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Some integral geometry problems for wave equations

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Abstract

We consider the Cauchy problem and the source problem for normally hyperbolic operators on the Minkowski spacetime, and study the determination of solutions from their integrals along light-like geodesics. For the Cauchy problem, we give a new proof of the stable determination result obtained by Vasy and Wang (2021 *Commun. Math. Phys.* **384** 503–32). For the source problem, we obtain stable determination for sources with space-like singularities. Our proof is based on the microlocal analysis of the normal operator of the light ray transform composed with the parametrix for strictly hyperbolic operators.

Keywords: light ray transform, wave equation, microlocal analysis

1. Introduction

Consider the $n+1$-dimensional Minkowski space $(\mathbb{R}^{n+1}, g), n \geqslant 2$ where $g = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$. Hereafter, we use $z = (z_0, z_1, \ldots, z_n) = (t, x_1, \ldots, x_n)$ for the coordinates on $\mathbb{R}^{n+1}$. Let $\Box = -\partial_0^2 + \sum_{j=1}^n \partial_j^2$ be the d'Alembertian where $\partial_j = \frac{\partial}{\partial z^j}, j = 0, 1, 2, \ldots, n$. The normally hyperbolic operators on $(\mathbb{R}^{n+1}, g)$ are of the form

$$P(z, \partial) = \Box + \sum_{j=0}^{n} A_j(z)\partial_j + B(z) \tag{1}$$

where A_j, B are real or complex valued smooth functions in z, see e.g. [2]. In this paper, we study the determination of solutions of the Cauchy problem and the source problem of (1) from their integrals along light-like geodesics on $(\mathbb{R}^{n+1}, g)$, called the light ray transform. In addition to their own interest, these integral geometry problems arise from some inverse problems in cosmology which concern the determination of primordial gravitational perturbations from the cosmic microwave background, see [14] for further discussions.

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To describe the light ray transform, we parametrize the future pointing light-like geodesics as follows: for $(x,\theta)\in\mathcal{C}\doteq\mathbb{R}^n\times\mathbb{S}^{n-1}$, the light-like geodesics from $(0,x)$ in the direction $(1,\theta)$ is given by $\gamma_{y,\theta}(s)=(s,x+s\theta), s\in\mathbb{R}$. Then the light ray transform is

$$Lf(x,\theta)=\int_{\mathbb{R}} f(s,x+s\theta)\mathrm{d}s,\quad f\in C_0^\infty(\mathbb{R}^{n+1}). \tag{2}$$

It is worth mentioning that the light ray transform depends on the choice of the parametrization, see [9, corollary 6.2]. Although light-like geodesics are preserved under conformal transformations, the light ray transform is not. It is also known that L is injective on $C_0^\infty(\mathbb{R}^{n+1})$ see for example [10] which is a result of Fourier slice theorem and the analyticity of the Fourier transform of f.

For $t_0<t_1$, we denote $\mathcal{M}=(t_0,t_1)\times\mathbb{R}^n$ and $\mathcal{S}=\{t_0\}\times\mathbb{R}^n$. For simplicity, we assume $t_0=0$. Consider the Cauchy problem

$$\begin{aligned} &P(z,\partial)u(z)=0,\ \text{on}\ \mathcal{M}\\ &u=f_1,\quad \partial_t u=f_2\ \text{on}\ \mathcal{S}. \end{aligned} \tag{3}$$

Here, u is a function on $\mathcal{M}$. We also consider u as a function on $\mathbb{R}^{n+1}$ by defining $u=0$ outside of $\mathcal{M}$. Let $\chi_0\geqslant 0$ be a smooth cut-off function in $C_0^\infty((t_0,t_1])$ not identically vanishing. Our main result is the stable determination of u from $L\chi_0 u$.

Theorem 1.1. *Let u be the solution of (3) on $\mathcal{M}$ with Cauchy data $f_1\in H^{s+1}(\mathcal{S}), f_2\in H^s(\mathcal{S}), s\geqslant 0$ supported in a compact set $\mathcal{V}$ of $\mathcal{S}$. Suppose that the coefficients $A_j(z)$ in (1) are real valued smooth functions. Then f_1, f_2 are uniquely determined by $L\chi_0 u$. Furthermore, there exists $C>0$ such that*

$$\|u\|_{H^{s+1}(\mathcal{M})}\leqslant C\|(f_1,f_2)\|_{H^{s+1}(\mathcal{S})\times H^s(\mathcal{S})}\leqslant C\|L\chi_0 u\|_{H^{s+n/2+\delta}(\mathcal{C})} \tag{4}$$

where $\delta=0$ for $n\geqslant 3$ and $\delta=-1/4$ for $n=2$.

The theorem for $n=3$ was proved in [14] when χ_0 is the characteristic function $\chi_{[t_0,t_1]}$ for $[t_0,t_1]$ in $\mathbb{R}$. Here, the result is generalized to $n\geqslant 2$ and the Sobolev order in the stability estimate (4) is improved. We decided to replace $\chi_{[t_0,t_1]}$ by χ_0 to avoid some technicalities (see the proof of lemma 3.1). In fact, by the continuity of L (see proposition 4.2), the difference of $L\chi_{[t_0,t_1]}u$ and $L\chi_0 u$ can be made arbitrarily small in a proper sense. Despite these differences, the main point of this note is to give a new proof which explores the microlocal structure of the light ray transform. It is expected that the new approach would work in more general settings. Let us recall the approach in [14] and point out the differences. For the Cauchy problem (3), one can use the parametrix E constructed by Duistermaat and Hörmander to represent the solution up to a smooth term. Roughly, we write $f=(f_1,f_2)$ and solution of (3) as $u=Ef$ thus $Lu=LEf$. In [14], it is shown that LE can be modified to an elliptic pseudo-differential operator on $\mathcal{S}$ modulo some lower order Fourier integral operators by integrating in θ variable. Then a microlocal parametrix can be constructed from which the stability estimate follows.

In this paper, we will look at the normal operator E^*L^*LE which seems natural to examine for an integral geometry problem. It turns out that the composition is not good as it stands. We first explain the issue in section 3 by using a model problem. In fact, the issue is related to the microlocal structure of the normal operator $N=L^*L$. As shown in [16] and reviewed in section 4, the Schwartz kernel of N is a paired Lagrangian distribution. By judicious use of the kernel on one of the Lagrangians, we show that the composition E^*NE can be slightly modified to behave well within the clean FIO calculus of Duistermaat and Guillemin, yielding a pseudo-differential operator on $\mathcal{S}$. The rest of the proof goes as in [14].

We remark that in [16], it is shown that for globally hyperbolic Lorentzian manifolds without conjugate points, the normal operator of the light ray transform also has a paired Lagrangian structure. The parametrix construction for the Cauchy problem for strictly hyperbolic operators works in this generality as well. Thus we believe that the new method would give stability estimates as in theorem 1.1 and injectivity of the light ray transform for functions with sufficiently small support in such settings. These will be pursued somewhere else.

In section 7, we analyze the source problem from the same point of view

$$\begin{aligned} P(z,\partial)u &= f, \quad \text{on } \mathcal{M} \\ u &= 0 \quad \text{for } t < t_0, \end{aligned} \tag{5}$$

that is we determine u on $\mathcal{M}$ from $L\chi_0 u$. As pointed out however not addressed in [14], this problem arises from the inverse Sachs–Wolfe problem when the entropy perturbation cannot be ignored in Bardeen's equation. For the source problem, there is a parametrix E constructed by Melrose and Uhlmann [13] whose Schwartz kernel is a paired Lagrangian distribution. As for the composition LE, our idea is to consider $L^*LE = NE$ which turns out to be a paired Lagrangian distribution in view of a composition result of Antoniano and Uhlmann [1]. We will show that by considering the information on the other Lagrangian of the pair, one can stably determine f when the wave front set is space-like, see theorem 7.1. The case of light-like singularities is unclear. In view of the results in [16], one may not be able to determine light-like singularities of f in a stably way. For sources with special type of singularities such as conormal, it is possible to recover light-like singularities as in [16] but the result may depend on the coefficients A_j in $P(z,\partial)$.

As pointed out by one of the referees, it would be interesting to study the problem of simultaneously determining both the Cauchy data and the source term, and information of the coefficients in (1). Problems of similar nature have been explored in various settings, see for instance [11, 12]. We believe that the method developed here will be useful at least for identifying the singularities.

The note is organized as follows. We analyze a model problem in sections 2 and 3 where we can use oscillatory integral representations to explain the idea of the proof. Then we examine the argument from the Lagrangian distribution point of view and prove theorem 1.1 in sections 4–6. Finally, we study the source problem in section 7.

2. A model problem

We start with a model problem for which we can give an elementary proof using oscillatory integrals. Another motivation to consider a simpler example first is that through the explicit calculation, we can explain some subtlety of the problem which helps to explain the treatment for general cases.

Theorem 2.1. *Let $n \geqslant 3$ be an odd integer and $s \geqslant 0$. Let u be the solution of the Cauchy problem*

$$\begin{aligned} \Box u &= 0, \quad \text{on } \mathcal{M} \\ u &= f_1, \quad \partial_t u = f_2, \text{ on } \mathcal{S} \end{aligned} \tag{6}$$

where $(f_1, f_2) \in H^{s+1}(\mathcal{S}) \times H^s(\mathcal{S})$ are supported in a compact set $\mathcal{V} \subset \mathcal{S}$. Then $L\chi_0 u$ uniquely determines u on $\mathcal{M}$ and f_1, f_2 on $\mathcal{S}$. Moreover, we have the estimate

$$\|f_1\|_{H^{s+1}(\mathcal{S})} + \|f_2\|_{H^s(\mathcal{S})} \leqslant C\|L\chi_0 u\|_{H^{s+n/2}(\mathcal{C})}$$

for $C > 0$ depending on n and $\mathscr{V}$.

In this section, we collect the oscillatory integral representations of the solution to the Cauchy problem and the normal operator of the light ray transform.

First, consider the solution of the Cauchy problem (6). It will become convenient to consider the Cauchy problem on a larger set

$$\begin{aligned} &\Box u = 0, \quad \text{on } \mathcal{N} = (-T, T) \times \mathbb{R}^n \\ &\quad u = f_1, \quad \partial_t u = f_2, \quad \text{on} \, \mathcal{S} \end{aligned} \tag{7}$$

where $T > t_1$. Let $(\tau, \xi), \xi \in \mathbb{R}^n$ be the dual variables in $T^*\mathcal{N}$ to $(t, x), x \in \mathbb{R}^n$. Using Fourier transform in the x variable, we get

$$\begin{aligned} u(t,x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x\cdot\xi + t|\xi|)} \hat{h}_1(\xi) \mathrm{d}\xi + (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x\cdot\xi - t|\xi|)} \hat{h}_2(\xi) \mathrm{d}\xi \\ &= E_+ h_1 + E_- h_2, \end{aligned} \tag{8}$$

where

$$\hat{h}_1 = \frac{1}{2}(\hat{f}_1 + \frac{1}{\mathrm{i}|\xi|}\hat{f}_2), \qquad \hat{h}_2 = \frac{1}{2}(\hat{f}_1 - \frac{1}{\mathrm{i}|\xi|}\hat{f}_2).$$

Here, h_1, h_2 are the re-parametrized Cauchy data for the Cauchy problem. Thus, $E_\pm$ are represented by oscillatory integrals

$$E_\pm f(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}((x-y)\cdot\xi \pm t|\xi|)} f(y) \mathrm{d}y \, \mathrm{d}\xi. \tag{9}$$

The phase functions are $\phi_\pm(t, x, y, \xi) = (x - y) \cdot \xi \pm t|\xi|$ and amplitude function $a(t, x, \xi) = 1$. In particular, $\chi_0 E_\pm : \mathcal{E}'(\mathcal{S}) \to \mathcal{D}'(\mathcal{N})$ are Fourier integral operators with canonical relations

$$C^\pm_{wv} = \{(t, x, \zeta_0, \zeta'; y, \xi) \in T^*\mathcal{N}\backslash 0 \times T^*\mathcal{S}\backslash 0 : y = x \pm t\xi/|\xi|, \zeta' = \xi, \zeta_0 = \pm|\xi|\}. \tag{10}$$

Following the standard notation for Fourier integral operators see e.g. [4], we have $\chi_0 E_\pm \in I^{-\frac{1}{4}}(\mathcal{N}, \mathcal{S}; C^\pm_{wv})$. It suffices to determine h_1, h_2 because we can easily find f_1, f_2 from

$$f_1 = h_1 + h_2, \qquad f_2 = \mathrm{i}\Delta^{\frac{1}{2}}(h_1 - h_2). \tag{11}$$

Next, consider the light ray transform. On $\mathcal{C} = \mathbb{R}^n \times \mathbb{S}^{n-1}$, we use the standard product measure. Let L^* be the adjoint of L. Consider the normal operator $N = L^*L$. It is computed in [10, theorem 2.1] that

$$Nf(t,x) = \int_{\mathbb{R}^{n+1}} K_N(t, x, t', x') f(t', x') \mathrm{d}t' \, \mathrm{d}x'$$

where the Schwartz kernel

$$K_N(t, x, t', x') = \frac{\delta(t - t' - |x - x'|) + \delta(t - t' + |x - x'|)}{|x - x'|^{n-1}}. \tag{12}$$

In particular, K_N can be written as an oscillatory integral

$$K_N(t, x, t', x') = \int_{\mathbb{R}^{n+1}} \mathrm{e}^{\mathrm{i}(t-t')\tau + \mathrm{i}(x-x')\cdot\xi} k(\tau, \xi) \mathrm{d}\tau \, \mathrm{d}\xi \tag{13}$$

where

$$k(\tau,\xi)=C_n\frac{(|\xi|^2-\tau^2)_+^{\frac{n-3}{2}}}{|\xi|^{n-2}},\quad C_n=2\pi|\mathbb{S}^{n-2}|. \tag{14}$$

Here, for $s\in\mathbb{R}$, s_+^a, $Re\, a>-1$ denotes the distribution defined by $s_+^a=s^a$ if $s>0$ and $s_+^a=0$ if $s\leqslant 0$. Below, we denote by $\Psi^m(\mathcal{X})$ the set of pseudo-differential operators of order m on a smooth manifold $\mathcal{X}$.

3. The composition as oscillatory integrals

Consider the determination of h_1,h_2 from $L\chi_0 u=L\chi_0E_+h_1+L\chi_0E_-h_2$. We will analyze the normal operator

$$E^*L^*L\chi_0E,\quad \text{where } E=E_\pm. \tag{15}$$

There are issues about the composition as it is, and we will fine tune the operator as follows. We choose a smooth cut-off function $\chi\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi\subset(t_1,T)$, $\chi\geqslant 0$ and not vanishing identically. Note that $\chi_0\chi=0$. Then we consider the composition $E^*\chi L^*L\chi_0E$. We will show that the operator is a pseudo-differential operator on $\mathcal{S}$. The principal symbol is non-vanishing so the operator can be microlocally inverted. The necessity of the cut-off function χ is demonstrated in the next result.

Lemma 3.1. *For $n\geqslant 3$ odd, the composition $\chi L^*L\chi_0E_\pm\in I^{-n/2+1/4}(\mathcal{N},\mathcal{S};C_{wv}^\pm)$ are elliptic Fourier integral operators.*

Proof. We prove for E_+ below because the treatment for E_- is identical. The Schwartz kernel of $\chi N\chi_0E_+$ is

$$\begin{aligned}K(t,x,z)&=(2\pi)^{-2n}\int_{\mathbb{R}^n}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}(t-t')\tau+\mathrm{i}(x-x')\cdot\xi}\,\mathrm{e}^{\mathrm{i}((x'-z)\cdot\eta+t'|\eta|)}\\&\quad\times\chi(t)\chi_0(t')k(\tau,\xi)\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}t'\,\mathrm{d}x'\\&=(2\pi)^{-n}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}(t-t')\tau+\mathrm{i}(x-z)\cdot\xi+\mathrm{i}t'|\xi|}\chi(t)\chi_0(t')k(\tau,\xi)\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}t'\end{aligned}$$

where we integrated in x',η. We make a change of variable $s=\tau-|\xi|$ so

$$k(s,\xi)=C_n\frac{s_-^{\frac{n-3}{2}}(s+2|\xi|)_+^{\frac{n-3}{2}}}{|\xi|^{n-2}}.$$

Then

$$\begin{aligned}K(t,x,z)&=(2\pi)^{-n}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}(t-t')s+\mathrm{i}(x-z)\cdot\xi+\mathrm{i}t|\xi|}\chi(t)\chi_0(t')k(s,\xi)\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}t'\\&=(2\pi)^{-n}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-z)\cdot\xi+\mathrm{i}t|\xi|}\chi(t)\chi_0(t')A(t-t',\xi)\mathrm{d}\xi\,\mathrm{d}t'\end{aligned} \tag{16}$$

where A is defined by

$$A(\sigma,\xi)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\sigma s}k(s,\xi)\mathrm{d}s=\int_{-2|\xi|}^{0}\mathrm{e}^{\mathrm{i}\sigma s}C_n\frac{s^{\frac{n-3}{2}}(s+2|\xi|)^{\frac{n-3}{2}}}{|\xi|^{n-2}}\,\mathrm{d}s$$
$$=C_n2^{n-2}\int_{-1}^{0}\mathrm{e}^{2\mathrm{i}\sigma|\xi|s}s^{\frac{n-3}{2}}(s+1)^{\frac{n-3}{2}}\,\mathrm{d}s. \tag{17}$$

For $n=3$,

$$A(\sigma,\xi)=2C_3\int_{-1}^{0}\mathrm{e}^{2\mathrm{i}\sigma|\xi|s}\mathrm{d}s=C_3\frac{1}{\mathrm{i}\sigma}(1-\mathrm{e}^{-2\mathrm{i}\sigma|\xi|})|\xi|^{-1}.$$

Then

$$K(t,x,z)=C_3\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-z)\cdot\xi+\mathrm{i}t|\xi|}\chi(t)\chi_0(t')\frac{1}{\mathrm{i}(t-t')|\xi|}\mathrm{d}\xi\,\mathrm{d}t'$$
$$+C_3\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-z)\cdot\xi-\mathrm{i}t|\xi|+2\mathrm{i}t'|\xi|}\chi(t)\chi_0(t')\frac{-1}{\mathrm{i}(t-t')|\xi|}\mathrm{d}\xi\,\mathrm{d}t'$$
$$=K_1(t,x,z)+K_2(t,x,z) \tag{18}$$

where K_1,K_2 denotes the first and second integral above. For K_1, because $\chi(t)$ is supported away from $[t_0,t_1]$, we see that $(t-t')^{-1}$ is integrable in t'. This is where we need the cut-off function! So we get

$$K_1(t,x,z)=\widetilde{C}_3\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-z)\cdot\xi+\mathrm{i}t|\xi|}\widetilde{\chi}(t)|\xi|^{-1}\mathrm{d}\xi$$

with a non-vanishing constant $\widetilde{C}_3$ and $\widetilde{\chi}(t)\geqslant 0$ not vanishing identically. This implies that $K_1(t,x,z)$ is the Schwartz kernel of an Fourier integral operator, denoted by K_1, associated with canonical relation C_{wv}^{+} with a symbol of order -1. Note that the symbol $|\xi|^{-1}$ is singular at $\xi=0$ but this can be removed by introducing a smooth cut-off function supported near $\xi=0$, which amounts to changing K_1 by a smoothing operator.

For K_2, we have

$$K_2(t,x,z)=C_3\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-z)\cdot\xi-\mathrm{i}t|\xi|+2\mathrm{i}t'|\xi|}\chi(t)\chi_0(t')\frac{-1}{\mathrm{i}(t-t')|\xi|}\mathrm{d}\xi\,\mathrm{d}t'.$$

As $t-t'>0$, we can use integration by parts (note that χ_0 is compactly supported in (t_0,t_1)) to conclude that K_2 is a smoothing. We remark that if χ_0 were the characteristic function $\chi_{[t_0,t_1]}$, one would obtain some additional Fourier integral operators as a result of the boundary terms from integration by parts. The operators are more regular because the symbols are of order -2. With some adjustments, one can handle these terms in the rest of the argument for proving theorem 2.1. However, we chose to avoid the technicality and used χ_0.

We conclude that K in (18) is a Fourier integral operator associated with C_{wv}^{+} with a symbol $a(t,\xi)$ of order -1. Thus K in (18) can be written as

$$K(t,x,z)=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-z)\cdot\xi+\mathrm{i}t|\xi|}a(t,\xi)\mathrm{d}\xi$$

where $a(t,\xi)$ is a symbol of order -1. The leading order term is

$$a_0(t,\xi)=C_3\int_{t_0}^{t_1}\chi(t)\chi_0(t')\frac{1}{\mathrm{i}(t-t')|\xi|}\,\mathrm{d}t'. \tag{19}$$

For $n \geqslant 5$ odd, we use (17) and apply integration by parts to get

$$\begin{aligned} A(\sigma,\xi) &= C_n \frac{2^{n-3}}{\mathrm{i}\sigma|\xi|}\int_{-1}^{0} \mathrm{e}^{\mathrm{i}\sigma 2|\xi|s}(s^{\frac{n-3}{2}}(s+1)^{\frac{n-3}{2}})'\mathrm{d}s \\ &= (-1)C_n \frac{2^{n-3}}{\mathrm{i}\sigma|\xi|}\int_{-1}^{0} \mathrm{e}^{\mathrm{i}\sigma 2|\xi|s}\frac{n-3}{2}\left(s^{\frac{n-3}{2}-1}(s+1)^{\frac{n-3}{2}}\right. \\ &\quad \left. + s^{\frac{n-3}{2}}(s+1)^{\frac{n-3}{2}-1}\right)\mathrm{d}s. \end{aligned}$$

Repeating the integration by part $\frac{n-3}{2}$ times, we get

$$\begin{aligned} A(\sigma,\xi) &= (-1)^{\frac{n-3}{2}} C_n \frac{1}{(2\mathrm{i}\sigma|\xi|)^{\frac{n-3}{2}}}(\frac{n-3}{2})!\left(\int_{-1}^{0} \mathrm{e}^{\mathrm{i}\sigma 2|\xi|s}(s+1)^{(n-3)/2}\,\mathrm{d}s\right. \\ &\quad \left. + \int_{-1}^{0} \mathrm{e}^{\mathrm{i}\sigma 2|\xi|s}s^{(n-3)/2}\,\mathrm{d}s\right) + \sum_{k,j\geqslant 1, k+j=(n-3)/2} c_{k,j} \\ &\quad \times \int_{-1}^{0} \mathrm{e}^{\mathrm{i}\sigma 2|\xi|s}s^{(n-3)/2-k}(s+1)^{(n-3)/2-j}\,\mathrm{d}s \end{aligned}$$

where $c_{k,j}$ are constants. So far, the boundary terms from integration by parts vanish. We continue with integration by parts to get

$$\begin{aligned} A(\sigma,\xi) &= (-1)^{\frac{n-3}{2}} C_n \frac{1}{(2\mathrm{i}\sigma|\xi|)^{\frac{n-3}{2}}(2\mathrm{i}\sigma|\xi|)}\left(\frac{n-3}{2}\right)! + \sum_{k=1}^{M} a_k(\sigma)|\xi|^{-\frac{n-3}{2}-1-k} \\ &\quad + (-1)^{\frac{n-3}{2}} C_n \frac{1}{(2\mathrm{i}\sigma|\xi|)^{\frac{n-3}{2}}(2\mathrm{i}\sigma|\xi|)}\left(\frac{n-3}{2}\right)!\mathrm{e}^{-\mathrm{i}\sigma 2|\xi|} \\ &\quad + \sum_{k=1}^{M} b_k(\sigma)\mathrm{e}^{-\mathrm{i}\sigma 2|\xi|}|\xi|^{-\frac{n-3}{2}-1-k} \end{aligned}$$

where a_k, b_k are smooth in σ for $\sigma \neq 0$, and M is some integer depending on n. We remark that for n even, integration by parts will eventually lead to singular integrals. This is why we cannot deal with n even at this point. We let

$$a_0(\sigma) = b_0(\sigma) = (-1)^{\frac{n-3}{2}} C_n \frac{1}{(2\mathrm{i}\sigma)^{\frac{n-3}{2}+1}}(\frac{n-3}{2})!$$

We see that

$$A(\sigma,\xi) = \sum_{k=0}^{M} a_k(\sigma)|\xi|^{-\frac{n-3}{2}-1-k} + \sum_{k=0}^{M} b_k(\sigma)\mathrm{e}^{-\mathrm{i}\sigma 2|\xi|}|\xi|^{-\frac{n-3}{2}-1-k}.$$

Finally, using (16), we get

$$K(t,x,z)=\int_{t_0}^{t_1}\int_{\mathbb{R}^n} e^{it|\xi|+i(x-z)\cdot\xi}\chi(t)\chi_0(t')\sum_{k=0}^{M}a_k(t-t')|\xi|^{-\frac{n-3}{2}-1-k}d\xi\,dt'$$
$$+\int_{t_0}^{t_1}\int_{\mathbb{R}^n} e^{-it|\xi|+2it'|\xi|+i(x-z)\cdot\xi}\chi(t)\chi_0(t')$$
$$\times\sum_{k=0}^{M}b_k(t-t')|\xi|^{-\frac{n-3}{2}-1-k}d\xi\,dt'. \tag{20}$$

By the same arguments as for $n=3$, we see that K is an FIO associated with $C^{\pm}_{wv}$ with a symbol of order $-(n-3)/2-1$. The leading order term of the symbol is

$$a_0(t,\xi)=C_n\int_{t_0}^{t_1}\chi(t)\chi_0(t')\frac{1}{i(t-t')|\xi|^{\frac{n-3}{2}+1}}\,dt'. \tag{21}$$

To summarize, for $n\geqslant 3$ odd, $\chi N\chi_0 E_+$ is an FIO with canonical relation $C^{\pm}_{wv}$ of order

$$-(n-3)/2-1+n/2-(2n+1)/4=-n/2+1/4.$$

The principal symbol is clearly non-vanishing. This completes the proof. □

Next, we prove

Lemma 3.2. *For $n\geqslant 3$ odd,*

(a) $E_+^*\chi N\chi_0 E_+$ and $E_-^*\chi N\chi_0 E_-$ are elliptic pseudo-differential operators in $\Psi^{-n/2+1/2}(\mathcal{S})$.
(b) $E_-^*\chi N\chi_0 E_+$ and $E_+^*\chi N\chi_0 E_-$ are smoothing operators on $\mathcal{S}$.

Proof. For (a), we consider $E_+^*\chi N\chi_0 E_+$. We know that the Schwartz kernel for E_+^* is

$$K_{E_+^*}(w,t,x)=(2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i(x-w)\cdot\eta-it|\eta|}\,d\eta.$$

Using the notations in lemma 3.1, the kernel of $E_+^*\chi N\chi_0 E_+$ is

$$K(w,z)=(2\pi)^{-2n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(x-w)\cdot\eta-it|\eta|}e^{i(x-z)\cdot\xi+it|\xi|}a(t,\xi)$$
$$\times\chi(t)d\xi\,dt\,dx\,d\eta$$
$$=(2\pi)^{-n}\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{i(w-z)\cdot\eta}a(t,\eta)\chi(t)d\eta\,dt$$
$$=(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(w-z)\cdot\eta}c(\eta)d\eta$$

where $c(\eta)=\int_{\mathbb{R}}a(t,\eta)\chi(t)dt$ is a symbol of order $-\frac{n-3}{2}-1$. Thus, the composition $E_+^*\chi N\chi_0 E_+$ is a pseudo-differential operator of order $-n/2+1/2$ on $\mathcal{S}$.

Let us find the leading order term of $c(\eta)$, denoted by $c_0(\eta)$. For $n\geqslant 3$ odd, we use $a_0(t,\eta)$ in (19) and (21) to get

$$c_0(\eta)=\int_{\mathbb{R}}\int_{t_0}^{t_1}C_n\frac{1}{i(t-t')^{\frac{n-3}{2}+1}}|\eta|^{-\frac{n-3}{2}-1}\chi(t)\chi_0(t')dt'\,dt.$$

In the integral, $t > t'$ so the integrand it positive. Thus, $c_0(\eta)$ is non-zero. The proof for $E_-^* \chi N \chi_0 E_-$ is identical.

For (b), let us consider $E_-^* \chi N \chi_0 E_+$. Then the Schwartz kernel is

$$\begin{aligned} K(w,z) &= (2\pi)^{-2n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}(x-w)\cdot\eta + \mathrm{i}t|\eta|} \mathrm{e}^{\mathrm{i}(x-z)\cdot\xi + \mathrm{i}t|\xi|} a(t,\xi) \\ &\quad \times \chi(t) \mathrm{d}\xi \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}\eta \\ &= (2\pi)^{-n} \int_{\mathbb{R}}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(w-z)\cdot\eta + 2\mathrm{i}t|\eta|} a(t,\eta)\chi(t) \mathrm{d}\eta \, \mathrm{d}t \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(w-z)\cdot\eta} c(\eta) \mathrm{d}\eta. \end{aligned}$$

Because $\chi(t)$ is supported on $t > t_1 > 0$, the integration in t implies that the symbol $c(\eta)$ decays to infinite order as $|\eta| \to \infty$. So the operator is smoothing. □

Using the two lemmas, we can finish the proof of theorem 2.1.

Proof of theorem 2.1. Let u be the solution of (6). We start with

$$\chi L^* L \chi_0 u = \chi L^* L \chi_0 E_+ h_1 + \chi L^* L \chi_0 E_- h_2.$$

We apply $E_\pm^*$ to get

$$\begin{aligned} E_+^* \chi N \chi_0 u &= E_+^* \chi N \chi_0 E_+ h_1 + E_+^* \chi N \chi_0 E_- h_2 \\ E_-^* \chi N \chi_0 u &= E_-^* \chi N \chi_0 E_+ h_1 + E_-^* \chi N \chi_0 E_- h_2. \end{aligned} \tag{22}$$

From lemma 3.2, $E_+^* \chi N \chi_0 E_+, E_-^* \chi N \chi_0 E_- \in \Psi^{-n/2+1/2}(\mathcal{S})$ are elliptic pseudo-differential operators. There are parametrices $Q_\pm \in \Psi^{n/2-1/2}(\mathcal{S})$ such that

$$Q_\pm E_\pm^* \chi N \chi_0 E_\pm = \mathrm{Id} + R_\pm$$

with $R_\pm$ smoothing operators. We also know from lemma 3.2 that $E_+^* \chi N \chi_0 E_-$, $E_-^* \chi N \chi_0 E_+$ are smoothing operators. So we get from (22) that

$$\begin{aligned} Q_+ E_+^* \chi N \chi_0 u &= h_1 + R_1 h_1 + R_2 h_2 \\ Q_- E_-^* \chi N \chi_0 u &= h_2 + R_3 h_1 + R_4 h_2 \end{aligned}$$

where $R_i, i = 1,2,3,4$ are smoothing operators.

Finally, for any $\rho \in \mathbb{R}$, we get the estimate

$$\begin{aligned} \|h_1\|_{H^s(\mathcal{S})} &\leqslant C \|Q_+ E_+^* \chi N \chi_0 u\|_{H^s(\mathcal{S})} + C_\rho \|h_1\|_{H^{s-\rho}(\mathcal{S})} + C_\rho \|h_2\|_{H^{s-\rho}(\mathcal{S})} \\ \|h_2\|_{H^s(\mathcal{S})} &\leqslant C \|Q_- E_-^* \chi N \chi_0 u\|_{H^s(\mathcal{S})} + C_\rho \|h_1\|_{H^{s-\rho}(\mathcal{S})} + C_\rho \|h_2\|_{H^{s-\rho}(\mathcal{S})} \end{aligned} \tag{23}$$

for $C > 0, C_\rho > 0$. We know that

$$Q_\pm : H^s_{\mathrm{comp}}(\mathcal{S}) \to H^{s-n/2+1/2}_{\mathrm{loc}}(\mathcal{S})$$

is bounded. For $E_\pm^*$, one can show directly using the oscillatory integral representations or using the clean FIO calculus see lemma 6.3 later, that $E_\pm^* E_\pm \in \Psi^0(\mathcal{S})$. Then we derive that

$E_{\pm}^{*}: H^{s}_{\mathrm{comp}}(\mathcal{N}) \to H^{s}_{\mathrm{loc}}(\mathcal{S})$ is bounded. Finally, $L^{*}: H^{s}_{\mathrm{comp}}(\mathcal{C}) \to H^{s+1/2}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is bounded for $n \geqslant 3$, see proposition 4.2 in section 4. We thus conclude that

$$E_{\pm}^{*}\chi L^{*}: H^{s}_{\mathrm{comp}}(\mathcal{C}) \to H^{s+1/2}_{\mathrm{loc}}(\mathcal{N})$$

is bounded. Therefore, from (23), we get

$$\begin{aligned} \|h_1\|_{H^s(\mathcal{S})} &\leqslant C\|L\chi_0 u\|_{H^{s+n/2-1}(\mathcal{C})} + C_\rho\|h_1\|_{H^{s-\rho}(\mathcal{S})} + C_\rho\|h_2\|_{H^{s-\rho}(\mathcal{S})} \\ \|h_2\|_{H^s(\mathcal{S})} &\leqslant C\|L\chi_0 u\|_{H^{s+n/2-1}(\mathcal{C})} + C_\rho\|h_1\|_{H^{s-\rho}(\mathcal{S})} + C_\rho\|h_2\|_{H^{s-\rho}(\mathcal{S})}. \end{aligned} \tag{24}$$

Now, as shown in [14, theorem 8.1], L is injective on $L^1_{\mathrm{comp}}(\mathbb{R}^{n+1})$ hence on $L^2_{\mathrm{comp}}(\mathbb{R}^{n+1})$. For $s \geqslant 0$, we know that u in (6) belongs to $L^2_{\mathrm{comp}}(\mathbb{R}^{n+1})$ hence the injectivity result can be applied. One can drop the last two terms in each of the inequalities in (24) by using the same argument in [14, theorem 1.1]. We get

$$\|h_1\|_{H^{s+1}(\mathcal{S})} \leqslant C\|L\chi_0 u\|_{H^{s+n/2}(\mathcal{C})}, \qquad \|h_2\|_{H^{s+1}(\mathcal{S})} \leqslant C\|L\chi_0 u\|_{H^{s+n/2}(\mathcal{C})}.$$

In terms of f_1, f_2 see (11), we get

$$\|f_1\|_{H^{s+1}(\mathcal{S})} + \|f_2\|_{H^s(\mathcal{S})} \leqslant C\|L\chi_0 u\|_{H^{s+n/2}(\mathcal{C})}.$$

This completes the proof of theorem 2.1. □

4. Representation of operators

To understand the mechanism behind the composition in lemmas 3.1 and 3.2, we will examine the arguments from the Lagrangian distribution point of view. Note that the signature of g is $(-,+,\ldots,+)$. On the dual space $\mathbb{R}^{n+1}_{(\tau,\xi)}$, we let $\Gamma^{tm}_{\pm} = \{(\tau,\xi) \in \mathbb{R}^{n+1} : \tau^2 > |\xi|^2, \pm\tau > 0\}$ be the set of future/past pointing time-like vectors, and $\Gamma^{tm} = \Gamma^{tm}_{+} \cup \Gamma^{tm}_{-}$. Let $\Gamma^{sp} = \{(\tau,\xi) \in \mathbb{R}^{n+1} : \tau^2 < |\xi|^2\}$ be the set of space-like vectors. Finally, let $\Gamma^{lt}_{\pm} = \{(\tau,\xi) \in \mathbb{R}^{n+1} : \tau^2 = |\xi|^2, \pm\xi_0 > 0\}$ be the set of future/past pointing light-like vectors. We also let $\Gamma^{lt} = \Gamma^{lt}_{+} \cup \Gamma^{lt}_{-}$. We see that in (14), the symbol $k(\tau,\xi)$ is supported in Γ^{sp}, is homogeneous of degree -1 in (τ,ξ) and smooth away from Γ^{lt}. Moreover, $k(\tau,\xi) \sim \mathrm{dist}((\tau,\xi),\Gamma^{lt})^{(n-3)/2}$, for (τ,ξ) space-like near Γ^{lt}. Therefore, $k(\tau,\xi)$ looks like a symbol for a pseudo-differential operator of order -1 with a conormal singularity at Γ^{lt}. This is an example of paired Lagrangian distribution introduced in [6], as proved in [16] for general globally hyperbolic Lorentzian manifolds. In this section, we briefly recall the notion of paired Lagrangian distributions and the construction for the Minkowski spacetime.

To define paired Lagrangian distributions, we first consider the following model problem. Let $\widetilde{\mathcal{X}} = \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}, 1 \leqslant k \leqslant n-1$, and use coordinates $x = (x', x'') \in \mathbb{R}^k, x'' \in \mathbb{R}^{n-k}$. Let $\widetilde{\Lambda}_0 = \{(x,\xi,x,-\xi) \in T^*(\widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}})\backslash 0 : \xi \neq 0\}$ be the punctured conormal bundle of Diag in $T^*(\widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}})$, and

$$\widetilde{\Lambda}_1 = \{(x,\xi,y,\eta) \in T^*(\widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}})\backslash 0 : x'' = y'', \xi' = \eta' = 0, \xi'' = \eta'' \neq 0\}$$

which is the punctured conormal bundle to $\{(x,y) \in \widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}} : x'' = y''\}$. The two Lagrangians intersect cleanly at $\widetilde{\Sigma} = \{(x,\xi,y,\eta) \in T^*(\widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}})\backslash 0 : x'' = y'', \xi'' = \eta'', x' = y', \xi' = \eta' = 0\}$

which is of codimension k. For this model pair, the paired Lagrangian distribution $I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \widetilde{\Lambda}_0, \widetilde{\Lambda}_1)$ consists of oscillatory integrals (see [3, section 5])

$$u(x,y) = \int e^{i[(x'-y')\cdot\eta' + (x''-y'')\cdot\eta'']} b(x,y,\eta) d\eta \tag{25}$$

modulo $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, where b satisfies the following estimates. We remark that the order here is different from that in [3] because we work on the product space. First, in the region $|\eta'| \leqslant C|\eta''|, |\eta''| \geqslant 1$, b satisfies

$$|(Qb)(x,y,\eta)| \leqslant C\langle\eta''\rangle^{p+k/2}\langle\eta'\rangle^{l-k/2}$$

for all Q which is a finite product of differential operators of the form $D_{\eta'}$, $\eta'_j D_{\eta'_m}$, $\eta''_j D_{\eta''_m}$. Second, in the region $|\eta''| \leqslant C|\eta'|, |\eta'| \geqslant 1$, b satisfies the standard regularity estimate

$$|(Qb)(x,y,\eta)| \leqslant C\langle\eta'\rangle^{p+l}$$

for all Q which is a finite product of differential operators of the form $\eta'_j D_{\eta'_m}$, $\eta'_j D_{\eta''_m}$. We use the notation $I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \widetilde{\Lambda}_0, \widetilde{\Lambda}_1)$ to denote the space of operators $A : \mathcal{E}'(\mathbb{R}^n; \Omega^{\frac{1}{2}}_{\mathbb{R}^n}) \to \mathcal{D}'(\mathbb{R}^n; \Omega^{\frac{1}{2}}_{\mathbb{R}^n})$ where $\Omega^{\frac{1}{2}}_{\mathbb{R}^n}$ denotes the line bundle of half-densities on $\mathbb{R}^n$, whose Schwartz kernel K_A is a paired Lagrangian distribution with values in $\Omega^{\frac{1}{2}}_{\mathbb{R}^n\times\mathbb{R}^n}$.

Let $\mathcal{X}$ be a C^∞ manifold of dimension n. Let Λ_0, Λ_1 be two conic Lagrangian submanifold of $T^*(\mathcal{X}\times\mathcal{X})\backslash 0$ such that $\Lambda_0 \cap \Lambda_1$ cleanly at a codimension k, $1 \leqslant k \leqslant 2n-1$ submanifold Σ. From [6, proposition 2.1], we know that all such intersecting pairs (Λ_0, Λ_1) are locally symplectic diffeomorphic to each other. Let $\chi : T^*(\mathcal{X}\times\mathcal{X})\backslash 0 \to T^*(\widetilde{\mathcal{X}}\times\widetilde{\mathcal{X}})\backslash 0$ be a canonical transformation such that $\chi(\Lambda_0) \subseteq \widetilde{\Lambda}_0, \chi(\Lambda_1) \subseteq \widetilde{\Lambda}_1$. Then the set of paired Lagrangian distributions $I^{p,l}(\mathcal{X}\times\mathcal{X}; \Lambda_0, \Lambda_1)$ are defined invariantly by conjugating elements of $I^{p,l}(\widetilde{\mathbb{R}}^n \times \widetilde{\mathbb{R}}^n; \widetilde{\Lambda}_0, \widetilde{\Lambda}_1)$ by Fourier integral operators with canonical relation χ, see [6] for more details. For any $u \in I^{p,l}(\mathcal{X}\times\mathcal{X}; \Lambda_0, \Lambda_1)$, u is an Fourier integral operator of order $p+l$ on $\Lambda_0\backslash\Sigma$ and an Fourier integral operator of order p on $\Lambda_1\backslash\Sigma$. The principal symbols satisfy certain compatibility conditions at Σ. In particular, the principal symbol of u on $\Lambda_1\backslash\Sigma$ is singular at Σ.

To see that the kernel (12) is a paired Lagrangian distribution, we make a symplectic change of variables on $T^*\mathbb{R}^{n+1}\backslash 0$

$$\widetilde{x} = x + t\xi/|\xi|, \quad \widetilde{t} = t, \quad s = \tau - |\xi|, \quad \xi = \xi.$$

We can choose an Fourier integral operator with symbol of order 0 which quantizes the symplectic change of variable and transform K_N to

$$K_N(\widetilde{t},\widetilde{x},\widetilde{t}',\widetilde{x}') = \int_{\mathbb{R}^{n+1}} e^{i(\widetilde{t}-\widetilde{t}')s + i(\widetilde{x}-\widetilde{x}')\cdot\xi} k(s,\xi) ds\, d\xi \tag{26}$$

modulo a smooth term, where

$$\begin{aligned} k(s,\xi) &= C_n \frac{s_-^{\frac{n-3}{2}}(s+2|\xi|)_+^{\frac{n-3}{2}}}{|\xi|^{n-2}} + C_n \frac{s_+^{\frac{n-3}{2}}(s+2|\xi|)_-^{\frac{n-3}{2}}}{|\xi|^{n-2}} \\ &= C_n \frac{s_-^{\frac{n-3}{2}}(s+2|\xi|)_+^{\frac{n-3}{2}}}{|\xi|^{n-2}}. \end{aligned} \tag{27}$$

The symbol $k(s,\xi)$ satisfies the product type estimate with $p=-n/2, l=n/2-1$. In fact, for $|\xi|\leqslant C|s|, |s|\geqslant 1$, we have

$$|k(s,\xi)|\leqslant C|s|^{-1}.$$

One can verify the same estimate for Qk where Q is the finite product of differential operators of the form sD_s, sD_{ξ_m}. For $|s|\leqslant C|\xi|, |\xi|\geqslant 1$, we have

$$|k(s,\xi)|\leqslant C\frac{|s|^{\frac{n-3}{2}}|\xi|^{\frac{n-3}{2}}}{|\xi|^{n-2}}\leqslant C|\xi|^{-n/2+1/2}|s|^{n/2-3/2}$$

and one can verify the estimate for Qk where Q is the finite product of differential operators of the form $D_s, sD_s, \xi_j D_{\xi_m}$. So K_N is a paired Lagrangian distribution. The two associated Lagrangians are

$$\Lambda_0=\{(t,x,\tau,\xi;t',x',\tau',\xi')\in T^*\mathbb{R}^{n+1}\backslash 0\times T^*\mathbb{R}^{n+1}\backslash 0: t=t', x=x', \tau=-\tau', \xi=-\xi'\} \tag{28}$$

which is the punctured conormal bundle of the diagonal in $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ and

$$\begin{aligned}\Lambda_1=\big\{(t,x,\tau,\xi;t',x',\tau',\xi')\in T^*\mathbb{R}^{n+1}\backslash 0\times T^*\mathbb{R}^{n+1}\backslash 0: x=x'+(t-t')\xi/|\xi|\\ \tau=\pm|\xi|, \tau'=-\tau, \xi'=-\xi\big\}.\end{aligned} \tag{29}$$

The two Lagrangians intersect cleanly at

$$\begin{aligned}\Sigma=\big\{(t,x,\tau,\xi;t',x',\tau',\xi')\in T^*\mathbb{R}^{n+1}\backslash 0\times T^*\mathbb{R}^{n+1}\backslash 0: t=t', x=x',\\ \tau=-\tau', \xi=-\xi', \tau^2=|\xi|^2\big\}.\end{aligned} \tag{30}$$

In fact, Λ_1 is the flow out of Σ under the Hamilton vector field H_f of $f(\tau,\xi)=\frac{1}{2}(\tau^2-|\xi|^2)$.

Theorem 4.1 (Theorem 3.1 of [16]). *For the Minkowski light ray transform L defined in (2), the Schwartz kernel of the normal operator $N=L^*L$ belongs to $I^{-n/2,n/2-1}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1};\Lambda_0,\Lambda_1)$, in which Λ_0,Λ_1 are two cleanly intersection Lagrangians defined in (28) and (29). The principal symbols of N on $\Lambda_1\backslash\Sigma$ are real valued and non-vanishing.*

The principal symbols of K_N on $\Lambda_0\backslash\Lambda_1$ and $\Lambda_1\backslash\Lambda_0$ can be found explicitly, see [16] for details. We only need the symbol on $\Lambda_1\backslash\Lambda_0$ where the kernel $K_N\in I^{-n/2}(\mathcal{M}\times\mathcal{M};\Lambda_1)$. To find the symbol, we can use (12) and write K_N as

$$\begin{aligned}K_N(t,x,t',x')=&\int_{\mathbb{R}}e^{i(t-t'-|x-x'|)\tau}(t-t')_+^{-(n-1)}d\tau\\ &+\int_{\mathbb{R}}e^{i(t-t'+|x-x'|)\tau}(t-t')_-^{-(n-1)}d\tau\end{aligned} \tag{31}$$

for $t\neq t'$. This gives another oscillatory integral representation of K_N with a real phase function valid for $t\neq t'$. The principal symbol is non-vanishing and positive in this representation.

Using the estimates for paired Lagrangian distributions for the flow out model one can derive the Sobolev estimates for L and L^*.

Proposition 4.2 (Corollary 3.2 of [16]). *The Minkowski light ray transform $L: H^s_{\mathrm{comp}}(\mathbb{R}^{n+1})\to H^{s+s_0/2}_{\mathrm{loc}}(\mathbb{R}^n\times\mathbb{S}^{n-1})$ and its adjoint $L^*: H^s_{\mathrm{comp}}(\mathbb{R}^n\times\mathbb{S}^{n-1})\to H^{s+s_0/2}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ are continuous for $s_0=1/2$ when $n=2$ and for $s_0=1$ when $n\geqslant 3$.*

5. Parametrix for the Cauchy problem

A linear differential operator $P : C^\infty(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1})$ of second order is called *normally hyperbolic* if the principal symbol $\mathcal{P}(z,\zeta) \doteq \sigma(P)(z,\zeta) = g^*(\zeta,\zeta), (z,\zeta) \in T^*M$, see [2, page 33]. Note that $P(z,\partial)$ in (1) is exactly the normally hyperbolic operator on $(\mathbb{R}^{n+1}, g)$. The operator is strictly hyperbolic of multiplicity one with respect to the Cauchy hypersurfaces $\mathcal{S}_t = \{t\} \times \mathbb{R}^n, t \in \mathbb{R}$, see definition 5.1.1 of [4]. This means that all bicharacteristic curves of P are transversal to $\mathcal{S}_t$ and for $(\bar{z},\bar{\zeta}) \in T^*\mathcal{S}_t\backslash 0$

$$\mathcal{P}(\bar{z},\zeta) = 0, \quad \zeta|_{T_{\bar{z}}\mathcal{S}} = \bar{\zeta}$$

has exactly one solution. As before, we also use $\mathcal{S}_0 = \mathcal{S}$. It is convenient to use $D_j = -\iota\partial_j$, $j = 0, 1, 2, \ldots, n$ in which $\iota^2 = -1$. Consider the Cauchy problem

$$\begin{aligned} &P(z,D)u(z) = 0, \text{ on } \mathcal{M} \\ &u = f_1, D_t u = f_2 \text{ on } \mathcal{S}. \end{aligned} \tag{32}$$

We use Duistermaat–Hörmander's parametrix construction, see e.g. [4]. The restriction operator $\rho_0 : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{S})$ is an FIO in $I^{1/4}(\mathcal{N},\mathcal{S};C_0)$ with canonical relation

$$C_0 = \{(z,\zeta,\bar{z},\bar{\zeta}) \in T^*\mathcal{N}\backslash 0 \times T^*\mathcal{S}\backslash 0 : \bar{z} = z, \bar{\zeta} = \zeta|_{T_{\bar{z}}\mathcal{S}}\}. \tag{33}$$

We consider the canonical relation C_{wv} defined by

$$\begin{aligned} C_{wv} = \big\{&(w,\iota,\bar{z},\bar{\zeta}) \in T^*\mathcal{N}\backslash 0 \times T^*\mathcal{S}\backslash 0 : (w,\iota) \text{ is on the bicharacteristic} \\ &\text{strip through some } (\bar{z},\zeta) \text{ such that } \bar{\zeta} = \zeta|_{T_{\bar{z}}\mathcal{S}} \text{ and } \mathcal{P}(\bar{z},\zeta) = 0\big\}. \end{aligned} \tag{34}$$

The next result is straight forward from theorem 5.1.2 of [4].

Proposition 5.1. *There exists $E_1 \in I^{-1/4}(\mathcal{N},\mathcal{S};C_{wv}), E_2 \in I^{-5/4}(\mathcal{N},\mathcal{S};C_{wv})$ such that*

$$\begin{aligned} P(z,D)E_k &\in C^\infty(\mathcal{N}), \quad k = 1,2 \\ \rho_0 E_1 - \mathrm{Id} &\in C^\infty(\mathcal{S}), \quad \rho_0 E_2 \in C^\infty(\mathcal{S}) \\ \rho_0 D_t E_1 &\in C^\infty(\mathcal{S}), \quad \rho_0 D_t E_2 - \mathrm{Id} \in C^\infty(\mathcal{S}). \end{aligned} \tag{35}$$

Now we can represent the solution of (32) as $u = E_1 f_1 + E_2 f_2$ modulo a smooth term. Note that this is not the same representation used in section 2 because the Cauchy data are not re-parametrized to h_1, h_2. It is natural to decompose C_{wv} as the disjoint union of C^+_{wv} and C^-_{wv} which are

$$\begin{aligned} C^\pm_{wv} = \big\{&(w,\iota,\bar{z},\bar{\zeta}) \in T^*\mathcal{N}\backslash 0 \times T^*\mathcal{S}\backslash 0 : \iota \text{ is future/past} \\ &\text{pointing light} - \text{like and lies on the bicharacteristic strip through} \\ &\text{some } (\bar{z},\zeta) \text{ such that } \bar{\zeta} = \zeta|_{T_{\bar{z}}\mathcal{S}} \text{ and } \mathcal{P}(\bar{z},\zeta) = 0\big\}. \end{aligned} \tag{36}$$

These are (10) under the parametrization in section 2. We can decompose (for $k = 1, 2$)

$$E_k = E_k^+ + E_k^-, \qquad E_k^\pm \in I^{1-k-1/4}(\mathcal{N},\mathcal{S};C^\pm_{wv}).$$

We will find the relation of the principal symbols of $E_1^{\pm}, E_2^{\pm}$. We remark that the Maslov bundle and the half density bundle can be trivialized because the Lagrangians involved allow global parametrization. We will not show these factors in the notations below.

Lemma 5.2. *Let $e_k^{\pm}, k = 1, 2$ be the principal symbol of $E_k^{\pm}$ on $\Lambda^{\pm} = (C_{wv}^{\pm})'$ respectively. Suppose that the sub-principal symbol of $P(z, D)$ is purely imaginary, in which case $P(z, D)$ is of the form*

$$P(z, D) = \Box + \sum_{j=0}^{n} \imath A_j(z) D_j + B(z) \tag{37}$$

where $A_j(z)$ are real valued smooth functions. Then $e_k^{\pm}, k = 1, 2$ are real valued and

$$e_1^+ > 0, \quad e_2^+ > 0, \quad e_1^- > 0, \quad e_2^- < 0.$$

Proof. We can find the principal symbols following the argument in [4, page 117]. For $E_k \in I^{1-k-1/4}(\mathcal{N}, \mathcal{S}; C_{wv}), k = 1, 2$, if e_k is the principal symbol of E_k, then it satisfies

$$\frac{1}{\imath} \mathcal{L}_{H_p} e_k + p_{\text{sub}} e_k = 0 \tag{38}$$

where $\mathcal{L}_{H_p}$ denotes the Lie derivative and p_{sub} denotes the sub-principal symbol of $P(z, D)$. This is a transport equation along the bicharacteristics. The initial conditions are determined as follows. For $(\bar{z}, \bar{\zeta}) \in T^*\mathcal{S}$, we have two cotangent vectors $(\bar{z}, \zeta^{\pm})$ corresponding to it in $T^*\mathcal{N}$ where (regarding $\bar{\zeta}$ as a covector on $\mathcal{S}$)

$$\zeta^+ = (\tau, \bar{\zeta}), \qquad \zeta^- = (-\tau, \bar{\zeta})$$

where $\tau = |\bar{\zeta}|$. From the initial conditions in (35), we have

$$\begin{aligned} e_1(\bar{z}, \zeta^+; \bar{z}, \bar{\zeta}) &= e_1(\bar{z}, \zeta^-; \bar{z}, \bar{\zeta}) > 0 \\ e_2(\bar{z}, \zeta^+; \bar{z}, \bar{\zeta}) &= -e_2(\bar{z}, \zeta^-; \bar{z}, \bar{\zeta}) > 0 \end{aligned} \tag{39}$$

which are all real valued. Let $\gamma^{\pm}(s), s \in \mathbb{R}$ be bicharactersitics such that $\gamma^{\pm}(0) = (\bar{z}, \zeta^{\pm}) \in T^*\mathcal{N}$. Along $\gamma^{\pm}(s)$, the equation (38) can be written as

$$\partial_s e_k(s) + a(s) e_k(s) = 0, \quad e_k(0) = e_k^{\pm}(\gamma^{\pm}(0)) \tag{40}$$

where $a(s) = \imath p_{\text{sub}}(\gamma^{\pm}(s))$. Solving (40), we obtain that

$$e_k(s) = e_k(0) \mathrm{e}^{-\int_0^s a(\beta) \mathrm{d}\beta}. \tag{41}$$

Consider the operator (37). In local coordinates, let $p_2(z, \zeta)$ be the symbol modulo $S^0(T^*\mathcal{N})$, namely

$$p_2(z, \zeta) = g(\zeta, \zeta) + \sum_{j=0}^{n} \imath A_j(z) \zeta_j$$

where $g(\zeta, \zeta) = -\tau^2 + |\xi|^2, \zeta = (\tau, \xi)$. We have modulo symbols of order 0

$$p_{\text{sub}}(z,\zeta)=\sum_{j=0}^{n}\imath A_j(z)\zeta_j-\frac{1}{2\imath}\sum_{j=0}^{n}\frac{\partial^2 p_2(z,\zeta)}{\partial z_j\partial\zeta_j}=\sum_{j=0}^{n}\imath A_j(z)\zeta_j$$
$$-\frac{1}{2\imath}\sum_{i,j=1}^{n}\frac{\partial^2(h_{ij}\zeta^i\zeta^j)}{\partial z_j\partial\zeta_j}.$$

If the subprincipal symbol $p_{\text{sub}}(z,\xi)$ is pure imaginary, the coefficients of the transport equation (40) are real valued. We can tell from (41) that $e_k^{\pm}$ are real valued and the signs are determined by the initial conditions in (39). This completes the proof. □

6. The composition as Lagrangian distributions

In this section, we re-examine the composition in section 3 from the point of view of Lagrangian distributions and complete the proof of theorem 1.1. Let us outline the main ingredients. We look at $E^*\chi N\chi_0 E$, with $E=E_k^{\pm}, k=1,2$ in proposition 5.1.

(a) As $\chi\cdot\chi_0=0$, from section 4, we know that $\chi N\chi_0\in I^{-n/2}(\mathbb{R}^{n+1},\mathbb{R}^{n+1};\Lambda_1)$. Note that the role of χ is to keep the kernel of N away from the diagonal Λ_0 where the principal symbol is singular!

(b) We will show that Λ_1 intersects $\Lambda_{\pm}$ cleanly with excess one so the composition $\chi N\chi_0 E\in I^*(\mathcal{N},\mathcal{S};C_{wv})$ as a result of Duistermaat–Guillemin's clean FIO calculus with the order $*$ to be determined.

(c) We can compose the operator in (b) with E^* by using clean FIO calculus again to conclude that $E^*\chi N\chi_0 E\in\Psi^*(\mathcal{S})$.

This is what is behind the calculations in section 3. In the follows, we carry out the details of the above arguments. In this section, we assume that $n\geqslant 2$ is an integer.

Lemma 6.1. *Consider Λ_1 defined in (29). Then Λ_1 intersects $\Lambda^{\pm}=(C_{wv}^{\pm})'$ cleanly with excess one.*

Proof. We check by the definition of clean intersection. We use the following parametrization for Λ_1

$$\Lambda_1=\big\{(t,x,\tau,\xi;t',x',\tau',\xi')\in T^*\mathbb{R}^{n+1}\backslash 0\times T^*\mathbb{R}^{n+1}\backslash 0: x=x'+(t-t')\xi/|\xi|,$$
$$\tau=\pm|\xi|,\tau'=-\tau,\xi'=-\xi\big\}.$$

For $\Lambda^{\pm}$, we use (10)

$$\Lambda^{\pm}=\{(\widetilde{t},\widetilde{x},\widetilde{\tau},\widetilde{\xi};z,\eta)\in T^*\mathbb{R}^{n+1}\backslash 0\times T^*\mathbb{R}^n\backslash 0:\widetilde{x}=z\pm\widetilde{t}\eta/|\eta|,\widetilde{\xi}=-\eta,\widetilde{\tau}=\mp|\eta|\}.$$

We consider Λ^+ below. The case for Λ^- is similar. Let $\mathcal{X}=\Lambda_1\times\Lambda^+$ and $\mathcal{Y}=T^*\mathcal{M}\times\text{Diag}(T^*\mathcal{M})\times T^*\mathcal{S}$. These are submanifolds of $T^*\mathcal{M}\times T^*\mathcal{M}\times T^*\mathcal{M}\times T^*\mathcal{S}$. We show that for $p\in\mathcal{X}\cap\mathcal{Y}$, $T_p\mathcal{X}\cap T_p\mathcal{Y}=T_p(\mathcal{X}\cap\mathcal{Y})$.

First of all, Λ_1 is parametrized by $(t,t',x',\xi')\in\mathcal{A}\doteq\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. Also, Λ^+ is parametrized by $(\widetilde{t},z,\eta)\in\mathcal{B}\doteq\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ as

$$\widetilde{x}=z+\widetilde{t}\eta/|\eta|,\quad\widetilde{\tau}=-|\eta|,\quad\widetilde{\xi}=-\eta.$$

Consider $p \in \mathcal{X} \cap \mathcal{Y}$ and if we write $p = (t, t', x', \xi', \widetilde{t}, z, \eta) \in \mathcal{A} \times \mathcal{B}$, we must have

$$t' = \widetilde{t}, \quad x' = z + \widetilde{t}\eta/|\eta|, \quad |\eta| = |\xi'|, \quad \xi' = -\eta. \tag{42}$$

Thus, $\mathcal{X} \cap \mathcal{Y}$ is parametrized by $(t, \widetilde{t}, z, \eta) \in \mathcal{D} \doteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ as

$$t, \quad x = z + t\eta/|\eta|, \quad \tau = -|\eta|, \quad \xi = \eta, \quad t' = \widetilde{t}, \quad x' = z + \widetilde{t}\eta/|\eta|,$$
$$\tau' = |\eta|, \quad \xi' = -\eta, \quad \widetilde{t}, \quad z, \quad \eta.$$

We find that tangent vector $\delta p \in T_p(\mathcal{X} \cap \mathcal{Y})$ is given by

$$\begin{aligned} \delta p = \big(&\delta t, \delta z + \eta/|\eta|\delta t + t\alpha\, \mathrm{d}\eta, +\eta/|\eta|\delta\eta, \delta\eta, \\ &\delta\widetilde{t}, \delta z + \eta/|\eta|\delta\widetilde{t} + \widetilde{t}\alpha\, \mathrm{d}\eta, -\eta/|\eta|\delta\eta, -\delta\eta, \\ &\delta\widetilde{t}, \delta z + \eta/|\eta|\delta\widetilde{t} + \widetilde{t}\alpha\, \mathrm{d}\eta, -\eta/|\eta|\delta\eta, -\delta\eta, \delta z, \delta\eta\big) \end{aligned} \tag{43}$$

where $\alpha = \partial_\eta(\eta/|\eta|)$. Next, we compute $T_p\mathcal{X}$ and $T_p\mathcal{Y}$ and find their intersection. For $\delta p \in T_p\mathcal{X}$, we use variables in $\mathcal{A}$ and $\mathcal{B}$ to get

$$\begin{aligned} \delta p = \big(&\delta t, \delta x' - (\delta t - \delta t')\xi'/|\xi'| - (t - t')\beta\, \mathrm{d}\xi', \mp\xi'/|\xi'|\delta\xi', -\delta\xi', \delta t', \delta x', \\ &\pm\xi'/|\xi'|\delta\xi', \delta\xi', \delta\widetilde{t}, \delta z + \eta/|\eta|\delta\widetilde{t} + \widetilde{t}\alpha\, \mathrm{d}\eta, -\eta/|\eta|\delta\eta, -\delta\eta, \delta z, \delta\eta\big) \end{aligned}$$

where $\beta = \partial_\eta(\eta/|\eta|)|_{\eta=\xi'}$. For $\delta p \in T_p\mathcal{Y}$, we see that

$$\delta t = \delta t', \quad \delta x' = \delta z + \eta/|\eta|\delta\widetilde{t} + \widetilde{t}\alpha\, \mathrm{d}\eta, \quad \pm\xi'/|\xi'|\delta\xi' = -\eta/|\eta|\delta\eta, \quad \delta\eta = -\delta\xi'.$$

Also, at the intersection we use (42) and $(t, \widetilde{t}, z, \eta)$ as variables to get

$$\begin{aligned} \delta p = \big(&\delta t, \delta z + \eta/|\eta|\delta t + t\alpha\, \mathrm{d}\eta, -\eta/|\eta|\delta\eta, \delta\eta, \delta\widetilde{t}, \delta z + \eta/|\eta|\delta\widetilde{t} + \widetilde{t}\alpha\, \mathrm{d}\eta, \\ &-\eta/|\eta|\delta\eta, -\delta\eta, \delta\widetilde{t}, \delta z + \eta/|\eta|\delta\widetilde{t} + \widetilde{t}\alpha\, \mathrm{d}\eta, -\eta/|\eta|\delta\eta, -\delta\eta, \delta z, \delta\eta\big). \end{aligned} \tag{44}$$

Comparing (43) and (44), we proved $T_p\mathcal{X} \cap T_p\mathcal{Y} = T_p(\mathcal{X} \cap \mathcal{Y})$.

To find the excess, we see that $\operatorname{codim}(\mathcal{X}) = 8n + 6 - (4n + 3) = 4n + 3$, $\operatorname{codim}(\mathcal{Y}) = (8n + 6) - (6n + 4) = 2n + 2$. Also, $\dim(\mathcal{X} \cap \mathcal{Y}) = 2n + 2$. So the excess (see e.g. [7, appendix C.3])

$$e = \operatorname{codim}(\mathcal{X}) + \operatorname{codim}(\mathcal{Y}) - \operatorname{codim}(\mathcal{X} \cap \mathcal{Y}) = 4n + 3 + 2n + 2 - (6n + 4) = 1.$$

This completes the proof of the lemma. □

Lemma 6.2. *The composition $\chi N \chi_0 E_k^{\pm} \in I^{-n/2+1/4+1-k}(\mathcal{N}, \mathcal{S}; C_{wv}^{\pm})$ and the principal symbol is non-vanishing.*

Proof. Because $\chi(t)\chi_0(t) = 0$, we know from section 4 that $\chi N \chi_0 \in I^{-n/2}(\mathcal{N}, \mathcal{N}; \Lambda_1)$. One can apply the clean calculus directly to see that $\chi N \chi_0 E_k^{\pm} \in I^{-n/2+1/4+1-k}(\mathcal{N}, \mathcal{S}; C_{wv}^{\pm})$ using lemma 6.1. For $p = (t, x, \tau, \xi, y, \eta) \in \Lambda^{\pm}$, let C_p be the fiber over p in $T^*\mathcal{M} \times T^*\mathcal{M} \times T^*\mathcal{S}$ which is connected and compact. Then the principal symbol of the composition at p is given by

$$\int_{C_p} \sigma(\chi N \chi_0)(t, x, \tau, \xi, t', x', \tau', \xi')\sigma(E_k^{\pm})(t', x', \tau', \xi', y, \eta) \tag{45}$$

where $\sigma(\chi N\chi_0), \sigma(E_k^{\pm})$ denote the principal symbols of $\chi N\chi_0, E_k^{\pm}$ respectively and the integration is over the fiber C_p, see [8, theorem 25.2.3]. Note that both symbols are real valued and non-vanishing on the fiber thus they do not change signs. Also, the Maslov factors are constant. We see that the principal symbol of the composition is real valued and non-vanishing. □

Lemma 6.3. *For $j,k=1,2$, we have*

(a) $E_j^{\pm,*}\chi N\chi_0 E_k^{\pm} \in \Psi^{-n/2+1/2+2-j-k}(\mathcal{S})$ are elliptic.
(b) $E_j^{+,*}\chi N\chi_0 E_k^{-}, E_j^{-,*}\chi N\chi_0 E_k^{+}$ are smoothing operators on $\mathcal{S}$.

Proof. First of all, $E_j^{\pm,*} \in I^{-1/4+1-j}(\mathcal{S},\mathcal{M};C_{wv}^{\pm,-1})$ and $\chi N\chi_0 E_k^{\pm} \in I^{-n/2+1/4+1-k}(\mathcal{M},\mathcal{S};C_{wv}^{\pm})$. Let $\Lambda^{\pm}=(C_{wv}^{\pm})'$ and $\Lambda^{\pm,-1}=(C_{wv}^{\pm,-1})'$. We first prove that $\Lambda^{\pm,-1}$ intersect $\Lambda^{\pm}$ cleanly with excess one.

We consider the plus sign. Recall that

$$\Lambda^{+}=\{(t,x,\tau,\xi;z,\eta)\in T^*\mathbb{R}^{n+1}\backslash 0\times T^*\mathbb{R}^n\backslash 0 : x=z+t\eta/|\eta|,\xi=-\eta,\tau=|\eta|\} \tag{46}$$

and it can be parametrized by $(t,z,\eta)\in\mathcal{B}=\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. Let $\mathcal{X}=\Lambda^{+,-1}\times\Lambda^{+}$ and $\mathcal{Y}=T^*\mathcal{S}\times \mathrm{Diag}(T^*\mathcal{M})\times T^*\mathcal{S}$. We see that

$$\mathcal{X}\cap\mathcal{Y}=\Big\{(\widetilde{z},\widetilde{\eta},\widetilde{t},\widetilde{x},\widetilde{\tau},\widetilde{\xi};t,x,\tau,\xi,z,\eta) : x=z+t\eta/|\eta|,\xi=-\eta,\tau=|\eta| \\ \eta=\widetilde{\eta},\tau=\widetilde{\tau},\xi=\widetilde{\xi},t=\widetilde{t},\widetilde{x}=x,\widetilde{z}=z\Big\}.$$

So this set is parametrized by t,z,η. For $p\in\mathcal{X}\cap\mathcal{Y}$, the tangent vector $\delta p\in T_p(\mathcal{X}\cap\mathcal{Y})$ is

$$\delta p=\big(\delta z,\delta\eta,\delta t,\delta z+\eta/|\eta|\delta t+t\partial_\eta(\eta/|\eta|)\delta\eta,\eta/|\eta|\delta\eta,-\delta\eta\delta t,\delta z+\eta/|\eta|\delta t \\ +t\partial_\eta(\eta/|\eta|)\delta\eta,\eta/|\eta|\delta\eta,-\delta\eta,\delta z,\delta\eta\big). \tag{47}$$

Next, for $p\in\mathcal{X}$ which is parametrized by $(t,z,\eta,\widetilde{t},\widetilde{z},\widetilde{\eta})$, the tangent vector is given by

$$\delta p=\big(\delta\widetilde{z},\delta\widetilde{\eta},\delta\widetilde{t},\delta\widetilde{z}+\widetilde{\eta}/|\widetilde{\eta}|\delta\widetilde{t}+\widetilde{t}\partial_{\widetilde{\eta}}(\widetilde{\eta}/|\widetilde{\eta}|)\delta\widetilde{\eta},\widetilde{\eta}/|\widetilde{\eta}|\delta\widetilde{\eta},-\delta\widetilde{\eta}\delta t,\delta z+\eta/|\eta|\delta t \\ +t\partial_\eta(\eta/|\eta|)\delta\eta,\eta/|\eta|\delta\eta,-\delta\eta,\delta z,\delta\eta\big). \tag{48}$$

If δp also belongs to $T_p\mathcal{Y}$, we see that

$$\delta t=\delta\widetilde{t},\quad \delta x=\delta\widetilde{x},\quad \delta\tau=\delta\widetilde{\tau},\quad \delta\eta=\delta\widetilde{\eta}$$

which implies $\delta z=\delta\widetilde{z}$, so (48) agree with (47). This shows the intersection is clean. To find the excess, we see that $\mathrm{codim}(T_p\mathcal{X})=(8n+4)-(4n+2)=4n+2$ and $\mathrm{codim}(T_p\mathcal{Y})=(8n+4)-(6n+2)=2n+2$. Also $\mathrm{codim}(T_p(\mathcal{X}\cap\mathcal{Y}))=(8n+4)-(2n+1)=6n+3$. Thus the excess

$$e=4n+2+2n+2-6n+3=1.$$

Now we can use the clean FIO calculus [8, theorem 25.2.3] to conclude that $E_j^{\pm,*}\chi N\chi_0 E_k^{\pm}\in\Psi^{-n/2+1/2+2-j-k}(\mathcal{S})$. Note that both principal symbols of $E_j^{\pm,*}$ and $\chi N\chi_0 E_k^{\pm}$ are real and non-vanishing hence they do not change signs. Also, the Maslov factors are constants. The principal of the composition is the integration of the product of principal symbols so is also non-vanishing. This proves part (a).

Part (b) can be seen from a wave front set analysis using e.g. [4, theorem 1.3.7]. □

Proof of theorem 1.1. The idea is similar to that for theorem 2.1, despite that the parametrization of the Cauchy data is different. We write the solution of (32) as

$$\chi_0 u = \chi_0 E_1^+ f_1 + \chi_0 E_2^+ f_2 + \chi_0 E_1^- f_1 + \chi_0 E_2^- f_2.$$

Next, we apply χL^* to $L\chi_0 u$ to get

$$\chi N \chi_0 u = \chi N \chi_0 E_1^+ f_1 + \chi N \chi_0 E_2^+ f_2 + \chi N \chi_0 E_1^- f_1 + \chi N \chi_0 E_2^- f_2. \tag{49}$$

Now we apply $E_1^{+,*}$ and use part (b) of lemma 6.3 to get

$$E_1^{+,*}\chi N \chi_0 u = E_1^{+,*}\chi N \chi_0 E_1^+ f_1 + E_1^{+,*}\chi N \chi_0 E_2^+ f_2 + R_1 f_1 + R_2 f_2 \tag{50}$$

with R_1, R_2 smoothing operators. In the following, we use R_1, R_2 to denote generic smoothing operators which may change line by line. From lemma 6.3 part (a), we see that $E_1^{+,*}\chi N \chi_0 E_1^+ \in \Psi^{-n/2+1/2}(\mathcal{S})$ and $E_1^{+,*}\chi N \chi_0 E_2^+ \in \Psi^{-n/2-1/2}(\mathcal{S})$.

On the other hand, we apply $E_1^{-,*}$ to (49) to get

$$E_1^{-,*}\chi N \chi_0 u = E_1^{-,*}\chi N \chi_0 E_1^- f_1 + E_1^{-,*}\chi N \chi_0 E_2^- f_2 + R_1 f_1 + R_2 f_2. \tag{51}$$

From lemma 6.3 part (a), we see that $E_1^{-,*}\chi N \chi_0 E_1^- \in \Psi^{-n/2+1/2}(\mathcal{S})$ and $E_1^{-,*}\chi N \chi_0 E_2^- \in \Psi^{-n/2-1/2}(\mathcal{S})$. It follows from lemma 5.2 and the composition results lemmas 6.2 and 6.3 that

$$\sigma(E_1^{+,*}\chi N \chi_0 E_1^+) > 0, \quad \sigma(E_1^{-,*}\chi N \chi_0 E_1^-) > 0$$
$$\sigma(E_1^{+,*}\chi N \chi_0 E_2^+) > 0, \quad \sigma(E_1^{-,*}\chi N \chi_0 E_2^-) < 0.$$

Let $Q^+, Q^- \in \Psi^{n/2-1/2}(\mathcal{S})$ be parametrices for $E_1^{+,*}\chi N \chi_0 E_1^+, E_1^{-,*}\chi N \chi_0 E_1^-$ respectively. We know that the principal symbols of $Q^\pm$ are positive. Applying $Q^\pm$ to (50) and (51), we get

$$Q^+ E_1^{+,*}\chi N \chi_0 u = f_1 + B_+ f_2 + R_1 f_1 + R_2 f_2 \tag{52}$$

$$Q^- E_1^{-,*}\chi N \chi_0 u = f_1 + B_- f_2 + R_1 f_1 + R_2 f_2 \tag{53}$$

where

$$B_+ = Q_+ E_1^{+,*}\chi N \chi_0 E_2^+, \qquad B_- = Q_- E_1^{-,*}\chi N \chi_0 E_2^-.$$

From (52), we get

$$Q^+ E_1^{+,*}\chi N \chi_0 u - Q^- E_1^{-,*}\chi N \chi_0 u = (B_+ - B_-) f_2 + R_1 f_1 + R_2 f_2.$$

Note that $B_\pm \in \Psi^{-1}(\mathcal{S})$ are elliptic. Also, the principal symbol of B_+ is positive but the principal symbol of B_- is negative. Thus $B_+ - B_- \in \Psi^{-1}(\mathcal{S})$ is elliptic. Let $W \in \Psi^1(\mathcal{S})$ be a parametrix for $B_+ - B_-$. We get

$$WQ^+ E_1^{+,*}\chi N \chi_0 u - WQ^- E_1^{-,*}\chi N \chi_0 u = f_2 + R_1 f_1 + R_2 f_2. \tag{54}$$

So we solved f_2 up to smooth terms. We can use f_2 for example in (52) to get

$$Q^+ E_1^{+,*}\chi N \chi_0 u - B_+(WQ^+ E_1^{+,*}\chi N \chi_0 u - WQ^- E_1^{-,*}\chi N \chi_0 u) = f_1 + R_1 f_1 + R_2 f_2. \tag{55}$$

From this point, we can follow the proof of theorem 2.1 line by line. In fact, $WQ^{\pm} \in \Psi^{n/2+1/2}(\mathcal{S})$ and $Q^{+}, B_{+}WQ^{\pm} \in \Psi^{n/2-1/2}(\mathcal{S})$ so

$$WQ^{\pm} : H^{s}_{\mathrm{comp}}(\mathcal{S}) \to H^{s-n/2-1/2}_{\mathrm{loc}}(\mathcal{S})$$
$$Q^{+}, B_{+}WQ^{\pm} : H^{s}_{\mathrm{comp}}(\mathcal{S}) \to H^{s-n/2+1/2}_{\mathrm{loc}}(\mathcal{S})$$

are bounded. For $E^{\pm,*}_{k}, k = 1, 2$, we have $E^{\pm,*}_{k} : H^{s}_{\mathrm{comp}}(\mathcal{N}) \to H^{s+1-k}_{\mathrm{loc}}(\mathcal{S})$ is bounded. Finally, $L^{*} : H^{s}_{\mathrm{comp}}(\mathcal{C}) \to H^{s+1/2}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is bounded for $n \geqslant 3$, see proposition 4.2. We obtain that

$$E^{\pm,*}_{1}\chi L^{*} : H^{s}_{\mathrm{comp}}(\mathcal{C}) \to H^{s+1/2}_{\mathrm{loc}}(\mathcal{N})$$

is bounded. Thus using (54) and (55), we get

$$\|f_1\|_{H^{s+1}(\mathcal{S})} \leqslant C\|L\chi_0 u\|_{H^{s+n/2}(\mathcal{C})} + C_\rho\|f_1\|_{H^{s-\rho}(\mathcal{S})} + C_\rho\|f_2\|_{H^{s-\rho}(\mathcal{S})}$$
$$\|f_2\|_{H^{s}(\mathcal{S})} \leqslant C\|L\chi_0 u\|_{H^{s+n/2}(\mathcal{C})} + C_\rho\|f_1\|_{H^{s-\rho}(\mathcal{S})} + C_\rho\|f_2\|_{H^{s-\rho}(\mathcal{S})}.$$

The rest of the proof are as in theorem 2.1. For $n = 2$, one just need to use that $L^{*} : H^{s}_{\mathrm{comp}}(\mathbb{R}^2 \times \mathbb{S}^1) \to H^{s+1/4}_{\mathrm{loc}}(\mathbb{R}^3)$ is bounded from proposition 4.2. □

We conclude the proof of theorem 1.1 with two remarks.

Remark 6.4. In the proof of theorem 1.1, we actually constructed operators A_1, A_2 such that

$$A_1 L\chi_0 u = f_1 + R_1 f_1 + R_2 f_2, \qquad A_2 L\chi_0 u = f_2 + R_1' f_1 + R_2' f_2$$

where R_1, R_2, R_1', R_2' are smoothing operators. The operators A_1, A_2 can be used to determine wave front set of f_1, f_2 from $L\chi_0 u$.

Remark 6.5. There are other ways to fine tune the normal operator $E^{*}L^{*}LE$ in (15) to prove theorem 1.1. We outline one possible construction and leave the details to interested readers. Instead of using χ compactly supported in (t_1, T), we let $\rho_{\widetilde{T}}$ be the restriction operator to $\mathcal{S}_{\widetilde{T}} = \{\widetilde{T}\} \times \mathbb{R}^n$ for some $\widetilde{T} \in (t_1, T)$. In particular, $\rho_{\widetilde{T}} \in I^{1/4}(\mathcal{S}_T, \mathcal{M}; C_0)$ in which

$$C_0 = \{(y, \eta, t, x, \tau, \xi) \in T^{*}\mathcal{S}_{\widetilde{T}}\backslash 0 \times T^{*}\mathcal{M}^{\circ}\backslash 0 : y = x, \eta = \xi\}$$

see (5.1.2) of [4]. Recall from lemma 6.2 that $\chi N\chi_0 E_{\pm} \in I^{-n/2+1/4}(\mathcal{N}, \mathcal{S}; C^{\pm}_{wv})$. Note that the composition $C^{\pm}_{wv,0} = C_0 \circ C^{\pm}_{wv}$ is given by

$$C^{\pm}_{wv,0} = \{(y, \eta, x, \xi) \in T^{*}\mathcal{S}_{\widetilde{T}}\backslash 0 \times T^{*}\mathcal{S}\backslash 0 : y = x \pm \widetilde{T}\xi, \eta = \xi\}$$

which is a canonical graph. One can show that the composition is clean as in lemma 6.2 and obtain that

$$\rho_{\widetilde{T}} N\chi_0 E_{\pm} \in I^{-n/2+1}(\mathcal{S}_{\widetilde{T}}, \mathcal{S}; C^{\pm}_{wv,0}). \tag{56}$$

In particular, the principal symbol is non-vanishing. Now, (56) is an elliptic FIO of canonical graph type. We can find parametrix $Q_{\pm} \in I^{n/2-1}(\mathcal{S}, \mathcal{S}_{\widetilde{T}}; C^{\pm,-1}_{wv,0})$ such that

$$Q_{\pm}\rho_{\widetilde{T}} N\chi_0 E_{\pm} = \mathrm{Id} + R_{\pm}$$

where $R_{\pm}$ are smoothing operators. The rest of the argument goes as in the proof of theorem 1.1.

7. The source problem

In this section, we consider the source problem

$$\begin{aligned} &P(z,D)u = f, \text{on } \mathcal{M} \\ &u = 0 \quad \text{for } t < t_0 \end{aligned} \tag{57}$$

where f is compactly supported in $\mathcal{M}$. Consider the determination of f from $L\chi_0 u$ where we assume in addition that $\operatorname{supp} f \subset \operatorname{supp} \chi_0$. Before stating the main result, we explain the difference to the Cauchy problem.

According to [13], there exists a parametrix E for (57) such that $P(z,D)E = \mathrm{Id}$ modulo a smoothing operator. The Schwartz kernel of E belongs to $I^{-3/2,-1/2}(\mathcal{M}\times\mathcal{M};\Lambda_0,\Lambda_1)$. It suffices to look at $L\chi_0 Ef$. It is natural to apply L^* and study $L^*L\chi_0 Ef = N\chi_0 Ef$. The Schwartz kernel of N belongs to $I^{-n/2,n/2-1}(\mathcal{M}\times\mathcal{M};\Lambda_0,\Lambda_1)$ so both N and E are paired Lagrangian distributions of the flow-out type. One can apply the composition result in [1] to conclude that $N\chi_0 E \in I^{-n/2-1,n/2-2}(\mathcal{M}\times\mathcal{M};\Lambda_0,\Lambda_1)$. It is possible to find a parametrix for $N\chi_0 E$ within the class of paired Lagrangian distributions, however the remainder term belongs to $I^{\mu}(\mathcal{M}\times\mathcal{M};\Lambda_1)$ for some $\mu\in\mathbb{R}$ rather than smooth, see [5, 15]. Moreover, although the parametrix is good for reconstructing space-like singularities, time-like singularities are lost and it is not clear whether one can determine light-like singularities of f. Below, we will assume that $\mathrm{WF}(f)$ is contained in Γ^{sp} and use the kernel of $N\chi_0 E$ on $\Lambda_0\backslash\Lambda_1$ to stably determine f. We remark that in general relativity, space-like singularities correspond to particles moving slower than the speed of light.

For $\delta>0$, let $\Gamma^{sp}_{\delta} = \{(t,x,\tau,\xi)\in T^*\mathcal{M} : \tau^2 - |\xi|^2 > \delta\}$.

Theorem 7.1. *Suppose that $f\in\mathcal{E}'(\mathcal{M})$ is supported in a compact set $\mathcal{V}$ of $\mathcal{M}$ and that $WF(f)\subset\Gamma^{sp}_{\delta}$ for some $\delta>0$. Let u be the solution of (57). Then there exists $C>0$ (depending on δ) such that*

$$\|f\|_{H^s(\mathcal{M})} \leqslant C\|L\chi_0 u\|_{H^{s+3-\frac{s_0}{2}}(\mathcal{C})}$$

with s_0 in proposition 4.2 and $s\geqslant 0$.

Proof. Because $\mathrm{WF}(f)\subset\Gamma^{sp}_{\delta}$, there exists an elliptic pseudo-differential operator $\chi(D)\in\Psi^0(\mathcal{M})$ whose symbol $\chi(x,\xi)$ is supported in $\Gamma^{sp}_{\delta/2}$ such that $\chi(D)f = f$ modulo a smooth term. Thus, $N\chi_0 Ef = N\chi_0 E\chi(D)f$ modulo a smooth term. Because $E\in I^{-3/2,-1/2}(\mathcal{M}\times\mathcal{M};\Lambda_0,\Lambda_1)$, we claim that $E\chi(D)\in\Psi^{-2}(\mathcal{M})$ with principal symbol $\chi(\tau,\xi)\sigma_0(E)(\tau,\xi)$ supported in $\Gamma^{sp}_{\delta/2}$. Here, $\sigma_0(E)$ denotes the principal symbol of E on Λ_0. To see this, we can split $E = E_0 + E_1$ such that $E_0\in\Psi^{-2}(\mathcal{M})$ and $\mathrm{WF}(E_1)$ is sufficiently close to Λ_1. Then we know that $E_0\chi(D)\in\Psi^{-3}(\mathcal{M})$ and $E_1\chi(D)$ is a smoothing operator as the result of a wave front analysis using e.g. theorem 1.3.7 of [4] because the symbol of $\chi(D)$ is supported away from Λ_1.

It follows from the same argument that $N\chi_0 E\chi(D)\in\Psi^{-3}(\mathcal{M})$ with principal symbol $\chi(\tau,\xi)\sigma_0(E)(\tau,\xi)\sigma_0(N\chi_0)(\tau,\xi)$ which is non-vanishing. Thus, we can find a parametrix $Q\in\Psi^3(\mathcal{M})$ of $N\chi_0 E\chi(D)$ such that

$$QN\chi_0 E\chi(D) = \mathrm{Id} + R$$

where R is a smoothing operator. For $f\in\mathcal{E}'(\mathcal{M})$ with $\mathrm{WF}(f)\subset\Gamma^{sp}_{\delta}$, we actually have $QN\chi_0 E = \mathrm{Id} + R$ where we changed R to another smoothing operator. Finally, we get that

for any $\rho \in \mathbb{R}$,

$$\|f\|_{H^s(\mathcal{M})} \leqslant C\|N\chi_0 Ef\|_{H^{s+3}(\mathcal{M})} + C_\rho\|f\|_{H^{s-\rho}(\mathcal{M})}$$

for some $C, C_\rho > 0$. Using the estimate of L^*, we arrive at

$$\|f\|_{H^s(\mathcal{M})} \leqslant C\|L\chi_0 Ef\|_{H^{s+3-\frac{s_0}{2}}(\mathcal{C})} + C_\rho\|f\|_{H^{s-\rho}(\mathcal{M})}$$

with s_0 in proposition 4.2. Let u be the solution of (57) with source f. We get

$$\|f\|_{H^s(\mathcal{M})} \leqslant C\|L\chi_0 u\|_{H^{s+3-\frac{s_0}{2}}(\mathcal{C})} + C_\rho\|f\|_{H^{s-\rho}(\mathcal{M})}. \tag{58}$$

By using the injectivity of L as in the proof of theorem 2.1, we can get rid of the last term as in theorem 1.1 of [14]. We denote by $H^s_{\mathcal{V}}(\mathcal{M})$ the function space consisting of $f \in H^s(\mathcal{M})$ supported in $\mathcal{V}$. Then the inclusion of $H^s_{\mathcal{V}}(\mathcal{M})$ into $H^{s-\rho}_{\mathcal{V}}(\mathcal{M})$, $\rho > 0$ is compact. We claim that

$$\|f\|_{H^s(\mathcal{M})} \leqslant C\|L\chi_0 u\|_{H^{s+3-\frac{s_0}{2}}(\mathcal{C})} \tag{59}$$

for f with $\mathrm{WF}(f) \subset \Gamma^{sp}_\delta$. We argue by contradiction. Assume the above is not true. We can get a sequence $f^{(j)}$, $j = 1, 2, \ldots$ with unit norm in $H^s_{\mathcal{V}}(\mathcal{M})$ and $\mathrm{WF}(f^{(j)}) \subset \Gamma^{sp}_\delta$ such that $Lu^{(j)}$ goes to 0 in $H^{s+3-\frac{s_0}{2}}(\mathcal{C})$ where $u^{(j)}$ is the solution of (57) with source $f^{(j)}$. By (58), we conclude that $1 = \|f^{(j)}\|_{H^s(\mathcal{M})} \leqslant C'_\rho\|f^{(j)}\|_{H^{s-\rho}(\mathcal{M})}$ for some constant C'_ρ. This gives a weak limit f in $H^s(\mathcal{M})$ along a subsequence, which thus converges strongly in $H^{s-\rho}(\mathcal{M})$. Therefore, $\|f\|_{H^{s-\rho}(\mathcal{M})}$ is bounded below by $1/C'_\rho$, thus non-zero. Now we use the regularity estimate of the source problem $\|u\|_{H^{s+1}(\mathcal{M})} \leqslant C\|f\|_{H^s(\mathcal{M})}$ to conclude that $L\chi_0 u = 0$ with u the solution of (57) with source f. By the injectivity of L we get $\chi_0 u = 0$ which gives $f = 0$ from the equation (57). We reached a contradiction which means (59) holds. This finishes the proof. □

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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