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Carleson estimates for the Green function on domains with lower dimensional boundaries ☆

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A B S T R A C T

In the present paper we consider an elliptic divergence form operator in $\mathbb{R}^n \setminus \mathbb{R}^d$ with $d < n - 1$ and prove that its Green function is almost affine, in the sense that the normalized difference between the Green function with a sufficiently far away pole and a suitable affine function at every scale satisfies a Carleson measure estimate. The coefficients of the operator can be very oscillatory, and only need to satisfy some condition similar to the traditional quadratic Carleson condition.

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1. Introduction and main results

In a recent paper [7], we showed that for a slightly larger class of elliptic operators than the Dahlberg-Kenig-Pipher operators on the upper half-space $\mathbb{R}^{d+1}_+$, the Green function is well approximated by affine functions. The current paper extends this result to higher co-dimensions. That is, we consider the Green function on $\mathbb{R}^n \setminus \mathbb{R}^d$, with $d < n-1$, for operators satisfying a condition analogous to the Dahlberg-Kenig-Pipher condition on $\mathbb{R}^n \setminus \mathbb{R}^d$ and show that it is close, in a suitable sense, to affine functions. There are multiple challenges specific to the higher-codimensional setting, but before discussing those, let us provide some context for this work.

There has been a wide success in establishing connections between the geometry of the boundary of $\Omega \subset \mathbb{R}^n$ and properties of solutions of an elliptic PDE on Ω ([11], [13], [12], [1], etc). However, when the boundary of Ω has dimension lower than $n-1$, results are relatively rare. Essentially the only characterization of the uniform rectifiability of a lower-dimensional set by a PDE property is the recent work [8]. However, it pertains to weak rather than strong estimates on the solutions and, in particular, yields qualitative rather than quantitative results. This not merely a technical obstacle: the proofs in [8], relying on the blow-up techniques, are not amenable to a more quantitative analysis. On the other hand, the free boundary results obtained in [8] are even stronger than perhaps is natural to expect. Specifically, the authors show that even weak estimates on the Green function imply uniform rectifiability, and hence, if one can show that the Green function is close to the distance to the boundary in a strong, quantifiable sense, this would furnish the first quantifiable PDE characterization of the lower-dimensional uniform rectifiability. The present paper is the first step in this direction.

Aside from the aforementioned weak results, it has two important pre-runners. In [7], we managed to prove that the Green function is close to the distance function in a precise, quantitative way in the upper half-space (that is, in co-dimension 1). In [6], a different in form but similar in spirit, quantitative estimate for the Green function is obtained on domains with uniformly rectifiable sets of dimension strictly less than $n-1$ using a completely different method. The goal of this paper is to obtain a precise estimate for the Green function for more general operators than the ones considered in [8] and [6] on domains with lower dimensional boundary. Roughly speaking, the operators considered in [8] and [6] are close to the analogues of the Laplacian. In the present paper, we consider operators with much more oscillatory coefficients, albeit trading off by considering only flat boundary. Let us be more precise.

Consider $\Omega = \mathbb{R}^n \setminus \Gamma$, where $\Gamma \subset \mathbb{R}^n$ is Ahlfors-regular of dimension $d < n-1$. This means that there is a constant $C_0 \geq 1$ such that

$$C_0^{-1} r^d \leq \mathcal{H}^d(\Gamma \cap B_r(x)) \leq C_0 r^d, \tag{1.1}$$

for all balls $B_r(x)$ centered on $x \in \Gamma$, with radius $r > 0$. Classical elliptic operators are not appropriate for boundary value problems on Ω, as their solutions cannot "see" the lower dimensional set Γ. To overcome this obstacle, the first and third authors of the present paper, together with J. Feneuil, developed an elliptic theory on such domains with degenerate elliptic operators ([2]). It was shown that the general results, such as the maximum principle, trace and extension theorems, existence of the harmonic measure and Green function, all hold for the operators

$$\mathcal{L} = -\operatorname{div}(\mathcal{A} \operatorname{dist}(\cdot, \Gamma)^{d+1-n} \nabla),$$

where $\operatorname{dist}(\cdot, \Gamma)$ is the Euclidean distance to the boundary, and $\mathcal{A}$ is a matrix of real, bounded, measurable functions that satisfies the usual ellipticity conditions. That is, there is some $\mu_0 > 1$ such that

$$\begin{aligned} \langle \mathcal{A}(X)\xi, \zeta\rangle &\le \mu_0 \, |\xi| \, |\zeta| \quad \text{for } X \in \Omega \text{ and } \xi, \eta \in \mathbb{R}^n, \\ \langle \mathcal{A}(X)\xi, \xi\rangle &\ge \mu_0^{-1} \, |\xi|^2 \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n. \end{aligned} \tag{1.2}$$

Some of the results in this general setting are included in Section 2.

For the purpose of this paper, we focus only on $\Gamma = \{(x, t) \in \mathbb{R}^n : t = 0\} \cong \mathbb{R}^d$, and our domain is $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d = \big\{(x, t) \in \mathbb{R}^d \times \mathbb{R}^{n-d} : t \neq 0\big\}$. Notice that in this case, for a point $X = (x, t) \in \mathbb{R}^n$, $\operatorname{dist}(X, \Gamma) = |t|$.

Before introducing our conditions on the operator, let us define Carleson measures on the upper half-space $\mathbb{R}^{d+1}_+$. We shall systematically use lower case letters for points in $\mathbb{R}^d$ and capital letters for points in $\mathbb{R}^n$. It will be necessary to distinguish a ball in $\mathbb{R}^n$ from a ball in $\mathbb{R}^{d+1}$, so we use the cumbersome notation $B_r^{(d+1)}(x)$ for a ball in $\mathbb{R}^{d+1}$ with radius r centered at $(x, 0) \in \mathbb{R}^{d+1}$. The main purpose of defining balls in $\mathbb{R}^{d+1}$ is to define Carleson balls in $\mathbb{R}^{d+1}_+$, that is, we let $T(x, r) = B_r^{(d+1)}(x) \cap \mathbb{R}^{d+1}_+$. Although we do not emphasize it in notation, $T(x, r)$ is $(d+1)$-dimensional. For $x \in \mathbb{R}^d$ and $r > 0$, we denote by $\Delta(x, r)$ the surface ball $B_r(x) \cap \Gamma$. Thus $\Delta(x, r)$ is a ball in $\mathbb{R}^d$, and $T(x, r)$ is a half ball in $\mathbb{R}^{d+1}_+$ over $\Delta(x, r)$. We may simply write T_Δ for a half ball over $\Delta \subset \mathbb{R}^d$.

Definition 1.3 *(Carleson measures on $\mathbb{R}^{d+1}_+$).* We say that a nonnegative Borel measure μ is a Carleson measure on $\mathbb{R}^{d+1}_+$, if its Carleson norm

$$\|\mu\|_{\mathcal{C}} := \sup_{\Delta \subset \mathbb{R}^d} \frac{\mu(T_\Delta)}{|\Delta|}$$

is finite, where the supremum is over all the surface balls Δ and $|\Delta|$ is the Lebesgue measure of Δ in $\mathbb{R}^d$. We use $\mathcal{C}$ to denote the set of Carleson measures on $\mathbb{R}^{d+1}_+$.

For any surface ball $\Delta_0 \subset \mathbb{R}^d$, we use $\mathcal{C}(\Delta_0)$ to denote the set of Borel measures satisfying the Carleson condition restricted to Δ_0, i.e., such that

$$\|\mu\|_{\mathcal{C}(\Delta_0)} := \sup_{\Delta \subset \Delta_0} \frac{\mu(T_\Delta)}{|\Delta|} < +\infty.$$

Next we want to define our conditions that say that the matrix $\mathcal{A} = \mathcal{A}(X)$ is often close to a "constant" coefficient matrix $\mathcal{A}_0$. But since our operators have a singular weight $|t|^{d+1-n}$, we need to impose some structural assumptions on the matrix $\mathcal{A}_0$ so that the operator $\mathcal{L}_0 := -\operatorname{div}(\mathcal{A}_0 |t|^{d+1-n} \nabla)$ behaves like a constant coefficient operator in $\mathbb{R}^n \setminus \mathbb{R}^d$.

It was observed in [4] that given an elliptic operator $L = -\operatorname{div}(\widetilde{A}\nabla)$ defined on $\mathbb{R}^{d+1}_+$, one can construct a degenerate elliptic operator $\mathcal{L} = -\operatorname{div}(\mathcal{A}\nabla)$ so that if v is a solution to $Lv = 0$ in $\mathbb{R}^{d+1}_+$, then the function u defined on $\mathbb{R}^n \setminus \mathbb{R}^d$ by $u(x,t) = v(x,|t|)$ is a solution to $\mathcal{L}u = 0$ on $\mathbb{R}^n \setminus \mathbb{R}^d$. The precise construction is the following. Consider a $(d+1) \times (d+1)$ matrix $\widetilde{A}$ written in a block form as

$$\widetilde{A} = \left[\begin{array}{c:c} A & \mathbf{b} \\ \hdashline \mathbf{c} & d \end{array}\right],$$

where A is a $d \times d$ matrix, $\mathbf{b}$ is a $d \times 1$ vector, $\mathbf{c}$ is a $1 \times d$ vector, and d is a scalar function. Then for $n > d+1$, the $n \times n$ matrix $\mathcal{A}$ is constructed from $\widetilde{A}$ as

$$\mathcal{A} = \left[\begin{array}{c:c} A & \dfrac{\mathbf{b}\,t}{|t|} \\ \hdashline \dfrac{t^T\mathbf{c}}{|t|} & dI_{n-d} \end{array}\right], \tag{1.4}$$

where I_{n-d} is the identity matrix of size $n-d$, t is seen as a horizontal vector in $\mathbb{R}^{n-d}$, and thus $\mathbf{b}\,t$ is a $d \times (n-d)$ matrix and $t^T\mathbf{c}$ is a $(n-d) \times d$ matrix.

Inspired by this observation, we fix the aforementioned class of matrices constructed from constant matrices in $\mathbb{R}^{d+1}$.

Definition 1.5 *(The class $\mathfrak{A}_0(\mu_0)$).* We define $\mathfrak{A}_0(\mu_0)$ to be the class of $n \times n$ matrices satisfying the ellipticity conditions (1.2) with constant μ_0 that can be written as the following block matrix

$$\mathcal{A}_0 = \mathcal{A}_0(x,t) = \left[\begin{array}{c:c} A_0 & \dfrac{\mathbf{b_0} t}{|t|} \\ \hdashline \dfrac{t^T\mathbf{c_0}}{|t|} & d_0 I_{n-d} \end{array}\right]. \tag{1.6}$$

Here, A_0 is a $d \times d$ constant matrix, $\mathbf{b_0}$ is a $d \times 1$ constant vector, $\mathbf{c_0}$ is a $1 \times d$ constant vector, d_0 is a real number.

The reason that this class of matrices plays the role of constant matrices for our purpose is actually different from the above observation made in [4]. We want them to relate back to constant-coefficient operators in $\mathbb{R}^{d+1}$, not the other way around. In fact, it is shown in Section 3 that for any $\mathcal{A}_0 \in \mathfrak{A}_0(\mu_0)$, any solution of $-\operatorname{div}(\mathcal{A}_0\nabla u) = 0$ can be transformed into a solution of an elliptic equation in $\mathbb{R}^{d+1}$. Notice that a solution $u(x,t)$ of $-\operatorname{div}(\mathcal{A}_0\nabla u) = 0$ is not necessarily radial in t, while a solution constructed from a solution of an elliptic equation in $\mathbb{R}^{d+1}_+$ as above is radial in t.

Now let us return to conditions on $\mathcal{A}$. Since we shall compare $\mathcal{A}$ and $\mathcal{A}_0 \in \mathfrak{A}_0(\mu_0)$ at every scale, we introduce Whitney regions in $\mathbb{R}^n$: for any $(x,r) \in \mathbb{R}^{d+1}_+$, define

$$W(x,r) := \left\{(y,t) \in \mathbb{R}^n : y \in \Delta(x,r),\ \frac{r}{2} \le |t| \le r\right\}. \tag{1.7}$$

Notice that $W(x,r)$ is an annular region in $\mathbb{R}^n$ whose distance to Γ is $r/2$.

The difference between $\mathcal{A}$ and some matrix $\mathcal{A}_0 \in \mathfrak{A}_0(\mu_0)$ at a given scale is measured by the following quantity. For $x \in \mathbb{R}^d$ and $r > 0$, define

$$\alpha(x,r) := \inf_{\mathcal{A}_0 \in \mathfrak{A}_0(\mu_0)} \left\{ \frac{1}{m(W(x,r))} \int_{(y,t)\in W(x,r)} |\mathcal{A}(y,t) - \mathcal{A}_0|^2 \frac{dydt}{|t|^{n-d-1}} \right\}^{1/2} \tag{1.8}$$

Here, $m(W(x,r))$ is the measure of $W(x,r)$ with weight $|t|^{-n+d+1}$.

Definition 1.9 *(Weak DKP condition).* We say that the coefficient matrix $\mathcal{A}$ satisfies the weak DKP condition with constant $M > 0$, if $\alpha(x,r)^2 \frac{dxdr}{r}$ is a Carleson measure on $\mathbb{R}^{d+1}_+$, with norm

$$\mathfrak{N}(\mathcal{A}) := \left\| \alpha(x,r)^2 \frac{dxdr}{r} \right\|_{\mathcal{C}} \le M. \tag{1.10}$$

The name comes from Dalhberg, Kenig and Pipher. In 1984, Dahlberg first conjectured that a Carleson condition on the coefficients, which is roughly that $|\nabla A|^2\, dxdr/r$ be a Carleson measure on $\mathbb{R}^{d+1}_+$, guarantees the absolute continuity of the elliptic measure with respect to the Lebesgue measure. In 2001, Kenig and Pipher [14] proved Dahlberg's conjecture.

The condition we consider here is weaker than the classical DKP condition in the following sense. Consider a matrix $\mathcal{A}$ of bounded, measurable functions defined on $\mathbb{R}^n$ that can be written as (1.4), but with the coefficients depending on x, t. Assume that A, $\mathbf{b}$, $\mathbf{c}$ and d all satisfy the usual DKP condition with Carleson norm M. That is,

$$\left\| \sup_{(y,t)\in W(x,r)} |\nabla A(y,t)|^2\, r dxdr \right\|_{\mathcal{C}} \le M,$$

and similarly for $\mathbf{b}$, $\mathbf{c}$ and d. One can verify that under this assumption, the matrix $\mathcal{A}$ satisfies the weak DKP condition with constant M. We point out that from our definition, a matrix $\mathcal{A}$ that satisfies the weak DKP condition does not have to be of the form (1.4). Moreover, we can always add to $\mathcal{A}$ a matrix $\mathcal{D}$ that satisfies

$$d\mu(x,r) = \sup_{(y,t)\in W(x,r)} \mathcal{D}(y,t)^2 \frac{dxdr}{r} \in \mathcal{C},$$

and the new matrix still satisfies the weak DKP condition if $\mathcal{A}$ does. We remark that our Definition 1.9 is the higher co-dimensional analogue of what we defined in [7], where we say that a $(d+1)\times(d+1)$ matrix satisfies the weak DKP condition with constant M, if (1.10) holds with $\mathcal{A}_0$ replaced by some constant $(d+1)\times(d+1)$ matrix in the definition (1.8) of $\alpha(x,r)$.

Let us now turn to the approximation of the Green function by affine functions in higher co-dimension. In [7], we showed that any solution in $T(x_0,R)$ that vanishes on $\Delta(x_0,R)$ is locally well approximated by affine functions in $T(x_0,R/2)$, with essentially uniform Carleson bounds. More precisely, we proved the following result.

Theorem 1.11 *([7] Theorem 1.13). Let $\widetilde{A}$ be a $(d+1)\times(d+1)$ matrix of real-valued functions on $\mathbb{R}^{d+1}_+$ satisfying the ellipticity conditions with constant μ_0. If $\widetilde{A}$ satisfies the weak DKP condition with some constant $M\in(0,\infty)$, and if we are given $x_0\in\mathbb{R}^d$, $R>0$, and a positive solution u of $Lu = -\operatorname{div}\left(\widetilde{A}\nabla u\right) = 0$ in $T(x_0,R)$, with $u=0$ on $\Delta(x_0,R)$, then for some C depending only on d and μ_0, there holds*

$$\left\|\beta_u(x,r)\frac{dxdr}{r}\right\|_{\mathcal{C}(\Delta(x_0,R/2))} \le C + CM,$$

where

$$\beta_u(x,r) = \frac{\fint_{T(x,r)} \left|\nabla\left(u(y,t) - \lambda_{x,r}(u)\,t\right)\right|^2 dydt}{\fint_{T(x,r)} |\nabla u(y,t)|^2 dydt},$$

and $\lambda_{x,r}(u) = \fint_{T(x,r)} \partial_t u(z,t)dzdt$.

In higher co-dimension, we want to measure in a similar way the closeness between a solution and an affine function in $\mathbb{R}^n\setminus\mathbb{R}^d$. Given a positive solution u of $\mathcal{L}u=0$ in a ball $B_r(x)$ centered on Γ, the best affine function that approximates u in $B_r(x)$ should be $\lambda_{x,r}(u)\,|t|$, where

$$\lambda_{x,r}(u) = \frac{1}{m(B_r(x))}\int_{B_r(x)} \frac{\nabla_t u(z,t)\cdot t}{|t|}\frac{dzdt}{|t|^{n-d-1}}. \tag{1.12}$$

In Section 3, we will see that this $\lambda_{x,r}(u)$ is indeed the best coefficient of $|t|$ to approximate u in $B_r(x)$, and that it is closely related to the best coefficient in the co-dimension one setting.

As in the co-dimension one case, the proximity of the two functions is measured by the weighted L^2 average of the difference of the gradients divided by the weighted local energy of u. That is, we set

$$J_u(x,r) := \frac{1}{m(B_r(x))} \int\limits_{B_r(x)} \left|\nabla_{y,t}\left(u(y,t) - \lambda_{x,r}(u)\,|t|\right)\right|^2 \frac{dydt}{|t|^{n-d-1}}, \tag{1.13}$$

and then divide by

$$E_u(x,r) := \frac{1}{m(B_r(x))} \int\limits_{B_r(x)} |\nabla u(y,t)|^2 \frac{dydt}{|t|^{n-d-1}}, \tag{1.14}$$

to get the number

$$\beta_u(x,r) := \frac{J_u(x,r)}{E_u(x,r)}. \tag{1.15}$$

The solutions considered here are all weak solutions in a weighted Sobolev space. Their values on the boundary $\Gamma = \mathbb{R}^d$ are considered in the trace sense. All this is made precise in Section 2, and also in Section 4.1. Our main result is the following.

Theorem 1.16. *Let $\mathcal{A}$ be an $n\times n$ matrix of bounded, real-valued functions on $\mathbb{R}^n$ satisfying the ellipticity conditions (1.2). If $\mathcal{A}$ satisfies the weak DKP condition with some constant $M \in (0,\infty)$, and if we are given $x_0 \in \mathbb{R}^d$, $R > 0$, and a nonnegative solution $u \in W_r(B_R(x_0))$ of $\mathcal{L}u = -\operatorname{div}_{x,t}\left(\mathcal{A}(x,t)\,|t|^{d+1-n}\,\nabla_{x,t}u\right) = 0$ in $B_R(x_0)\setminus\Gamma$, with $Tu = 0$ on $\Gamma \cap B_R(x_0)$, then the function β_u defined by (1.15) satisfies a Carleson condition in $T(x_0, R/2)$, and more precisely*

$$\left\|\beta_u(x,r)\frac{dxdr}{r}\right\|_{\mathcal{C}(\Delta(x_0,R/2))} \leq C + CM \tag{1.17}$$

where C depend only on d, n and μ_0.

The next theorem is an improvement of Theorem 1.16, which says that we can have Carleson norms for β_u that are as small as we want, provided that we take a small DKP constant and a suitably large ball where u is a positive solution that vanishes on the boundary.

Theorem 1.18. *Let $x_0 \in \mathbb{R}^d$, $R > 0$, $\mu_0 > 0$ be given, let u satisfy the assumptions of Theorem 1.16, and let $\mathcal{A}$ satisfy the weak DKP condition in $\Delta(x_0,R)$. Then for $\tau \leq 1/2$*

$$\left\|\beta_u(x,r)\frac{dxdr}{r}\right\|_{\mathcal{C}(\Delta(x_0,\tau R))} \leq C\tau^a + C\left\|\alpha_2(x,r)^2\frac{dxdr}{r}\right\|_{\mathcal{C}(\Delta(x_0,R))}, \tag{1.19}$$

where C and $a > 0$ depends only on d, n and μ_0.

Finally, let us comment that our results are essentially optimal. In [7], we constructed an example that shows that $\beta_{G_L^\infty}(x,r)\frac{dxdr}{r}$ may not be a Carleson measure if an operator

$L = -\operatorname{div}(\widetilde{A}\nabla)$ does not satisfy the DKP condition. Here, G_L^∞ is the Green function with pole at infinity for L on the upper half-plane $\mathbb{R}^2_+$. Construct an operator $\mathcal{L} = -\operatorname{div}(\mathcal{A}\nabla)$ from the 2- dimensional operator L as in (1.4). One can show that this operator does not satisfy the DKP condition either. Moreover, the corresponding Green function is $G_{\mathcal{L}}^\infty(x,t) = G_L^\infty(x,|t|)$, and a similar computation as in the co-dimensional one setting shows that $\beta_{G_{\mathcal{L}}^\infty}(x,r)\frac{dxdr}{r}$ cannot be a Carleson measure on $\mathbb{R}^2_+$.

The main differences in the proof, compared to the setting of co-dimension 1, lie in the decay estimates for the non-affine part of solutions to equations with a coefficient matrix in the class $\mathfrak{A}_0(\mu_0)$. In the co-dimension one case, we have good estimates for the second derivatives of solutions to equations with constant coefficients. This enables us to control the oscillations of the gradient of solutions. However, in the higher co-dimensional setting, the coefficients have a singular weight $|t|^{-n+d+1}$, which prevents us from getting an estimate for the second derivatives of solutions. To overcome this difficulty, we split the solution into one part which is radial in t, and the other part which is purely rotational in t. The radial part can be treated similarly to the co-dimension one case, while the rotational part requires a compactness argument and other properties of solutions. The entire Section 4 is devoted to implementing this idea. The decay estimate is proved in the key lemma (Lemma 4.13).

The rest of the paper is organized as follows. In Section 2, we collect some results that will be used frequently in the rest of the paper; most of them are proved in [2]. In Section 3, we relate the n-dimensional operator $\mathcal{L}_0$ back to a $d+1$- dimensional operator L, and transform solutions of $\mathcal{L}_0 u = 0$ into solution of $Lv = 0$. Also, we study the properties of $\lambda_{x,r}$ in that section. In Section 5, we show how to generalize the decay estimates from operators with a coefficient matrix in $\mathfrak{A}_0(\mu_0)$ to weak DKP operators. The ideas in that section are similar to those in the co-dimensional one case, and we mainly illustrate the modifications needed in the higher co-dimension. We give a proof of the reverse Hölder inequalities for the gradient of solutions, where we have to address the issue of mixed-dimensional boundaries.

2. Preliminaries

In this section we recall, mostly from [2], how to extend standard results for elliptic PDE's in the upper half space (or NTA domains) to the setting of co-dimension > 1. The familiar reader can probably jump to Section 3 and return to this section when needed.

Consider $\Omega = \mathbb{R}^n \setminus \Gamma$, where $\Gamma \subset \mathbb{R}^n$ is Ahlfors-regular of dimension $d < n-1$. In all the other sections, Γ will be simply $\mathbb{R}^d$. For $X \in \Omega$, write $\delta(X) := \operatorname{dist}(X,\Gamma)$. Define the weight function $w(X) := \delta(X)^{-n+d+1}$, and a measure $dm(X) = w(X)dX$. Denote by $B_r(X)$ the open ball in $\mathbb{R}^n$ centered at X with radius r. One can show that

$$m(B_r(X)) \approx r^n w(X) \qquad \text{if } \delta(X) \geq 2r, \tag{2.1}$$

$$m(B_r(X)) \approx r^{d+1} \qquad \text{if } \delta(X) \leq 2r. \tag{2.2}$$

In particular, this implies that m is a doubling measure. See [2], Chapter 2 for details.

Denote by $W = \dot{W}^{1,2}_w(\Omega)$ the weighted Sobolev space of functions $f \in L^1_{loc}(\Omega)$ whose distributional gradient in Ω lies in $L^2(\Omega, w)$:

$$W := \left\{ f \in L^1_{loc}(\Omega) : \nabla f \in L^2(\Omega, w) \right\} = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \nabla f \in L^2(\mathbb{R}^n, w) \right\}, \tag{2.3}$$

and set $\|f\|_W = \left(\int_\Omega |\nabla f(X)|^2 \, w(X) dX \right)^{1/2}$ for $f \in W$. Here, the identity (i.e., the fact that the distribution derivative of f on Ω can also be used as a derivative on $\mathbb{R}^n$) is proved in [2], Lemma 3.2. We shall also use the following local version of the space W. Let $O \subset \mathbb{R}^n$ be an open set, then

$$W_r(O) := \left\{ f \in L^1_{loc}(O) : \varphi f \in W \text{ for any } \varphi \in C_0^\infty(O) \right\}. \tag{2.4}$$

Note that $W_r(O) = \left\{ f \in L^1_{loc}(O) : \nabla f \in L^2_{loc}(O, w) \right\}$; see [2] Chapter 8 for details.

For functions in W or $W_r(O)$, it is shown in [2] that there exists a well-defined trace on Γ, or $\Gamma \cap O$, respectively. The trace of $u \in W$ is such that for $\mathcal{H}^d$-almost every $x \in \Gamma$,

$$Tu(x) = \lim_{r \to 0} \fint_{B(x,r)} u(Y) dY := \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(Y) dY. \tag{2.5}$$

For $u \in W_r(O)$, the trace is defined in the same way for $\mathcal{H}^d$-almost every $x \in \Gamma \cap O$.

Consider the divergence-form operator $\mathcal{L} = -\operatorname{div}_X(\mathcal{A}(X) w(X) \nabla_X)$, where $\mathcal{A}$ is an $n \times n$ matrix of real, bounded, measurable functions defined in Ω, that satisfies the ellipticity conditions (1.2).

Definition 2.6. We say that $u \in W$ is a (weak) solution of $\mathcal{L}u = 0$ in Ω if for any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \mathcal{A} \nabla u \cdot \nabla \varphi \, dm = 0.$$

Let $O \subset \mathbb{R}^n$ be an open set. We say that $u \in W_r(O)$ is a (weak) solution of $\mathcal{L}u = 0$ in O if for any $\varphi \in C_0^\infty(O)$, $\int_O \mathcal{A} \nabla u \cdot \nabla \varphi dm = 0$. We say that $u \in W_r(O)$ is a subsolution (respectively, supersolution) in O if for any $\varphi \in C_0^\infty(O)$ such that $\varphi \geq 0$, $\int_O \mathcal{A} \nabla u \cdot \nabla \varphi dm \leq 0$ (respectively, ≥ 0).

We collect some basic properties for functions in W and solutions of $\mathcal{L}u = 0$ in this section. The constant C below might be different from line to line, but depends only on d, n, the Ahlfors constant C_0, and the ellipticity constant μ_0 unless otherwise stated.

Lemma 2.7 *(Poincaré inequality ([2], Lemma 4.2)). Let $p \in [1, \frac{2n}{n-2}]$ (or $p \in [1, +\infty)$ if $n = 2$). Then for any $u \in W$, any ball $B \subset \mathbb{R}^n$ with radius $r > 0$, there is some constant C depending only on n, d and C_0, such that*

$$\left(\frac{1}{m(B)}\int_B |u-u_B|^p\,dm\right)^{1/p} \le Cr\left(\frac{1}{m(B)}\int_B |\nabla u|^2\,dm\right)^{1/2}$$

where u_B denotes either $\fint_B u$ or $m(B)^{-1}\int_B u dm$. If B is centered on Γ and if, in addition, $Tu = 0$ on $\Gamma \cap B$, then

$$\left(\frac{1}{m(B)}\int_B |u|^p\,dm\right)^{1/p} \le Cr\left(\frac{1}{m(B)}|\nabla u|^2\,dm\right)^{1/2}.$$

Remark 2.8. One also has (see the proof of Lemma 4.2 in [2])

$$\left(\frac{1}{m(B)}\int_B |u-u_B|^2\,dm\right)^{1/2} \le Cr\left(\frac{1}{m(B)}\int_B |\nabla u|^{\frac{2n}{n+2}}\,dm\right)^{\frac{n+2}{2n}}. \tag{2.9}$$

Moreover, if B is centered on Γ and if, in addition, $Tu = 0$ on $\Gamma \cap B$, then

$$\left(\frac{1}{m(B)}\int_B |u|^2\,dm\right)^{1/2} \le Cr\left(\frac{1}{m(B)}\int_B |\nabla u|^{\frac{2n}{n+2}}\,dm\right)^{\frac{n+2}{2n}}. \tag{2.10}$$

To see (2.10), write

$$\left(\frac{1}{m(B)}\int_B |u|^2\,dm\right)^{1/2} \le C\left(\frac{1}{m(B)}\int_B \left|u-\fint_B u\right|^2 dm\right)^{1/2} + C\fint_B |u(X)|\,dX.$$

By Lemma 4.1 of [2] and Hölder's inequality,

$$\fint_B |u(X)|\,dX \le Cr\left(\frac{1}{r^{d+1}}\int_B |\nabla u|^{\frac{2n}{n+2}}\,dm\right)^{\frac{n+2}{2n}}.$$

Note that since B is centered on Γ, $m(B) \approx r^{d+1}$. Thus, (2.10) follows from the above observation and (2.9).

Lemma 2.11 *(Interior Caccioppoli inequality ([2], Lemma 8.6)). Let B be a ball of radius r such that $2B \subset \Omega$ and $u \in W_r(2B)$ is a nonnegative subsolution of $\mathcal{L}$ in $2B$. Then there exists a constant $C > 0$ depending only on d, n, C_0 and μ_0, such that for any constant $c \in \mathbb{R}$,*

$$\int_B |\nabla u|^2\,dm \le Cr^{-2}\int_{\frac{3}{2}B} |u-c|^2\,dm.$$

Lemma 2.12 *(Caccioppoli inequality on the boundary ([2] Lemma 8.11)). Let $B \subset \mathbb{R}^n$ be a ball of radius r centered on Γ, and let $u \in W_r(2B)$ be a nonnegative subsolution in $2B \setminus \Gamma$ such that $Tu = 0$ on $2B \cap \Gamma$. Then*

$$\int_B |\nabla u|^2 \, dm \leq Cr^{-2} \int_{\frac{3}{2}B} u^2 dm.$$

Lemma 2.13 *(Moser estimates on the boundary ([2] Lemma 8.12)). Let B and u be as in Lemma 2.12. Then*

$$\sup_B u \leq C \left(m(B)^{-1} \int_{\frac{3}{2}B} u^2 dm \right)^{1/2}.$$

Here, the constant C depends only on d, n and μ_0 as usual.

Let B be a ball centered on Γ with radius r. We say that a point X_B is a *corkscrew point* for B if $X_B \in B$ and $\delta(X_B) \geq \varepsilon r$ for some ε depending only on d, n and the Ahlfors constant C_0 of Γ.

Lemma 2.14 *(Boundary Harnack's inequality ([2], Lemma 11.8)). Let $x_0 \in \Gamma$ and $r > 0$ be given, and let X_r be a corkscrew point for $B_r(x_0)$. Let $u \in W_r(B_{2r}(x_0))$ be a nonnegative solution of $\mathcal{L}u = 0$ in $B_{2r}(x_0) \setminus \Gamma$, such that $Tu = 0$ on $B_{2r}(x_0) \cap \Gamma$. Then*

$$u(X) \leq Cu(X_r) \qquad \text{for } X \in B_r(x_0).$$

Lemma 2.15. *Let $x_0 \in \Gamma$ and $R > 0$ be given. Suppose $u \in W_r(B_R(x_0))$ is a nonnegative solution of $\mathcal{L}u = 0$ in $B_R(x_0) \setminus \Gamma$ with $Tu = 0$ on $B_R(x_0) \cap \Gamma$. Then for all $0 < r < R/2$,*

$$\frac{u^2(X_r)}{r^2} \approx \frac{1}{m(B_r(x_0))} \int_{B_r(x_0)} |\nabla u|^2 \, dm,$$

where X_r is a corkscrew point of $B_r(x_0)$.

Proof. By translation invariance, we may assume that the origin is on Γ and that x_0 is the origin. To see the less than or equal to direction, we say that $Tu = 0$ on the boundary and use Lemma 2.13 followed by Sobolev's inequality to get

$$\frac{u^2(X_r)}{r^2} \leq \frac{Cr^{-2}}{m(B_r)} \int_{B_r} u^2 dm \leq \frac{C}{m(B_r)} \int_{B_r} |\nabla u|^2 \, dm.$$

To see the other direction, we use the boundary Caccioppoli and boundary Harnack inequalities to get

$$\frac{1}{m(B_{2r})}\int_{B_r}|\nabla u|^2\,dm \le \frac{C}{r^2}\frac{1}{m(B_{2r})}\int_{B_{2r}}|u|^2\,dm \le \frac{C}{r^2}u^2(X_r). \quad \square$$

Lemma 2.16 *(Comparison principle ([2], Theorem 11.17)). Let $x_0 \in \Gamma$ and $r > 0$, and let X_r be a corkscrew point. Let $u, v \in W_r(B_{2r}(x_0))$ be two nonnegative, not identically zero, solutions of $\mathcal{L}u = \mathcal{L}v = 0$ in $B_{2r}(x_0) \setminus \Gamma$, such that $Tu = Tv = 0$ on $\Gamma \cap B_{2r}(x_0)$. Then*

$$C^{-1}\frac{u(X_r)}{v(X_r)} \le \frac{u(X)}{v(X)} \le C\frac{u(X_r)}{v(X_r)} \qquad \text{for all } X \in B_r(x_0) \setminus \Gamma,$$

where $C > 1$ depends only on n, d, C_0 and μ_0.

Corollary 2.17 *([3], Corollary 6.4). Let u, v, r, x_0 be as in Lemma 2.16. There exists $C > 0$ and $\gamma \in (0,1)$ depending only on n, d, C_0 and μ_0, such that*

$$\left|\frac{u(X)v(Y)}{u(Y)v(X)} - 1\right| \le C\left(\frac{\rho}{r}\right)^{\gamma}$$

for all X, $Y \in B_\rho(x_0) \setminus \Gamma$, as long as $\rho < r/4$.

We have the following reverse Hölder inequality for the gradient of solutions.

Lemma 2.18. *Let $B \subset \mathbb{R}^n$ be a ball centered on Γ. Let $u \in W_r(3B)$ be a solution of $\mathcal{L}u = 0$ in $3B \setminus \Gamma$ with $Tu = 0$ on $3B \cap \Gamma$. Then there exist $p > 2$ depending only on d, n, C_0 and μ_0, and $C > 0$ depending on d, n, C_0, μ_0 and p, such that*

$$\left(\frac{1}{m(B)}\int_B |\nabla u|^p\,dm\right)^{1/p} \le C\left(\frac{1}{m(2B)}\int_{2B}|\nabla u|^2\,dm\right)^{1/2}. \tag{2.19}$$

If in addition, $u \ge 0$ in $3B$, then

$$\left(\frac{1}{m(B)}\int_B |\nabla u|^p\,dm\right)^{1/p} \le C\left(\frac{1}{m(B)}\int_{B}|\nabla u|^2\,dm\right)^{1/2}. \tag{2.20}$$

To prove Lemma 2.18, we first derive the following inequality

Lemma 2.21. *Let $X \in \mathbb{R}^n$ and $r > 0$ be given. Let $u \in W_r(B_{4r}(X))$ be a solution of $\mathcal{L}u = 0$ in $B_{4r}(X) \setminus \Gamma$, with $Tu = 0$ on $B_{4r}(X) \cap \Gamma$ if $B_{4r}(X) \cap \Gamma$ is not empty. Then*

$$\left(\frac{1}{m(B_r(X))}\int_{B_r(X)}|\nabla u|^2\,dm\right)^{1/2} \le C\left(\frac{1}{m(B_{3r}(X))}\int_{B_{3r}(X)}|\nabla u|^{\frac{2n}{n+2}}\,dm\right)^{\frac{n+2}{2n}}. \tag{2.22}$$

Proof. Case 1: $\delta(X) \le \frac{5}{4}r$. Then there exists $x_0 \in \Gamma$ so that $B_r(X) \subset B_{\frac{9}{4}r}(x_0)$. Hence, by Caccioppoli's inequality on the boundary and (2.10),

$$\left(\frac{1}{m(B_r(X))}\int_{B_r(X)} |\nabla u|^2\, dm\right)^{1/2} \lesssim \left(\frac{1}{m(B_{9r/4}(x_0))}\int_{B_{9r/4}(x_0)} |\nabla u|^2\, dm\right)^{1/2}$$
$$\lesssim \left(\frac{1}{m(B_{5r/2}(x_0))}\int_{B_{5r/2}(x_0)} |\nabla u|^{\frac{2n}{n+2}}\, dm\right)^{\frac{n+2}{2n}}.$$

Then (2.22) follows from the fact that $B_{5r/2}(x_0) \subset B_{3r}(X)$.

Case 2: $\delta(X) > \frac{5}{4}r$. Then $B_{5r/4}(X) \subset \mathbb{R}^n \setminus \Gamma$. By the interior Caccioppoli inequality and the Poincaré inequality (2.9),

$$\Big(\frac{1}{m(B_r(X))}\int_{B_r(X)} |\nabla u|^2\, dm\Big)^{\frac{1}{2}} \lesssim \frac{1}{r}\Big(\frac{1}{m(B_{\frac{5r}{4}}(X))}\int_{B_{\frac{5r}{4}}(X)} \left|u - u_{B_{5r/4}(X)}\right|^2 dm\Big)^{\frac{1}{2}}$$
$$\lesssim \Big(\frac{1}{m(B_{\frac{5r}{4}}(X))}\int_{B_{\frac{5r}{4}}(X)} |\nabla u|^{\frac{2n}{n+2}}\, dm\Big)^{\frac{n+2}{2n}}. \quad \square$$

Sketch of proof of Lemma 2.18. One can deduce (2.19) in Lemma 2.18 from Lemma 2.21 and a modification of the argument in [9] (Theorem 1.2, Chapter V). Thanks to the fact that m is a doubling measure, the argument in [9] carries over. The only modification is that one should choose parameters everywhere in the argument in [9] according to the doubling constant of m instead of that of Lebesgue measure in $\mathbb{R}^n$. Once we obtain (2.19) and assume additionally u is an nonnegative solution, (2.20) follows immediately from Lemma 2.15 and Harnack's inequality. $\square$

3. Connection with the co-dimensional one case: an analogue of constant-coefficient operators

From now on, we focus only on $\Omega = \mathbb{R}^n \setminus \Gamma$ with $\Gamma = \{(x,t) \in \mathbb{R}^n : t = 0\} \cong \mathbb{R}^d$. Notice that in this setting, for a point $(x,t) \in \mathbb{R}^n$, its distance to Γ is simply $|t|$. Therefore, we can simply define the weight function w as a function in $\mathbb{R}^{n-d}$. That is, for $t \in \mathbb{R}^{n-d}$, define

$$w(t) := |t|^{-n+d+1}.$$

Recall that $B_r(X)$ denotes the ball in $\mathbb{R}^n$ with radius r centered at $X \in \mathbb{R}^n$. For $x \in \mathbb{R}^d$, we write $B_r(x) := B_r(x,0)$, the ball in $\mathbb{R}^n$ with radius r centered at $(x,0) \in \mathbb{R}^n$. Recall also that for a set E in $\mathbb{R}^n$, $m(E) = \int_E w(t)\, dxdt$. As the following computation shows, for a ball in $\mathbb{R}^n$ centered on Γ, its m measure is equal to the Lebesgue measure of a Carleson ball in $\mathbb{R}^{d+1}$ multiplied by the surface area of the unit $(n-d-1)$-dimensional sphere:

$$\begin{aligned} m(B_r(x_0)) &= \int\limits_{B_r(x_0)} w(t)\, dxdt = \int\limits_{B_r(x_0)} |t|^{-n+d+1}\, dxdt \\ &= \int\limits_{|x-x_0|\le r} \int\limits_{\rho=0}^{\sqrt{r^2-|x-x_0|^2}} \int\limits_{\omega\in S^{n-d-1}} d\omega d\rho dx \\ &= |T(x_0,r)|\, \sigma(S^{n-d-1}) = c_{n,d} r^{d+1}. \end{aligned} \tag{3.1}$$

Let $\mathcal{L} = -\operatorname{div}_{x,t}(\mathcal{A}(x,t)w(t)\nabla_{x,t})$ be an operator defined in $\mathbb{R}^n \setminus \Gamma$, where $\mathcal{A}(x,t) = [a_{ij}(x,t)]$ is an $n \times n$ matrix of real-valued, measurable functions on $\mathbb{R}^n$, which satisfies the ellipticity conditions (1.2). We shall systematically use $\mathcal{A}_0$ to denote an $n \times n$ matrix in the class $\mathfrak{A}_0(\mu_0)$, and write $\mathcal{L}_0 = -\operatorname{div}_{x,t}(\mathcal{A}_0 w(t)\nabla_{x,t})$.

The main benefit of taking $\mathcal{A}_0$ in this particular form is that the solutions of $\mathcal{L}_0 u = 0$ can be converted to solutions of a constant-coefficient equation in $\mathbb{R}^{d+1}$. Let us introduce the $(d+1)$- dimensional constant-coefficient elliptic operator

$$L_0 := -\operatorname{div}_{x,\rho}(\widetilde{A}\nabla_{x,\rho}), \qquad \text{with } \widetilde{A} = \left[\begin{array}{c:c} A_0 & \mathbf{b_0} \\ \hdashline \mathbf{c_0} & d_0 \end{array}\right], \tag{3.2}$$

where A_0, $\mathbf{b_0}$, $\mathbf{c_0}$ and d_0 are the same as in (1.6). Alternatively, we can write

$$L_0 = -\operatorname{div}_x(A_0\nabla_x) - \operatorname{div}_x(\mathbf{b_0}\partial_\rho) - \partial_\rho(\mathbf{c_0}\nabla_x) - d_0\partial_\rho^2. \tag{3.3}$$

To relate solutions of $\mathcal{L}_0 u = 0$ to those of $L_0 v = 0$, let us first give some definitions.

Definition 3.4. Let $f = f(x,t)$ be a function defined on $\mathbb{R}^n$. Write $t = \rho\omega$ in polar coordinates, with $\rho \in \mathbb{R}_+$ and $\omega \in S^{n-d-1}$. We still denote the function in polar coordinates as f, that is, $f(x,t)$ and $f(x,\rho\omega)$ are the same function in different coordinates. For any $(x,\rho) \in \mathbb{R}^{d+1}_+$, define

$$f_\theta(x,\rho) := \fint\limits_{S^{n-d-1}} f(x,\rho\omega) d\omega. \tag{3.5}$$

For any $(x,t) \in \mathbb{R}^n$, define

$$\widetilde{f}_\theta(x,t) := f_\theta(x,|t|) = \fint\limits_{S^{n-d-1}} f(x,|t|\omega) d\omega. \tag{3.6}$$

In particular, $\widetilde{f}_\theta$ is a function of n variables and is radial in t.

Lemma 3.7. *With the definitions above, the following statements hold:*

(1) If $u \in W$, then $u_\theta \in L^1_{loc}(\mathbb{R}^{d+1}_+)$, $\nabla u_\theta \in L^2(\mathbb{R}^{d+1}_+)$, and $\widetilde{u}_\theta \in W$.
(2) Let $x_0 \in \Gamma$ and $r > 0$. If $u \in W_r(B_r(x_0))$, then

$$u_\theta \in W^{1,2}_{loc}(T(x_0, r)) = \left\{ f \in L^2_{loc}(T(x_0, r)) : \nabla f \in L^2_{\text{loc}}\ (T(x_0, r)) \right\},$$

and $\widetilde{u}_\theta \in W_r(B_r(x_0))$.

Proof. (1) We first show that $u_\theta \in L^1_{loc}(\mathbb{R}^{d+1}_+)$. Let K be a compact set in $\mathbb{R}^{d+1}_+$. Then we can find $x_0 \in \mathbb{R}^d$, $r > 0$ and $\varepsilon > 0$ so that $K \subset \{(x, \rho) \in T(x_0, r) : \rho \geq \varepsilon\}$. By translation invariance, we can assume that x_0 is the origin. Then we have

$$\begin{aligned}
\int_K |u_\theta(x,\rho)|\, d\rho dx &\leq \varepsilon^{-n+d+1} \int_{|x|\leq r} \int_{\varepsilon}^{\sqrt{r^2-|x|^2}} |u_\theta(x,\rho)|\, \rho^{n-d-1} d\rho dx \\
&\leq C_{\varepsilon,n,d} \int_{|x|\leq r} \int_{\varepsilon}^{\sqrt{r^2-|x|^2}} \int_{S^{n-d-1}} |u(x,\rho\omega)|\, \rho^{n-d-1} d\omega d\rho dx \\
&= C_{\varepsilon,n,d} \int_{B_r} |u(x,t)|\, dxdt < \infty,
\end{aligned}$$

where we have used $u \in L^1_{loc}(\mathbb{R}^n)$ to get the finiteness of the last term. This shows $u_\theta \in L^1_{loc}(\mathbb{R}^{d+1}_+)$.

Now we compute the L^2 integral of $|\nabla u_\theta|$ over a Carleson ball T_r centered at the origin. Observe that by the definition of u_θ and $\widetilde{u}_\theta$, expressing the gradient in polar coordinates, we have $|\nabla_{x,\rho} u_\theta(x,\rho)| = |\nabla_{x,t} \widetilde{u}_\theta(x,t)|$, for $\rho = |t|$. Hence,

$$|\nabla_{x,\rho} u_\theta(x,\rho)|^2 = \left| \nabla_{x,t} \fint_{S^{n-d-1}} u(x, |t|\,\omega) d\omega \right|^2 \leq \fint_{S^{n-d-1}} |\nabla_{x,t} u(x, |t|\,\omega)|^2\, d\omega. \tag{3.8}$$

Let $s = |t|\,\omega$, then $|s| = |t|$, and $\frac{\partial s_k}{\partial t_j} = \frac{t_j}{|t|}\omega_k$, for $k, j = 1, 2, \ldots, n-d$. Thus,

$$|\nabla_t u(x, |t|\,\omega)|^2 = \sum_{j=1}^{n-d} \left(\partial_{t_j} u(x,s)\right)^2 = \sum_{j=1}^{n-d} \left(\sum_{k=1}^{n-d} \partial_{s_k} u(x,s) \frac{t_j}{|t|} \omega_k \right)^2 \leq |\nabla_s u(x,s)|^2 .$$

Combining this with (3.8), we obtain

$$|\nabla_{x,\rho} u_\theta(x,\rho)|^2 \leq \frac{1}{\sigma(S^{n-d-1})} \int\limits_{\rho S^{n-d-1}} |\nabla_{x,s} u(x,s)|^2 \rho^{-n+d+1} ds.$$

Therefore, integrating in polar coordinates, we can control the L^2 integral of $|\nabla u_\theta|$ as follows.

$$\begin{aligned}
\int\limits_{T_r} |\nabla_{x,\rho} u_\theta(x,\rho)|^2 \, dx d\rho &\leq C_{n,d} \int\limits_{|x|\leq r} \int\limits_{\rho=0}^{\sqrt{r^2-|x|^2}} \int\limits_{\rho S^{n-d-1}} |\nabla_{x,s} u(x,s)|^2 \frac{ds d\rho dx}{\rho^{n-d-1}} \\
&= C_{n,d} \int\limits_{|x|\leq r} \int\limits_{|t|\leq\sqrt{r^2-|x|^2}} |\nabla_{x,t} u(x,t)|^2 \frac{dt dx}{|t|^{n-d-1}} \\
&= C_{n,d} \int\limits_{B_r} |\nabla_{x,t} u(x,t)|^2 \, w(t) \, dx dt \leq C_{n,d} \|\nabla u\|^2_{L^2(\mathbb{R}^n,w)} . \quad (3.9)
\end{aligned}$$

Letting r go to infinity we obtain $\nabla u_\theta \in L^2(\mathbb{R}^{d+1})$ with

$$\|\nabla u_\theta\|_{L^2(\mathbb{R}^{d+1}_+)} \leq C_{n,d} \|\nabla u\|_{L^2(\mathbb{R}^n,w)} .$$

As for $\widetilde{u}_\theta$, let us fix any $r > 0$ and evaluate the integral of $\widetilde{u}_\theta$ over the ball B_r.

$$\begin{aligned}
\int\limits_{B_r} |\widetilde{u}_\theta(x,t)|^2 \, dx dt &= \int\limits_{|x|\leq r} \int\limits_{0}^{\sqrt{r^2-|x|^2}} \int\limits_{S^{n-d-1}} |u_\theta(x,\rho)| \, \rho^{n-d-1} d\omega d\rho dx \\
&\leq \int\limits_{|x|\leq r} \int\limits_{0}^{\sqrt{r^2-|x|^2}} \int\limits_{S^{n-d-1}} |u(x,\rho\omega')| \, d\omega' \rho^{n-d-1} d\rho dx = \int\limits_{B_r} |u(x,t)| \, dx dt.
\end{aligned}$$

This shows that $\widetilde{u}_\theta \in L^1_{loc}(\mathbb{R}^n)$. Finally, we compute

$$\begin{aligned}
\int\limits_{B_r} |\nabla_{x,t} \widetilde{u}_\theta(x,t)|^2 \, w(t) \, dx dt &= \int\limits_{|x|\leq r} \int\limits_{0}^{\sqrt{r^2-|x|^2}} \int\limits_{S^{n-d-1}} |\nabla_{x,\rho} u_\theta(x,\rho)|^2 \, d\omega d\rho dx \\
&= \sigma(S^{n-d-1}) \int\limits_{T_r} |\nabla_{x,\rho} u_\theta(x,\rho)|^2 \, d\rho dx.
\end{aligned}$$

This and (3.9) give $\nabla \widetilde{u}_\theta \in L^2(\mathbb{R}^n, w)$. Thus, $\widetilde{u}_\theta \in W$.

(2) By translation and dilation invariance, we can assume that x_0 is the origin and $r = 1$. By a similar argument as in (1), one sees that if $u \in W_r(B_1)$, then $u_\theta \in L^1_{loc}(T_1)$, $\nabla u_\theta \in L^2_{loc}(T_1)$, and $\widetilde{u}_\theta \in W_r(B_1)$. So it remains to show $u_\theta \in L^2_{loc}(T_1)$. Notice however

that u_θ lives in the upper half space, where there is no disturbing weight w; then we can apply the usual Poincaré estimate, in the homogeneous space of locally integrable functions f such that $\nabla f \in L^2$, and indeed get that $u_\theta \in L^2_{loc}(T_1)$. This gives $u_\theta \in W^{1,2}_{loc}(T_1)$. $\square$

Now we can show how solutions of equations in $\mathbb{R}^{d+1}_+$ and $\mathbb{R}^n \setminus \mathbb{R}^d$ are related.

Lemma 3.10. *Let B be a ball centered on Γ. If $u \in W_r(B)$ is a solution of $\mathcal{L}_0 u = 0$ in $B \setminus \Gamma$, where $\mathcal{A}_0$ is in block form (1.6), then $u_\theta \in W^{1,2}_{loc}(T)$ is a solution of $L_0 u_\theta = 0$ in T, and $\widetilde{u}_\theta \in W_r(B)$ is a solution of $\mathcal{L}_0 \widetilde{u}_\theta = 0$ in $B \setminus \Gamma$.*

Proof. Let us assume that u is a C^2 function and thus a strong solution. Writing out the derivatives, we see that

$$\mathcal{L}_0 = \frac{-1}{|t|^{n-d-1}} \left(\operatorname{div}_x(A_0 \nabla_x) + \operatorname{div}_x \left(\frac{\mathbf{b_0} t \cdot \nabla_t}{|t|} \right) + \operatorname{div}_t \left(\frac{t^T \mathbf{c_0} \nabla_x}{|t|} \right) + \operatorname{div}_t (d_0 \nabla_t) \right) \\ + \frac{n-d-1}{|t|^{n-d}} \mathbf{c_0} \nabla_x + \frac{n-d-1}{|t|^{n-d+1}} d_0 t \cdot \nabla_t \,.$$

Fortunately, some of these terms cancel. In fact,

$$\operatorname{div}_t \left(\frac{t^T \mathbf{c_0} \nabla_x}{|t|} \right) = \frac{(n-d-1) \mathbf{c_0} \nabla_x}{|t|} + \frac{t \cdot \nabla_t (\mathbf{c_0} \nabla_x)}{|t|},$$

and thus

$$\mathcal{L}_0 = -\frac{1}{|t|^{n-d-1}} \left(\operatorname{div}_x(A_0 \nabla_x) + \operatorname{div}_x \left(\frac{\mathbf{b_0} t \cdot \nabla_t}{|t|} \right) + \frac{t \cdot \nabla_t (\mathbf{c_0} \nabla_x)}{|t|} + d_0 \Delta_t \right) \\ + \frac{n-d-1}{|t|^{n-d+1}} d_0 t \cdot \nabla_t \,.$$

Changing to polar coordinates $t = \rho\,\omega$, we have $t \cdot \nabla_t = \rho \partial_\rho$, and

$$\Delta_t = \partial_\rho^2 + (n-d-1) \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \Delta_\omega,$$

where Δ_ω is the Laplacian on the sphere S^{n-d-1}. Then $\mathcal{L}_0$ can be simplified as

$$\mathcal{L}_0 = -\frac{1}{\rho^{n-d-1}} \left(\operatorname{div}_x(A_0 \nabla_x) + \operatorname{div}_x(\mathbf{b_0} \partial_\rho) + \partial_\rho (\mathbf{c_0} \nabla_x) + d_0 \partial_\rho^2 \right) - \frac{d_0}{\rho^{n-d+1}} \Delta_\omega. \tag{3.11}$$

Notice that the quantity in the first parenthesis is exactly what we have for L_0 in (3.3). Now since $\widetilde{u}_\theta$ is radial in t, $\Delta_\omega \widetilde{u}_\theta = 0$, and thus,

$$\mathcal{L}_0 \widetilde{u}_\theta = \frac{1}{\rho^{n-d-1}} L_0 u_\theta. \tag{3.12}$$

Notice that $\int_{S^{n-d-1}} \Delta_\omega u(x, \rho\,\omega) d\omega = 0$ by the divergence theorem, so that we can add this term for free and get the following.

$$\mathcal{L}_0 \widetilde{u}_\theta = -\frac{1}{\rho^{n-d-1}} L_0 u_\theta + \frac{d_0}{\rho^{n-d+1}} \fint_{S^{n-d-1}} \Delta_\omega u(x, \rho\,\omega) d\omega.$$

Exchanging the order of integration and differentiation, we obtain

$$\mathcal{L}_0 \widetilde{u}_\theta = -\frac{1}{\rho^{n-d-1}} \fint_{S^{n-d-1}} \left(L_0 + \frac{d_0}{\rho^2} \Delta_\omega \right) u(x, \rho\,\omega) d\omega = \fint_{S^{n-d-1}} \mathcal{L}_0 u \, d\omega = 0.$$

This and (3.12) show that $\mathcal{L}_0 \widetilde{u}_\theta = 0 = L_0 u_\theta$.

The smoothness assumption on solutions is harmless. First, we have checked in Lemma 3.7 that given $u \in W_r(B)$, u_θ and $\widetilde{u}_\theta$ are in the right spaces stated in the lemma. Now if $u \in W_r(B)$ is a weak solution, it is a strong solution in any compact set in $B \setminus \Gamma$. This is because on these sets, $|t| \geq \delta$ for some $\delta > 0$, and thus the coefficients are smooth. Then we use our results for strong solutions and conclude that u_θ and $\widetilde{u}_\theta$ are strong solutions in any compact set in T and $B \setminus \Gamma$, respectively. Then they are of course weak solutions in these compact sets. But this is all we need as in the weak formulation of equations, the test functions are compactly supported in T (for u_θ) and in $B \setminus \Gamma$ (for $\widetilde{u}_\theta$). $\square$

Remark 3.13. Writing $\mathcal{L}_0$ in polar coordinates as in (3.11), one immediately sees $\mathcal{L}_0 \, |t| = 0$ in $\mathbb{R}^n \setminus \mathbb{R}^d$. We shall use this property of $|t|$ in the future.

We now turn to the quantity $\lambda_{x,r}(u)$. First we show that $\lambda_{x,r}(u)\,|t|$ is the best approximation of a given function u in $B_r(x)$ by a multiple of $|t|$.

Lemma 3.14 *(Orthogonality). For any $(x, r) \in \mathbb{R}^{d+1}_+$, for any function $u(x, t)$,*

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} \nabla(u(y, t) - \lambda_{x,r}(u)\,|t|) \cdot \nabla\,|t| \; w(t) dy dt = 0. \tag{3.15}$$

Moreover,

$$\inf_{\lambda \in \mathbb{R}} \frac{1}{m(B_r(x))} \int_{B_r(x)} |\nabla(u(y, t) - \lambda\,|t|)|^2 \, w(t) dy dt = J_u(x, r), \tag{3.16}$$

where $J_u(x, r)$ is defined in (1.13).

Proof. For any $\lambda \in \mathbb{R}$, we compute

$$\nabla(u - \lambda\,|t|) \cdot \nabla\,|t| = \sum_{i=1}^{n-d} \left(\partial_{t_i} u - \frac{\lambda t_i}{|t|} \right) \frac{t_i}{|t|} = \frac{\nabla_t u \cdot t}{|t|} - \lambda. \tag{3.17}$$

By the definition of $\lambda_{x,r}(u)$ in (1.12), $\frac{\nabla_t u \cdot t}{|t|} - \lambda_{x,r}(u)$ is orthogonal to constants in $L^2(B_r(x), w)$. Therefore, using (3.17) with $\lambda = \lambda_{x,r}(u)$, one sees that (3.15) holds. Turning to (3.16), we see that for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
&\frac{1}{m(B_r(x))} \int\limits_{B_r(x)} |\nabla(u(y,t) - \lambda\,|t|)|^2\, w(t) dy dt \\
&\qquad = \frac{1}{m(B_r(x))} \int\limits_{B_r(x)} |\nabla(u(y,t) - \lambda_{x,r}(u)\,|t|)|^2\, w(t) dy dt \\
&\qquad\quad + \frac{|\lambda_{x,r}(u) - \lambda|^2}{m(B_r(x))} \int\limits_{B_r(x)} |\nabla\,|t||^2\, w(t) dy dyt \\
&\qquad = J_u(x,r) + |\lambda_{x,r}(u) - \lambda|^2 \geq J_u(x,r),
\end{aligned}$$

where in the first equality we have used (3.15). $\square$

It follows from (3.16) that $J_u(x,r) \leq E_u(x,r)$, which implies

$$\beta_u(x,r) \leq 1 \qquad \text{for any } (x,r) \in \mathbb{R}^{d+1}_+. \tag{3.18}$$

The following lemma shows that the best approximation of u by a multiple of $|t|$ in $B_r(x)$ is the same as the best approximation of u_θ in $T(x,r)$.

Lemma 3.19. *Let $x \in \mathbb{R}^d$, $r > 0$, and u be as in Lemma 3.14. Define u_θ as in Definition 3.4. Then*

$$\lambda_{x,r}(u) = \fint\limits_{T(x,r)} \partial_\rho u_\theta(y,\rho) dy d\rho.$$

Proof. Without loss of generality, we may assume that x is the origin and $r = 1$. Passing to polar coordinates $t = \rho\omega$, and noticing that $\frac{t \cdot \nabla_t u}{|t|} = \frac{\rho \partial_\rho u}{\rho}$, we have

$$\frac{1}{m(B)} \int\limits_B \frac{\nabla_t u \cdot t}{|t|} w(t)\, dx dt = \frac{1}{m(B)} \int\limits_{|x| \leq 1} \int\limits_{|t| \leq \sqrt{1 - |x|^2}} \frac{\nabla_t u(x,t) \cdot t}{|t|} w(t)\, dx dt$$

$$= \frac{1}{m(B)} \int_{|x|\leq 1} \int_0^{\sqrt{1-|x|^2}} \int_{S^{n-d-1}} \partial_\rho u(x, \rho\omega) d\omega d\rho dx.$$

Exchanging the order of integration and differentiation,

$$\frac{1}{m(B)} \int_B \frac{\nabla_t u \cdot t}{|t|} w(t)\, dx dt = \frac{1}{m(B)} \int_{T_1} \left(\partial_\rho \int_{S^{n-d-1}} u(x, \rho\omega) d\omega \right) d\rho dx$$

$$= \fint_{T_1} \left(\partial_\rho \fint_{S^{n-d-1}} u(x, \rho\omega) d\omega \right) d\rho dx = \fint_{T_1} \partial_\rho u_\theta(x, \rho) dx d\rho,$$

because $|T_1| = m(B)\sigma(S^{n-d-1})$ and as desired. $\square$

4. Estimates for solutions of $\mathcal{L}_0 u = 0$

4.1. More about function spaces

When proving estimates for (weak) solutions, it is useful to allow test functions that lie in a bigger space than C_0^∞. For this reason, we now define some new spaces.

Definition 4.1. Let $O \subset \mathbb{R}^n$ be an open, bounded set. Define

$$W(O) := \left\{ u \in L^1_{loc}(O) : \nabla u \in L^2(O, w) \right\}. \tag{4.2}$$

Here $L^1_{loc}(O)$ is for the Lebesgue measure, which is more natural if we want to see u as a distribution and talk about its gradient. Equip $W(O)$ with the seminorm $\|f\|_{W(O)} = \left(\int_O |\nabla f|^2 \, dm \right)^{1/2}$. Define $W_0(O)$ to be the closure of $C_0^\infty(O)$ under $\|\cdot\|_{W(O)}$.

As we shall see, $W(O)$ plays the same role as the usual Sobolev space $W^{1,2}(O)$, and $W_r(O)$ should be compared with $W^{1,2}_{loc}(O)$.

For the purposes of this paper, we are only interested in the simple case when O is a ball B centered on Γ, or $O = B \setminus \Gamma$.

Lemma 4.3. *Let $B \subset \mathbb{R}^n$ be a ball centered on Γ. Then*

(1) $W(B \setminus \Gamma) = W(B) = \left\{ u \in L^1(B) : \nabla u \in L^2(B, w) \right\}$;
(2) $W(B) \subset W^{1,2}(B) = \{u \in L^2(B, dX) : \nabla u \in L^2(B, w)$;
(3) If $u \in W_r(2B)$, then $u \in W(B)$.

Proof. (1) Let $u \in W(B \setminus \Gamma)$ be given. By definition, $u \in L^1_{loc}(B \setminus \Gamma, dX) = L^1_{loc}(B \setminus \Gamma, w dX)$, and by Lemma 3.2 in [2], $u \in L^1_{loc}(B, dX)$. So $u \in W(B)$; we still need to check that $u \in L^1(B, dX)$. However Poincaré's inequality (Lemma 2.7, with $p = 1$) says that $u \in L^1(B, wdX)$, and then an easy estimate ((2.13) in [2]) shows that $u \in L^1(B, dX)$. Notice that although our assumptions, for instance in Lemma 2.7, appear to be global, we never use the values of u outside of B.

(2) For $u \in W(B)$, we now apply Poincaré's inequality (Lemma 2.7), now with $p = 2$, to find that

$$\int_B |u - u_B|^2 \, wdX \leq C \int_B |\nabla u|^2 \, dm,$$

Then again $u \in L^2(B, dX)$ by (2.13) in [2], this time applied to $g = |u - u_B|^2$.

(3) follows immediately from (1) and the definition (2.4). □

Next we claim that if $u \in W(B)$ is a (weak) solution of $\mathcal{L}u = 0$ in $B \setminus \Gamma$, we can take test functions in the space $W_0(B \setminus \Gamma)$. That is,

$$\int_B \mathcal{A}\nabla u \cdot \nabla \varphi \, dm = 0 \quad \text{for every } \varphi \in W_0(B \setminus \Gamma). \tag{4.4}$$

In fact, since $\varphi \in W_0(B \setminus \Gamma)$ we can find a sequence $\{\varphi_k\}$ in $C_0^\infty(B \setminus \Gamma)$ that converges to φ in $W(B \setminus \Gamma)$. Then

$$\left| \int_B \mathcal{A}\nabla u \cdot \nabla \varphi_k dm - \int_B \mathcal{A}\nabla u \cdot \nabla \varphi dm \right| \leq \mu_0 \left(\int_B |\nabla u|^2 \, dm \right)^{1/2} \left(\int_B |\nabla \varphi_k - \nabla \varphi|^2 \, dm \right)^{1/2}.$$

The right-hand side is finite and vanishes as k go to infinity. So (4.4) follows from taking limits.

Let us also discuss the trace on $\partial(B \setminus \Gamma) = \partial B \cup (\Gamma \cap B)$. Since $W(B)$ is a subset of $W_r(B)$, for $u \in W(B)$, its trace Tu on $B \cap \Gamma$ can be defined by (2.5) for almost every $x \in B \cap \Gamma$, and $Tu \in L^1_{loc}(B \cap \Gamma, dx)$. Moreover, by slightly modifying the proof of [2], Theorem 3.4, one can show that

$$\|Tu\|_{L^2(B \cap \Gamma, dx)} \lesssim \|u\|_{L^1(B)} + \|\nabla u\|_{L^2(B,w)}.$$

For $u \in W(B)$, we can define its trace on ∂B by

$$Tu(X) := \lim_{r \to 0} \fint_{B_r(X) \cap B} u(Y) dY \qquad \text{for } X \in \partial B,$$

and one can show that $\|Tu\|_{L^2(\partial B)} \lesssim \|u\|_{L^1(B)} + \|\nabla u\|_{L^2(B,w)}$. Alternatively, since we proved that $W(B) \subset W^{1,2}(B)$, the trace theorem for Sobolev spaces applies. We remark that in [5], a trace theorem is developed in a much more general setting and is different from what we have discussed here. But for the purposes of this paper, this simpler approach suffices.

4.2. Decay estimates for the non-affine part of solutions

We want to show that for a solution of $\mathcal{L}_0 u = 0$ that vanishes on $\Gamma = \mathbb{R}^d$, its non-affine part $J_u(x,r)$ decreases in r. In the case when $d = n-1$, this property can be obtained from Moser estimates for solutions on the boundary. We state it in $T_1 = T(0,1)$, for the constant coefficient operator L_0 that was defined in (3.2), to simplify the notation.

Lemma 4.5 *($d = n-1$ case, [7], Lemma 3.4). Let $u \in W^{1,2}(T_1)$ be a solution of $L_0 u = 0$ in T_1 with $u = 0$ on Δ_1. Then there exists some constant C depending only on d and μ_0, such that for $0 < r < 1/2$,*

$$\fint_{T_r} |\nabla (u(x,t) - \lambda_r(u)\, t)|^2 \, dxdt \le Cr^2 \fint_{T_1} |\nabla(u(x,t) - \lambda_1(u)\, t)|^2 \, dxdt, \tag{4.6}$$

where $\lambda_r(u) = \fint_{T_r} \partial_s u(y,s) dyds$.[1]

The way we show this decay estimate is by controlling the non-affine part of the solution in T_r by the oscillation of the derivative of some solution in T_r, which is further controlled by the energy of the solution in T_1 multiplied by r^2. The bound on the oscillation of the first derivative of solutions is essentially a consequence of estimates for the *second* derivatives of solutions. However, when $d < n-1$, we do not have a good estimate for the second derivatives because the coefficients involve $|t|^{-n+d+1}$, which is singular on the boundary. Fortunately, we still have an analogue of Lemma 4.5 in the case of $d < n-1$. The first step is to show that solutions of $\mathcal{L}_0 u = 0$ with a vanishing trace on Γ are roughly Lipschitz in t near the boundary. To be precise, we have the following.

Lemma 4.7. *Let B be a ball centered on Γ and let $u \in W_r(2B)$ be a solution of $\mathcal{L}_0 u = 0$ in $2B \setminus \Gamma$, with $Tu = 0$ on $\Gamma \cap 2B$. Then there is some constant $C > 0$ depending only on d, n and μ_0, such that*

$$|u(x,t)| \le C \left(\frac{1}{m(B)} \int_B |\nabla u|^2 \, dm \right)^{1/2} |t|, \qquad \text{for all } (x,t) \in B. \tag{4.8}$$

Proof. Observe that if u is *nonnegative*, then (4.8) simply follows from the comparison principle and the fact that $|t|$ is a solution of $\mathcal{L}_0 |t| = 0$ that vanishes on Γ. In fact, by the comparison principle (Lemma 2.16), for $(x,t) \in B \setminus \Gamma$,

[1] Note that we are using the same notation $\lambda_r(u)$ to denote different quantities in $d = n-1$ and $d < n-1$.

$$\frac{u(x,t)}{|t|} \leq C\frac{u(X_B)}{r(B)} \leq C\left(\frac{1}{m(B)}\int_B |\nabla u|^2\,dm\right)^{1/2},$$

where X_B is a corkscrew point for B, $r(B)$ denotes its radius, and the second inequality is due to Lemma 2.15.

If u changes signs in $2B$, we write $u = u_1 - u_2$, with $u_1 = \sup\{u, 0\}$ and $u_2 = \sup\{-u, 0\}$. Notice that by Lemma 4.3 (3), $u \in W(B)$. Then by [2], Lemma 6.1, $u_i \in W(B)$ for $i = 1, 2$, with

$$\int_B |\nabla u_i|^2\,dm \leq \int_B |\nabla u|^2\,dm, \quad \text{and } Tu_i = 0 \text{ on } \Gamma\cap B, \quad i = 1, 2.$$

Moreover, the Hölder continuity of solutions (see [2], Lemma 8.8 and Lemma 8.16) implies that $u \in C(\overline{B})$, and thus $u_i \in C(\overline{B})$ for $i = 1, 2$.

We want to look at the solutions v_i to $\mathcal{L}_0 v_i = 0$ in $B\setminus\Gamma$, with data u_i on $\partial(B\setminus\Gamma)$ (and in a suitable weak sense). First, the nonhomogeneous problem $\mathcal{L}_0\widetilde{v}_i = -\mathcal{L}_0 u_i$ in $B\setminus\Gamma$ has a unique solution $\widetilde{v}_i \in W_0(B\setminus\Gamma)$ due to the Lax-Milgram Theorem. Setting $v_i = \widetilde{v}_i + u_i$, one sees that $v_i \in W(B)$ and verifies

$$\begin{cases} \mathcal{L}_0 v_i = 0 & \text{in } B\setminus\Gamma, \\ v_i - u_i \in W_0(B\setminus\Gamma). \end{cases} \tag{4.9}$$

We claim that the $W(B)$ seminorm of v_i is controlled by that of u_i. To see this, take $v_i - u_i$ as a test function for $\mathcal{L}_0 v_i = 0$, which is allowed because $v_i \in W(B)$ and $v_i - u_i \in W_0(B\setminus\Gamma)$ (see the remark around (4.4)). Then

$$\int_B \mathcal{A}_0\nabla v_i\cdot\nabla(v_i - u_i)dm = 0.$$

Therefore, using the ellipticity conditions and the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_B |\nabla v_i|^2\,dm &\leq \mu_0\int_B \mathcal{A}_0\nabla v_i\cdot\nabla v_i\,dm = \mu_0\int_B \mathcal{A}_0\nabla v_i\cdot\nabla u_i\,dm \\ &\leq \mu_0^2\left(\int_B |\nabla v_i|^2\,dm\right)^{1/2}\left(\int_B |\nabla u_i|^2\,dm\right)^{1/2},\end{aligned}$$

which implies that

$$\int_B |\nabla v_i|^2\,dm \leq \mu_0^4\int_B |\nabla u_i|^2\,dm \leq \mu_0^4\int_B |\nabla u|^2\,dm, \qquad i = 1, 2. \tag{4.10}$$

Next, v_i is nonnegative in B, for $i = 1, 2$. To see this, we first show that v_i is continuous in $\overline{B}$. Since $Tu_i = Tv_i = 0$ on $\Gamma \cap B$, the Poincaré inequality implies that their weighted $L^2(B, w)$ norm is controlled by their $W(B)$ seminorm. Therefore, both of them belong to the weighted Sobolev space $W^{1,2}(B, w)$. In particular, $u_i \in W^{1,2}(B, w) \cap C(\overline{B})$. Notice that $w(t)$ is an A_2 weight with respect to the Lebesgue measure on $\mathbb{R}^n$. That is, there holds

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B |t|^{-n+d+1} \, dxdt \right) \left(\frac{1}{|B|} \int_B |t|^{n-d-1} \, dxdt \right) < \infty.$$

So we can apply [10], Theorems 6.27 and 6.31, to get that for any $X \in \partial B$, $\lim_{X \to X_0} v_i(X) = u_i(X_0)$. This takes care of continuity on ∂B, so it remains to treat the interior and $\Gamma \cap B$. But since $Tv_i = 0$ on $\Gamma \cap B$, Hölder estimates for solutions ([2] Lemma 8.8 and Lemma 8.16) guarantee that $v_i \in C(B)$. So we conclude that $v_i \in C(\overline{B})$, $i = 1, 2$. Next, we show $v_i \geq 0$ in B, using a standard argument. Set $v^i_\varepsilon = \min\{v_i, -\varepsilon\} + \varepsilon$. Then $v^i_\varepsilon \leq 0$ in $\overline{B}$. Since $v_i \in C(\overline{B})$ is nonnegative on $\partial(B \setminus \Gamma)$, v^i_ε is compactly supported in $B \setminus \Gamma$. Moreover,

$$\nabla v^i_\varepsilon = \begin{cases} \nabla v_i & v_i < -\varepsilon \\ 0 & v_i \geq -\varepsilon. \end{cases} \tag{4.11}$$

We take v^i_ε as a test function and get

$$0 = \int_B \mathcal{A}_0 \nabla v \cdot \nabla v_\varepsilon dm = \int_B \mathcal{A}_0 \nabla v^i_\varepsilon \cdot \nabla v^i_\varepsilon dm \geq \mu_0^{-1} \int_B \left| \nabla v^i_\varepsilon \right|^2 dm.$$

This implies that $\nabla v^i_\varepsilon = 0$ a.e. in B, and, since it is compactly supported in $B \setminus \Gamma$, we get that $v^i_\varepsilon = 0$ a.e. and $v_i \geq -\varepsilon$ in B. Since $\varepsilon > 0$ is arbitrary, we obtain $v_i \geq 0$ in B for $i = 1, 2$, as desired.

Now we can apply the result for *nonnegative* solutions to v_i, and use (4.10) to conclude that

$$v_i(x, t) \leq C \left(\frac{1}{m(B)} \int_B |\nabla u|^2 \, dm \right)^{1/2} |t|, \qquad \text{for } (x, t) \in B, \qquad i = 1, 2. \tag{4.12}$$

Finally, let $v = v_1 - v_2$. Then $v = u$ on $\partial(B \setminus \Gamma)$ (both in pointwise sense and in $W_0(B \setminus \Gamma)$ sense), and so the uniqueness of the solution implies $u = v$ in B. The desired estimate for $|u(x, t)|$ follows from (4.12) and the fact that $|u| \leq v_1 + v_2$ in B. $\square$

Now we derive the decay estimate for solutions of $\mathcal{L}_0 u = 0$, which is an analogue of Lemma 4.5 for $d < n - 1$. By the translation and dilation invariance of the problem, we only need to consider the problem on the unit ball. We shall use $J_u(r)$ to denote $J_u(0,r)$. Similarly, $E_u(r)$ and $\beta_u(r)$ are shorthand for $E_u(0,r)$ and $\beta_u(0,r)$, respectively.

Lemma 4.13 *(Key lemma). For any $\theta_0 \in (0,1)$, there exists $r_0 = r_0(n,d,\mu_0,\theta_0) \in (0,1)$ such that for any solution $u \in W(B_1)$ of $\mathcal{L}_0 u = 0$ in $B_1 \setminus \Gamma$, with $Tu = 0$ on $\Gamma \cap B_1$, there holds*

$$J_u(r) \le \theta_0 J_u(1), \qquad \textit{for} \quad 0 < r \le r_0. \tag{4.14}$$

Proof. We first show that for any $r_1 \in (0,1)$ and any $\theta_1 \in (0,1)$, there exists $r_0 = r_0(\theta_1, r_1, n, d) < r_1$, such that for any solution $u \in W(B_1)$ of $\mathcal{L}_0 u = 0$ in $B_1 \setminus \Gamma$, with $Tu = 0$ on $\Gamma \cap B_1$, there holds

$$\frac{1}{m(B_r)} \int_{B_r} |\nabla (u - \widetilde{u}_\theta)|^2 \, dm \le \frac{\theta_1}{m(B_{r_1})} \int_{B_{r_1}} |\nabla (u - \widetilde{u}_\theta)|^2 \, dm \qquad \text{for } r \le r_0, \tag{4.15}$$

where $\widetilde{u}_\theta$ is defined in (3.6). We prove (4.15) by contradiction. If the statement is not true, then there is a $\theta_1 \in (0,1)$, a sequence of operators $\mathcal{L}_0^{(k)} \in \mathfrak{A}_0(\mu_0)$, a sequence $\{r_k\}_{k=1}^\infty$ decreasing to 0, and a sequence of solutions $\{u^{(k)}\}_{k=1}^\infty \subset W(B_1)$ verifying $\mathcal{L}_0^{(k)} u^{(k)} = 0$ in $B_1 \setminus \Gamma$ and $Tu^{(k)} = 0$ on $B_1 \cap \Gamma$, such that

$$\frac{1}{m(B_{r_k})} \int_{B_{r_k}} \left|\nabla (u^{(k)} - \widetilde{u}_\theta^{(k)})\right|^2 dm > \frac{\theta_1}{m(B_{r_1})} \int_{B_{r_1}} \left|\nabla (u^{(k)} - \widetilde{u}_\theta^{(k)})\right|^2 dm, \tag{4.16}$$

for $k = 1, 2, \dots$. Define

$$v_k = \frac{u^{(k)} - \widetilde{u}_\theta^{(k)}}{\left(m(B_{r_1})^{-1} \int_{B_{r_1}} \left|\nabla \left(u^{(k)} - \widetilde{u}_\theta^{(k)}\right)\right|^2 dm\right)^{1/2}}.$$

Notice that we do not need to worry about the denominator being equal to 0 because in that case, both sides of (4.16) are 0, making the inequality false. By Lemma 3.10, $\widetilde{u}_\theta^{(k)}$ verifies $\mathcal{L}_0^{(k)} \widetilde{u}_\theta^{(k)} = 0$, and thus v_k verifies $\mathcal{L}_0^{(k)} v_k = 0$ in $B_1 \setminus \Gamma$, with $Tv_k = 0$ on $B_1 \cap \Gamma$. Moreover, v_k is constructed in a way that guarantees the following properties:

$$\begin{gathered} \fint_{\partial B(0,r)} v_k \, d\omega = 0 \quad \text{for } 0 < r \le 1, \\ \frac{1}{m(B_{r_1})} \int_{B_{r_1}} |\nabla v_k|^2 \, dm = 1, \\ \frac{r_k^{-2}}{m(B_{2r_k})} \int_{B_{2r_k}} |v_k|^2 \, dm \ge \theta_1 / C, \end{gathered} \tag{4.17}$$

where the last inequality follows from (4.16) and the Caccioppoli inequality on the boundary.

Set $V_k(X) := \frac{1}{r_k} v_k(r_k X)$. Then $\mathcal{L}_0^{(k)} V_k = 0$ in $B_{1/r_k} \setminus \Gamma$, with $TV_k = 0$ on $B_{1/r_k} \cap \Gamma$. Moreover,

$$\fint_{\partial B(0,r)} V_k \, d\omega = 0 \quad \text{for } 0 < r \leq 1/r_k, \tag{4.18}$$

$$m(B_2)^{-1} \int_{B_2} |V_k|^2 \, dm \geq \theta_1 / C. \tag{4.19}$$

Notice that (4.19) implies that there exists $(x_k, t_k) \in B_2$ such that

$$|V_k(x_k, t_k)| \geq \sqrt{\theta_1 / C}. \tag{4.20}$$

Observe that by (4.17),

$$\frac{1}{m(B_{\frac{r_1}{r_k}})} \int_{B_{\frac{r_1}{r_k}}} |\nabla V_k|^2 \, dm = 1. \tag{4.21}$$

By (4.21) and Lemma 4.7, there is some constant $c > 0$ depending only on d, n and μ_0, such that

$$|V_k(x,t)| \leq c\,|t| \quad \text{for all} \quad (x,t) \in B_{\frac{r_1}{2r_k}}. \tag{4.22}$$

Now (4.20) and (4.22) imply that the t_k in (4.20) has to satisfy

$$2 \geq |t_k| \geq C' \theta_1^{1/2}. \tag{4.23}$$

Moreover, on any compact set in $\mathbb{R}^n$, (4.22) implies that the sequence $\{V_k\}_{k=1}^\infty$ is uniformly bounded, and the regularity of solutions implies that $\{V_k\}_{k=1}^\infty$ is equicontinuous. Therefore, there is a subsequence of $\{V_k\}$, still denoted by $\{V_k\}$, converges pointwise to a V_∞. We can also find a limit $\mathcal{L}_0 \in \mathfrak{A}_0(\mu_0)$ of the $\mathcal{L}_0^{(k)}$, and it is easy to verify that $V_\infty \in W_r(\mathbb{R}^n)$ is a solution of $\mathcal{L}_0 V_\infty = 0$ in $\mathbb{R}^n \setminus \Gamma$, with $V_\infty(x, 0) = 0$ on Γ. For sure there is a convergent subsequence of $\{(x_k, t_k)\}$ in B_2; let us denote the limit point by $(x_\infty, t_\infty) \in B_2$. Then by (4.20) and (4.23),

$$|t_\infty| > C' \theta_1^{1/2}, \qquad |V_\infty(x_\infty, t_\infty)| \geq \sqrt{\theta_1 / C}. \tag{4.24}$$

By (4.22) (and the fact that r_k tends to 0), $2c\,|t| - V_\infty(x,t) > 0$ everywhere. So $2c\,|t| - V_\infty(x,t) \in W_r(\mathbb{R}^n)$ is a positive solution in $\mathbb{R}^n \setminus \Gamma$ that vanishes on Γ. On the other hand, $|t| \in W_r(\mathbb{R}^n)$ is also a positive solution in $\mathbb{R}^n \setminus \Gamma$ that vanishes on Γ. Therefore, we can apply the Corollary 2.17 to $2c\,|t| - V_\infty(x,t)$ and $|t|$, and obtain

$$\left|\frac{2c\,|t| - V_\infty(x,t)}{\alpha\,|t|} - 1\right| \le C\left(\frac{|(x,t)-(0,1)|}{R}\right)^\gamma \qquad \text{for all } R \ge 2,$$

where $\alpha = 2c - V_\infty(0,1) > 0$. Letting $R \to \infty$ one sees that $2c\,|t| - V_\infty(x,t) = \alpha\,|t|$, and thus $V_\infty(x,t) = \alpha'\,|t|$ for $(x,t) \in \mathbb{R}^n$. Thanks to (4.24), $\alpha' \neq 0$. Therefore, $⨍_{S^{n-d-1}} V_\infty d\omega \neq 0$, which is impossible since (4.18) holds for all k. This proves (4.15).

Now we show (4.14). Fix $r_1 \in (0,1/2)$ and $\theta_1 \in (0,1)$ to be determined later, and let $r_0 = r_0(\theta_1, r_1, n, d) < r_1$ be as in (4.15). Then for any $0 < r \le r_0$, we write

$$\begin{aligned} J_u(r) &= \frac{1}{m(B_r)}\int_{B_r} |\nabla(u(x,t) - \lambda_r(u)\,|t|)|^2\,w(t)\,dxdt \\ &\le \frac{2}{m(B_r)}\int_{B_r} |\nabla(u-\widetilde{u}_\theta)|^2\,dm + \frac{2}{m(B_r)}\int_{B_r} |\nabla(\widetilde{u}_\theta - \lambda_r(u)\,|t|)|^2\,w(t)\,dxdt, \end{aligned}$$

where we recall from (1.12) that

$$\lambda_r(u) = \lambda_{0,r}(u) = \frac{1}{m(B_r)}\int_{B_r} \frac{\nabla_t u(x,t)\cdot t}{|t|} w(t)\,dxdt.$$

Apply (4.15) to get

$$J_u(r) \le \frac{2\theta_1}{m(B_{r_1})}\int_{B_{r_1}} |\nabla(u-\widetilde{u}_\theta)|^2\,dm + \frac{2}{m(B_r)}\int_{B_r} |\nabla(\widetilde{u}_\theta - \lambda_r(u)\,|t|)|^2\,w(t)\,dxdt.$$

Inserting $\lambda_{r_1}(u)\,|t|$ in the first integral on the right-hand side,

$$\begin{aligned} J_u(r) \le &\frac{4\theta_1}{m(B_{r_1})}\int_{B_{r_1}} |\nabla(u - \lambda_{r_1}(u)\,|t|)|^2\,w(t)\,dxdt \\ &+ \frac{4\theta_1}{m(B_{r_1})}\int_{B_{r_1}} |\nabla(\widetilde{u}_\theta - \lambda_{r_1}(u)\,|t|)|^2\,w(t)\,dxdt \\ &+ \frac{2}{m(B_r)}\int_{B_r} |\nabla(\widetilde{u}_\theta - \lambda_r(u)\,|t|)|^2\,w(t)\,dxdt. \end{aligned} \tag{4.25}$$

We estimate the last two terms in (4.25) using decay estimates for the case $d = n-1$. First, changing to polar coordinates as in (3.1), one sees that

$$\frac{1}{m(B_r)}\int_{B_r} |\nabla(\widetilde{u}_\theta(x,t) - \lambda_r(u)\,|t|)|^2\,w(t)\,dxdt$$

$$= \frac{1}{m(B_r)} \int\limits_{|x|\le r} \int\limits_0^{\sqrt{r^2-|x|^2}} \left|\nabla_{x,\rho}\left(u_\theta(x,\rho) - \lambda_r(u)\rho\right)\right|^2 \left(\int\limits_{S^{n-d-1}} d\omega \right) d\rho dx$$

$$= \fint\limits_{T_r} \left|\nabla_{x,\rho}(u_\theta(x,\rho) - \lambda_r(u)\rho)\right|^2 d\rho dx. \quad (4.26)$$

Recall from Lemma 3.19 that $\lambda_r(u) = \fint_{T_r} \partial_\rho u_\theta(y,\rho) dy d\rho$. Since u_θ verifies $L_0 u_\theta = 0$ (see Lemma 3.10), we can apply Lemma 4.5 to u_θ and get

$$\fint\limits_{T_r} \left|\nabla_{x,\rho}(u_\theta(x,\rho) - \lambda_r(u)\rho)\right|^2 d\rho dx \le Cr^2 \fint\limits_{T_1} \left|\nabla_{x,\rho}(u_\theta(x,\rho) - \lambda_1(u)\rho)\right|^2 d\rho dx. \quad (4.27)$$

Notice that

$$\left|\nabla_{x,\rho}\left(u_\theta(x,\rho) - \lambda_1(u)\rho\right)\right|^2 = \left|\nabla_{x,t}\left(\widetilde{u}_\theta(x,t) - \lambda_1(u)\,|t|\right)\right|^2$$

$$\le \fint\limits_{S^{n-d-1}} \left|\nabla_{x,t}\left(u(x,|t|\,\omega) - \lambda_1(u)\,|t|\right)\right|^2 d\omega.$$

By a computation similar to that in the proof of Lemma 3.7, this yields

$$\fint\limits_{T_1} \left|\nabla_{x,\rho}(u_\theta(x,\rho) - \lambda_1(u)\rho)\right|^2 d\rho dx \le \frac{1}{m(B_1)} \int\limits_{B_1} \left|\nabla(u(x,t) - \lambda_1(u)\,|t|)\right|^2 w(t)\, dx dt.$$

Combining this with (4.27) and (4.26), we obtain

$$\frac{1}{m(B_r)} \int\limits_{B_r} \left|\nabla(\widetilde{u}_\theta(x,t) - \lambda_r(u)\,|t|)\right|^2 dw(t)\, dx dt$$

$$\le \frac{Cr^2}{m(B_1)} \int\limits_{B_1} \left|\nabla(u(x,t) - \lambda_1(u)\,|t|)\right|^2 w(t)\, dx dt. \quad (4.28)$$

Now we return to the first term in the right-hand side of (4.25). Since λ_r is a minimizer (see (3.16)),

$$\frac{4\theta_1}{m(B_{r_1})} \int\limits_{B_{r_1}} \left|\nabla(u - \lambda_{r_1}(u)\,|t|)\right|^2 w(t) \le \frac{4\theta_1}{m(B_{r_1})} \int\limits_{B_{r_1}} \left|\nabla(u - \lambda_1(u)\,|t|)\right|^2 w(t).$$

Enlarging the ball, the right-hand side is bounded by

$$\frac{4\theta_1}{r_1^{d+1}} \frac{1}{m(B_1)} \int\limits_{B_1} \left|\nabla(u(x,t) - \lambda_1\,|t|)\right|^2 w(t)\, dx dt.$$

This estimate, together with (4.28) and (4.25), gives

$$\frac{1}{m(B_r)}\int_{B_r} |\nabla(u(x,t)-\lambda_r(u)\,|t|)|^2\, w(t)\,dxdt$$
$$\leq \left(\frac{4\theta_1}{r_1^{d+1}} + C\theta_1 r_1^2 + Cr_1^2\right)\frac{1}{m(B_1)}\int_{B_1} |\nabla(u-\lambda_1(u)\,|t|)|^2\, w(t)\,dxdt$$

for $0 < r \leq r_0$. Now we only need to choose θ_1 and r_1 properly. Let for instance, $\theta_1 = r_1^{d+2}$, and then choose $r_1 = r_1(\theta_0, n, d, \mu_0) \in (0,1)$ sufficiently small so that $4r_1 + Cr_1^{d+4} + Cr_1^2 \leq \theta_0$. Recall that r_0 is determined by θ_1 and r_1, and thus depends only on θ_0, n, d, and μ_0. This completes the proof of the key lemma. $\square$

Ultimately, we want to derive a decay estimate for the *normalized* non-affine part of the local energy of u, i.e. $\beta_r(u)$. So we need to compare the local energy of positive solutions of $\mathcal{L}_0 u = 0$ for different scales.

Lemma 4.29. *Let $u \in W(B_1)$ be a positive solution of $\mathcal{L}_0 u = 0$ in $B_1 \setminus \Gamma$ with $Tu = 0$ on $\Gamma \cap B_1$. Then*

$$E_u(r) \geq C(1 - C' r^2) E_u(1) \qquad \textit{for } 0 < r < \frac{1}{2},$$

where C and C' are positive constants depending only on d, n and μ_0.

Proof. Recall that by Lemma 3.10, u_θ is a solution of the $(d+1)$ dimensional operator L_0, and that by Lemma 3.19, $\lambda_r(u) = \fint_{T_r} \partial_\rho u_\theta(x,\rho)dxd\rho$. So by the boundary regularity of the solutions of constant-coefficient operator L_0 in $\mathbb{R}^{d+1}_+$ (see [7] Lemma 2.10),

$$|\lambda_r(u) - \lambda_s(u)| \leq \operatorname*{osc}_{T_r} \partial_\rho u_\theta \leq Cr\left(\fint_{T_1} |\nabla_{x,\rho} u_\theta(x,\rho)|^2\, dxd\rho\right)^{1/2}$$
$$\leq Cr\left(\frac{1}{m(B_1)}\int_{B_1} |\nabla u(x,t)|^2\, w(t)\,dxdt\right)^{1/2} \tag{4.30}$$

for $0 < s < r < 1/2$. Hence $\lambda_0(u) = \lim_{s\to 0}\lambda_s(u)$ exists, and since we even have a bound on $\operatorname*{osc}_{T_r} \partial_\rho u_\theta$, we see that $\lambda_0(u) = \partial_\rho u_\theta(0,0)$. Since $\mathcal{L}_0\,|t| = 0$, we can apply the comparison principle (Lemma 2.16) to get that

$$\frac{u(x,t)}{|t|} \approx u(x,t_0) \qquad \text{for all } (x,t)\in B_{1/2} \quad \text{and any } t_0 \text{ such that } |t_0| = \frac{1}{2}.$$

So by Lemma 2.15,

$$\frac{u(x,t)}{|t|} \approx \left(\frac{1}{m(B_1)} \int_{B_1} |\nabla u|^2 \, dm \right)^{1/2} \qquad \text{for all } (x,t) \in B_{1/2},$$

which implies that

$$\frac{u_\theta(x,\rho)}{\rho} = \frac{\fint_{S^{n-d-1}} u(x,\rho\omega) d\omega}{\rho} \approx \left(\frac{1}{m(B_1)} \int_{B_1} |\nabla u|^2 \, dm \right)^{1/2} \tag{4.31}$$

for any $(x,\rho) \in T_{1/2}$. Letting $\rho \to 0$, this yields a bound

$$\lambda_0(u) = \partial_\rho u_\theta(0,0) \gtrsim \left(\frac{1}{m(B_1)} \int_{B_1} |\nabla u|^2 \, dm \right)^{1/2}.$$

Combining it with (4.30), we get

$$\begin{aligned} \frac{1}{m(B_r)} \int_{B_r} |\nabla u|^2 \, dm \geq \lambda_r^2(u) &\geq \frac{\lambda_0^2(u)}{2} - (\lambda_r(u) - \lambda_0(u))^2 \\ &\geq \left(C - C' r^2\right) \frac{1}{m(B_1)} \int_{B_1} |\nabla u|^2 \, dm, \end{aligned}$$

as desired. □

5. Extension to a general operator $\mathcal{L}$

5.1. Decay estimates

In this subsection, we shall follow the approach that is used in [7] to obtain a decay estimate for the normalized non-affine part of the energy of solutions of $\mathcal{L}u = 0$. Namely, we shall approximate $\beta_u(r)$ by $\beta_{u_0}(r)$, with u_0 verifying $\mathcal{L}_0 u_0 = 0$, and show that the error is a Carleson measure. Since the strategy is the same as in the $d = n-1$ setting, we shall focus less on motivation but more on technical details that are different from the co-dimension 1 case. For the same reason, many proofs will be omitted if they can be borrowed from [7] without substantial changes.

We start with comparing solutions of $\mathcal{L}u = 0$ and solutions of $\mathcal{L}_0 u_0 = 0$ with the same boundary data. The following two lemmas hold for any matrix $\mathcal{A}_0$ satisfying the ellipticity conditions (1.2). Ultimately, we will apply them to $\mathcal{A}_0 \in \mathfrak{A}_0(\mu_0)$.

Lemma 5.1. *Let $u \in W(B_1)$ be a solution to $\mathcal{L}u = 0$ in $B_1 \setminus \Gamma$ with $Tu = 0$ on $\Gamma \cap B_1$. Let $u^0 \in W(B_1)$ be a solution to $\mathcal{L}_0 u^0 = 0$ in $B_1 \setminus \Gamma$ with $u^0 - u \in W_0(B_1 \setminus \Gamma)$. Then there is a constant $C > 0$ depending only on the ellipticity constant μ_0, d and n, such that*

$$\int_{B_1} \left|\nabla(u-u^0)\right|^2 dm \le \mu_0^2 \min\left\{\int_{B_1} |\mathcal{A}-\mathcal{A}_0|^2 |\nabla u|^2 dm, \int_{B_1} |\mathcal{A}-\mathcal{A}_0|^2 \left|\nabla u^0\right|^2 dm\right\}. \quad (5.2)$$

Proof. First of all, the existence of u^0 is guaranteed by the Lax-Milgram Theorem. Taking $u - u^0 \in W_0(B_1 \setminus \Gamma)$ as a test function in the equation $\mathcal{L}u = 0$, using ellipticity conditions and Young's inequality, we can get

$$\begin{aligned}
\mu_0^{-1} \int_{B_1} \left|\nabla(u-u^0)\right|^2 dm &\le \int_{B_1} A\nabla(u-u^0)\cdot\nabla(u-u^0)dm \\
&= -\int_{B_1} A\nabla u^0 \cdot \nabla(u-u^0)dm = \int_{B_1} (A_0 - A)\nabla u^0 \cdot \nabla(u-u^0)dm \\
&\le \frac{\mu_0}{2}\int_{B_1} |A-A_0|^2 \left|\nabla u^0\right|^2 dm + \frac{1}{2\mu_0}\int_{B_1} \left|\nabla(u-u^0)\right|^2 dm.
\end{aligned}$$

This yields

$$\int_{B_1} \left|\nabla(u-u^0)\right|^2 dm \le \mu_0^2 \int_{B_1} |\mathcal{A}-\mathcal{A}_0|^2 \left|\nabla u^0\right|^2 dm.$$

Interchanging the roles of u and u^0, and $\mathcal{A}$ and $\mathcal{A}_0$, we also obtain the other bound. □

Lemma 5.3. *Let u and u^0 be as in Lemma 5.1. Then*

$$C^{-1} \int_{B_1} \left|\nabla u^0\right|^2 dm \le \int_{B_1} |\nabla u|^2 dm \le C \int_{B_1} \left|\nabla u^0\right|^2 dm,$$

where $C = \mu_0^4$.

The triangle inequality would almost give this directly; the proof (with $C = \mu_0^4$) is the same as when $d = n - 1$ and is thus omitted; see [7], Lemma 3.13.

Define

$$\gamma(x,r) = \inf_{\mathcal{A}_0 \in \mathfrak{A}_0(\mu_0)} \left\{ m(B(x,r))^{-1} \int_{(y,t)\in B(x,r)} |\mathcal{A}(y,t) - \mathcal{A}_0|^2 w(t)\, dydt \right\}^{1/2}. \quad (5.4)$$

Notice that the domain of integration is larger than what we have in (1.8).

Lemma 5.5. *If the matrix-valued function $\mathcal{A}$ satisfies the weak DKP condition of Definition 1.9, with constant $\varepsilon > 0$, then $\gamma(x,r)^2 \frac{dxdr}{r}$ is a Carleson measure on $\mathbb{R}^{d+1}_+$, with the norm*

$$\left\| \gamma(x,r)^2 \frac{dxdr}{r} \right\|_{\mathcal{C}} \leq C\mathfrak{N}(\mathcal{A}) \leq C\varepsilon, \tag{5.6}$$

where $\mathfrak{N}(\mathcal{A}) = \left\|\alpha(x,r)^2 \frac{dxdr}{r}\right\|_{\mathcal{C}}$ is as in (1.8) - (1.10), and

$$\gamma(x,r)^2 \leq C\mathfrak{N}(\mathcal{A}) \leq C\varepsilon \quad \text{for } (x,r) \in \mathbb{R}^{d+1}_+. \tag{5.7}$$

Here, C depends only on d, n, and μ_0.

Proof. This lemma can be proved quite similarly as the $d = n-1$ case. Here, we only mention some modifications and refer the readers to [7], Section 4.1, for details.

We want to show $\gamma(x,r)^2 \frac{dxdr}{r}$ is a Carleson measure on $\mathbb{R}^{d+1}_+$. Let $\Delta_0 = \Delta(x_0, r_0)$ be given. We claim that we can control $\gamma(x_0, r_0)$ in terms of α as in the case of $d = n-1$. That is, we want to show that

$$\gamma(x_0,r_0)^2 \leq C\alpha_2(x_0,r_0)^2 + C\sum_{m\geq 0} \sigma^{\frac{m}{2}} \fint_{\Delta_0'} \alpha_2(y, \sigma^m r_0)^2 dy, \tag{5.8}$$

where $\sigma = \frac{4}{5}$, and $\Delta_0' = \Delta(x_0, 3r_0/2)$. To this end, for each pair (x,r), choose a $\mathcal{A}_{x,r} \in \mathfrak{A}_0(\mu_0)$ such that

$$m(W(x,r))^{-1} \int_{W(x,r)} |\mathcal{A}(y,t) - \mathcal{A}_{x,r}|^2 \, w(t) \, dydt = \alpha(x,r)^2.$$

Let $\mathcal{A}_0 = \mathcal{A}_{x_0,r_0}$. Then

$$\begin{aligned}\gamma(x_0,r_0)^2 &\leq m(B(x_0,r_0))^{-1} \int_{(y,t)\in B(x_0,r_0)} |\mathcal{A}(y,t) - \mathcal{A}_0|^2 w(t) dydt \\ &\leq \frac{1}{m(B(x_0,r_0))} \int_{y\in\Delta_0} \int_{|t|\leq r_0} |\mathcal{A}(y,t) - \mathcal{A}_0|^2 w(t) dtdy.\end{aligned}$$

Let $Q_0 = \{(x,t) : x \in \Delta_0, |t| \leq r_0\}$. As in the case of $d = n-1$, we cut Q_0 into horizontal slices H_m associated to radii $r_m = \sigma^m r_0$, $m \geq 0$. The only difference is that now these slices are annular regions. That is, $H_m = \{(x,t) : x \in \Delta_0, \, r_{m+1} < |t| \leq r_m\}$. Once we have set this up, (5.8) can be obtained by showing that

$$\int_{H_m} |\mathcal{A}(y,t) - \mathcal{A}_0|^2 \, w(t) dydt \leq C r_m \alpha_2(x_0,r_0)^2 |\Delta_0| + C r_m \int_{\Delta_0'} \Big\{ \sum_{j=0}^{m} \alpha_2(y, r_j) \Big\}^2 dy$$

as for $d = n - 1$.

Now (5.6) and (5.7) can be obtained verbatim from the proof in the $d = n - 1$ case, since we have (5.8) and both γ and α are functions on $\mathbb{R}^{d+1}_+$. $\square$

The following estimate on ∇u can be proved similarly to in the case $d = n - 1$. One only needs to replace Carleson balls in $\mathbb{R}^{d+1}_+$ with balls centered on Γ in $\mathbb{R}^n$. One needs to use the reverse Hölder estimate Lemma 2.18, which gives an exponent greater than 2 that depends only on d, n and μ_0. We refer readers to [7], Lemma 3.19, for details of the proof.

Lemma 5.9. *Let $u \in W_r(B_5)$ be a positive solution to $\mathcal{L}u = 0$ in $B_5 \setminus \Gamma$, such that $Tu = 0$ on $\Gamma \cap B_5$. Choose a matrix $\mathcal{A}_0 \in \mathfrak{A}_0(\mu_0)$ that attains the infimum in the definition (5.4) for $\gamma(0,1)$, and let u^0 be the solution from Lemma 5.1 (with this choice of $\mathcal{A}_0$). Then for any $\delta > 0$,*

$$\int_{B_1} \left|\nabla u - \nabla u^0\right|^2 dm \le \left(\delta + C_\delta \gamma(0,1)^2\right) E_u(1), \tag{5.10}$$

where C_δ depends on d, n, μ_0, and δ.

We can now derive the decay estimates for the non-affine part of solutions u. The following is an analogue of Lemma 4.13, and should be compared to Lemma 3.24 in [7] for the case $d = n - 1$.

Lemma 5.11. *Let $u \in W(B_1)$ be a solution to $\mathcal{L}u = 0$ in $B_1 \setminus \Gamma$ with $Tu = 0$ on $\Gamma \cap B_1$. Then there exist constants $p = p(d, n, \mu_0) \in (2, \infty)$, $C = C(d, n, \mu_0) \in (0, \infty)$ such that for any $\theta_0 \in (0, 1)$, there exists $r_0 = r_0(\theta_0, d, n, \mu_0) \in (0, 1/4)$, such that*

$$J_u(r) \le C \left(\theta_0 + K^{\frac{2-p}{2}} r^{-d-1}\right) J_u(1) + \frac{C_K}{r^{d+1}} \gamma(0,1)^2 E_u(1) \tag{5.12}$$

for any $0 < r \le r_0$, and any $K > 0$. Here, C_K depends on K, and d, n, μ_0.

Proof. In what follows, we shall follow rather closely the proof of the $d = n - 1$ case, and refer to [7] for an occasional missing detail. We shall choose a u_0 verifying $\mathcal{L}_0 u_0 = 0$, use the decay estimates for $J_{u_0}(r)$ to get a decay estimate for $J_u(r)$ with an error (5.16). Then using some reverse Hölder estimates, we shall control the error by terms on the right-hand side of (4.14).

We write u as an affine part plus its complement on B_1, i.e.

$$u(x,t) = v(x,t) + \lambda_1(u)\,|t|\,.$$

Notice that $E_v(1) = J_u(1)$ by the definitions near (1.12), and in addition

$$\lambda_1(u)^2 \leq \frac{1}{m(B_1)} \int_{B_1} |\nabla_t u|^2 \, w(t)\, dxdt \leq E_u(1) \tag{5.13}$$

Choose a matrix $\mathcal{A}_0$ in the compact set $\mathfrak{A}_0(\mu_0)$, that attains the infimum in the definition (5.4) of $\gamma(0,1)$, and let $\mathcal{L}_0 = -\operatorname{div}(\mathcal{A}_0 w(t)\nabla)$ as usual.

Now consider the $\mathcal{L}_0$-harmonic extension u_0 of the restriction of u to $\partial(B_{1/2}\setminus\Gamma)$, that is, the unique solution $u_0 \in W(B_{1/2})$ to $\mathcal{L}_0 u_0 = 0$ in $B_{1/2}\setminus\Gamma$, with $u_0 - u \in W_0(B_{1/2}\setminus\Gamma)$. Write

$$u_0(x,t) = v_0(x,t) + \lambda_1(u)\,|t|\,. \tag{5.14}$$

Since $\mathcal{L}_0\,|t| = 0$, $v_0 \in W(B_{1/2})$ verifies

$$\mathcal{L}_0 v_0 = 0 \quad \text{in } B_{1/2}\setminus\Gamma \quad \text{and } v_0 - v \in W_0(B_{1/2}\setminus\Gamma). \tag{5.15}$$

In particular, $Tv_0 = Tv = 0$ on $B_{1/2}\cap\Gamma$.

We claim that for any $\theta_0 \in (0,1)$, there exists $r_0 \in (0,1/4)$ depending on θ_0, d, n and μ_0, and a constant C depending only on d, n, μ_0, such that

$$J_u(r) \leq C\theta_0 J_u(1) + \frac{C(\theta_0 + r^{-d-1})}{m(B_{1/2})} \int_{B_{1/2}} |\mathcal{A} - \mathcal{A}_0|^2\, |\nabla u_0|^2\, dm, \tag{5.16}$$

for any $0 < r \leq r_0$.

To see this, we use the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ to write

$$\begin{aligned} J_u(r) \leq \frac{3}{m(B_r)} \int_{B_r} |\nabla(u_0 - \lambda_r(u_0)\,t)|^2\, dm + \frac{3}{m(B_r)} \int_{B_r} |\nabla(u-u_0)|^2\, dm \\ + \frac{3}{m(B_r)} \int_{B_r} |\nabla(\lambda_r(u_0)\,|t| - \lambda_r(u)\,|t|)|^2\, dm. \end{aligned} \tag{5.17}$$

The last integral can be controlled by the second integral on the right-hand side of (5.17), as follows:

$$\begin{aligned} \frac{1}{m(B_r)} \int_{B_r} |\nabla(\lambda_r(u_0)\,|t| - \lambda_r(u)\,|t|)|^2\, dm = (\lambda_r(u_0) - \lambda_r(u))^2 \\ = \left(\frac{1}{m(B_r)} \int_{B_r} \frac{\nabla_t(u-u_0)\cdot t}{|t|} dm \right)^2 \leq \frac{1}{m(B_r)} \int_{B_r} |\nabla(u-u_0)|^2\, dm. \end{aligned} \tag{5.18}$$

For the second integral on the right-hand side of (5.17), we enlarge B_r and apply Lemma 5.1 to get

$$\frac{1}{m(B_r)}\int_{B_r}|\nabla(u-u_0)|^2\,dm \le \frac{r^{-(d+1)}}{m(B_{1/2})}\int_{B_{1/2}}|\nabla(u-u_0)|^2\,dm$$
$$\le \frac{Cr^{-(d+1)}}{m(B_{1/2})}\int_{B_{1/2}}|\mathcal{A}-\mathcal{A}_0|^2\,|\nabla u_0|^2\,dm. \tag{5.19}$$

Finally, by Lemma 4.13, for any fixed $\theta_0\in(0,1)$, there is some $r_0=r_0(\theta_0,n,d,\mu_0)\in(0,1/4)$ such that the first integral in (5.17) is bounded by $\frac{\theta_0}{3}J_{u_0}(1/2)$. On the other hand, the same sort of computation as above gives

$$J_{u_0}(1/2)\le 3J_u(1/2)+\frac{3}{m(B_{1/2})}\int_{B_{1/2}}|\nabla(u-u_0)|^2\,dm+3(\lambda_{1/2}(u)-\lambda_{1/2}(u_0))^2$$
$$\le 3J_u(1/2)+\frac{C}{m(B_{1/2})}\int_{B_{1/2}}|\mathcal{A}-\mathcal{A}_0|^2\,|\nabla u_0|^2\,dm.$$

Combining this with (5.17), (5.18) and (5.19), we obtain

$$J_u(r)\le\theta_0 J_u(1/2)+\frac{C(\theta_0+r^{-d-1})}{m(B_{1/2})}\int_{B_{1/2}}|\mathcal{A}-\mathcal{A}_0|^2\,|\nabla u_0|^2\,dm,$$

which is almost (5.16). To show (5.16), we only need to observe that by the minimizing property of $\lambda_{1/2}(u)$ (see (3.16)),

$$J_u(1/2)\le m(B_{1/2})^{-1}\int_{B_{1/2}}|\nabla(u(x,t)-\lambda_1(u)\,|t|)|^2\,w(t)\,dxdt\le CJ_u(1).$$

This finishes the proof of (5.16).

Now it suffices to control the second term on the right-hand side of (5.16). We use the decomposition of u_0 as in (5.14), as well as (5.13) to write

$$m(B_{1/2})^{-1}\int_{B_{1/2}}|\mathcal{A}-\mathcal{A}_0|^2\,|\nabla u_0|^2\,dm$$
$$\le\frac{2}{m(B_{1/2})}\int_{B_{1/2}}|\mathcal{A}-\mathcal{A}_0|^2\,|\nabla v_0|^2\,dm+\frac{2\lambda_1(u)^2}{m(B_{1/2})}\int_{B_{1/2}}|\mathcal{A}-\mathcal{A}_0|^2\,|\nabla\,|t||^2\,dm$$
$$\le\frac{2}{m(B_{1/2})}\int_{B_{1/2}}|\mathcal{A}-\mathcal{A}_0|^2\,|\nabla v_0|^2\,dm+2E_u(1)\gamma(0,1)^2. \tag{5.20}$$

We claim that $m(B_{1/2})^{-1} \int_{B_{1/2}} |\mathcal{A} - \mathcal{A}_0|^2 |\nabla v_0|^2 \, dm$ can be estimated as in the $d = n-1$ case as long as one has the following reverse Hölder type estimates.

For some $p = p(d, n, \mu_0) > 2$ sufficiently close to 2,

$$\left(\fint_{B_{1/2}} |\nabla v_0|^p \, dm \right)^{1/p} \lesssim \left(\fint_{B_{1/2}} |\nabla v_0|^2 \, dm \right)^{1/2} + \left(\fint_{B_{1/2}} |\nabla v|^p \, dm \right)^{1/p}, \tag{5.21}$$

and

$$\left(\fint_{B_{1/2}} |\nabla v|^p \, dm \right)^{1/p} \lesssim \left(\fint_{B_1} |\nabla v|^2 \, dm \right)^{1/2} + |\lambda_1(u)| \left(\fint_{B_1} |\mathcal{A} - \mathcal{A}_0|^p \, dm \right)^{1/p}, \tag{5.22}$$

where the implicit constants depend on d, n, μ_0 and p. We postpone the proof of these two inequalities to Section 5.2.

Now fix any $K > 0$. Assuming (5.21) and (5.22), we can control the contribution from the set

$$B_{\frac{1}{2}} \setminus \left\{ X \in B_{\frac{1}{2}} \setminus \Gamma : |\nabla v_0(X)|^2 \le K E_u(1) \right\}$$

to the integral, much as in the case $d = n - 1$, and finally obtain

$$\fint_{B_{1/2}} |\mathcal{A} - \mathcal{A}_0|^2 |\nabla v_0|^2 \, dm \le C K^{\frac{2-p}{2}} J_u(1) + C \left(K + K^{\frac{2-p}{2}} \right) \gamma(0,1)^2 E_u(1).$$

From this and (5.20), the desired estimate (5.12) follows. □

Using Lemma 4.29, Lemma 5.9 and Lemma 5.3, one obtains the following analogue of Lemma 4.29 for positive solutions of $\mathcal{L}u = 0$.

Lemma 5.23. *Let $u \in W_r(B_5)$ be a positive solution of $\mathcal{L}u = 0$ in $B_5 \setminus \Gamma$, with $Tu = 0$ on $\Gamma \cap B_5$. Then for any $\delta > 0$ and $0 < r < 1/2$,*

$$E_u(r) \ge \left(\frac{1 - C' r^2}{C} - \frac{C'' \left(\delta + C_\delta \gamma(0,1)^2 \right)}{r^{d+1}} \right) E_u(1), \tag{5.24}$$

where C, C', C'' are positive constants depending only on d, n and μ_0.

As before, we will only find this useful when the parenthesis is under control.

With Lemma 5.11, Lemma 5.23, and Lemma 5.5 at hand, we are finally ready to prove the decay estimate for $\beta_u(x,r)$, the normalized non-affine energy of solutions of $\mathcal{L}u = 0$.

Let u be as in Lemma 5.23. We first choose a $\theta_0 \in (0,1)$ so that $C\theta_0 < \frac{1}{16C}$ in Lemma 5.11. By Lemma 5.11, this choice of θ_0 gives an $r_0 \in (0,1/4)$ such that (5.12) holds for any $r \le r_0$. Now we choose $r = \tau_0 \le r_0$ so that $C'r^2 < 1/2$ in (5.24). Then we require

$$\gamma(0,1)^2 \le \varepsilon_0, \tag{5.25}$$

and choose ε_0 and $\delta > 0$ sufficiently small (depending on τ_0) so that

$$C''\left(\delta + C_\delta \varepsilon_0\right) \tau_0^{-d-1} < \frac{1}{4C}$$

in (5.24). This way, (5.24) implies that

$$E_u(r) \ge \frac{1}{4C} E_u(1). \tag{5.26}$$

We divide both sides of (5.12) by $E_u(r)$ and get that

$$\beta_u(0,r) \le C\left(\theta_0 + K^{\frac{2-p}{2}} r^{-d-1}\right) \frac{J_u(1)}{E_u(r)} + \frac{C_K}{r^{d+1}} \gamma(0,1)^2 \frac{E_u(1)}{E_u(r)}. \tag{5.27}$$

Then we choose $K > 0$ sufficiently small (depending on τ_0) so that $CK^{\frac{2-p}{2}} \tau_0^{-d-1} < \frac{1}{16C}$. Now assuming (5.25), our choice of θ, ε_0, δ and K guarantees that we can apply (5.26) and deduce from (5.27) that

$$\beta_u(0,\tau_0) \le \frac{1}{2}\beta_u(1) + C_{\tau_0}\gamma(0,1)^2. \tag{5.28}$$

We recapitulate what we obtained in the next corollary. Of course, by translation and dilation invariance, what was done on the unit ball B_1 can also be done for any other $B_R(x)$, $x \in \Gamma$, $R > 0$. We use this opportunity to state the general case, which of course can easily be deduced from the case of B_1 by homogeneity.

Corollary 5.29. *There exist constants $\tau_0 \in (0, 10^{-1})$ and $C > 0$ which depend only on d, n and μ_0, such that if $u \in W_r(B_{5R}(x))$ is a positive solution of $\mathcal{L}u = 0$ in $B_{5R}(x) \setminus \Gamma$, with $Tu = 0$ on $\Gamma \cap B_{5R}(x)$, then*

$$\beta_u(x,\tau_0 R) \le \frac{1}{2}\beta_u(x,R) + C\gamma(x,R)^2. \tag{5.30}$$

Proof. The discussion above gives the result under the additional condition that $\gamma(x,R) \le \varepsilon_0$. But we now have chosen τ_0 and ε_0, and if $\gamma(x,R) > \varepsilon_0$, (5.30) holds trivially (maybe with a larger constant), because $\beta_u(x,\tau_0 R) \le 1$ by (3.18). □

Finally, Theorems 1.16 and 1.18 can be deduced from the decay estimate (5.30) exactly as what was done in the $d = n - 1$ case, as $\beta_u(x, r)$ is a function in $\mathbb{R}^{d+1}_+$ and the goal is to prove a Carleson estimate in $\mathbb{R}^{d+1}_+$. We refer readers to Section 4.2 in [7] for details.

5.2. Proof of the reverse Hölder inequalities

Proof of (5.21)**.** The idea of the proof is essentially from [9], Chapter V. However, we need to treat the boundary estimates more carefully as this time the boundary is of mixed co-dimensions.

Recall that $\mathcal{L}_0 v_0 = 0$ in $B_{1/2} \setminus \Gamma$, with $v_0 - v \in W_0(B_{1/2} \setminus \Gamma)$. Since $v \in W(B_{1/2})$ with $Tv = 0$ on $B_1 \cap \Gamma$, $Tv_0 = 0$ on $B_{1/2} \cap \Gamma$. Let $R_0 = 10^{-2} n^{-1/2}$. Set

$$Q_R(X) := \{Y \in \mathbb{R}^n : |Y_i - X_i| < R \ \text{ for } i = 1, 2, \dots, n\}, \quad R > 0.$$

We claim that there exists $p = p(d, n, \mu_0) > 2$ such that

$$\begin{aligned} &\left(m(Q_{R_0/2}(X_0))^{-1} \int\limits_{Q_{R_0/2}(X_0) \cap B_{1/2}} |\nabla v_0|^p \, dm \right)^{1/p} \\ &\qquad \lesssim \left(m(Q_{R_0}(X_0))^{-1} \int\limits_{Q_{R_0}(X_0) \cap B_{1/2}} |\nabla v_0|^2 \, dm \right)^{1/2} \\ &\qquad\qquad + \left(m(Q_{R_0}(X_0))^{-1} \int\limits_{Q_{R_0}(X_0) \cap B_{1/2}} |\nabla v|^p \, dm \right)^{1/p} \end{aligned} \tag{5.31}$$

for any $Q_{R_0}(X_0) \subset \mathbb{R}^n$ with $Q_{R_0}(X_0) \cap B_{1/2} \neq \emptyset$. Notice that the first integral concerns the cube $Q_{R_0/2}(X_0)$, while the two other ones are on the larger $Q_{R_0}(X_0)$; this will allow the localization argument below. Once this is proved, one can obtain the desired estimate (5.21) by covering $B_{1/2}$ with finitely many cubes $Q_{R_0}(X_0)$.

Fix $Q_{R_0}(X_0)$ with $Q_{R_0/2}(X_0) \cap B_{1/2} \neq \emptyset$. Let $X \in Q_{R_0}(X_0)$ be given, and pick any radius $R < \frac{1}{12} \operatorname{dist}(X, \partial Q_{R_0}(X_0))$. We need to introduce R because we will apply a local result soon.

Let $q := \frac{2n}{n+2}$. There are three possibilities: (1) $Q_{3R}(X) \subset B_{1/2}$, (2) $Q_{3R}(X) \cap B_{1/2} \neq \emptyset$ and $Q_{3R}(X) \cap {B_{1/2}}^{\mathsf{c}} \neq \emptyset$, (3) $Q_{3R}(X) \subset {B_{1/2}}^{\mathsf{c}}$. The last situation is trivial.

If $Q_{3R}(X) \subset B_{1/2}$, then we can apply Lemma 2.21 to get

$$m(Q_R(X))^{-1} \int\limits_{Q_R(X)} |\nabla v_0|^2 \, dm \leq \frac{C}{m(Q_{3R}(X))} \int\limits_{Q_{3R}(X)} |\nabla v_0|^q \, dm. \tag{5.32}$$

We will see later how to continue in this case, but let us first discuss (2). If $Q_{3R}(X) \cap B_{1/2} \neq \emptyset$ and $Q_{3R}(X) \cap {B_{1/2}}^{\mathsf{c}} \neq \emptyset$, choose $\eta \in C_0^\infty(Q_{3R}(X))$ with $\eta = 1$ on $Q_R(X)$ and $|\nabla \eta| \lesssim \frac{1}{R}$. Taking $(v - v_0)\eta^2 \in W_0(B_{1/2} \setminus \Gamma)$ as a test function in $\mathcal{L}_0 v_0 = 0$, and using the ellipticity conditions on $\mathcal{A}_0$, and then the Cauchy-Schwarz inequality, one can get the estimate

$$\int_{Q_R(X)\cap B_{1/2}} |\nabla v_0|^2 \, dm \leq C \int_{Q_{3R}(X)\cap B_{1/2}} |\nabla v|^2 \, dm + \frac{C}{R^2} \int_{Q_{3R}(X)\cap B_{1/2}} |v_0 - v|^2 \, dm. \quad (5.33)$$

We want to control $\int_{Q_{3R}(X)\cap B_{1/2}} |v_0 - v|^2 \, dm$ using the Poincaré inequality. Extend $v_0 - v$ by zero outside $B_{1/2}$ and denote by h the extended function. We need to discuss two cases.

Case 1: $Q_{4R}(X) \cap \Gamma = \emptyset$. Then $\delta(X) \geq 4R$, where $\delta(X) = \operatorname{dist}(X, \Gamma)$ as usual. Since for any $Z \in Q_{3R}(X)$, $\delta(X) - 3R \leq \delta(Z) \leq \delta(X) + 3R$, we have $\frac{1}{4} \leq \frac{\delta(Z)}{\delta(X)} \leq \frac{7}{4}$. This implies that

$$C_{n,d} w(X) \leq w(Z) \leq C_{n,d} w(X) \qquad \text{for } Z \in Q_{3R}(X),$$

and thus

$$\int_{Q_{3R}(X)} |h(Z)|^2 \, w(Z) dZ \leq C_{n,d} w(X) \int_{Q_{3R}(X)} |h(Z)|^2 \, dZ.$$

Since $\partial B_{1/2}$ is smooth, $Q_{3R}(X) \cap {B_{1/2}}^{\mathsf{c}} \neq \emptyset$ implies that $\left|Q_{7R/2}(X) \setminus B_{1/2}\right| \geq \gamma \left|Q_{7R/2}(X)\right|$ for some $\gamma > 0$. Recalling that $h = 0$ in ${B_{1/2}}^{\mathsf{c}}$, we can apply the Sobolev inequality to get

$$\begin{aligned}\int_{Q_{3R}(X)} |h(Z)|^2 \, w(Z) dZ &\leq C w(X) \Big(\int_{Q_{7R/2}(X)} |\nabla h|^q \, dZ \Big)^{\frac{2}{q}} \\ &\leq C w(X)^{-\frac{2}{n}} \Big(\int_{Q_{7R/2}(X)} |\nabla h|^q \, w(Z) dZ \Big)^{\frac{2}{q}}.\end{aligned}$$

Notice that by (2.1), $m(Q_{3R}(X)) \approx m(Q_{7R/2}(X)) \approx R^n w(X)$. Hence,

$$\frac{1}{m(Q_{3R}(X))} \int_{Q_{3R}(X)} h^2 dm \leq C R^2 \Big(m(Q_{7R/2}(X))^{-1} \int_{Q_{7R/2}(X)} |\nabla h|^q \, w(Z) dZ \Big)^{\frac{2}{q}}.$$

Case 2: $Q_{4R}(X) \cap \Gamma \neq \emptyset$. Then there is $x_0 \in \Gamma$ so that $Q_{3R}(X) \subset Q_{7R}(x_0) \subset Q_{11R}(X)$. Enlarging $Q_{3R}(X)$ and applying (2.10), one has

$$\frac{1}{m(Q_{3R}(X))}\int\limits_{Q_{3R}(X)} h^2 dm \le \frac{C}{m(Q_{7R}(x_0))}\int\limits_{Q_{7R}(x_0)} h^2 dm$$
$$\le CR^2\Big(m(Q_{7R}(x_0))^{-1}\int\limits_{Q_{7R}(x_0)} |\nabla h|^q\, dm\Big)^{2/q}$$
$$\le CR^2\Big(m(Q_{11R}(X))^{-1}\int\limits_{Q_{11R}(X)} |\nabla h|^q\, dm\Big)^{2/q}.$$

To summarize, in both Case 1 and Case 2, we have

$$\frac{1}{m(Q_{3R}(X))}\int\limits_{Q_{3R}(X)\cap B_{1/2}} |v_0 - v|^2\, dm$$
$$\le CR^2\Big(\frac{1}{m(Q_{11R}(X))}\int\limits_{Q_{11R}(X)\cap B_{1/2}} |\nabla(v_0 - v)|^q\, dm\Big)^{\frac{2}{q}}. \quad (5.34)$$

Notice that we have chosen $R < \frac{1}{12}\operatorname{dist}(X, \partial Q_{R_0}(X_0))$ to make sure $Q_{11R}(X) \subset Q_{R_0}(X_0)$. Set

$$g(X) = \begin{cases} |\nabla v_0(X)|^q & \text{for } X \in Q_{R_0}(X_0)\cap B_{1/2}, \\ 0 & \text{otherwise,} \end{cases}$$

$$f(X) = \begin{cases} |\nabla v(X)|^q & \text{for } X \in Q_{R_0}(X_0)\cap B_{1/2}, \\ 0 & \text{otherwise.} \end{cases}$$

By (5.34), (5.33) and (5.32), we obtain

$$\frac{1}{m(Q_R)}\int\limits_{Q_R} g^r dm \le \frac{C}{m(Q_{3R})}\int\limits_{Q_{3R}} f^r dm$$
$$+ C\Big(\frac{1}{m(Q_{11R})}\int\limits_{Q_{11R}} g dm\Big)^r + C\Big(\frac{1}{m(Q_{11R})}\int\limits_{Q_{11R}} f dm\Big)^r$$
$$\le \frac{C}{m(Q_{11R})}\int\limits_{Q_{3R}} f^r dm + C\Big(\frac{1}{m(Q_{11R})}\int\limits_{Q_{11R}} g dm\Big)^r,$$

where $r = \frac{n+2}{n}$. As we noted in the proof of Lemma 2.18, we can still apply Proposition 1.1 in Chapter V of [9] when the Lebesgue measure is replaced with the doubling measure m. Then (5.31) follows. $\square$

Proof of (5.22). The proof is similar to that in the $d = n - 1$ case. We present the proof for the sake of completeness.

Set $R_0 = 10^{-2}n^{-1/2}$. For any $X_0 = (x_0, t_0) \in B_{1/2} \setminus \Gamma$ and $0 < R \leq R_0$, choose $\eta \in C_0^\infty(Q_R(X_0))$, with $\eta \equiv 1$ in $Q_{2R/3}(X_0)$, $|\nabla\eta| \lesssim 1/R$. Here, $Q_R(X) = \{Y \in \mathbb{R}^n : |Y_i - X_i| < R \quad i = 1, 2, \ldots, n\}$ as before. We shall write Q_R for $Q_R(X_0)$ when this does not cause a confusion. Since $u \in W(B_{1/2})$ verifies $\mathcal{L}u = 0$ in $B_1 \setminus \Gamma$, we can take any $\varphi \in W_0(B_1 \setminus \Gamma)$ as test function (see (4.4)). Moreover, recall that $v(x,t) = u(x,t) - \lambda|t|$, with $\lambda = \lambda_1(u)$, and that $\mathcal{L}_0|t| = 0$. Therefore, for any $\varphi \in W_0(B_1 \setminus \Gamma)$,

$$
\begin{aligned}
0 = \int_{B_1} \mathcal{A}\nabla u \cdot \nabla\varphi dm &= \int_{B_1} \mathcal{A}\nabla v \cdot \nabla\varphi dm + \int_{B_1} \mathcal{A}\nabla(\lambda|t|) \cdot \nabla\varphi dm \\
&= \int_{B_1} \mathcal{A}\nabla v \cdot \nabla\varphi dm + \int_{B_1} (\mathcal{A} - \mathcal{A}_0)\nabla(\lambda|t|) \cdot \nabla\varphi dm. \quad (5.35)
\end{aligned}
$$

When $|t_0| \leq R$, we choose $\varphi(X) = v(X)\eta^2(X)$; when instead $|t_0| > R$, we take $\varphi(X) = (v(X) - v_{Q_R})\eta^2(X)$, with $v_{Q_R} = m(Q_R)^{-1}\int_{Q_R} v dm$. One can check that in both cases $\varphi \in W_0(B_1 \setminus \Gamma)$. As in the proof of (3.34) in [7], we plug φ into (5.35), compute the derivatives, estimate some terms brutally, and finally use Cauchy-Schwarz inequality, and get the following estimates.

Case 1: $|t_0| \leq R$. In this case, we obtain

$$
\int_{Q_{2R/3}} |\nabla v|^2 dm \leq \frac{C_{\mu_0}}{R^2}\int_{Q_R} v^2 dm + C_{\mu_0}|\lambda|^2 \int_{Q_R} |\mathcal{A} - \mathcal{A}_0|^2 dm.
$$

There is $x_0 \in \Gamma$ such that $Q_R \subset Q_{2R}(x_0) \subset Q_{3R}$. Since $Tv = 0$ on $\Gamma \cap B_1$, we can enlarge Q_R and apply (2.10) to control $\int_{Q_R} v^2 dm$ and deduce from the above that

$$
\begin{aligned}
&\frac{1}{m(Q_{2R/3})}\int_{Q_{2R/3}} |\nabla v|^2 dm \\
&\qquad \leq C\left(\frac{1}{m(Q_{3R})}\int_{Q_{3R}} |\nabla v|^{\frac{2n}{n+2}} dm\right)^{\frac{n+2}{n}} + \frac{C|\lambda|^2}{m(Q_R)}\int_{Q_R} |\mathcal{A} - \mathcal{A}_0|^2 dm. \quad (5.36)
\end{aligned}
$$

Case 2: $|t_0| > R$. The same computation as in Case 1 gives

$$
\int_{Q_{2R/3}} |\nabla v|^2 dm \leq \frac{C}{R^2}\int_{Q_R} |v - v_{Q_R}|^2 dm + C|\lambda|^2 \int_{Q_R} |\mathcal{A} - \mathcal{A}_0|^2 dm.
$$

Then by Lemma 2.7, (5.36) holds.

Now it follows from [9] V, Proposition 1.1 that

$$\frac{1}{m(Q_{R_0/2})}\int\limits_{Q_{R_0/2}} |\nabla v|^p \, dm \le C \left(\frac{1}{m(Q_{R_0})}\int\limits_{Q_{R_0}} |\nabla v|^2 \, dm \right)^{\frac{p}{2}} + \frac{C\,|\lambda|^p}{m(Q_{R_0})}\int\limits_{Q_{R_0}} |\mathcal{A}-\mathcal{A}_0|^p \, dm,$$

for some $p = p(d, n, \mu_0) > 2$, which implies the desired reverse Hölder type estimate since $B_{1/2}$ can be covered by finitely many $Q_{R_0/2}$. $\square$

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