bigcup_{k\geq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$ in Lemma 2.7, the vertical velocities of the nearby traveling waves have simple asymptotic behavior as seen in (2.2). However, if $c_0 \in \{u_{\min}, u_{\max}\}$, then the asymptotic behavior of vertical velocities might be complicated, see Remark 5.2. The proofs of Lemmas 2.5-2.7 are given in Section 5.

By Lemma 2.7, for any set of traveling waves near $(u, 0)$ with traveling speeds converging to c_0, c_0 must be an isolated real eigenvalue of the linearized Euler operator (besides $u_{\min}$ and $u_{\max}$). By Lemma 2.5, every isolated real eigenvalue is contiguous to the speeds of nearby traveling waves. As the minimal periods of traveling waves in x can be less than $2\pi/\alpha$, there might be two or more sets of traveling wave solutions near $(u, 0)$ with traveling speeds converging to a same isolated real eigenvalue. For example, if $((i+1)\alpha)^2 = -\lambda_{n_i}(c_0)$, λ_{n_i} is monotone near c_0 for $i = 1, 2$, and $(k\alpha)^2 \neq \lambda_n(c_0)$ for $k \notin \{2, 3\}$ and $n \notin \{n_1, n_2\}$, then an application to Lemma 2.5 (see Case 1 in its proof) gives two sets of traveling wave solutions, which has minimal periods π/α and $2\pi/(3\alpha)$ respectively, near $(u, 0)$ with traveling speeds converging to c_0. Moreover, traveling wave solutions could be bifurcated from nearby shear flows, which might induce more sets of traveling wave solutions near $(u, 0)$ with traveling speeds converging to c_0. This suggests us to define a traveling wave family near $(u, 0)$ by an equivalence class as follow:

Definition 2.8. A traveling wave family near $(u, 0)$ is defined by an equivalence class under $\sim$, where if $\{\boldsymbol{u}_{i,\varepsilon}\left(x - c_{i,\varepsilon}t, y\right) = (u_{i,\varepsilon}\left(x - c_{i,\varepsilon}t, y\right), v_{i,\varepsilon}\left(x - c_{i,\varepsilon}t, y\right))|\varepsilon \in (0, \varepsilon_i)\}$, $i = 1, 2$, are two sets of traveling wave solutions near $(u, 0)$ with traveling speeds converging to $c_i \notin \text{Ran}(u)$, then $\{\boldsymbol{u}_{1,\varepsilon}\left(x - c_{1,\varepsilon}t, y\right)|\varepsilon \in (0, \varepsilon_1)\}$ and $\{\boldsymbol{u}_{2,\varepsilon}\left(x - c_{2,\varepsilon}t, y\right)|\varepsilon \in (0, \varepsilon_2)\}$ are equivalent, $\{\boldsymbol{u}_{1,\varepsilon}\left(x - c_{1,\varepsilon}t, y\right)|\varepsilon \in (0, \varepsilon_1)\} \sim \{\boldsymbol{u}_{2,\varepsilon}\left(x - c_{2,\varepsilon}t, y\right)|\varepsilon \in (0, \varepsilon_2)\}$, if $c_1 = c_2$.

By Lemma 2.7, there exists $\varphi_i \in \ker(\mathcal{G}_{c_i})$ such that $\tilde{v}_{i,\varepsilon} \longrightarrow \varphi_i$ in $H^2(D_T)$, where $\tilde{v}_{i,\varepsilon} = v_{i,\varepsilon}/\|v_{i,\varepsilon}\|_{L^2(D_T)}$ and $v_{i,\varepsilon}$, $\varepsilon \in (0, \varepsilon_i)$, are given in Definition 2.8. By Lemmas 2.5 and 2.7 , we obtain the exact number of traveling wave families near a flow $u \in H^4(y_1, y_2)$ in Theorem 2.1.

Thus, the number of isolated real eigenvalues of the linearized Euler operator plays an important role in counting the traveling wave families near the shear flow. In terms of the stream function ψ, (1.4) can be written as $\partial_t \Delta\psi + u\partial_x \Delta\psi + (\beta - u'')\partial_x \psi = 0$. By taking Fourier transform in x, we have that

$$(\partial_y^2 - \alpha^2)\partial_t \widehat{\psi} = i\alpha((u'' - \beta) - u(\partial_y^2 - \alpha^2))\widehat{\psi}. \tag{2.4}$$

For $\alpha > 0$ and $\beta \in \mathbf{R}$, the linearized Euler operator is given by

$$\mathcal{R}_{\alpha,\beta} := -(\partial_y^2 - \alpha^2)^{-1}((u'' - \beta) - u(\partial_y^2 - \alpha^2)). \tag{2.5}$$

Then (2.4) becomes $-\frac{1}{i\alpha}\partial_t \widehat{\psi} = \mathcal{R}_{\alpha,\beta}\widehat{\psi}$. Recall that $\sigma_e(\mathcal{R}_{\alpha,\beta}) = \mathrm{Ran}(u)$. Then the set of isolated real eigenvalues $\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R} \subset (-\infty, u_{\min}) \cup (u_{\max}, \infty)$. Moreover, it is well-known that $\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R} = \emptyset$ if $\beta = 0$; $\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (u_{\max}, \infty) = \emptyset$ if $\beta > 0$; and $\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min}) = \emptyset$ if $\beta < 0$, see [22,27,35,42]. Therefore, we only need to study $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min}))$ for $\beta > 0$ and $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (u_{\max}, \infty))$ for $\beta < 0$. We mainly study $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min}))$ for $\beta > 0$, since the other is similar. $c \in \sigma_d(\mathcal{R}_{\alpha,\beta})$ if and only if its corresponding eigenfunction ψ_c satisfies the Rayleigh-Kuo BVP

$$\mathcal{L}_c\phi := -\phi'' + \frac{u'' - \beta}{u - c}\phi = \lambda\phi, \quad \phi(y_1) = \phi(y_2) = 0, \tag{2.6}$$

where $\phi \in H_0^1 \cap H^2(y_1, y_2)$ and $\lambda = -\alpha^2$. This equation is formulated by KUO [22]. For $c < u_{\min}$, it follows from [39] that the n-th eigenvalue of (2.6) is

$$\begin{aligned} \lambda_n(c) &= \inf_{\dim V_n = n} \sup_{\phi \in H_0^1, \phi \in V_n, \|\phi\|_{L^2} = 1} \langle \mathcal{L}_c\phi, \phi \rangle \qquad (2.7) \\ &= \inf_{\dim V_n = n} \sup_{\phi \in H_0^1, \phi \in V_n, \|\phi\|_{L^2} = 1} \int_{y_1}^{y_2} \left(|\phi'|^2 + \frac{u'' - \beta}{u - c}|\phi|^2 \right) \mathrm{d}y. \end{aligned}$$

In this way, we have that

$$\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min}) = \bigcup_{n \geqq 1} \{c < u_{\min} : \lambda_n(c) = -\alpha^2\}.$$

To determine whether $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min}))$ is finite, we need to study the number of solutions $c < u_{\min}$ such that $\lambda_n(c) = -\alpha^2$ for $n \geqq 1$. Since $\lim_{c \to -\infty} \lambda_n(c) = \frac{n^2}{4}\pi^2 > 0$ by Proposition 4.2 in [27] and $\lambda_n(c)$ is real-analytic on $(-\infty, u_{\min})$, the only possibility such that $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) = \infty$ is that there exists a sequence $\{c_j(\alpha, \beta)\}_{j=1}^{\infty} \subset \bigcup_{n \geqq 1} \{c < u_{\min} : \lambda_n(c) = -\alpha^2\}$ such that $c_j(\alpha, \beta) \to u_{\min}^-$. Thus, the key is to study the asymptotic behavior of $\lambda_n(c)$ as $c \to u_{\min}^-$. We divide that it into two steps.

Step 1. We study how many n's such that $\lambda_n(c) \to -\infty$ as $c \to u_{\min}^-$. We determine a transitional β value such that the number $\sharp\{n \geqq 1 : \lambda_n(c) \to -\infty$ as $c \to u_{\min}^-\}$ changes suddenly from finite to infinite when β passes through it.

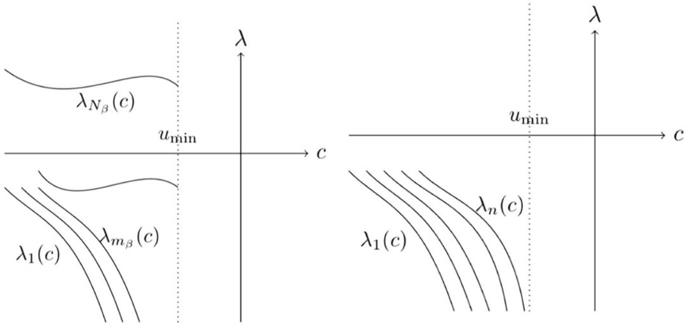
Theorem 2.9. *Let u satisfy* (**H1**). (1) *Let $0 < \beta \leqq \frac{9}{8}\kappa_+$. Then*

(i) $\lim_{c \to u_{\min}^-} \lambda_n(c) = -\infty, \quad 1 \leqq n \leqq m_\beta$;

(ii) $M_\beta < \infty$;

(iii) there exists an integer $N_\beta > m_\beta$ such that $\inf_{c \in (-\infty, u_{\min})} \lambda_{N_\beta}(c) \geqq 0$.

(2) *Let $\frac{9}{8}\kappa_- \leqq \beta < 0$. Then*



$0 < \beta \le \frac{9}{8}\kappa_+$ $\qquad\qquad$ $\beta > \frac{9}{8}\kappa_+$

Fig. 1. The transitional value $\beta = \frac{9}{8}\kappa_+$

(i) $\lim_{c \to u_{\max}^+} \lambda_n(c) = -\infty, \quad 1 \leqq n \leqq m_\beta$;
(ii) $M_\beta < \infty$;
(iii) there exists an integer $N_\beta > m_\beta$ *such that* $\inf_{c \in (u_{\max}, \infty)} \lambda_{N_\beta}(c) \geqq 0$.

(3) *Let* $\beta > \frac{9}{8}\kappa_+$. *Then* $\lim_{c \to u_{\min}^-} \lambda_n(c) = -\infty$ *for* $n \geqq 1$.
(4) *Let* $\beta < \frac{9}{8}\kappa_-$. *Then* $\lim_{c \to u_{\max}^+} \lambda_n(c) = -\infty$ *for* $n \geqq 1$.
Here, $\kappa_\pm$, m_β *and* M_β *are defined in (1.6)–(1.8), (1.15) and (1.16), respectively.*

The transitional value $\beta = \frac{9}{8}\kappa_+$ is illustrated in Fig. 1. We give a simple example to explain why such a transitional value exists. Consider the flow $u = \frac{1}{2}y^2$ on [0, 1] and $\beta > 0$. If $c < 0$ is very close to 0, then the energy quadratic form in (2.7) roughly looks like this:

$$\langle \mathcal{L}_c \phi, \phi \rangle \sim \int_0^1 |\phi'|^2 + \frac{2 - 2\beta}{y^2} |\phi|^2 \, \mathrm{d}y.$$

Thus, if $2 - 2\beta > -\frac{1}{4} \Leftrightarrow \beta < \frac{9}{8}$, by Hardy type inequality (Lemma 3.1) we have $\langle \mathcal{L}_c \phi, \phi \rangle$ is bounded from below for any test functions ϕ with $\phi(0) = 0$. From this formal observation, we may expect $\lambda_1(c)$ is bounded from below. If $2 - 2\beta < -\frac{1}{4} \Leftrightarrow \beta > \frac{9}{8}$, $\langle \mathcal{L}_c \phi, \phi \rangle$ is unbounded from below by looking at the test functions $y^{\frac{1}{2}+\varepsilon}$ with $\varepsilon \to 0^+$. We will construct test functions motivated by the function $y^{\frac{1}{2}}$ to show that all the eigenvalues are unbounded from below.

For general flows, the main idea in the proof of Theorem 2.9 (1)–(2) is to control $\int \frac{u'' - \beta}{u - c} |\phi|^2 \, \mathrm{d}y$ using the L^2 norm of ϕ' near a singular point (see Lemma 3.2), which involves very delicate and careful localized analysis. The transitional β values $\frac{9}{8}\kappa_\pm$ are essentially due to the optimal Hardy type inequality (3.1). The idea in the proof of Theorem 2.9 (3)–(4) is to construct suitable test functions such that the functional in (2.7) converges to $-\infty$ as $c \to u_{\min}^-$ or $u_{\max}^+$, see (3.20). This is inspired by the "eigenfunction" $y^{\frac{1}{2}}$ for the optimal Hardy type equality and a support-separated technique. The proof of Theorem 2.9 is given in Section 3.

Then we give sharp criteria for $\lambda_1(c) \to -\infty$ as $c \to u_{\min}^-$ if $\beta \in [\frac{9}{8}\kappa_-, \frac{9}{8}\kappa_+]$. By Theorem 2.1, the number of traveling wave families is to count the union of $\sharp(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$ for all $k \geqq 1$. Thus, the number of traveling wave families is infinity provided that $\lambda_1(c) \to -\infty$ as $c \to u_{\min}^-$. By Theorem 2.9 (1)–(2), we get the sharp criteria for $\lambda_1(c) \to -\infty$ as $c \to u_{\min}^-$.

Corollary 2.10. *Let u satisfy* (**H1**).

(1) If $\{u = u_{\min}\} \cap (y_1, y_2) \neq \emptyset$, then a transitional β value $\min\{\frac{9}{8}\kappa_+, \mu_+\}$ *exists in* $(0, \frac{9}{8}\kappa_+]$ *such that* $\inf_{c\in(-\infty,u_{\min})} \lambda_1(c) > -\infty$ *for* $\beta \in (0, \min\{\frac{9}{8}\kappa_+, \mu_+\}]$ *and*
$\lim_{c\to u_{\min}^-} \lambda_1(c) = -\infty$ *for* $\beta \in (\min\{\frac{9}{8}\kappa_+, \mu_+\}, \frac{9}{8}\kappa_+]$.

(2) If $\{u = u_{\max}\} \cap (y_1, y_2) \neq \emptyset$, then a transitional β value $\max\{\frac{9}{8}\kappa_-, \mu_-\}$ *exists in* $[\frac{9}{8}\kappa_-, 0)$ *such that* $\inf_{c\in(u_{\max},\infty)} \lambda_1(c) > -\infty$ *for* $\beta \in [\max\{\frac{9}{8}\kappa_-, \mu_-\}, 0)$ *and* $\lim_{c\to u_{\max}^+} \lambda_1(c) = -\infty$ *for* $\beta \in [\frac{9}{8}\kappa_-, \max\{\frac{9}{8}\kappa_-, \mu_-\})$.

(3) If $\{u = u_{\min}\} \cap (y_1, y_2) = \emptyset$, then $\inf_{c\in(-\infty,u_{\min})} \lambda_1(c) > -\infty$ *for* $\beta \in (0, \frac{9}{8}\kappa_+]$.

(4) If $\{u = u_{\max}\} \cap (y_1, y_2) = \emptyset$, then $\inf_{c\in(u_{\max},\infty)} \lambda_1(c) > -\infty$ *for* $\beta \in [\frac{9}{8}\kappa_-, 0)$.

Here, $\kappa_\pm$ and $\mu_\pm$ are defined in (1.6)–(1.10).

Here, a key point for Corollary 2.10 (1) and (3) is that $\inf_{c\in(-\infty,u_{\min})} \lambda_1(c) > -\infty$ if and only if $m_\beta = 0$ and $\beta \leqq \frac{9}{8}\kappa_+$.

Step 2. We rule out the oscillation of $\lambda_n(c)$ as $c \to u_{\min}^-$ (or $c \to u_{\max}^+$). By Theorem 2.9 (1), we get that for $1 \leqq n \leqq m_\beta$, $\lambda_n(c) = -\alpha^2$ has only finite number of solutions c on $(-\infty, u_{\min})$. Moreover, if $n \geqq N_\beta$, no solutions exist for $\lambda_n(c) = -\alpha^2$ on $c \in (-\infty, u_{\min})$. Now, we consider whether $\sharp(\{\lambda_n(c) = -\alpha^2, c \in (-\infty, u_{\min})\}) < \infty$ for $m_\beta < n < N_\beta$. Indeed, we rule out the oscillation of $\lambda_n(c)$ under the spectral assumption ($\mathbf{E}_\pm$), or under the "good" endpoints assumption (i.e. one of the conditions (i)–(iii) in Theorem 2.2), or for flows in class $\mathcal{K}^+$. The oscillation of $\lambda_n(c)$ is illustrated in Fig. 2.

Case 1. Under the spectral assumption, the main argument to rule out oscillation is to prove uniform H^1 bound for corresponding eigenfunctions, and the proof is in Subsection 4.1. In this case, $\frac{9}{8}\kappa_\pm$ are also transitional β values for the number of isolated real eigenvalues of the linearized Euler operator if $|\kappa_\pm| < \infty$.

Theorem 2.11. *Assume that u satisfies* (**H1**) *and* $\alpha > 0$.

(1) If $0 < \beta < \frac{9}{8}\kappa_+$, $0 < \alpha^2 \leqq M_\beta$ *and* ($\mathbf{E}_+$) *holds for this α, then*

$$m_\beta \leqq \sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) < \infty. \tag{2.8}$$

If $0 < \beta \leqq \frac{9}{8}\kappa_+$, *then (2.9) holds for* $\alpha^2 > M_\beta$.

(2) If $\frac{9}{8}\kappa_- < \beta < 0$, $0 < \alpha^2 \leqq M_\beta$ *and* ($\mathbf{E}_-$) *holds for this α, then*

$$m_\beta \leqq \sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (u_{\max}, \infty)) < \infty. \tag{2.9}$$

If $\frac{9}{8}\kappa_- \leqq \beta < 0$, *then (2.9) holds for* $\alpha^2 > M_\beta$.

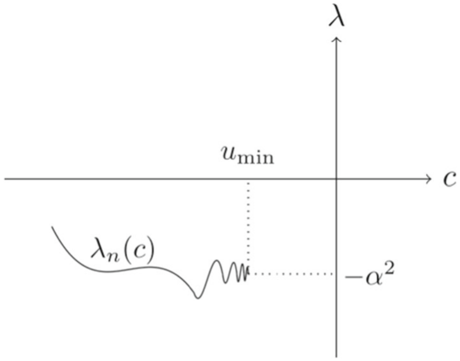


Fig. 2. The oscillation of $\lambda_n(c)$

(3) If $\beta > \frac{9}{8}\kappa_+$, then $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) = \infty$.
(4) If $\beta < \frac{9}{8}\kappa_-$, then $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (u_{\max}, \infty)) = \infty$.

Here, $\kappa_\pm$, $(\mathbf{E}_\pm)$, m_β and M_β are defined in (1.6)–(1.8), (1.14)–(1.14), (1.15) and (1.16), respectively.

In fact, by Theorem 2.9 we have $M_\beta < \infty$ for $0 < \beta \leqq \frac{9}{8}\kappa_+$ or $\frac{9}{8}\kappa_- \leqq \beta < 0$. Here, we focus on sufficient conditions of (2.9) and (2.9), it is unclear whether (2.9) is true for the case $\beta = \frac{9}{8}\kappa_+$ with $0 < \alpha^2 \leqq M_\beta$, or the case $0 < \beta < \frac{9}{8}\kappa_+$ with $0 < \alpha^2 \leqq M_\beta$ but no assumption $(\mathbf{E}_+)$.

Note that Theorem 2.11 (3)–(4) is a direct consequence of Theorem 2.9 (3)–(4).

Case 2. Under the "good" endpoints assumption (i.e. one of the conditions (i)–(iii) in Theorem 2.2), a delicate analysis near the endpoints is involved to rule out oscillation, and the proof is in Subsection 4.2. In this case, we get that no transitional β values exist if $|\kappa_\pm| = \infty$.

Theorem 2.12. *Let $\alpha > 0$ and u satisfy* (**H1**). *Assume that $u(y_1) \neq u(y_2)$, and one of the conditions* (i)–(iii) *in Theorem 2.2 holds. Then*

(1) $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}) < \infty$ for all $\beta \in (0, \infty)$ if and only if $\{u' = 0\} \cap \{u = u_{\min}\} = \emptyset$;
(2) $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}) < \infty$ for all $\beta \in (-\infty, 0)$ if and only if $\{u' = 0\} \cap \{u = u_{\max}\} = \emptyset$.

Consequently, $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}) < \infty$ for all $\beta \in \mathbf{R}$ if and only if $\{u' = 0\} \cap \{u = u_{\min}\} = \{u' = 0\} \cap \{u = u_{\max}\} = \emptyset$.

Note that if $\alpha > \sqrt{M_\beta}$, then (2.9) and (2.9) are true, and the "good" endpoints assumption (i.e. one of the conditions (i)–(iii) in Theorem 2.2) is not needed in Theorem 2.12. Consequently, if $\frac{2\pi}{T} > \sqrt{M_\beta}$, then Theorem 2.2 (3)–(4) hold true without this assumption (see their proof).

Let u be an analytic monotone flow and $\alpha > 0$. Then $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}) < \infty$ for $\beta \neq 0$. This is a corollary of Theorem 2.12, and can also be deduced by the method used in Lemma 3.2 and Theorem 4.1 of [40].

Case 3. For flows in class $\mathcal{K}^+$, the main tools to rule out oscillation are Hamiltonian structure and index formula, and the proof is in Subsection 4.3. This is also the main reason that the spectral and "good" endpoints assumptions can be removed in Theorem 1.2.

Theorem 2.13. *Let u be in class $\mathcal{K}^+$ and $\alpha > 0$. Then $m_\beta \leqq \sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) < \infty$ for $0 < \beta \leqq \frac{9}{8}\kappa_+$; $m_\beta \leqq \sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (u_{\max}, \infty)) < \infty$ for $\frac{9}{8}\kappa_- \leqq \beta < 0$; and $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}) = \infty$ for $\beta \notin [\frac{9}{8}\kappa_-, \frac{9}{8}\kappa_+]$. Here, $\kappa_\pm$ and m_β are defined in (1.6)–(1.8) and (1.15), respectively.*

The idea of the proof is as follows. The linearized equation has Hamiltonian structure and the energy quadratic form has finite negative directions. The key observation is that oscillation of $\lambda_n(c)$ brings infinite times of sign-changes of $\lambda_n'(c)$. This contributes infinite negative directions of quadratic form for non-resonant neutral modes, which is a contradiction to the index formula. Thus, the oscillation of $\lambda_n(c)$ can be ruled out unconditionally for flows in class $\mathcal{K}^+$.

3. Transitional values $\beta = \frac{9}{8}\kappa_\pm$ for the n-th eigenvalue of Rayleigh-Kuo BVP

We begin to study the asymptotic behavior of the n-th eigenvalue $\lambda_n(c)$ of Rayleigh-Kuo BVP. In this section, we focus on the number $\sharp\{n \geqq 1 : \lambda_n(c) \to -\infty$ as $c \to u_{\min}^-$ (or $c \to u_{\max}^+$)$\}$. We prove that the number is finite for $\beta \in [\frac{9}{8}\kappa_-, \frac{9}{8}\kappa_+]$ and it is infinite for $\beta \notin [\frac{9}{8}\kappa_-, \frac{9}{8}\kappa_+]$, which is stated precisely in Theorem 2.9.

3.1. Finite number for $\beta \in [\frac{9}{8}\kappa_-, \frac{9}{8}\kappa_+]$

The optimal constant in the following Hardy type inequality plays an important role in discovering the transitional values $\beta = \frac{9}{8}\kappa_\pm$.

Lemma 3.1. *Let $\phi \in H^1(a, b)$ and $\phi(y_0) = 0$ for some $y_0 \in [a, b]$. Then*

$$\left\| \frac{\phi}{y - y_0} \right\|_{L^2(a,b)}^2 + \frac{\int_a^b |y - y_0|^{-1}\phi(y)^2\, \mathrm{d}y}{\max(b - y_0, y_0 - a)} \leqq 4\|\phi'\|_{L^2(a,b)}^2. \tag{3.1}$$

Here the constant 4 is optimal.

Proof. Suppose that ϕ is real-valued without loss of generality. Let $\varepsilon = 1/\max(b - y_0, y_0 - a)$. First, we consider the integration on $[y_0, b]$ (if $y_0 < b$). Since

$$\left| \frac{\phi(y)^2}{y - y_0} \right| \leqq \frac{\|\phi'\|_{L^2(y_0,y)}^2 (y - y_0)}{y - y_0} = \|\phi'\|_{L^2(y_0,y)}^2 \to 0, \quad as \quad y \to y_0^+, \tag{3.2}$$

we have that

$$\left\|\frac{\phi}{y-y_0}\right\|^2_{L^2(y_0,b)} = -\int_{y_0}^b \phi(y)^2 d\left(\frac{1}{y-y_0}\right) = -\frac{\phi(y)^2}{y-y_0}\Big|_{y_0}^b + \int_{y_0}^b \frac{2\phi(y)\phi'(y)}{y-y_0}\,\mathrm{d}y,$$

$$\int_{y_0}^b 2\phi(y)\phi'(y)\,\mathrm{d}y = \phi(y)^2\Big|_{y_0}^b = \phi(b)^2,$$

$$\left\|\frac{\phi}{y-y_0}\right\|^2_{L^2(y_0,b)} + \left\|\frac{\phi}{y-y_0} - \varepsilon\phi - 2\phi'\right\|^2_{L^2(y_0,b)}$$
$$= -2\frac{\phi(b)^2}{b-y_0} + 2\varepsilon\phi(b)^2 - 2\varepsilon\int_{y_0}^b \frac{\phi(y)^2}{y-y_0}\,\mathrm{d}y + \varepsilon^2\|\phi\|^2_{L^2(y_0,b)} + 4\|\phi'\|^2_{L^2(y_0,b)}$$
$$\leqq -\varepsilon\int_{y_0}^b \frac{\phi(y)^2}{y-y_0}\,\mathrm{d}y + 4\|\phi'\|^2_{L^2(y_0,b)}.$$

Here we used $\varepsilon \leqq \frac{1}{b-y_0}$ and $\frac{\phi(y)^2}{y-y_0}\big|_{y_0}^b = \frac{\phi(b)^2}{b-y_0}$. Thus, $\left\|\frac{\phi}{y-y_0}\right\|^2_{L^2(y_0,b)} + \varepsilon\int_{y_0}^b \frac{\phi(y)^2}{y-y_0}\,\mathrm{d}y \leqq 4\|\phi'\|^2_{L^2(y_0,b)}$. Similarly, $\left\|\frac{\phi}{y-y_0}\right\|^2_{L^2(a,y_0)} + \varepsilon\int_a^{y_0} \frac{\phi(y)^2}{y_0-y}\,\mathrm{d}y \leqq 4\|\phi'\|^2_{L^2(a,y_0)}$. This gives (3.1). Letting $y_0 = a$, $\phi(y) = (y-a)^{\frac{1}{2}+\varepsilon_1}$ and sending $\varepsilon_1 \to 0^+$, we see that the constant 4 is optimal. □

For other versions of Hardy type inequality, the readers are referred to [15,32]. To study the lower bound of the n-th eigenvalue $\lambda_n(c)$ of Rayleigh-Kuo BVP for c close to $u_{\min}^-$, it is important to estimate the energy expression (2.7) near singular points. To this end, we need the following lemma:

Lemma 3.2. *Assume that* $a \in [y_1, y_2]$, $u(a) = u_{\min}$, $\phi \in H_0^1(y_1, y_2)$, $c < u_{\min}$ *and* $\beta > 0$. *Then there exists a constant* $\delta_0 > 0$ *(depending only on u and a) such that for* $0 < \delta \leqq \delta_0$,

(1) if (i) $u'(a) \neq 0$ *or* (ii) $u'(a) = 0$, $\beta \leqq 9u''(a)/8$, $\phi(a) = 0$, *then*

$$\int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right)\mathrm{d}y \geqq 0; \tag{3.3}$$

(2) if (iii) $a \in (y_1, y_2)$, $u'(a) = 0$, $\beta \leqq u''(a)$, *then*

$$\int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right)\mathrm{d}y \geqq -C_\delta\|\phi\|^2_{L^2([a-\delta,a+\delta]\cap[y_1,y_2])}. \tag{3.4}$$

Here C_δ *is a positive constant depending only on* u, a *and* δ.

Proof. First, we assume (i). Then we have $a \in \{y_1, y_2\}$ and thus $\phi(a) = 0$. Without loss of generality, we assume that $a = y_1$. In this case, $u'(y_1) > 0$. Choose $\delta_1 \in (0, y_2 - y_1)$ small enough such that $u'(y) > \frac{u'(y_1)}{2}$ for $y \in (y_1, y_1 + \delta_1)$, and thus, there exists $\xi_y \in (y_1, y)$ such that $u(y) - c > u(y) - u_{\min} = u'(\xi_y)(y - y_1) > \frac{u'(y_1)}{2}(y - y_1) > 0$ for $c < u_{\min}$. Note that, for $y \in (y_1, y_1 + \delta)$,

$$|\phi(y)|^2 = \left|\int_{y_1}^y \phi'(s)\,\mathrm{d}s\right|^2 \leqq \|\phi'\|^2_{L^2(y_1,y)}(y - y_1).$$

Now we take $\delta_0 \in (0, \delta_1)$ to be small enough such that $\int_{y_1}^{y_1+\delta_0} \frac{2|u''-\beta|}{u'(y_1)}\,\mathrm{d}y \leqq 1$. Then, for $0 < \delta \leqq \delta_0$,

$$\left|\int_{y_1}^{y_1+\delta} \frac{u''-\beta}{u-c}|\phi|^2\,\mathrm{d}y\right| \leqq \|\phi'\|^2_{L^2(y_1,y_1+\delta)} \int_{y_1}^{y_1+\delta} \frac{2|u''-\beta|}{u'(y_1)}\,\mathrm{d}y \leqq \|\phi'\|^2_{L^2(y_1,y_1+\delta)},$$

which implies (3.3) since $[a-\delta, a+\delta] \cap [y_1, y_2] = [y_1, y_1+\delta]$.

Now we assume (ii), then $u''(a) > 0$. Let $\delta_1 \in (0, \max(y_2 - a, a - y_1))$ be small enough such that $u''(y) > \frac{u''(a)}{2} > 0$ for $y \in [a-\delta_1, a+\delta_1] \cap [y_1, y_2]$. Since $u \in H^4(y_1, y_2) \subset C^3([y_1, y_2])$, we have $|u''(y) - u''(a)| \leqq C|y-a|$ and $|1/u''(y) - 1/u''(a)| \leqq C|y-a|$ for $y \in [a-\delta_1, a+\delta_1] \cap [y_1, y_2]$. Then there exists $\xi_y \in \{z : |z-a| < |y-a|\}$ such that $u(y) - c > u(y) - u_{\min} = \frac{u''(\xi_y)}{2}(y-a)^2 > \frac{2\beta}{9}(y-a)^2 > 0$ for $c < u_{\min}$ and $y \in [a-\delta_1, a+\delta_1] \cap [y_1, y_2]$, and thus

$$0 < \frac{1}{u(y)-c} < \frac{2}{u''(\xi_y)(y-a)^2} \leqq \frac{2 + C|\xi_y - a|}{u''(a)(y-a)^2} \leqq \frac{2 + C|y-a|}{u''(a)(y-a)^2},$$

$$u''(y) - \beta \geqq u''(a) - C|y-a| - 9u''(a)/8 = -u''(a)/8 - C|y-a|,$$

$$\begin{aligned}\frac{u''(y)-\beta}{u(y)-c} &\geqq \frac{-u''(a)/8 - C|y-a|}{u(y)-c} \geqq -\frac{u''(a)}{8}\frac{2+C|y-a|}{u''(a)(y-a)^2} - \frac{C|y-a|}{(2\beta/9)(y-a)^2}\\ &\geqq -\frac{1 + C_0|y-a|}{4(y-a)^2}.\end{aligned}$$

Now we take $\delta_0 = \min(\delta_1, C_0^{-1}) > 0$. For $0 < \delta \leqq \delta_0$, we have that

$$\begin{aligned}&-\int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \frac{u''-\beta}{u-c}|\phi|^2\,\mathrm{d}y \leqq \int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \frac{1 + C_0|y-a|}{4(y-a)^2}|\phi|^2\,\mathrm{d}y \\ &\leqq \int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \frac{1 + \delta^{-1}|y-a|}{4(y-a)^2}|\phi|^2\,\mathrm{d}y \leqq \left\|\phi'\right\|^2_{L^2([a-\delta,a+\delta]\cap[y_1,y_2])},\end{aligned} \tag{3.5}$$

which implies (3.3). Here we used Lemma 3.1 in the last step.

Next, we assume (iii). In this case, $u'(a) = 0$. Let $\widetilde{\beta} = u''(a) > 0$. Then $\widetilde{\beta} \geqq \beta$. Let $\delta_1 \in (0, \min(y_2 - a, a - y_1))$ be small enough such that $u''(y) > \frac{u''(a)}{2} > 0$ and $|u(y) - u_{\min}| \leqq 1/2$ for $y \in [a-\delta_1, a+\delta_1]$. Then $0 < u(y) - c < 1$ for $y \in [a-\delta_1, a+\delta_1]$ and $c \in (u_{\min} - 1/2, u_{\min})$. Now we assume $0 < \delta \leqq \delta_1$. Direct computation implies

$$\begin{aligned}&\int_{a-\delta}^{a+\delta} \frac{u''-\widetilde{\beta}}{u-c}|\phi|^2\,\mathrm{d}y = \int_{a-\delta}^{a+\delta} \frac{u''-\widetilde{\beta}}{u'}|\phi|^2 d(\ln(u-c))\\ &= \left(\frac{u''-\widetilde{\beta}}{u'}|\phi|^2(\ln(u-c))\right)\Big|_{a-\delta}^{a+\delta} - \int_{a-\delta}^{a+\delta} \ln(u-c)\left(\frac{u''-\widetilde{\beta}}{u'}|\phi|^2\right)'\mathrm{d}y\\ &= I_{c,\delta}(\phi) + II_{c,\delta}(\phi).\end{aligned}$$

Since $\widetilde{\beta} - u''(a) = 0$, it follows from the proof of Lemma 3.7 in [46] that $\frac{u''-\widetilde{\beta}}{u'} \in H^1(a-\delta_1, a+\delta_1)$. By interpolation, we have $\|\phi\|_{L^\infty(a-\delta,a+\delta)} \leqq C_\delta \|\phi\|_{L^2(a-\delta,a+\delta)} + \|\phi'\|_{L^2(a-\delta,a+\delta)}$, and thus that

$$\begin{aligned}
&\left|\left(\frac{u''-\widetilde{\beta}}{u'}|\phi|^2\right)(a+\delta) - \left(\frac{u''-\widetilde{\beta}}{u'}|\phi|^2\right)(a-\delta)\right| \\
&\leqq C \left\|\left(\frac{u''-\widetilde{\beta}}{u'}|\phi|^2\right)'\right\|_{L^2(a-\delta,a+\delta)} \delta^{\frac{1}{2}} \\
&\leqq C \left\|\left(\frac{u''-\widetilde{\beta}}{u'}\right)'|\phi|^2 + \left(\frac{u''-\widetilde{\beta}}{u'}\right)2\phi\phi'\right\|_{L^2(a-\delta,a+\delta)} \delta^{\frac{1}{2}} \\
&\leqq C \left(\|\phi\|^2_{L^\infty(a-\delta,a+\delta)} + \|\phi\|_{L^\infty(a-\delta,a+\delta)}\|\phi'\|_{L^2(a-\delta,a+\delta)}\right)\delta^{\frac{1}{2}} \\
&\leqq (C_\delta\|\phi\|^2_{L^2(a-\delta,a+\delta)} + C\|\phi'\|^2_{L^2(a-\delta,a+\delta)})\delta^{\frac{1}{2}}
\end{aligned}$$

and $\left|\left(\frac{u''-\widetilde{\beta}}{u'}|\phi|^2\right)(a-\delta)\right| \leqq C\|\phi\|^2_{L^\infty(a-\delta,a+\delta)} \leqq C_\delta\|\phi\|^2_{L^2(a-\delta,a+\delta)} + C\|\phi'\|^2_{L^2(a-\delta,a+\delta)}$. Then

$$\begin{aligned}
I_{c,\delta}(\phi) &= \left(\frac{u''-\widetilde{\beta}}{u'}|\phi|^2\right)\Big|_{a-\delta}^{a+\delta} \ln(u(a+\delta)-c) + \left(\frac{u''-\widetilde{\beta}}{u'}|\phi|^2\right)\Big|_{a-\delta} \ln(u-c)\Big|_{a-\delta}^{a+\delta} \\
&\leqq (C_\delta\|\phi\|^2_{L^2(a-\delta,a+\delta)} + C\|\phi'\|^2_{L^2(a-\delta,a+\delta)})(\delta^{\frac{1}{2}}|\ln(u(a+\delta)-c)| + |\ln(u-c)|_{a-\delta}^{a+\delta}|).
\end{aligned}$$

Note that $C'(y-a)^2 \leqq |u(y)-u(a)| \leqq C''(y-a)^2$ for $y \in [a-\delta, a+\delta]$. Then $|\ln(u(y)-c)| = -\ln(u(y)-c) \leqq -\ln(u(y)-u(a)) \leqq -\ln(C'(y-a)^2)$ and $|\ln(u(a+\delta)-c)| \leqq -\ln(C'\delta^2) \leqq C(|\ln\delta|+1)$ for $|c-u_{\min}| < 1/2$ and $y \in [a-\delta, a+\delta]$. Let $u_+ = \max(u(a+\delta), u(a-\delta))$ and $u_- = \min(u(a+\delta), u(a-\delta))$. Then $u_+ \geqq u_- > u(a)$, and

$$|\ln(u-c)|_{a-\delta}^{a+\delta}| = \ln\frac{u_+-c}{u_--c} \leqq \ln\frac{u_+-u(a)}{u_--u(a)} = \left|\ln\frac{u(a+\delta)-u(a)}{u(a-\delta)-u(a)}\right|$$

for $c < u_{\min} = u(a)$. Thus,

$$|I_{c,\delta}(\phi)| \leqq (C_\delta\|\phi\|^2_{L^2(a-\delta,a+\delta)} + C\|\phi'\|^2_{L^2(a-\delta,a+\delta)})\Psi(\delta) \tag{3.6}$$

for $c \in (u_{\min} - 1/2, u_{\min})$, where

$$\Psi(\delta) = \delta^{\frac{1}{2}}(|\ln\delta|+1) + \left|\ln\frac{u(a+\delta)-u(a)}{u(a-\delta)-u(a)}\right|.$$

Note that

$$\lim_{\delta\to 0^+}\left|\ln\frac{u(a+\delta)-u(a)}{u(a-\delta)-u(a)}\right| = \lim_{\delta\to 0^+}\left|\ln\frac{u''(\xi_{a+\delta})}{u''(\xi_{a-\delta})}\right| = 0,$$

where $\xi_{a+\delta} \in (a, a+\delta)$ and $\xi_{a-\delta} \in (a-\delta, a)$. Therefore $\lim_{\delta\to 0^+}\Psi(\delta) = 0$.

Next, we claim that $\ln(u-c)$, $c \in (u_{\min} - 1/2, u_{\min})$, is uniformly bounded in $L^p(a-\delta, a+\delta)$ for $1 < p < \infty$. The proof is similar as that in Lemma 3.7

of [46]. Note that $|\ln(u(y)-c)| \leqq -\ln(C'(y-a)^2)$ for $|c-u_{\min}|<1/2$ and $y \in [a-\delta, a+\delta]$. Therefore,

$$\begin{aligned}\int_{a-\delta}^{a+\delta} |\ln(u-c)|^p \, \mathrm{d}y \leqq & C \int_{a-\delta}^{a+\delta} (|\ln(y-a)^2|^p + 1) \, \mathrm{d}y \\ & \leqq C \int_{-\delta}^{\delta} (|\ln|z|^2|^p + 1) \, \mathrm{d}z \leqq C.\end{aligned}$$

Now, we consider $II_{c,\delta}(\phi)$.

$$\begin{aligned}|II_{c,\delta}(\phi)| \leqq & (2\delta)^{\frac{1}{4}} \|\ln(u-c)\|_{L^4(a-\delta,a+\delta)} \left\| \left(\frac{u''-\beta}{u'} |\phi|^2 \right)' \right\|_{L^2(a-\delta,a+\delta)} \\ \leqq & (C_\delta \|\phi\|^2_{L^2(a-\delta,a+\delta)} + C\|\phi'\|^2_{L^2(a-\delta,a+\delta)}) \delta^{\frac{1}{4}}.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we get for $c \in (u_{\min} - 1/2, u_{\min})$,

$$\left| \int_{a-\delta}^{a+\delta} \frac{u''-\widetilde{\beta}}{u-c} |\phi|^2 \, \mathrm{d}y \right| \leqq (C_\delta \|\phi\|^2_{L^2(a-\delta,a+\delta)} + C_1 \|\phi'\|^2_{L^2(a-\delta,a+\delta)})(\delta^{\frac{1}{4}} + \Psi(\delta)).$$

Since $\lim_{\delta\to0^+} \Psi(\delta) = 0$, we have $\lim_{\delta\to0^+} (\delta^{\frac{1}{4}} + \Psi(\delta)) = 0$, and there exists $\delta_0 \in (0, \delta_1)$ such that $C_1(\delta^{\frac{1}{4}} + \Psi(\delta)) < 1$. Then for $0 < \delta \leqq \delta_0$ and $c \in (u_{\min} - 1/2, u_{\min})$, we have

$$\begin{aligned}&\int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \frac{u''-\beta}{u-c} |\phi|^2 \, \mathrm{d}y \geqq \int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \frac{u''-\widetilde{\beta}}{u-c} |\phi|^2 \, \mathrm{d}y \\ \geqq & -(C_\delta \|\phi\|^2_{L^2(a-\delta,a+\delta)} + C_1 \|\phi'\|^2_{L^2(a-\delta,a+\delta)})(\delta^{\frac{1}{4}} + \Psi(\delta)) \\ \geqq & -C_\delta \|\phi\|^2_{L^2(a-\delta,a+\delta)} (\delta^{\frac{1}{4}} + \Psi(\delta)) - \|\phi'\|^2_{L^2(a-\delta,a+\delta)},\end{aligned}$$

which implies (3.4) since $[a-\delta, a+\delta] \cap [y_1, y_2] = [a-\delta, a+\delta]$. On the other hand, for $0 < \delta \leqq \delta_0$ and $c \leqq u_{\min} - 1/2$, we have that

$$\begin{aligned}&\int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \frac{u''-\beta}{u-c} |\phi|^2 \, \mathrm{d}y \geqq \int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \frac{u''-\widetilde{\beta}}{u-c} |\phi|^2 \, \mathrm{d}y \\ \geqq & -2 \int_{[a-\delta,a+\delta]\cap[y_1,y_2]} |u''-\widetilde{\beta}||\phi|^2 \, \mathrm{d}y \geqq -C\|\phi\|^2_{L^2(a-\delta,a+\delta)},\end{aligned}$$

which implies (3.4). This completes the proof. □

Now, we are ready to prove Theorem 2.9 (1)–(2).

Proof of Theorem 2.9 (1)–(2). We first give the proof of (1), and (2) can be proved similarly. Consider $1 \leqq n \leqq m_\beta$. It suffices to show that $\lim_{c\to u_{\min}^-} \lambda_{m_\beta}(c) = -\infty$. Let

$$\{a \in (y_1, y_2) : u = u_{\min}, u''(a) - \beta < 0\} = \{a_1, \cdots, a_{m_\beta}\}, \tag{3.8}$$

and

$$\eta(x) = \begin{cases} \mu \exp\left(\frac{-1}{1-x^2}\right), & x \in (-1, 1), \\ 0, & x \notin (-1, 1), \end{cases} \tag{3.9}$$

where $\mu > 0$ is a constant such that $\int_{-1}^{1} \eta(x)^2 \, dx = 1$. Then $\eta \in C_0^\infty(\mathbf{R})$. Define

$$\varphi_i(y) = \delta_0^{-\frac{1}{2}} \eta\left(\frac{y - a_i}{\delta_0}\right), \quad y \in [y_1, y_2],$$

where $1 \leqq i \leqq m_\beta$, and $\delta_0 > 0$ is small enough such that $(a_i - \delta_0, a_i + \delta_0) \cap (a_j - \delta_0, a_j + \delta_0) = \emptyset$ for $i \neq j$ and $u''(y) - \beta < 0$ for all $y \in \cup_{1 \leqq i \leqq m_\beta}(a_i - \delta_0, a_i + \delta_0) \subset (y_1, y_2)$. Then $\|\varphi_i\|_{L^2(y_1, y_2)} = 1$ and supp $(\varphi_i) = (a_i - \delta_0, a_i + \delta_0)$. Thus, $\varphi_i \perp \varphi_j$ in the L^2 sense for $i \neq j$. Let $V_{m_\beta} = \text{span}\{\varphi_1, \cdots, \varphi_{m_\beta}\}$. Then $V_{m_\beta} \subset H_0^1(y_1, y_2)$. By (2.7), there exist $b_{i,c} \in \mathbf{R}$, $i = 1, \cdots, m_\beta$, with $\sum_{i=1}^{m_\beta} |b_{i,c}|^2 = 1$ such that $\varphi_c = \sum_{i=1}^{m_\beta} b_{i,c} \varphi_i \in V_{m_\beta}$ with $\|\varphi_c\|_{L^2}^2 = 1$, and

$$\begin{aligned} \lambda_{m_\beta}(c) &\leqq \sup_{\|\phi\|_{L^2} = 1, \phi \in V_{m_\beta}} \int_{y_1}^{y_2} \left(|\phi'|^2 + \frac{u'' - \beta}{u - c}|\phi|^2\right) dy \\ &= \int_{y_1}^{y_2} \left(|\varphi_c'|^2 + \frac{u'' - \beta}{u - c}|\varphi_c|^2\right) dy \\ &= \sum_{i=1}^{m_\beta} |b_{i,c}|^2 \int_{a_i - \delta_0}^{a_i + \delta_0} \left(|\varphi_i'|^2 + \frac{u'' - \beta}{u - c}|\varphi_i|^2\right) dy \\ &\leqq \max_{1 \leqq i \leqq m_\beta} \int_{a_i - \delta_0}^{a_i + \delta_0} \left(|\varphi_i'|^2 + \frac{u'' - \beta}{u - c}|\varphi_i|^2\right) dy \to -\infty \text{ as } c \to u_{\min}^-. \end{aligned} \tag{3.10}$$

Next, we prove (ii). Let $\delta_1 > 0$ be a sufficiently small constant such that $(a - \delta_1, a + \delta_1) \subset [y_1, y_2]$ for $a \in \{u = u_{\min}\} \setminus \{y_1, y_2\}$, and $|a - b| > 2\delta_1$ for $a, b \in \{u = u_{\min}\}$ and $a \neq b$. There are four cases for zeros of $a \in \{u = u_{\min}\}$ as follows:

Case 1. $a \in \{y_1, y_2\}$ and $u'(a) \neq 0$;

Case 2. $a \in \{y_1, y_2\}$, $u'(a) = 0$ (thus $\beta \leqq \frac{9}{8}\kappa_+ \leqq 9u''(a)/8$);

Case 3. $a \in (y_1, y_2)$ and $\beta \leqq u''(a)$;

Case 4. $a \in (y_1, y_2)$ and $u''(a) < \beta \leqq 9u''(a)/8$.

Then we divide our proof into four cases as above. In fact, for Cases 1–2, by Lemma 3.2 (1) there exists $\delta(a) > 0$ such that for $0 < \delta \leqq \delta(a)$, $c < u_{\min}$ and $\phi \in H_0^1$,

$$\int_{[a-\delta, a+\delta] \cap [y_1, y_2]} \left(|\phi'|^2 + \frac{u'' - \beta}{u - c}|\phi|^2\right) dy \geqq 0. \tag{3.11}$$

For Case 3, by Lemma 3.2 (2) there exists $\delta(a) > 0$ such that for $0 < \delta \leqq \delta(a)$, $c < u_{\min}$ and $\phi \in H_0^1$,

$$\int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right) \mathrm{d}y \geqq -C(\delta,a) \int_{[a-\delta,a+\delta]\cap[y_1,y_2]} |\phi|^2 \,\mathrm{d}y. \tag{3.12}$$

Here $C(\delta,a)$ depends only on u, a, δ. Moreover, if $\phi(a) = 0$, then by Lemma 3.2 (1),

$$\int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right) \mathrm{d}y \geqq 0. \tag{3.13}$$

For Case 4, we have $a \in \{a_1, \cdots, a_{m_\beta}\}$. By Lemma 3.2 (1), there exists $\delta(a) > 0$ such that for $0 < \delta \leqq \delta(a)$, $c < u_{\min}$, $\phi \in H_0^1$ and $\phi(a) = 0$,

$$\int_{[a-\delta,a+\delta]\cap[y_1,y_2]} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right) \mathrm{d}y \geqq 0. \tag{3.14}$$

Now let $\delta_0 = \min(\{\delta(a) : a \in \{u = u_{\min}\}\} \cup \{\delta_1\})$. Define

$$(q_1(y), q_1^0(y)) = \begin{cases} \left(\frac{u''(y)-\beta}{u(y)-c}, 1\right) & y \in [y_1,y_2] \setminus \cup_{a\in\{u=u_{\min}\}}(a-\delta_0, a+\delta_0), \\ (0,0) & y \in \cup_{a\in\{u=u_{\min}\}} \left((a-\delta_0,a+\delta_0)\cap[y_1,y_2]\right), \end{cases}$$

$$(q_2(y), q_2^0(y)) = \left(\frac{u''(y)-\beta}{u(y)-c} - q_1(y), 1 - q_1^0(y)\right) \ \ y \in [y_1,y_2].$$

Then there exists $C_0 > 0$ such that for $c < u_{\min}$,

$$|q_1(y)| \leqq C_0 \ \text{ for } \ y \in [y_1,y_2].$$

For $\phi \in H_0^1$ and $\|\phi\|_{L^2} = 1$,

$$\begin{aligned} &\int_{y_1}^{y_2} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right) \mathrm{d}y \\ &= \int_{y_1}^{y_2} \left(q_1^0|\phi'|^2 + q_1|\phi|^2\right) \mathrm{d}y + \int_{y_1}^{y_2} \left(q_2^0|\phi'|^2 + q_2|\phi|^2\right) \mathrm{d}y = I_c(\phi) + II_c(\phi). \end{aligned}$$

Let us first consider $I_c(\phi)$. For $\phi \in H_0^1$ and $\|\phi\|_{L^2} = 1$, we have

$$I_c(\phi) \geqq \int_{y_1}^{y_2} \left(-C_0|\phi|^2\right) \mathrm{d}y \geqq -C_0 \tag{3.15}$$

for $c < u_{\min}$. We proceed to consider $II_c(\phi)$.

$$\begin{aligned} II_c(\phi) &= \sum_{a\in\{u=u_{\min}\}} \int_{[a-\delta_0,a+\delta_0]\cap[y_1,y_2]} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right) \mathrm{d}y \qquad (3.16) \\ &= \left(\sum_{\text{Case 1}} + \cdots + \sum_{\text{Case 4}}\right) \int_{[a-\delta_0,a+\delta_0]\cap[y_1,y_2]} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right) \mathrm{d}y. \end{aligned}$$

Recall that $a_1, \cdots, a_{m_\beta}$ are defined in (3.8). For any $(m_\beta+1)$-dimensional subspace $V = \mathrm{span}\{\psi_1, \cdots, \psi_{m_\beta+1}\}$ in $H_0^1(y_1, y_2)$, there exists $0 \neq (\xi_1, \cdots, \xi_{m_\beta+1}) \in \mathbf{R}^{m_\beta+1}$ such that $\xi_1\psi_1(a_i) + \cdots + \xi_{m_\beta+1}\psi_{m_\beta+1}(a_i) = 0, i = 1, \cdots, m_\beta$. Define $\tilde{\psi} = \xi_1\psi_1 + \cdots + \xi_{m_\beta+1}\psi_{m_\beta+1}$. Then $\tilde{\psi}(a_i) = 0, i = 1, \cdots, m_\beta$. We normalize $\tilde{\psi}$ such that $\|\tilde{\psi}\|_{L^2(y_1,y_2)} = 1$. Then by (3.11), (3.12), (3.14) and (3.16), we have that

$$\begin{aligned} II_c(\tilde{\psi}) &\geqq \sum_{\text{Case 3}} \int_{[a-\delta_0,a+\delta_0]\cap[y_1,y_2]} \left(|\tilde{\psi}'|^2 + \frac{u''-\beta}{u-c}|\tilde{\psi}|^2\right) \mathrm{d}y \\ &\geqq -\sum_{\text{Case 3}} C(\delta_0, a) \int_{[a-\delta_0,a+\delta_0]\cap[y_1,y_2]} |\tilde{\psi}|^2 \,\mathrm{d}y \geqq -\max_{\text{Case 3}} C(\delta_0, a). \end{aligned}$$

This, along with (2.7) and (3.15), yields that $\inf_{c\in(-\infty,u_{\min})} \lambda_{m_\beta+1}(c) \geqq -\max_{\text{Case 3}} C(\delta_0, a) - C_0$. This proves (ii).

Finally, we prove (iii). Let $q_1, q_2, I_c(\phi), II_c(\phi)$ and C_0 be defined as in (ii). Let $\mu_1([a, b])$ be the principal eigenvalue of

$$-\phi'' = \lambda\phi, \quad \phi(a) = \phi(b) = 0.$$

Then we have $\mu_1([a, b]) = |\pi/(b-a)|^2$, and

$$\int_a^b |\phi'|^2 \,\mathrm{d}y \geqq \mu_1([a, b]) \int_a^b |\phi|^2 \,\mathrm{d}y \quad \text{for } \phi \in H_0^1(a, b). \tag{3.17}$$

Let $\delta_2 = \pi/C_0^{\frac{1}{2}}$. Then we have $\mu_1([a, b]) \geqq C_0$ for $0 < b - a \leqq \delta_2$. Let

$$M = (\{n\delta_2 : n \in \mathbf{Z}\} \cup \{a + b : a \in \{u = u_{\min}\}, b \in \{-\delta_0, 0, \delta_0\}\} \cup \{y_1, y_2\}) \cap [y_1, y_2].$$

Then M is a finite set, and we can write its elements in the increasing order

$$M = \{a_0', \cdots, a_{N_\beta}'\}, \ y_1 = a_0' < \cdots < a_{N_\beta}' = y_2.$$

Then $0 < a_{k+1}' - a_k' \leqq \delta_2$ and $\mu_1([a_k', a_{k+1}']) \geqq C_0$ for $0 \leqq k < N_\beta$. Let

$$M_1 = \{k \in \mathbf{Z} : 0 \leqq k < N_\beta, \ [a_k', a_{k+1}'] \cap (a - \delta_0, a + \delta_0) = \emptyset, \ \forall\, a \in \{u = u_{\min}\}\}.$$

Then we have $[y_1, y_2] \setminus (\cup_{a\in\{u=u_{\min}\}}(a - \delta_0, a + \delta_0)) = \cup_{k\in M_1}[a_k', a_{k+1}']$.

For any N_β-dimensional subspace $V = \mathrm{span}\{\psi_1, \cdots, \psi_{N_\beta}\}$ in $H_0^1(y_1, y_2)$, there exists $0 \neq (\xi_1, \cdots, \xi_{N_\beta}) \in \mathbf{R}^{N_\beta}$ such that $\xi_1\psi_1(a_i') + \cdots + \xi_{N_\beta}\psi_{N_\beta}(a_i') = 0, i = 1, \cdots, N_\beta - 1$. Define $\tilde{\psi} = \xi_1\psi_1 + \cdots + \xi_{N_\beta}\psi_{N_\beta}$. Then $\tilde{\psi}(a_i') = 0, i = 1, \cdots, N_\beta - 1$. We normalize $\tilde{\psi}$ such that $\|\tilde{\psi}\|_{L^2(y_1,y_2)} = 1$. Since $\tilde{\psi} \in H_0^1(y_1, y_2)$, we also have $\tilde{\psi}(a_0') = \tilde{\psi}(y_1) = 0$, $\tilde{\psi}(a_{N_\beta}') = \tilde{\psi}(y_2) = 0$, and thus $\tilde{\psi}(a_i') = 0$ for $i = 0, \cdots, N_\beta$, i.e. $\tilde{\psi}|_M = 0$. By (3.17), we have

$$\int_{a_k'}^{a_{k+1}'} |\tilde{\psi}'|^2 \,\mathrm{d}y \geqq \mu_1([a_k', a_{k+1}']) \int_{a_k'}^{a_{k+1}'} |\tilde{\psi}|^2 \,\mathrm{d}y \geqq C_0 \int_{a_k'}^{a_{k+1}'} |\tilde{\psi}|^2 \,\mathrm{d}y, \ k = 0, \cdots, N_\beta - 1. \tag{3.18}$$

First, we consider $I_c(\tilde{\psi})$. By (3.18), we have that

$$I_c(\tilde{\psi}) \geqq \int_{[y_1,y_2]\setminus(\cup_{a\in\{u=u_{\min}\}}(a-\delta_0,a+\delta_0))} \left(|\tilde{\psi}'|^2 - C_0|\tilde{\psi}|^2\right) \mathrm{d}y \tag{3.19}$$
$$= \sum_{k\in M_1} \int_{a'_k}^{a'_{k+1}} \left(|\tilde{\psi}'|^2 - C_0|\tilde{\psi}|^2\right) \mathrm{d}y \geqq 0.$$

Next, we consider $II_c(\tilde{\psi})$. For $a \in \{u = u_{\min}\}$, we have $a \in M$ and $\tilde{\psi}(a) = 0$. Then by (3.11), (3.13), (3.14) and (3.16), we have $II_c(\tilde{\psi}) \geqq 0$. This, along with (2.7) and (3.19), yields that $\inf_{c\in(-\infty,u_{\min})} \lambda_{N_\beta}(c) \geqq 0$. This proves (iii). □

3.2. Infinite number for $\beta \notin [\frac{9}{8}\kappa_-, \frac{9}{8}\kappa_+]$

In this subsection, we prove Theorem 2.9 (3)–(4). The proof is based on construction of suitable test functions such that the energy in (2.7) converges to $-\infty$ as $c \to u_{\min}^-$ or $c \to u_{\max}^+$.

Proof of Theorem 2.9 (3)–(4). We only prove Theorem 2.9 (3), since (4) can be proved similarly. Let $\beta > \frac{9}{8}\kappa_+$. Then there exists $a \in [y_1, y_2]$ such that $\beta/u''(a) > 9/8$, $u'(a) = 0$ and $u(a) = u_{\min}$. If $a \in [y_1, y_2)$, our analysis is completely on $[a, a+\delta] \subset [y_1, y_2)$ for $\delta > 0$ small enough. If $a = y_2$, the analysis is only on $[a-\delta, a]$ and the proof is similar as $a \in [y_1, y_2)$. Now we assume that $a = 0 \in [y_1, y_2)$. Then $u''(0) > 0$ and there exists $\varepsilon_0 > 0$ such that $u''(z) > 0$ and

$$\frac{2(u''(y)-\beta)}{u''(z)} < -\frac{1}{4-\varepsilon_0}$$

for $y, z \in [0, \delta] \subset [y_1, y_2)$ and $\delta > 0$ small enough. Let $\nu_0 = \min_{z\in[0,\delta]}\{u''(z)\} > 0$ and $J(x) = \eta(2x-1)$, $x \in \mathbf{R}$, where η is defined in (3.9). Define

$$\varphi_{i,R}(y) = \begin{cases} y^{\frac{1}{2}} J\left(\frac{\ln y}{R} + i + 1\right), & y \in (0, y_2], \\ 0, & y \in [y_1, 0], \end{cases}$$

where $i = 1, \cdots, n$, and R is large enough such that $e^{-R} < \delta$. Then $\varphi_{i,R} \in H_0^1(y_1, y_2)$ and $\operatorname{supp}\varphi_{i,R} = [e^{-(i+1)R}, e^{-iR}]$, $i = 1, \cdots, n$. Thus, $\varphi_{i,R} \perp \varphi_{j,R}$ in the L^2 sense for $i \neq j$. Note that $u''(y) - \beta < 0$ for $y \in [0, \delta]$. For $1 \leqq i \leqq n$, we define

$$\tilde{\varphi}_{i,R} = \frac{1}{\|\varphi_{i,R}\|_{L^2}}\varphi_{i,R}, \quad \text{and} \quad \tilde{V}_{n,R} = \operatorname{span}\{\tilde{\varphi}_{1,R}, \cdots, \tilde{\varphi}_{n,R}\}.$$

Choose $R > 0$ such that $2(u_{\min} - c) = \frac{\varepsilon_0\nu_0 e^{-2(n+1)R}}{8}$. Then $c \to u_{\min}^- \Leftrightarrow R \to \infty$. We shall show that for $1 \leqq i \leqq n$,

$$\lim_{c\to u_{\min}^-} \int_{y_1}^{y_2} \left(|\tilde{\varphi}'_{i,R}|^2 + \frac{u''-\beta}{u-c}|\tilde{\varphi}_{i,R}|^2\right) \mathrm{d}y = -\infty. \tag{3.20}$$

Assume that (3.20) is true. Similar to (3.10), there exist $d_{i,c} \in \mathbf{R}, i = 1, \cdots, n$, with $\sum_{i=1}^{n} |d_{i,c}|^2 = 1$ such that

$$\lambda_n(c) \leqq \sum_{i=1}^{n} |d_{i,c}|^2 \int_{y_1}^{y_2} \left(|\tilde{\varphi}_{i,R}'|^2 + \frac{u'' - \beta}{u - c} |\tilde{\varphi}_{i,R}|^2 \right) \mathrm{d}y \to -\infty \quad \text{as} \quad c \to u_{\min}^{-}. \tag{3.21}$$

Now we prove (3.20). Direct computation gives that

$$\begin{aligned} \frac{u''(y) - \beta}{u(y) - c} =& \frac{u''(y) - \beta}{u(y) - u_{\min} + u_{\min} - c} = \frac{2(u''(y) - \beta)}{u''(\xi_y)y^2 + 2(u_{\min} - c)} < \frac{2(u''(y) - \beta)}{\left(u''(\xi_y) + \frac{\varepsilon_0 u''(\xi_y)}{8}\right) y^2} \\ <& \frac{-\frac{1}{4-\varepsilon_0} u''(\xi_y)}{\left(u''(\xi_y) + \frac{\varepsilon_0 u''(\xi_y)}{8}\right) y^2} = -\frac{1}{(4 - \varepsilon_0)\left(1 + \frac{\varepsilon_0}{8}\right) y^2} = -\frac{1}{(4 - \varepsilon_1) y^2} \end{aligned}$$

for $y \in [e^{-(i+1)R}, e^{-iR}]$ and $2(u_{\min} - c) = \frac{\varepsilon_0 \nu_0 e^{-2(n+1)R}}{8}$, where $\xi_y \in (0, y)$ and $\varepsilon_1 = \frac{\varepsilon_0}{2} + \frac{\varepsilon_0^2}{8}$. Then

$$\begin{aligned} \int_{y_1}^{y_2} \left(|\varphi_{i,R}'|^2 + \frac{u'' - \beta}{u - c} |\varphi_{i,R}|^2 \right) \mathrm{d}y &= \int_{e^{-(i+1)R}}^{e^{-iR}} \left(|\varphi_{i,R}'|^2 + \frac{u'' - \beta}{u - c} |\varphi_{i,R}|^2 \right) \mathrm{d}y \\ &< \int_{e^{-(i+1)R}}^{e^{-iR}} \left(|\varphi_{i,R}'|^2 - \frac{1}{(4 - \varepsilon_1) y^2} |\varphi_{i,R}|^2 \right) \mathrm{d}y. \end{aligned} \tag{3.22}$$

Note that, for $y \in [e^{-(i+1)R}, e^{-iR}]$,

$$\begin{aligned} |\varphi_{i,R}'(y)|^2 =& \left| J(x) \frac{-(2x - 1)}{4x^2(x - 1)^2} \frac{1}{Ry} y^{\frac{1}{2}} + \frac{1}{2} y^{-\frac{1}{2}} J(x) \right|^2 \\ =& J(x)^2 \frac{(2x - 1)^2}{16x^4(x - 1)^4} \frac{1}{R^2 y} - J(x)^2 \frac{2x - 1}{4x^2(x - 1)^2} \frac{1}{Ry} + \frac{1}{4y} J(x)^2, \end{aligned}$$

where $x = \frac{\ln y}{R} + i + 1$. Since $\left| J(x)^2 \frac{(2x-1)^2}{16x^4(x-1)^4} \right| \leqq C$ and $\left| J(x)^2 \frac{2x-1}{4x^2(x-1)^2} \right| \leqq C$ for $x \in [0, 1]$, we get that

$$\begin{aligned} &\left| \int_{e^{-(i+1)R}}^{e^{-iR}} \left(J(x)^2 \frac{(2x - 1)^2}{16x^4(x - 1)^4} \frac{1}{R^2 y} - J(x)^2 \frac{2x - 1}{4x^2(x - 1)^2} \frac{1}{Ry} \right) \mathrm{d}y \right| \\ \leqq& \frac{C}{R^2} \int_{e^{-(i+1)R}}^{e^{-iR}} \frac{1}{y} \mathrm{d}y + \frac{C}{R} \int_{e^{-(i+1)R}}^{e^{-iR}} \frac{1}{y} \mathrm{d}y = \frac{C}{R} + C \leqq C. \end{aligned}$$

Then we infer from (3.22) that

$$\begin{aligned} &\int_{y_1}^{y_2} \left(|\varphi_{i,R}'|^2 + \frac{u'' - \beta}{u - c} |\varphi_{i,R}|^2 \right) \mathrm{d}y \\ &\leqq C + \int_{e^{-(i+1)R}}^{e^{-iR}} \left(\frac{1}{4y} J(x)^2 - \frac{1}{(4 - \varepsilon_1) y^2} |\varphi_{i,R}|^2 \right) \mathrm{d}y \end{aligned} \tag{3.23}$$

$$=C+\int_{e^{-(i+1)R}}^{e^{-iR}}\frac{-\varepsilon_1}{4(4-\varepsilon_1)}J(x)^2\frac{1}{y}\,\mathrm{d}y=C-\frac{\varepsilon_1 R}{8(4-\varepsilon_1)}<0$$

when R is large enough. Direct computation gives that

$$\|\varphi_{i,R}\|_{L^2}^2=\int_{e^{-(i+1)R}}^{e^{-iR}}yJ\left(\frac{\ln y}{R}+i+1\right)^2\mathrm{d}y=\int_0^1 Re^{2R(x-i-1)}J(x)^2\,\mathrm{d}x$$
$$\leqq CR\int_0^1 e^{2R(x-i-1)}\,\mathrm{d}x=C(e^{-2iR}-e^{-2(i+1)R})\leqq Ce^{-2iR}. \tag{3.24}$$

Combining (3.23) and (3.24), we have that

$$\int_{y_1}^{y_2}\left(|\tilde\varphi_{i,R}'|^2+\frac{u''-\beta}{u-c}|\tilde\varphi_{i,R}|^2\right)\mathrm{d}y=\frac{1}{\|\varphi_{i,R}\|_{L^2}^2}\int_{y_1}^{y_2}\left(|\varphi_{i,R}'|^2+\frac{u''-\beta}{u-c}|\varphi_{i,R}|^2\right)\mathrm{d}y$$
$$\leqq\left(C-\frac{\varepsilon_1 R}{8(4-\varepsilon_1)}\right)\frac{1}{\|\varphi_{i,R}\|_{L^2}^2}\leqq\left(C-\frac{\varepsilon_1 R}{8(4-\varepsilon_1)}\right)\frac{e^{2iR}}{C}\to-\infty$$

as $R\to\infty$. This proves (3.20). □

4. Rule out oscillation of the n-th eigenvalue of Rayleigh-Kuo BVP

Let $\beta\in[\frac{9}{8}\kappa_-,\frac{9}{8}\kappa_+]$. By Theorem 2.9 (1)–(2), $\lambda_n(c)=-\alpha^2$ has only finite number of solutions c outside the range of u for $1\leqq n\leqq m_\beta$, and no solutions exist for $n\geqq N_\beta$. It is non-trivial to study whether the number of solutions is finite for $m_\beta<n<N_\beta$. Recall that N_β is obtained in Theorem 2.9 such that $\inf\limits_{c\in(-\infty,u_{\min})}\lambda_{N_\beta}(c)\geqq 0$ for $0<\beta\leqq\frac{9}{8}\kappa_+$, $\inf\limits_{c\in(u_{\max},\infty)}\lambda_{N_\beta}(c)\geqq 0$ for $\frac{9}{8}\kappa_-\leqq\beta<0$. The main difficulty is that $\lambda_n(c)$ might oscillate when c is close to $u_{\min}$ or $u_{\max}$. In this section, we rule out the oscillation in the following three cases:

4.1. Rule out oscillation under the spectral assumption

We rule out the oscillation of $\lambda_n(c)$ under the spectral assumption ($\mathbf{E}_\pm$), which is stated in Theorem 2.11 (1)–(2). To this end, we first consider the compactness near a class of singular points.

Lemma 4.1. *Let $c\in Ran(u)$, $y_0\in u^{-1}\{c\}\cap(y_1,y_2)$, $u'(y_0)=0$ and $\delta>0$ so that $(u''(y_0)-\beta)(u''(y)-\beta)>0$ on $[y_0-\delta,y_0+\delta]\subset[y_1,y_2]$ and $[y_0-\delta,y_0+\delta]\cap u^{-1}\{c\}=\{y_0\}$. Assume that $\beta/u''(y_0)<9/8$. Let $\phi_n,\omega_n\in H^1(y_0-\delta,y_0+\delta)$ and $c_n\in\mathbf{C}$ so that $c_n^i>0$, $c_n\to c$, $\phi_n\rightharpoonup 0$, $\omega_n\to 0$ in $H^1(y_0-\delta,y_0+\delta)$ and*

$$(u-c_n)(\phi_n''-\alpha^2\phi_n)-(u''-\beta)\phi_n=\omega_n$$

holds on $[y_0-\delta,y_0+\delta]$. Then $\phi_n\to 0$ in $H^1(y_0-\delta,y_0+\delta)$.

Here $c_n^i = \mathrm{Im}(c_n)$. The proof of Lemma 4.1 is the same as that of Lemma 3.4 in [46], where we only used the condition $\beta/u''(y_0) < 9/8$ rather than the stronger condition:

$$(\mathbf{H}) \quad u \in H^4(y_1, y_2), \; u''(y_c) \neq 0, \; \beta/u''(y_c) < 9/8 \text{ at critical points } u'(y_c) = 0.$$

Since otherwise, we can construct $\widetilde{u}$ such that $\widetilde{u} \in H^4(y_1, y_2)$, $\widetilde{u}|_{[y_0-\delta, y_0+\delta]} = u|_{[y_0-\delta, y_0+\delta]}$ and $(\widetilde{u}')^{-1}\{0\} = \{y_0\}$. Recall that all the conditions and conclusions depend only on $u|_{[y_0-\delta, y_0+\delta]}$. Then we prove the uniform H^1 bound for the eigenfunctions. More precisely, we have the following result:

Proposition 4.2. *Let $0 < \beta < \frac{9}{8}\kappa_+$. Assume that $m_\beta < n < N_\beta$, $\{c_k\} \subset (-\infty, u_{\min})$, $c_k \to u_{\min}^-$ and $-\lambda_n(c_k) = \alpha^2 > 0$, where m_β and N_β are given in (1.15) and Theorem 2.9, and $\lambda_n(c_k)$ is the n-th eigenvalue of*

$$-\psi_k'' + \frac{u'' - \beta}{u - c_k}\psi_k = \lambda_n(c_k)\psi_k, \quad \psi_k(y_1) = \psi_k(y_2) = 0 \tag{4.1}$$

with the L^2 normalized eigenfunction ψ_k. Then

$$\|\psi_k\|_{H^1(y_1,y_2)} \leqq C, \quad k \geqq 1. \tag{4.2}$$

Proof. Suppose that (4.2) is not true. Up to a subsequence, we can assume that $\|\psi_k\|_{H^1(y_1,y_2)} \geqq k$. Let $\hat{\psi}_k = \frac{\psi_k}{\|\psi_k\|_{H^1(y_1,y_2)}}$ on $[y_1, y_2]$. Then $-\hat{\psi}_k{}'' + \frac{u''-\beta}{u-c_k}\hat{\psi}_k = -\alpha^2\hat{\psi}_k$ on $[y_1, y_2]$, $\|\hat{\psi}_k\|_{H^1(y_1,y_2)} = 1$ and $\|\hat{\psi}_k\|_{L^2(y_1,y_2)} = 1/\|\psi_k\|_{H^1(y_1,y_2)} \leqq 1/k \to 0$. Thus, $\hat{\psi_k} \rightharpoonup 0$ in $H^1(y_1, y_2)$.

Similar to Lemma 3.1 in [46], we have $\hat{\psi}_k \to 0$ in $H^1((a-\delta, a+\delta)\cap[y_1, y_2])$ for $a \in \{u = u_{\min}\} \cap \{u' \neq 0\}$. Similar to Lemma 3.5 and Remark 3.6 in [46], we have $\hat{\psi}_k \to 0$ in $H^1((a-\delta, a+\delta)\cap[y_1, y_2])$ for $a \in \{u = u_{\min}\}\cap\{y_1, y_2\}\cap\{u' = 0\} \cap \{u'' \neq \beta\}$. Similar to Lemma 3.7 and Remark 3.8 in [46], we have $\hat{\psi}_k \to 0$ in $H^1((a-\delta, a+\delta)\cap[y_1, y_2])$ for $a \in \{u = u_{\min}\} \cap \{u' = 0\} \cap \{u'' = \beta\}$. The main difference is that $c_k \in \mathbf{R}$ rather than $\mathrm{Im}(c_k) > 0$, and we can overcome this difficulty by perturbation of c_k as in the next case.

If $a \in \{u = u_{\min}\}\cap(y_1, y_2)\cap\{u'' \neq \beta\}$, then $0 < \beta < \frac{9}{8}\kappa_+ \leqq 9u''(a)/8$. Take $\delta \in (0, \min(y_2 - a, a - y_1))$ small enough so that $(u''(a) - \beta)(u''(y) - \beta) > 0$ on $[a - \delta, a + \delta] \subset [y_1, y_2]$ and $[a - \delta, a + \delta] \cap \{u = u_{\min}\} = \{a\}$. Noting that $\hat{\psi}_k, \frac{u''-\beta}{u-c_k} \in H^1(a-\delta, a+\delta)$, we have $\hat{\psi}_k{}'' - \alpha^2\hat{\psi}_k = \frac{u''-\beta}{u-c_k}\hat{\psi}_k \in H^1(a-\delta, a+\delta)$, and there exists $\epsilon_k > 0$ such that $\epsilon_k(1 + \|\hat{\psi}_k{}'' - \alpha^2\hat{\psi}_k\|_{H^1(a-\delta,a+\delta)}) \to 0$. Let $\widetilde{c}_k = c_k + i\epsilon_k$ and $\omega_k = -i\epsilon_k(\hat{\psi}_k{}'' - \alpha^2\hat{\psi}_k)$. Then we have $(u - \widetilde{c}_k)(\hat{\psi}_k'' - \alpha^2\hat{\psi}_k) - (u'' - \beta)\hat{\psi}_k = \omega_k$, $\|\omega_k\|_{H^1(a-\delta,a+\delta)} \to 0$, $\widetilde{c}_k \to u_{\min}$, $\mathrm{Im}(\widetilde{c}_k) > 0$ and $\hat{\psi_k} \rightharpoonup 0$ in $H^1(a-\delta, a+\delta)$. By Lemma 4.1, we have $\hat{\psi}_k \to 0$ in $H^1((a-\delta, a+\delta)\cap[y_1, y_2])$. Note that $\hat{\psi}_k \to 0$ in $C^2_{loc}([y_1, y_2]\backslash\{u = u_{\min}\})$. Therefore, $\hat{\psi}_k \to 0$ in $H^1(y_1, y_2)$, which contradicts $\|\hat{\psi}_k\|_{H^1(y_1,y_2)} = 1$. Thus, (4.2) is true.

Following Definition 3.10 in [46], we call $u_{\min}$ (or $u_{\max}$) to be an embedding eigenvalue of $\mathcal{R}_{\alpha,\beta}$ if there exists a nontrivial $\psi \in H_0^1(y_1, y_2)$ such that for any $\varphi \in H_0^1(y_1, y_2)$ and $\operatorname{supp}\varphi \subset (y_1, y_2) \setminus \{y \in (y_1, y_2) : u(y) = u_{\min}, u''(y) \neq \beta\}$,

$$\int_{y_1}^{y_2} (\psi'\varphi' + \alpha^2\psi\varphi)\,\mathrm{d}y + p.v. \int_{y_1}^{y_2} \frac{(u'' - \beta)\psi\varphi}{u - u_{\min}}\,\mathrm{d}y = 0.$$

Equivalently, $u_{\min}$ (or $u_{\max}$) is an embedding eigenvalue of the linearized operator of (1.1) (in velocity form) defined on $L^2 \times L^2$. In fact, $\boldsymbol{v} = (\psi', -i\alpha\psi) \neq 0$ is the corresponding eigenfunction.

We are now in a position to prove Theorem 2.11 (1)–(2).

Proof of Theorem 2.11 (1)–(2). First, we prove Theorem 2.11 (1). Suppose $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) = \infty$. Then by Theorem 2.9, there exist $m_\beta < n < N_\beta$ and $\{c_k\} \subset (-\infty, u_{\min})$ with $c_k \to u_{\min}^-$ such that $\lambda_n(c_k) = -\alpha^2$ is the n-th eigenvalue of (4.1) with the L^2 normalized eigenfunction ψ_k. By the definition of M_β we have $\alpha^2 = -\lambda_n(c_k) \leqq M_\beta$, which implies the second statement of Theorem 2.11 (1). To prove the first statement, we now assume that $0 < \alpha^2 \leqq M_\beta$ and $0 < \beta < \frac{9}{8}\kappa_+$. By Proposition 4.2, up to a subsequence, there exists $\psi_0 \in H_0^1(y_1, y_2)$ such that $\psi_k \rightharpoonup \psi_0$ in $H_0^1(y_1, y_2)$. In a manner similar to (3.28) in [46],

$$\lim_{k\to\infty} \int_{a-\delta}^{a+\delta} \frac{(u'' - \beta)\psi_k\varphi}{u - c_k}\,\mathrm{d}y = p.v. \int_{a-\delta}^{a+\delta} \frac{(u'' - \beta)\psi_0\varphi}{u - u_{\min}}\,\mathrm{d}y$$

for any $a \in \{u = u_{\min}\} \cap (y_1, y_2) \cap \{u'' = \beta\}$ and $\varphi \in H_0^1(a - \delta, a + \delta)$. Since $\frac{(u''-\beta)\psi_k\varphi}{u-c_k} \to \frac{(u''-\beta)\psi_0\varphi}{u-u_{\min}}$ in $C^0_{\mathrm{loc}}((y_1, y_2) \setminus \{u = u_{\min}\})$, taking limits in

$$\int_{y_1}^{y_2} (\psi_k'\varphi' + \alpha^2\psi_k\varphi) + \frac{(u'' - \beta)\psi_k\varphi}{u - c_k}\,\mathrm{d}y = 0$$

for any $\varphi \in H_0^1(y_1, y_2)$ and $\operatorname{supp}\varphi \subset (y_1, y_2) \setminus \{y \in (y_1, y_2) : u(y) = u_{\min}, u''(y) \neq \beta\}$, we get

$$\int_{y_1}^{y_2} (\psi_0'\varphi' + \alpha^2\psi_0\varphi)\,\mathrm{d}y + p.v. \int_{y_1}^{y_2} \frac{(u'' - \beta)\psi_0\varphi}{u - u_{\min}}\,\mathrm{d}y = 0.$$

If ψ_0 is nontrivial, $u_{\min}$ is an embedding eigenvalue of $\mathcal{R}_{\alpha,\beta}$, which is a contradiction. Therefore, $\psi_k \rightharpoonup \psi_0 \equiv 0$ in $H^1(y_1, y_2)$, which contradicts that $\|\psi_k\|_{L^2(y_1,y_2)} = 1$, $k \geqq 1$. Thus, $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) < \infty$. Theorem 2.11 (2) can be proved similarly. □

4.2. Rule out oscillation under "good" endpoints assumption

We rule out the oscillation of $\lambda_n(c)$ under the "good" endpoints assumption (i.e. one of the conditions (i)–(iii) in Theorem 2.2). The statement is given in Theorem 2.12. To this end, we need the following two lemma:

Lemma 4.3. *Let* $u \in C^2([y_1, y_2])$, $u(y_1) = u_{\min}$ *and* $u'(y_1) \neq 0$. *For fixed* $\gamma \in (0, 1/2]$, *there exist constants* $C > 0$ *and* $\delta \in (0, y_2 - y_1)$ *such that if* $\delta_1 \in (0, \delta]$, $z = y_1 + \delta_1$, $0 < u_{\min} - c < 1$, $\phi \in C^2([y_1, z])$ *and* $\phi'' = F$, *then*

$$|\phi|_{L^\infty(z)} \leqq C(\delta_1^\gamma |(u-c)^{2-\gamma} F|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma |\phi'(z)|), \tag{4.3}$$

$$|(u-c)^{1-\gamma}\phi|_{L^\infty(z)} + \delta_1^\gamma |(u-c)^{2-2\gamma}\phi'|_{L^\infty(z)} \tag{4.4}$$
$$\leqq C(\delta_1^\gamma |(u-c)^{3-2\gamma} F|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma |\phi'(z)|),$$

$$|(u-c)^{\gamma-1}\phi|_{L^\infty(z)} \tag{4.5}$$
$$\leqq C(|(u-c)^{\gamma+1} F|_{L^\infty(z)} + |(u_{\min}-c)^{\gamma-1}\phi(y_1)| + |\phi(z)| + |\phi'(z)|),$$

$$|(u-c)^{-1}\phi|_{L^\infty(z)} \leqq C(\delta_1^\gamma |(u-c)^{1-\gamma} F|_{L^\infty(z)} + |(u_{\min}-c)^{-1}\phi(y_1)| + |\phi'(z)|), \tag{4.6}$$

where $|f|_{L^\infty(z)} := \sup_{y \in [y_1, z]} |f(y)|$.

Proof. Since $0 < u_{\min} - c < 1$ and $-C \leqq u_{\min} \leqq u(y) \leqq C$, we have $0 < u(y) - c \leqq C$ for $y \in [y_1, z]$. Let $A_\mu = |(u-c)^\mu F|_{L^\infty(z)}$ and $B_\mu = |(u-c)^\mu \phi'|_{L^\infty(z)}$ for $\mu \in \mathbf{R}$. Let $\delta > 0$ be small enough such that $u'(y) > \frac{u'(y_1)}{2} > 0$ for $y \in [y_1, z] \subset [y_1, y_1 + \delta] \subset [y_1, y_2]$. Then

$$\int_y^z (u(s) - c)^{-\mu}\,\mathrm{d}s \leqq \frac{2}{u'(y_1)} \int_y^z u'(s)(u(s) - c)^{-\mu}\,\mathrm{d}s$$
$$= \frac{2(u(s)-c)^{1-\mu}|_{s=z}^{s=y}}{u'(y_1)(\mu - 1)} \leqq \frac{2(u(y)-c)^{1-\mu}}{u'(y_1)(\mu - 1)}$$

for fixed $\mu > 1$ and $y \in [y_1, z]$, and thus

$$\int_y^z (u(s) - c)^{-\mu}\,\mathrm{d}s \leqq \frac{2(u(y)-c)^{1-\mu}}{u'(y_1)(\mu - 1)} \leqq C(u(y) - c)^{1-\mu}. \tag{4.7}$$

Similarly, for fixed $\mu < 1$ and $y \in [y_1, z]$, we have that

$$\int_{y_1}^y (u(s) - c)^{-\mu}\,\mathrm{d}s \leqq \frac{2(u(s)-c)^{1-\mu}|_{s=y_1}^{s=y}}{u'(y_1)(1 - \mu)} \leqq C(u(y) - c)^{1-\mu}. \tag{4.8}$$

Since $u(s) - c \geqq u(s) - u_{\min} \geqq u'(y_1)(s - y_1)/2$ and $u(s) - c \geqq u(y) - c$ for $y_1 \leqq y \leqq s \leqq z$, we have for fixed $\mu \geqq 1 - \gamma$,

$$\int_y^z (u(s) - c)^{-\mu}\,\mathrm{d}s \leqq \int_y^z (u'(y_1)(s - y_1)/2)^{\gamma-1}(u(y) - c)^{1-\gamma-\mu}\,\mathrm{d}s \tag{4.9}$$
$$\leqq C(u(y) - c)^{1-\gamma-\mu} \int_y^z (s - y_1)^{\gamma-1}\,\mathrm{d}s \leqq C(u(y) - c)^{1-\gamma-\mu}(z - y_1)^\gamma$$
$$= C\delta_1^\gamma (u(y) - c)^{1-\gamma-\mu}.$$

For fixed $\mu > 1$, using (4.7) and the definition of A_μ, we have, for $y \in [y_1, z]$,

$$|\phi'(y)-\phi'(z)| \leqq \int_y^z |\phi''(s)|\,\mathrm{d}s = \int_y^z |F(s)|\,\mathrm{d}s \leqq \int_y^z (u(s)-c)^{-\mu}A_\mu\,\mathrm{d}s$$
$$\leqq C(u(y)-c)^{1-\mu}A_\mu,$$

and

$$|\phi'(y)| \leqq |\phi'(z)| + C(u(y)-c)^{1-\mu}A_\mu,$$
$$|(u(y)-c)^{\mu-1}\phi'(y)| \leqq (u(y)-c)^{\mu-1}|\phi'(z)| + CA_\mu \leqq C|\phi'(z)| + CA_\mu.$$

Then by the definition of B_μ, we have

$$B_{\mu-1} = \sup_{y\in[y_1,z]} |(u(y)-c)^{\mu-1}\phi'(y)| \leqq CA_\mu + C|\phi'(z)| \text{ for fixed } \mu > 1. \tag{4.10}$$

Similarly, for fixed $\mu \geqq 1-\gamma$, using (4.9) and the definition of A_μ, we have for $y \in [y_1, z]$,

$$|\phi(y)-\phi(z)| \leqq \int_y^z |\phi'(s)|\,\mathrm{d}s \leqq \int_y^z (u(s)-c)^{-\mu}B_\mu\,\mathrm{d}s \leqq C\delta_1^\gamma (u(y)-c)^{1-\gamma-\mu}B_\mu.$$

This implies

$$|(u-c)^{\mu+\gamma-1}\phi|_{L^\infty(z)} \leqq C\delta_1^\gamma B_\mu + C|\phi(z)| \text{ for fixed } \mu \geqq 1-\gamma. \tag{4.11}$$

Using (4.11) for $\mu = 1-\gamma$ and (4.10) for $\mu = 2-\gamma$, we have

$$|\phi|_{L^\infty(z)} \leqq C\delta_1^\gamma B_{1-\gamma} + C|\phi(z)| \leqq C\delta_1^\gamma (A_{2-\gamma} + |\phi'(z)|) + C|\phi(z)|,$$

which implies (4.3) by recalling the definition of A_μ. Using (4.11) for $\mu = 2-2\gamma$ and (4.10) for $\mu = 3-2\gamma$, we have

$$|(u-c)^{1-\gamma}\phi|_{L^\infty(z)} + \delta_1^\gamma B_{2-2\gamma} \leqq C\delta_1^\gamma B_{2-2\gamma} + C|\phi(z)|$$
$$\leqq C\delta_1^\gamma (A_{3-2\gamma} + |\phi'(z)|) + C|\phi(z)|,$$

which implies (4.4) by recalling the definition of A_μ and B_μ.

For fixed $\mu < 1$, using (4.8) and the definition of A_μ, we have for $y \in [y_1, z]$,

$$|\phi(y)-\phi(y_1)| \leqq \int_{y_1}^y |\phi'(s)|\,\mathrm{d}s \leqq \int_{y_1}^y (u(s)-c)^{-\mu}B_\mu\,\mathrm{d}s \leqq C(u(y)-c)^{1-\mu}B_\mu,$$

and

$$|\phi(y)| \leqq |\phi(y_1)| + C(u(y)-c)^{1-\mu}B_\mu,$$
$$|(u(y)-c)^{\mu-1}\phi(y)| \leqq (u(y)-c)^{\mu-1}|\phi(y_1)| + CB_\mu \leqq (u_{\min}-c)^{\mu-1}|\phi(y_1)| + CB_\mu,$$

where we used $u(y)-c \geqq u_{\min}-c > 0$ and $\mu - 1 < 0$. Thus,

$$|(u-c)^{\mu-1}\phi|_{L^\infty(z)} \leqq (u_{\min}-c)^{\mu-1}|\phi(y_1)| + CB_\mu \text{ for fixed } \mu < 1. \tag{4.12}$$

Using (4.12) for $\mu = \gamma$ and (4.10) for $\mu = 1+\gamma$, we have

$$|(u-c)^{\gamma-1}\phi|_{L^\infty(z)} \leqq (u_{\min}-c)^{\gamma-1}|\phi(y_1)| + CB_\gamma$$
$$\leqq (u_{\min}-c)^{\gamma-1}|\phi(y_1)| + C(A_{1+\gamma} + |\phi'(z)|),$$

which implies (4.5) by recalling the definition of A_μ. Using (4.9) for $\mu = 1-\gamma$ and the definition of A_μ, we have for $y \in [y_1, z]$,

$$|\phi'(y)-\phi'(z)| \leqq \int_y^z |\phi''(s)|ds$$
$$= \int_y^z |F(s)|\,\mathrm{d}s \leqq \int_y^z (u(s)-c)^{\gamma-1}A_{1-\gamma}\,\mathrm{d}s \leqq C\delta_1^\gamma A_{1-\gamma}.$$

Then by the definition of B_μ, we have that

$$B_0 = \sup_{y\in[y_1,z]} |\phi'(y)| \leqq C\delta_1^\gamma A_{1-\gamma} + C|\phi'(z)|. \tag{4.13}$$

Using (4.12) for $\mu = 0$ and (4.13), we have that

$$|(u-c)^{-1}\phi|_{L^\infty(z)} \leqq (u_{\min}-c)^{-1}|\phi(y_1)| + CB_0 \leqq (u_{\min}-c)^{-1}|\phi(y_1)| + C(\delta_1^\gamma A_{1-\gamma} + |\phi'(z)|),$$

which implies (4.6) by recalling the definition of A_μ.

Lemma 4.4. *Let $u \in C^2([y_1, y_2])$, $u(y_1) = u_{\min}$ and $u'(y_1) \neq 0$. For fixed $\gamma \in (0, 1/2]$, there exist constants $C > 0$ and $\delta_1 > 0$ such that if $z = y_1 + \delta_1$, $0 < u_{\min} - c < 1$, $\phi \in C^2([y_1, z])$ and*

$$-\phi'' + \frac{u''-\beta}{u-c}\phi = -\alpha^2\phi - F \quad on\ [y_1, z], \tag{4.14}$$

then the inequalities (4.3)–(4.6) are still true.

Proof. Let $\widetilde{F} = \frac{u''-\beta}{u-c}\phi + \alpha^2\phi + F$ and $\delta_1 \in (0, \delta]$ be given in Lemma 4.3. Then $\phi'' = \widetilde{F}$ on $[y_1, z]$. By Lemma 4.3, (4.3)–(4.6) are still true with F replaced by $\widetilde{F}$. As $|u''-\beta| \leqq C$ and $|u-c| \leqq C$, we have $|F-\widetilde{F}| \leqq C|\phi/(u-c)|$ for $y \in [y_1, z]$. Thus, for $\mu \in \mathbf{R}$, we have that

$$\begin{aligned}|(u-c)^\mu \widetilde{F}|_{L^\infty(z)} &\leqq |(u-c)^\mu F|_{L^\infty(z)} + |(u-c)^\mu (F-\widetilde{F})|_{L^\infty(z)} \\ &\leqq |(u-c)^\mu F|_{L^\infty(z)} + C|(u-c)^{\mu-1}\phi|_{L^\infty(z)}.\end{aligned} \tag{4.15}$$

Using (4.3) with F replaced by $\widetilde{F}$ and (4.15) for $\mu = 2-\gamma$, we have that

$$\begin{aligned}&|\phi|_{L^\infty(z)} \leqq C(\delta_1^\gamma|(u-c)^{2-\gamma}\widetilde{F}|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma|\phi'(z)|) \\ &\leqq C\delta_1^\gamma|(u-c)^{1-\gamma}\phi|_{L^\infty(z)} + C(\delta_1^\gamma|(u-c)^{2-\gamma}F|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma|\phi'(z)|) \\ &\leqq C_1\delta_1^\gamma|\phi|_{L^\infty(z)} + C(\delta_1^\gamma|(u-c)^{2-\gamma}F|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma|\phi'(z)|).\end{aligned} \tag{4.16}$$

Using (4.4) with F replaced by $\widetilde{F}$ and (4.15) for $\mu = 3-2\gamma$, we have that

$$\begin{aligned}&|(u-c)^{1-\gamma}\phi|_{L^\infty(z)} + \delta_1^\gamma|(u-c)^{2-2\gamma}\phi'|_{L^\infty(z)} \\ &\leqq C(\delta_1^\gamma|(u-c)^{3-2\gamma}\widetilde{F}|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma|\phi'(z)|) \\ &\leqq C(\delta_1^\gamma|(u-c)^{3-2\gamma}F|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma|\phi'(z)|) + C\delta_1^\gamma|(u-c)^{2-2\gamma}\phi|_{L^\infty(z)} \\ &\leqq C(\delta_1^\gamma|(u-c)^{3-2\gamma}F|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma|\phi'(z)|) + C_1\delta_1^\gamma|(u-c)^{1-\gamma}\phi|_{L^\infty(z)}.\end{aligned} \tag{4.17}$$

Using (4.6) with F replaced by $\widetilde{F}$ and (4.15) for $\mu = 1 - \gamma$, we have that

$$\begin{aligned}
&|(u-c)^{-1}\phi|_{L^\infty(z)} \leqq C(\delta_1^\gamma|(u-c)^{1-\gamma}\widetilde{F}|_{L^\infty(z)} + |(u_{\min}-c)^{-1}\phi(y_1)| + |\phi'(z)|) \quad (4.18)\\
\leqq & C(\delta_1^\gamma|(u-c)^{1-\gamma}F|_{L^\infty(z)} + |(u_{\min}-c)^{-1}\phi(y_1)| + |\phi'(z)|) + C\delta_1^\gamma|(u-c)^{-\gamma}\phi|_{L^\infty(z)}\\
\leqq & C(\delta_1^\gamma|(u-c)^{1-\gamma}F|_{L^\infty(z)} + |(u_{\min}-c)^{-1}\phi(y_1)| + |\phi'(z)|) + C_1\delta_1^\gamma|(u-c)^{-1}\phi|_{L^\infty(z)}.
\end{aligned}$$

Here, $C_1 > 0$ is a constant depending only on $\gamma,\ \alpha,\ \beta,\ u,\ \delta$ (and independent of δ_1). Taking $\delta_1 \in (0, \delta]$ small enough such that $C_1\delta_1^\gamma \leqq 1/2$ in (4.16)–(4.18), we obtain (4.3), (4.4) and (4.6).

Note that $\gamma > 0$ and $2-\gamma \geqq \gamma+1$. Using (4.15) for $\mu = \gamma+1$ and (4.3), we have that

$$\begin{aligned}
&|(u-c)^{\gamma+1}\widetilde{F}|_{L^\infty(z)} \leqq |(u-c)^{\gamma+1}F|_{L^\infty(z)} + C|(u-c)^\gamma\phi|_{L^\infty(z)} \qquad (4.19)\\
\leqq & |(u-c)^{\gamma+1}F|_{L^\infty(z)} + C|\phi|_{L^\infty(z)}\\
\leqq & |(u-c)^{\gamma+1}F|_{L^\infty(z)} + C(\delta_1^\gamma|(u-c)^{2-\gamma}F|_{L^\infty(z)} + |\phi(z)| + \delta_1^\gamma|\phi'(z)|)\\
\leqq & C(|(u-c)^{\gamma+1}F|_{L^\infty(z)} + |\phi(z)| + |\phi'(z)|).
\end{aligned}$$

Using (4.5) with F replaced by $\widetilde{F}$ and (4.19), we have that

$$\begin{aligned}
&|(u-c)^{\gamma-1}\phi|_{L^\infty(z)}\\
\leqq & C(|(u-c)^{\gamma+1}\widetilde{F}|_{L^\infty(z)} + |(u_{\min}-c)^{\gamma-1}\phi(y_1)| + |\phi(z)| + |\phi'(z)|)\\
\leqq & C(|(u-c)^{\gamma+1}F|_{L^\infty(z)} + |(u_{\min}-c)^{\gamma-1}\phi(y_1)| + |\phi(z)| + |\phi'(z)|).
\end{aligned}$$

Thus, (4.5) is also true. □

We are now in a position to prove Theorem 2.12.

Proof of Theorem 2.12. We only prove (1), and the proof of (2) is similar. If $\{u' = 0\} \cap \{u = u_{\min}\} \neq \emptyset$, then $0 < \kappa_+ < \infty$. By Theorem 2.11 (3), $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}) = \infty$ for $\beta > \frac{9}{8}\kappa_+$. If $\{u' = 0\} \cap \{u = u_{\min}\} = \emptyset$, then $\{u = u_{\min}\} \subset \{y_1, y_2\}$ (i.e. $u = u_{\min}$ can be achieved only at the endpoints). We assume that $u(y_1) = u_{\min}$. Then $u(y_2) > u(y_1)$ and $u'(y_1) > 0$. By taking $\delta \in (0, y_2 - y_1)$ smaller, we can assume that $u' > \frac{u'(y_1)}{2}$ on $y \in [y_1, y_1+\delta]$. Let $\psi_c, c \in \mathbf{C} \setminus \mathrm{Ran}(u)$, be the solution of

$$-\partial_y^2\psi_c + \frac{u''-\beta}{u-c}\psi_c = -\alpha^2\psi_c \ \text{ on } \ [y_1, y_2],\ \ \psi_c(y_2) = 0,\ \ \partial_y\psi_c(y_2) = 1. \tag{4.20}$$

Note that for $c \in (-\infty, u_{\min})$, $c \in \sigma_d(\mathcal{R}_{\alpha,\beta})$ if and only if $\psi_c(y_1) = 0$. Suppose that $\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min}) = \{c_k\}_{k=1}^\infty$. Then $c_k \to u_{\min}^-$. Note that if (iii) is true for $i = 1$, then $\beta_1 \leqq 0$ and thus $\beta \neq \beta_1$ for all $\beta > 0$. So we divide the discussion into two cases.

Case 1. $\beta = \beta_1$ and (i) holds for $i = 1$.

Case 2. $\beta = \beta_1$ and (ii) holds for $i = 1$; or $\beta \neq \beta_1$.

If Case 1 is true, then $u'' - \beta = 0$ on $[y_1, y_1+\delta]$. By (4.20), ψ_c can be extended to an analytic function in $\mathbf{C} \setminus u([y_1+\delta, y_2])$. Since $u_{\min} \notin u([y_1+\delta, y_2])$, $\psi_c(y_1)$

has a finite number of zeros in a neighborhood of $c = u_{\min}$, which contradicts that $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) = \infty$.

Now, we assume Case 2 is true. If $\beta \neq \beta_1$, define $m = 0$; if $\beta = \beta_1$ and (ii) is true, define $m = m_1$. Then $m \geqq 0$, by taking $\delta > 0$ smaller, we can assume that

$$C^{-1}|y - y_1|^m \leqq |u''(y) - \beta| \leqq C|y - y_1|^m \quad \text{for} \quad y \in [y_1, y_1 + \delta]. \tag{4.21}$$

As $|u''(y) - \beta|/|y - y_1|^m \in C([y_1 + \delta, y_2])$, $|u''(y) - \beta| \leqq C|y - y_1|^m$ is also true for $y \in [y_1 + \delta, y_2]$. Since $u \in C^1([y_1, y_2])$, we have $u(y) - u_{\min} = u(y) - u(y_1) = \int_{y_1}^{y} u'(z)\,\mathrm{d}z = (y - y_1)v(y)$, here $v(y) = \int_0^1 u'(y_1 + s(y - y_1))\,\mathrm{d}s$ and $v \in C([y_1, y_2])$. If $y \in (y_1, y_2]$, we have $u(y) > u_{\min}$ and $v(y) > 0$. If $y = y_1$, we have $v(y) = u'(y_1) > 0$. Thus, $v(y) > 0$ in $[y_1, y_2]$ and there exists a constant $C > 1$ such that $C^{-1} \leqq v(y) \leqq C$, which implies

$$C^{-1}|y - y_1| \leqq u(y) - u_{\min} \leqq C|y - y_1|, \tag{4.22}$$

$$C^{-1}|y - y_1| \leqq u(y) - c \quad \text{for} \quad c \leqq u_{\min}. \tag{4.23}$$

Let $n \in \mathbf{N}$ and $\psi_{c,n} = \partial_c^n \psi_c$. By Rolle's Theorem, there exists $\{c_{k,n}\}_{k=1}^{\infty} \subset (-\infty, u_{\min})$ such that $\psi_{c_{k,n},n}(y_1) = 0$ and $c_{k,n} \to u_{\min}^-$ as $k \to \infty$. For fixed $c < u_{\min}$, let $k > 0$ be large enough such that $c_{k,n} \in (c, u_{\min})$. Then $\psi_{c,n}(y_1) = \int_{c_{k,n}}^{c} \partial_s \psi_{s,n}(y_1)\,\mathrm{d}s = \int_{c_{k,n}}^{c} \psi_{s,n+1}(y_1)\,\mathrm{d}s$, and

$$|\psi_{c,n}(y_1)| \leqq \int_c^{c_{k,n}} |\psi_{s,n+1}(y_1)|\,\mathrm{d}s \leqq \int_c^{u_{\min}} |\psi_{s,n+1}(y_1)|\,\mathrm{d}s. \tag{4.24}$$

Moreover, $\psi_{c,n}$ satisfies

$$\begin{aligned} &-\partial_y^2 \psi_{c,n} + \frac{u'' - \beta}{u - c}\psi_{c,n} = -\alpha^2 \psi_{c,n} - F_{c,n}, \\ &F_{c,n} = \sum_{k=1}^{n} \frac{n!}{(n-k)!} \frac{u'' - \beta}{(u - c)^{k+1}} \psi_{c,n-k}, \end{aligned} \tag{4.25}$$

where $\psi_{c,0} = \psi_c$ and $F_{c,0} = 0$. Note that $\psi_c(y)$ is continuous on $(\mathbf{C} \setminus \mathrm{Ran}(u)) \times [y_1, y_2]$, and analytic in c. Let $u_+(y) = \inf u([y, y_2])$. Then $\psi_c(y)$ can be extended to a continuous function on $D_1 := \{(c, y) : c < u_+(y),\ y \in [y_1, y_2]\}$, still satisfying (4.20) in D_1. Moreover, u_+ is increasing and continuous on $[y_1, y_2]$, $u_+(y_1) = u_{\min}$ and $u_+(y) > u_{\min}$ for $y \in (y_1, y_2]$. By standard theory of ODE, $\partial_c^n \psi_c = \psi_{c,n} \in C^2(D_1)$ and $\psi_{c,n}|_{D_1}$ is real-valued for $n \in \mathbf{N}$. Using this extension, $\psi_c|_{c=u_{\min}}$ is well-defined and satisfies (4.20) for $y \in (y_1, y_2]$. For fixed $\delta_1 \in (0, \delta]$ and $n \in \mathbf{N}$, we have

$$|\psi_{c,n}(y)| + |\partial_y \psi_{c,n}(y)| \leqq C(n, \delta_1) \text{ for } y \in [y_1 + \delta_1, y_2] \text{ and } 0 \leqq u_{\min} - c \leqq 1, \tag{4.26}$$

since a continuous function is bounded in a compact set. Let $m_0 \in \mathbf{Z}$ be such that $m_0 - 1 < m \leqq m_0$ and $\gamma = (m + 1 - m_0)/2$. Then $m_0 \geqq 0$ and $\gamma \in (0, 1/2]$. We claim that the uniform bounds

$$|\psi_{c,n}| \leqq C, \ |\psi_{c,0}| \leqq C|u-c|, \ |\psi_{c,1}| \leqq C|u-c|^{1-\gamma}, \ |\partial_y \psi_{c,m_0+2}| \leqq C|u-c|^{2\gamma-2} \tag{4.27}$$

hold for $0 < u_{\min} - c < 1$, $y \in [y_1, y_2]$ and $n \in \mathbf{Z} \cap [0, m_0+1]$. Assume that the uniform bounds (4.27) are true, which will be verified later. Let $W_{c,n} = \partial_y \psi_{c,n} \psi_c - \partial_y \psi_c \psi_{c,n}$. Then we get by (4.25) that $\partial_y W_{c,n} = F_{c,n} \psi_c$. By (4.27) and using $u(y_1) = u_{\min}$, we have for $0 < u_{\min} - c < 1$,

$$|\psi_c(y_1)| \leqq \int_c^{u_{\min}} |\psi_{s,1}(y_1)| \,\mathrm{d}s \leqq C \int_c^{u_{\min}} |u_{\min} - s|^{1-\gamma} \,\mathrm{d}s \leqq C|u_{\min} - c|^{2-\gamma},$$

and thus

$$\begin{aligned} |\partial_y \psi_{c,m_0+2} \psi_c|(y_1) &= |\partial_y \psi_{c,m_0+2}(y_1)||\psi_c(y_1)| \\ &\leqq C|u_{\min} - c|^{2\gamma-2}|u_{\min} - c|^{2-\gamma} = C|u_{\min} - c|^{\gamma}, \end{aligned}$$

which implies $\lim\limits_{c \to u_{\min}^-} \partial_y \psi_{c,m_0+2} \psi_c(y_1) = 0$. Since $\psi_{c_{k,m_0+2},m_0+2}(y_1) = 0$ for $k \geqq 1$ and $c_{k,m_0+2} \to u_{\min}^-$, we have $\liminf\limits_{c \to u_{\min}^-} |\partial_y \psi_c \psi_{c,m_0+2}|(y_1) = 0$, and thus $\liminf\limits_{c \to u_{\min}^-} |W_{c,m_0+2}|(y_1) = 0$.

Since $\psi_c(y_2) = 0$, $\partial_y \psi_c(y_2) = 1$ and recall that $\psi_{c,n} = \partial_c^n \psi_c$, we have $\psi_{c,n}(y_2) = 0$, $\partial_y \psi_{c,n}(y_2) = 0$ and $W_{c,n}(y_2) = 0$ for $n > 0$. Thus, $-W_{c,n}(y_1) = \int_{y_1}^{y_2} \partial_y W_{c,n}(y) \,\mathrm{d}y = \int_{y_1}^{y_2} F_{c,n} \psi_c(y) \,\mathrm{d}y$. Note that

$$F_{c,n} - n! \frac{u'' - \beta}{(u-c)^{n+1}} \psi_c = \sum_{k=1}^{n-1} \frac{n!}{(n-k)!} \frac{u'' - \beta}{(u-c)^{k+1}} \psi_{c,n-k}.$$

If $n \in \mathbf{Z} \cap [2, m_0+2]$, by (4.27) and (4.21), we have for $0 < u_{\min} - c < 1$ and $y \in [y_1, y_2]$,

$$\begin{aligned} \left| F_{c,n} - \frac{n!(u''-\beta)\psi_c}{(u-c)^{n+1}} \right| &\leqq \sum_{k=1}^{n-1} \frac{n!}{(n-k)!} \frac{|u''-\beta|}{(u-c)^{k+1}} |\psi_{c,n-k}| \\ &\leqq C \sum_{k=1}^{n-2} \frac{|y-y_1|^m}{(u-c)^{k+1}} + C \frac{|y-y_1|^m |u-c|^{1-\gamma}}{(u-c)^n} \leqq C \frac{|y-y_1|^m}{|u-c|^{n-1+\gamma}}, \\ \left| F_{c,n}\psi_c - \frac{n!(u''-\beta)\psi_c^2}{(u-c)^{n+1}} \right| &\leqq C \frac{|y-y_1|^m |\psi_c|}{|u-c|^{n-1+\gamma}} \leqq C \frac{|y-y_1|^m |u-c|}{|u-c|^{n-1+\gamma}} \\ &= C \frac{|y-y_1|^m}{|u-c|^{n-2+\gamma}}, \end{aligned}$$

and thus using $m - m_0 = 2\gamma - 1$ and (4.23), we have that

$$\begin{aligned} \left| F_{c,m_0+2}\psi_c - \frac{(m_0+2)!(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}} \right| &\leqq C \frac{|y-y_1|^m}{|u-c|^{m_0+\gamma}} \leqq C \frac{|y-y_1|^m}{|y-y_1|^{m_0+\gamma}} \\ &= C|y-y_1|^{\gamma-1}. \end{aligned}$$

Integrating it on $[y_1, y_2]$ and using $-W_{c,n}(y_1) = \int_{y_1}^{y_2} F_{c,n}\psi_c(y)\,\mathrm{d}y$, we have for $0 < u_{\min} - c < 1$,

$$\left|\int_{y_1}^{y_2} \frac{(m_0+2)!(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y\right| \leqq \left|\int_{y_1}^{y_2} F_{c,m_0+2}\psi_c\,\mathrm{d}y\right| + C\int_{y_1}^{y_2} |y-y_1|^{\gamma-1}\,\mathrm{d}y$$
$$\leqq |W_{c,m_0+2}|(y_1) + C,$$

and as $\liminf\limits_{c\to u_{\min}^-} |W_{c,m_0+2}|(y_1) = 0$, we have that

$$\liminf_{c\to u_{\min}^-}\left|\int_{y_1}^{y_2} \frac{(m_0+2)!(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y\right| \leqq C, \quad \liminf_{c\to u_{\min}^-}\left|\int_{y_1}^{y_2} \frac{(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y\right| \leqq C.$$

Since $u''-\beta$ is continuous and real-valued, and $C^{-1}|y-y_1|^m \leqq |u''(y)-\beta|$, it does not change sign on $y \in [y_1, y_1+\delta]$. Then for $0 < u_{\min} - c < 1$,

$$\left|\int_{y_1}^{y_1+\delta} \frac{(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y\right| = \int_{y_1}^{y_1+\delta} \frac{|u''-\beta|\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y \geqq C^{-1}\int_{y_1}^{y_1+\delta} \frac{|y-y_1|^m\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y.$$

As $|\psi_c| \leqq C,\ |u''-\beta| \leqq C, u-c \geqq u-u_{\min} \geqq C^{-1}$ for $y \in [y_1+\delta, y_2]$, and $0 < u_{\min} - c < 1$, we have that

$$\left|\int_{y_1+\delta}^{y_2} \frac{(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}} dy\right| \leqq \int_{y_1+\delta}^{y_2} C\,\mathrm{d}y \leqq C.$$

Thus,

$$\liminf_{c\to u_{\min}^-}\int_{y_1}^{y_1+\delta} \frac{|y-y_1|^m\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y \leqq C\liminf_{c\to u_{\min}^-}\left|\int_{y_1}^{y_1+\delta} \frac{(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y\right|$$
$$\leqq C\liminf_{c\to u_{\min}^-}\left|\int_{y_1}^{y_2} \frac{(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y\right| + C\limsup_{c\to u_{\min}^-}\left|\int_{y_1+\delta}^{y_2} \frac{(u''-\beta)\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y\right| \leqq C.$$

Since $\psi_c \in C(D_1)$, we have for fixed $y \in (y_1, y_1+\delta]$,

$$\lim_{c\to u_{\min}^-} \psi_c(y) = \psi_{u_{\min}}(y), \qquad \lim_{c\to u_{\min}^-} \frac{|y-y_1|^m\psi_c^2}{(u-c)^{m_0+3}} = \frac{|y-y_1|^m\psi_{u_{\min}}^2}{(u-u_{\min})^{m_0+3}}.$$

Thus, by Fatou's Lemma, we have that

$$\int_{y_1}^{y_1+\delta} \frac{|y-y_1|^m\psi_{u_{\min}}^2}{(u-u_{\min})^{m_0+3}}\,\mathrm{d}y = \int_{y_1}^{y_1+\delta} \lim_{c\to u_{\min}^-} \frac{|y-y_1|^m\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y$$
$$\leqq \liminf_{c\to u_{\min}^-}\int_{y_1}^{y_1+\delta} \frac{|y-y_1|^m\psi_c^2}{(u-c)^{m_0+3}}\,\mathrm{d}y \leqq C.$$

By (4.22), we have that

$$\frac{|y-y_1|^m\psi_{u_{\min}}^2}{(u-u_{\min})^{m_0+3}} \geqq \frac{|y-y_1|^m\psi_{u_{\min}}^2}{(C(y-y_1))^{m_0+3}} \geqq \frac{C^{-1}\psi_{u_{\min}}^2}{|y-y_1|^{m_0-m+3}} \geqq \frac{C^{-1}\psi_{u_{\min}}^2}{|y-y_1|^3}$$

for $y \in (y_1, y_1 + \delta]$, where we used $m \leqq m_0$. Thus,

$$\int_{y_1}^{y_1+\delta} \frac{\psi_{u_{\min}}^2}{|y - y_1|^3}\,dy \leqq C \int_{y_1}^{y_1+\delta} \frac{|y - y_1|^m \psi_{u_{\min}}^2}{(u - u_{\min})^{m_0+3}}\,dy \leqq C.$$

Now we take $\varphi = \psi_{u_{\min}}$. Then φ is real-valued and for $y \in (y_1, y_2]$, it satisfies

$$-\varphi'' + \frac{u'' - \beta}{u - u_{\min}}\varphi = -\alpha^2\varphi,\ \varphi(y_2) = 0,\ \varphi'(y_2) = 1,\ \int_{y_1}^{y_1+\delta} \frac{\varphi^2}{|y - y_1|^3}\,dy \leqq C. \tag{4.28}$$

Thus, $\varphi, \varphi/|y - y_1| \in L^2(y_1, y_1 + \delta)$. By (4.22), we have that for $y \in (y_1, y_1 + \delta]$,

$$\left|\frac{u'' - \beta}{u - u_{\min}}\varphi\right| \leqq \frac{C|\varphi|}{u - u_{\min}} \leqq \frac{C|\varphi|}{|y - y_1|}, \quad \frac{u'' - \beta}{u - u_{\min}}\varphi \in L^2(y_1, y_1 + \delta).$$

Thus, $\varphi'' \in L^2(y_1, y_1 + \delta)$, $\varphi \in H^2(y_1, y_1 + \delta)$ and $\varphi \in C^1([y_1, y_1 + \delta])$ by defining $\varphi(y_1) = \lim_{y\to y_1^+} \varphi(y)$. If $\varphi(y_1) \neq 0$, then there exists $\delta_1 \in (0, \delta]$ such that $|\varphi(y)| \geqq |\varphi(y_1)|/2 \geqq C^{-1}|y - y_1|$ for $y \in (y_1, y_1 + \delta_1]$. If $\varphi(y_1) = 0$ and $\varphi'(y_1) \neq 0$, then there exists $\delta_1 \in (0, \delta]$ such that $|\varphi'(y)| \geqq |\varphi'(y_1)|/2$ for $y \in (y_1, y_1 + \delta_1]$, and $|\varphi(y)| = |\varphi(y) - \varphi(y_1)| = |(y - y_1)\varphi'(\xi_y)| \geqq |y - y_1||\varphi'(y_1)|/2$ for $\xi_y \in (y_1, y)$ and $y \in (y_1, y_1 + \delta_1]$. Therefore, if $\varphi(y_1) \neq 0$ or $\varphi'(y_1) \neq 0$, then there exists $\delta_1 \in (0, \delta]$ and $C > 0$ such that $|\varphi(y)| \geqq C^{-1}|y - y_1|$ for $y \in (y_1, y_1 + \delta_1]$, and

$$\int_{y_1}^{y_1+\delta} \frac{\varphi^2}{|y - y_1|^3}\,dy \geqq C^{-2} \int_{y_1}^{y_1+\delta_1} \frac{|y - y_1|^2}{|y - y_1|^3}\,dy = +\infty,$$

which contradicts (4.28). Thus, we must have $\varphi(y_1) = \varphi'(y_1) = 0$. Then by the proof of Lemma 3 in [27], we have $\varphi \equiv 0$ on $[y_1, y_2]$, which contradicts $\varphi'(y_2) = 1$. This proves (1) for Case 2.

It remains to prove (4.27). Let $\delta_1 \in (0, \delta]$ be fixed such that Lemma 4.4 is true. Recall that $z = y_1 + \delta_1$ and $|f|_{L^\infty(z)} = \sup_{y\in[y_1,z]} |f(y)|$. By (4.26) we know that (4.27) is true for $y \in [z, y_2]$. Now we assume that $y \in [y_1, z]$, $0 < u_{\min} - c < 1$, and that (4.3)–(4.6) are used for F satisfying (4.14) (i.e. the condition in Lemma 4.4). The proof of (4.27) for $y \in [y_1, z]$ is divided into 7 steps as follow:

Step 1. $|\psi_{c,n}| \leqq C$ for $y \in [y_1, z]$ and $n \in \mathbf{Z} \cap [0, m_0]$.

For $n = 0$, by (4.25), (4.3), (4.26) and $F_{c,0} = 0$, we have $|\psi_{c,0}|_{L^\infty(z)} \leqq C(|\psi_{c,0}(z)| + \delta_1^\gamma |\psi'_{c,0}(z)|) \leqq C$, and thus $|\psi_{c,0}| \leqq C$ for $y \in [y_1, z]$. Now, we prove the result by induction. Assume that $n \in \mathbf{Z} \cap (0, m_0]$ and $|\psi_{c,k}| \leqq C$ for $k \in \mathbf{Z} \cap [0, n)$ and $y \in [y_1, z]$. Then by (4.25), (4.21), (4.23) and $m - m_0 = 2\gamma - 1$, we have, for $y \in [y_1, z]$, that

$$|F_{c,n}| \leqq C \sum_{k=1}^{n} \frac{|y - y_1|^m |\psi_{c,n-k}|}{(u - c)^{k+1}} \leqq C \sum_{k=1}^{n} \frac{|y - y_1|^m}{(u - c)^{k+1}} \leqq C \frac{(u - c)^m}{(u - c)^{n+1}}, \tag{4.29}$$

$$|(u - c)^{2-\gamma} F_{c,n}| \leqq C(u - c)^{2-\gamma+m-n-1} \leqq C(u - c)^{m-m_0+1-\gamma}$$

$$= C(u-c)^{\gamma} \leqq C. \tag{4.30}$$

By (4.3), (4.26) and (4.30), we have that

$$|\psi_{c,n}|_{L^\infty(z)} \leqq C(\delta_1^{\gamma}|(u-c)^{2-\gamma}F_{c,n}|_{L^\infty(z)} + |\psi_{c,n}(z)| + \delta_1^{\gamma}|\psi'_{c,n}(z)|) \leqq C, \tag{4.31}$$

which means $|\psi_{c,n}| \leqq C$ for $y \in [y_1, z]$. Thus, the result in Step 1 is true.

Step 2. $|\psi_{c,n}| \leqq C|u-c|^{\gamma-1}$ for $y \in [y_1, z]$ and $n = m_0 + 1$.

Let $n = m_0 + 1$. By Step 1, we know that $|\psi_{c,k}| \leqq C$ for $y \in [y_1, z]$ and $k \in \mathbf{Z} \cap [0, n)$. Thus, (4.29) is still true and for $y \in [y_1, z]$,

$$|(u-c)^{3-2\gamma}F_{c,n}| \leqq C(u-c)^{3-2\gamma+m-n-1} = C(u-c)^{m-m_0+1-2\gamma} = C,$$

which, along with (4.4) and (4.26), implies that

$$|(u-c)^{1-\gamma}\psi_{c,n}|_{L^\infty(z)} \leqq C(\delta_1^{\gamma}|(u-c)^{3-2\gamma}F_{c,n}|_{L^\infty(z)} + |\psi_{c,n}(z)| + \delta_1^{\gamma}|\psi'_{c,n}(z)|) \leqq C.$$

Then $|(u-c)^{1-\gamma}\psi_{c,n}| \leqq C$, and thus $|\psi_{c,n}| \leqq C|u-c|^{\gamma-1}$ for $y \in [y_1, z]$.

Step 3. $|\psi_{c,0}| \leqq C|u-c|^{1-\gamma}$ for $y \in [y_1, z]$.

If $m_0 = 0$, then $m = 0$ and $\gamma = 1/2$. By Step 2, we have $|\psi_{c,1}| \leqq C|u-c|^{\gamma-1} = C|u-c|^{-\gamma}$ for $y \in [y_1, z]$. If $m_0 > 0$, then by Step 1, we have $|\psi_{c,1}| \leqq C \leqq C|u-c|^{-\gamma}$ for $y \in [y_1, z]$. Thus, $|\psi_{c,1}| \leqq C|u-c|^{-\gamma}$ is always true for $y \in [y_1, z]$. Then by (4.24) and $u(y_1) = u_{\min}$, we have $|\psi_{c,0}(y_1)| \leqq \int_c^{u_{\min}} |\psi_{s,1}(y_1)|\, ds \leqq C\int_c^{u_{\min}} |u_{\min} - s|^{-\gamma}\, ds \leqq C|u_{\min} - c|^{1-\gamma}$. By (4.5), (4.26) and $F_{c,0} = 0$, we have $|(u-c)^{\gamma-1}\psi_{c,0}|_{L^\infty(z)} \leqq C(|(u_{\min}-c)^{\gamma-1}\psi_{c,0}(y_1)| + |\psi_{c,0}(z)| + |\psi'_{c,0}(z)|) \leqq C$. Then $|(u-c)^{\gamma-1}\psi_{c,0}| \leqq C$, and thus $|\psi_{c,0}| \leqq C|u-c|^{1-\gamma}$ for $y \in [y_1, z]$.

Step 4. $|\psi_{c,n}| \leqq C$ for $y \in [y_1, z]$ and $n = m_0 + 1$.

Let $n = m_0 + 1$. By (4.25), (4.21), (4.22), Step 1 and Step 3, we have for $y \in [y_1, z]$, that

$$\begin{aligned} |F_{c,n}| &\leqq C\sum_{k=1}^{n} \frac{|y-y_1|^m |\psi_{c,n-k}|}{(u-c)^{k+1}} \leqq C\sum_{k=1}^{n-1} \frac{|y-y_1|^m}{(u-c)^{k+1}} + C\frac{|y-y_1|^m|\psi_{c,0}|}{(u-c)^{n+1}} \\ &\leqq C\frac{(u-c)^m}{(u-c)^n} + C\frac{|y-y_1|^m|u-c|^{1-\gamma}}{(u-c)^{n+1}} \leqq C(u-c)^{m-\gamma-n}, \\ &|(u-c)^{2-\gamma}F_{c,n}| \leqq C(u-c)^{2-2\gamma+m-n} = C(u-c)^{m-m_0+1-2\gamma} = C. \end{aligned}$$

Here, we used $n = m_0 + 1$ and $m - m_0 = 2\gamma - 1$. Thus, (4.31) is still true for $n = m_0 + 1$, i.e. $|\psi_{c,n}| \leqq C$ for $y \in [y_1, z]$.

Step 5. $|\psi_{c,0}| \leqq C|u-c|$ for $y \in [y_1, z]$.

By Step 1 and Step 4, we have $|\psi_{c,1}| \leqq C$ for $y \in [y_1, z]$. Then by (4.24), we have $|\psi_{c,0}(y_1)| \leqq \int_c^{u_{\min}} |\psi_{s,1}(y_1)|\, ds \leqq C\int_c^{u_{\min}} ds \leqq C|u_{\min} - c|$. By (4.6), (4.26) and $F_{c,0} = 0$, we have $|(u-c)^{-1}\psi_{c,0}|_{L^\infty(z)} \leqq C(|(u_{\min}-c)^{-1}\psi_{c,0}(y_1)| + |\psi'_{c,0}(z)|) \leqq C$, which gives $|(u-c)^{-1}\psi_{c,0}| \leqq C$ and $|\psi_{c,0}| \leqq C|u-c|$ for $y \in [y_1, z]$.

Step 6. $|\partial_y\psi_{c,n}| \leqq C|u-c|^{2\gamma-2}$ and $|\psi_{c,n}| \leqq C|u-c|^{\gamma-1}$ for $y \in [y_1, z]$ and $n = m_0 + 2$.

Since $n = m_0 + 2$ and $m - m_0 = 2\gamma - 1$, we have, by (4.25), (4.21), (4.23), Step 1 and Steps 4-5, that for $y \in [y_1, z]$,

$$|F_{c,n}| \leqq C\sum_{k=1}^{n}\frac{|y-y_1|^m|\psi_{c,n-k}|}{(u-c)^{k+1}} \leqq C\sum_{k=1}^{n-1}\frac{|y-y_1|^m}{(u-c)^{k+1}} + C\frac{|y-y_1|^m|\psi_{c,0}|}{(u-c)^{n+1}}$$
$$\leqq C\frac{(u-c)^m}{(u-c)^n} + C\frac{|y-y_1|^m|u-c|}{(u-c)^{n+1}} \leqq C(u-c)^{m-n} = C(u-c)^{m-m_0-2},$$
$$|(u-c)^{3-2\gamma}F_{c,n}| \leqq C(u-c)^{1-2\gamma+m-m_0} = C.$$

Then by (4.4) and (4.26), we have that

$$|(u-c)^{1-\gamma}\psi_{c,n}|_{L^\infty(z)} + \delta_1^{\gamma}|(u-c)^{2-2\gamma}\partial_y\psi_{c,n}|_{L^\infty(z)}$$
$$\leqq C(|(u-c)^{3-2\gamma}F_{c,n}|_{L^\infty(z)} + |\psi_{c,n}(z)| + |\partial_y\psi_{c,n}(z)|) \leqq C.$$

Therefore, $|\partial_y\psi_{c,n}| \leqq C\delta_1^{-\gamma}|u-c|^{2\gamma-2} \leqq C|u-c|^{2\gamma-2}$ and $|\psi_{c,n}| \leqq C|u-c|^{\gamma-1}$ for $y \in [y_1, z]$.

Step 7. $|\psi_{c,1}| \leqq C|u-c|^{1-\gamma}$ for $y \in [y_1, z]$.

If $m_0 = 0$, then $m = 0$ and $\gamma = 1/2$. By Step 6, we have $|\psi_{c,2}| \leqq C|u-c|^{\gamma-1} = C|u-c|^{-\gamma}$ for $y \in [y_1, z]$. If $m_0 > 0$, then by Step 1 and Step 4, we have $|\psi_{c,2}| \leqq C \leqq C|u-c|^{-\gamma}$ for $y \in [y_1, z]$. Thus, $|\psi_{c,2}| \leqq C|u-c|^{-\gamma}$ is always true for $y \in [y_1, z]$. Then by (4.24) and $u(y_1) = u_{\min}$, we have $|\psi_{c,1}(y_1)| \leqq \int_c^{u_{\min}} |\psi_{s,2}(y_1)|\,\mathrm{d}s \leqq C\int_c^{u_{\min}} |u_{\min} - s|^{-\gamma}\,\mathrm{d}s \leqq C|u_{\min} - c|^{1-\gamma}$. By (4.25), (4.21) and Step 5, we have, for $y \in [y_1, z]$, that

$$|F_{c,1}| \leqq C\frac{|y-y_1|^m|\psi_{c,0}|}{(u-c)^2} \leqq C\frac{|y-y_1|^m|u-c|}{(u-c)^2} = C\frac{|y-y_1|^m}{u-c},$$
$$|(u-c)^{\gamma+1}F_{c,1}| \leqq C(u-c)^{\gamma}|y-y_1|^m \leqq C.$$

Then by (4.5) and (4.26), we have that

$$|(u-c)^{\gamma-1}\psi_{c,1}|_{L^\infty(z)}$$
$$\leqq C(|(u-c)^{\gamma+1}F_{c,1}|_{L^\infty(z)} + |(u_{\min}-c)^{\gamma-1}\psi_{c,1}(y_1)| + |\psi_{c,1}(z)| + |\partial_y\psi_{c,1}(z)|) \leqq C,$$

which gives $|(u-c)^{\gamma-1}\psi_{c,1}| \leqq C$ and $|\psi_{c,1}| \leqq C|u-c|^{1-\gamma}$ for $y \in [y_1, z]$.

By Step 1 and Steps 4-7 we know that (4.27) is true for $y \in [y_1, z] = [y_1, y_1 + \delta_1]$. This completes the proof of (4.27) and thus Case 2.

4.3. Rule out oscillation for flows in class $\mathcal{K}^+$

We rule out the oscillation of $\lambda_n(c)$ for flows in class $\mathcal{K}^+$, which is stated in Theorem 2.13. The proof is based on Hamiltonian structure and index theory.

Proof of Theorem 2.13. The assumption (**H1**) is satisfied for a flow u in class $\mathcal{K}^+$. By Theorem 2.11, it suffices to prove $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) < \infty$ for $0 < \alpha^2 \leqq M_\beta$ and $0 < \beta \leqq \frac{9}{8}\kappa_+$. Similar proof is valid for $\frac{9}{8}\kappa_- \leqq \beta < 0$. First, we consider $\beta \in \mathrm{Ran}(u) \cap (0, \frac{9}{8}\kappa_+]$. Define the non-shear space

$$X := \{\omega \in L^2(D_T) : \int_0^T \omega(x, y)\,\mathrm{d}x = 0, \text{T-periodic in } x\}.$$

Note that as $\omega = \partial_x v_2 - \partial_y v_1$, $\int_0^T \omega(x, y)\,\mathrm{d}x = 0$ is equivalent to $\int_0^T v_1(x, y)\,\mathrm{d}x$ =constant. Thus, $\int_0^T v_1(x, y)\,\mathrm{d}x = 0$ implies $\int_0^T \omega(x, y)\,\mathrm{d}x = 0$.

The linearized equation (1.4) has a Hamiltonian structure in the traveling frame $(x - u_\beta t, y, t)$:

$$\omega_t = -(\beta - u'')\partial_x \left(\omega/K_\beta - \psi\right) = JL\omega.$$

Here $J = -(\beta - u'')\partial_x : X^* \to X, L = 1/K_\beta - (-\Delta)^{-1} : X \to X^*$. Let $J_\alpha = -i\alpha(\beta - u'')$ and $L_\alpha = \frac{1}{K_\beta} - (-\frac{d^2}{dy^2} + \alpha^2)^{-1}$ on $L^2_{\frac{1}{K_\beta}}$. It follows from Theorem 3 in [27] that

$$k_c + k_r + k_i^{\leqq 0} = n^-(L_\alpha),$$

where $n^-(L_\alpha)$ is the Morse index of L_α, k_r is the sum of algebraic multiplicities of positive eigenvalues of $J_\alpha L_\alpha$, k_c is the sum of algebraic multiplicities of eigenvalues of $J_\alpha L_\alpha$ in the first and the fourth quadrants and $k_i^{\leqq 0}$ is the total number of non-positive dimensions of $\langle L_\alpha \cdot, \cdot\rangle$ restricted to the generalized eigenspaces of nonzero purely imaginary eigenvalues of $J_\alpha L_\alpha$.

Suppose that $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) = \infty$. Then it follows from Theorem 2.9 that there exists $m_\beta < n < N_\beta$ such that $\sharp(\{\lambda_n(c) = -\alpha^2, c < u_{\min}\}) = \infty$. Let $c^* < u_{\min}$ be a solution of $\lambda_n(c) = -\alpha^2$ with eigenfunction ϕ^*. c^* can be chosen sufficiently close to $u_{\min}$. Then $-i\alpha(c^* - u_\beta)$ is a purely imaginary eigenvalue of $J_\alpha L_\alpha$ with eigenfunction $\omega^* = -\phi^{*\prime\prime} + \alpha^2\phi^*$. By Theorem 4 in [27],

$$\langle L_\alpha \omega^*, \omega^*\rangle = -(c^* - u_\beta)\lambda_n'(c^*).$$

Note that $c - u_\beta$ does not change sign when $c < u_{\min}$ is sufficiently close to $u_{\min}$. Then

$$\sharp(\{-(c - u_\beta)\lambda_n'(c) \leqq 0, c < u_{\min}\} \cap \{\lambda_n(c) = -\alpha^2, c < u_{\min}\}) = \infty.$$

Hence, $k_i^{\leqq 0} = \infty$. This contradicts that

$$k_i^{\leqq 0} \leqq n^-(L_\alpha) = n^-(\tilde{L}_0 + \alpha^2) \leqq n^-(\tilde{L}_0) < \infty,$$

where $\tilde{L}_0 = -\frac{d^2}{dy^2} - K_\beta : H^2 \cap H_0^1 \to L^2$. Therefore, $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) < \infty$.

Then, we consider $\beta \in (0, \frac{9}{8}\kappa_+] \setminus \mathrm{Ran}(u'')$. By Corollary 1 in [27], $\lambda_n(c)$ is decreasing on $c \in (-\infty, u_{\min})$ for any fixed $n \geqq 1$. By Theorem 2.9, $\sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, u_{\min})) < N_\beta$. $\square$

5. Relations between a traveling wave family and an isolated real eigenvalue

In this section, we establish the correspondence between a traveling wave family near a shear flow and an isolated real eigenvalue of $\mathcal{R}_{k\alpha,\beta}$. For a given isolated real eigenvalue c_0, we prove that there exists a set of traveling wave solutions near $(u, 0)$ with traveling speeds converging to c_0, which is stated precisely in Lemma 2.5.

We assume $k, k_0 \in \mathbf{Z}$.

Proof of Lemma 2.5. We assume that $\beta > 0$, and the case for $\beta < 0$ is similar. Since $c_0 \in \sigma_d(\mathcal{R}_{k_0\alpha,\beta}) \cap \mathbf{R}$ for some $k_0 \geqq 1$, we have $c_0 < u_{\min}$ and we choose $\delta_0 > 0$ such that $c_0 + \delta_0 < u_{\min}$. By (1.3), $\boldsymbol{u}\,(x - ct, y)$ is a solution of (1.1)–(1.2) if and only if (ψ, c) solves

$$\frac{\partial\,(\omega + \beta y, \psi - cy)}{\partial\,(x, y)} = 0, \tag{5.1}$$

and ψ takes constant values on $\{y = y_i\}$, where $i = 1, 2$, $\omega = \operatorname{curl} \boldsymbol{u}$ and $\boldsymbol{u} = (\partial_y \psi, -\partial_x \psi)$. Let ψ_0 be a stream function associated with the shear flow $(u, 0)$, i.e., $\psi_0' = u$. Since $u - c > 0$ for $c \in [c_0 - \delta_0, c_0 + \delta_0]$, $\psi_0 - cy$ is increasing on $[y_1, y_2]$. Let $I_c = \{\psi_0\,(y) - cy : y \in [y_1, y_2]\}$ for $c \in [c_0 - \delta_0, c_0 + \delta_0]$, and then we can define a function $\tilde{f}_c \in C^2(I_c)$ such that

$$\tilde{f}_c\,(\psi_0\,(y) - cy) = \omega_0\,(y) + \beta y = -\psi_0''\,(y) + \beta y. \tag{5.2}$$

Moreover,

$$\tilde{f}_c'\,(\psi_0\,(y) - cy) = \frac{\beta - u''\,(y)}{u\,(y) - c} =: \mathcal{K}_c\,(y)$$

for $c \in [c_0 - \delta_0, c_0 + \delta_0]$. We extend $\tilde{f}_c$ to $f_c \in C_0^2\,(\mathbf{R})$ such that $f_c = \tilde{f}_c$ on I_c and $\partial_z^2 \partial_c f_c(z)$ is continuous for $c \in [c_0 - \delta_0, c_0 + \delta_0]$ and $z \in \mathbf{R}$. Taking c as the bifurcation parameter, we now construct steady solutions $\boldsymbol{u} = \left(\partial_y \psi, -\partial_x \psi\right)$ near $(u, 0)$ by solving the elliptic equations

$$-\Delta \psi + \beta y = f_c\,(\psi - cy) \tag{5.3}$$

with the boundary conditions that ψ takes constant values on $\{y = y_i\}$, $i = 1, 2$.

Define the perturbation of the stream function by

$$\phi\,(x, y) = \psi\,(x, y) - \psi_0\,(y)\,.$$

Then by (5.2)–(5.3), we have that

$$-\Delta\phi - (f_c(\phi + \psi_0 - cy) - f_c\,(\psi_0 - cy)) = 0.$$

Define the spaces

$$B = \{\varphi \in H^4(D_T) : \ \varphi(x, y_i) = 0, \ i = 1, 2, \ \varphi \text{ is even and } T\text{-periodic in } x\}$$

and

$$C = \left\{\varphi \in H^2(D_T) : \ T\text{-periodic in } x\right\},$$

where $T = 2\pi/\alpha$. Consider the mapping

$$F : B \times [c_0 - \delta_0, c_0 + \delta_0] \longrightarrow C,$$
$$(\phi, c) \longmapsto -\Delta\phi - \left(f_c(\phi + \psi_0 - cy) - f_c(\psi_0 - cy)\right).$$

Then $F(0, c) = 0$ for $c \in [c_0 - \delta_0, c_0 + \delta_0]$. We study the bifurcation near the trivial solution $(0, c_0)$ of the equation $F(\phi, c) = 0$ in B, whose solutions give steady flows of

For fixed $c \in [c_0 - \delta_0, c_0 + \delta_0]$, by linearizing F around $\phi = 0$, we have that

$$\partial_\phi F(0, c) = -\Delta - f_c'(\psi_0 - cy) = -\Delta - \mathcal{K}_c = \mathcal{G}_c|_B,$$

where $\mathcal{G}_c|_B$ is the restriction of $\mathcal{G}_c$ in B and $\mathcal{G}_c$ is defined in (2.3). Then we divide the discussion of bifurcation near $(0, c_0)$ of the equation $F(\phi, c) = 0$ into three cases. Since $c_0 \in \sigma_d(\mathcal{R}_{k_0\alpha,\beta}) \cap \mathbf{R}$, there exists $n_0 \geqq 1$ such that $(k_0\alpha)^2 = -\lambda_{n_0}(c_0)$, where $\lambda_{n_0}(c_0)$ is the n_0-th eigenvalue of $\mathcal{L}_{c_0}$ and $\mathcal{L}_{c_0}$ is defined in (2.6). Let

$$k_* = \max_{k \geqq 1}\{k : \text{there exists } n_k \geqq 1 \text{ such that } -(k\alpha)^2 = \lambda_{n_k}(c_0)\}. \tag{5.4}$$

Then k_* exists by our assumption and $1 \leqq k_0 \leqq k_* < \infty$. Now we denote $n_* = n_{k_*}$.

Case 1. $\lambda_{n_*}'(c_0) \neq 0$ (the transversal crossing condition) and $c_0 \notin \sigma_d(\mathcal{R}_{0,\beta}) \cap \mathbf{R}$.

In this case, we have $0 \notin \sigma(\mathcal{L}_{c_0})$. Let $B_* = \{\varphi \in B : \frac{2\pi}{k_*\alpha}\text{-periodic in } x\}$ and $C_* = \{\varphi \in C : \frac{2\pi}{k_*\alpha}\text{-periodic in } x\}$. Consider the restriction $F|_{B_*}$ and $\mathcal{G}_c|_{B_*}$. Then by the definition of k_*, we have

$$\ker(\mathcal{G}_{c_0}|_{B_*}) = \operatorname{span}\{\phi_{c_0,n_*}(y)\cos(k_*\alpha x)\} \quad \text{and} \quad \dim(\ker(\mathcal{G}_{c_0}|_{B_*})) = 1, \tag{5.5}$$

where ϕ_{c_0,n_*} is a real-valued eigenfunction of $\lambda_{n_*}(c_0) \in \sigma(\mathcal{L}_{c_0})$. Note that

$$\partial_c\partial_\phi F(0, c_0)\left(\phi_{c_0,n_*}(y)\cos(k_*\alpha x)\right) = -\frac{\beta - u''}{(u - c_0)^2}\phi_{c_0,n_*}(y)\cos(k_*\alpha x).$$

Then by Lemma 11 in [27], we have that

$$\int_0^T \int_{y_1}^{y_2} \phi_{c_0,n_*}(y)\cos(k_*\alpha x)\left[\partial_c\partial_\phi F(0, c_0)\left(\phi_{c_0,n_*}(y)\cos(k_*\alpha x)\right)\right] \mathrm{d}y\,\mathrm{d}x$$
$$= -\int_0^T \int_{y_1}^{y_2} \frac{\beta - u''}{(u - c_0)^2}|\phi_{c_0,n_*}(y)|^2\cos^2(k_*\alpha x)\,\mathrm{d}y\,\mathrm{d}x = \frac{\pi}{\alpha}\lambda_{n_*}'(c_0) \neq 0,$$

where we used that ϕ_{c_0,n_*} is real-valued. By (5.5), we have $\phi_{c_0,n_*}(y)\cos(k_*\alpha x) \in \ker(\mathcal{G}_{c_0}|_{B_*})$ and thus, $\partial_c\partial_\phi F(0, c_0)\left(\phi_{c_0,n_*}(y)\cos(k_*\alpha x)\right) \notin \operatorname{Ran}(\mathcal{G}_{c_0}|_{B_*})$. Then by Theorem 1.7 in [7], there exist $\delta > 0$ and a nontrivial C^1 bifurcating curve $\{\left(\phi_\gamma, c(\gamma)\right), \gamma \in (-\delta, \delta)\}$ of $F(\phi, c) = 0$, which intersects the trivial curve $(0, c)$ at $c = c_0$, such that

$$\phi_\gamma(x, y) = \gamma\phi_{c_0,n_*}(y)\cos(k_*\alpha x) + o(|\gamma|).$$

So the stream functions take the form

$$\psi_\gamma(x, y) = \psi_0(y) + \phi_\gamma(x, y) = \psi_0(y) + \gamma\phi_{c_0,n_*}(y)\cos(k_*\alpha x) + o(|\gamma|).$$

Let the velocity $\boldsymbol{u}_\gamma = (u_{(\gamma)}, v_{(\gamma)}) = \left(\partial_y\psi_\gamma, -\partial_x\psi_\gamma\right)$. Since $c_0 < u_{\min}$, we have that

$$\begin{aligned} &u_{(\gamma)}(x,y) - c(\gamma) = \partial_y\psi_\gamma(x,y) - c(\gamma) \\ =&u(y) - c(\gamma) + \gamma\phi'_{c_0,n_*}(y)\cos(k_*\alpha x) + o(|\gamma|) > 0, \end{aligned} \tag{5.6}$$

and

$$v_{(\gamma)}(x,y) = -\partial_x\psi_\gamma(x,y) = k_*\alpha\gamma\phi_{c_0,n_*}(y)\sin(k_*\alpha x) + o(|\gamma|) \neq 0 \tag{5.7}$$

when γ is small. Moreover, $\|(u_{(\gamma)}, v_{(\gamma)}) - (u,0)\|_{H^3(D_T)} + |c(\gamma) - c_0| \leqq C_0\gamma$ for some constant $C_0 > 0$ large enough. Thus, we can take $\delta > 0$ smaller and $\varepsilon_0 = C_0\delta$ such that for $\varepsilon \in (0, \varepsilon_0)$, $(u_\varepsilon, v_\varepsilon, c_\varepsilon) := (u_{(\gamma)}, v_{(\gamma)}, c(\gamma))|_{\gamma=\varepsilon/C_0}$ satisfies that $\|(u_\varepsilon, v_\varepsilon) - (u,0)\|_{H^3(D_T)} \leqq \varepsilon, c_\varepsilon \to c_0, u_\varepsilon(x,y) - c_\varepsilon > 0$ and $\|v_\varepsilon\|_{L^2(D_T)} \neq 0$. By (5.7), $\tilde{v}_\varepsilon \longrightarrow \sqrt{\alpha/\pi}\phi_{c_0,n_*}(y)\sin(k_*\alpha x)$ in $H^2(D_T)$, where $\tilde{v}_\varepsilon = v_\varepsilon/\|v_\varepsilon\|_{L^2(D_T)}$.

Case 2. $\lambda'_{n_*}(c_0) = 0$ and $c_0 \notin \sigma_d(\mathcal{R}_{0,\beta}) \cap \mathbf{R}$.

In this case, there exist $\delta_1 \in (0, \delta_0]$ and $a \in \{\pm 1\}$ such that $a\lambda_{n_*}$ is increasing in $[c_0, c_0 + \delta_1]$, and thus,

$$a\lambda_{n_*}(c) > a\lambda_{n_*}(c_0) = -a(k_*\alpha)^2, \quad \forall\, c \in (c_0, c_0 + \delta_1]. \tag{5.8}$$

Let $\zeta_1 \in C^\infty([y_1, y_2])$ be a positive function, u_1 be a solution of the regular ODE

$$u_1''(u - c_0) - (u'' - \beta)u_1 = \zeta_1 \quad \text{on} \quad [y_1, y_2], \tag{5.9}$$

and $\tau_0 > 0$ be such that $[c_0, c_0 + \delta_1] \cap \mathrm{Ran}(u + \tau u_1) = \emptyset$ for $\tau \in [-\tau_0, \tau_0]$. Since $u \in H^4(y_1, y_2)$ and $\zeta_1 \in C^\infty([y_1, y_2])$, we have $u_1 \in H^4(y_1, y_2)$. Let $\lambda_n(c, \tau)$ denote the n-th eigenvalue of $\mathcal{L}_{c,\tau} : H^2 \cap H_0^1(y_1, y_2) \longrightarrow L^2(y_1, y_2)$ defined by

$$\mathcal{L}_{c,\tau}\phi = -\phi'' + \frac{u'' + \tau u_1'' - \beta}{u + \tau u_1 - c}\phi$$

for $c \in [c_0, c_0 + \delta_1]$ and $\tau \in [-\tau_0, \tau_0]$. Then by (5.9) and the fact that ζ_1 is a positive function, we have that

$$\begin{aligned} \partial_\tau\lambda_{n_*}(c_0, 0) &= \int_{y_1}^{y_2} \partial_\tau\left(\frac{u'' + \tau u_1'' - \beta}{u + \tau u_1 - c_0}\right)\Big|_{\tau=0}\phi_{n_*,c_0}^2\,\mathrm{d}y \\ &= \int_{y_1}^{y_2} \frac{u_1''(u - c_0) - (u'' - \beta)u_1}{(u - c_0)^2}\phi_{n_*,c_0}^2\,\mathrm{d}y \\ &= \int_{y_1}^{y_2} \frac{\zeta_1}{(u - c_0)^2}\phi_{n_*,c_0}^2\,\mathrm{d}y > 0, \end{aligned}$$

where ϕ_{n_*,c_0} is a L^2 normalized eigenfunction of $\lambda_{n_*}(c_0) \in \sigma(\mathcal{L}_{c_0})$. By the definition of k_*, $-(k\alpha)^2 \notin \sigma(\mathcal{L}_{c_0,0})$ for $k > k_*$. Since $c_0 \notin \sigma_d(\mathcal{R}_{0,\beta}) \cap \mathbf{R}$, we have $0 \notin \sigma(\mathcal{L}_{c_0,0})$. By the continuity of $\partial_\tau\lambda_{n_*}$ and the small perturbation of $\sigma(\mathcal{L}_{c,\tau})$, we can take $\tau_0 > 0$ and $\delta_1 > 0$ smaller such that $\partial_\tau\lambda_{n_*}(c, \tau) > 0$ and

$$0 \notin \sigma(\mathcal{L}_{c,\tau}) \quad \text{and} \quad -(k\alpha)^2 \notin \sigma(\mathcal{L}_{c,\tau}), \quad \forall\, k > k_* \tag{5.10}$$

for $(c,\tau) \in [c_0, c_0+\delta_1] \times [-\tau_0, \tau_0]$. By taking $\delta_1 > 0$ smaller and the Implicit Function Theorem, there exists $\widetilde{\gamma} \in C^1([c_0, c_0+\delta_1])$ such that $\lambda_{n_*}(c, \widetilde{\gamma}(c)) = \lambda_{n_*}(c_0, 0) = -(k_*\alpha)^2$, $\widetilde{\gamma}(c_0) = 0$ and $|\widetilde{\gamma}(c)| \leqq \tau_0$ for $c \in [c_0, c_0+\delta_1]$. By (5.8), we have $\lambda_{n_*}(c, \widetilde{\gamma}(c)) = \lambda_{n_*}(c_0, 0) \neq \lambda_{n_*}(c, 0)$ and $\widetilde{\gamma}(c) \neq 0$ for $c \in (c_0, c_0+\delta_1]$. Then for fixed $\tau \in (0, \tau_0]$, there exists $c_\tau \in [c_0, c_0+\delta_1]$ such that $\widetilde{\gamma}'(c_\tau) \neq 0$ and $|\widetilde{\gamma}(c_\tau)| \leqq \tau$. Note that $0 = \frac{d}{dc}[\lambda_{n_*}(c, \widetilde{\gamma}(c))] = \partial_c \lambda_{n_*}(c, \widetilde{\gamma}(c)) + \widetilde{\gamma}'(c)\partial_\tau \lambda_{n_*}(c, \widetilde{\gamma}(c))$. Let $\tau_1 = \widetilde{\gamma}(c_\tau)$. Then we have $\partial_c \lambda_{n_*}(c_\tau, \tau_1) = -\widetilde{\gamma}'(c_\tau)\partial_\tau \lambda_{n_*}(c_\tau, \tau_1) \neq 0$.

Fix any $\varepsilon \in (0, 1)$. Then we can choose $\tau \in (0, \tau_0]$ and $\delta_1 > 0$ smaller such that for $\tau_1 = \widetilde{\gamma}(c_\tau)$,

$$\|(u + \tau_1 u_1, 0) - (u, 0)\|_{H^3(y_1, y_2)} \leqq \tau_1 \|u_1\|_{H^3(y_1, y_2)} < \frac{\varepsilon}{2} \quad \text{and} \quad |c_\tau - c_0| < \delta_1 < \frac{\varepsilon}{2}. \tag{5.11}$$

By (5.10), $\lambda_{n_*}(c_\tau, \tau_1) = -(k_*\alpha)^2$ and $\partial_c \lambda_{n_*}(c_\tau, \tau_1) \neq 0$, we can apply Case 1 to the shear flow $(u + \tau_1 u_1, 0)$: there exists a traveling wave solution $(u_\varepsilon(x - c_\varepsilon t, y), v_\varepsilon(x - c_\varepsilon t, y))$ to (1.1)–(1.2) which has period $T = 2\pi/\alpha$ in x,

$$\|(u_\varepsilon, v_\varepsilon) - (u + \tau_1 u_1, 0)\|_{H^3(D_T)} \leqq \frac{\varepsilon}{2} \quad \text{and} \quad |c_\varepsilon - c_\tau| \leqq \frac{\varepsilon}{2}, \tag{5.12}$$

$u_\varepsilon(x, y) - c_\varepsilon \neq 0$ and $\|v_\varepsilon\|_{L^2(D_T)} \neq 0$. Then by (5.11)–(5.12), we have $\|(u_\varepsilon, v_\varepsilon) - (u, 0)\|_{H^3(D_T)} < \varepsilon$ and $|c_\varepsilon - c_0| < \varepsilon$.

Case 3. $c_0 \in \sigma_d(\mathcal{R}_{0,\beta}) \cap \mathbf{R}$.

In this case, $0 \in \sigma(\mathcal{L}_{c_0})$ and there exists $j_0 > n_0 \geqq n_*$ such that $\lambda_{j_0}(c_0) = 0$. There exist $\delta_1 \in (0, \delta_0]$ and $a, b \in \{\pm 1\}$ such that both $a\lambda_{n_*}$ and $b\lambda_{j_0}$ are decreasing in $[c_0, c_0+\delta_1]$.

Since $\phi^2_{n_*,c_0}$ is linearly independent of $\phi^2_{j_0,c_0}$, there exists $\xi_1 \in C^\infty([y_1, y_2])$ such that

$$\int_{y_1}^{y_2} \xi_1 \frac{\phi^2_{n_*,c_0}}{(u - c_0)^2}\, \mathrm{d}y = a \quad \text{and} \quad \int_{y_1}^{y_2} \xi_1 \frac{\phi^2_{j_0,c_0}}{(u - c_0)^2}\, \mathrm{d}y = -b. \tag{5.13}$$

Let u_1 be a solution of (5.9) with $\zeta_1 = \xi_1$, and $\tau_0 > 0$ be such that $[c_0, c_0+\delta_1] \cap \mathrm{Ran}(u + \tau u_1) = \emptyset$ for $\tau \in [-\tau_0, \tau_0]$. Then by (5.13), we have $a\partial_\tau \lambda_{n_*}(c_0, 0) = a^2 > 0$ and $b\partial_\tau \lambda_{j_0}(c_0, 0) = -b^2 < 0$. As in Case 2, we can take $\tau_0 > 0$ and $\delta_1 > 0$ smaller such that

$$a\partial_\tau \lambda_{n_*}(c, \tau) > 0 \quad \text{and} \quad -(k\alpha)^2 \notin \sigma(\mathcal{L}_{c,\tau}), \quad \forall\, k > k_*$$

for $(c, \tau) \in [c_0, c_0+\delta_1] \times [-\tau_0, \tau_0]$. Note that $\lambda_{j_0-1}(c_0, 0) < \lambda_{j_0}(c_0, 0) = 0 < \lambda_{j_0+1}(c_0, 0)$. By the continuity of $\partial_\tau \lambda_{j_0}$, λ_{j_0-1} and λ_{j_0+1}, we can choose $\tau_0 > 0$ and $\delta_1 > 0$ smaller such that

$$b\partial_\tau \lambda_{j_0}(c, \tau) < 0 \quad \text{and} \quad \lambda_{j_0-1}(c, \tau) < 0 < \lambda_{j_0+1}(c, \tau) \tag{5.14}$$

for $(c, \tau) \in [c_0, c_0+\delta_1] \times [-\tau_0, \tau_0]$.

As $a\partial_\tau \lambda_{n_*}(c_0, 0) > 0$ and $a\lambda_{n_*}(\cdot, 0)$ is decreasing in $[c_0, c_0+\delta_1]$, we can choose $\tau_1 \in (0, \tau_0]$ such that $a\lambda_{n_*}(c_0+\delta_1, \tau_1) < a\lambda_{n_*}(c_0, 0) = -a(k_*\alpha)^2 < a\lambda_{n_*}(c_0, \tau_1)$. Then there exists $c_{\tau_1} \in (c_0, c_0+\delta_1)$ such that $\lambda_{n_*}(c_{\tau_1}, \tau_1) = -(k_*\alpha)^2$.

Since $b\partial_\tau \lambda_{j_0}(c,\tau) < 0$ and $b\lambda_{j_0}(\cdot,0)$ is decreasing in $[c_0, c_0+\delta_1]$, we have $b\lambda_{j_0}(c,\tau) < b\lambda_{j_0}(c,0) < b\lambda_{j_0}(c_0,0) = 0$, which, along with (5.14), gives that

$$b\lambda_{j_0+b}(c,\tau) > 0 > b\lambda_{j_0}(c,\tau), \quad (c,\tau) \in (c_0, c_0+\delta_1] \times (0,\tau_0].$$

Since $(c_{\tau_1}, \tau_1) \in (c_0, c_0+\delta_1] \times (0,\tau_0]$, we have $0 \notin \sigma(\mathcal{L}_{c_{\tau_1},\tau_1})$.

Now, we can construct a desired traveling wave solution $(u_\varepsilon(x-c_\varepsilon t, y), v_\varepsilon(x-c_\varepsilon t, y))$ by first perturbing the shear flow $(u,0)$ to $(u+\tau_1 u_1, 0)$ and then applying Case 1 or Case 2 to $(u+\tau_1 u_1, 0)$ as in (5.11)–(5.12).

To prove Corollary 2.6, we only need to modify the spaces B and C from H^4 and H^2 to H^{s+1} and H^{s-1} in the proof of Lemma 2.5. We also use the fact that $\tilde{f}_c \in C^\infty(I_c)$, $f_c \in C_0^\infty(\mathbf{R})$ and $u_1 \in C^\infty([y_1,y_2])$ due to the assumption that $u \in C^\infty([y_1,y_2])$.

Conversely, for a set of traveling wave solutions near $(u,0)$ with traveling speeds converging to c_0, we show that c_0 is an isolated real eigenvalue besides $u_{\min}$ and $u_{\max}$, which is given in Lemma 2.7.

Proof of Lemma 2.7. It suffices to show that if $c_0 \notin \{u_{\max}, u_{\min}\}$, then $c_0 \in \bigcup_{k\geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$ and (2.2) holds.

Note that $(u_\varepsilon, v_\varepsilon)$ solves

$$(u_\varepsilon - c_\varepsilon)\partial_x \omega_\varepsilon + v_\varepsilon \partial_y \omega_\varepsilon + \beta v_\varepsilon = 0. \tag{5.15}$$

Moreover,

$$\|\omega_\varepsilon - \omega_0\|_{H^2(D_T)} \leqq C\|(u_\varepsilon, v_\varepsilon) - (u,0)\|_{H^3(D_T)} \leqq C\varepsilon.$$

By taking $\varepsilon_0 > 0$ smaller,

$$|u_\varepsilon - c_\varepsilon| \geqq |u - c_\varepsilon| - |u - u_\varepsilon| \geqq C^{-1} \tag{5.16}$$

for $\varepsilon \in (0,\varepsilon_0)$ and $y \in [y_1,y_2]$. Note that $\frac{\pi}{y_2-y_1}\|v_\varepsilon\|_{L^2(D_T)} \leqq \|\nabla v_\varepsilon\|_{L^2(D_T)}$. By Sobolev embedding, we have that

$$\|v_\varepsilon\|_{L^4(D_T)} \leqq C\|v_\varepsilon\|_{H^1(D_T)} \leqq C\|\nabla v_\varepsilon\|_{L^2(D_T)}, \tag{5.17}$$

$$\|\partial_y(\omega_\varepsilon - \omega_0)\|_{L^4(D_T)} \leqq C\|\partial_y(\omega_\varepsilon - \omega_0)\|_{H^1(D_T)} \leqq C\|\omega_\varepsilon - \omega_0\|_{H^2(D_T)} \leqq C\varepsilon. \tag{5.18}$$

Since $\partial_x \omega_\varepsilon = \partial_x(\partial_x v_\varepsilon - \partial_y u_\varepsilon) = \Delta v_\varepsilon$, we get by (5.15) that

$$\Delta \tilde{v}_\varepsilon + \frac{\partial_y(\omega_\varepsilon - \omega_0)}{u_\varepsilon - c_\varepsilon}\tilde{v}_\varepsilon + \frac{\beta - u''}{u_\varepsilon - c_\varepsilon}\tilde{v}_\varepsilon = 0, \tag{5.19}$$

where $\tilde{v}_\varepsilon = v_\varepsilon/\|v_\varepsilon\|_{L^2(D_T)}$. By (5.16), we have $\left|\frac{\beta-u''}{u_\varepsilon-c_\varepsilon}\right| \leqq C$ for $y \in [y_1,y_2]$ and

$$\begin{aligned}\|\Delta\tilde{v}_\varepsilon\|_{L^2(D_T)} &\leqq C\|\partial_y(\omega_\varepsilon-\omega_0)\|_{L^4(D_T)}\|\tilde{v}_\varepsilon\|_{L^4(D_T)} + C\|\tilde{v}_\varepsilon\|_{L^2(D_T)} \\ &\leqq C\varepsilon\|\tilde{v}_\varepsilon\|_{H^1(D_T)} + C\|\tilde{v}_\varepsilon\|_{L^2(D_T)} \leqq C\varepsilon\|\tilde{v}_\varepsilon\|^{1/2}_{H^2(D_T)} + C,\end{aligned}$$

where we used (5.17)–(5.18) and $\|\tilde{v}_\varepsilon\|_{H^1(D_T)} \leqq C\|\tilde{v}_\varepsilon\|^{1/2}_{H^2(D_T)}\|\tilde{v}_\varepsilon\|^{1/2}_{L^2(D_T)}$. Since $v_\varepsilon(x, y_i) = 0$ for $i = 1, 2$, we have $\|\tilde{v}_\varepsilon\|_{H^2(D_T)} \leqq C\|\Delta\tilde{v}_\varepsilon\|_{L^2(D_T)}$. Thus, $\|\tilde{v}_\varepsilon\|_{H^2(D_T)} \leqq C$ and

$$\|\tilde{v}_\varepsilon\|_{C^0([0,T]\times[y_1,y_2])} \leqq C\|\tilde{v}_\varepsilon\|_{H^2(D_T)} \leqq C, \tag{5.20}$$

$$\|\partial_x\tilde{v}_\varepsilon\|_{L^4(D_T)} + \|\partial_y\tilde{v}_\varepsilon\|_{L^4(D_T)} \leqq C\|\tilde{v}_\varepsilon\|_{H^2(D_T)} \leqq C. \tag{5.21}$$

Up to a subsequence, there exists $\tilde{v}_0 \in H^2(D_T)$ such that $\tilde{v}_\varepsilon \rightharpoonup \tilde{v}_0$ in $H^2(D_T)$, $\tilde{v}_\varepsilon \to \tilde{v}_0$ in $H^1(D_T)$ and $\|\tilde{v}_0\|_{L^2(D_T)} = 1$. Taking derivative in (5.19) with respect to x and y, we get by (5.18) and (5.20)–(5.21) that

$$\begin{aligned}
&\|\partial_x\Delta\tilde{v}_\varepsilon\|_{L^2(D_T)}\\
\leqq&\left\|\partial_x\left(\frac{\partial_y(\omega_\varepsilon-\omega_0)}{u_\varepsilon-c_\varepsilon}\right)\tilde{v}_\varepsilon+\frac{\partial_y(\omega_\varepsilon-\omega_0)}{u_\varepsilon-c_\varepsilon}\partial_x\tilde{v}_\varepsilon\right.\\
&\qquad\left.+\partial_x\left(\frac{\beta-u''}{u_\varepsilon-c_\varepsilon}\right)\tilde{v}_\varepsilon+\frac{\beta-u''}{u_\varepsilon-c_\varepsilon}\partial_x\tilde{v}_\varepsilon\right\|_{L^2(D_T)}\\
\leqq&C\left(\|\partial_{xy}(\omega_\varepsilon-\omega_0)\|_{L^2(D_T)}+\|\partial_y(\omega_\varepsilon-\omega_0)\|_{L^2(D_T)}\right)\|\tilde{v}_\varepsilon\|_{C^0([0,T]\times[y_1,y_2])}+\\
&C\|\partial_y(\omega_\varepsilon-\omega_0)\|_{L^4(D_T)}\|\partial_x\tilde{v}_\varepsilon\|_{L^4(D_T)}+C\|\tilde{v}_\varepsilon\|_{L^2(D_T)}+C\|\partial_x\tilde{v}_\varepsilon\|_{L^2(D_T)}\leqq C,
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_y\Delta\tilde{v}_\varepsilon\|_{L^2(D_T)}\\
\leqq&\left\|\partial_y\left(\frac{\partial_y(\omega_\varepsilon-\omega_0)}{u_\varepsilon-c_\varepsilon}\right)\tilde{v}_\varepsilon+\frac{\partial_y(\omega_\varepsilon-\omega_0)}{u_\varepsilon-c_\varepsilon}\partial_y\tilde{v}_\varepsilon\right.\\
&\qquad\left.+\partial_y\left(\frac{\beta-u''}{u_\varepsilon-c_\varepsilon}\right)\tilde{v}_\varepsilon+\frac{\beta-u''}{u_\varepsilon-c_\varepsilon}\partial_y\tilde{v}_\varepsilon\right\|_{L^2(D_T)}\\
\leqq&C\left(\|\partial_y^2(\omega_\varepsilon-\omega_0)\|_{L^2(D_T)}+\|\partial_y(\omega_\varepsilon-\omega_0)\|_{L^2(D_T)}\right)\|\tilde{v}_\varepsilon\|_{C^0([0,T]\times[y_1,y_2])}+\\
&C\|\partial_y(\omega_\varepsilon-\omega_0)\|_{L^4(D_T)}\|\partial_y\tilde{v}_\varepsilon\|_{L^4(D_T)}+C\|\tilde{v}_\varepsilon\|_{L^2(D_T)}+C\|\partial_y\tilde{v}_\varepsilon\|_{L^2(D_T)}\leqq C,
\end{aligned}$$

which implies that $\|\tilde{v}_\varepsilon\|_{H^3(D_T)} \leqq C$ and thus, $\tilde{v}_\varepsilon \longrightarrow \tilde{v}_0$ in $H^2(D_T)$. For any $\phi \in H^1(D_T)$ with periodic boundary condition in x and Dirichlet boundary condition in y, we have that

$$\int_0^T\int_{y_1}^{y_2}\left(-\nabla\tilde{v}_\varepsilon\cdot\nabla\phi+\frac{\partial_y(\omega_\varepsilon-\omega_0)}{u_\varepsilon-c_\varepsilon}\tilde{v}_\varepsilon\phi+\frac{\beta-u''}{u_\varepsilon-c_\varepsilon}\tilde{v}_\varepsilon\phi\right)\mathrm{d}y\,\mathrm{d}x=0. \tag{5.22}$$

Since $\|\tilde{v}_\varepsilon\|_{L^4(D_T)} \leqq C\|\tilde{v}_\varepsilon\|_{H^1(D_T)} \leqq C$, we have by (5.16) and (5.18) that

$$\begin{aligned}
&\left|\int_0^T\int_{y_1}^{y_2}\frac{\partial_y(\omega_\varepsilon-\omega_0)}{u_\varepsilon-c_\varepsilon}\tilde{v}_\varepsilon\phi\,\mathrm{d}y\,\mathrm{d}x\right|\leqq C\int_0^T\int_{y_1}^{y_2}|\partial_y(\omega_\varepsilon-\omega_0)||\tilde{v}_\varepsilon||\phi|\,\mathrm{d}y\,\mathrm{d}x\\
\leqq&C\|\partial_y(\omega_\varepsilon-\omega_0)\|_{L^4(D_T)}\|\tilde{v}_\varepsilon\|_{L^4(D_T)}\|\phi\|_{L^2(D_T)}\leqq C\varepsilon\|\phi\|_{L^2(D_T)}\longrightarrow 0\quad\text{as }\varepsilon\to 0^+.
\end{aligned}$$

Noting that $\tilde{v}_\varepsilon \longrightarrow \tilde{v}_0$ in $H^2(D_T)$ and sending $\varepsilon \to 0^+$ in (5.22), we have that

$$\int_0^T \int_{y_1}^{y_2} \left(-\nabla \tilde{v}_0 \cdot \nabla \phi + \frac{\beta - u''}{u - c_0} \tilde{v}_0 \phi \right) \mathrm{d}y\, \mathrm{d}x = 0.$$

Thus, $\tilde{v}_0 \in H^2(D_T)$ is a weak solution of

$$\mathcal{G}_{c_0} \tilde{v}_0 = -\Delta \tilde{v}_0 - \frac{\beta - u''}{u - c_0} \tilde{v}_0 = 0. \tag{5.23}$$

Since $c_0 \notin \mathrm{Ran}(u)$, we have $\left| \frac{\beta - u''}{u - c_0} \right| \leqq C$ for $y \in [y_1, y_2]$. Then by elliptic regularity theory, we have that $\tilde{v}_0$ is a classical solution of (5.23). Thus, $\varphi_{c_0} := \tilde{v}_0 \in \ker(\mathcal{G}_{c_0})$. Since $-\Delta \phi = 0$ has no nontrivial solutions satisfying the boundary conditions, we have $|c_0| < \infty$.

Since $\varphi_{c_0} = \sum_{k \in \mathbf{Z}} \widehat{\varphi}_{c_0,k}(y) e^{ik\alpha x} \neq 0$ solves (5.23), there exists $k_0 \in \mathbf{Z}$ such that $\widehat{\varphi}_{c_0,k_0} \neq 0$ solves

$$-\widehat{\varphi}''_{c_0,k_0} + (k_0 \alpha)^2 \widehat{\varphi}_{c_0,k_0} - \frac{\beta - u''}{u - c_0} \widehat{\varphi}_{c_0,k_0} = 0,$$

with $\widehat{\varphi}_{c_0,k_0}(y_1) = \widehat{\varphi}_{c_0,k_0}(y_2) = 0$. Now we show that $k_0 \neq 0$. Let $P_0 f(x, y) = \frac{1}{T} \int_0^T f(x, y)\, \mathrm{d}x$ for $f \in L^2(D_T)$. Then P_0 is a bounded linear operator on $L^2(D_T)$. Since $\tilde{v}_\varepsilon = v_\varepsilon / \|v_\varepsilon\|_{L^2(D_T)} = -\partial_x \psi_\varepsilon / \|v_\varepsilon\|_{L^2(D_T)}$, we have $P_0 \tilde{v}_\varepsilon = P_0 v_\varepsilon = 0$. Taking limit as $\varepsilon \to 0^+$, we have $P_0 \varphi_{c_0} = P_0 \tilde{v}_0 = 0$ and thus, $\widehat{\varphi}_{c_0,0}(y) = P_0 \varphi_{c_0}(x, y) \equiv 0$, which implies that $k_0 \neq 0$. Thus, $c_0 \in \bigcup_{k \neq 0} (\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}) = \bigcup_{k \geqq 1} (\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$, where we used the fact that $\sigma_d(\mathcal{R}_{k\alpha,\beta}) = \sigma_d(\mathcal{R}_{-k\alpha,\beta})$.

We give two remarks to Lemma 2.7: the first is to study the Fourier expansion of the limit function φ_{c_0}, and the second is to show that the asymptotic behavior of L^2 normalized vertical velocities $\tilde{v}_\varepsilon$ might be complicated if $c_\varepsilon \to c_0 \in \{u_{\min}, u_{\max}\}$. □

Remark 5.1. The function φ_{c_0} in Lemma 2.7 is a superposition of finite normal modes. In fact, since $0 \neq \varphi_{c_0}(x, y) = \sum_{k \in \mathbf{Z}} \widehat{\varphi}_{c_0,k}(y) e^{ik\alpha x} \in \ker(\mathcal{G}_{c_0})$ and $\inf \sigma(\mathcal{L}_{c_0}) > -\infty$, we have $n_* := \sharp(\{k \in \mathbf{Z} : -(k\alpha)^2 \in \sigma(\mathcal{L}_{c_0})\}) \in [1, \infty)$. Let $\{k \in \mathbf{Z} : -(k\alpha)^2 \in \sigma(\mathcal{L}_{c_0})\} = \{k_n : 1 \leqq n \leqq n_*\}$. Then

$\varphi_{c_0}(x, y) = \sum_{n=1}^{n_*} \widehat{\varphi}_{c_0,k_n}(y) e^{ik_n \alpha x}$ and $\widehat{\varphi}_{c_0,k_n}$ is an eigenfunction of $-(k_n\alpha)^2 \in \sigma(\mathcal{L}_{c_0})$.

Remark 5.2. Consider a flow u satisfying **(H1)**, $\{u' = 0\} \cap \{u = u_{\min}\} \neq \emptyset$, $\alpha > 0$ and $\beta > \frac{9}{8}\kappa_+$. By Theorem 2.9 (3), there exists $\{c_n\}_{n=1}^{\infty} = \sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}$ such that $c_n \to u_{\min}^-$, $c_{n+1} > c_n$ and $\alpha^2 = -\lambda_n(c_n)$ for $n \geqq 1$. By Lemma 2.5, we can choose $\varepsilon_n \to 0^+$ and nearby traveling wave solutions $\boldsymbol{u}_{\varepsilon_n}\left(x - c_{\varepsilon_n} t, y\right) = \left(u_{\varepsilon_n}\left(x - c_{\varepsilon_n} t, y\right), v_{\varepsilon_n}\left(x - c_{\varepsilon_n} t, y\right)\right)$ with period $2\pi/\alpha$ in x such that $\|(u_{\varepsilon_n}, v_{\varepsilon_n}) - (u, 0)\|_{H^3(D_T)} \leqq \varepsilon_n$, $|c_{\varepsilon_n} - c_n| \to 0^+$, $\|v_{\varepsilon_n}\|_{L^2(D_T)} \neq 0$ and there exists $\varphi_{c_n} \in \ker(\mathcal{G}_{c_n})$ such that $\left\|\tilde{v}_{\varepsilon_n} - \varphi_{c_n}\right\|_{C^0}$ is small enough for large n, where $\tilde{v}_{\varepsilon_n} = v_{\varepsilon_n} / \|v_{\varepsilon_n}\|_{L^2(D_T)}$. Then $c_{\varepsilon_n} \to u_{\min}^-$. A possible case is that there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ such that $c_{n_j} \notin \bigcup_{k>1} (\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$ for $j \geqq 1$. In this case,

$\varphi_{c_{n_j}}(x,y) = \sqrt{\alpha/\pi}\phi_{c_{n_j}}(y)\sin(\alpha x)$ since it is odd in x (see the construction in Lemma 2.5), where $\phi_{c_{n_j}}$ is a L^2 normalized eigenfunction of $\lambda_{n_j}(c_{n_j}) = -\alpha^2 \in \sigma(\mathcal{L}_{c_{n_j}})$. Since $\phi_{c_{n_j}}$ has $n_j - 1$ sign-changed zeros in (y_1, y_2), $\tilde{v}_{\varepsilon_{n_j}}$ oscillates frequently in the y-direction for large j.

The minimal period of any nearby traveling wave solution in x can be determined under the following condition:

Lemma 5.3. *Under the assumption of Lemma 2.7, if*

$$c_0 \in \sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R} \quad \textit{and} \quad c_0 \notin \bigcup_{k \geqq 2} (\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}), \tag{5.24}$$

then $\boldsymbol{u}_\varepsilon\,(x - c_\varepsilon t, y)$ has minimal period $2\pi/\alpha$ in x for $\varepsilon > 0$ small enough.

Proof. Let the minimal horizontal period of the traveling wave solution $\boldsymbol{u}_\varepsilon\,(x - c_\varepsilon t, y)$ be T/n_ε for $\varepsilon \in (0, \varepsilon_0)$, where $n_\varepsilon \in \mathbf{Z}^+$ and $T = 2\pi/\alpha$. Fix $\varepsilon \in (0, \varepsilon_0)$ and let $\tilde{v}_\varepsilon = v_\varepsilon / \|v_\varepsilon\|_{L^2(D_T)}$. Since $\tilde{v}_\varepsilon(x,y) = \tilde{v}_\varepsilon(x + T/n_\varepsilon, y)$ for $x \in \mathbf{R}$ and $y \in [y_1, y_2]$, we have $\int_0^T e^{i\alpha x}\tilde{v}_\varepsilon(x,y)\,\mathrm{d}x = e^{i\alpha T/n_\varepsilon}\int_0^T e^{i\alpha x}\tilde{v}_\varepsilon(x,y)\,\mathrm{d}x$. Thus, if $n_\varepsilon > 1$, then $e^{i\alpha T/n_\varepsilon} = e^{2\pi i/n_\varepsilon} \neq 1$ and $\int_0^T e^{i\alpha x}\tilde{v}_\varepsilon(x,y)\,\mathrm{d}x = 0$ for $y \in [y_1, y_2]$.

Suppose that there exists a sequence $\{\varepsilon_k : k \geqq 1\} \subset (0, \varepsilon_0)$ such that $\varepsilon_k \to 0^+$ and $\boldsymbol{u}_{\varepsilon_k}\left(x - c_{\varepsilon_k} t, y\right)$ has minimal period $T/n_{\varepsilon_k} < T$ in x for $k \geqq 1$. Then $\int_0^T e^{i\alpha x}\tilde{v}_{\varepsilon_k}(x,y)\,\mathrm{d}x = 0$ for $k \geqq 1$ and $y \in [y_1, y_2]$. By Lemma 2.7, $\tilde{v}_{\varepsilon_k} \longrightarrow \varphi_{c_0}$ in $H^2\,(D_T)$, where $\varphi_{c_0} \in \ker(\mathcal{G}_{c_0})$ and $\mathcal{G}_{c_0}$ is defined in (2.3). Then

$$\int_0^T e^{i\alpha x}\varphi_{c_0}(x,y)\,\mathrm{d}x = \lim_{k\to\infty}\int_0^T e^{i\alpha x}\tilde{v}_{\varepsilon_k}(x,y)\,\mathrm{d}x = 0 \tag{5.25}$$

for $y \in [y_1, y_2]$. By (5.24) and $\widehat{\varphi}_{c_0,0} = 0$, we have $\varphi_{c_0}(x,y) = \widehat{\varphi}_{c_0,1}(y)e^{i\alpha x} + \overline{\widehat{\varphi}_{c_0,1}(y)}e^{-i\alpha x}$, where $\widehat{\varphi}_{c_0,1} \neq 0$ is an eigenfunction of $-\alpha^2 \in \sigma(\mathcal{L}_{c_0})$. On the other hand, we have that

$$\int_0^T e^{i\alpha x}\varphi_{c_0}(x,y)\,\mathrm{d}x = T\overline{\widehat{\varphi}_{c_0,1}(y)} \not\equiv 0,$$

which contradicts (5.25). $\square$

Remark 5.4. (1) Let $c_0 \in \bigcup_{k\geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})$ and k_* be defined in (5.4). It follows from Lemma 5.3 that under the assumption of Lemma 2.7, if $\boldsymbol{u}_\varepsilon\,(x - c_\varepsilon t, y)$ has period $2\pi/(k_*\alpha)$ in x for $\varepsilon \in (0, \varepsilon_0)$, then the period $2\pi/(k_*\alpha)$ is minimal for $\varepsilon > 0$ small enough.

(2) In Proposition 7 of [27], it should be corrected that the minimal period of constructed traveling wave solutions in x might be less than $2\pi/\alpha_0$, since it is possible that $(c_0, k\alpha_0, \beta, \phi_k)$ is a non-resonant neutral mode for some $k \geqq 2$ and $\phi_k \in H_0^1 \cap H^2(y_1, y_2)$.

Consequently, the minimal period of constructed traveling wave solutions near the sinus profile in Theorem 7 (i) of [27] might be less than $2\pi/\alpha_0$, see Example

7.1 for systematic study of traveling wave families near the sinus profile. If (5.24) holds true for $\alpha = \alpha_0$, then the minimal period of these traveling wave solutions in Proposition 7 and Theorem 7 (i) of [27] is $2\pi/\alpha_0$.

6. The number of traveling wave families near a shear flow

In this section, we prove the main theorems-Theorems 2.2 and 1.2 . The proof is based on the study on the number of isolated real eigenvalues of the linearized Euler operator in Sections 3-4, and correspondence between a traveling wave family near the shear flow and an isolated real eigenvalue in Section 5. We only prove Theorem 2.2, since the other is similar.

Proof of Theorem 2.2. Let the number of traveling wave families near $(u, 0)$ be denoted by θ. By Theorem 2.1, $\theta = \sharp(\bigcup_{k\geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}))$. Here $\alpha = 2\pi/T$.

Proof of (1): Since $\{u' = 0\} \cap \{u = u_{\min}\} \neq \emptyset$, we have $0 < \kappa_+ < \infty$. First, let $\{u = u_{\min}\} \cap (y_1, y_2) \neq \emptyset$ and we divide the discussion into two cases.

Case 1a. $\beta \in (0, \min\{\frac{9}{8}\kappa_+, \mu_+\})$.

By Corollary 2.10 (1),

$$\inf_{c\in(-\infty,u_{\min})} \lambda_1(c) > -\infty. \tag{6.1}$$

Thus, there exists $1 \leqq k_0 < \infty$ such that

$$\bigcup_{k>k_0} (\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}) = \emptyset. \tag{6.2}$$

By Theorem 2.11 (1), we have that

$$\theta = \sharp\Big(\bigcup_{k\geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})\Big) \leqq \sum_{k\geqq 1} \sharp(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}) = \sum_{k=1}^{k_0} \sharp(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}) < \infty. \tag{6.3}$$

Case 1b. $\beta \in (\min\{\frac{9}{8}\kappa_+, \mu_+\}, \infty)$.

By Corollary 2.10 (1) for $\beta \in (\min\{\frac{9}{8}\kappa_+, \mu_+\}, \frac{9}{8}\kappa_+]$ and Theorem 2.9 (3) for $\beta \in (\frac{9}{8}\kappa_+, \infty)$, we have that

$$\lim_{c\to u_{\min}^-} \lambda_1(c) = -\infty. \tag{6.4}$$

Thus, there exists $c_k \in \sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}$ such that $c_k < c_{k+1} < u_{\min}$ for every $k \geqq 1$, and $c_k \to u_{\min}^-$. Then $\theta = \sharp(\bigcup_{k\geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R})) = \infty$.

Next, let $\{u = u_{\min}\} \cap (y_1, y_2) = \emptyset$ and we separate the proof into two cases.

Case 2a. $\beta \in (0, \frac{9}{8}\kappa_+)$.

By Corollary 2.10 (3), we obtain (6.1). Thus, there exists $1 \leqq k_0 < \infty$ such that (6.2) holds. By Theorem 2.11 (1), we obtain (6.3).

Case 2b. $\beta \in (\frac{9}{8}\kappa_+, \infty)$.

By Theorem 2.9 (3), we have that

$$\lim_{c \to u_{\min}^-} \lambda_n(c) = -\infty, \quad n \geqq 1. \tag{6.5}$$

Using (6.5) for $n = 1$ and the fact that $\theta = \sharp(\bigcup_{k \geqq 1}(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}))$, it can be proved that $\theta = \infty$ by a similar way as in Case 1b. This completes the proof of (1). The proof of (2) is similar.

Proof of (3): Since $\{u' = 0\} \cap \{u = u_{\min}\} = \emptyset$, we have $\{u = u_{\min}\} \cap (y_1, y_2) = \emptyset$ and $\kappa_+ = \infty$. Fix $\beta \in (0, \infty)$. By Corollary 2.10 (3), we obtain (6.1), and thus, there exists $1 \leqq k_0 < \infty$ such that (6.2) holds. By Theorem 2.12 (1), we obtain (6.3). This completes the proof of (3). The proof of (4) is similar. □

Remark 6.1. In Cases 1*b* and 2*b* of the above proof, the infinitely many traveling wave families are produced by the asymptotic behavior of the first eigenvalue $\lambda_1(c)$ of $\mathcal{L}_c$, see (6.4). There could be many other traveling wave families in general, which are produced by the asymptotic behavior of $\lambda_n(c)$ for $n \geqq 2$, see (6.5). In fact, if $\beta \in (\frac{9}{8}\kappa_+, \infty)$, for fixed $n \geqq 2$, there exists $c_{k,n} \in \sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}$ such that $\lambda_n(c_{k,n}) = -(k\alpha)^2$ for every $k \geqq 1$. If $\{c_{k,n} : k \geqq 1, n \geqq 2\} \setminus \{c_k : k \geqq 1\} \neq \emptyset$, then a simple application of Theorem 2.1 yields other traveling wave families.

7. Application to the sinus profile

In this section, we apply our main results to the sinus profile. Moreover, we calculate the explicit number of isolated real eigenvalues of $\mathcal{R}_{\alpha,\beta}$ and traveling wave families near sinus profile.

Example 7.1. The sinus profile is $u(y) = \frac{1+\cos(\pi y)}{2}$, $y \in [-1, 1]$. We determine $\sharp\big(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}\big)$ and the number of traveling wave families for the sinus profile on the (α, β)'s region. For the sinus profile, we have $u_{\min} = 0$, $u_{\max} = 1$, $\{u' = 0\} \cap \{u = u_{\min}\} = \{\pm 1\}$, $\{u' = 0\} \cap \{u = u_{\max}\} = \{0\}$, $\kappa_+ = u''(\pm 1) = \frac{1}{2}\pi^2$ and $\kappa_- = u''(0) = -\frac{1}{2}\pi^2$. We divide the plane into nine parts as follows:

In Fig. 3,

$$I = \{(\alpha, \beta) | \alpha > 0, \beta < -\frac{9}{16}\pi^2\},$$

$$II = \{(\alpha, \beta) | 0 < \alpha < \pi\sqrt{-r^2 - r + \frac{3}{4}}, -\frac{9}{16}\pi^2 \leqq \beta < -\frac{1}{2}\pi^2, r \in [\frac{1}{4}, \frac{1}{2})\},$$

$$III = \{(\alpha, \beta) | \alpha \geqq \pi\sqrt{-r^2 - r + \frac{3}{4}}, -\frac{9}{16}\pi^2 \leqq \beta < -\frac{1}{2}\pi^2, r \in [\frac{1}{4}, \frac{1}{2})\} \cup$$

$$\{(\alpha, \beta) | 0 < \alpha < \frac{\sqrt{3}}{2}\pi, \beta = -\frac{1}{2}\pi^2\},$$

$$IV = \{(\alpha, \beta) | 0 < \alpha < \sqrt{\Lambda_\beta}, -\frac{1}{2}\pi^2 < \beta < \beta_l\},$$

$$V = \{(\alpha, \beta) | \pi\sqrt{1 - r^2} < \alpha < \sqrt{\Lambda_\beta}, \frac{\sqrt{3} - 1}{4}\pi^2 < \beta < \frac{1}{2}\pi^2\},$$

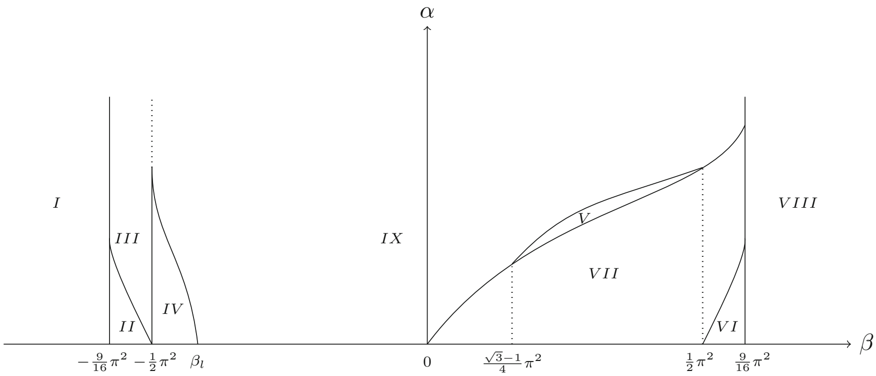


Fig. 3. Nine parts of the plane

$$VI = \{(\alpha,\beta)|0<\alpha<\pi\sqrt{-r^2-r+\frac{3}{4}},\ \frac{1}{2}\pi^2<\beta\leqq\frac{9}{16}\pi^2, r\in[\frac{1}{4},\frac{1}{2})\},$$

$$VII = \{(\alpha,\beta)|0<\alpha<\sqrt{\Lambda_\beta},\ 0<\beta\leqq\frac{\sqrt{3}-1}{4}\pi^2\}\cup$$
$$\{(\alpha,\beta)|0<\alpha\leqq\pi\sqrt{1-r^2},\ \frac{\sqrt{3}-1}{4}\pi^2<\beta<\frac{1}{2}\pi^2, r\in(\frac{1}{2},\frac{\sqrt{3}}{2})\}\cup$$
$$\{(\alpha,\beta)|\pi\sqrt{-r^2-r+\frac{3}{4}}\leqq\alpha<\pi\sqrt{1-r^2},\ \frac{1}{2}\pi^2\leqq\beta\leqq\frac{9}{16}\pi^2, r\in[\frac{1}{4},\frac{1}{2}]\},$$

$$VIII = \{(\alpha,\beta)|\alpha>0,\ \beta>\frac{9}{16}\pi^2\},$$

$$IX = \{(\alpha,\beta)|\alpha\geqq\frac{\sqrt{3}}{2}\pi,\ \beta=-\frac{1}{2}\pi^2\}\cup\{(\alpha,\beta)|\alpha>\sqrt{\Lambda_\beta},$$
$$-\frac{1}{2}\pi^2<\beta<\frac{1}{2}\pi^2\}\cup\{(\alpha,\beta)|$$
$$\alpha=\sqrt{\Lambda_\beta},\ 0<\beta\leqq\frac{\sqrt{3}-1}{4}\pi^2\}\cup\{(\alpha,\beta)|\alpha\geqq\pi\sqrt{1-r^2},\ \frac{1}{2}\pi^2\leqq\beta$$
$$\leqq\frac{9}{16}\pi^2, r\in[\frac{1}{4},\frac{1}{2}]\},$$

where $\Lambda_\beta = \sup_{c\notin(0,1)}\max\{-\lambda_1(c),0\}$, $r=\frac{1}{4}+\sqrt{\frac{9}{16}+\frac{\beta}{\pi^2}}$ for $-\frac{9}{16}\pi^2\leqq\beta<0$, $r=\frac{1}{4}+\sqrt{\frac{9}{16}-\frac{\beta}{\pi^2}}$ for $0<\beta\leqq\frac{9}{16}\pi^2$, and β_l is given by Theorem 6 of [27]. Moreover, $m_\beta=0$ and $\max\{M_\beta,0\}=\Lambda_\beta$ for $\beta\in[-\frac{1}{2}\pi^2,0)\cup(0,\frac{9}{16}\pi^2]$, and $m_\beta=1$ and $M_\beta=(-r^2-r+\frac{3}{4})\pi^2$ for $\beta\in[-\frac{9}{16}\pi^2,-\frac{1}{2}\pi^2)$.

The explicit number $\sharp\left(\sigma_d(\mathcal{R}_{\alpha,\beta})\cap\mathbf{R}\right)$ is given as follows:

$$(\alpha,\beta)\in IX \Longrightarrow \sharp\left(\sigma_d(\mathcal{R}_{\alpha,\beta})\cap\mathbf{R}\right)=0;$$
$$(\alpha,\beta)\in III\cup VII \Longrightarrow \sharp\left(\sigma_d(\mathcal{R}_{\alpha,\beta})\cap\mathbf{R}\right)=1;$$
$$(\alpha,\beta)\in IV\cup V\cup VI \Longrightarrow \sharp\left(\sigma_d(\mathcal{R}_{\alpha,\beta})\cap\mathbf{R}\right)=2; \qquad (7.1)$$

$$(\alpha, \beta) \in II \Longrightarrow \sharp\left(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}\right) = 3;$$
$$(\alpha, \beta) \in I \cup VIII \Longrightarrow \sharp\left(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}\right) = \infty.$$

In addition, $\sharp\left(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}\right) = 1$ if $(\alpha, \beta) \in \Gamma := \{(\alpha, \beta)|\alpha = \sqrt{\Lambda_\beta}, -\frac{1}{2}\pi^2 < \beta < \beta_l$ or $\frac{\sqrt{3}-1}{4}\pi^2 < \beta < \frac{\pi^2}{2}\}$. Now we fix $\alpha = 2\pi/T$.

The number (denoted by θ) of traveling wave families near the sinus profile is given by

$$(\alpha, \beta) \in IX \Longrightarrow \theta = 0; \tag{7.2}$$
$$(\alpha, \beta) \in VII \Longrightarrow 1 \leqq \theta < \infty;$$
$$(\alpha, \beta) \in IV \cup V \cup VI \Longrightarrow 2 \leqq \theta < \infty;$$
$$(\alpha, \beta) \in I \cup II \cup III \cup VIII,\ \beta \neq -\frac{\pi^2}{2} \Longrightarrow \theta = \infty.$$

In addition, $\theta = \sharp(\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R}) = 1$ if $(\alpha, \beta) \in \Gamma$. Moreover,

$$(\alpha, \beta) \in III \cup IV \cup V \cup VII \Longrightarrow \theta = \sum_{k \geqq 1} \sharp(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}). \tag{7.3}$$

If $(\alpha, \beta) \in III \cup IV \cup V \cup VII \cup \Gamma$, then

$$c_0 \in \sigma_d(\mathcal{R}_{\alpha,\beta}) \cap \mathbf{R} \Longrightarrow \boldsymbol{u}_\varepsilon\,(x - c_\varepsilon t, y) \text{ has minimal period } 2\pi/\alpha \text{ in } x \tag{7.4}$$

for $\varepsilon > 0$ small enough, where $\boldsymbol{u}_\varepsilon\,(x - c_\varepsilon t, y)$ satisfies the assumption of Lemma 2.7.

To prove (7.1)–(7.4), we need the following asymptotic behavior, signatures and monotonicity of λ_n.

(1) $\lim_{c\to\pm\infty} \lambda_n(c) = \frac{n^2}{4}\pi^2 > 0$ for $\beta \in \mathbf{R}$;

(2) $\lim_{c\to 0^-} \lambda_n(c) = -\infty$ and λ_n is decreasing on $(-\infty, 0)$ for $\beta \in (\frac{9}{16}\pi^2, \infty)$;

(3) $\lim_{c\to 0^-} \lambda_n(c) = \lambda_n(0)$, $\lambda_1(0) < \lambda_2(0) \leqq 0$, $\lambda_3(0) > 0$ and λ_n is decreasing on $(-\infty, 0)$ for $\beta \in [\frac{1}{2}\pi^2, \frac{9}{16}\pi^2]$;

(4) $\lim_{c\to 0^-} \lambda_n(c) = \lambda_n(0)$, $\lambda_1(0) < 0$, $\lambda_2 > 0$ on $(-\infty, 0]$, there exist $c_1 < c_2 \in (-\infty, 0)$ such that $\lambda_1(c) \geqq \lambda_1(c_1) = 0$ for $c \in (-\infty, c_1)$, λ_1 is decreasing on (c_1, c_2), $\lambda_1(c_2) = \inf_{c\in(-\infty,0)} \lambda_1(c)$ and λ_1 is increasing on $(c_2, 0)$ for $\beta \in (\frac{\sqrt{3}-1}{4}\pi^2, \frac{1}{2}\pi^2)$;

(5) $\lim_{c\to 0^-} \lambda_n(c) = \lambda_n(0)$, $\lambda_1(0) < 0$, $\lambda_2 > 0$ on $(-\infty, 0]$, there exists $c_1 \in (-\infty, 0)$ such that $\lambda_1(c) \geqq \lambda_1(c_1) = 0$ for $c \in (-\infty, c_1)$ and λ_1 is decreasing on $(c_1, 0)$ for $\beta \in (0, \frac{\sqrt{3}-1}{4}\pi^2]$;

(6) $\lambda_1 \geqq 0$ on $(1, \infty)$ for $\beta \in [\beta_l, 0)$;

(7) $\lim_{c\to 1^+} \lambda_n(c) = \lambda_n(1)$, $\lambda_1(1) > 0$, $\lambda_2 > 0$ on $(1, \infty)$, there exist $c_1 < c_2 < c_3 \in (1, \infty)$ such that $\lambda_1(c) \geqq \lambda_1(c_1) = \lambda_1(c_3) = 0$ for $c \in (1, c_1) \cup (c_3, \infty)$, λ_1 is decreasing on (c_1, c_2), $\lambda_1(c_2) = \inf_{c\in(1,\infty)} \lambda_1(c)$ and λ_1 is increasing on (c_2, c_3) for $\beta \in (-\frac{1}{2}\pi^2, \beta_l)$;

(8) $\lim_{c\to 1^+} \lambda_n(c) = \lambda_n(1)$, $\lambda_1(1) < 0$, $\lambda_2(1) = 0$ and λ_n is increasing on $(1, \infty)$ for $\beta = -\frac{1}{2}\pi^2$;

(9) $\lim_{c\to 1^+} \lambda_1(c) = -\infty$, $\lim_{c\to 1^+} \lambda_{n+1}(c) = \lambda_n(1)$, $\lambda_1(1) = \lambda_2(1) < 0$, $\lambda_3(1) > 0$ and λ_n is increasing on $(1, \infty)$ for $\beta \in [-\frac{9}{16}\pi^2, -\frac{1}{2}\pi^2)$;

(10) $\lim_{c\to 1^+}\lambda_n(c) = -\infty$ and λ_n is increasing on $(1,\infty)$ for $\beta \in (-\infty, -\frac{9}{16}\pi^2)$, where $n \geqq 1$, $\lambda_n(0) = \left(\left(r+\frac{n-1}{2}\right)^2 - 1\right)\pi^2$ for $\beta \in (0, \frac{9}{16}\pi^2]$, $\lambda_n(1) = \left(\left(r-\frac{1}{2}+\lceil\frac{n}{2}\rceil\right)^2 - 1\right)\pi^2$ for $\beta \in [-\frac{9}{16}\pi^2, -\frac{1}{2}\pi^2) \cup (-\frac{1}{2}\pi^2, 0)$, and $\lambda_n(1) = (\frac{n^2}{4}-1)\pi^2$ for $\beta = -\frac{1}{2}\pi^2$ by Proposition 1 in [27].

Assertions (1)–(10) provide pictures of the negative eigenvalues of $\mathcal{L}_c$ for fixed β. Assume that (1)–(10) are true. Note that $\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (1,\infty) = \emptyset$ if $\alpha \in \mathbf{R}$ and $\beta > 0$, and $\sigma_d(\mathcal{R}_{\alpha,\beta}) \cap (-\infty, 0) = \emptyset$ if $\alpha \in \mathbf{R}$ and $\beta < 0$. Then we claim that

$$(\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R}) \cap (\sigma_d(\mathcal{R}_{j\alpha,\beta}) \cap \mathbf{R}) = \emptyset, \tag{7.5}$$

$$\sigma_d(\mathcal{R}_{k\alpha,\beta}) \cap \mathbf{R} = \{c \in \mathbf{R} \setminus [0,1] | \lambda_1(c) = -(k\alpha)^2\} \tag{7.6}$$

for $k, j \in \mathbf{Z}^+$, $k \neq j$ and $(\alpha,\beta) \in III \cup IV \cup V \cup VII \cup \Gamma$. In fact, (7.6) implies (7.5). If $(\alpha,\beta) \in III$, then by (8)–(9) we have $\lambda_2 > \lambda_2(1) \geqq -\alpha^2$ on $(1,\infty)$, which gives (7.6). If $(\alpha,\beta) \in IV \cup \Gamma$ with $\beta < 0$, then by (7) we have $\lambda_2 > 0$ on $(1,\infty)$ and thus, $\mathcal{L}_c$ has at most one negative eigenvalue for $c \in (1,\infty)$, which gives (7.6). Similarly, we can prove (7.6) for $(\alpha,\beta) \in V \cup VII \cup \Gamma$ with $0 < \beta < \frac{\pi^2}{2}$ by (4)–(5). If $(\alpha,\beta) \in VII$ with $\beta \geqq \frac{\pi^2}{2}$, then by (3) we have $\lambda_2 > \lambda_2(0) \geqq -\alpha^2$ on $(-\infty, 0)$, which gives (7.6).

By applying Theorem 2.1, we get (7.1)–(7.3). (7.4) is a direct consequence of Lemma 5.3. Here, (7.5) is used in the proof of (7.3)–(7.4).

Using (7.3) we can evaluate θ for $(\alpha,\beta) \notin VI$ as follow:

$$\begin{aligned}
\beta \in (-\infty, -\frac{1}{2}\pi^2) \cup (\frac{9}{16}\pi^2, +\infty) &\Longrightarrow \theta = \infty;\\
\beta = -\frac{1}{2}\pi^2 &\Longrightarrow \theta = \lceil\frac{\sqrt{3}\pi}{2\alpha}\rceil - 1;\\
-\frac{1}{2}\pi^2 < \beta < \beta_l &\Longrightarrow \theta = \lfloor\frac{\sqrt{\Lambda_\beta}}{\alpha}\rfloor + \lceil\frac{\sqrt{\Lambda_\beta}}{\alpha}\rceil - 1;\\
\beta_l \leqq \beta \leqq 0 &\Longrightarrow \theta = 0;\\
0 < \beta \leqq \frac{\sqrt{3}-1}{4}\pi^2 &\Longrightarrow \theta = \lceil\frac{\sqrt{\Lambda_\beta}}{\alpha}\rceil - 1;\\
\frac{\sqrt{3}-1}{4}\pi^2 < \beta < \frac{1}{2}\pi^2 &\Longrightarrow \theta = \lfloor\frac{\sqrt{\Lambda_\beta}}{\alpha}\rfloor + \lceil\frac{\sqrt{\Lambda_\beta}}{\alpha}\rceil\\
&\qquad - \lfloor\frac{\pi\sqrt{1-r^2}}{\alpha}\rfloor - 1;\\
\pi\sqrt{-r^2-r+\frac{3}{4}} \leqq \alpha,\ \frac{1}{2}\pi^2 \leqq \beta \leqq \frac{9}{16}\pi^2 &\Longrightarrow \theta = \lceil\frac{\pi\sqrt{1-r^2}}{\alpha}\rceil - 1.
\end{aligned} \tag{7.7}$$

The case $(\alpha,\beta) \in VI$ is more complicated. By (3), we have $\theta = \sharp(A_1 \cup A_2)$ with $A_i = \{c < 0 | \lambda_i(c) = -(k\alpha)^2,\ k \in \mathbf{Z}^+\}$. Then by (3) and the expression of $\lambda_i(0)$, $i = 1, 2$, we have $\sharp(A_1) = \lceil\frac{\pi\sqrt{1-r^2}}{\alpha}\rceil - 1$, $\sharp(A_2) = \lceil\frac{\pi\sqrt{-r^2-r+\frac{3}{4}}}{\alpha}\rceil - 1$, and $\sharp(A_1) \leqq \theta \leqq \sharp(A_1) + \sharp(A_2)$, i.e. $\lceil\frac{\pi\sqrt{1-r^2}}{\alpha}\rceil - 1 \leqq \theta \leqq \lceil\frac{\pi\sqrt{1-r^2}}{\alpha}\rceil + \lceil\frac{\pi\sqrt{-r^2-r+\frac{3}{4}}}{\alpha}\rceil - 2$. In fact, $\theta = \sharp(A_1) + \sharp(A_2) - \sharp(A_1 \cap A_2)$, but it seems difficult to give an explicit formula if $A_1 \cap A_2 \neq \emptyset$.

Now, we prove (1)–(10). (1) and (4)–(7) are a summary of spectral results in Section 4 of [27]. Monotonicity of λ_n for $\beta \in (-\infty, -\frac{1}{2}\pi^2] \cup [\frac{1}{2}\pi^2, \infty)$ is due to Corollary 1 in [27]. Asymptotic behavior of λ_n in (2) and (10) is obtained by Corollary 2.10. Signatures of λ_n in (3) and (8)–(9) are due to Proposition 1 in [27] and simple computation.

The rest is to prove the asymptotic behavior of λ_n in (3) and (8)–(9). First, we consider $\beta = \pm\frac{1}{2}\pi^2$. We only prove that $\lim_{c \to 0^-} \lambda_n(c) = \lambda_n(0)$ for $\beta = \frac{1}{2}\pi^2$ and $n \geqq 1$. Note that $\frac{\beta - u''}{u} = \pi^2$ and $\left\| \frac{\beta - u''}{u-c} - \frac{\beta - u''}{u} \right\|_{L^1(-1,1)} = \pi^2 \left\| \frac{c}{u-c} \right\|_{L^1(-1,1)}$ for $c < 0$. Let $0 < \delta < 1$. Then

$$\left\| \frac{c}{u-c} \right\|_{L^1(1-\delta,1)} = \frac{2}{\pi} \int_0^{\cos(\frac{\pi}{2}(1-\delta))} \frac{-c}{z^2 - c} \frac{1}{\sqrt{1-z^2}} \, \mathrm{d}z \leqq C_\delta \sqrt{-c} \to 0$$

as $c \to 0^-$. Similarly, $\left\| \frac{c}{u-c} \right\|_{L^1(-1,-1+\delta)} \to 0$. Clearly, $\left\| \frac{c}{u-c} \right\|_{L^1(-1+\delta,1-\delta)} \to 0$. Then $\left\| \frac{c}{u-c} \right\|_{L^1(-1,1)} \to 0$. It follows from Theorem 2.1 in [21] that $\lim_{c \to 0^-} \lambda_n(c) = \lambda_n(0)$ for $n \geqq 1$.

We then consider $\beta \in (\frac{1}{2}\pi^2, \frac{9}{16}\pi^2)$. We use the eigenfunctions of $\lambda_n(\beta, 0)$ in Proposition 1 (iii) of [27]. Here, we rewrite $\lambda_n(\beta, c) = \lambda_n(c)$ to indicate its dependence on β if necessary. There exist $\phi_n(y) = \phi_n^{(\beta,0)}(y) = \cos^{2r}(\frac{\pi}{2}y) P_{n-1}(\sin(\frac{\pi}{2}y))$, $n \geqq 1$, satisfying

$$-\phi_n'' - \frac{\beta - u''}{u}\phi_n = \lambda_n(0)\phi_n \quad \text{on} \quad (-1, 1), \quad \phi_n(\pm 1) = 0.$$

Here, $\lambda_n(\beta, 0) = \left(\left(r + \frac{n-1}{2}\right)^2 - 1\right)\pi^2$, $r = \frac{1}{4} + \sqrt{\frac{9}{16} - \frac{\beta}{\pi^2}}$ and $P_{n-1}(\cdot)$ is a polynomial with order $n-1$. Moreover, $\phi_n \in H_0^1(-1, 1)$ is real-valued, and we normalize it such that $\|\phi_n\|_{L^2(-1,1)} = 1$. Then we have, for $m, n \geqq 1$, that

$$\int_{-1}^{1} \phi_n \phi_m \, \mathrm{d}y = \delta_{mn} = \begin{cases} 1 \text{ if } n = m, \\ 0 \text{ if } n \neq m, \end{cases} \quad \int_{-1}^{1} \left(\phi_n' \phi_m' + \frac{u'' - \beta}{u} \phi_n \phi_m \right) \mathrm{d}y = \lambda_n(0)\delta_{mn}.$$

Note that ϕ_n has $n-1$ zeros in $(-1, 1)$, and we denote $Z_n := \{y \in (-1, 1) : \phi_n(y) = 0\} = \{a_{n,1}, \cdots, a_{n,n-1}\}$. For any n-dimensional subspace $V = \text{span}\{\psi_1, \cdots, \psi_n\}$ in $H_0^1(-1, 1)$, there exists $0 \neq (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$ such that $\xi_1\psi_1(a_{n,i}) + \cdots + \xi_n\psi_n(a_{n,i}) = 0$, $i = 1, \cdots, n-1$. Define $\tilde{\psi} = \xi_1\psi_1 + \cdots + \xi_n\psi_n$. Then $\tilde{\psi}(a_{n,i}) = 0$, $i = 1, \cdots, n-1$, i.e. $\tilde{\psi}|_{Z_n} = 0$. We normalize $\tilde{\psi}$ such that $\|\tilde{\psi}\|_{L^2(-1,1)} = 1$. Since $\tilde{\psi} \in H_0^1(-1, 1)$, we have $\tilde{\psi}(\pm 1) = 0$. Similar to (3.2), we have $|\tilde{\psi}(y)|^2 \phi_n'(y)/\phi_n(y) \to 0$ as $y \to a_{n,i}$ or $y \to -1^+$ or $y \to 1^-$, where $1 \leqq i \leqq n-1$. Integration by parts gives

$$\left\|\tilde{\psi}' - \tilde{\psi}\frac{\phi_n'}{\phi_n}\right\|^2_{L^2(-1,1)} = \int_{-1}^{1}\left(|\tilde{\psi}'|^2 + \frac{\phi_n''}{\phi_n}|\tilde{\psi}|^2\right)\mathrm{d}y$$
$$= \int_{-1}^{1}\left(|\tilde{\psi}'|^2 - \frac{\beta - u''}{u}|\tilde{\psi}|^2 - \lambda_n(0)|\tilde{\psi}|^2\right)\mathrm{d}y.$$

If $c < 0$, then using $\beta - u'' > 0$ and $u - c > 0$ for $y \in [-1, 1]$, we have that

$$\int_{-1}^{1}\left(|\tilde{\psi}'|^2 - \frac{\beta - u''}{u - c}|\tilde{\psi}|^2\right)\mathrm{d}y \geqq \int_{-1}^{1}\left(|\tilde{\psi}'|^2 - \frac{\beta - u''}{u}|\tilde{\psi}|^2\right)\mathrm{d}y$$
$$\geqq \int_{-1}^{1}\lambda_n(0)|\tilde{\psi}|^2\,\mathrm{d}y = \lambda_n(0).$$

This, along with (2.7), yields that $\inf_{c\in(-\infty,0)}\lambda_n(c) \geqq \lambda_n(0)$. Now, we consider the upper bound. Let $V_n = \mathrm{span}\{\phi_1, \cdots, \phi_n\}$. Then $V_n \subset H_0^1(-1, 1)$. By (2.7), there exist $b_{i,c} \in \mathbf{R}$, $i = 1, \cdots, n$, with $\sum_{i=1}^n |b_{i,c}|^2 = 1$ such that $\varphi_c = \sum_{i=1}^n b_{i,c}\phi_i \in V_n$ with $\|\varphi_c\|^2_{L^2} = 1$, and

$$\lambda_n(c) \leqq \sup_{\|\phi\|_{L^2}=1,\phi\in V_n}\int_{-1}^{1}\left(|\phi'|^2 + \frac{u'' - \beta}{u - c}|\phi|^2\right)\mathrm{d}y$$
$$= \int_{-1}^{1}\left(|\varphi_c'|^2 + \frac{u'' - \beta}{u - c}|\varphi_c|^2\right)\mathrm{d}y$$
$$= \sum_{i=1}^{n}|b_{i,c}|^2\int_{-1}^{1}\left(|\phi_i'|^2 + \frac{u'' - \beta}{u - c}|\phi_i|^2\right)\mathrm{d}y$$
$$+ \sum_{1\leqq i<j\leqq n}2b_{i,c}b_{j,c}\int_{-1}^{1}\left(\phi_i'\phi_j' + \frac{u'' - \beta}{u - c}\phi_i\phi_j\right)\mathrm{d}y$$
$$\leqq \max_{1\leqq i\leqq n}\int_{-1}^{1}\left(|\phi_i'|^2 + \frac{u'' - \beta}{u - c}|\phi_i|^2\right)\mathrm{d}y$$
$$+ \sum_{1\leqq i<j\leqq n}\left|\int_{-1}^{1}\left(\phi_i'\phi_j' + \frac{u'' - \beta}{u - c}\phi_i\phi_j\right)\mathrm{d}y\right|$$
$$\to \max_{1\leqq i\leqq n}\int_{-1}^{1}\left(|\phi_i'|^2 + \frac{u'' - \beta}{u}|\phi_i|^2\right)\mathrm{d}y$$
$$+ \sum_{1\leqq i<j\leqq n}\left|\int_{-1}^{1}\left(\phi_i'\phi_j' + \frac{u'' - \beta}{u}\phi_i\phi_j\right)\mathrm{d}y\right|$$
$$= \max_{1\leqq i\leqq n}\lambda_i(0) + \sum_{1\leqq i<j\leqq n}0 = \lambda_n(0), \text{ as } c \to 0^-.$$

Combining the upper and lower bounds, we have $\lim_{c\to 0^-}\lambda_n(c) = \lambda_n(0)$ for $\beta \in (\frac{1}{2}\pi^2, \frac{9}{16}\pi^2)$.

Now, we consider $\beta = \frac{9}{16}\pi^2$. By Corollary 1 (i) in [27], we have for fixed $c < 0$, $\lambda_n(\beta, c) \leqq \lambda_n(\beta', c)$ if $\beta' < \beta$. As $\lambda_n(\beta', c) \geqq \lambda_n(\beta', c')$ if $c < c' < 0$

(see Corollary 1 (iv) in [27]) and $\lim_{c\to 0^-}\lambda_n(\beta',c)=\lambda_n(\beta',0)$, we have for fixed $\beta'\in(\frac{1}{2}\pi^2,\frac{9}{16}\pi^2)$, $\lambda_n(\beta',c)\geqq\lambda_n(\beta',0)$ if $c<0$. Then

$$\lim_{c\to 0^-}\lambda_n(\beta,c)\leqq\liminf_{\beta'\to\beta^-}\lim_{c\to 0^-}\lambda_n(\beta',c)=\liminf_{\beta'\to\beta^-}\lambda_n(\beta',0)=\lambda_n(\beta,0),$$
$$\lim_{c\to 0^-}\lambda_n(\beta,c)=\lim_{c\to 0^-}\lim_{\beta'\to\beta^-}\lambda_n(\beta',c)\geqq\lim_{\beta'\to\beta^-}\lambda_n(\beta',0)=\lambda_n(\beta,0).$$

Here, we used the left continuity of $\lambda_n(\cdot,0)$ at $\beta=\frac{9}{16}\pi^2$. Thus, $\lim_{c\to 0^-}\lambda_n(\beta,c)=\lambda_n(\beta,0)$.

Next, we consider $\beta\in(-\frac{9}{16}\pi^2,-\frac{1}{2}\pi^2)$. By Proposition 1 (iv) in [27], there exists $\phi_n(y)=\phi_n^{(\beta,1)}(y)=|\sin(\frac{\pi}{2}y)|^{2r}P_n(\cos(\frac{\pi}{2}y))$ if n is odd; there exists $\phi_n(y)=\phi_n^{(\beta,1)}(y)=\operatorname{sign}(y)\,|\sin(\frac{\pi}{2}y)|^{2r}P_{n-1}(\cos(\frac{\pi}{2}y))$ if n is even; and, for $n\geqq 1$,

$$-\phi_n''-\frac{\beta-u''}{u-1}\phi_n=\lambda_n(1)\phi_n\quad\text{on}\quad(-1,1)\setminus\{0\},\quad\phi_n(\pm 1)=0.$$

Here, $\lambda_n(\beta,1)=\left(\left(r-\frac{1}{2}+\lceil\frac{n}{2}\rceil\right)^2-1\right)\pi^2$ and $r=\frac{1}{4}+\sqrt{\frac{9}{16}+\frac{\beta}{\pi^2}}$.

Moreover, $\phi_n\in H_0^1(-1,1)$ is real-valued and we normalize it such that $\|\phi_n\|_{L^2(-1,1)}=1$. Then for $m,n\geqq 1$,

$$\int_{-1}^1\phi_n\phi_m dy=\delta_{mn},\quad\int_{-1}^1\left(\phi_n'\phi_m'+\frac{u''-\beta}{u-1}\phi_n\phi_m\right)\mathrm{d}y=\lambda_n(1)\delta_{mn}.$$

If $n\geqq 1$ is odd, then ϕ_n has n zeros in $(-1,1)$, and $0\in Z_n=\{y\in(-1,1):\phi_n(y)=0\}=\{a_{n,1},\cdots,a_{n,n}\}$. For any $(n+1)$-dimensional subspace $V=\operatorname{span}\{\psi_1,\cdots,\psi_{n+1}\}$ in $H_0^1(-1,1)$, there exists $0\neq(\xi_1,\cdots,\xi_{n+1})\in\mathbf{R}^{n+1}$ such that $\xi_1\psi_1(a_{n,i})+\cdots+\xi_{n+1}\psi_{n+1}(a_{n,i})=0$, $i=1,\cdots,n$. Define $\tilde\psi=\xi_1\psi_1+\cdots+\xi_{n+1}\psi_{n+1}$. Then $\tilde\psi(a_{n,i})=0$, $i=1,\cdots,n$, i.e. $\tilde\psi|_{Z_n}=0$. We normalize $\tilde\psi$ such that $\|\tilde\psi\|_{L^2(-1,1)}=1$. Since $\tilde\psi\in H_0^1(-1,1)$, we have $\tilde\psi(\pm1)=0$. Integration by parts gives that

$$\left\|\tilde\psi'-\tilde\psi\frac{\phi_n'}{\phi_n}\right\|_{L^2(-1,1)}^2=\int_{-1}^1\left(|\tilde\psi'|^2+\frac{\phi_n''}{\phi_n}|\tilde\psi|^2\right)\mathrm{d}y$$
$$=\int_{-1}^1\left(|\tilde\psi'|^2-\frac{\beta-u''}{u-1}|\tilde\psi|^2-\lambda_n(1)|\tilde\psi|^2\right)\mathrm{d}y.$$

If $c>1$, then using $\beta-u''<0$ and $u-c<0$ for $y\in[-1,1]$, we have

$$\int_{-1}^1\left(|\tilde\psi'|^2-\frac{\beta-u''}{u-c}|\tilde\psi|^2\right)\mathrm{d}y\geqq\int_{-1}^1\left(|\tilde\psi'|^2-\frac{\beta-u''}{u-1}|\tilde\psi|^2\right)\mathrm{d}y$$
$$\geqq\int_{-1}^1\lambda_n(1)|\tilde\psi|^2\,\mathrm{d}y=\lambda_n(1).$$

This, along with (2.7), yields that $\inf_{c\in(1,+\infty)}\lambda_{n+1}(c)\geqq\lambda_n(1)$. If $n\geqq 1$ is even, then $\lambda_{n+1}(c)\geqq\lambda_n(c)\geqq\lambda_{n-1}(1)=\lambda_n(1)$. Thus, $\inf_{c\in(1,+\infty)}\lambda_{n+1}(c)\geqq\lambda_n(1)$ is always true. Now, we consider the upper bound. Let $V_{n+1} = \operatorname{span}\{\phi_0, \phi_1, \cdots, \phi_n\}$, here $\phi_0 = \eta|_{[-1,1]}$ is defined in (3.9), and $\phi_1, \cdots, \phi_n$ are L^2 normalized eigenfunctions. Then $V_{n+1} \subset H_0^1(-1,1)$. By (2.7), for $c > 1$, there exists $\varphi_c \in V_{n+1}$ with $\|\varphi_c\|_{L^2}^2 = 1$, and

$$\begin{aligned}\lambda_{n+1}(c) &\leqq \sup_{\|\phi\|_{L^2}=1,\phi\in V_{n+1}} \int_{-1}^{1} \left(|\phi'|^2 + \frac{u''-\beta}{u-c}|\phi|^2\right) \mathrm{d}y \\ &= \int_{-1}^{1} \left(|\varphi_c'|^2 + \frac{u''-\beta}{u-c}|\varphi_c|^2\right) \mathrm{d}y.\end{aligned}$$

Since $V_{n+1} \subset H_0^1(-1,1)$ is finite dimensional, there exist $\varphi_1 \in V_{n+1}$ and $c_m \to 1^+$ such that $\varphi_{c_m} \to \varphi_1$ in $H_0^1(-1,1)$. Then $\|\varphi_1\|_{L^2}^2 = 1$, $\int_{-1}^1 |\varphi_{c_m}'|^2 \,\mathrm{d}y \to \int_{-1}^1 |\varphi_1'|^2 \,\mathrm{d}y$ and

$$\lambda_{n+1}(c_m) \leqq \int_{-1}^{1} \left(|\varphi_{c_m}'|^2 + \frac{u''-\beta}{u-c_m}|\varphi_{c_m}|^2\right) \mathrm{d}y.$$

Since $u'' - \beta > 0$ and $u - c < 0$ for $y \in [-1,1]$ and $c > 1$, by Fatou's Lemma, we have

$$\begin{aligned}\limsup_{m\to\infty} \int_{-1}^{1} \frac{u''-\beta}{u-c_m}|\varphi_{c_m}|^2 \,\mathrm{d}y &\leqq \int_{-1}^{1} \limsup_{m\to\infty} \frac{u''-\beta}{u-c_m}|\varphi_{c_m}|^2 \,\mathrm{d}y \\ &= \int_{-1}^{1} \frac{u''-\beta}{u-1}|\varphi_1|^2 \,\mathrm{d}y.\end{aligned}$$

In particular, if $\varphi_1(0) \neq 0$, then

$$\begin{aligned}&\int_{-1}^{1} \frac{u''-\beta}{u-1}|\varphi_1|^2 \,\mathrm{d}y = -\infty, \quad \limsup_{m\to\infty} \int_{-1}^{1} \frac{u''-\beta}{u-c_m}|\varphi_{c_m}|^2 \,\mathrm{d}y \\ &= -\infty, \quad \limsup_{m\to\infty} \lambda_{n+1}(c_m) = -\infty.\end{aligned}$$

If $\varphi_1(0) = 0$, then $\varphi_1 \in \operatorname{span}\{\phi_1, \cdots, \phi_n\}$ and

$$\limsup_{m\to\infty} \lambda_{n+1}(c_m) \leqq \int_{-1}^{1} \left(|\varphi_1'|^2 + \frac{u''-\beta}{u-1}|\varphi_1|^2\right) \mathrm{d}y.$$

As $\|\varphi_1\|_{L^2}^2 = 1$, there exist $b_i \in \mathbf{R}$, $i = 1, \cdots, n$, with $\sum_{i=1}^n |b_i|^2 = 1$ such that $\varphi_1 = \sum_{i=1}^n b_i\phi_i \in V_{n+1}$ and

$$\begin{aligned}\int_{-1}^{1} \left(|\varphi_1'|^2 + \frac{u''-\beta}{u-1}|\varphi_1|^2\right) \mathrm{d}y &= \sum_{i=1}^{n} |b_i|^2 \int_{-1}^{1} \left(|\phi_i'|^2 + \frac{u''-\beta}{u-1}|\phi_i|^2\right) \mathrm{d}y \\ &= \sum_{i=1}^{n} |b_i|^2 \lambda_i(1) \\ &\leqq \max_{1\leqq i\leqq n} \lambda_i(1) = \lambda_n(1).\end{aligned}$$

Therefore, if $\varphi_1(0) = 0$, then $\limsup_{m\to\infty} \lambda_{n+1}(c_m) \leqq \lambda_n(1)$; if $\varphi_1(0) \neq 0$, this is clearly true since the limit is $-\infty$ in this case. By monotonicity of λ_n, we have $\lim_{c\to 1^+} \lambda_{n+1}(c) \leqq \lambda_n(1)$. Combining the upper and lower bounds, we have $\lim_{c\to 1^+} \lambda_{n+1}(c) = \lambda_n(1)$ for $\beta \in (-\frac{9}{16}\pi^2, -\frac{1}{2}\pi^2)$.

For $\beta = -\frac{9}{16}\pi^2$, the limits $\lim_{c\to 1^+} \lambda_{n+1}(c) = \lambda_n(1)$, $n \geqq 1$, can be proved similarly as in the case $\beta = \frac{9}{16}\pi^2$. Finally, the limit $\lim_{c\to 1^+} \lambda_1(c) = -\infty$ for $\beta \in [-\frac{9}{16}\pi^2, -\frac{1}{2}\pi^2)$ follows from Theorem 2.9 (2).

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