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Approximation of Green functions and domains with uniformly rectifiable boundaries of all dimensions [☆]

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ABSTRACT

The present paper concerns divergence form elliptic and degenerate elliptic operators in a domain $\Omega \subset \mathbb{R}^n$, and establishes the equivalence between the uniform rectifiability of the boundary $E = \partial\Omega$ and weak Carleson condition on the good approximation of the Green function G by affine, or distance, functions. There are two main original contexts for the results, elliptic operators in a non-tangential access domain with an $n-1$ dimensional boundary and degenerate elliptic operators adapted to a domain with an Ahlfors regular boundary of larger co-dimension. In both cases necessary and sufficient conditions are given, in the form of Carleson packing conditions on the collection of balls centered on E where G is not well approximated.

(1) This is the first time the underlying property of the control of the Green function by affine functions, or by the distance to the boundary, in the sense of the Carleson prevalent sets, appears in the literature; some results established here are new even in the half space;

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(2) the results are optimal, providing a full characterization of uniform rectifiability under the (standard) mild topological assumptions;
(3) to the best of the authors' knowledge, even in traditional domains with $(n-1)$-dimensional boundaries, this is the first free boundary result applying to all elliptic operators, without any restriction on the coefficients (the direct one assumes the standard, and necessary, Carleson measure condition);
(4) this is the first free boundary result in higher co-dimensional setting and as such, the first PDE characterization of uniform rectifiability for a set of dimension d, $d < n-1$, in $\mathbb{R}^n$.

The paper offers a general way to deal with related issues considerably beyond the scope of the aforementioned theorem, including the question of approximability of the *gradient* of the Green function, and the comparison of the Green function to a certain version of the distance to the original set rather than distance to the hyperplanes.

Contents

1. Introduction

We consider elliptic operators, or their generalizations, on a domain $\Omega \subset \mathbb{R}^n$. The main goal of the present paper is to show that Ω is reasonably regular (uniformly rectifiable) if and only if the Green function can be well approximated by distances to planes, or by a certain distance to the boundary $\partial\Omega$. Formally speaking, the result is one of the end-points of the big quest of establishing sharp, optimal connections between the geometric and PDE properties of sets – see the work by J. Azzam, J. Garnett, S. Hofmann, P. Le, J.-M. Martell, M. Mourgoglou, K. Nyström, X. Tolsa, T. Toro, Z. Zhao, and their collaborators [2,17,18,21,19,3].

The present result is somewhat different both philosophically and in terms of the involved techniques. The "PDE" side of our result is that for a certain class of elliptic operators, the Green function is morally affine in the sense of Carleson prevalent sets (a weak condition). The idea that the Green function is almost affine whenever the set is reasonably flat permeates virtually all approaches to free boundary problems, including the work of Alt and Caffarelli, Kenig and Toro, Lewis and Vogel, Hofmann, Martell, Tolsa, Garnett, Nystrom, J. Azzam, and others. Even in the precise context of elliptic measure, Azzam [1] has a recent characterization of rectifiability in terms (among other) of Green functions. The story is different here for two reasons. First, we pass from the weak-type condition on the Green function to the strong-type conclusion (uniform rectifiability), and in fact, we prove their equivalence. The reader should not be too surprised that this is possible: since [14,15], it is known that the uniform rectifiability of sets can often be derived from weak conditions. Secondly, we treat more general operators, even in co-dimension 1, because we need weaker conditions on the coefficients. The "free boundary" result, stating that the desired approximability of the Green function by distances to planes implies uniform rectifiability is, to the best of our knowledge, the first free boundary result in this context that pertains to all elliptic operators, without any additional restrictions on the coefficients of the equation. All of this is due to the fact that we employ different techniques, not relying on the integrations by parts traditional in this context.

Finally – and secretly this was one of our main incentives – this paper is the first PDE characterization of the lower-dimensional uniformly rectifiable sets. The efforts in this direction have spanned now about a decade and could roughly be split into two principal directions: the work on reflectionless measures by F. Nazarov, B. Jaye, and their collaborators, which unfortunately still hinges on the problem of proving the reflectionless property for the key operators at hand, and the work by the authors of the present paper and J. Feneuil which identifies an appropriate PDE context and proves a number of "direct" results, but also faces a mysterious problem in the "free boundary" direction. Indeed, we discovered that in domains with lower dimensional boundaries there exist elliptic operators for which the elliptic measure is proportional to the Hausdorff measure on any AR set, independently of any sort of regularity or uniform rectifiability, which shatters the natural conjectures inspired by the co-dimension one case. This is what forced us to turn to the Green function and its proximity to the affine functions, or to the distance: such a condition seems to control more accurately the torsion of the solution around a low-dimensional boundary than the more traditional estimates on the harmonic measure. It appears, for all the reasons outlined above, that in a certain sense it is stronger and more appropriate in the classical domains with $n-1$ dimensional boundary as well. Let us now turn to the details.

In all this paper, Ω will be a domain in $\mathbb{R}^n$ whose boundary $E = \partial\Omega$ is Ahlfors regular of some dimension $d < n$. This means that there is a measure μ, supported on E, and a constant $C \geq 1$ such that

$$C^{-1} r^d \leq \mu(B(x,r)) \leq C r^d \quad \text{for } x \in E \text{ and } 0 < r < +\infty. \tag{1.1}$$

It is well known that in this case μ is equivalent to $\mathcal{H}^d_{|E}$, the restriction to E of the Hausdorff measure of dimension d. In this paper we assume that Ω and E are unbounded;

the case of bounded domains would be similar but need a little more notation. Note that a priori we do not assume that d is integer and allow, for instance, Ahlfors regular snowflakes and Cantor sets. Instead, we will prove that the desired approximation of the Green function (when meaningful) implies that the dimension is integer and the set is rectifiable.

When $d \geq n-1$, we shall systematically add the assumption that Ω is a one-sided NTA domain, which means that we assume some quantitative openness and connectedness in the form of the existence of corkscrew points and Harnack chains in Ω; see the beginning of Section 2 for the definitions. This is a traditional topological background hypothesis in this context. Fortunately, we do not need to impose this explicitly when $d < n-1$, because then Ω satisfies the corkscrew and Harnack chain conditions automatically.

Turning to the operators, we will split the discussion into two cases. When $n-2 < d < n$, we will concentrate on the classical Laplacian Δ, or more generally elliptic operators $L = -\operatorname{div} A\nabla$, with bounded measurable, not necessarily symmetric, coefficients – see (2.4)–(2.5) for the (usual) definition of ellipticity.

When $d \leq n-2$, the classical elliptic operators are not appropriate. Their solutions do not “see” the lower dimensional sets, and for instance, a bounded harmonic function in $\mathbb{R}^n \setminus \mathbb{R}^d$ is indistinguishable from a harmonic function in $\mathbb{R}^n$. Over the recent years, the authors of the present paper, together with J. Feneuil, M. Engelstein, and other collaborators, developed a rather complete elliptic theory on such domains which identifies a certain class of degenerate elliptic operators as a proper substitute to standard elliptic operators in this setting [7–10,13,16]. It was shown that the general results, such as the maximum principle, trace and extension theorems, existence of the harmonic measure and Green function, all hold for the operators

$$L = -\operatorname{div} A \operatorname{dist}(\cdot, E)^{d+1-n}\nabla,$$

where A is the usual elliptic matrix as above and $\operatorname{dist}(\cdot, E)$ is the Euclidean distance to the boundary. Note that for $d = n-1$ these operators coincide with the classical elliptic ones.

It turns out, perhaps surprisingly, that the analogue of the emblematic “Laplacian” case is not simply $L_{euc} = -\operatorname{div} \operatorname{dist}(\cdot, E)^{d+1-n}\nabla$. This is because the Euclidean distance is not always smooth enough for our initial techniques to be applied directly. So one was led to replacing $\operatorname{dist}(\cdot, E)$ with the smoother distance function D_α below, which also turned to be interesting for other reasons [6]. Eventually, it may be that L_{euc} can be controlled in terms of L_α below, using DKP conditions on $\operatorname{dist}(\cdot, E)^{d+1-n}$, but we did not check.

Our favorite degenerate elliptic operator is finally

$$L = L_\alpha = -\operatorname{div} D_\alpha^{d+1-n}\nabla, \tag{1.2}$$

where $\alpha > 0$ is a parameter and D_α is the smoother distance function defined by

$$D_\alpha(X) = D_{\alpha,\mu}(X) = R_{\alpha,\mu}(X)^{-1/\alpha}, \tag{1.3}$$

with

$$R_{\alpha,\mu}(X) = \int_E |X-y|^{-d-\alpha} d\mu(y), \tag{1.4}$$

and where μ is any Ahlfors regular (AR for short) measure on $E = \partial\Omega$, i.e., any measure that satisfies (1.1). This is the best substitute for the Laplacian that we could find; in particular, we have showed that for such an operator the elliptic measure is an A^∞ weight for any domain with a uniformly rectifiable boundary [13,16].

As we have mentioned above, our free boundary results will hold in the general setting of elliptic operators as above. The "direct" results are, by necessity, restricted to operators which are morally similar to the Laplacian or, in the case of lower dimensional boundaries, to (1.2). In either setting, the resulting class of operators is of the nature of the best possible. In fact, it is slightly more general than anything previously considered in this context, but morally speaking we impose the "usual" Carleson measure condition on the coefficients, whose failure is known to produce abundant counterexamples [24,25], see also Remark 3.2. To state this more precisely, we need some definitions.

We shall focus in this paper on so-called weak properties, whose model is always the same. We choose some desirable property (either geometric or related to the Green function), often stated in terms of some parameters like $\varepsilon > 0$, then consider the set of balls $B(x,r)$ for which it fails, and require that this bad set satisfies a Carleson packing condition (we will also say that the complement is a Carleson-prevalent set), as follows.

Definition 1.5. Let $\mathcal{B}$ be a subset of $E \times (0,+\infty)$. We say that $\mathcal{B}$ satisfies a Carleson packing condition when there is a constant $C \geq 0$ such that for every $x \in E$ and $r > 0$,

$$\int_{y\in E\cap B(x,r)} \int_{0<t<r} \mathbb{1}_{\mathcal{B}}(y,t) \frac{d\mu(y)dt}{t} \leq Cr^d. \tag{1.6}$$

We say that $\mathcal{G} \subset E \times (0,+\infty)$ is a Carleson-prevalent set when $E \times (0,+\infty) \setminus \mathcal{G}$ satisfies a Carleson packing condition.

The term prevalent is new in this context (we think it will help to give a name to the good sets too), and has no intended relation with existing uses of prevalence in other domains of mathematics. Here we chose an Ahlfors regular measure μ on E, but if we use a different one, for instance $\mathcal{H}^d_{|E}$, we obtain the same notion (but a different Carleson constant C).

Given an elliptic operator $L = -\operatorname{div} A\nabla$ we say that L is sufficiently close locally to a constant coefficient elliptic operator, if the following weak Carleson measure condition holds. For each choice of constants $\tau > 0$ (small) and $K \geq 1$ (large), denote by $\mathcal{G}_{cc}(\tau, K)$ the set of pairs $(x,r) \in E \times (0,+\infty)$ such that there is a constant matrix $A_0 = A_0(x,r)$

such that

$$\int_{X \in W_K(x,r)} |A(X) - A_0| dX \leq \tau r^n, \tag{1.7}$$

where $W_K(x,r)$ is a large Whitney region associated to $B(x,r)$, defined by

$$W_K(x,r) = \big\{X \in \Omega \cap B(x, Kr)\,;\ \mathrm{dist}(X, E) \geq K^{-1} r\big\}. \tag{1.8}$$

Our condition is that

$$\text{for every choice of } \tau > 0 \text{ and } K \geq 1,\ \mathcal{G}_{cc}(\tau, K) \text{ is a Carleson prevalent set} \tag{1.9}$$

(as in Definition 1.5), or in other terms the corresponding bad set $\mathcal{B}_{cc}(\tau, K) = E \times (0, +\infty) \setminus \mathcal{G}_{cc}(\tau, K)$ satisfies a Carleson packing condition.

Incidentally, we can assume that A_0 is elliptic with the same ellipticity constant as the $A(X)$, because if (1.7) holds for any A_0, it also holds (with a worse constant) with $A_0 = A(X_0)$, for some $X_0 \in W_K$ chosen by Chebyshev.

The reader might perceive the condition of being "sufficiently close locally to a constant coefficient elliptic operator" as too restrictive. This is a little misleading because the constant coefficient operators in questions can change from scale to scale, and in fact our condition is (slightly) weaker than any other one previously used in this context. In particular, it would be easy to check that this condition is weaker than the standard Dahlberg-Kenig-Pipher condition where one requires that A be locally Lipschitz in Ω, with

$$\nabla A(X)\, \mathrm{dist}(X, E) \in L^\infty(\Omega), \tag{1.10}$$

and that $|\nabla A(X)|^2 \delta(X)$ satisfy a Carleson measure condition, i.e., that there be a constant $C_M \geq 0$ such that

$$\int_{\Omega \cap B(x,r)} |\nabla A(X)|^2\, \mathrm{dist}(X, E) dX \leq C_M r^{n-1} \tag{1.11}$$

for $x \in E$ and $0 < r < +\infty$. As we shall see, we can manage with the weaker condition (1.9) because we don't integrate by parts as usual, and also we don't aim for very precise estimates.

Our preferred geometric condition on E, uniform rectifiability, can also be defined using Definition 1.5, via the property called Bilateral Weak Geometric Lemma (BWGL). The BWGL is known to imply apparently stronger rectifiability properties, such as the existence of Big Pieces of Lipschitz Images of $\mathbb{R}^d$ inside of E; we refer to [15] for lots of information about uniform rectifiability, which is not the issue here. To define the BWGL, we will use the following local version of the Hausdorff distance between two sets $E, F \subset \mathbb{R}^n$: we set

$$d_{x,r}(E,F) = \frac{1}{r}\sup\big\{\operatorname{dist}(y,F)\,;\, y \in E\cap B(x,r)\big\} \\ + \frac{1}{r}\sup\big\{\operatorname{dist}(y,E)\,;\, y \in F\cap B(x,r)\big\} \qquad (1.12)$$

for $x \in \mathbb{R}^n$ and $r > 0$; when $F \cap B(x,r) = \emptyset$, we take $\sup\big\{\operatorname{dist}(y,E)\,;\, y \in F\cap B(x,r)\big\} = 0$, and similarly decide that $\sup\big\{\operatorname{dist}(y,F)\,;\, y \in E\cap B(x,r)\big\} = 0$ when $E \cap B(x,r) = \emptyset$, but we will probably not need to use this convention.

Definition 1.13. Let E be an Ahlfors regular set of dimension d in $\mathbb{R}^n$, where $d \in (0,n)$ is an integer. We say that E is uniformly rectifiable when for every $\varepsilon > 0$, the set $\mathcal{G}_{ur}(\varepsilon)$ of pairs $(x,r) \in E \times (0,+\infty)$ such that $d_{x,r}(E,P) \leq \varepsilon r$ for some d-plane $P = P(x,r)$, is Carleson-prevalent.

As always with this type of conditions, we allow the Carleson constant for $\mathcal{B}_{ur}(\varepsilon)$ to depend on ε in any brutal way, but in the case of Definition 1.13 it turns out that we only need a single ε, chosen small enough, depending on n, d, and the AR (Ahlfors regularity) constant for E.

Finally, turning to the properties of the Green function, we shall find it more convenient to state our results in terms of G^∞, the Green function with a pole at infinity. We will review the construction in Section 2 (see also [6,10]). Since our approximation property appears to be new, to motivate the forthcoming definition, let us first describe a situation which we judge perfect, and our conditions will try to measure how far we are from this situation.

In the case of Δ (or a constant coefficient operator), the perfect situation is when $d = n-1$ and E is a d-plane. In this case the harmonic measure ω^∞ (with a pole at ∞) is a multiple of the Lebesgue measure on E, and the Green function G^∞ is a multiple of the distance to E. In the case of our favorite operator $L_\alpha = -\operatorname{div} D_\alpha^{d+1-n}\nabla$ from (1.2), the perfect situation is when d is an integer, E is a d-plane, and μ is a multiple of the Hausdorff measure $\mathcal{H}^d_{|E}$ on E. In this case again the harmonic measure ω_L^∞ is a multiple of $\mathcal{H}^d_{|E}$ and the Green function G^∞ is a multiple of the distance to E, which is also proportional to each D_β, $\beta > 0$.

Definition 1.14. We say that G^∞ is prevalently close to the distance to a d-plane when for each choice of $\varepsilon > 0$ and $M \geq 1$, the set $\mathcal{G}_{Gd}(\varepsilon, M)$ of pairs $(x,r) \in E \times (0,+\infty)$ such that there exists a d-plane $P(x,r)$ and a positive constant $c > 0$, with

$$|\operatorname{dist}(X,P) - cG^\infty(X)| \leq \varepsilon r \text{ for } X \in \Omega \cap B(x,Mr), \qquad (1.15)$$

is Carleson-prevalent.

Similarly, given $\beta > 0$, we say that G^∞ is prevalently close to D_β when for each choice of $\varepsilon > 0$ and $M \geq 1$, the set $\mathcal{G}_{GD_\beta}(\varepsilon, M)$ of pairs $(x,r) \in E \times (0,+\infty)$ such that there exists a positive constant $c > 0$, with

$$|D_\beta - cG^\infty(X)| \leq \varepsilon r \text{ for } X \in \Omega \cap B(x, Mr), \tag{1.16}$$

is Carleson-prevalent.

Here it is convenient to have a constant $c > 0$ that we don't need to compute, especially since our function G^∞ is only defined modulo a multiplicative constant. The definitions are easier to understand when we allow M to be large, but in terms of Carleson-prevalent sets, taking $M = 1$ and compensating with ε would in fact give an equivalent result. The main positive result of this paper is as follows.

Theorem 1.17. *Assume that $\Omega \subset \mathbb{R}^n$ is a domain with an unbounded d-Ahlfors regular boundary $E = \partial\Omega$, d integer. When $d = n-1$, assume, in addition, that Ω is 1-sided NTA. Let $L = -\operatorname{div} A\nabla$ be an elliptic operator with bounded measurable coefficients when $d = n-1$, and more generally, given by*

$$L = -\operatorname{div} A D_{\alpha,\mu}^{d+1-n} \nabla, \quad 0 < d < n,$$

where $D_{\alpha,\mu}$ is the smooth distance (1.3) associated to some $\alpha > 0$ and some AR measure μ on E, and A is any elliptic matrix.

If G^∞ is prevalently close to the distance to a d-plane, in the sense of Definition 1.14, then E is uniformly rectifiable.

Let us assume, in addition, that A is locally close to a constant coefficient operator when $d = n-1$, that is, (1.7)–(1.9) holds, or, when $d < n-1$, that A is locally close to the identity matrix, that is, (1.7)–(1.9) holds with $A_0 = I$.

Then, conversely, G^∞ is prevalently close to the distance to a d-plane whenever E is uniformly rectifiable.

The theorem is a combination of Theorems 3.1, 3.43, and 6.1. As we discussed above, in both directions the conditions on the coefficients are of the nature of the best possible. The reader could perhaps be surprised that in domains with lower dimensional boundaries we require that A is locally close to identity, rather than just a constant coefficient operator. This has to do with the underlying "perfect" situation. Indeed, when $d = n-1$, the distance to a flat boundary is an affine function, which furnishes a solution for any constant coefficient operator. When $d < n-1$, there is a more delicate cancellation at place. Even in $\mathbb{R}^n \setminus \mathbb{R}^d$ the norm of t (the component of X in $\mathbb{R}^{n-d}$), is a solution for $-\operatorname{div} |t|^{d-n+1}\nabla$, but not necessarily for operators with a more general, even if constant, matrix of coefficients. Similarly, even if E is a hyperplane, the distance to the boundary is only a solution for the emblematic operator (1.2). One could maybe reach some more general results further generalizing the concept of a distance, but we chose not to pursue this direction in the present paper.

Notice that when we prove that G is prevalently close to $\operatorname{dist}(X, P)$, for instance, this does not imply that it is Lipschitz, because the constant c in (1.15) may depend on $B(x, r)$. For instance, if E is a Lipschitz domain with a corner at 0, we know that G may behave like $\mathrm{dist}(X,E)^{\alpha}$, $\alpha \neq 1$, near 0. The balls centered at the origin probably lie in the complement of the good set $\mathcal{G}_{Gd}(\varepsilon, M)$, but there are many good balls $B(y,t)$ near 0, where E is flat and (1.15) or (1.16) holds, and for these balls the constants $c = c(y,t)$ will typically tend to 0 or $+\infty$ as (y,t) tends to $(0,0)$.

Going further, let us discuss the "weak" nature of the results. The way we formulate our theorems, via a weak Carleson packing condition, or one could also say, without a precise control of the constants, feels surprising, stronger than expected in the "free boundary" direction. This has to do with a self-improvement of scale invariant estimates which takes place in uniformly rectifiable sets. Indeed, in the context of uniform rectifiability for AR sets, the Bilateral Weak Geometric Lemma of Definition 1.13 is known to be equivalent to stronger and more precise definitions, for instance, the aforementioned existence of big pieces of Lipschitz images uniformly at all scales [15]. Similarly, here when we say that G^{∞} is prevalently close to $\mathrm{dist}(X,P)$, we prove that $\mathcal{G}_{GD}(\varepsilon, M)$ is prevalent for each choice of M, ε, but we do not give the rate of convergence of the Carleson packing constant for $E \times (0,+\infty) \setminus \mathcal{G}_{GD}(\varepsilon, M)$ in terms of ε, cf. [12]. It is worth mentioning that a version of such stronger Carleson measure bounds on the Green function for the Laplacian and more general operators coefficients satisfying the L^1 analogue of the Carleson measure condition (1.11) appeared "between the lines" of [20], that is, could be deduced from their arguments. However, (1.11) and even more so its weak analogue treated in this paper was not handled, even though it is an obvious optimal condition. This is not a technicality: the integration-by-parts arguments from [20] could not be extended to more general operators and in that sense even our "direct" results in traditional sets with $n-1$ dimensional boundaries are the first ones to treat Carleson measure Green function estimates for the optimal class of the operators. Similarly, a recent paper of J. Azzam [1], Theorem VI, did exactly this: characterize uniformly rectifiable boundaries by the regularity of the Green function for the Laplace operator. Our operators are more general (he only treats the Laplacian), his condition on the derivative of the Green function is stronger (more in line with [12,20] than with the present paper), but his domains don't need to be Ahlfors regular, and the techniques are different and independent. We find the two approaches nicely complementing each other, and it is curious that they appeared independently roughly at the same time.

For now we have only discussed approximation of the Green function by the distance to a plane, that is, (1.15), and only in integer dimensions. The second main question in the present paper concerns the approximation by a distance to the initial boundary E, as in (1.16). Despite an apparent similarity to (1.15), it is actually more intricate at least in one direction, for even the fact that G^{∞} is on the spot equal to a multiple of the distance to E, D_{β}, might or might not ensure that the boundary is nice. Even the fact that the dimension is integer has to be proved and, in some circumstances, can fail. For brevity, we will formulate all results in the context of operators which are locally close to the constant coefficient ones, even though some results are still true in the full generality of all elliptic operators – see Section 7. We start with the classical case.

Theorem 1.18. *Assume that $\Omega \subset \mathbb{R}^n$ is a domain with an unbounded d-Ahlfors regular boundary $E = \partial\Omega$, $n-2 < d < n$. When $d \geq n-1$, assume, in addition, that Ω is 1-sided NTA. Let $L = -\operatorname{div} A\nabla$ be an elliptic operator with bounded measurable coefficients which are locally close to constant coefficient matrices in the sense of* (1.7)–(1.9).

If E is uniformly rectifiable, then for every choice of $\beta > 0$ and any AR measure μ on E, G is prevalently close to $D_{\beta,\mu}$.

Conversely, if the Green function G^∞ is prevalently close to $D_{\alpha,\mu}$ for some $\alpha > 0$ and some AR measure μ of dimension d on E then d is integer and E is uniformly rectifiable.

This is a combination of Theorems 3.1, 7.1, and Proposition 7.11, the latter being valid for all elliptic operators.

The situation for domains with lower dimensional boundaries is trickier, due to the aforementioned rather mysterious fact that for a certain very special choice of coefficients the situation could be "perfect" in terms of PDEs without any regularity or flatness. This is one of the main discoveries in [6], and since that paper, we refer to it as a "magic α" case. Indeed, if E is any unbounded d-Ahlfors regular set and $\alpha = n-d-2$, R_α is just the convolution of μ with a multiple of the Laplace's fundamental solution in $\mathbb{R}^n$, so it is harmonic, and a direct computation shows that $L_\alpha D_{\alpha,\mu} = 0$ in Ω for L_α given by (1.2). Hence, by uniqueness, $D_{\alpha,\mu}$ is a multiple of G^∞, and $C^{-1}\mu \leq \omega^\infty \leq C\mu$. In fact, if d is an integer, and E is rectifiable (not necessarily uniformly rectifiable) with $\mu = \mathcal{H}^d_{|E}$, then the harmonic measure ω^∞ with pole at infinity is proportional to μ. In all the other cases, we only have that $C^{-1}\mu \leq \omega^\infty \leq C\mu$, essentially because we cannot say that $\frac{\partial\omega^\infty}{\partial\mu}$ is the normal derivative of G^∞ when E is unrectifiable, but we can argue that getting the measure μ point blank makes less sense in this case, because the density of μ does not exist. Thus, for such a "magic α" the elliptic measure is absolutely continuous with respect to the Hausdorff measure, with a density given by an A^∞ weight, on any Ahlfors regular set. Notice that we need $d < n-2$ for this to happen as $\alpha > 0$, but we do not need d to be an integer, and certainly we do not need E to be uniformly rectifiable. This is a strangely degenerate case which does not resonate with anything we know about the standard domains with $n-1$ dimensional boundaries, but it also turns out to be immensely useful as one basically gets to use the distance D_α as the Green function, getting the best of both worlds: a solution to a PDE and an explicit, easy-to handle formula. This is a rare luxury as typically we do not know explicitly the Green function – see, e.g., [16] for some applications. However, in the context of this paper this case is certainly not amenable to any free boundary results akin to Theorem 1.18. We hope, however, that this is an isolated miraculous cancellation and generally the fact that $G^\infty = CD_{\alpha,\mu}$ implies that the set is flat and μ is a multiple of a flat measure. Not to extend the introduction any further, let us refer the reader to Section 8 for a detailed statement of this condition, which we will denote Υ_{flat}.

Theorem 1.19. *Assume that $\Omega \subset \mathbb{R}^n$ is a domain with an unbounded d-Ahlfors regular boundary $E = \partial\Omega$. When $d \geq n-1$, assume, in addition, that Ω is 1-sided NTA. Let*

$$L = -\operatorname{div} A D_{\alpha,\mu}^{d+1-n} \nabla, \quad 0 < d < n,$$

where $D_{\alpha,\mu}$ is the smooth distance (1.3) associated to some $\alpha > 0$ and some AR measure μ on E, and A is any elliptic matrix with bounded measurable coefficients which are locally close to the identity in the sense that (1.7)–(1.9) holds with $A_0 = I$.

If E is uniformly rectifiable, then for every choice of $\beta > 0$ and any AR measure ν on E (possibly different from μ), G is prevalently close to $D_{\beta,\nu}$.

Conversely, if the condition $\Upsilon_{\rm flat}(d, \alpha, \Delta)$ of Definition 8.1 holds and the Green function G^∞ is prevalently close to $D_{\alpha,\mu}$ then d is integer and E is uniformly rectifiable.

This is a combination of Theorems 3.43 and Theorem 8.8. One can find more related results and an extended discussion of the condition $\Upsilon_{\rm flat}$ in Section 8. For now, let us wrap up the introduction and send an interested reader to the body of the paper.

The authors wish to thank the referee for a careful reading and many useful suggestions.

2. The Green function at infinity and the basic result about limits

We first remind the reader of how the Green function G^∞ is constructed. We have two main cases in mind, which we rapidly describe now. Let us start with the geometric assumptions.

Throughout the paper, Ω is a domain in $\mathbb{R}^n$ such that $E = \partial\Omega$ is an unbounded Ahlfors regular set of dimension $d \in (0, n)$, so that (1.1) is satisfied. There are natural counterparts of our results for bounded domains, but for simplicity of notation we will concentrate on the unbounded case.

When $d \geq n - 1$, we also demand that Ω is a uniform (*aka* a one-sided NTA) domain, i.e., that

$$\Omega \text{ has interior corkscrew points and Harnack chains.} \tag{2.1}$$

Let us say what this means. First, we require the existence of corkscrew points: for $x \in E$ and $r > 0$, we demand the existence of a point $A = A_{x,r}$ such that

$$A \in \Omega \cap B(x, r) \ \text{ and } \ \operatorname{dist}(A, E) \geq C^{-1} r \tag{2.2}$$

(we shall call A a corkscrew point for $B(x, r)$).

Secondly, we require that for each $M \geq 1$, we can find an integer $N = N(M)$ such that when $X, Y \in \Omega$ are two points with the property that $|X - Y| \leq M \min(\operatorname{dist}(X, E), \operatorname{dist}(Y, E))$, we can find a (Harnack) chain of balls $B_1, \ldots, B_N$ such that B_1 contains X, B_N contains Y, each B_i, $2 \leq i \leq N$, meets B_{i-1}, and the $2B_i$ are all contained in Ω. Let us not play useless games here; we also require that all the B_i of the chain have a radius at least $C^{-1} \min(\operatorname{dist}(X, E), \operatorname{dist}(Y, E))$.

Another way to formulate the Harnack chain condition would be to require the existence of a path in Ω that goes from X to Y, stays at distance at least $C^{-1}\min(\operatorname{dist}(X,E), \operatorname{dist}(Y,E))$ from E, and has a length at most $C|X-Y|$.

We shall refer to C in (1.1) as an AR constant of E, and to C and M is the corkscrew and Harnack chain conditions above as 1-sided NTA constants.

A uniform domain with an Ahlfors regular boundary is sometimes referred to as a 1-sided chord-arc domain, but we will rarely use this terminology.

The interior corkscrew and Harnack chain conditions are merely quantitative openness and path connectedness of the domain Ω. They guarantee a reasonable behavior of the Green function at the level of fundamental estimates and so we prefer to assume that the domains are uniform when $d \geq n-1$ throughout the paper. When $d < n-1$, it turns out that interior corkscrew and Harnack chain conditions are automatically satisfied for any domain with an Ahlfors regular boundary $E = \partial\Omega$ of dimension d, essentially because E is so thin. This is checked in [8].

Let us turn to the elliptic operators considered in the present paper. We split into two cases, essentially corresponding to the dimension of the boundary of the set, although they are not completely exclusive.

In **Case 1**, which we will also call the classical case, we are given a domain Ω with $E = \partial\Omega$, an unbounded Ahlfors regular set E of dimension d, with $n-2 < d < n$. When $d \geq n-1$, we assume, in addition, that the domain is uniform, i.e., the interior corkscrew and Harnack chain conditions are satisfied. We consider an elliptic divergence form operator

$$L = -\operatorname{div} A\nabla, \tag{2.3}$$

where the matrix A of coefficients is measurable and satisfies the usual ellipticity properties

$$|\langle A(X)\xi,\zeta\rangle| \leq C_e|\xi||\eta| \text{ for } X \in \Omega,\ \xi \in \mathbb{R}^n, \text{ and } \zeta \in \mathbb{R}^n, \tag{2.4}$$

$$\langle A(X)\xi,\xi\rangle \geq C_e^{-1}|\xi|^2 \text{ for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{2.5}$$

and some constant $C_e \geq 1$.

In **Case 2**, which is not exclusive of Case 1, Ω is still a domain in $\mathbb{R}^n$ such that $E = \partial\Omega$ is an unbounded Ahlfors regular of dimension d, but now we allow any dimension $d \in (0,n)$. When $d \geq n-1$, we require that Ω is uniform, i.e., that (2.1) is satisfied, as before. And now we consider our favorite operator

$$L = L_\alpha = -\operatorname{div} D_{\alpha,\mu}^{d+1-n}\nabla \tag{2.6}$$

of (1.1), where $\alpha > 0$ and $D_{\alpha,\mu}$ is given by (1.3) and (1.4), for some AR measure μ on E, or its generalization

$$L = -\operatorname{div} A\, D_{\alpha,\mu}^{d+1-n}\nabla, \tag{2.7}$$

where A satisfies (2.4)–(2.5). Clearly, at such a level of generality we could as well say $L = -\operatorname{div} A \operatorname{dist}^{d+1-n}(\cdot, E)\nabla$, but writing L as above will occasionally be somewhat more convenient. Clearly, so far Case 1 and Case 2 coincide when $d = n-1$ and in fact, we could further generalize Case 2 to cover Case 1 completely, but we prefer to keep them separate for reasons that will become evident a little later: the additional assumptions down the road will start to deviate.

Let us review the definition of the Green function with a pole at ∞, and then state the main result about limits that will be behind all our proofs by compactness.

First recall that in both cases, we can associate to Ω and L a collection of probability measures $\omega^X = \omega_L^X$, referred to as the harmonic (or elliptic) measure with pole at $X \in \Omega$, and that have standard properties (such as doubling) that will be recalled when we need them. There are quite a few papers listing these fundamental estimates in Case 1, at least when $d = n-1$ (and perhaps more generally). Essentially, one can say that the program outlined in [22] for 2-sided NTA domains still applies. In Case 2, this is done in [8] when $d < n-1$. Finally, both cases for all relevant dimensions are covered in [10]. In Section 3 of [10] it is explained how the present assumptions are covered by the hypotheses of [10]. The setting of [10] is much more general than what we need here, but it is convenient to have all the relevant references in one place and so we will mainly use [10] for fundamental elliptic theory: Caccioppoli, De Giorgi-Nash-Moser, definition of elliptic measure, Green functions, etc. In particular, the reader may find in the same source a construction of the Green function G^X for L, at the pole $X \in \Omega$. Again these functions enjoy the usual properties of Green functions in NTA domains, and we will recall these properties (and those of solutions of $Lu = 0$) when we need them.

Next we say a few words about the Green function G^∞ with pole at ∞, which we construct as a limit of functions G^X. We will also use the opportunity to recall some notation and estimates that will be used later.

First of all, we define a weight w on Ω by

$$w(X) = 1 \text{ in Case 1 and } w(X) = \operatorname{dist}(X, E)^{d+1-n} \text{ in Case 2.} \tag{2.8}$$

Then we let $W = W(\Omega)$ denote the Hilbert space of functions $u \in L^1_{loc}(\Omega)$ whose derivative (in the sense of distributions) lies in $L^2(\Omega, w(X)dX)$; thus

$$||u||_W^2 = \int_\Omega |\nabla u(X)|^2 w(X) dX \tag{2.9}$$

is finite. We also need local versions $W_r(B)$ of W, which are defined in Section 8 of [8] or Section 10 of [10], as follows. Given an open set $H' \subset \mathbb{R}^n$ and $H := H' \cap \overline{\Omega}$, we let

$$W_r(H) := \{u \in L^1_{loc}(H \cap \Omega, m) : \varphi u \in W \text{ for all } \varphi \in C_0^\infty(H')\}.$$

It is natural to call this space $W_r(H)$, as opposed to $W_r(H')$, because it does not depend on the part of H' that lives away from $\overline{\Omega}$. It does or does not carry information about the behavior of u near the boundary depending on whether H is properly contained in the domain Ω. Clearly, $\nabla u \in L^2(B, w(X)dX)$ when $u \in W_r(B)$. The Green function G^X with pole at $X \in \Omega$ lies in $W_r(B)$ for any ball $B \subset \mathbb{R}^n$ such that $X \notin 2B$ (see Lemma 10.2 in [8] or Lemma 14.60 in [10]).

We say that the function u is a weak solution to $Lu = -\operatorname{div} A\nabla u = 0$ in Ω when it lies in all the local spaces $W_r(B)$ and for every $\varphi \in C_0^\infty(\Omega)$

$$\int_{\Omega} A\nabla u \cdot \nabla \varphi = 0. \tag{2.10}$$

Here, A can be as in (2.4)–(2.5) in Case 1 or (2.6)–(2.7) in Case 2.

We are now ready to describe how we construct the Green function G^∞ with pole at ∞. Fix any ball $B_0 = B(x_0, r_0)$ centered on E, choose a corkscrew point A_0 for B_0 (see (2.2)), take any sequence $\{X_k\}$ in $\Omega \setminus B_0$ such that $\lim_{k\to+\infty} |X_k| = +\infty$, and consider the functions

$$g_k(Y) = \frac{G^{X_k}(Y)}{G^{X_k}(A_0)}. \tag{2.11}$$

Notice that g_k is nonnegative, L-harmonic on $\Omega \cap B(x_0, |X_k - x_0|/2)$, and $g_k \in W_r(B)$ for every ball B with constants that are uniform in k, as soon as k is so large that $X_k \notin 2B$. It also has a vanishing trace at the boundary, by definition of any G^X.

By Harnack's principle and the normalization $g_k(A_0) = 1$, we see that for each compact set $K \subset \Omega$, there is a constant $C_K \geq 1$ such that

$$C_K^{-1} \leq g_k(Y) \leq C_K \text{ for } Y \in K, \tag{2.12}$$

as soon as k is large enough (depending on K), so that X_k no longer lies in the union of a (finite) collection of Harnack chains that connect A_0 to any point of K.

Similar estimates hold near E, i.e., on $\Omega \cap B$, where B is any ball centered on $x_0 \in E$. Indeed, the functions g_k are Hölder continuous at the boundary. With this in mind, let us extend g_k by zero to the rest of $\mathbb{R}^n$. We use Lemma 11.50 in [8] or Lemma 15.14 in [10], which says that $g_k(Y) \leq C g_k(A^B)$ for $Y \in B$ (and k large enough), where A^B is a corkscrew point for B. But $g_k(A^B)$ is controlled by (2.12), so we get that for each large ball B centered at x_0, there is a constant C_B such that for k large,

$$g_k(Y) \leq C_B \text{ for } Y \in B. \tag{2.13}$$

Then we can use the Caccioppoli estimate at the boundary (Lemma 8.47 in [8] or Lemma 11.15 in [10]), and get that for B as above (and k large)

$$\int_{B\cap\Omega} |\nabla g_k|^2 w(X)dX \leq C(B) \int_{CB\cap\Omega} |g_k|^2 w(X)dX \leq C(B). \tag{2.14}$$

Here, as usual the fact that g_k vanishes on E and $g_k \in W_r(B)$ for any given B (and for k large) was used to check the assumptions. Finally, we can also use the Hölder continuity at the boundary (Lemma 8.106 in [8] or Theorem 1.6 in [10]) to show that

$$g_k \text{ is Hölder continuous on } B\text{, with exponent } \beta \text{ and constant } C(B). \tag{2.15}$$

Of course the important point for the moment is that none of the constants β, C_B, and $C(B)$ depends on k, even though they are only valid for k large, depending on B.

Return to the g_k. Because of (2.15) it is easy to extract a subsequence so that $\{g_k\}$ converges uniformly on compact subsets of $\mathbb{R}^n$ to a limit G^∞. In fact, using again the Caccioppoli estimate at the boundary and passing to a subsequence, we also get that for each ball B, ∇g_k converges weakly in $L^2(\Omega \cap B, w)$ to ∇G^∞, and then that G^∞ is a (weak) solution of L (we will showcase the details in similar, but more delicate, arguments soon). Since g_k is Hölder continuous and vanishing on the boundary, G^∞ also vanishes at the boundary. And finally $G^\infty \in W_r(B)$ for every B (using the definition of W_r, weak convergence, and the estimates above).

Now, G^∞ is also unique, modulo a multiplicative constant. That is, if G is positive on Ω, lies in all the spaces $W_r(B)$, it is L-harmonic on Ω, and has a vanishing trace on E (the latter is well defined when $G \in W_r(B)$), then $G = cG^\infty$ for some $c > 0$. This follows from the comparison principle (Theorem 11.146 in [8] or Theorem 1.16 in [10]), plus some algebraic manipulations on the oscillation of G/G^∞ (similar to what one does for the Hölder continuity at the boundary) that the reader may find in [6], Corollary 6.4 and Lemma 6.5. This procedure coherently and uniquely defines G^∞ (modulo a multiplicative constant), which we will refer to as the Green function with the pole at infinity.

We are ready for our main result about limits. We assume that we have a sequence of open sets Ω_k, bounded by Ahlfors regular sets E_k, and operators L_k, that all satisfy the assumptions of this paper (for either Case 1 or Case 2), with uniform estimates.

We assume (for convenience) that all the E_k contain the origin, and that $\{E_k\}$ converges to a closed set E_∞ and $\{\Omega_k\}$ converges to an open set Ω_∞, in the sense that with the notation of (1.12),

$$\lim_{k\to+\infty} d_{x,r}(E_\infty, E_k) = 0 \quad \text{and} \quad \lim_{k\to+\infty} d_{x,r}(\Omega_\infty, \Omega_k) = 0 \tag{2.16}$$

for every choice of $x \in \mathbb{R}^n$ and $r > 0$ (we may also restrict to $x \in E_\infty$; this would be equivalent).

We shall check soon that E_∞ is the boundary of Ω_∞, E_∞ is Ahlfors regular of dimension d, and Ω_∞ satisfies our one-sided NTA condition (2.1), but let us continue our description.

We also need the operators $L_k = -\operatorname{div} A_k \nabla$ to converge to a limit $L_\infty = -\operatorname{div} A_\infty \nabla$ (and similarly for the operators of Case 2), and we require that

$$\lim_{k\to+\infty} ||A_k - A_\infty||_{L^1(B)} = 0 \ \text{ for every ball } B \text{ such that } 2B \subset \Omega_\infty. \tag{2.17}$$

Our proof would be a little simpler under the stronger local L^∞ convergence of coefficients, namely when

$$\lim_{k\to+\infty} ||A_k - A_\infty||_{L^\infty(B)} = 0 \text{ for every ball } B \text{ such that } 2B \subset \Omega_\infty, \tag{2.18}$$

which would probably be more reasonable if we wanted better quantitative results, but the fact that we are only interested in weak results allows us to use the weaker condition (2.17).

Finally we choose a corkscrew point A_0 (relative to Ω_∞) for some ball B_0 centered on E_∞, which will be used to normalize the Green functions.

The next Theorem shows that with the assumptions above, the functions G_k^∞ associated to the Ω_k and the L_k and normalized by $G_k^\infty(A_0) = 1$ converge to the Green functions G_∞^∞ associated to Ω_∞ and L_∞ and normalized by $G_\infty^\infty(A_0) = 1$.

Theorem 2.19. *Let the Ω_k be domains in $\mathbb{R}^n$ with unbounded d-dimensional Ahlfors regular boundaries $E_k = \partial\Omega_k$, $d \in (0, n)$, corresponding to AR measures μ_k, and, in addition, satisfying the interior Harnack chain and corkscrew conditions (2.1) when $d \geq n-1$, with all AR and 1-sided NTA constants uniform in k. Assume that the origin belongs to all the E_k, that the domains E_k converge to a closed set E_∞, and that the Ω_k converge to an open set Ω_∞ in the sense of (1.2). Then E_∞ is the boundary of Ω_∞, E_∞ is Ahlfors regular of dimension d, and Ω_∞ is uniform, i.e., satisfies (2.1). If μ_k is an AR measure on E_k (with uniform bounds) and μ is any weak-* limit of the μ_k, then μ is an AR measure on E_∞.*

Assume furthermore that either

$$L_k = -\operatorname{div} A_k\nabla, \quad L_\infty = -\operatorname{div} A_\infty\nabla, \quad n-2 < d < n, \tag{2.20}$$

or

$$L_k = -\operatorname{div} A_k D_{\alpha,\mu_k}^{d+1-n}\nabla, \quad L_\infty = -\operatorname{div} A_\infty D_{\alpha,\mu}^{d+1-n}\nabla, \quad 0 < d < n \tag{2.21}$$

(with $\mu_k \rightharpoonup \mu$ as above), in both cases subject to (2.17).

Then the functions G_k^∞ associated to the Ω_k and the L_k and normalized by $G_k^\infty(A_0) = 1$ converge, uniformly on every compact subset of $\mathbb{R}^n$, and in $W_{loc}^{1,2}(\Omega_\infty)$, to the Green functions G_∞^∞ associated to Ω_∞ and L_∞ and normalized by $G_\infty^\infty(A_0) = 1$.

Here and everywhere, we say that the sequence of Radon measures μ_k on $\mathbb{R}^n$ weak-* converges to some Radon measure μ, and write $\mu_k \rightharpoonup \mu$, if

$$\int f\, d\mu_k \to \int f\, d\mu \quad \text{for any} \quad f \in C_c(\mathbb{R}^n).$$

Notice that there always exist weak-* limits μ, as in the statement.

As usual, (2.20) and (2.21) will be referred to as Case 1 and Case 2.

Remark 2.22. Formally speaking, the Green functions are only defined in their corresponding domains, but using their Hölder continuity inside and at the boundary, we silently extend them by zero to the entire $\mathbb{R}^n$, and hence we can justifiably talk about the uniform convergence on compacta in $\mathbb{R}^n$.

Remark 2.23. The same statement also holds if we replace the Green functions G_k^∞ with Green functions $G_k = c_k G_k^{Y_k}$ computed at poles Y_k and normalized so that $G_k(A_0) = 1$, if we also assume that $\lim_{k\to+\infty} |Y_k| = +\infty$. We will track this case too along the proof.

Proof of Theorem 2.19. Let us first check that $(\Omega_\infty, E_\infty)$ satisfies our desired geometric properties. We shall only highlight some elements of the proof, as the reader can consult [19] for the details in the case of $d = n-1$ which transfer virtually verbatim to our more general situation.

The fact that $\partial\Omega_\infty = E_\infty$ will be easy. Maybe we should notice first that (2.16) also implies that $d_{x,r}(\mathbb{R}^n \setminus \overline{\Omega}_k, \mathbb{R}^n \setminus \overline{\Omega}_\infty)$ also tends to 0 for all pairs (x, r): for instance, if $z \in \mathbb{R}^n \setminus \overline{\Omega}_k$, it cannot be far from $\mathbb{R}^n \setminus \overline{\Omega}_\infty$ because otherwise it lies in the middle of Ω_∞, hence also of Ω_k). Next, if $z \in \partial\Omega_\infty$, then for each $\varepsilon > 0$ the ball $B(z, \varepsilon)$ meets both Ω_∞ and $\mathbb{R}^n \setminus \Omega_\infty$. Then for k large, $B(z, \varepsilon)$ also meets Ω_k and $\mathbb{R}^n \setminus \Omega_k$, hence also E_k. Now $\{E_k\}$ converges to E_∞, and E_∞ is closed; it follows that $z \in E_\infty$. Finally, let $z \in E_\infty$ be given; for each $\varepsilon > 0$ we can find points $z_k \in E_k \cap B(z, \varepsilon)$ for k large, hence also points $x_k \in \Omega_k \cap B(z, \varepsilon)$ and $y_k \in B(z, \varepsilon) \setminus \overline{\Omega}_k$, and by the extension of (2.16), points $x_k' \in \Omega_\infty \cap B(z, 2\varepsilon)$ and $y_k' \in B(z, 2\varepsilon) \setminus \overline{\Omega}_\infty$. So $z \in \partial\Omega_\infty$ and $\partial\Omega_\infty = E_\infty$.

From here, we can show that E_∞ supports a measure which is a weak limit of μ_k and which is Ahlfors regular too. Moreover, (2.1) also holds for Ω_∞. For instance, if $X, Y \in \Omega_\infty$ are given, then for k large we can find $X_k, Y_k \in \Omega_k$, as close as we want to X and Y, and a Harnack chain for X_k and Y_k in Ω_k will also work, with very minor modifications, for X and Y in Ω_∞. Again, the reader can consult [19] for more details. For us, the main point of this verification was that we can apply the results of [22,8,10] to L_∞ and Ω_∞. In particular, we can talk about G_∞^∞ and use the above uniqueness result for G_∞^∞.

We are ready for the PDE part of the argument. Set $G_k = G_k^\infty$ for all k to save notation. We can run the same limiting argument as the one which allowed us to define G^∞ on a given domain. Indeed, (2.13) still holds in any ball $B \subset \mathbb{R}^n$ (not necessarily contained in Ω_∞) with a constant uniform in k as long as G_k is extended by zero to the complement of Ω_k. (If we take a limit of $G_k = c_k G_k^{Y_k}$ as in Remark 2.23 instead, we also have to make sure that k is large enough for Y_k to stay away from B, but the constants are still uniform in k). Furthermore, the analogue of (2.14) holds:

$$\int_{B\cap\Omega_k} |\nabla G_k|^2 w_k(X)dX \leq C(B) \int_{CB\cap\Omega_k} |G_k|^2 w_k(X)dX \leq C(B), \tag{2.24}$$

with uniform constants, which again yields (2.15). Here, w_k is as in (2.8). Thus, much as before, we can extract a subsequence which converges uniformly on compacta in $\mathbb{R}^n$ to some G_∞, continuous and equal to zero on the complement of Ω_∞, and such that ∇G_k converges weakly to ∇G_∞ in $L^2_{loc}(\Omega_\infty)$ (notice that the weights are irrelevant when we stay away from the boundary). Eventually, we will prove the strong convergence of G_k in $W^{1,2}_{loc}(\Omega_\infty)$, but for now let us continue.

The function G_∞ is in $W_r(B)$ for any ball $B \subset \mathbb{R}^n$. Indeed, it is sufficient to consider B centered on E_∞. For a single G_k, (2.24) holds with uniform in k constants. Then we can look at the contribution to (2.24) of any fixed compact subset H of $\Omega_\infty \cap B$, let k tend to $+\infty$ in (2.24), and get that

$$\int_H |\nabla G_\infty|^2 w_\infty(X) dX \leq C(B), \tag{2.25}$$

for instance by Fatou, or the weak convergence in $W^{1,2}_{loc}(\Omega_\infty)$. Then we use the fact that $C(B)$ does not depend on H, take a supremum, and get that

$$\int_{\Omega_\infty \cap B} |\nabla G_\infty|^2 w_\infty(X) dX \leq C(B). \tag{2.26}$$

This (together with the local Hölder continuity of G and the local integrability of the weight w_∞) is actually stronger than the fact that $G \in W_r(B)$ for every ball, which requires that φG lies in the Hilbert space W of (2.9) for every $\varphi \in C_0^\infty$ (see the beginning of Section 8 in [8]).

We also need to show that G is a weak solution of $L_\infty G = 0$. In Case 1, we write $\int_\Omega A_\infty \nabla G_\infty \cdot \nabla\varphi$ as

$$\int A_\infty(\nabla G_\infty - \nabla G_k) \cdot \nabla\varphi + \int (A_\infty - A_k)\nabla G_k \cdot \nabla\varphi + \int A_k \nabla G_k \cdot \nabla\varphi \tag{2.27}$$

for every $\varphi \in C_0^\infty(\Omega_\infty)$ and any k such that Ω_k contains the support of φ. The first integral converges to zero by the weak convergence of ∇G_k in $L^2_{loc}(\Omega_\infty)$, the second one is bounded by $\sup |\nabla\varphi| \int_{\operatorname{supp}\varphi} |A_\infty - A_k||\nabla G_k|$, and hence, it converges to zero by (2.17) and (2.24), and the last one is zero because G_k is a solution to L_k. Hence, G_∞ is a solution to L_∞, as desired.

A similar argument also handles Case 2. Indeed, we can write (2.27) with $A_\infty D^{d+1-n}_{\alpha,\mu}$ in place of A_∞ and $A_k D^{d+1-n}_{\alpha,\mu_k}$ in place of A_k. The weights are harmless since we are away from the boundary, we only have to show that $A_\infty D^{d+1-n}_{\alpha,\mu_k}$ converge to $A_k D^{d+1-n}_{\alpha,\mu}$ to handle the analogue of the second term on the right-hand side of (2.27). However, D^{d+1-n}_{α,μ_k} converge to $D^{d+1-n}_{\alpha,\mu}$ uniformly on any compact set in Ω_∞. Indeed, on any set away from the boundary it is sufficient to show that R_{α,μ_k} converge to $R_{\alpha,\mu}$ (see (1.4)) which follows from the weak-* convergence of μ_k to μ and the Ahlfors regularity of the measures μ_k and μ (the latter allowing us to restrict attention to a compactly supported approximation of $|X-y|^{-d-\alpha}$). Hence, writing

$$A_\infty D_{\alpha,\mu}^{d+1-n} - A_k D_{\alpha,\mu_k}^{d+1-n} = (A_\infty - A_k) D_{\alpha,\mu_k}^{d+1-n} + A_\infty (D_{\alpha,\mu}^{d+1-n} - D_{\alpha,\mu_k}^{d+1-n}), \qquad (2.28)$$

we see that (2.17) holds with $A_\infty D_{\alpha,\mu}^{d+1-n}$ in place of A_∞ and $A_k D_{\alpha,\mu_k}^{d+1-n}$ in place of A_k.

So G is a solution for L_∞ that vanishes on E_∞, and it lies in the correct spaces $W_r(B)$. Since Ω_∞ and A_∞ satisfy the same assumptions as the Ω_k and A_k, we can apply the result of uniqueness for the Green function, and we find out that G is a Green function for L_∞. The normalization $G(A_0)$ is correct too, since $G_k(A_0) = 1$ by construction.

We are almost finished now. We started from a sequence $\{G_k\} = \{G_k^\infty\}$, then extracted a subsequence so that, in particular, G_k^∞ converges to some limit G, and then proved that G is the desired Green function. The same thing would happen if we started from any other subsequence, and we would always get the same limit. Since we can always extract convergent subsequences, this means that the limit existed already for our initial sequence, and is the desired Green function.

It remains only to show that G_k converges (strongly) in $W_{loc}^{1,2}(\Omega_\infty)$.

Lemma 2.29. *Keep the notations and assumptions of Theorem 2.19. Then for each compact subset K of Ω_∞,*

$$\lim_{k\to+\infty} \int_K |\nabla G_k - \nabla G_\infty|^2 = 0. \qquad (2.30)$$

We do not need to worry about putting our weight $w(X)$ in the integral or not, because it is bounded from above and below on any compact subset of Ω. Similarly, as long as we stay in a compact subset K of Ω_∞, our degenerate elliptic operators coincide on K with multiples of a standard elliptic operator, with bounds on the ellipticity that depend only on K and coefficients that in any case satisfy (2.17) – see the discussion near (2.28). That is, Case 1 and Case 2 are identical as far as the estimates strictly inside the domain are concerned, and we will treat them as such.

Proof of Lemma 2.29. Of course it is enough to prove (2.30) when K is the closure of a ball $B = B(x,r)$ such that $4B \subset \Omega_\infty$. The functions G_k and G are bounded in $W^{1,2}(3B)$ uniformly in k (for k sufficiently large) by a constant which we will denote by $C(B)$ (see, in particular, (2.24) and (2.26).

Let us show that $\{G_k\}$ is a Cauchy sequence in $W^{1,2}(B)$. For this we fix k and l large, and we want to estimate $\nabla(G_k - G_l)$. We will introduce an intermediate function U_{kl}, which coincides with G_l on the sphere $S_\rho = \partial B_\rho = \partial B(x,\rho)$ for some $\rho \in (r, 2r)$ but satisfies the same equation $L_k U_{kl} = 0$ as G_k in B_ρ.

First we claim that for almost every radius $\rho \in (r, 2r)$, the restriction g_k of G_k to $S_\rho = \partial B(x,\rho)$ lies in the space $W^{1,2}(S_\rho, d\sigma)$ of functions of $L^2(S_\rho, d\sigma)$, which have a distribution gradient $\nabla_T g_k$ in $L^2(S_\rho, d\sigma)$, and in addition, that the gradient $\nabla_T g_k$ of g_k is given, almost everywhere on S_ρ, by the restriction of ∇G_k to S_ρ. Here σ denotes the surface measure on σ, and before we start we know that the restriction of ∇G_k is defined σ-almost-everywhere on almost every S_ρ by Fubini. There is no real doubt as to what we mean by restriction, because G_k is Hölder continuous on $2B$, but in general we could use Fubini to say that g_k is defined almost-everywhere on almost every sphere S_ρ, by Fubini.

The claim is not hard. We can use spherical coordinates to reduce matters to the case of a function in $W^{1,2}(R)$ for some parallelepiped R, and use the classical fact that then the restriction to almost every hyperplane P parallel to the axes lies in $W^{1,2}(P \cap R)$, with partial derivatives given by the restriction to P of the partial derivatives of the function. So we skip the details, but refer to the proof of Corollary 14.28 in [5] for a similar computation. We choose $\rho \in (r, 2r)$ so that in addition to the property above and its analogue for the restriction g_l of G_l to S_ρ, we have that

$$\begin{aligned} \int_{S_\rho} (|\nabla_T g_k|^2 + |\nabla_T g_l|^2) d\sigma &\leq \int_{S_\rho} (|\nabla G_k|^2 + |\nabla G_l|^2) d\sigma \\ &\leq C r^{-1} \int_{2B} |\nabla G_k|^2 + |\nabla G_l|^2 \leq C(B). \end{aligned} \tag{2.31}$$

See the beginning of the proof of the Lemma where we discuss the uniform bound $C(B)$ and recall that while we do not care how $C(B)$ depends on B, it does not depend on k or l.

Going further, denote by $W_\rho = \dot{W}^{1,2}(B_\rho)$ the space of functions $F \in L^2(B_\rho)$, with $\nabla F \in L^2(B_\rho)$, equipped with the homogeneous norm $||F||_\rho = \left(\int_{B_\rho} |\nabla F|^2\right)^{1/2}$ for $F \in W_\rho$. We also use the (homogeneous) space $H_\rho = \dot{H}^{1/2}(S_\rho)$ of functions $f \in L^2(S_\rho, d\sigma)$ which have half a derivative in $L^2(S_\rho)$, once again, equipped with the corresponding homogeneous norm. We need the following classical facts about H_ρ. First, every $F \in W_\rho$ has a trace $Tr(F)$ in H_ρ, with $||Tr(F)||_{H_\rho} \leq C||F||_\rho$. In the other direction, every $f \in H_\rho$ has an extension $F \in W_\rho$, with $||F||_\rho \leq C||f||_{H_\rho}$ and $Tr(F) = f$. Using these results and the Lax-Milgram's theorem, we know that for each elliptic operator $L = -\operatorname{div} A\nabla$, each $f \in H_\rho$ has a unique L-harmonic extension F to B_ρ, i.e., a function $F \in W_\rho$ such that $LF = 0$ (weakly as in (2.10)) and $Tr(F) = f$. In addition, $||F||_\rho \leq C||f||_{H_\rho}$ (where now C depends also on r and the ellipticity constants for A).

Interpolating between $L^2(S_\rho)$ and the space of functions with tangential derivatives in $L^2(S_\rho)$, we deduce that

$$||f||_{H_\rho}^2 \leq C||f||_{L^2(S_\rho)} ||\nabla_T f||_{L^2(S_\rho)}. \tag{2.32}$$

This is a rather simple result of interpolation between Sobolev spaces, but since we are on a ball, we can provide an even more direct argument. Because we are working with a ball, we can also give a description of H_ρ and its norm in terms of spherical harmonics. If $f = \sum_j f_j$ is the decomposition of f into spherical harmonics, where f_j is the part that comes from harmonics of degree j, (and we do not need to further decompose f_j into polynomials), we know that $||f||^2_{L^2(S_\rho)} = \sum_j ||f_j||_2^2$, while $||\nabla_T f||^2_{L^2(S_\rho)} = C\sum_j j^2||f_j||_2^2$, and the H_ρ-norm of f is given by $||f||^2_{H_\rho} = \sum_j j||f_j||_2^2$. Again, we do not need to know how the constants depend on ρ, as long as they are uniform in $\rho \in (r, 2r)$. Thus by Cauchy-Schwarz, (2.32) holds.

Let us apply this to the operator L_k and the restriction g_l of G_l to S_ρ; we obtain a function $U_{kl} \in W_\rho$ such that

$$L_k U_{kl} = 0 \text{ on } B_\rho, \;\; Tr(U_{kl}) = g_l, \tag{2.33}$$

and

$$||U_{kl}||_\rho \leq C||g_l||_{H_\rho} \leq C||\nabla G_l||_{L^2(S_\rho)} \leq C(B) \tag{2.34}$$

because (2.32) holds, G_l is continuous and bounded on $2B$, hence $||g_l||_{L^2(S_\rho)} \leq C$, and then by definition of ρ and (2.31). We said that g_l is the restriction of G_l to S_ρ, and this makes sense because G_k is continuous, but it is also the trace of G, in the sense of the operator above (recall that $G_l \in W_\rho$), and the same is true for g_k.

Then $U_{kl} - G_k$ is L_k-harmonic, lies in the space $W^{1,2}(B)$, and the trace of U_{kl} is $f = g_l - g_k$. Hence,

$$\begin{aligned} ||U_{kl} - G_k||_\rho &\leq C||f||_{H_\rho} \leq C||f||^{1/2}_{L^2(S_\rho)} \, ||\nabla_T(g_l - g_k)||^{1/2}_{L^2(S_\rho)} \\ &\leq C||G_l - G_k||^{1/2}_{L^\infty(2B)} \, ||\nabla_T(g_l - g_k)||^{1/2}_{L^2(S_\rho)} \leq C||G_l - G_k||^{1/2}_{L^\infty(2B)} \end{aligned} \tag{2.35}$$

as in (2.34), (2.32). We used the fact that $G_l - G_k$ is continuous on $2B$, so $||f||_{L^2(S_\rho)} \leq C||G_l - G_k||_{L^\infty(2B)}$, and then (2.31). This will be sufficient to ensure that $||U_{kl} - G_k||_\rho$ tends to zero since $||G_l - G_k||_{L^\infty(2B)}$ tends to 0: G_k tends to G_∞ uniformly on compact subsets of Ω_∞.

We now take care of $||G_l - U_{kl}||_\rho$, with an argument that comes from the calculus of variations, but which we had to modify because maybe the L_k are not symmetric and the solutions of $L_k U_{kl} = 0$ do not minimize an energy. We first observe that the trace of $G_l - U_{kl}$ on S_ρ vanishes, by definition of U and because the trace of G_l is g_l. Now we claim that

$$\int_{B_\rho} \langle A_l \nabla G_l, \nabla(G_l - U_{kl})\rangle = 0. \tag{2.36}$$

If $G_l - U_{kl}$ were a test function with compact support in B_ρ, this would just be the definition that G_l is a weak solution of L_l, as in (2.10). The fact that (2.36) also holds now follows from the fact that the test functions with compact support in B_ρ are dense in the subspace $W_{\rho,0} \subset W_\rho$ of functions with a vanishing trace. The same reasoning also yields that

$$\int_{B_\rho} \langle A_k \nabla U_{kl}, \nabla(G_l - U_{kl}) \rangle = 0, \tag{2.37}$$

because $L_k U_{kl} = 0$ weakly in B_ρ. Now

$$\begin{aligned}\int_{B_\rho} |\nabla(G_l - U_{kl})|^2 &\leq C \int_{B_\rho} \langle A_k \nabla(G_l - U_{kl}), \nabla(G_l - U_{kl}) \rangle \\ &= C \int_{B_\rho} \langle A_k \nabla G_l, \nabla(G_l - U_{kl}) \rangle = C \int_{B_\rho} \langle (A_k - A_l) \nabla G_l, \nabla(G_l - U_{kl}) \rangle \end{aligned} \tag{2.38}$$

because A_k is elliptic, and then by (2.37) and (2.36). We apply Cauchy-Schwarz, simplify, and get that

$$||G_l - U_{kl}||_\rho \leq C \Big\{ \int_{B_\rho} |A_k - A_l|^2 |\nabla G_l|^2 \Big\}^{1/2}. \tag{2.39}$$

If we had (2.18), we could conclude that the expression above tends to zero simply pulling out the $L^\infty(B)$ norm of $A_k - A_l$ and using that G_l is uniformly bounded in $W^{1,2}(B)$.

Under the weaker assumption (2.17), we use the reverse L^p inequality for the gradients of solutions. Indeed, there exists $p > 2$, that depends on n and the ellipticity constants for the L_l, such that

$$\Big\{ \int_{B_\rho} |\nabla G_l|^p \Big\}^{2/p} \leq \int_{2B} |\nabla G_l|^2 \leq C(B). \tag{2.40}$$

In particular, the power p and a constant $C(B)$ do not depend on l (see, e.g., [23], Lemma 1.1.12). This, together with (2.17), shows that (2.39) tends to zero:

$$\begin{aligned}||G_l - U_{kl}||_\rho^2 &\leq C \int_{B_\rho} |A_k - A_l|^2 |\nabla G_l|^2 \\ &\leq C \Big\{ \int_{B_\rho} |A_k - A_l|^{2q} \Big\}^{1/q} \Big\{ \int_{B_\rho} |\nabla G_l|^p \Big\}^{1/2p} \leq C \Big\{ \int_{B_\rho} |A_k - A_l|^{2q} \Big\}^{1/q}, \end{aligned} \tag{2.41}$$

where q is the dual exponent of $p/2$. The last expression above tends to 0 because $A_k - A_l$ tends to 0 in L^1_{loc} and is bounded. The rest of the proof stays as before, so Lemma 2.29 follows. This finishes the proof of Theorem 2.19. $\square$

3. Prevalent approximation of G^∞ when E is uniformly rectifiable

The assumptions for the next theorem concern the classical situation, i.e., Case 1 above. We are given an unbounded domain $\Omega \subset \mathbb{R}^n$, assume that $E = \partial\Omega$ is Ahlfors regular of dimension $n-1$ (see the definition (1.1)) and uniformly rectifiable (Definition 1.13), and that Ω has interior corkscrew points and Harnack chains, as in (2.1). We assume, furthermore, that the operator is close to a constant coefficient one, in the sense of (1.9).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain with an $n-1$ dimensional uniformly rectifiable boundary $E = \partial\Omega$, and satisfying the interior corkscrew and Harnack chain conditions* (2.1)*. Assume that L is an elliptic divergence form operator that satisfies* (1.9)*, and denote by $G = G^\infty$ the Green function for L, with pole at ∞. Then G is prevalently close to the distance to a plane (as in Definition 1.14) and for every choice of $\beta > 0$ and any AR measure μ on E, G is prevalently close to $D_{\beta,\mu}$.*

In this statement G is only determined modulo a multiplicative constant, but the conditions (1.15) and (1.16) do not see this constant.

Remark 3.2. Something like (1.9) is needed for this statement. Otherwise consider the falsely 2-dimensional example where $\Omega = \{(x,y) \in \mathbb{R}^2\,;\, y > 0\} \subset \mathbb{R}^2$ and $A(x,y) = a(y)I$ for $(x,y) \in \Omega$, where I denotes the identity matrix and for instance $a(y) = 1$ when $y \in [2^{2k}, 2^{2k+1})$ and $a(y) = 2$ when $y \in [2^{2k+1}, 2^{2k+2})$, $k \in \mathbb{Z}$. The Green function for L is easy to compute: take $G(x,y) = Cg(y)$, where $g(0) = 0$ and $g'(y) = a(y)^{-1}$; this is clearly a solution of $LG = 0$, and the uniqueness of G^∞ does the rest. We can see that G is not prevalently close to a distance function because g spends its time oscillating between two affine functions. Yet notice that for this example the harmonic measure ω^∞ is (proportional to) the Lebesgue measure on E.

Moreover, there are more complicated examples [4,24,25] which show that without the Dahlberg-Kenig-Pipher condition above, the absolute continuity of the harmonic measure with respect to the Hausdorff measure could fail even in the half-space.

For all these reasons our conditions on the coefficients are morally optimal.

Proof of Theorem 3.1. Part I: comparison to $\mathrm{dist}(X,P)$**.** Let us define some good sets. Let $M \geq 1$ and $\varepsilon > 0$ be given, and consider the set $\mathcal{G}(\varepsilon, M)$ of all pairs $(x,r) \in E \times (0,+\infty)$ such that all the following properties are satisfied. First, there is a hyperplane $P = P(x,r)$ through x, such that

$$d_{x,2Mr}(E,P) \leq \varepsilon \tag{3.3}$$

(where $d_{x,r}$ is as in (1.12)), and also, for one of the two half spaces H bounded by P,

$$d_{x,Mr}(\Omega,H) \leq 2\varepsilon. \tag{3.4}$$

We claim that each $\mathcal{G}(\varepsilon, M)$ is a Carleson-prevalent set, with a constant that may depend (wildly) on ε, M, and the various constants implicit in our geometric assumptions.

First consider the pairs (x,r) such that we cannot find a plane P as in (3.3). If (x,r) is such a pair, it is easy to see that $(x, 2Mr) \notin \mathcal{G}_{ur}(2M\varepsilon)$, where $\mathcal{G}_{ur}$ is the good set of Definition 1.13. This set of bad pairs satisfies a Carleson packing condition, by Definition 1.13 and because $\big\{(x,r)\,;\,(x,Ar) \in \mathcal{B}\big\}$ satisfies a Carleson packing condition when $\mathcal{B}$ does.

Now we claim that if P satisfies (3.3), and if ε is small enough and M is large enough (which we may assume) then (3.4) automatically holds for one of the two half spaces bounded by P. That is, the case when $\Omega \cap B(x, Mr)$ is nearly empty, or on the contrary very close to the full $B(x,r)$, is excluded by our NTA assumptions. In fact, since we assume that Ω is a 1-sided NTA domain with a uniformly rectifiable boundary, it possesses exterior corkscrew points as well. Then, on one hand, we can find a corkscrew point Z for $B(x,r)$, this point lies far from E, hence far from P too, then we select the half space H bounded by P that contains Z, and already (3.3) says that every point $Y \in H \cap B(x, Mr)$ such that $\operatorname{dist}(Y, P) > \varepsilon Mr$ lies in Ω, because the line segment from Y to Z does not meet E. Similarly, all the points $Y \in B(x, Mr) \setminus H$ such that $\operatorname{dist}(Y, P) > \varepsilon Mr$ lie in the same component of $\mathbb{R}^n \setminus E$ as Z', the image of Z by the reflection across P. We just need to see that this component is not Ω, or in other words that $Z' \notin \Omega$. But if $Z' \in \Omega$, there is a Harnack chain from Z to Z', and by definition of a Harnack chain it stays inside of $B(x, Mr)$ if M is large enough. One of the balls B_j of the chain meets $P \cap B(x, Mr)$, and this is impossible if ε is small enough because $2B_j \subset \Omega$ and $\operatorname{diam} B_j \geq C^{-1} \min(\operatorname{dist}(Z, E), \operatorname{dist}(Z', E)) \geq C^{-1} r$. So (3.4) comes for free.

Lemma 3.5. *For any $\varepsilon > 0$ and $M > 0$, we can find $\varepsilon_1 > 0$, $M_1 \geq 1$, $\kappa > 0$, and $K \geq 1$, depending on n, ε, M, the AR constant for E, and the one sided NTA constants from (2.1), such that if $(x,r) \in \mathcal{G}(\varepsilon_1, M_1) \cap \mathcal{G}_{cc}(\tau, K)$, then (x,r) lies in the good set $\mathcal{G}_{Gd}(\varepsilon, M)$ of Definition 1.14.*

Proof of Lemma 3.5. Let the various constants and ε, M, be given, and suppose the lemma fails for these constants. Then there exists an example (Ω_k, E_k, L_k) where all the assumptions are satisfied, and for some pair $(x_k, r_k) \in E_k \times (0, +\infty)$, we have that $(x_k, r_k) \in \mathcal{G}(\varepsilon_k, M_k) \cap \mathcal{G}_{cc}(\tau_k, K_k)$, say, with $\varepsilon_k = \tau_k = 2^{-k}$ and $K_k = M_k = 2^k$, and yet $(x_k, r_k) \notin \mathcal{G}_{Gd}(\varepsilon, M)$.

As the reader guessed, we want to take a limit and derive a contradiction. By translation, dilation, and rotation invariance of the problem, we may assume that $x_k = 0$, $r_k = 1$, and we can take a fixed hyperplane P through the origin and a fixed half space H bounded by P that work in (3.3) and (3.4) for all k. By (3.3) and (3.4), E_k tends to P and Ω_k tends to H, as in (2.16). Pick a corkscrew point X_0, for instance the point of H that lies at distance 1 from both 0 and P. Then let G_k denote the Green function for Ω_k, with the pole at ∞, and normalized by $G_k(X_0) = 1$. We want to apply Theorem 2.19, so we check the last hypothesis (2.17). The assumption that $(x_k, r_k) \in \mathcal{G}_{cc}(\tau_k, K_k)$ gives us a constant matrix $A_{k,0}$ and, modulo extracting a subsequence, we may assume that $A_{k,0}$ tends to a limit $A_{0,0}$. Now (2.17) holds, with $A_\infty = A_{0,0}$, because if B is a ball such that $2B \subset \Omega_\infty$, then for k large $B \subset W_{K_k}(0,1) = W_{K_k}(x_k, r_k)$, and

$$\int_B |A_k - A_\infty| \leq |B||A_{k,0} - A_{0,0}| + \int_B |A_k - A_{k,0}| \leq |B||A_{0,k} - A_{0,0}| + \tau_k,$$

which tends to 0.

So Theorem 2.19 says that $\{G_k\}$ converges, uniformly on compact subsets of $\mathbb{R}^n$, to the Green function G_∞ for $L_\infty = -\operatorname{div} A_{0,0}\nabla$ on $H = \Omega_\infty$, with a pole at ∞ and normalized at X_0. Now $G(X) = \operatorname{dist}(X,P)$ is a solution to $L_\infty G = 0$, lies in all the correct $W_r(B)$, vanishes along P, and has the same normalization, so $G_\infty(X) = \operatorname{dist}(X,P)$ on $\Omega_\infty = H$. Thus the G_k converge uniformly, on the compact subsets of $\mathbb{R}^n$, to the distance function. Therefore, for k large,

$$|\operatorname{dist}(X,P) - G_k(X)| \leq \varepsilon \text{ for } X \in \Omega \cap B(0,M), \tag{3.6}$$

which contradicts the fact that (1.15) fails for G_k and the unit ball. This completes the proof of Lemma 3.5. $\square$

The first part of Theorem 3.1, with the distance function to a plane, follows at once, by a combination of definitions and because the union $\mathcal{G}(\varepsilon_1, M_1) \cap \mathcal{G}_{cc}(\tau, K)$ of two prevalent sets is prevalent.

Proof of Theorem 3.1. Part II: comparison to $D_{\beta,\mu}(X)$. We now intend to show that G is prevalently close to any given $D_{\beta,\mu}$. To this end, we will use the result above, and compare $D_{\beta,\mu}$ to $\operatorname{dist}(X,P)$. That is, we will show that prevalently, the restriction of $D_{\beta,\mu}$ to Ω is close to some $\operatorname{dist}(X,P)$, and then we will deal with organization issues, for instance making sure that we use the same plane in the two descriptions.

The estimates that follow are similar to the proofs that were done in [6], but they are a little simpler here because we can content ourselves with weak estimates. We intend to use a result of X. Tolsa [27] on the good approximation of μ by flat measures, and for this we need some notation. These results will be used later in full generality, and for the time being we assume that E is an Ahlfors regular set E dimension d, with any $d \in (0,n)$, and μ is any AR measure of dimension d whose support is E.

Denote by $\mathcal{F}_d$ the set of flat measures of dimension d in $\mathbb{R}^n$, i.e., measures $\sigma = c\mathcal{H}^d_{|P}$, where $c > 0$ and P is an affine d-plane. For $x \in \mathbb{R}^n$ and $r > 0$, denote by $\Lambda(x,r)$ the set of Lipschitz functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi(x) = 0$ on $\mathbb{R}^n \setminus B(x,r)$ and the Lipschitz norm of φ is less than or equal to 1. Then define a sort of Wasserstein distance between two measures μ and σ by

$$D_{x,r}(\mu,\sigma) = r^{-d-1} \sup_{\varphi \in \Lambda(x,r)} \Big| \int \varphi d\mu - \int \varphi d\sigma \Big|, \tag{3.7}$$

and finally set

$$\alpha(x,r) = \inf_{\sigma \in \mathcal{F}_d} D_{x,r}(\mu,\sigma). \tag{3.8}$$

The normalization is such that when μ is an AR measure, $\alpha(x,r) \leq C$ systematically, but when E is uniformly rectifiable and μ is any AR measure on E, $\alpha(x,r)$ is often much smaller than this. In fact, it is proved in [27] that $\alpha^2(x,r)\frac{d\mu(x)dr}{r}$ is a Carleson measure on $E \times (0,+\infty)$. Here we shall only use the corresponding weak estimate, which follows by Chebyshev's inequality, that says that for each choice of $\eta > 0$ and $N \geq 1$,

$$\mathcal{G}_\alpha(\eta,N) = \big\{(x,r) \in E \times (0,+\infty)\,;\, \alpha(x,Nr) \leq \eta\big\} \text{ is a Carleson-prevalent set.} \tag{3.9}$$

Lemma 3.10. *Let E and μ be Ahlfors regular of dimension $d \in (0,n)$ in $\mathbb{R}^n$. Let $(x,r) \in \mathcal{G}_\alpha(\eta,N) \subset E \times (0,+\infty)$, and let $\sigma = c\mathcal{H}^d_{|P}$ be a flat measure such that*

$$D_{x,Nr}(\mu,\sigma) \leq 2\eta. \tag{3.11}$$

If N is large enough (depending on n, d, and the AR constant for μ) and η is small enough (depending on the aforementioned parameters and also on N), then

$$\mathrm{dist}(z,P) \leq \eta_1 r \quad \textit{for } z \in E \cap B(x,Nr/3), \tag{3.12}$$

and

$$\mathrm{dist}(z,E) \leq \eta_1 r \quad \textit{for } z \in P \cap B(x,Nr/3), \tag{3.13}$$

where we set $\eta_1 = C_1 N \eta^{1/(d+1)}$, with a constant C_1 that depends on the AR constant for μ, d and n. Furthermore, for the distance function $D_{\beta,\mu}$,

$$|D_{\beta,\mu}(X) - (ca_\beta)^{-1/\beta}\,\mathrm{dist}(X,P)| \leq C_2 N \eta^{1/(d+2+\beta)} r + C_2\,\mathrm{dist}(X,P)^{1+\beta} N^{-\beta} r^{-\beta} \tag{3.14}$$

for $z \in B(x,Nr/5)$, where C_2 depends on n, d, the AR constant for μ, and β, and a_β is a geometric constant that depends on β.

Proof. The definition of $D_{x,Nr}(\mu,\sigma)$ says that

$$\Big|\int \varphi\big(d\mu - d\sigma\big)\Big| \leq (Nr)^{d+1} D_{x,r}(\mu,\sigma) \leq 2\eta N^{d+1} r^{d+1} \tag{3.15}$$

for every $\varphi \in \Lambda(x,Nr)$. We intend to use this for various functions φ to get relevant information on μ and σ. Let us first check that

$$P \text{ meets } B(x,Nr/4) \text{ and } C^{-1} \leq c \leq C. \tag{3.16}$$

Here and throughout the proof, C is a constant which depends on d, n, and the AR constants of μ only, and whose value may change from line to line. Take a first bump function ψ_1 such that $0 \leq \psi_1 \leq 1$ everywhere, $\psi_1(z) = 0$ on the complement of $B(x, Nr/4)$, and $\psi_1(z) = 1$ on $B(x, Nr/8)$. We can manage to do this with $|\nabla\psi| \leq (Nr/8)^{-1}$, so $(Nr/8)\psi_1 \Lambda(x, Nr)$ and (3.15) says that

$$\Big| \int \psi_1 \big(d\mu - d\sigma\big) \Big| \leq 16\eta N^d r^d. \tag{3.17}$$

Since $\psi_1(z) = 1$ on $B(x, Nr/8)$ and $\mu(B(x, Nr/8) \geq C^{-1} N^d r^d$, we see that (if η is small enough) $\sigma(B(x, Nr/4)) \geq (2C)^{-1} N^d r^d$ (because $\psi_1(z) = 0$ on $\mathbb{R}^n \setminus B(x, Nr/4)$), and hence P meets $B(x, Nr/4)$ and $c \geq C^{-1}$.

Similarly, if now ψ is a similar function, but with $0 \leq \psi \leq 1$ everywhere, $\psi(z) = 0$ on $\mathbb{R}^n \setminus B(x, Nr)$, $\psi(z) = 1$ on $B(x, Nr/2)$, and $|\nabla\psi| \leq (Nr/2)^{-1}$, we get that

$$\Big| \int_{B(x,Nr)} \psi \big(d\mu - d\sigma\big) \Big| \leq 4\eta N^d r^d. \tag{3.18}$$

But $\mu(B(x, Nr)) \leq C N^d r^d$, so $\int_{B(x,Nr)} \psi d\sigma \leq 2C N^d r^d$. Moreover, since $\int_{B(x,Nr)} \psi d\sigma \geq \sigma(B(x, Nr/2)) = c\mathcal{H}^d(P \cap B(x, Nr/2))$ and P meets the ball $B(x, Nr/4)$, we get that $c \leq C$ too; (3.16) follows.

Next we want to control the distance to P. We keep the same function ψ, but try the product $\varphi(z) = \psi(z) \operatorname{dist}(z, P)$. We still have that $\varphi = 0$ on $\mathbb{R}^n \setminus B(x, Nr)$, as required in the definition of $\Lambda(x, Nr)$, but now

$$|\nabla\varphi(z)| \leq \psi(z) + |\nabla\psi(z)| \operatorname{dist}(z, P) \leq 1 + (Nr/2)^{-1} \operatorname{dist}(z, P) \leq 5 \tag{3.19}$$

because P meets $B(x, Nr/4)$. So $\varphi/5 \in \Lambda(x, Nr)$, and (3.15) yields

$$\Big| \int \psi(z) \operatorname{dist}(z, P) \big(d\mu(z) - d\sigma(z)\big) \Big| \leq 10\eta N^{d+1} r^{d+1}. \tag{3.20}$$

Now $\int \psi(z) \operatorname{dist}(z, P) d\sigma = 0$ because σ is supported on P, and since $\psi(z) = 1$ on $B(x, Nr/2)$, we get that

$$\int_{B(x,Nr/2)} \operatorname{dist}(z, P) d\mu(z) \leq 10\eta N^{d+1} r^{d+1}. \tag{3.21}$$

We still need to transform this into an L^∞ bound. So we assume that (3.12) fails, so that $\operatorname{dist}(z, P) \geq \eta_1 r$ for some $z \in E \cap B(x, Nr/3)$. Then $\operatorname{dist}(z, P) \geq \eta_1 r/2$ on $B_1 = B(z, \eta_1 r/2)$, and since $\eta_1 < Nr/6$ if η is small enough, $B_1 \subset B(x, Nr/2)$ and (3.21) yields

$$10\eta N^{d+1}r^{d+1} \geq (\eta_1 r/2)\mu(B_1) \geq C^{-1}(\eta_1 r)^{d+1} = C^{-1}(C_1 N\eta^{1/(d+1)})^{d+1}r^{d+1} \tag{3.22}$$

by Ahlfors regularity and the definition of η_1. We now choose C_1 so large that (3.22) is impossible, and this contradiction completes the proof of (3.12).

For (3.13) we proceed exactly the same way, but with $\psi(z)\,\mathrm{dist}(z,E)$, which of course vanishes on the support of μ. Maybe we need to make C_1 larger because we now use the Ahlfors regularity of σ, but this is all right.

Now we estimate $D_{\beta,\mu}(X) - (ca_\beta)^{-1/\beta}\,\mathrm{dist}(X,P)$. Recall that $D_{\beta,\mu} = D_\beta(X) = R_\beta(X)^{-1/\beta}$, with

$$R_\beta(X) = \int\limits_E |X-y|^{-d-\beta}d\mu(y), \tag{3.23}$$

so we take care of R_β first. Let $R(X) = \int_E |X-y|^{-d-\beta}d\sigma(y)$ denote the analogue of $R_\beta(X)$, but for the measure σ. A direct computation shows that

$$R(X) = ca_\beta\,\mathrm{dist}(X,P)^{-\beta}, \tag{3.24}$$

just because $\sigma = c\mathcal{H}^d_{|P}$. Let ψ be a smooth cut-off function such that $0 \leq \psi \leq 1$ everywhere, $\psi = 0$ outside of $B(x, Nr/4)$, $\psi = 1$ on $B(x, Nr/8)$, $|\nabla\psi| \leq 16(Nr)^{-1}$ and $|\nabla^2\psi| \leq C(Nr)^{-2}$ (to be used later). We first study the main part of $R_\beta(X) - R(X)$, namely

$$A(X) = \int |X-y|^{-d-\beta}\psi(y)[d\mu(y) - d\sigma(y)] \tag{3.25}$$

[here and for the rest of the proof of the lemma, A no longer denotes a coefficient matrix]. Also let $\eta_2 > \eta_1$ be small, to be chosen later, and then introduce a smooth cut-off function ξ such that $0 \leq \xi \leq 1$ on $\mathbb{R}^n$, $\xi(X) = 1$ when $\mathrm{dist}(X,P) \leq \eta_2 r$, $\xi(X) = 0$ when $\mathrm{dist}(X,P) \geq 2\eta_2 r$, $|\nabla\xi| \leq 2(\eta_2 r)^{-1}$, and $|\nabla^2\xi| \leq C(\eta_2 r)^{-2}$. Notice that $\xi(y) = 1$ when $y \in B(x, Nr/4)$ lies in the support of σ or μ, by (3.12), so

$$A(X) = \int |X-y|^{-d-\beta}\psi(y)\xi(y)[d\mu(y) - d\sigma(y)]. \tag{3.26}$$

We are going to restrict our attention to the case when

$$X \in H = \big\{X \in B(x, Nr/5)\,;\ \mathrm{dist}(X,P) \geq 3\eta_2 r\big\}, \tag{3.27}$$

because in this region the distance from X to the support of ξ is at least $\eta_2 r$. Set $\varphi_X(y) = |X-y|^{-d-\beta}\psi(y)\xi(y)$ for $X \in H$. Notice that φ_X is supported on $B(x, Nr)$ because of ψ, and

$$\begin{aligned}|\nabla\varphi_X(y)| &\leq C|X-y|^{-d-\beta-1} + C(\eta_2 r)^{-1}|X-y|^{-d-\beta} + 16(Nr)^{-1}|X-y|^{-d-\beta} \\ &\leq C(\eta_2 r)^{-1}|X-y|^{-d-\beta} \leq C(\eta_2 r)^{-d-\beta-1}\end{aligned} \tag{3.28}$$

because $|X-y| \geq \eta_2 r$. Normally we should cut the integral into annuli and get a better result, but let us not bother. We apply (3.15) to a multiple of φ_X and get that

$$\begin{aligned}|A(X)| = \Big|\int \varphi_X[d\mu - d\sigma]\Big| &\leq 2\eta(Nr)^{d+1}||\nabla\varphi_X||_\infty \\ &\leq C\eta(Nr)^{d+1}(\eta_2 r)^{-d-\beta-1} = C\eta N^{d+1}\eta_2^{-d-\beta-1}r^{-\beta}.\end{aligned} \tag{3.29}$$

This will be good enough (because we can choose η last). For the rest of the integral, set $B(X) = R_\beta(X) - R(X) - A(X)$, and notice that

$$\begin{aligned}|B(X)| &= \Big|\int |X-y|^{-d-\beta}(1-\psi(y))[d\mu(y) - d\sigma(y)]\Big| \\ &\leq \int\limits_{\mathbb{R}^n\setminus B(x,Nr/8)} |X-y|^{-d-\beta}(1-\psi(y))[d\mu(y) + d\sigma(y)] \leq C(Nr)^{-\beta}\end{aligned} \tag{3.30}$$

by the Ahlfors regularity of both measures, and a simple argument where one cuts the domain into annuli of size $2^l Nr$. We add this to (3.29) and get that

$$|R_\beta(X) - R(X)| \leq C[\eta N^{d+1}\eta_2^{-d-\beta-1} + N^{-\beta}]r^{-\beta}. \tag{3.31}$$

Set $D(X) = R(X)^{-1/\beta} = (ca_\beta)^{-1/\beta}\,\mathrm{dist}(X,P)$ by (3.24), and notice that

$$D_{\beta,\mu}(X) - D(X) = R_\beta(X)^{-1/\beta} - R(X)^{-1/\beta}. \tag{3.32}$$

Set $t = R_\beta(X)$ and $u = R(X)$ for a minute, and observe that since the derivative of $t \mapsto t^{-1/\beta}$ is $-\frac{1}{\beta}t^{-\frac{1+\beta}{\beta}}$, we get that $|t^{-1/\beta} - u^{-1/\beta}| \leq C|t-u|(t^{-\frac{1+\beta}{\beta}} + u^{-\frac{1+\beta}{\beta}})$ (including in the case when $|u-t|$ is not small compared to u and t). Here $C^{-1}\,\mathrm{dist}(X,P)^{-\beta} \leq R(X) \leq C\,\mathrm{dist}(X,P)^{-\beta}$ by direct calculation, and the same thing holds for $R_\beta(X)$, because μ is Ahlfors regular and $\mathrm{dist}(X,P) \sim \mathrm{dist}(X,E)$ for $X \in H$, so t and u are both comparable to $\mathrm{dist}(X,P)^{-\beta}$, and (3.31) yields

$$|D_{\beta,\mu}(X) - D(X)| \leq C\,\mathrm{dist}(X,P)^{1+\beta}[\eta N^{d+1}\eta_2^{-d-\beta-1} + N^{-\beta}]r^{-\beta}. \tag{3.33}$$

Let us choose $\eta_2 = C_1 N\eta^{1/(d+2+\beta)}$. This way our constraint that $\eta_2 \geq \eta_1$ is clearly satisfied (we just decreased the power a little bit), and (3.33) simplifies to

$$\begin{aligned}|D_{\beta,\mu}(X) - D(X)| &\leq C\,\mathrm{dist}(X,P)^{1+\beta}[\eta^{1-\frac{d+1+\beta}{d+2+\beta}}N^{-\beta} + N^{-\beta}]r^{-\beta} \\ &\leq C\,\mathrm{dist}(X,P)^{1+\beta}N^{-\beta}r^{-\beta}.\end{aligned} \tag{3.34}$$

This is good enough for (3.14), but we only get it for $X \in H$. When $x \in B(r, Nr/5) \setminus H$, we know that $\mathrm{dist}(X, P) \leq 3\eta_2 r$ by definition of H, and then $\mathrm{dist}(X, E) \leq 4\eta_2 r$ by (3.13) and because $\eta_2 \geq \eta_1$, hence $D_{\beta,\mu}(X) \leq C\,\mathrm{dist}(X, E) \leq C\eta_2 r$ (because $D_\beta(X)$ is always comparable to $\mathrm{dist}(X, E)$). So

$$|D_{\beta,\mu}(X) - D(X)| \leq C\eta_2 r = CN\eta^{1/(d+2+\beta)} r \tag{3.35}$$

for $X \in B(r, Nr/5) \setminus H$. Now (3.14) follows from this and (3.34), and this completes our proof of Lemma 3.10. □

Remark 3.36. The proof of (3.31) also shows that

$$|\nabla R_\beta(X) - \nabla R(X)| \leq C[\eta N^{d+1}\eta_2^{-d-\beta-2} + N^{-\beta-1}]r^{-\beta-1}. \tag{3.37}$$

This is because when we compute ∇R or ∇R_β, we replace the kernel $|X - y|^{-d-\beta}$ with a kernel of size $|X - y|^{-d-\beta-1}$; we can then follow the computations above with this different power. That is, the estimates are the same as they would be with $D_{\beta+1}$. Then the proof of (3.34) yields

$$|\nabla D_\beta(X) - \nabla D(X)| \leq C\,\mathrm{dist}(X, P)^{2+\beta} N^{-\beta-1} r^{-\beta-2}, \tag{3.38}$$

for $X \in B(x, Nr/5)$ such that $\mathrm{dist}(X, P) \geq C\eta_2' r$, with $\eta_2' = C_1 N\eta^{1/(d+3+\beta)}$.

We may now return to the proof of prevalence for the good approximation of G by $D_{\beta,\mu}$, that we need to complete our proof of Theorem 3.1. Let $\varepsilon > 0$, $M \geq 1$ and $(x, r) \in E \times (0, +\infty)$ be given. We want to show that prevalently, (x, r) lies in the good set $\mathcal{G}_{GD_\beta}(\varepsilon, M)$ of Definition 1.14. We may assume that $(x, r) \in \mathcal{G}_{Gd}(\varepsilon, M)$ because we already proved that this happens prevalently, or rather that (1.15) holds with $\varepsilon/3$. We may also assume that $(x, r) \in \mathcal{G}_\alpha(\eta, N)$, with values of η and N that will be chosen soon, because (3.9) says that this happens prevalently. Then Lemma 3.10 says that (3.12)-(3.14) hold, but maybe this happens with a different plane P_1. So we will need to see whether P_1 is close enough to P.

First observe that in the proof of prevalence for $\mathcal{G}_{Gd}(\varepsilon/3, M)$, we can actually find P such that in addition to (1.15), we also have that

$$d_{x,Mr}(E, P) \leq (10M)^{-1}\varepsilon; \tag{3.39}$$

see the proof of (3.3) and (3.4), using the definition of uniform rectifiability. Now (3.12) and (3.13) also imply that

$$\begin{aligned} d_{x,Mr}(E, P_1) &\leq (N/3)M^{-1} d_{x,Nr/3}(E, P_1) \leq M^{-1}\eta_1 \\ &\leq CNM^{-1}\eta^{1/(d+1)} \leq (10M)^{-2}\varepsilon \end{aligned} \tag{3.40}$$

because we can pick $N > 3M$, and choose η small enough, depending on M, N, and ε. So P_1 and P are close to each other, and

$$|\operatorname{dist}(X, P_1)| - \operatorname{dist}(X, P)| \leq \varepsilon r/3 \tag{3.41}$$

for $X \in B(x, M)$. Finally, (3.14) says that for $X \in B(x, Mr) \subset B(x, Nr/3)$,

$$\begin{aligned}|(ca_\beta)^{-1/\beta} D_{\beta,\mu}(X) - \operatorname{dist}(X, P_1)| &\leq CN\eta^{1/(d+2+\beta)}r \\ &+ C\operatorname{dist}(X, P_1)^{1+\beta}N^{-\beta}r^{-\beta} \leq CN\eta^{1/(d+2+\beta)}r + C(Mr)^{1+\beta}N^{-\beta}r^{-\beta}\end{aligned} \tag{3.42}$$

(recall that $c \geq C^{-1}$ by (3.16)). We choose so N large, depending on M and β, that $CM^{1+\beta}N^{-\beta} < \varepsilon/6$, and then η so small, depending on M and N, that $CN\eta^{1/(d+2+\beta)} < \varepsilon/6$. Then $|(ca_\beta)^{-1/\beta} D_{\beta,\mu}(X) - \operatorname{dist}(X, P_1)| \leq \varepsilon r/3$, and (1.16) follows from (3.41) and (1.15). That is, $(x, r) \in \mathcal{G}_{GD_\beta}(\varepsilon, M)$. This completes the proof of Theorem 3.1. □

The next statement is the version of Theorem 3.1 in Case 2.

Theorem 3.43. *Let Ω be a domain in $\mathbb{R}^n$, whose boundary $E = \partial\Omega$ is an Ahlfors regular and uniformly rectifiable set of (integer) dimension $d \in (0, n)$. If $d = n - 1$, assume, in addition, that E satisfies the interior corkscrew point and Harnack chain conditions, as in (2.1). Let $\alpha > 0$ be given, pick any AR measure μ on E (as in (1.1)), and define D_α and $L = -\operatorname{div} AD_\alpha^{d+1-n}\nabla$ by (1.4), (1.3), and (2.7), with A satisfying (2.4), (2.5), and (1.9) with $A_0 \equiv I$. Denote by $G = G^\infty$ the Green function for L, with pole at ∞. Then G is prevalently close to the distance to a plane (as in Definition 1.14) and for every $\beta > 0$ and every AR measure ν on E, G is prevalently close to $D_{\beta,\nu}$.*

As was observed in the comments below Theorem 1.17, the special operators L of (1.2) are built with specific distance functions D_α that respect the rotation invariance of $\mathbb{R}^n$; this is the reason why we restrict to $A_0 = I$ here, while we authorized other constant coefficient operators in Theorem 3.1.

See Section 2 for the definition and some information on G^∞. Notice that the measure ν used to compute the distance $D_{\beta,\nu}$ does not need to be the same as the measure μ used to define D_α and L because the two measures are used in different places and never interact.

Proof. The proof will be nearly the same as for Theorem 3.1. We made sure, when we treated Part II (about $D_\beta = D_{\beta,\nu}$) not to use the specific dimension of E, and even less the precise form of the operator. In particular Lemma 3.10 is still valid in the present situation, and this means that when $(x, r) \in \mathcal{G}_\alpha(\eta, N)$ (for the measure ν), and η and N are chosen correctly, we can find a d-plane P such that

$$d_{x,M_1 r}(E, P) \leq (10M)^{-2}\varepsilon_1, \tag{3.44}$$

as in (3.39), and where ε_1 and M_1 are given in advance. We also require that $(x, r) \in \mathcal{G}_\alpha(\eta, N)$ for the measure μ and we claim that (if η and N are chosen correctly again), then automatically

$$(x, r) \in \mathcal{G}_{Gd}(\varepsilon, M), \tag{3.45}$$

i.e., we have a good approximation of G by $\operatorname{dist}(X, P)$ as required in (1.15), in fact with the same P as in (3.44).

We prove this as we did for Lemma 3.5, by contradiction and compactness. If we cannot get (3.45), there exist examples $(\Omega_k, E_k, \nu_k, L_k)$ where all the general assumptions are satisfied, yet for some pair $(x_k, r_k) \in E_k \times (0, +\infty)$, we have that (3.44) holds with $M = 2^k$ and $\varepsilon = 2^{-k}$, $(x_k, r_k) \in \mathcal{G}_\alpha(2^{-(d+2)k}, 2^k)$, but $(x_k, r_k) \notin \mathcal{G}_{Gd}(\varepsilon, M)$.

We may assume that $x_k = 0$, $r_k = 1$, and that all the planes P_k coming from (3.44) are parallel to a same plane P through the origin (we did not require P_k to contain x_k). Then by (3.44), $\{E_k\}$ converges to P. If $d = n-1$, we can proceed as in the proof of Theorem 3.1 to extract a subsequence (or turn the domains) so that Ω_k converges to a fixed half space H bounded by P. If $d \leq n-2$, a direct inspection shows that Ω_k converges to $\mathbb{R}^n \setminus P$.

We required that $(x_k, r_k) \in \mathcal{G}_\alpha(2^{-(d+2)k}, 2^k)$ (for ν_k) because this way, we get the existence of a flat measure σ_k such that by (3.7)

$$\left| \int \varphi d\nu_k - \int \varphi d\sigma_k \right| \leq 2^{(d+1)k} D_{0,2^k}(\nu_k, \sigma_k) \leq 2^{-k} \tag{3.46}$$

for every 1-Lipschitz function φ supported on $B(0, 2^k)$. By construction (and in particular the proof of (3.44)), we could make sure that σ_k is supported by P_k, but even if we were not that cautious, it follows from the proof of (3.12) and (3.13) that the support P'_k of σ_k tends to P too. In other words, the limiting measure of ν_k guaranteed by Theorem 2.19 is, in fact, flat.

Similarly to the argument of Theorem 2.19 in the paragraph right after (2.27), it follows easily from (3.46) (and the uniform convergence of the integrals in (1.3) at ∞) that the functions R_{α,ν_k} of (1.3) (but associated to ν_k) converge, uniformly on every compact subset of $\mathbb{R}^n \setminus P$, to the function $R_{\alpha,\sigma}$ associated to the limit of the σ_k (maybe after extraction of a subsequence so that the coefficients c_k converge).

Now look at the operators. We also required that $(x, r) \in \mathcal{G}_\alpha(\eta, N)$ for the measure μ, and so we may assume, by extracting a subsequence again, that μ_k converges to a limit μ_∞ which is a flat measure on the same plane P as before. Then by Theorem 2.19 the (properly normalized) Green functions with the pole at infinity of $L_k = -\operatorname{div} D_{\alpha,\mu_k}^{d+1-n}\nabla$ and $L_\infty = -\operatorname{div} D_{\alpha,\mu_\infty}^{d+1-n}\nabla$ converge uniformly on compact subsets of $\mathbb{R}^n$, to the Green function with the pole at infinity for $L_\infty = -\operatorname{div} A_k D_{\alpha,\mu_\infty}^{d+1-n}\nabla$ on the domain Ω_∞. Here μ_∞ is a flat measure of a plane P, Ω_∞ is H if $d = n-1$ and $\mathbb{R}^n \setminus P$ otherwise, and so the Green function with the pole at infinity for the limiting operator is $c \operatorname{dist}(X, P)$. This contradicts our assumption that

$$|c_k G_k(X_k) - \operatorname{dist}(X, P)| \geq \varepsilon \tag{3.47}$$

for some $X_k \in \Omega_k \cap B(0, M)$ and for any c_k. It is here that we use $A_0 = I$ in the approximation of A_k in order to ensure that the solution of the limiting equation in the exterior of $\mathbb{R}^d$ is the distance to the boundary.

So the Green function G is prevalently close to the distance to a plane, and now the same argument as for Theorem 3.1 shows that it is also prevalently close to any $D_{\beta,\nu}$. Theorem 3.43 follows. □

4. Approximation of ∇G^∞ when E is uniformly rectifiable

We want to extend the positive results of Section 3 to a more precise notion of approximation of G with distance functions, where we also control the first derivative of G. This time, we will not control ∇G all the way to the boundary, so our notion of approximation will use the Whitney regions associated to balls $B(x, r)$ centered on E, and defined as in (1.8) by

$$W_M(x, r) = \big\{X \in \Omega \cap B(x, Mr)\,;\ \operatorname{dist}(X, E) \geq M^{-1} r\big\}. \tag{4.1}$$

Also, we only control (a little bit more than) the L^2_{loc} norm of ∇G, so we will use that norm. Except for this, the following definition is similar to Definition 1.14 above.

Definition 4.2. Let G^∞ denote the Green function for the operator L in the domain Ω bounded by an Ahlfors regular set E of dimension $d < n$. We say that ∇G^∞ is prevalently close to the gradient of the distance to a plane when for each choice of $\varepsilon > 0$ and $M \geq 1$, the set $\mathcal{G}_{\nabla G d}(\varepsilon, M)$ of pairs $(x, r) \in E \times (0, +\infty)$ such that there exists a d-plane $P(x, r)$ and a positive constant $c > 0$, with

$$\int_{W_M(x,r)} |\nabla \operatorname{dist}(X, P) - c\nabla G^\infty(X)|^2 \leq \varepsilon r^n, \tag{4.3}$$

is a Carleson-prevalent set.

If in addition we are given an AR measure ν on E and an exponent $\beta > 0$, and $D_{\beta,\nu}$ is defined in (1.3)-(1.4), we say that ∇G^∞ is prevalently close to $\nabla D_{\beta,\nu}$ when for any $\varepsilon > 0$ and $M \geq 1$, the set $\mathcal{G}_{\nabla G D_\beta}(\varepsilon, M)$ of pairs $(x, r) \in E \times (0, +\infty)$ such that there exists a positive constant $c > 0$, with

$$\int_{W_M(x,r)} |\nabla D_{\beta,\nu}(X) - c\nabla G^\infty(X)|^2 dX \leq \varepsilon r^n, \tag{4.4}$$

is Carleson-prevalent.

In Definition 1.14, we were able to require the approximation to go all the way to the boundary, because, in particular, we had the uniform convergence of Green functions to their limit on compacta of $\mathbb{R}^n$ rather than just compacta in Ω_∞, but this is no longer the case with the gradients, and we shall content ourselves with estimates on the values of ∇G on Whitney cubes. We expect better estimates to be valid, but we shall not pursue this issue here because the point of this paper is rather to prove weak estimates with compactness arguments. It will be interesting to investigate more the precise way our approximations quantify, including near the boundary.

Theorem 4.5. *Let Ω, E, and L satisfy the assumptions of Theorem 3.1 or Theorem 3.43, and let G denote the Green function for L in Ω, with pole at ∞. Then ∇G^∞ is prevalently close to the gradient of the distance to a plane and to $\nabla D_{\beta,\nu}$, for any $\beta > 0$ and any AR measure ν on E.*

Proof. We will keep the same structure for the proofs, but will need to replace some estimates. We start as in the proof of Theorem 3.1, but need to replace $\mathcal{G}_{Gd}(\varepsilon, M)$ with $\mathcal{G}_{\nabla Gd}(\varepsilon, M)$ in Lemma 3.5. This means that instead of (3.6), we now want to prove

$$\int_{W_M(x,r)} |\nabla \operatorname{dist}(X,P) - \nabla G_k(X)|^2 \leq \varepsilon r^n. \tag{4.6}$$

In the proof of Theorem 3.1, the desired estimate comes from the uniform convergence in Theorem 2.19; in the present case, Lemma 2.29 provides exactly the L^2 convergence estimate that we need (because $G_\infty(X) = \operatorname{dist}(X,P)$ here).

Then we stay in Case 1, as in Theorem 3.1, but approximate ∇G with $\nabla D_{\beta,\mu}$. We start the same way, but need to replace our main estimates for the difference between $D_{\beta,\mu}(X)$ by $C \operatorname{dist}(X,P)$ by estimates on the difference of gradients. That is, we need to replace (3.14), which itself comes from estimates for differences of integrals, which culminate with (3.34), at least in the region H of (3.27). Here we don't care about what happens on $B(r, Nr/5) \setminus H$ which was treated at the end of the argument, because this region does not meet the given Whitney region $W_M(x,r)$ if N is large enough and η_2 is small enough. That is, if we only care about $W_M(x,r)$, (3.34) is enough for the corresponding analogue of (3.14).

Now Remark 3.36 says that instead of (3.34) we can prove (3.38), which is just similar but controls the difference of gradients. That is, repeating (3.38), we claim that we have the better estimate

$$|\nabla D_{\beta,\mu}(X) - \nabla D(X)| \leq C \operatorname{dist}(X,P)^{2+\beta} N^{-\beta-1} r^{-\beta-2}, \tag{4.7}$$

where $D(X) = (ca_\beta)^{-1/\beta} \operatorname{dist}(X,P)$ as needed (see the line below (3.31)), and which is valid in the region

$$H(N, \eta_2') = \left\{X \in B(x, Nr/5)\,;\ \operatorname{dist}(X,P) \geq C\eta_2' r\right\} \tag{4.8}$$

where $\eta_2' = C_1 N \eta^{1/(d+3+\beta)}$ and $\eta_1 = CN\eta^{1/(d+1)}$. This estimate will be a good replacement for (3.14), but let us check that we do not go wrong with the management of Whitney regions.

Recall that we want to prove that ∇G is prevalently close to $\nabla D_{\beta,\mu}$. We give ourselves $\varepsilon > 0$, $M \geq 1$, and we need to show that for $(x,r) \in E \times (0,+\infty)$, $(x,r) \in \mathcal{G}_{\nabla G D_\beta}(\varepsilon, M)$ prevalently, which means that we want to prove (4.4). We may assume that $(x,r) \in \mathcal{G}_{\nabla G d}(\varepsilon/2, M)$, because this is a prevalent condition. So we have (4.3), and we only need to check that

$$\int_{W_M(x,r)} |\nabla \operatorname{dist}(X,P) - c\nabla D_\beta(X)|^2 \leq \varepsilon r^n/2. \tag{4.9}$$

We may also assume that $(x,r) \in \mathcal{G}_\alpha(\eta, N)$, with values of η and N that we can choose, and then we get (4.7), but maybe for some other d-plane P' that comes from the α-number. Let us first assume that $P' = P$, and check that (4.7) then implies (4.9). In particular we need to make sure that in (4.7), $C \operatorname{dist}(X,P)^{2+\beta} N^{-\beta-1} r^{-\beta-1} \leq \varepsilon/2$ when $X \in B(x, Mr)$; we can do this by taking N large enough, depending on M and ε. We also need $H(N, \eta_2')$ above to contain $W_M(x,r)$, which is easy to arrange by taking N large and then η small. So (4.7) would imply (4.9) if we had the same plane.

Maybe P, that we get for our proof that $(x,r) \in \mathcal{G}_{\nabla G d}(\varepsilon/2, M)$ prevalently, and P', that we choose in terms of α-numbers (because $(x,r) \in \mathcal{G}_\alpha(\eta, N)$), are not the same. But we can choose them as close to each other as we want, in particular because the α-numbers also control the flatness, by (3.12) and (3.13). See an analogous argument in the proof of Theorem 3.1. Then we can control the difference between $\nabla \operatorname{dist}(X,P)$ and $\nabla D(X)$, where D is the distance associated to $\operatorname{dist}(X,P')$, and this is enough for (4.9). This ends the proof for Case 1.

Now we need to follow the proof of Theorem 3.43 to take care of Case 2. In this case, we can follow the proof above and, when it comes to applying Theorem 2.19, we observe as in Case 1 that Lemma 2.29 gives the L^2 convergence of the ∇G_k to G^∞ away from $E_\infty = P$. The approximation by $\nabla D_{\beta,\nu}$ is treated exactly as in Case 1 (we made sure not to use anything specific, just the good approximation of $\nabla D_{\beta,\nu}(X)$ by $\nabla \operatorname{dist}(X,P)$). □

Remark 4.10. Assume that $A_k = I$ in Case 2 (this is the center of our interests anyway). Then the coefficients of our operator $L_{\alpha,\mu}$ are smooth away from E, so we could replace the L^2 norm in (4.3) and (4.4) by a L^∞ norm, and even write prevalent approximation properties with higher gradients. We decided not to check any details here, and anyway the best way to prove such estimates is probably to start from the fact that G or its first gradient is close to $D(X) = \operatorname{dist}(X,P)$, or ∇D, and then follow the same route as for Lemma 2.29. That is, we would say that in balls $B(Y,R) \subset \Omega_\infty$, G is prevalently close to (some) D, that $L_{\alpha,\mu} G = 0$ while $\Delta D = 0$, that these two operators are prevalently close to each other (including for the derivatives of coefficients), and use some integrations by parts to conclude that the derivatives of G are also close to the derivatives of D in, say, $B(Y, R/2)$. The difference between the derivatives of $D_{\beta,\nu}$ and those of D can be handled by Remark 3.36.

5. Local variants with G^X

Maybe the reader does not like too much the function G^∞, and prefers to use Green functions G^Y with poles at finite distance. In this section we prove local estimates with G^Y, which will follow from the previous proofs because the difference between G^Y and G^∞ is quite small, in particular in small balls $B(x,r) \subset B(x_0,r_0)$, $r \ll r_0$. We need a local notion of prevalence.

Definition 5.1. Let $E \subset \mathbb{R}^n$ be Ahlfors regular of dimension d, and let $B_0 = B(x_0, r_0)$ be a ball centered on E. We say that $\mathcal{G} \subset (E \cap B_0) \times (0, r_0)$ is locally prevalent in B_0 when its complement relative to B_0, $(E \cap B_0) \times (0, r_0) \setminus \mathcal{G}$, satisfies the Carleson Packing condition (1.6). Then the (best) constant C in (1.6) is also called the prevalence constant for $\mathcal{G}$ in B_0.

It may happen that $\mathcal{G}$ comes from a globally defined subset of $E \times (0, +\infty)$, but this will not always be the case, typically because $\mathcal{G}$ depends on the choice of a pole that we take far enough from $E \cap B_0$, but anyway when this is the case we simply forget about that part of $\mathcal{G}$ that does not lie in $(E \cap B_0) \times (0, r_0)$.

More importantly, we usually ask for uniform estimates, by which we mean that the Carleson constant C in (1.6) does not depend on B_0 (and the associated set $\mathcal{G}$).

Now what sets $\mathcal{G}$ (associated to B_0) shall we consider? Let $C_0 \geq 1$ be given; we shall only consider poles Y such that

$$\operatorname{dist}(Y, E \cap B_0) \geq C_0^{-1} r_0 \tag{5.2}$$

and then, for the local prevalent approximation of G^Y by a distance function, we select a pole Y such that (5.2) holds, and use the good set $\mathcal{G}_{Gd}^{B_0,Y}(\varepsilon, M)$ of pairs $(x,r) \in (E \cap B_0) \times (0, r_0)$ such that there exists a d-plane $P(x,r)$ and a positive constant $c > 0$, with

$$|\operatorname{dist}(X, P) - cG^Y(X)| \leq \varepsilon r \text{ for } X \in \Omega \cap B(x, Mr). \tag{5.3}$$

For the approximation of G^Y by a distance function $D_{\beta,\nu}$, we use instead the good set $\mathcal{G}_{GD_\beta}^{B_0,Y}(\varepsilon, M)$ defined the same way, but with (5.3) replaced by

$$|D_{\beta,\nu} - cG^Y(X)| \leq \varepsilon r \text{ for } X \in \Omega \cap B(x, Mr). \tag{5.4}$$

Finally we define the prevalent approximation property in the way that will suit our results.

Definition 5.5. Let $E \subset \mathbb{R}^n$ and L be as above. We say that the G^Y are locally prevalently close to the distance to a plane (respectively, locally prevalently close to the function $D_{\beta,\nu}$) when for each ball $B_0 = B(x_0, r_0)$ and each $Y \in \Omega$ such that (5.2) holds, the set $\mathcal{G}_{Gd}^{B_0,Y}(\varepsilon, M)$ (respectively, $\mathcal{G}_{GD_\beta}^{B_0,Y}(\varepsilon, M)$) is locally prevalent in B_0, with constants that do not depend on B_0 or on Y satisfying (5.2).

Notice that this definition depends on C_0; this will not disturb us, because we will be able to prove the desired result for any C_0, and with Carleson constants that do not even depend too badly on C_0. In fact, with a little bit of manipulations of the definitions, it is easy to check that different constants C_0 give the same notion of local prevalence (but different constants). The proof below gives a good idea of that.

Theorem 5.6. *Retain the assumptions of Theorem 3.1 or 3.43. Denote by G^Y, $Y \in \Omega$, the Green function associated to L in Ω with pole at Y. Then the G^Y are locally prevalently close to the distance to a plane, and to any function $D_{\beta,\nu}$ where $\beta > 0$ and ν is an AR measure on E.*

As before, in the second statement, we are even allowed to define $D_{\beta,\nu}$ with a different AR measure ν than the one in the definition of L_α.

The reader should not be too surprised that we also allow poles Y that are far from $E \cap B_0$ but close to (faraway parts of) E; the comparison principle says that the corresponding Green functions do not behave in a different way than if we forced Y_k to be corkscrew points in large balls centered at x_0. Of course we may need to normalize differently, i.e., choose very different constants c in (5.3) or (5.4), but this is all right.

We do not expect to have a good approximation of G^Y in the balls $B(x, r)$ such that $x \in E \cap 2B_0$ and $r \geq C^{-1} r_0$, even when E is a plane and $L = \Delta$, but those balls will satisfy a Carleson packing condition. Indeed, once ε and M are given (as in (5.3) or (5.4)), our first action is to remove the set

$$\mathcal{B}_0 = \big\{(x, r)\,;\, x \in E \cap B_0 \text{ and } \tau r_0 \leq r \leq r_0\big\} \tag{5.7}$$

for some very small $\tau \in (0, 1)$ that we allow to depend on ε, M, and C_0. We can do this because $\mathcal{B}_0$ satisfies a Carleson packing condition (with a constant that depends on τ and the AR constants).

Now we pick $(x, r) \in (E \cap B_0) \times (0, \tau r_0)$, and follow the same argument as in the proofs above. Essentially, we use the fact that r is so small that Y will appear as very far, and seen from $B(x, r)$, G^Y will look a lot as G^∞. Let us for instance consider Case 1, for the approximation by distances to planes. There is a moment, in the proof of Lemma 3.5, where we want to show by compactness that if (x, r) satisfies some prevalently good properties, then $(x, r) \in \mathcal{G}_{Gd}^{B_0,Y}(\varepsilon, M)$. We approach it by contradiction and find a sequence of counterexamples $(\Omega_k, E_k, L_k, x_k, y_k)$, and now we also need to add a larger ball $B_{0,k} = B(x_{0,k}, r_{0,k})$ and a pole Y_k such that $\operatorname{dist}(Y_k, E_k \cap B_{0,k}) \geq C_0^{-1} r_{0,k}$.

We also let τ_k tend to 0, and recall that $r_k \leq \tau_k r_{0,k}$. Finally, G_k is no longer a Green function at ∞, but rather $G_k = G_{L_k}^{Y_k}$, with the pole Y_k.

As before we normalize things so that $x_k = 0$ and $r_k = 1$. Then $|Y_k| \geq C^{-1} r_{0,k} \geq C_0^{-1}\tau_k^{-1}$ tends to $+\infty$, and by Theorem 2.19 and Remark 2.23, the functions $c_k G_k^{Y_k}$ (correctly normalized by constants c_k) converge to the Green function G_∞^∞ for Ω_∞ and L_∞. From there on, we can keep the same proof. Case 2 is similar, and for the approximation with our special distances $D_{\beta,\nu}$, we only need to observe that since $D_{\beta,\nu}$ and the distance to a plane are prevalently close to each other, the result stays true locally, just by definitions. We leave the details to the reader. □

Theorem 5.8. *Retain the assumptions of Theorem 3.1 or 3.43. Denote by G^Y, $Y \in \Omega$, the Green function associated to L in Ω with pole at Y. Then the ∇G^Y are locally prevalently close to the gradient of the distance to a plane, and to any $\nabla D_{\beta,\nu}$ where $\beta > 0$ and ν is an AR measure on E.*

The proof would follow the proof of Theorem 4.5, using the fact that Lemma 2.29 remains valid in the conditions of Remark 2.23. We could also deduce the approximation of the gradients from the approximation of the functions, as in the proof of Lemma 2.29 itself.

6. The basic converse, with the distance to a d-plane

We want to say that when $\Omega \subset \mathbb{R}^n$ is bounded by an unbounded AR set $E = \partial\Omega$ of integer dimension d, and the Green function G^∞ associated to one of our operators is prevalently close to the distance to a d-plane, then E is uniformly rectifiable.

Let us state the assumptions of our theorem in advance. We are given a domain $\Omega \subset \mathbb{R}^n$, whose boundary $E = \partial\Omega$ is an (unbounded) AR set of integer dimension $d \leq n-1$. When $d = n-1$, we also assume that Ω contains corkscrew balls and Harnack chains, as in (2.1). Recall that when $d < n-1$, we don't need to ask for (2.1), because it is always true.

Then we are also given an operator $L = -\operatorname{div} A\nabla$, and we distinguish two cases. In Case 1, we assume that $d = n-1$ and that A satisfies the ellipticity conditions (2.4) and (2.5). In particular we do not need to know that L is close to a constant coefficient operator in any way: as in the next case, we just want to make sure that there is a reasonable definition of the Green function G^∞, as in Section 2, with some basic properties.

In Case 2, we allow any integer $0 < d \leq n-1$, and any operator (2.7), or, equivalently, $L = -\operatorname{div} A \operatorname{dist}(X,E)^{d-n+1}\nabla$, such that A satisfies the ellipticity conditions (2.4) and (2.5). This includes the good operators L_α of (1.2)-(1.3), but also any degenerate elliptic operator in the class that was studied in [8]. Again we need something like this because we want to quote general results concerning Harnack's inequality and the Green function, but no delicate geometry will be used concerning L.

Since d is integer in this Theorem, Case 2 automatically includes Case 1.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain bounded by an unbounded d-AR set $E = \partial\Omega$, $0 < d < n$ integer, and assume that when $d = n-1$ the domain Ω contains corkscrew balls and Harnack chains, as in (2.1). Let $L = -\operatorname{div} A \operatorname{dist}(X,E)^{d-n+1}\nabla$, with A elliptic as per (2.4), (2.5). Assume that the Green function G^∞ associated to Ω, L, with a pole at ∞, is prevalently close to the distance to a d-plane. Then E is uniformly rectifiable.*

Proof. Our assumption is that for every choice of $\varepsilon > 0$ and $M \geq 1$, the good set $\mathcal{G}_{Gd}(\varepsilon, M)$ of Definition 1.14 is prevalent. We want to show that for each $\tau > 0$, the other good set $\mathcal{G}_{ur}(\tau)$ of Definition 1.13 is prevalent, and the simplest way is to show that if M and ε are chosen correctly, $\mathcal{G}_{Gd}(\varepsilon, M) \subset \mathcal{G}_{ur}(\tau)$. In the present case, taking $M = 10$ will be enough.

So we pick $(x,r) \in \mathcal{G}_{Gd}(\varepsilon, 10)$ and by definition there is a d-plane P and a constant $c > 0$ such that, as in (1.15),

$$|\operatorname{dist}(X,P) - cG^\infty(X)| \leq \varepsilon r \text{ for } X \in \Omega \cap B(x,10r). \tag{6.2}$$

Recall that G^∞ is Hölder continuous on $\Omega \cap B(x,10r)$, so we can safely say that $\lim_{X\to y} G^\infty(X) = G^\infty(y) = 0$ when $y \in E \cap B(x,10r)$ without thinking about the notion of trace, and by (6.2)

$$\operatorname{dist}(y,P) \leq \varepsilon r \text{ for } y \in E \cap B(x,10r). \tag{6.3}$$

If we have this property alone, we get what is known as the weak geometric lemma, which is somewhat weaker than the uniform rectifiability of E, so we have to continue the argument and show that conversely,

$$\operatorname{dist}(X,E) \leq \tau r/2 \text{ for } X \in P \cap B(x,r). \tag{6.4}$$

Obviously, if we prove (6.4) and take $\varepsilon < \tau/2$, we get that $d_{x,r}(E,P) \leq \tau$, and hence $(x,r) \in \mathcal{G}_{ur}(\tau)$ as needed.

Set $u = cG^\infty$. We will derive (6.4) via estimates on the size of u. Since (6.3) says that $\operatorname{dist}(y,P) \leq \varepsilon r$ for $y \in E \cap B(x,10r)$, we now get that

$$u(X) \leq \operatorname{dist}(X,P) + \varepsilon r \leq 11r \text{ for } X \in \Omega \cap B(x,10r), \tag{6.5}$$

and we shall now find a point A_1 such that $u(A_1)$ is not too small. First take a corkscrew point A_0 for $B(x,r)$. That is, $A_0 \in B(x,r) \cap \Omega$ and $\operatorname{dist}(A_0,E) \geq C^{-1}r$, and then (6.3) allows us to find a new point A_1 such that $|A_1 - A_0| \leq (2C)^{-1}r$ (and so $A_1 \in \Omega \cap B(x,2r)$) and in addition $\operatorname{dist}(A_1,P) \geq (2C)^{-1}r$. Thus we can also use A_1 as a corkscrew point, and now

$$u(A_1) \geq (2C)^{-1}r - \varepsilon r \geq (3C)^{-1}r \tag{6.6}$$

by (6.2) (and if ε is chosen small enough). Now we claim that

$$u(X) \geq C[r^{-1}\,\mathrm{dist}(X,E)]^{\gamma}\,u(A_1) \geq c'r[r^{-1}\,\mathrm{dist}(X,E)]^{\gamma} \ \text{ for } X \in \Omega \cap B(x,r), \tag{6.7}$$

where $C \geq 1$, $\gamma > 0$, and then $c' > 0$ depend only on the geometric constants for Ω and the ellipticity (or degenerate ellipticity) constant for L. This will be basically due to the fact that the length of the Harnack chain from A_1 to X is $C\log(\,\mathrm{dist}(X,E)^{-1}r)$.

Indeed let $X \in \Omega \cap B(x,r)$, and pick $y \in E \cap B(x,2r)$ such that $|X-y| = \mathrm{dist}(X,E)$. Then pick for $k \geq 0$ a corkscrew point X_k for $B(y, 2^k\,\mathrm{dist}(X,E))$. We can take $X_0 = X$, and we stop as soon as $2^k\,\mathrm{dist}(X,E) \geq r$. Thus our final X_k lies in $B(y,2r) \subset B(x,4r)$, and its distance to E is more than $C^{-1}r$. By Harnack's inequality, $u(A_1) \leq Cu(X_k)$. We can also apply Harnack's inequality to consecutive points of the sequence, and get that $u(X_{j+1}) \leq Cu(X_j)$ for $0 \leq j < k$. Altogether, $u(A_1) \leq C^{k+1}u(X_0) = C^{k+1}u(X)$; (6.7) follows because $k+1 \sim \log_2(\,\mathrm{dist}(X,E)^{-1}r)$.

Now set

$$H = \big\{X \in B(x,2r)\,;\ \mathrm{dist}(X,P) \leq 2\varepsilon r\big\} \tag{6.8}$$

and let $X \in H \cap \Omega$ be given. By (6.2),

$$u(X) = cG^{\infty}(X) \leq \mathrm{dist}(X,P) + \varepsilon r \leq 3\varepsilon r \quad \text{for all} \quad X \in H.$$

Then by (6.7),

$$c'r[r^{-1}\,\mathrm{dist}(X,E)]^{\gamma} \leq u(X) \leq 3\varepsilon r,$$

hence $r^{-1}\,\mathrm{dist}(X,E) \leq C\varepsilon^{1/\gamma}$; we choose ε so small that $C\varepsilon^{1/\gamma} \leq \tau/3$, and get that

$$\mathrm{dist}(X,E) \leq \tau r/3 \ \text{ for } X \in H \cap \Omega \cap B(x,2r). \tag{6.9}$$

When $d < n-1$, E is so small that $\mathbb{R}^n \setminus E$ is connected, so $\mathbb{R}^n = \Omega \cup E$. Then for any $X \in P \cap B(x,2r)$, either $X \in \Omega$ and $\mathrm{dist}(X,E) \leq \tau r/3$ by (6.9), or else $X \in E$ and $\mathrm{dist}(X,E) = 0$. So (6.4) holds in this case.

We are left with the case when $d = n-1$. Recall that $\mathrm{dist}(A_1,P) \geq (2C)^{-1}r$, so A_1 lies in $B(x,2r) \setminus H$. Denote by V the component of $B(x,2r) \setminus H$ that contains A_1; then $V \subset \Omega$ because it meets Ω and not E (by (6.3) and (6.8)).

Now let $X \in P \cap B(x,r)$ be given. There exists $Y \in V$ such that $|Y - X| \leq 3\varepsilon$, and by (6.9) $\mathrm{dist}(Y,E) \leq \tau r/3$; if ε is small enough, this implies that $\mathrm{dist}(X,E) \leq \tau r/2$; this proves (6.4) and Theorem 6.1 follows. $\square$

Remark 6.10. Notice that in the course of the proof of Theorem 6.1 we hardly use the fact that G^{∞} is a Green function (we only use the fact that it vanishes on E and the nondegeneracy property (6.7)). This, however, is not shocking – see, e.g., [6], where the free boundary results also rely very moderately on the particular properties of the distance function. The situation may be even more peculiar for $d = n-1$ because it could happen (but the authors did not manage to prove this) that if E is Ahlfors regular of dimension $n-1$ and satisfies the Weak Geometric Lemma (see for instance [15]), and, in addition, Ω contains corkscrew balls and Harnack chains, then E is uniformly rectifiable. If this were true, then we could stop the argument as soon as we proved (6.3), and the only properties of G^∞ that we would use are the prevalent good approximation and the fact that $G^\infty = 0$ on E.

Remark 6.11. Theorem 6.1 also holds when we assume that the Green functions G^Y are locally prevalently close to distances to a d-plane (see Definition 5.5). The proof is the same; that is, we only used the fact that the function $u = cG^\infty$ is L-harmonic in $\Omega \cap B(x, 10r)$, and reducing to the case when the pole Y lies outside of $B(x, 10r)$ is just a matter of skipping the large balls $B(x, r)$, $r \geq r_0/C$, i.e., restricting our attention to pairs $(x, r) \in (E \cap B(x_0, r_0)) \times (0, r_0/C)$, in Definition 5.1 of local prevalence.

Remark 6.12. Theorem 6.1 also holds when we replace $\mathrm{dist}(X, E)$ with any power $\mathrm{dist}(X, E)^\alpha$, with the same proof. This is a quite natural condition as the Green function is expected to be only Hölder continuous at the boundary, that is, to behave as a power of the distance.

Remark 6.13. In this section and the next ones, we give statements with distance functions, and not their gradients. The results of these sections should also hold with the prevalent approximation with the gradients, but we decided not to check them. Note however that in principle conditions on the good approximation of ∇G are stronger than those on G itself, because G can essentially be computed from its gradient.

7. The converse with the distance D_α

We are now interested in finding out whether if the Green function G^∞ associated to one of our operators is prevalently close to some D_α (and later in this section, to some power of D_α), then d is an integer and E is uniformly rectifiable. The situation is more challenging than with the distance to d-planes, first because we allow dimensions that are not integers (this made little sense when discussing the distance to a plane), and also, due to the fact that D_α vanishes on E, and not on a d-plane as in Section 6, we cannot simply use the fact that D_α is small near E; that is, the answer is no longer hidden in the question.

Contrary to the previous section, we are now restricted to our special classes of elliptic operators, satisfying (1.9) or built from the distance functions. We start with the most classical case of co-dimension 1.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be such that $E = \partial\Omega$ is Ahlfors regular of dimension $n-1$ and assume, in addition, that the domain Ω has interior corkscrew balls and Harnack chains, that is,* (2.1) *holds. Let $L = -\operatorname{div} A\nabla$ satisfy* (2.4), (2.5), *and* (1.9). *Suppose that the Green function G^∞ for L on Ω is prevalently close to $D_{\alpha,\mu}$ for some $\alpha > 0$ and some AR measure μ of dimension $n-1$ on E. Then E is uniformly rectifiable.*

Proof. This time we shall not use Definition 1.13 to prove the uniform rectifiability, but rather show that, in addition to being 1-sided NTA, Ω has exterior corkscrew points. This is also known as <u>Condition B</u>. We say that the (unbounded) AR set E of dimension $n-1$ satisfies Condition B when there is a constant $c > 0$ such that for $x \in E$ and $r > 0$,

$$\begin{aligned} &\exists\, X_1, X_2 \in B(x,r), \text{ that lie in different connected components} \\ &\quad \text{of } \mathbb{R}^n \setminus E, \text{ and such that } \operatorname{dist}(X_i, E) \geq cr \text{ for } i = 1,2. \end{aligned} \tag{7.2}$$

This condition was introduced by S. Semmes in [26], and is known to imply the uniform rectifiability of E (see [26] in the smooth case and, probably for the simplest proof, [11]).

This property can be also written in terms of prevalent sets. Let $\mathcal{G}_{CB}(c)$ denote the set of $(x,r) \in E \times (0,+\infty)$ such that (7.2) holds. The definition requires that $\mathcal{G}_{CB}(c) = E \times (0,+\infty)$, but we claim that if the set $\mathcal{G}_{CB}(c)$ is prevalent for some $c > 0$, then E satisfies Condition B (with a worse constant). Indeed, if $\mathcal{G}_{CB}(c)$ is prevalent, we claim that there is a constant $a > 0$ such that for each $(x,r) \in E \times (0,+\infty)$, we can find $y \in E \cap B(x,r/2)$ and $t \in (ar, r/2)$ such that $(y,t) \in \mathcal{G}_{CB}(c)$. Otherwise the set $E \cap B(x,r/2) \times (ar, r/2)$ is contained in the bad set $\mathcal{B} = E \times (0,+\infty) \setminus \mathcal{G}_{CB}(c)$, and then

$$\begin{aligned} \int\limits_{y \in E \cap B(x,r)} \int\limits_{0<t<r} \mathbb{1}_{\mathcal{B}}(y,t) \frac{d\mu(y)dt}{t} \quad & \int\limits_{y \in E \cap B(x,r/2)} \int\limits_{ar<t<r/2} \frac{d\mu(y)dt}{t} \\ & \mu(E \cap B(x,r/2) \int\limits_{ar<t<r/2} \frac{dt}{t} \geq C^{-1} r^d \ln(1/(2a)), \end{aligned} \tag{7.3}$$

which contradicts (1.6) if a is chosen small enough, depending on the Carleson constant for $\mathcal{B}$. But once we obtain that pair (y,t), the two points X_1 and X_2 that we get from (7.2) for $B(y,t)$ also work for $B(x,r)$, although with the worse constant ac.

Return to the theorem, and let Ω, E, L satisfy the assumptions. We only need to find $c > 0$ such that $\mathcal{G}_{CB}(c)$ is prevalent. Since we know that for each choice of ε and M, the sets $\mathcal{G}_{GD_\alpha}(\varepsilon, M)$ and $\mathcal{G}_{cc}(\varepsilon, M)$ are prevalent, it is enough to show that for some choice of $c > 0$, M, and ε

$$\mathcal{G}_{GD_\alpha}(\varepsilon, M) \cap \mathcal{G}_{cc}(\varepsilon, M) \subset \mathcal{G}_{CB}(c). \tag{7.4}$$

We proceed by contradiction. Assume that (7.4) fails for $c_k = \varepsilon_k = M_k^{-1} = 2^{-k}$, and pick an open set Ω_k bounded by an AR set E_k, an operator L_k, and a pair $(x_k, r_k) \in$

$E \times (0, +\infty)$ that all satisfy the assumptions, and for which $(x_k, r_k) \in \mathcal{G}_{GD_\alpha}(\varepsilon_k, M_k) \cap \mathcal{G}_{cc}(\varepsilon_k, M_k) \setminus \mathcal{G}_{CB}(c_k)$.

By translation and dilation invariance, we may assume that $x_k = 0$ and $r_k = 1$. We may also extract a subsequence so that Ω_k converges to a limit Ω_∞, E_k converges to E_∞, and per Theorem 2.19, Ω_∞ is bounded by E_∞ and Ω_∞ satisfies the assumption (2.1). Moreover, we can extract a further subsequence so that the AR measure μ_k given on E_k converges weakly to a measure μ, and this limit is automatically AR. Because $(0,1) \in \mathcal{G}_{cc}(2^{-k}, 2^k)$, the matrix A_k of L_k converges, in $L^1_{loc}(\Omega_\infty)$, to a constant matrix A_0.

By Theorem 2.19, the Green function $G_k = G_k^\infty$ for L_k in Ω_k, correctly normalized, converges to the Green function $G = G_\infty^\infty$ for L_0 on Ω_∞, uniformly on compact sets of Ω_∞. Here L_0 is the constant coefficients operator $L_0 = -\operatorname{div} A_0 \nabla$. Hence, using the facts that $(0,1) \in \mathcal{G}_{GD_\alpha}(2^{-k}, 2^k)$ and μ_k stays uniformly AR and converges to μ, we can conclude that $G = CD_{\alpha,\mu}$.

We shall start many of our "free boundary" compactness arguments in this fashion, but now we need to use the specific assumption of co-dimension 1. Why does the fact that $G = CD_{\alpha,\mu}$ imply something nice about the domain? The simplest path would be to brutally use a known fact that on 1-sided NTA domains if the harmonic measure (associated to a constant coefficient elliptic operator on Ω) is absolutely continuous with respect to μ (any AR measure on E) and given by an A_∞ weight, then E is uniformly rectifiable. This is essentially due to [21]. A careful reader might notice that formally speaking, they treat the Laplacian only, but this extends to all constant coefficients symmetric elliptic operators, and then a symmetrization argument could be used to get rid of the assumption of symmetry. See, e.g., [19] where an analogous result is obtained in much bigger generality (so it could serve as a reference by itself), but in particular, the symmetrization is discussed in the beginning of Proof of Theorem 1.6.

Our limiting domain Ω_∞, with the constant coefficients operator L_0, admits the Green function $G = CD_{\alpha,\mu}$, and in particular

$$C^{-1} \operatorname{dist}(X, E) \leq G(X) \leq C \operatorname{dist}(X, E) \text{ for } X \in \Omega_\infty. \tag{7.5}$$

We claim that this implies that the harmonic measure ω^∞, with a pole at ∞, is also comparable to $\mathcal{H}^{n-1}_{|E_\infty}$, in the sense that

$$C^{-1} \mathcal{H}^{n-1}_{|E_\infty} \leq \omega^\infty \leq C \mathcal{H}^{n-1}_{|E_\infty} \quad \text{on} \quad E_\infty.$$

Once this claim is established, we can apply the aforementioned result and obtain that

$$E_\infty \text{ is uniformly rectifiable.} \tag{7.6}$$

The claim is due to the following estimate for the Green function, valid on any 1-sided NTA domain with an unbounded d-dimensional Ahlfors regular boundary when $d \geq n-1$, and on any domain with an unbounded d-dimensional Ahlfors regular boundary when

$d < n-1$. We consider the Green function with pole at a faraway point Y for any elliptic operator L on E, with (2.4) and (2.5), or the operator given by (2.6). Under these assumptions, there is a constant $C \geq 1$, that depends on the dimensions and the constants in our assumptions on Ω and L, such that if $B = B(x,r)$ is a ball centered on $E = \partial\Omega$, and $Z^B \in B \cap \Omega$ denotes a corkscrew point for B,

$$C^{-1} r^{n-2} G^Y(Z^B) \leq \omega^Y(E \cap B) \leq C r^{n-2} G^Y(Z^B) \tag{7.7}$$

as soon as the pole Y lies out of $2B$. We refer to Lemma 15.28 in [10], but in co-dimension 1 this is of course a classical result. To be careful with the normalization of the Green functions, let us consider balls B that are contained in B_0 for some fixed ball B_0 and pick any $Y \in \Omega \setminus 2B_0$. The comparison principle says that $C^{-1} G^Y \leq G^\infty \leq C G^Y$, with a uniform constant, if for instance we normalize G^∞ by $G^\infty(Z^0) = G^Y(Z^0)$ for a corkscrew point Z^0 for B_0. Then by (7.5)

$$C^{-1} r \leq c\, G^Y(Z^B) \leq Cr, \tag{7.8}$$

where the normalizing constant c depends on B_0, but not C. By (7.7), this yields

$$C^{-1} r^{n-1} \leq c\, \omega^Y(E \cap B) \leq C r^{n-1} \quad \text{for } B \subset B_0, \tag{7.9}$$

which implies by an easy covering argument that ω^Y is essentially proportional to $\mathcal{H}^{n-1}$ on $E \cap \frac{1}{2} B_0$. So we may apply [21] and get (7.6).

Recall that we want to use (7.6) to derive a contradiction, in fact with our assumption that $(0,1) = (x_k, r_k) \notin \mathcal{G}_{CB}(c_k)$.

Let $\varepsilon > 0$ small, to be chosen soon. We know that the set $\mathcal{G}_{ur}(\varepsilon)$ associated to E_∞ is prevalent, so by the same argument as in (7.3) we can find $y \in E_\infty \cap B(0, 1/2)$ and $t \in (a, 1/2)$ such that $d_{y,t}(E_\infty, P) \leq \varepsilon$ for some hyperplane P.

Denote by H_+ and H_- the two connected components of $H = \big\{X \in B(y,a);\ \mathrm{dist}(X,P) \geq 2a\varepsilon r\big\}$, and also let Z^0 be a corkscrew point for Ω_∞ in $B(0,a)$; we know that it exists because $0 \in E_\infty$ and Ω_∞ satisfies (2.1). Then $\mathrm{dist}(Z^0, P) \geq C^{-1} a$ because $d_{y,t}(E_\infty, P) \leq \varepsilon$, and so $Z^0 \in H_+$, say.

For k large, H_+ and H_- do not meet E, because $d_{y,t}(E_\infty, P) \leq \varepsilon$ and E_∞ is the limit of E_k; then we know that $H_+ \subset \Omega_k$ (because $Z^0 \in \Omega_k$ for k large), and we have two options. The pleasant one is when H_- is contained in some other component of $\mathbb{R}^n \setminus \Omega_k$, because this contradicts immediately our contradiction assumption that $(0,1) \notin \mathcal{G}_{CB}(c_k)$ (relative to Ω_k): the two points Z^0 and its symmetric relative to P would fit in (7.2). The other option is that H_- meets Ω_k, hence $H_- \subset \Omega_k$ (because it does not meet E_k). This is impossible too, this time because if ε is small enough, we contradict the existence of Harnack chains in Ω_k; see the argument below (3.4).

We finally proved (7.4), and Theorem 7.1 follows. $\square$

Remark 7.10. It seems that we should not be forced to use [21,19]: this is a rather indirect route, and we have a detailed information about the Green function at hand, in particular, we know that it is a multiple of a suitably defined distance to the boundary, not just the fact that it satisfies the bounds from above and below (7.5). Yet, these matters are highly non-trivial, both for the Green function and for the harmonic measure. For instance, it is known that there are cones E, other than hyperplanes, such that ω^∞ is proportional to $\mathcal{H}^{n-1}_{|E}$. We do not know (but did not try to make a computation) whether the Green function will be a multiple of D_α for these examples. It could happen that in the situation of $G = CD_{\mu,\alpha}$, μ really must be a flat measure (instead of being merely uniformly rectifiable, as in (7.6). We will return to this issue in the next section.

We now switch to the question of prevalent approximation by $D_{\alpha,\mu}$ of the Green functions for elliptic operators in dimensions $d \in (n-2,n) \setminus \{n-1\}$. We want to say that this does not happen. That is, the fact that the Green function is prevalently close to $D_{\alpha,\mu}$ implies that the dimension is integer, for the operators in Theorem 7.1, and, more generally, for all elliptic operators.

Proposition 7.11. *Let $\Omega \subset \mathbb{R}^n$ be such that $E = \partial\Omega$ is Ahlfors regular of dimension $d \in (n-2,n) \setminus \{n-1\}$. If $d > n-1$, suppose in addition that Ω contains corkscrew balls and Harnack chains, as in (2.1). Then let $L = -\operatorname{div} A\nabla$ be an elliptic operator (as in (2.4) and (2.5)), and denote by G^∞ the Green function for L on Ω. Then G^∞ is not prevalently close to $D_{\alpha,\mu}$ for any choice of $\alpha > 0$ and an AR measure μ of dimension d on E.*

This was demoted to the rank of proposition, because it will appear soon that the approximation fails just because $D_{\alpha,\mu}$ does not have the right homogeneity.

Proof. Suppose the Proposition fails, so that we can find an example (Ω, E, L) for which G^∞ is prevalently close to some $D_{\alpha,\mu}$. This means in particular that we can find pairs $(x,r) \in \mathcal{G}_{GD_\alpha}(\varepsilon, M)$, with arbitrary values of ε and M.

Pick such a pair; by translation and dilation invariance we may assume that $(x,r) = (0,1)$, so we deduce from (1.16) that there is a constant $c > 0$ such that

$$|D_{\alpha,\mu}(X) - cG^\infty(X)| \leq \varepsilon \ \text{ for } X \in \Omega \cap B(0,M). \tag{7.12}$$

In particular, recalling that $D_{\alpha,\mu}(X) \sim \operatorname{dist}(X,E)$ and if ε is small enough, we get that

$$C^{-1}\operatorname{dist}(X,E) \leq cG^\infty(X) \leq C\operatorname{dist}(X,E) \tag{7.13}$$

for $X \in \Omega \cap B(0,M)$ such that $\operatorname{dist}(X,E)) \geq 1$. Now pick a pole $Y \in \Omega \setminus B(0,10M)$ and a corkscrew point X_0 for $B(0,M)$, then set

$$\lambda = G^Y(X_0)/G^\infty(X_0) \tag{7.14}$$

and apply the comparison principle to the two function G^∞ and G^Y, say, in $\Omega \cap B(0, 2M)$. We get that

$$C^{-1}\lambda \leq \frac{G^Y(X)}{G^\infty(X)} \leq C\lambda \quad \text{for } X \in \Omega \cap B(0, M) \tag{7.15}$$

and, because of (7.13),

$$C^{-1} \operatorname{dist}(X, E) \leq \frac{c}{\lambda} G^Y(X) \leq C \operatorname{dist}(X, E) \tag{7.16}$$

when in addition $\operatorname{dist}(X, E)) \geq 1$. Now consider a ball $B = B(x, r) \subset B(0, M)$, and choose a corkscrew point Z^B for B. We restrict to the case when $r \geq C_0$, where C_0 is chosen so that (7.16) always holds for Z^B and yields

$$G^Y(Z^B) \sim \frac{\lambda}{c} \operatorname{dist}(X, E) \sim \frac{\lambda}{c} r.$$

Then apply (7.7) to B; this yields

$$C^{-1} \frac{\lambda}{c} r^{n-1} \leq \omega^Y(E \cap B(x, r)) \leq C \frac{\lambda}{c} r^{n-1}. \tag{7.17}$$

We claim that this is incompatible with the Ahlfors regularity of E. We first take $B_0 = B(0, M/2)$ and find that $\omega^Y(E \cap B_0) \sim \frac{\lambda}{c} M^{n-1}$.

First assume that $d > n - 1$; we can find more than $C^{-1}M^d$ disjoint balls B of radius C_0 that are contained in B_0, and hence (7.17) yields

$$\frac{\lambda}{c} M^{n-1} \gtrsim \omega^Y(E \cap B_0) \geq \sum_B \omega^Y(E \cap B_0) \geq C^{-1} \frac{\lambda}{c} M^d,$$

a contradiction if M is large enough and $d > n - 1$. Observe that $\frac{\lambda}{c}$ disappears as it should (we could have replaced G^∞ by a multiple so that $\frac{\lambda}{c} = 1$ anyway).

If instead $d < n - 1$ we can cover B_0 by less than CM^d balls B of radius C_0 that are contained in $B(0, M)$, and (7.17) yields

$$\frac{\lambda}{c} M^{n-1} \lesssim \omega^Y(E \cap B_0) \leq \sum_B \omega^Y(E \cap B_0) \leq C \frac{\lambda}{c} M^d,$$

a contradiction again for M large enough.

This completes our proof of Proposition 7.11. $\square$

Remark 7.18. The fact that proving Proposition 7.11 was a question of homogeneity should convince us to try something different in our Case 1 (elliptic operators): approximate G^∞ with the power $D_{\alpha,\mu}(X)^{d+2-n}$, because this is the only one for which (7.7) leaves us a fighting chance. (Notice that when $d = n - 1$ we keep the power 1, as in

Theorem 7.1, and this issue does not arise). It turns out such a power is indeed natural in a much more general context, but unfortunately – perhaps for this reason – the corresponding results are much more intricate as well. We turn to this question in the next section.

8. The condition $\Upsilon_{\mathbf{flat}}$, higher co-dimension, and more exotic free boundary results.

Many known free boundary results rely on some "zero-level" flatness statement, that is, a degenerate version of the hypothesis entails that the set is a hyperplane. In our context, such a degenerate hypothesis is an equality between the Green function and the distance of our choice. This was not a problem in Section 6: if the Green function is assumed to be the distance to a plane, then of course the boundary is a plane simply because the Green function ought to vanish on it. However, when working with D_α, an analogous statement is not as clear: both the Green function and D_α vanish on the boundary E, whatever E is. We were lucky not to have to prove any "zero-level" flatness in Section 7, by virtue of the results in [21], and, as far as the harmonic measure goes, a straightforward zero-level flatness statement would not even be true – see Remark 7.10, but as we pointed out in the same remark, it seems reasonable to expect that such a statement would nonetheless hold for the Green function. Unfortunately, we do not manage to prove it at this point for the desired class of operators at hand, and will simply state it as a hypothesis.

Definition 8.1. Let the dimension $0 < d < n$, with $d \neq n-2$, $\alpha > 0$, and a constant coefficient elliptic operator $L_0 = -\operatorname{div} A_0 \nabla$ be given. We say that the property $\Upsilon_{\rm flat}(d, \alpha, L_0)$ is true when for each domain $\Omega \subset \mathbb{R}^n$ such that $E = \partial\Omega$ is AR of dimension d and (2.1) is satisfied when $d \geq n-1$, the following holds. If μ is an AR measure of dimension d on E and

$$L_0 D_{\alpha,\mu}^{d+2-n} = 0 \quad \text{on } \Omega, \tag{8.2}$$

then d is an integer, E is a d-plane, and μ is a flat measure.

Remark 8.3. Our forthcoming results will primarily rely on the case when $L_0 = -\Delta$. In fact, when L is associated to a constant coefficient operator $L_0 \neq -\Delta$, one could object that our definition of $D_{\alpha,\mu}$ is not as appropriate as when $L_0 = -\Delta$. That is, maybe we should use different distances like the D_α, but adapted to the operators L_0 that we intend to get at the limit. This could be more appropriate with respect to the rotation invariance, but it looks a little too far-fetched for this paper (and we do not want to have too many definitions).

Let us now discuss some particular cases and equivalent reformulations of the condition $\Upsilon_{\rm flat}(d, \alpha, L_0)$. Even when $d = n-1$ and $L_0 = -\Delta$, we do not know whether $\Upsilon_{\rm flat}(n-$

$1, \alpha, L_0)$ is valid. When $d \in (n-2, n) \setminus \{n-1\}$, the power in (8.2) is exactly the one that naturally arises in the proof of Proposition 7.11, see Remark 7.18. We could try to define an analogue of $\Upsilon_{\mathrm{flat}}(n-2, \alpha, L_0)$ (notice that the case $d = n-2$, as is, does not make much sense), but at this time we will not try to do so. When $d < n-2$, the exponent $d+2-n$ in Definition 8.1 is negative, and $D_{\alpha,\mu}^{d+2-n}$ becomes a function with a singularity near E.

To show that Definition 8.1 is reasonable we have to verify, at the very least, that (8.2) is satisfied if d is an integer, E is a d-plane, and μ is a flat measure. And indeed, if d is an integer and $\mu = c\mathcal{H}^d_{|P}$ is a flat measure, the scale invariance yields $R_{\alpha,\mu}(X) = C\,\mathrm{dist}(X,E)^{-1/\alpha}$, then $D_{\alpha,\mu}(X) = C\,\mathrm{dist}(X,E)$, and $D_{\alpha,\mu}^{d+2-n} = C\,\mathrm{dist}(X,E)^{d+2-n}$. When $d = n-1$ and Ω is a half-space, $D_{\alpha,\mu}^{d+2-n} = Ct$ (t denoting the coordinate perpendicular to the boundary), which is a solution to any constant coefficient elliptic equation, as desired. When $d < n-2$, we denote by t the projection of $X \in \mathbb{R}^n$ on $P^\perp \simeq \mathbb{R}^{n-d}$, and we see that $D_{\alpha,\mu}^{d+2-n} = C|t|^{d+2-n}$ is a function of t only. We can recognize a multiple of the Green function in $\mathbb{R}^{n-d}$. So $D_{\alpha,\mu}^{d+2-n}$ is indeed harmonic in that case. Alternatively, one could make a direct computation to check that $\Delta_{x,t}|t|^{d+2-n} = 0$ for $t \in \mathbb{R}^{n-d} \setminus 0$. These computations ensure that the condition Υ_{flat} is, at least, coherent. Does it have a reasonable chance to be true?

Let us look at an equivalent reformulation of the condition Υ_{flat}.

Lemma 8.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a domain with an unbounded d-Ahlfors regular boundary E, $0 < d < n$, and μ is a d-AR measure on E. Then for any $\alpha > 0$*

$$\Delta D_{\alpha,\mu}^{d+2-n} = 0 \text{ on } \Omega \text{ if and only if } L_{\alpha,\mu} D_{\alpha,\mu} = 0 \text{ on } \Omega, \tag{8.5}$$

where $L_{\alpha,\mu}$ is given by (1.2) with $D_\alpha = D_{\alpha,\mu}$.

In particular, for $0 < d < n$, with $d \neq n-2$, and $\alpha > 0$, the property $\Upsilon_{\mathrm{flat}}(d, \alpha, \Delta)$ is equivalent to the following. For each domain $\Omega \subset \mathbb{R}^n$ such that $E = \partial\Omega$ is AR of dimension d and (2.1) holds if $d \geq n-1$, if μ is an AR measure of dimension d on E and

$$L_{\alpha,\mu} D_{\alpha,\mu} = 0 \quad \text{on } \Omega, \tag{8.6}$$

then d is an integer, E is a d-plane, and μ is a flat measure.

Proof. The fact that (8.5) holds is the result of a direct computation. Notice that all these functions are smooth on Ω, so we can talk about strong solutions. The verification is easy, because

$$\nabla(D_{\alpha,\mu}^{d+2-n}) = (d+2-n) D_{\alpha,\mu}^{d+1-n} \nabla D_{\alpha,\mu},$$

so $\Delta D_{\alpha,\mu}^{d+2-n} = 0$ if and only if $\operatorname{div} D_{\alpha,\mu}^{d+1-n} \nabla D_{\alpha,\mu} = 0$; then we just need to compare with (1.2). Hence, $\Upsilon_{\rm flat}(d, \alpha, \Delta)$ says that $D_{\alpha,\mu}$ is only a solution of $L_{\alpha,\mu}$ in the trivial case of a flat measure, which yields (1) in the statement of the lemma. □

There is one emblematic case when Lemma 8.4's version of $\Upsilon_{\rm flat}(d, \alpha, \Delta)$ fails miserably; this is when $\alpha = n-d-2$, the "magic α" case that we discussed in the introduction. In this case, both conditions in (8.5) hold on any d-Ahlfors regular set, which does not even need to have an integer dimension, much less possess any regularity or uniform rectifiability. In the context of this paper this case is certainly not amenable to any free boundary results besides Theorem 6.1, not only because of the failure of $\Upsilon_{\rm flat}$, but because the Green function of $L = -\operatorname{div} D_{\alpha,\mu}^{-(n-d-1)} \nabla$ is a multiple of $D_{\alpha,\mu}$ and hence, the harmonic measure is equivalent to the Hausdorff measure of the boundary for any d-AR set E [6]. It remains to be seen whether $\Upsilon_{\rm flat}(n-2, \alpha, \Delta)$ holds in all other instances.

At this point, let us pass to our main results assuming Condition $\Upsilon_{\rm flat}$. As usual, we start with Case 1.

Theorem 8.7. *Let $d \in (n-2, n) \setminus \{n-1\}$ and $\alpha > 0$ be given, and suppose that the property $\Upsilon_{\rm flat}(d, \alpha, L_0)$ is true for every constant coefficient elliptic operator $L_0 = -\operatorname{div} A_0 \nabla$. Assume that E is Ahlfors regular of dimension d, L satisfies (2.4), (2.5), and (1.9), and Ω satisfies (2.1) if $d > n-1$. Then the Green function G^∞ for L on Ω is not prevalently close to $D_{\alpha,\mu}^{d+2-n}$ for any AR measure μ of dimension d on E.*

The question is already interesting when L is the Laplacian, and then of course Property $\Upsilon_{\rm flat}(d, \alpha, \Delta)$ is enough (see also Remark 8.3).

As we mentioned above, when $d = n-1$, we do not know whether $\Upsilon_{\rm flat}(n-1, \alpha, L_0)$ is true or not, but we proved Theorem 7.1 by different means.

Proof. We shall again prove this by contradiction and compactness. If G^∞ is prevalently close to $D_{\alpha,\mu}^{d+2-n}$, this means that we can find pairs (x, r) in the analogue $\mathcal{G}^*(\varepsilon, M)$ of $\mathcal{G}_{GD_\alpha}(\varepsilon, M)$ but with the power $d+2-n$, with arbitrary values of ε and M. We can even choose $(x, r) \in \mathcal{G}_{cc}(\varepsilon, M)$ too, because one of our assumptions, (1.9), says that this condition is prevalent. We want to show that this is impossible, and in fact we will prove a little more: for each choice of constants in the assumptions of the theorem, we can find ε and M such that if Ω, E, L satisfy the assumptions, $\mathcal{G}^*(\varepsilon, M) \cap \mathcal{G}_{cc}(\varepsilon, M)$ is empty.

We prove this by contradiction, so we suppose that for each $k \geq 0$, we can find a counterexample (Ω_k, E_k, L_k) where this fails for $\varepsilon_k = 2^{-k}$ and $M_k = 2^k$, and we want to derive a counterexample.

We are given a pair $(x_k, r_k) \in \mathcal{G}^*(\varepsilon_k, M_k) \cap \mathcal{G}_{cc}(\varepsilon_k, M_k)$, and by translation and dilation invariance we may assume that $x_k = 0$ and $r_k = 1$. Then we proceed as in Theorem 7.1, for instance, and extract a subsequence for which (Ω_k, E_k, L_k) converges to a triple $(\Omega_\infty, E_\infty, L_0)$. As before, this triple satisfies the assumptions of the theorem (because the (Ω_k, E_k, L_k) satisfy it uniformly), L_0 is a constant coefficient elliptic operator (because

$(0,1) \in \mathcal{G}_{cc}(\varepsilon_k, M_k))$, and we even know that after renormalization the G_k^∞ converge, uniformly on compact subsets of Ω_∞, to a Green function G for L_0 on Ω_∞. Finally, since $(0,1) \in \mathcal{G}^*(\varepsilon_k, M_k)$, we see that $G = D_{\alpha,\mu_\infty}^{d+2-n}$ for some d-AR measure μ_∞ that we obtain as a weak limit of the μ_k that were used to define $\mathcal{G}^*(\varepsilon_k, M_k)$ on Ω_k.

But our assumption $\Upsilon_{\text{flat}}(d, \alpha, L_0)$ says that this cannot happen (recall that $d \neq n-1$, so flat measures are not allowed). Theorem 8.7 follows. $\square$

Now we pass to Case 2. As before, we consider the question of whether the Green function for $L_{\alpha,\mu}$ is prevalently close to distance functions, and the most natural ones in this instance are clearly the distances $D_{\alpha,\mu}$ associated to the same $\alpha > 0$ and μ (see (8.6)).

The natural conjecture is now that $\Upsilon_{\text{flat}}(d, \alpha, \Delta)$ holds, unless $d + \alpha = n - 2$ or $d = n - 2$ (where it is not defined). At this time we have some additional information on this question, but no full proof, and for the moment we just state the consequence of a positive result.

Theorem 8.8. *Let $d \in (0, n) \setminus \{n-2\}$ and $\alpha > 0$, $\alpha \neq n-2-d$, be given, and suppose that the property $\Upsilon_{\text{flat}}(d, \alpha, \Delta)$ holds. Let furthermore Ω be a domain in $\mathbb{R}^n$, $E = \partial\Omega$ be d-Ahlfors regular, and assume that (2.1) holds if $d \geq n-1$. Let $L = L_{\alpha,\mu} = -\operatorname{div} A D_{\alpha,\mu}^{d+1-n} \nabla$ be the degenerate elliptic operator associated to an AR measure μ on E as in (1.3) and (1.4), with the matrix of coefficients A satisfying (2.4), (2.5), and (1.9) with $A_0 \equiv I$. If the Green function G^∞ for $L_{\alpha,\mu}$, with pole at ∞, is prevalently close to the distance function $D_{\alpha,\mu}$ (with the same α and μ) then d is an integer and E is uniformly rectifiable.*

Again $d + \alpha = n - 2$ is excluded here, because in this case $D_{\alpha,\mu}$ is the Green function no matter what.

We already know that the result holds when $d = n-1$, because in this case $L_{\alpha,\mu} = \Delta$ and we have Theorem 7.1, which fortunately does not need $\Upsilon_{\text{flat}}(d, \alpha, \Delta)$.

Proof. We prove the theorem by compactness as usual, and we start as in Theorem 7.1. Let Ω, E, L satisfy the assumptions. We want to prove that d is an integer and that E is uniformly rectifiable, i.e. that for all $\tau > 0$, the set $\mathcal{G}_{ur}(\tau)$ (associated to the integer d) is prevalent. By assumption, $\mathcal{G}_{GD_{\alpha,\mu}}(\varepsilon, M)$ is prevalent, and we added μ in the notation because this time we really need $D_{\alpha,\mu}$ to be defined in terms of μ. As before, it will be enough to prove that if ε and M^{-1} are chosen small enough, depending on n, d, τ, the AR constant for μ, and the constant in (2.1) when $d \geq n-1$,

$$\mathcal{G}_{GD_{\alpha,\mu}}(\varepsilon, M) = \emptyset \text{ unless } d \text{ is an integer} \tag{8.9}$$

and, when d is an integer,

$$\mathcal{G}_{GD_{\alpha,\mu}}(\varepsilon, M) \subset \mathcal{G}_{ur}(\tau). \tag{8.10}$$

We proceed by contradiction, assume that (8.9) or (8.10) fails for $\varepsilon_k = 4^{-k}$ and $M_k = 2^k$, and pick a counterexample for each k. This means, we find an open set Ω_k bounded by an AR set E_k, an AR measure μ_k on E_k (with uniform AR bounds), and finally a pair $(x_k, r_k) \in E_k \times (0, +\infty)$ such that the assumptions above hold (and in particular $(x_k, r_k) \in \mathcal{G}_{GD_{\alpha,\mu}}(4^{-k}, 2^k)$), but not the conclusions.

By translation and dilation invariance, we may assume that $x_k = 0$ and $r_k = 1$. We may also extract a subsequence so that Ω_k converges to a limit Ω_∞, E_k converges to E_∞, and μ_k converges weakly to a limit μ_∞ on E_∞. As in the proof of Theorem 2.19, Ω_∞ is bounded by E_∞, Ω_∞ satisfies the assumption (2.1) when needed, and μ_∞ is automatically an AR measure on E_∞.

The coefficients of the operator L_k converge in $L^1_{loc}(\Omega_\infty)$ to the coefficients $D^{d+1-n}_{\alpha,\mu_\infty}$ of the operator L_∞ associated to D_{α,μ_∞} on Ω_∞ (see the argument near (2.28): we are now proceeding as in the proof of Theorem 3.43, but with different assumptions). In particular, we have (2.17) with $A_\infty = I$, and we are allowed to apply Theorem 2.19. Hence the Green function G_k^∞ for L_k converges, uniformly on compact subsets of Ω_∞ to a multiple of the Green function G for L_∞.

But our assumption that $(0, 1) \in \mathcal{G}_{GD_{\alpha,\mu}}(4^{-k}, 2^k)$ implies that

$$|D_{\alpha,\mu_k}(X) - c_k G_k^\infty(X)| \leq 2^{-k} \text{ for } X \in \Omega_k \cap B(0, 2^k) \tag{8.11}$$

(compare with (1.16)), and since D_{α,μ_k} converges to D_{α,μ_∞}, we see that $G = CD_{\alpha,\mu_\infty}$ on Ω_∞. In particular $L_\infty D_{\alpha,\mu_\infty} = 0$, and now (8.5) and our assumption $\Upsilon_{\text{flat}}(d, \alpha, \Delta)$ imply that d is an integer, μ_∞ is a flat measure, and of course E_∞ is a d-plane.

So (8.9) did not fail, and the fact that E_k tends to E_∞ implies that $d_{0,1}(E, E_k)$ tends to 0, so (8.10) is also true for k large. This contradiction completes our proof of (8.9) and (8.10), and Theorem 8.8 follows. □

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