Advances in Mathematics 390 (2021) 107946

Contents lists available at ScienceDirect

Advances in Mathematics

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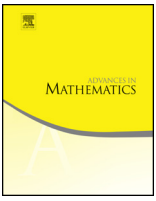

The landscape law for the integrated density of states

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ARTICLE INFO

Article history:
Received 5 June 2021
Received in revised form 29 June 2021
Accepted 1 July 2021
Available online 5 August 2021
Communicated by C. Fefferman

Keywords:
Integrated density of states
Localization
Weyl law
Landscape

ABSTRACT

The present paper establishes non-asymptotic estimates from above and below on the integrated density of states of the Schrödinger operator $L = -\Delta + V$, using a counting function for the minima of the localization landscape, a solution to the equation $Lu = 1$.

RÉSUMÉ

Dans cet article on établit des bornes inférieures et supérieures sur la densité d'états intégrée pour l'opérateur de Schrödinger $L = -\Delta + V$, à l'aide d'une fonction comptant les minimas de la fonction paysage, la solution de $Lu = 1$ avec des conditions au bord adaptées. Dans le cas des potentiels désordonnés, on en déduit les meilleures estimations connues sur la densité d'états intégrée dans le modèle d'Anderson sur $\mathbb{R}^d$.

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https://doi.org/10.1016/j.aim.2021.107946
0001-8708/

Contents

1. Introduction

The density of states of the Schrödinger operator $-\Delta + V$ is one of the main characteristics defining the physical properties of the matter. At this point, most of the known estimates for the integrated density of states pertain to two *asymptotic* regimes, each carrying restrictions on the underlying potentials. The first one stems from the Weyl law and its improved version due to the Fefferman-Phong uncertainty principle [7]. It addresses the energies or eigenvalues $\lambda \to +\infty$ and deteriorates for the potentials oscillating at a wide range of scales. The second one concentrates on the asymptotics as λ tends to 0 for disordered potentials, the so-called Lifschitz tails, and takes advantage of probabilistic arguments and the random nature of the disordered potentials. The goal of the present paper is to establish new bounds on the integrated density of states via the counting function of the so-called localization landscape [9]. The main theorem can be viewed as a new version of the uncertainty principle, which, contrary to the above, applies uniformly across the entire spectrum and covers all potentials bounded from below irrespectively of their nature.

To set the stage, let us consider the spectrum of the Schrödinger operator $L = -\Delta + V$ on a domain $\Omega \subset \mathbb{R}^d$. We shall assume for the time being that Ω is a cube in $\mathbb{R}^d$ of sidelength $R_0 \in \mathbb{N}$ and make sure that the estimates that we seek do not depend on the size of the domain, so that we can pass to the limit of infinite domain whenever it is desired and appropriate.

Assume furthermore that V is a bounded non-negative function on Ω and $L = -\Delta + V$ (once again, the boundedness assumption on V is, at this point, cosmetic: the resulting estimates do not depend on the maximum value and we can include more general potentials into consideration). We denote by N the (normalized) integrated density of states of the operator L with periodic boundary conditions on $\partial\Omega$, i.e.,

$$N(\mu) := \frac{1}{|\Omega|} \times \{\text{the number of eigenvalues } \lambda \text{ such that } \lambda \leq \mu\}. \tag{1.1}$$

As usual, eigenvalues are counted with multiplicity. It is known that the operator L above, with periodic boundary conditions on $\partial\Omega$, has a discrete spectrum consisting of positive eigenvalues and hence, the definition is coherent.

In 1911, Hermann Weyl proposed what became later known as the Weyl law for the asymptotics of $N(\mu)$, as $\mu \to +\infty$, for the Laplace-Beltrami operator with the Dirichlet boundary conditions in a bounded domain. In his setting, the law gives an asymptotic of

a multiple of $\mu^{d/2}$ as $\mu \to +\infty$. Perhaps much more importantly than the result itself, it gave a general approach to the asymptotics of the density of states of an elliptic operator, and in particular, the rule of thumb traditionally used in physics is

$$N(\mu) \sim \frac{1}{(2\pi)^d |\Omega|} \iint\limits_{|\xi|^2+V(x)<\mu} dx d\xi, \quad \text{as } \mu \to \infty. \tag{1.2}$$

It is simultaneously impossible to list all the directions in which the Weyl law has been extended over the years and to give a sharp class of V to which it applies, with nice control of the asymptotic errors.[1] However, the oscillations of V at the scales smaller than $\mu^{-1/2}$ can easily destroy the validity of the volume-counting (1.2) for the corresponding μ. In fact, the Weyl law prediction (1.2) fails even for systems as simple as two uncoupled harmonic oscillators, that is, the potential $V(x_1, x_2) = x_1^2 + \varepsilon x_2^2$ with a small ε (see, e.g., [7], p. 143).

An obvious shortcoming of the "classical" Weyl law is the emphasis on the volume counting itself, as an eigenfunction cannot occupy an arbitrarily shaped volume in the phase space. This issue has been alleviated with the celebrated Uncertainty Principle of Fefferman and Phong ultimately reaching out to the problem of stability of matter [7]. Instead of the volume-counting of (1.2), Fefferman and Phong suggested to estimate the number of disjoint cubes with sidelength $\mu^{-1/2}$ and such that $\left(⨍_Q |V|^p \, dx \right)^{1/p} \le C \, \mu$, smoothing the oscillations of V at the correct scales. The resulting bounds on $N(\mu)$ were proved when V is a polynomial and $p = \infty$ in [7] and for $V \ge 0$ in a suitable reverse Hölder case by Shen [23,24], and were also extended to estimates on a number of negative eigenvalues for general $V \le 0$. Overall, these ideas have brought a number of fascinating results – their goals and achievements, stemming from a new diagonalization of pseudodifferential operators, are beyond the scope of our review. But in the particular context of interest in this paper, they also fall short in some respects. First, searching for the aforementioned collection of optimal cubes for every μ can be computationally very challenging. Secondly, and this is exactly the reason for the restrictions on the potential and/or asymptotic nature of the results, the sharp estimates from above and below for positive potentials are only available when V behaves not too violently at the corresponding scales. This rends them formally inapplicable for the Anderson or other disordered potentials, and more generally whenever V is very different from its average on a cube. The Landscape Law proposed in this paper addresses both of these issues. The landscape "determines" the correct cubes and exhibits precisely the correct oscillation, in some sense creating a perfect effective potential for the Fefferman-Phong-type counting from any initial V. One can even be more precise and say that in the context of the disordered potentials some similar ideas have been used at the bottom of the spectrum, but much as in the Fefferman-Phong set-up, proving an estimate from above which

[1] The estimate from above is due to Cwickel, Lieb and Rosenblum [25].

would be desirably close to the estimate from below is challenging and requires different techniques.

The recent few years have brought a new tool, the so-called localization landscape, which yielded astonishingly precise non-asymptotic estimates on the density of states for both periodic and certain Anderson-type potentials throughout multiple numerical and physical experiments [9,2,3]. However, so far no rigorous mathematical results have supported these findings and, in particular, it was not clear what are the exact bounds, what is the range of potentials to which the theory could be applied, whether the results are generic or governed by the particular choice of examples, whether one can truly furnish localization landscape theory in the context of Anderson localization. In the present paper we prove that a counting function arising from the landscape provides sharp estimates from above and below on the density of states *for any non-negative potential* in the Schrödinger operator. As a by-product, we derive new estimates on the integrated density of states for the Anderson-type potentials. However, the latter is only a particular instance of our theory – our main results are deterministic.

The concept of *localization landscape* was pioneered by the second and third authors of the present paper in [9]. The landscape is the solution to $(-\Delta + V)\, u = 1$, with the same boundary conditions as the original operator in question. When applied to the Laplacian rather than the Schrödinger operator and equipped with the Dirichlet boundary conditions, the landscape is nothing else than the classical torsion function, however, its role in our theory and its character in the presence of a potential are very different, and we will continue using the *landscape* terminology which seems to be more illustrative under the circumstances.

First numerical [3] and then rigorous mathematical results [1] have demonstrated the relationship between the landscape and the location and shape of localized eigenfunctions, including the pattern of their exponential decay. One of the key observations underpinning these works is that the operator $L = -\Delta + V$ has exactly the same spectrum as a conjugated operator

$$-\frac{1}{u^2}\operatorname{div} u^2 \nabla + \frac{1}{u}$$

which brings up $1/u$ as an effective potential. This is a consequence of the identity

$$\int |\nabla f|^2 + V f^2 \, dx = \int u^2 \left| \nabla \left(\frac{f}{u} \right) \right|^2 + \frac{1}{u} f^2 \, dx, \tag{1.3}$$

valid for all f in the corresponding Sobolev space $W^{1,2}(\Omega)$ and proved in [1]. However, not only $1/u$ plays the role of a potential, but it exhibits decisively better properties than the original V. The reduced kinetic energy, which is the first term on the right-hand side of (1.3), is small in many typical examples, at least at the bottom of the spectrum, and hence $1/u$ "absorbs" the information about both kinetic and potential energy of the

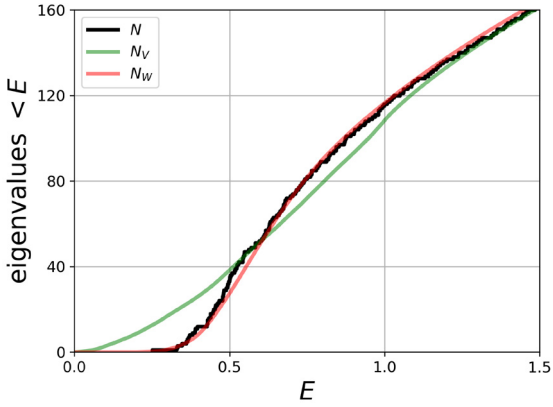

Fig. 1. [2] The IDOS N (in black), the original Weyl law approximation N_V from the right-hand side of (1.2) (in green), and the approximation using the landscape function, N_W, $W = 1/u$, from the right-hand side of (1.5) (in red) for a random uniform potential in one dimension on an interval of length 512. The quantities are not normalized by volume. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

original system, in some sense, yielding a stronger form of the Uncertainty Principle than those discussed above.

Motivated by these considerations, we were led to investigate the information about the spectrum of L encoded in $1/u$, and the numerical experiments brought surprising results, in fact, exceeding original expectations [3,2]. In generic samples of Anderson-type potentials in finite one- and two-dimensional domains one could observe two strongly emerging patterns. First, the eigenvalues at the bottom of the spectrum are essentially dimensional multiples of local minima of $1/u$. That is, independently of the potential, we observe an almost equality

$$\left(1 + \frac{d}{4}\right) \left(\min \frac{1}{u}\right)_j \sim \lambda_j \tag{1.4}$$

where the eigenvalues and minima are indexed in nondecreasing order. Secondly, a version of the Weyl law governed by the potential $1/u$

$$N(\mu) \sim \frac{1}{(2\pi)^d |\Omega|} \iint\limits_{|\xi|^2 + \frac{1}{u(x)} < \mu} dx d\xi \tag{1.5}$$

yields, contrary to (1.2), an approximation of the density of states throughout the spectrum, for all values of μ, albeit working a little worse than minima (1.4) at the very bottom. Fig. 1, taken from [2], shows the advantage of using the landscape rather than the original V in the predictor (1.5).

Both observations have been immediately adopted by physicists, for Schrödinger and Poisson-Schrödinger (Hartree-Fock) systems [10,21,4,27], and for Dirac equation [18]; however, even rigorous mathematical conjectures remained beyond reach, particularly if

aiming for non-asymptotic statements. Indeed, one can rather easily construct counterexamples about taking (1.4) or (1.5) as near identities [5,8], and the numerical evidence was initially restricted to dimensions 1 and 2, either Anderson-type or periodic potentials, and reasonably small domains, especially in dimension 2. The latter point, in particular, could raise doubts on the applicability of these approximations in the limit of infinite domain.

The present paper is the first mathematical treatment of a rigorous connection between the landscape function and the eigenvalues of L in the entire range of λ. We show that a counting function of the minima of $1/u$ yields sharp deterministic estimates *from above and below* on the integrated density of states, without restrictions on the underlying potential.

Passing to the statements of the results, recall that Ω is a cube in $\mathbb{R}^d$ of sidelength $R_0 \in \mathbb{N}$. For any $r > 0$ such that R_0 is an integer multiple of r, we denote by $\{Q\}_r$ a disjoint collection of cubes of sidelength r, such that every Q_r is contained in Ω and $\bigcup_{Q\in\{Q\}_r} \bar{Q} = \bar{\Omega}$. Our cubes are always open unless stated otherwise. We shall work with functions satisfying periodic boundary conditions on $\partial\Omega$ and, slightly abusing the notation, will often identify Ω with the torus $(\mathbb{R}/R_0\,\mathbb{Z})^d$. As in the beginning of the introduction, V is a bounded nonegative function on Ω, $L = -\Delta + V$ is the Schrödinger operator on Ω, which we take with the periodic boundary conditions, and the integrated density of states is defined by (1.1). Going further, let u be the solution to $Lu = 1$ on Ω, also with periodic boundary conditions. Then it is known (and easy to prove) that u is positive and bounded, and we define

$$N_u(\mu) := \frac{1}{|\Omega|} \times \left\{ \text{the number of cubes } Q \in \{Q\}_{\kappa\,\mu^{-1/2}} \text{ such that } \min_Q \frac{1}{u} \le \mu \right\}, \quad (1.6)$$

where by convention $1 \le \kappa < 2$ (depending on μ) is the smallest number such that R_0 is an integer multiple of $\kappa\mu^{-1/2}$.

Theorem 1.7 *(The Landscape law). Retain the definitions above. There exist constants C_i, $i = 1, ..., 4$, depending on the dimension only, such that*

$$C_1\alpha^d N_u(C_2\alpha^{d+2}\mu) - C_3 N_u(C_2\alpha^{d+4}\mu) \le N(\mu) \le N_u(C_4\mu) \quad (1.8)$$

for every $\alpha < 2^{-4}$ and every $\mu > 0$.

The strength of Theorem 1.7 lies in its generality compared to all previously available results:

- Theorem 1.7 is not asymptotic, the estimate (1.8) is valid throughout the spectrum, with constants independent of μ.
- The constants in (1.8) do not depend on smoothness or oscillations of V, nor on the possible probability law beyond its construction (or lack of thereof), nor, in fact, on

the L^∞ norm of V or the size of the domain R_0. If one allows the dependence on $\|V\|_{L^\infty(\Omega)}$, the situation for large μ is of course rather trivial (both the density of states and $N_u(\mu)$ roughly behave as those of the Laplacian), and similarly the scales bigger than R_0 would be easy to handle. We emphasize the lack of dependence on any of these parameters, which makes it possible to apply the theorem to the limit of an infinite potential or an infinite domain.

Looking at (1.8), one obviously faces the question of the polynomial correction in the estimate from below. And indeed, in applications (1.8) often transforms into the even stronger estimate

$$N_u(C_2' \mu) \leq N(\mu) \leq N_u(C_4 \mu)$$

by taking α small. There are (at least) two mechanisms to achieve this, which are fortunately roughly complementary. The first one is to prove a doubling condition for the landscape u.

Theorem 1.9 *(The doubling case). Retain the definitions above. If, in addition, u^2 is a doubling weight at relatively small scales, specifically, if there is a constant $C_D \geq 1$ such that*

$$\int_{Q_{2s}} u^2 \, dx \leq C_D \left(\int_{Q_s} u^2 \, dx + s^{d+4} \right) \tag{1.10}$$

for every cube Q_s of sidelength $s > 0$ then

$$N_u(C_2' \mu) \leq N(\mu) \leq N_u(C_4 \mu) \quad \text{for every } \mu > 0, \tag{1.11}$$

where C_4 is as in Theorem 1.7 and C_2' depends only on C_D and the dimension.

In the doubling condition and everywhere below, we interpret u as a function on the torus, that is, if the cubes intersect the boundary, it is understood that one uses the periodic extension of u.

There is a certain dichotomy between the range of applicability of Theorem 1.9 and its complement, in particular, disordered systems. Notice that (1.8) transforms into (1.11) if $N_u(\mu)$ decays sufficiently fast as μ tends to 0. This would not be the case, e.g., in the realm of periodic potentials, when one expects that both the integrated density of states and $N_u(\mu)$ behave as $\mu^{d/2}$. Fortunately, in this case u^2 is a doubling weight, (1.10) is satisfied, and hence we can directly apply Theorem 1.9.

A similar situation occurs when V is sufficiently well-behaved. For instance, for $d \geq 3$, if V satisfies the Kato condition

$$\sup_{z\in\mathbb{R}^d,\, r>0} \int\limits_{B_r(z)\cap\Omega} \frac{V(x)}{|x-z|^{d-2}}\, dx =: K < \infty, \tag{1.12}$$

then (1.10) is verified and hence, the integrated density of states satisfies (1.11) directly. This can be seen as a combination of results from Theorem 1.3 in [16], which guarantee that for non-negative supersolutions to $(-\Delta+V)u \geq 0$ there exists $\delta > 0$ such that u^δ is doubling, and classical Moser inequalities for subsolutions to $-\Delta u \leq 1$, which allow one to bound $\sup_{Q_s} u$ by $\left(⨍_{Q_s} u^2\, dx\right)^{1/2} + r^2$ (cf. [12], Theorem 4.14). We observe that this includes, on finite domains, even singular potentials weaker than $1/|x|^2$, but as usual, one has to pay attention to emerging constants: if (1.12) is used, the resulting constant in (1.11) may depend on K, which might or might not be suitable for the problem at hand. In fact, if V is regular itself, (1.10) could be easier to check directly, without involving (1.12), but for now let us move to the case when (1.10) can fail.

The Anderson model pertains to disordered potentials when V is, for instance, a linear combination of bumps with random amplitudes taking values between 0 and 1 according to some probability law. We shall give a detailed definition and some related history of the subject below, but for now let us just say that it is a setting of the Anderson localization – a famous phenomenon when such a system, in the limit of an infinite domain, could display pure point spectrum and exponentially decaying eigenfunctions. A certain pre-runner of Anderson localization (in fact, a simpler phenomenon of rare big regions) manifests itself through the so-called Lifschitz or Urbach tails: as $\mu \to 0$, $N(\mu)$ behaves roughly as $e^{-c\mu^{-d/2}}$ contrary to the more usual behavior $\mu^{d/2}$ observed in non-disordered systems (compare to the Weyl law above). We underline that this, once again, is an asymptotic result, now at the edge $\mu \to 0$, with a limited understanding of errors and the range where the asymptotic is precise.

A typical example of potential that destroys (1.10) is any of the Anderson-type potentials. The latter is a subclass of disordered potentials where V is, for instance, a linear combination of bumps with random amplitudes taking values between 0 and 1 according to some probability law. It is a setting of the Anderson localization – a famous phenomenon when such a system, in the limit of an infinite domain, could display pure point spectrum and exponentially decaying eigenfunctions. We shall see that in this case, although (1.10) fails, fortunately $N_u(\mu)$ has exponential growth as $\mu \to 0$, and hence (1.8) implies (1.11) because the exponential behavior suppresses polynomial corrections. In the terminology of [20], such is the situation near fluctuation boundaries generally exhibited by Schrödinger operators with random (disordered) potentials. Hence, any fluctuating boundary would yield (1.11). Here we just isolate one result.

Theorem 1.13. *Retain the definitions of Ω and L from Section 1.*

Let $\varphi \in C_0^\infty(B_{1/10}(0))$ be a nontrivial bump function supported in the ball centered at 0 of radius $1/10$, with $0 \leq \varphi \leq 1$, and set

$$V = V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j \varphi(x-j) \quad \text{for } x \in \Omega,$$

where the ω_j are i.i.d. variables taking values in $[0,1]$, whose probability distribution

$$F(\delta) = \mathbb{P}\{\omega \le \delta\}, \quad 0 \le \delta \le 1,$$

is not trivial, i.e., not concentrated at one point, and such that 0 is the infimum of its support. Denote by $N_u^E(\mu) = \mathbb{E}\, N_u(\mu)$ the expectation of the counting function of minima of $1/u$, as defined in (1.6) and by $N^E(\mu) = \mathbb{E}\, N(\mu)$ the expectation of the density of states, as defined in (1.1).

Then there exist constants $C_5, C_6 > 0$ depending on the dimension and the expectation of the random variables ω_j only, and a constant $C_4 > 0$, depending on the dimension only, such that

$$C_5 N_u^E(C_6\,\mu) \le N^E(\mu) \le N_u^E(C_4\,\mu), \tag{1.14}$$

for every $\mu > 0$.

Since 0 is the infimum of the support of F, we have $F(\delta) > 0$ for $\delta > 0$; also, the measure is not a Dirac mass at the origin, so $\lim_{\delta\to 0} F(\delta) < 1$. This implies that the common expectation $\mathbb{E}(\omega)$ of the ω_j lies in $(0,1)$, and we claim that d and $\mathbb{E}(\omega)$ alone control our constants. We will see in Theorems 3.1 and 3.5 that both numbers $N^E(\mu)$ and $N_u^E(\mu)$ are related to the behavior of the distribution function $F(\delta)$, and in particular its asymptotics when δ tends to 0, which may be complicated; here we say that the constants in these relations depend only on d and $\mathbb{E}(\omega)$.

We underline – yet again – that Theorem 1.13 is not an asymptotic result, and multiple numerical experiments [2] show the strength of this estimate in the intermediate regime where μ is neither large nor small, as well as its applicability to the potentials where V is disordered but unbounded and thus, no other results for large μ are readily available. Moreover, even in the asymptotic regimes, (1.14) offers more precision than the traditional Lifschitz tail estimates, in particular, encompassing faithfully the differences between individual choices of the disordered potentials; this will be discussed more thoroughly in Section 3; also see [6] for a detailed numerical study of the Landscape Law and its comparison to the available results in the presence of disorder. In conclusion, we would like to zoom back out from the specific applications and to reiterate that the Main Theorem should be viewed as a form of the Uncertainty Principle whose generality is not inhibited by properties of the potential or range of the energies, a "black box" which gives good bounds on the density of states irrespectively of the physical nature of the initial system.

Acknowledgments. We thank Douglas Arnold and David Jerison for uncountable inspiring conversations on the subject and the joint work [3,2] which lies at the foundation

of the results in this paper. The third author would also like to thank T. Spencer and L. Pastur for many stimulating discussions, and W. König, Z. Shen, and W. Kirsch, for sharing some references and the historical perspective. Finally, we thank S. Zhang for careful reading of the paper and useful comments.

David is supported in part by the H2020 grant GHAIA 777822, and Simons Foundation grant 601941, GD. Filoche is supported in part by Simons Foundation grant 601944, MF. Mayboroda is supported in part by the NSF grants DMS 1344235, DMS 1839077, and Simons Foundation grant 563916, SM.

2. Main estimates: doubling and non-doubling scenario

We start with the *Proof of Theorems 1.7 and 1.9.*

Step I: the upper bound. We start with the upper bound on $N(\mu)$. The estimate $N(\mu) \leq N$ is valid if we can find H_N, a codimension N subspace of H (where H is the space of periodic functions in $W^{1,2}(\Omega)$), such that

$$\frac{\langle Lv, v\rangle}{\|v\|^2_{L_2(\Omega)}} := \frac{\int_\Omega |\nabla v|^2 + V\, v^2\, dx}{\int_\Omega v^2\, dx} > \mu \quad \text{for all } v \in H_N.$$

To this end, denote

$$\mathcal{F} := \left\{ Q \in \{Q\}_{\kappa\,(C_4\mu)^{-1/2}} \text{ such that } \min_Q \frac{1}{u} \leq C_4\mu \right\},$$

with C_4 to be defined below, and $1 \leq \kappa < 2$ (depending on μ) is the smallest number such that R_0 is an integer multiple of $\kappa\mu^{-1/2}$. Then let H_N be the space of $v \in H$ such that $\int_Q v\, dx = 0$ for every $Q \in \mathcal{F}$. Since the cubes $Q \in \mathcal{F}$ are disjoint, it is evident that H_N has co-dimension $N = \operatorname{Card} \mathcal{F}$, simply taking the bumps on Q's as an orthogonal complement of H_N.

We recall from [1], Lemma 4.1, that

$$\int_\Omega |\nabla v|^2 + V\, v^2\, dx \geq \int_\Omega \frac{1}{u}\, v^2\, dx \quad \text{for all } v \in H$$

and hence,

$$2\int_\Omega |\nabla v|^2 + V\, v^2\, dx \geq \int_\Omega |\nabla v|^2 + \frac{1}{u}\, v^2\, dx \quad \text{for all } v \in H.$$

Thus, it is enough to prove that

$$\int_\Omega |\nabla v|^2 + \frac{1}{u}\, v^2\, dx > 2\mu \int_\Omega v^2\, dx \quad \text{for all } v \in H_N \setminus \{0\}. \tag{2.1}$$

On the part of Ω corresponding to any $Q \in \{Q\}_{\kappa\,(C_4\mu)^{-1/2}}$ such that $Q \notin \mathcal{F}$ the bound (2.1) is valid provided that $C_4 > 2$ because $\min_Q \frac{1}{u} \geq C_4\mu$ on such cubes. For $Q \in \mathcal{F}$, we use the Poincaré inequality to write

$$\int\limits_Q |\nabla v|^2\,dx \geq C_P\,C_4\mu \int\limits_Q |v - v_Q|^2\,dx = C_P\,C_4\mu \int\limits_Q v^2\,dx,$$

where $C_4\mu$ comes from the size of Q and we used the fact that $v_Q = \int_Q v\,dx = 0$ by the definition of H_N. Here C_P is the Poincaré constant and depends on the dimension only. Choosing C_4 so large that $C_P\,C_4 > 2$, we arrive at the desired estimate.

Step II: the lower bound in the doubling case. In this direction, in order to prove that $M \leq N(\mu)$, we need to find H_M, a subspace of H of dimension M, such that

$$\frac{\langle Lv, v\rangle}{\|v\|^2_{L_2(\Omega)}} := \frac{\int_\Omega |\nabla v|^2 + V\,v^2\,dx}{\int_\Omega v^2\,dx} \leq \mu \quad \text{for all } v \in H_M. \tag{2.2}$$

To this end, let

$$\mathcal{F}' := \left\{ Q \in \{Q\}_{\kappa\,(C_2\mu)^{-1/2}} \text{ such that } \min_Q \frac{1}{u} \leq C_2\mu \right\}, \tag{2.3}$$

where C_2 will be chosen below. Let H_M be the linear span of the functions $u\chi_Q$, $Q \in \mathcal{F}'$, picked such that $\chi_Q \in C_0^\infty(Q)$, $\chi_Q = 1$ on $Q/2$, $0 \leq \chi \leq 1$ on Q, and $|\nabla\chi_Q| \leq 4l(Q)^{-1}$.

Since $-\Delta u \leq 1$, the Moser-Harnack inequality ([12], Theorem 4.14) yields

$$\sup_Q u \leq C_H \left(\frac{1}{|Q|} \int\limits_{2Q} u^2 \right)^{1/2} + C_H l(Q)^2, \tag{2.4}$$

where C_H depends on the dimension only. In particular, using also the doubling condition three times,

$$\sup_Q u \leq C_H C_D^{3/2} \sup_{Q/4} u + C' l(Q)^2, \tag{2.5}$$

where $C' = C'(C_D, C_H)$ is a constant depending on C_D, C_H, and the dimension only.

We use (1.3), the definition of χ_Q, (2.4) for $Q/4$, and (2.5)

$$\frac{\langle L(u\chi_Q), u\chi_Q\rangle}{\|u\chi_Q\|^2_{L_2(\Omega)}} = \frac{\int u^2|\nabla\chi_Q|^2 + u\chi_Q^2\,dx}{\int (u\chi_Q)^2\,dx} \leq \frac{16\,l(Q)^{-2}\int_Q u^2\,dx + \int_Q u\,dx}{\int_{Q/2} u^2\,dx}$$

$$\leq \frac{16\,l(Q)^{-2}\sup_Q u^2 + \sup_Q u}{4^{-d}\left(\frac{1}{C_H}\sup_{Q/4} u - \frac{1}{16}l(Q)^2\right)^2} \leq \frac{4^{d+2}\,l(Q)^{-2}\sup_Q u^2 + 4^d \sup_Q u}{\left(\frac{1}{C_H^2 C_D^{3/2}}\sup_Q u - (\frac{1}{16} + \frac{C'}{C_H^2 C_D^{3/2}})l(Q)^2\right)^2}. \tag{2.6}$$

We temporarily choose κ small enough in terms of C_D and C_H so that

$$\frac{1}{2C_H^2 C_D^{3/2}} \sup_Q u \geq \left(\frac{1}{16} + \frac{C'}{C_H^2 C_D^{3/2}}\right) l(Q)^2,$$

and then, for some constants $C'_{d,5}$, $C''_{d,5}$, $C_{d,5}$ depending on the dimension, C_D, and C_H we have

$$\frac{\langle L(u\chi_Q), u\chi_Q\rangle}{\|u\chi_Q\|^2_{L_2(\Omega)}} \leq C'_{d,5}\, l(Q)^{-2} + C''_{d,5}\, \frac{1}{\sup_Q u} \leq C_{d,5} C_2 \mu, \tag{2.7}$$

where the last inequality comes from the definition (2.3) of $\mathcal{F}'$. Having fixed κ as above, we now choose C_2 such that $C_{d,5}C_2 = 1$ and arrive at the desired estimate. To be precise, we only showed the desired inequality on the elements of the basis of H_M but since the cubes Q are disjoint, we immediately get it for any element of H_M as well. The only difference with what we want is that the estimate we achieved is in terms of the cardinality of a set $\mathcal{F}'$ defined with an artificially small κ.

However, if we increase the κ to our usual fork $1 \leq \kappa < 2$, the cardinality of the resulting set $\mathcal{F}$ becomes even smaller, and our basis H_M has less elements than expected, as desired.

Step III: the lower bound in the non-doubling case. Our goal, once again, is to establish (2.2) for some subspace H_M of dimension M. This time, we pick any $\alpha \in (0, 1/16]$ and consider cubes of sidelength $R = \kappa\,(C^*\alpha^{d+4}\mu)^{-1/2}$. For $Q \in \{Q\}_R$, denote by $\check{Q} = Q_r$ the cube concentric with Q but with the smaller sidelength $r = \alpha R = \kappa\,(C^*\alpha^{d+2}\mu)^{-1/2}$. Now take

$$\mathcal{F}' := \left\{ Q \in \{Q\}_R \text{ such that } \min_{\check{Q}} \frac{1}{u} \leq C^*\alpha^{d+2}\mu \text{ and } \min_{Q} \frac{1}{u} \geq C^*\alpha^{d+4}\mu \right\}, \tag{2.8}$$

and let H_M be the linear span of the functions $u\chi_Q$, $Q \in \mathcal{F}'$, where we pick $\chi_Q \in C_0^\infty(Q)$, $0 \leq \chi_Q \leq 1$, such that $\chi_Q = 1$ on $2\check{Q}$ and $|\nabla\chi_Q| \leq CR^{-1}$. As before, we want to estimate

$$\frac{\langle L(u\chi_Q), u\chi_Q\rangle}{\|u\chi_Q\|^2_{L_2(\Omega)}} = \frac{\int u^2|\nabla\chi_Q|^2 + u\chi_Q^2\, dx}{\int (u\chi_Q)^2\, dx} \tag{2.9}$$

(by (1.3)). By definition of $\mathcal{F}'$, $u \leq (C^*\alpha^{d+4}\mu)^{-1}$ on Q, so the numerator is at most $C^2R^{-2}\int_Q u^2 + \int_Q u \leq (C^*\alpha^{d+4}\mu)^{-1}|Q|\big(C^2\kappa^{-2}+1\big)$. For the denominator D, we first apply the Moser-Harnack inequality (2.4) to $\check{Q}$, then the definition of $\mathcal{F}'$, to get that

$$D \geq \int_{2\check{Q}} u^2 \geq |\check{Q}|\big[C_H^{-1}\sup_{\check{Q}} u - \ell(\check{Q})^2\big]^2 = \alpha^d|Q|\big[C_H^{-1}\sup_{\check{Q}} u - \alpha^2R^2\big]^2$$
$$\geq \alpha^d|Q|\big[C_H^{-1}(C^*\alpha^{d+2}\mu)^{-1} - \kappa^2\alpha^2(C^*\alpha^{d+4}\mu)^{-1}\big]^2 = \alpha^d|Q|(C^*\alpha^{d+2}\mu)^{-2}[C_H^{-1} - \kappa^2]^2.$$

We chose $\kappa^2 \le \frac{1}{2C_H}$; then the first term dominates the second one and the expression in (2.9) is bounded by

$$\frac{C_{d,6}\,(C^*\alpha^{d+4}\mu)^{-1}}{C_{d,7}\alpha^d(C^*\alpha^{d+2}\mu)^{-2}} \le C_{d,8}C^*\mu = \mu, \tag{2.10}$$

provided that we choose $C^* = C_{d,8}^{-1}$. Then, using the orthogonality of the χ_Q, we get that

$$N(\mu) \ge \mathrm{Card}\Big\{Q \in \{Q\}_R\,;\, \min_{\check{Q}} \frac{1}{u} \le C^*\alpha^{d+2}\mu \ \text{ and } \min_{Q} \frac{1}{u} \ge C^*\alpha^{d+4}\mu\Big\} \ge N_1 - N_2,$$

where

$$N_1 = \mathrm{Card}\Big\{Q \in \{Q\}_R\,;\, \min_{\check{Q}} \frac{1}{u} \le C^*\alpha^{d+2}\mu\Big\},$$

$$N_2 = \mathrm{Card}\Big\{Q \in \{Q\}_R\,;\, \min_{Q} \frac{1}{u} \le C^*\alpha^{d+4}\mu\Big\}.$$

Notice that the cubes $\check{Q} = Q_r$ in this argument are smaller and do not cover Ω, so N_1 is probably not as large as $N_1' = \mathrm{Card}\,\big\{R \in \{Q\}_r\,;\, \min_R \frac{1}{u} \le C^*\alpha^{d+2}\mu\big\}$. However, keeping in mind that we can treat Ω as a torus, we can do the estimate above for a collection of translations of our cubes Q by a collection of at most $C\alpha^{-d}$ small vectors e_j, $j \in J$, so that when we take the cubes $Q = Q_R$ as above, the smaller cubes $\check{Q} + e_j$, $Q \in \{Q\}_R$ and $j \in J$, cover Ω. This implies that the sum of the corresponding numbers N_1 is at least $C^{-1}N_u(C^*\alpha^{d+2}\mu)$, where N_u is defined in (1.6) and C accounts for a slight difference between r and the official radius $\kappa(C^*\alpha^{d+2}\mu)^{-1/2}$ associated to $C^*\alpha^{d+2}\mu$. Let us pick a nearly optimal translation e_j, so that $N_1 \ge C^{-1}\alpha^d N_u(C^*\alpha^{d+2}\mu)$.

Similarly, $N_2 \le CN_u(C^*\alpha^{d+4}\mu)$, and thus by the estimate above

$$N(\mu) \ge C^{-1}\alpha^d N_u(C^*\alpha^{d+2}\mu) - CN_u(C^*\alpha^{d+4}\mu).$$

This is precisely the bound (1.8). □

It is important to point out that Theorem 1.7 does not rely on the condition $V \in L^\infty(\Omega)$ and there is no dependence in constants on $\|V\|_{L^\infty(\Omega)}$ or on the size of the domain R_0. This is one of the main features of our estimates. If instead one allows our estimates to depend on $\|V\|_{L^\infty(\Omega)}$, the situation for large μ is of course rather trivial, as both the density of states and $N_u(\mu)$ roughly behave as those for the Laplacian. In particular, there exist constants C_5, C_2, C_4 depending on the dimension only, such that (1.11) is valid for all $\mu > C_5\|V\|_{L^\infty(\Omega)}$. We will use an enhanced version of this statement in the next section.

3. Anderson-type potential

We start this section with estimates on the expectation of the counting function $N_u(\mu)$ associated to the landscape as in (1.6).

Theorem 3.1. *Let Ω and $L = -\Delta + V$ be as in Theorem 1.13. In particular, let $\varphi \in C_0^\infty(B_{1/10}(0))$ be such that $0 \le \varphi \le 1$, and set*

$$V = V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j\, \varphi(x-j), \quad x \in \Omega, \tag{3.2}$$

where the ω_j are i.i.d. variables taking values $\omega_j \in [0,1]$, with a probability distribution

$$F(\delta) = \mathbb{P}\{\omega \le \delta\}, \quad 0 \le \delta \le 1, \tag{3.3}$$

which is not concentrated at one point, and such that 0 is the infimum of its support. Denote by $N_u^E(\mu) = \mathbb{E}\, N_u(\mu)$ the expectation of the counting function of the minima of $1/u$, as defined in (1.6). Then there exist constants $\mu^*, c_P, \gamma_1, \gamma_2$, depending on the dimension and the common expectation of the random variables ω_j only, and constants $m, \widetilde{c}_P, \gamma_3, \gamma_4$, depending on the dimension only, such that

$$\gamma_3\, \mu^{d/2} F(\widetilde{c}_P \mu)^{\gamma_4 \mu^{-d/2}} \le N_u^E(\mu) \le \gamma_1\, \mu^{d/2} F(c_P \mu)^{\gamma_2 \mu^{-d/2}}, \tag{3.4}$$

whenever $\mu < \mu^*$ and $R_0 > (\mu m)^{-1/2}$.

Let us put this Theorem into the context of known results for the Lifschitz tails. On the way to our ultimate goals, we will show the following by-product of Theorem 1.13.

Theorem 3.5. *Retain the notation and assumptions of Theorem 1.13. Then there exist constants $\mu^*, m, c_P, \gamma_1, \gamma_2$, depending on the dimension and the expectation of the random variable only, and constants $\widetilde{c}_P, \gamma_3, \gamma_4$ depending on the dimension only, such that*

$$\gamma_3\, \mu^{d/2} F(\widetilde{c}_P \mu)^{\gamma_4 \mu^{-d/2}} \le N_u^E(\mu) \le \gamma_1\, \mu^{d/2} F(c_P \mu)^{\gamma_2 \mu^{-d/2}}, \tag{3.6}$$

$$\gamma_3\, \mu^{d/2} F(\widetilde{c}_P \mu)^{\gamma_4 \mu^{-d/2}} \le N^E(\mu) \le \gamma_1\, \mu^{d/2} F(c_P \mu)^{\gamma_2 \mu^{-d/2}} \tag{3.7}$$

whenever $\mu < \mu^*$ and $R_0 > (\mu m)^{-1/2}$.

This result, and in particular the traditionally sought-after estimate (3.7), is in itself stronger than formally known asymptotics of the density of states, particularly *for the continuous model*, although it is fair to say that (3.7) would be expected by specialists in the subject and perhaps could even be addressed by other methods than those in the present paper. Let us explain the situation in the currently available literature.

The literature devoted to Lifschitz tails is extensive, particularly if one includes Poisson and other models, and we do not thrive here to give a comprehensive list of references or methodology – see, e.g., [14,15,20] for surveys of related results. Here we just provide some pointers which will highlight the novelties of (3.7) (silently passing to the limit of infinite domain and removing the superscript E).

The early literature, by now considered classical, and many modern textbooks treat the case when $F(\delta) \geq C\delta^\beta$ for some $C, \beta > 0$, and provide the asymptotics

$$\lim_{\mu\to 0} \frac{\log|\log N(\mu)|}{\log \mu} = -\frac{d}{2}, \tag{3.8}$$

see, for instance, [13,26]. The quantity

$$L := \lim_{\mu\to 0} \frac{\log|\log N(\mu)|}{-\log \mu}$$

is generally known as a Lifschitz exponent, and, in addition to the results above, it is proved in [20] that

$$\lim_{\mu\to 0} \frac{\log(-\log F(\mu))}{-\log \mu} = a > 0 \quad \Longrightarrow \quad L = d/2 + a.$$

Theorems 1.13 and 3.5 ascertain that for any non-trivial F such that $F(\delta) > \delta$ for $\delta > 0$, we can recover the Lifschitz exponent from the behavior of the landscape counting function

$$L \equiv L_u, \quad \text{where} \quad L_u := \lim_{\mu\to 0} \frac{\log|\log N_u(\mu)|}{-\log \mu} \tag{3.9}$$

(assuming for simplicity that the limit exists) and in particular,

$$L = \frac{d}{2} + \lim_{\mu\to 0} \frac{\log(-\log F(\mu))}{-\log \mu}, \tag{3.10}$$

without any a priori restrictions on F. This formally recovers and generalizes the results mentioned above. In the context of our methods, however, such statements lose much of the precision exhibited in (1.14), (3.6), (3.7).

Indeed, the problem of (3.8) is not only, or not so much, the restricted class of the potentials to which it applies, but rather the notorious imprecision of double-logarithmic asymptotics. The underlying method of proof in [13,26] factually gives

$$\gamma_3\, \mu^{d/2} F(\widetilde{c}_P \mu)^{\gamma_4 \mu^{-d/2}} \leq N^E(\mu) \leq e^{-\gamma' \mu^{-d/2}}.$$

In general, the upper bound is larger than the lower bound and does not give sufficient precision to improve the double logarithm – see the discussion and the related conjectures in [13].

This is a well-known problem. The subtle difference between refined asymptotics roughly speaking asserting that $N(\mu) \sim e^{-c\mu^{-d/2}}$ and those with the logarithmic correction $N(\mu) \sim e^{c\mu^{-d/2}\log\mu}$ has not been overlooked in the literature. However, the refined estimates turned out to be much more challenging. At this point they are only available in $\mathbb{Z}^d$ rather than $\mathbb{R}^d$ and under various additional constraints on the probability distribution – see [15] and [19].[2] The proofs pass through the parabolic Anderson model – an approach not yet developed, to the best of our knowledge, in the context of the alloy Anderson model on $\mathbb{R}^d$ considered in the present paper. And, even in $\mathbb{Z}^d$, the situation has been far from well-understood. Both the conditions on the potential and the results in [15] and [19] are quite technical, so we will not provide the detailed statements. Let us just mention that they appeal to various cases according to the behavior of the scale function

$$S(\lambda,t) = (\lambda t)^{-1}G(\lambda t) - t^{-1}G(t), \quad \text{where} \quad G(t) = \log \mathbb{E}(\exp(-tV(0)),$$

(whether $S \sim C(\lambda^\rho - 1)t^\rho$ with C, ρ positive or negative, or $S \sim C\log\lambda$, or $S \sim -C(\lambda t)^{-1}\log t$) and draw the asymptotics in terms of $I(\mu) = \sup_{t>0}(\mu t - G(t))$. Such is the presentation in [19], and [15] gives somewhat different statements, also with a dependence on the features of a certain implicitly defined scale function. The strength of these results compared to Theorem 3.5 is that, at least in some cases, they provide actual asymptotics rather than the estimates from above and below and feature a number of cases that we did not explicitly consider, such as unbounded potentials. The weakness is that their coverage does not encompass all potentials, even among the bounded ones, and at this point is completely restricted to $\mathbb{Z}^d$.

By contrast, Theorem 3.5 provides a simple and universal law, covering all bounded potentials at once, clearly identifying the source of the logarithmic correction, the "Pastur tails" (3.10), the exact transition from the classical to quantum regime. Below are just a few examples of applications of (3.7):

(1) V is a Bernoulli potential: ω takes values 0 or 1 with probability 1/2. Then

$$\gamma_3\,\mu^{d/2}e^{-\gamma_4\mu^{-d/2}} \le N^E(\mu) \le \gamma_1\,\mu^{d/2}e^{-\gamma_2\mu^{-d/2}}.$$

(2) V is given by a uniform distribution on $[0,1]$ or any other F such that $F(\delta)$ is bounded from above and below by some positive power of δ. This leads to logarithmic correctors predicted in the physics literature [17,22]

$$\gamma_3\,\mu^{d/2}e^{-\gamma_4\,\mu^{-d/2}\log\mu} \le N^E(\mu) \le \gamma_1\,\mu^{d/2}e^{-\gamma_2\,\mu^{-d/2}\log\mu}.$$

[2] We are using here the review of these results from [14]. Unfortunately, the dissertation [19] has never been published and so we cannot attest to the validity of the proofs or to exact statements beyond what has been quoted [14].

(3) V is given by the probability distribution with $F(\delta) \sim e^{-C\delta^{-a}}$, $a > 0$. Then

$$\gamma_3\, \mu^{d/2} e^{\gamma_4\, \mu^{-d/2-a}} \le N^E(\mu) \le \gamma_1\, \mu^{d/2} e^{\gamma_2\, \mu^{-d/2-a}}.$$

This is an example of (3.10).

With this, we return to the proof of Theorem 3.1. Our initial lemma is purely deterministic.

Lemma 3.11. *Let Ω and L be as in Section 1, with V defined as follows. Let $\varphi \in C_0^\infty(B_{1/10}(0))$ be such that $0 \le \varphi \le 1$, and set*

$$V = V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j\, \varphi(x - j), \quad x \in \Omega,$$

where the sequence $\omega = \{\omega_j\}_{j \in \mathbb{Z}^d}$ takes values in $[0, 1]$. For $r \in [\sqrt{d}, R_0]$, where we recall that R_0 is the scale of Ω, let us denote by $Q = Q_r$ the maximal cube consisting of unit cubes centered on $\mathbb{Z}^d$ (and with edges parallel to the axes) which is contained in $B_{r/2}(0)$. Since $r \ge \sqrt{d}$, Q_r contains at least one unit cube.

Assume that $r \in [3\sqrt{d}, R_0]$ is such that

$$\text{Card}\left\{j \in Q_r \cap \mathbb{Z}^d : \omega_j \ge c_P r^{-2}\right\} \ge \lambda\, |Q_r|, \tag{3.12}$$

for some $c_P, \lambda > 0$.

If c_P is large enough, depending on λ and the dimension only, then there exist $\varepsilon = \varepsilon(\lambda, d) > 0$ (small) and $M = M(\varepsilon, \lambda, d) > 0$ (large) such that if $\xi_0 \in \overline{B}_{r/3}(0)$ is such that

$$u(\xi_0) \ge M r^2 \tag{3.13}$$

then

$$u(\xi) \ge (1 + \varepsilon)\, u(\xi_0) \quad \text{for some point} \quad \xi \in \overline{B_{\sqrt{1+\varepsilon}\, r}(\xi_0)}. \tag{3.14}$$

Again this is a deterministic statement, for which we do not care where the ω_j are coming from and probabilistic considerations are irrelevant. That is, at this point V could be any realization, even extremely unlikely, of the construction described in Theorem 3.1, even if we intend to show later that our assumption (3.12) is quite probable in some circumstances.

Here we gave a statement for a point $\xi_0 \in \overline{B}_{r/3}(0)$ so that we can take Q_r centered at the origin, but a similar statement for any $\xi_0 \in \Omega$ would be easy to obtain, because we could use the translation invariance of our problem by $\mathbb{Z}^d$ to apply the result to $\xi_0 - \overline{\xi}_0$, where $\overline{\xi}_0 \in \mathbb{Z}^d$ is such that $\xi_0 - \overline{\xi}_0 \in \overline{B}_{r/3}(0)$; we assumed $r \ge 3\sqrt{d}$ only to guarantee that we can find $\overline{\xi}_0$. We will use this comment about other centers ξ_0 later in the proof.

Proof. Because of the periodic nature of Ω and L, we may assume that Ω is centered at the origin; we do not assume that $\xi_0 = 0$ because $\mathbb{Z}^d$ plays a special role in the definition of V.

Step I. Let $\xi_0 \in \Omega$ be given, set $B_\rho = B_\rho(\xi_0)$ (for computations on u, we like to think that ξ_0 is the origin) and denote by $m(\rho) = m(\xi_0, \rho)$ the average of u on the sphere centered at ξ_0 with radius ρ. That is, when $d \geq 2$ we set

$$m(\rho) = \fint_{\partial B_\rho} u \, d\sigma, \quad \rho > 0,$$

where $d\sigma$ is the $(d-1)$ dimensional surface measure on ∂B_r, and when $d = 1$

$$m(\rho) = \frac{u(\xi_0 + \rho) + u(\xi_0 - \rho)}{2}, \quad \rho > 0.$$

For brevity, we set $m(0) = u(\xi_0)$; this makes sense because u is continuous on Ω. We claim that

$$m(\rho) \leq m(r) + r^2 - \rho^2 \quad \text{for} \quad 0 \leq \rho < r < \operatorname{dist}(\xi_0, \partial\Omega), \tag{3.15}$$

and in particular,

$$m(r) \geq m(0) - r^2. \tag{3.16}$$

This can be seen, for instance, by comparison with harmonic functions. Let v be a solution to $-\Delta v = 0$ in B_r that coincides with u on ∂B_r and set $w(y) := v(y) + r^2 - |y - \xi_0|^2$ for $y \in B_r$. Then $-\Delta w = 2d \geq 1 \geq -\Delta u$ in B_r (because $Lu = -\Delta u + Vu = 1$ and $V \geq 0$) and $w = v = u$ on ∂B_r. Hence, $w \geq u$ by the maximum principle, so that

$$m(\rho) = \fint_{\partial B_\rho} u \, d\sigma \leq \fint_{\partial B_\rho} w \, d\sigma = \fint_{\partial B_\rho} v \, d\sigma + r^2 - \rho^2 = m(r) + r^2 - \rho^2,$$

where we used the mean value property for harmonic functions in the last equality. The estimates (3.15)–(3.16) follow.

Furthermore, when $d \geq 2$, the Poisson formula for a harmonic function v in B_r yields

$$v(y) = \frac{r^2 - |y - \xi_0|^2}{d\alpha_d r} \fint_{\partial B_r} \frac{v(z)}{|z - y|^d} \, d\sigma_z \,,$$

where α_d is the volume of a unit ball in $\mathbb{R}^d$. Hence there exists a dimensional constant c_1 such that $v(y) \leq c_1 m(r)$ for all $y \in B_{2r/3}$. The same is of course true when $d = 1$, because harmonic functions on $\mathbb{R}$ are affine. Moreover, since

$$u(y) \le w(y) = v(y) + r^2 - |y - \xi_0|^2,$$

we get that

$$u(y) \le c_1 m(r) + r^2 \quad \text{for } y \in B_{2r/3}. \tag{3.17}$$

Notice that c_1 can be taken equal to 1 when $y = x$, according to (3.16).

Step II. Now we want to use the size of V. Integrating by parts against the Green function in a ball, we get for $d \ge 3$

$$\begin{aligned} m(r) = m(0) + c_2 \int_{B_r} \Delta u(y) \left(|y - \xi_0|^{2-d} - r^{2-d}\right) dy \\ = m(0) + c_2 \int_{B_r} (Vu - 1) \left(|y - \xi_0|^{2-d} - r^{2-d}\right) dy \end{aligned} \tag{3.18}$$

for some dimensional constant $c_2 > 0$ and as usual assuming $\overline{B}_r \subset \Omega$.

Now assume that $0 \le r \le R$ and $\overline{B}_R \subset \Omega$, and subtract (3.18) for R from this; we get that

$$\begin{aligned} m(R) - m(r) = c_2 \int_{B_R \setminus B_r} (Vu - 1) \left(|y - \xi_0|^{2-d} - R^{2-d}\right) dy \\ + c_2 \int_{B_r} (Vu - 1) \left(r^{2-d} - R^{2-d}\right) dy \end{aligned} \tag{3.19}$$

Recall that we are interested in $\xi_0 \in \overline{B}_{r/3}(0)$, so that since $Q_r \subset B_{r/2}(0)$, it is contained in $B_r = B_r(\xi_0)$. We will only keep the contribution of V on Q_r (because we want to use its simpler structure), and since

$$\int_{B_R \setminus B_r} \left(|y - \xi_0|^{2-d} - R^{2-d}\right) dy + \int_{B_r} \left(r^{2-d} - R^{2-d}\right) dy \le CR^d (r^{2-d} - R^{2-d}) \le C(R^2 - r^2)$$

(3.19) yields

$$m(R) - m(r) \ge -c_3(R^2 - r^2) + c_2 \left(r^{2-d} - R^{2-d}\right) \int_{Q_r} Vu\, dy. \tag{3.20}$$

In dimension $d = 2$ one has

$$m(r) = m(0) + c_2 \int_{B_r} (Vu - 1) \log \frac{r}{|y - \xi_0|}\, dy \tag{3.21}$$

in place of (3.18), and since

$$\int\limits_{B_R\setminus B_r} \log\frac{r}{|y-\xi_0|}dy + \int\limits_{B_r} \log\frac{R}{r}dy \le C(R^2-r^2) + Cr^2\log\frac{R}{r} \le C(R^2-r^2),$$

we obtain

$$m(R) - m(r) \ge -c_3(R^2-r^2) + c_2\log\frac{R}{r}\int\limits_{Q_r} Vu\,dy \tag{3.22}$$

in place of (3.20). In dimension $d=1$, (3.18) becomes

$$m(r) = m(0) + c_2\int\limits_{B_r} u''(y)\,(r-|y-\xi_0|)\,dy = m(0) + c_2\int\limits_{B_r}(Vu-1)\,(r-|y-\xi_0|)\,dy \tag{3.23}$$

and hence we have

$$m(R) - m(r) \ge -c_3(R^2-r^2) + c_2\,(R-r)\int\limits_{Q_r} Vu\,dy \tag{3.24}$$

in place of (3.20).

Step III. Write $Q_r = \bigcup_{j\in J} R_j$, where R_j is the cube of unit sidelength centered at $j\in\mathbb{Z}^d$, and $J = \mathbb{Z}^d\cap Q_r$ precisely corresponds to the cubes R_j that are contained in Q_r. Then set

$$J_V := \big\{j\in J\,:\,\omega_j \ge c_P r^{-2}\big\}. \tag{3.25}$$

Observe that since $V(x) = \sum\omega_j\,\varphi(x-j)$, with $\varphi\in C_0^\infty(B_{1/10}(0))$, we have that $⨍_{R_j} V = \omega_j\,⨍_{R_0}\varphi$, where R_0 (exceptionally) denotes the unit cube centered at 0. Thus

$$J_V := \big\{j\in J\,:\,⨍\limits_{R_j} V \ge c'_P r^{-2}\big\}, \tag{3.26}$$

with $c'_P = c_P\,⨍_{R_0}\varphi$.

Denote by m_r the average of u on the ball $B_r(\xi_0)$ (notice the difference with $m(r)$ which is an average on the sphere) and let $u_j := \inf_{R_j} u$. Now pick some $\eta>0$ (a dimensional constant to be chosen below) and let

$$J_\eta = \{j\in J_V\,:\,u_j < \eta\,m_r\}. \tag{3.27}$$

Step IV. We start with the case when

$$\operatorname{Card} J_\eta \ge \frac{\lambda}{2}\,|Q_r|.$$

By Harnack's inequality at scale 1 (see, [11], Theorem 8.18),

$$\fint_{R_j} u\,dx \le 2^d \fint_{2R_j} u\,dx \le C\left(\inf_{R_j} u + 1\right).$$

Since, in addition, $u \ge 1$ on Ω (recall that $0 \le V \le 1$ here, and see [1], Proposition 3.2), we have

$$\fint_{R_j} u\,dx \le C'_H \inf_{R_j} u,$$

for some constant C'_H depending on the dimension only. Therefore,

$$\int_{R_j} u\,dx = \fint_{R_j} u\,dx \le C'_H \eta m_r \quad \text{for any } \ j \in J_\eta.$$

Then

$$\int_{B_r \setminus \bigcup_{j\in J_\eta} R_j} u\,dx \ge |B_r| m_r - C'_H \eta m_r \operatorname{Card} J_\eta$$

and

$$\fint_{B_r \setminus \bigcup_{j\in J_\eta} R_j} u\,dx \ge \frac{|B_r| - C'_H \eta \operatorname{Card} J_\eta}{|B_r| - \operatorname{Card} J_\eta}\, m_r = \Big(1 + \frac{(1 - C'_H\eta)\operatorname{Card} J_\eta}{|B_r| - \operatorname{Card} J_\eta}\Big) m_r \ge (1 + c_3\lambda)\, m_r,$$

for $\eta = (2\,C'_H)^{-1}$ and a suitable dimensional constant c_3. We conclude that there exists a point $\xi \in B_r$ such that

$$\begin{aligned} u(\xi) \ge (1 + c_3\lambda)\, m_r &\ge (1 + c_3\lambda)(m(0) - r^2) \\ &\ge m(0) + c_3\lambda m(0) - m(0)\,(1 + c_3\lambda)/M \end{aligned} \tag{3.28}$$

where we integrated (3.16) for the second inequality and used the fact that $m(0) = u(\xi_0) \ge Mr^2$ by (3.13) in the third one. If we fix

$$M \ge \frac{c_4}{\lambda} \tag{3.29}$$

then there exists a point $\xi \in B_r$ such that

$$u(\xi) \ge (1 + c_5\lambda)\, m(0), \tag{3.30}$$

where as usual all c_i depend on the dimension only. Hence, choosing

$$\varepsilon < c_5 \lambda, \tag{3.31}$$

we arrive at (3.14).

Step V. Assume now that, on the contrary,

$$\operatorname{Card} J_\eta \le \frac{\lambda}{2} |Q_r|.$$

Let $R = \sqrt{1+\varepsilon}\, r$, $\varepsilon < 1/2$. First assume that $d \ge 3$; then by (3.20),

$$\begin{aligned} m(R) - m(r) &\ge -c_3(R^2 - r^2) + c_2 \left(r^{2-d} - R^{2-d}\right) \int\limits_{Q_r} V u \, dy \\ &\ge -c_3(R^2 - r^2) + c_2 \left(r^{2-d} - R^{2-d}\right) \sum_{j \in J_V \setminus J_\eta} \int\limits_{R_j} V u \, dy. \end{aligned}$$

But for such j, $\int_{R_j} V u \, dy \ge u_j \int_{R_j} V \, dy \ge \eta m_r \int_{R_j} V = \eta m_r \fint_{R_j} V \ge \eta m_r c'_P r^{-2}$ by various definitions including (3.26) and (3.27). Thus, since $R = \sqrt{1+\varepsilon}\, r$,

$$\begin{aligned} m(R) - m(r) &\ge -c_3(R^2 - r^2) + c_2 \left(r^{2-d} - R^{2-d}\right) c'_P r^{-2} \eta m_r \left(\operatorname{Card} J_V - \operatorname{Card} J_\eta\right) \\ &\ge -c_3 \varepsilon r^2 + c_6 \varepsilon c'_P m_r \, \lambda. \end{aligned}$$

When $d = 1, 2$, we use (3.22) and (3.24) instead of (3.20), and get the same final estimate, namely

$$m(R) - m(r) \ge -c_3 \varepsilon r^2 + c_6 \varepsilon c'_P m_r \, \lambda$$

(possibly further adjusting c_3 and c_6 still depending on dimension only). Using (3.16) and its integrated version for m_r, and then the fact that $m(0) \ge M r^2$ by (3.13), we obtain that

$$\begin{aligned} m(R) &\ge m(0) - r^2 - c_3 \varepsilon r^2 + c_6 \varepsilon c'_P \, \lambda \big(m(0) - r^2\big) \\ &\ge m(0) \Big(1 + c_6 \varepsilon c'_P \, \lambda \big(1 - \frac{1}{M}\big) - \frac{1 + c_3 \varepsilon}{M}\Big). \end{aligned}$$

Choosing c_P so large that

$$c'_P \ge \frac{4}{c_6 \lambda} \tag{3.32}$$

(recall Step III) and M such that

$$M > c_7 \max\left\{1, \frac{1}{\varepsilon}, \frac{1}{\lambda}\right\} \tag{3.33}$$

(the third part takes care of (3.29)) we ensure that the second term in the parentheses above is larger than 2ε and the third term smaller than ε, so that

$$m(R) \geq m(0)\,(1+\varepsilon)$$

and hence, (3.14) holds for some $\xi \in \partial B_R$, as needed for (3.14). □

Lemma 3.34. *Let Ω and $L = -\Delta + V$ be as in Theorems 1.13 and 3.1. In particular V is a random potential governed by a probability measure, as in (3.2) and (3.3). Fix $0 < \lambda < 1$. Then choose $c_P = c_P(\lambda, d)$ large enough, $\varepsilon = \varepsilon(\lambda, d) > 0$ small enough, and $M = M(\varepsilon, \lambda, d) > 0$ large enough, as in Lemma 3.11.*

Recall that $\Omega = \mathbb{R}^d/R_0\,\mathbb{Z}^d$ and, for $r \in [3\sqrt{d}, R_0]$, let Q_r denote as before the maximal cube consisting of unit cubes centered on $\mathbb{Z}^d$ which is contained in $B_{r/2}(0)$. Then let

$$\mathbb{P}_r := \mathbb{P}\big(\big\{\mathrm{Card}\,\{j \in Q_r \cap \mathbb{Z}^d \,:\, \omega_j \leq c_P r^{-2}\} \geq (1-\lambda)\,|Q_r|\big\}\big). \tag{3.35}$$

Also define a similar quantity for the whole domain, i.e.,

$$\mathbb{P}_\Omega = \mathbb{P}\Big(\big\{\mathrm{Card}\,\{j \in \Omega \cap \mathbb{Z}^d \,:\, \omega_j \leq c_P R_0^{-2}\} \geq \big(1-\lambda\big)\,|\Omega|\Big\}\Big). \tag{3.36}$$

Finally, for $3\sqrt{d} \leq r < R_0$, set $r_k = (1+\varepsilon)^{k/2}\,r$ for $0 \leq k \leq k_{max}$, where k_{max} is the largest integer such that $r_k < R_0$. Then

$$\mathbb{P}\Big(\Big\{\sup_{\xi \in \overline{B}_{r/3}(0)} u(\xi) \geq Mr^2\Big\}\Big) \leq \mathbb{P}_\Omega + C\varepsilon^{-d} \sum_{0 \leq k \leq k_{max}} \mathbb{P}_{r_k}, \tag{3.37}$$

where C depends only on the dimension.

Here we shall not even need our assumption that the probability distribution F of (3.3) is not concentrated at one point and $F(\delta) > 0$ for $\delta > 0$; we will evaluate the probabilities later.

We wrote our estimates with all the cubes Q_ρ, and our test ball $\overline{B}_{r/3}(0)$, all centered at 0, but since the ω_j are i.i.d. variables and our problem is invariant under translations by $\mathbb{Z}^d$, the various probabilities mentioned in the statement would be the same with all the cubes (and the test ball) centered anywhere else on $\mathbb{Z}^d$. We will also use this invariance during the proof.

Proof. The idea is to repeatedly use Lemma 3.11 and stop when the resulting ball exceeds the size of Ω.

Let r be given, suppose that $\sup_{\xi \in \overline{B}_{r/3}(0)} u(\xi) \geq Mr^2$; we pick $\xi_0 \in \overline{B}_{r/3}(0)$ such that $u(\xi_0) \geq Mr^2$, and try to use Lemma 3.11 repeatedly to find points ξ_j with $u(\xi_j)$ always larger. Set (for later coherence of notation) $Q_0 = Q_r$. One possibility is that (3.12) fails (with this choice of Q_r); we call this event $\mathfrak{A}_0$. But suppose not; then Lemma 3.11 gives

a point $\xi_1 \in \overline{B_{r_1}}(\xi_0)$, with $r_1 = (1+\varepsilon)^{1/2} r$ as above, such that $u(\xi_1) \geq (1+\varepsilon) u(\xi_0)$, as in (3.14).

Notice that $u(\xi_1) \geq M r_1^2$, so we can try to apply Lemma 3.11 again. This time, it could be that $\xi_1 \notin \overline{B}_{r_1/3}(0)$, so we choose $\overline{\xi}_1 \in \mathbb{Z}^d$ such that $\xi_1 - \overline{\xi}_1 \in \overline{B}_{r_1/3}(0)$, and apply the lemma after translating by $\overline{\xi}_1$. We will need to be more specific later about how we choose $\overline{\xi}_1$, but for the moment let us not bother. This means that the role of Q_r is now played by $Q_1 = \overline{\xi}_1 + Q_{r_1}$. One possibility is that (3.12) fails for Q_1; we call this event $\mathfrak{A}_1$. But we assume not for the moment, and the lemma gives a new point $\xi_2 \in \overline{B_{r_2}}(\xi_1)$ such that $u(\xi_2) \geq (1+\varepsilon) u(\xi_1)$, as in (3.14). Then $u(\xi_2) \geq M r_2^2$ and we can try to apply Lemma 3.11 again.

We continue as long as we do not encounter an event $\mathfrak{A}_k$ where (3.12) fails for Q_k, and then we end with a last application for k_{max}, which gives a point $\xi_{k_{max}+1}$ such that $u(\xi_{k_{max}+1}) \geq M r_{k_{max}+1}^2 \geq M R_0^2$. Let $\xi_\infty \in \Omega$ be such that $u(\xi_\infty) = ||u||_\infty$, and notice that $u(\xi_\infty) \geq M R_0^2$. We now try to apply Lemma 3.11 one last time, to the point ξ_∞, but for this it will be convenient to enlarge our domain.

Suppose for definiteness that our fundamental domain Ω (we abuse notation a little, and give it the same name as $\mathbb{R}^d / R_0 \mathbb{Z}^d$) is the cube of sidelength R_0 centered at the origin; we know that, due to our periodic conditions, other choices would be equivalent, but with this choice we were able to state and prove Lemma 3.11 without crossing the boundary. Pick an odd integer N larger than $4\sqrt{d}$, and denote by $\widetilde{\Omega}$ the cube centered at the origin and with sidelength $N R_0$; thus $\widetilde{\Omega}$ is composed of Ω, plus a certain number of translated copies. Extend V and u to be $R_0 \mathbb{Z}^d$-periodic. Then the extension of u still satisfies $Lu = 1$ on $\widetilde{\Omega}$, and by uniqueness it is the landscape function associated to $\widetilde{\Omega}$ and periodic boundary conditions. We apply Lemma 3.11 with this new, larger domain, and the radius $r = 2\sqrt{d} R_0$, so that the corresponding cube Q_r is precisely Ω. Our choice of N is large enough for this to be possible, and also we may assume, since our problem is invariant by translations from $\mathbb{Z}^d$, that $\xi_\infty \in \overline{B}_{r/3}(0)$. Our last bad event $\mathfrak{A}_{k_{max}+1}$ is when (3.12) fails for $Q_r = \Omega$, and if this does not happen, we get a new point $\xi \in \widetilde{\Omega}$ such that $u(\xi) \geq (1+\varepsilon) u(\xi_\infty)$. This is impossible, because $u(\xi_\infty) = ||u||_\infty$ and u takes the same values on $\widetilde{\Omega}$ as on Ω.

At this point we proved that if the event of the left-hand side of (3.37) occurs (i.e., we can find ξ_0 as above, with $u(\xi_0) \geq M r^2$), then one of the bad events $\mathfrak{A}_k$ occurs. What we just need to do now is check that the probability of each event $\mathfrak{A}_k$ is at most the corresponding term of the right-hand side of (3.37). In particular, we do not need to check anything about the independence of these events, we just add their probability.

In our last case we made sure that $Q_r = \Omega$ precisely, and so this is almost the definition (compare (3.36) with (3.12)); there is a small discrepancy, due to the fact that since $r = 2\sqrt{d} R_0$ here, we should have said $\omega_j \leq C_P (2\sqrt{d} R_0)^{-2}$ rather than $\omega_j \leq C_P R_0^{-2}$, but the difference only amounts to making C_P a little larger, which is not a problem, and we prefer the less sharp, but simpler form in (3.36).

For $0 \leq k \leq k_{max}$, we need to evaluate the probability of the event $\mathfrak{A}_k$, but we have to be a little careful, because we only know that (3.12) fails for the translated cube

$Q_k = \overline{\xi}_k + Q_{r_k}$, but a priori we do not know which cube this is. Given the position of $\xi_0 \in \overline{B}(r/3)(0)$, and the fact that for $0 \le m < k$, $|\xi_{m+1} - \xi_m| \le r_{m+1}$, we see that $|\xi_k| \le \sum_{m=1}^{k} r_m \le C\varepsilon^{-1} r_k$. We need to find $\overline{\xi}_k \in \mathbb{Z}^d$ such that $|\xi_k - \overline{\xi}_k| \le r_k/3$, so we can choose $\overline{\xi}_k$ in some set Ξ_k, known in advance, with less than $C\varepsilon^{-d}$ elements. Our event $\mathfrak{A}_k$ can only happen if (3.12) fails for one of the cubes $\overline{\xi}+Q_{r_k}$, $\overline{\xi} \in \Xi_k$, and the total probability that this happens is at most $C\varepsilon^{-d}\mathbb{P}_{r_k}$ (all the smaller events associated to a single $\overline{\xi} \in \Xi_k$ have the same probability $\mathbb{P}_{r_k}$, because our ω_j are i.i.d.). This completes the proof of Lemma 3.34. □

Lemma 3.38. *Let Q be some cube in $\mathbb{R}^d$ and assume that the ω_j, $j \in \mathbb{Z}^d \cap Q$, are i.i.d. variables taking values $0 \le \omega_j \le 1$, with a probability distribution*

$$F(\delta) = \mathbb{P}\{\omega \le \delta\}, \quad 0 \le \delta \le 1,$$

which is not trivial, i.e., not concentrated at one point, and such that 0 is the infimum of the support.

Fix $0 < \mu < 1$, $c_P^* > 0$, and consider $r > 0$ such that $\mu - F(c_P^* r^{-2}) > 0$. Then such that

$$\mathbb{P}\left(\left\{\mathrm{Card}\left\{j \in Q \cap \mathbb{Z}^d : \omega_j \le c_P^* r^{-2}\right\} \ge \mu\,\mathrm{Card}\left\{Q \cap \mathbb{Z}^d\right\}\right\}\right) \le \left(H(\mu)F(c_P^* r^{-2})^{\mu}\right)^{\mathrm{Card}\{Q\cap\mathbb{Z}^d\}} \tag{3.39}$$

with $H(\mu) = \left(\mu^{\mu}(1-\mu)^{1-\mu}\right)^{-1}$.

While we intend to use the Lemma for $\mathbb{P}_r$ and $\mathbb{P}_\Omega$ from Lemma 3.34, we chose to state it in full generality to emphasize explicit dependence on the constants which could be useful in other contexts. Also, observe that

$$\lim_{\mu \to 1} H(\mu) = 1; \tag{3.40}$$

we will be able to choose μ so close to 1, depending on $\mathbb{E}(\omega)$ and the dimension only, that $H(\mu)F(c_P^* r^{-2})^{\mu} < F(c_P^* r^{-2})^{1/2}$, at least for r sufficiently large, also depending on $\mathbb{E}(\omega)$ and the dimension only.

Proof. Let P denote the left-hand side of (3.39), and define the random variables ζ_j equal to 1 when $\omega_j \le c_P^* r^{-2}$ and 0 otherwise. By our assumptions the ζ_j are independent and identically distributed. Furthermore,

$$P = \mathbb{P}\Big(\Big\{\sum_{j \in Q\cap\mathbb{Z}^d} \zeta_j \ge \mu\,\mathrm{Card}\left\{Q \cap \mathbb{Z}^d\right\}\Big\}\Big),$$

hence for any $t > 0$,

$$P = \mathbb{P}\left(\left\{e^{t\sum_{j\in Q\cap\mathbb{Z}^d}\zeta_j} \geq e^{t\mu\,\mathrm{Card}\,\{Q\cap\mathbb{Z}^d\}}\right\}\right) \leq e^{-t\mu\,\mathrm{Card}\,\{Q\cap\mathbb{Z}^d\}}A \tag{3.41}$$

by Chebyshev's inequality, and where A is the expectation of the product of independent identically distributed variables $e^{t\zeta_j}$, hence $A = A_0^{\mathrm{Card}\,\{Q\cap\mathbb{Z}^d\}}$, where A_0 is the expectation of any of the $e^{t\zeta_j}$. That is,

$$A_0 = e^t\mathbb{P}(\{\omega_j \leq c_P^* r^{-2}\}) + \mathbb{P}(\{\omega_j > c_P^* r^{-2}\}) = e^t F(c_P^* r^{-2}) + 1 - F(c_P^* r^{-2})$$

and, by (3.41),

$$P \leq \exp\left(-\mathrm{Card}\,\{Q\cap\mathbb{Z}^d\}\big(t\mu - \log A_0\big)\right)$$

for every $t > 0$. We now want to optimize in t, but let us introduce notation before we compute. Set $N = \mathrm{Card}\,\{Q\cap\mathbb{Z}^d\}$, $F = F(c_P^* r^{-2})$ (two constants) and, for $t > 0$,

$$f(t) := t\mu - \log A_0 = t\mu - \log(e^t F + 1 - F).$$

Thus $P \leq e^{-Nf(t)}$, and we study f. First, $f(0) = 0$, and $f'(t) = \mu - \frac{e^t F}{e^t F + 1 - F}$. Thus $f'(0) = \mu - F = \mu - F(c_P^* r^{-2}) > 0$ by our assumptions, and hence f is increasing near 0. In fact, f' only vanishes at the point t^* such that

$$e^{t^*} = \frac{\mu}{1-\mu}\,\frac{1-F}{F}$$

(notice that this last value is > 1 since $\mu > F$). Since we strongly expect $f(t)$ to be minimal for $t = t^*$, we decide to take $t = t^*$ in the inequality above. This yields

$$\begin{aligned} P \leq e^{-Nf(t^*)} &= e^{-Nt^*\mu + N\log(e^{t^*}F + 1 - F))} \\ &= \exp\left(-N\mu\log\left(\frac{\mu}{1-\mu}\,\frac{1-F}{F}\right) + N\log\left(e^{t^*}F + 1 - F\right)\right) \\ &= \exp\left(-N\mu\log\left(\frac{\mu}{1-\mu}\,\frac{1-F}{F}\right) + N\log\left(\frac{1-F}{1-\mu}\right)\right) \\ &= \exp\left(-N\log\left(\frac{\mu^\mu(1-\mu)^{1-\mu}}{F^\mu\,(1-F)^{1-\mu}}\right)\right) = \left(\frac{F^\mu\,(1-F)^{1-\mu}}{\mu^\mu(1-\mu)^{1-\mu}}\right)^N. \end{aligned} \tag{3.42}$$

We may drop $(1-F)^{1-\mu} \leq 1$, and now this is the same thing as (3.39); Lemma 3.38 follows. □

Corollary 3.43. *Let Ω, L, and V be as in Theorem 3.1. There exist constants R^*, c_P, M, γ_1, γ_2, depending only on the dimension and the common expectation of the random variables ω_j, such that*

$$\mathbb{P}\left\{u(\xi_0) \geq Mr^2\right\} \leq \gamma_1 F(c_P\, r^{-2})^{\gamma_2\, r^d} \tag{3.44}$$

for any $\xi_0 \in \Omega$ *and any* $r \in (R^*, R_0]$.

Proof. This will follow from a combination of Lemmas 3.34 and 3.38. First recall our assumption that the measure associated to F (call it ν) is nontrivial. Let $\mathbb{E}(\omega)$ denote the expectation of our random variables; then

$$0 < \mathbb{E}(\omega) < 1, \tag{3.45}$$

where the first inequality holds because ν is not a Dirac mass at the origin, and second one holds because the support of ν touches 0 and is contained in $[0, 1]$.

Furthermore notice that $\mathbb{E}(\omega) = \int_{[0,1]} \delta d\nu(\delta) = \int_{(0,1]} \delta d\nu(\delta) \leq 1 - \nu(\{0\})$ by Chebyshev's inequality, so $F(0) = \nu(\{0\}) \leq 1 - \mathbb{E}(\omega) < 1$. Clearly, $F(c_p r^{-2})$ decays as r grows. We choose a value of $F(c_p r^{-2})$ that we don't want to exceed, half of the way between $1-\mathbb{E}(\omega)$ and 1, i.e., $F_0 = \frac{2-\mathbb{E}(\omega)}{2} < 1$, choose (we shall see why soon) $a = \frac{\mathbb{E}(\omega)}{2-\mathbb{E}(\omega)} \in (0, 1)$, and check now that

$$F(a) \leq F_0 = \frac{2 - \mathbb{E}(\omega)}{2}. \tag{3.46}$$

Indeed $\mathbb{E}(\omega) = \int_{[0,1]} x d\nu(x) \leq a\nu([0, a]) + \nu((a, 1] = aF(a) + 1 - F(a)$, hence $F(a)(1-a) \leq 1 - \mathbb{E}(\omega)$ and since $1 - a = \frac{2-2\mathbb{E}(\omega)}{2-\mathbb{E}(\omega)}$, we get (3.46).

Now let $\mu \in (3/4, 1)$ be given, to be chosen soon in terms of F_0, very close to 1. Also set $\lambda = 1 - \mu$ (small), and with this λ, define $c_P = c_P(\lambda, d)$ large enough, as in Lemma 3.11, and choose $\varepsilon = \varepsilon(\lambda, d) > 0$ small enough, and $M = M(\varepsilon, \lambda, d)$ large enough, again as in Lemma 3.11. Those choices also work for Lemma 3.34, so we will be able to apply these two lemmas with these constants.

We choose R^* so large that $c_P(R^*)^{-2} \leq a$; R^* depends on λ and μ, but soon we will be able to choose μ (and hence, λ), that depends only on $\mathbb{E}(\omega)$ and the dimension, so eventually R^* will depend only on $\mathbb{E}(\omega)$ and the dimension as well. With this choice of R^*, and since we shall always restrict to radii $r \geq R^*$, we will get that

$$F(c_P r^{-2}) \leq F(c_P(R^*)^{-2}) \leq F(a) \leq F_0 := \frac{2 - \mathbb{E}(\omega)}{2}. \tag{3.47}$$

The whole point of Lemma 3.38 was to give a bound on the probability $\mathbb{P}_r$ of (3.35), and this bound is

$$\mathbb{P}_r \leq \left(H(\mu)F(c_P r^{-2})^{\mu}\right)^N, \tag{3.48}$$

with $N = \text{Card}\,\{Q \cap \mathbb{Z}^d\}$. Notice that we can take $c_P^* = c_P$, and the assumption that $F(c_P r^{-2}) < \mu$ is satisfied by (3.47) if we take $\mu > F_0$. We also take $\mu > 3/4$, so that $F(c_P r^{-2})^{\mu-1/2} \leq F_0^{\mu-1/2} \leq F_0^{1/4}$ and use (3.40) to finally choose μ so close to 1 that

$H(\mu)F_0^{1/4} < 1$. This way (3.48) implies that $\mathbb{P}_r \le F(c_P r^{-2})^{N/2}$, which will be good enough for us.

Now let $r \ge R^*$ be given, and let us evaluate the probability (call it P) of (3.44). Notice that P is smaller than the probability of having $u(\xi) \ge Mr^2$ for some point of a cube S of size roughly $(10\sqrt{d})^{-1}r$, say, that contains ξ_0. This probability does not depend on S (by invariance), and can be estimated as in Lemma 3.34. Thus we get that

$$P \le \mathbb{P}_\Omega + C\varepsilon^{-d} \sum_{0\le k\le k_{max}} \mathbb{P}_{r_k},$$

with $r_k = (1+\varepsilon)^{k/2}r$. We use (the consequence of) (3.48) to estimate $\mathbb{P}_{r_k}$, noticing that $F(c_P r_k^{-2}) \le F(c_P r^{-2})$ and each set $Q_{r_k} \cap \mathbb{Z}^d$ has at least one more point than the previous one. That is, $N_k = \mathrm{Card}\{Q_{r_k} \cap \mathbb{Z}^d\}$ is at least $N+k$, where $N = \mathrm{Card}\{Q_r \cap \mathbb{Z}^d\}$. Then $\mathbb{P}_{r_k} \le F(c_P r_k^{-2})^{N_k/2} \le F(c_P r^{-2})^{(N+k)/2} \le F_0^{k/2} F(c_P r^{-2})^{N/2}$.

We have a similar estimate for $\mathbb{P}_\Omega$ (which is of the same type as $\mathbb{P}_{r_k}$, with $r_k \sim R_0$). So we can sum the geometric series, and get the more precise estimate

$$P \le \gamma_1 F(c_P r^{-2})^{\mathrm{Card}\{Q_r\cap\mathbb{Z}^d\}/2} \le \gamma_1 F(c_P r^{-2})^{\gamma_2 r^d} \tag{3.49}$$

with constants γ_1 and γ_2 that depend on d and $\mathbb{E}(\omega)$ (through our choice of F_0, a, μ, and then the various constants that ensue, including ε). As was said earlier, we can then compute R^*, depending on these constants. Corollary 3.43 follows. □

Corollary 3.50. *Let Ω, L, and V be as in Theorem 3.1, in particular V is a random potential governed by i.i.d. random variables ω_j. Then there exist constants $\mu^*, M, c_P, \gamma_3, \gamma_4$, depending only on the dimension and the expectation of the ω_j,*

$$N_u^E(\mu) \le \gamma_3 \mu^{d/2}\, F(Mc_P\,\mu)^{\gamma_4\,\mu^{-d/2}}, \tag{3.51}$$

whenever $\mu < \mu^*$ and $R_0 > (\mu M)^{-1/2}$.

Proof. Recall from (1.6) and the statement of Theorem 1.13 that

$$N_u^E(\mu) = \frac{1}{|\Omega|} \times \mathbb{E}\left\{\text{the number of cubes } Q \in \{Q\}_{\kappa\,\mu^{-1/2}} \text{ such that } \min_Q \frac{1}{u} \le \mu\right\},$$

where $1 \le \kappa < 2$ (depending on μ) is the smallest number such that R_0 is an integer multiple of $\kappa\mu^{-1/2}$. The expectation of the number of cubes is less than the sum of expectations (by the triangle inequality), so

$$N_u^E(\mu) \le \frac{1}{|\Omega|}\,\frac{|\Omega|}{(\kappa\mu^{-1/2})^d} \sup_{Q\in\{Q\}_{\kappa\,\mu^{-1/2}}} \mathbb{P}\left\{\min_Q \frac{1}{u} \le \mu\right\}.$$

We want to apply our estimate in (3.49), coming from Lemma 3.34. This one gives the probability that the infimum of $\frac{1}{u}$ on $B_{r/3}(0)$ is at most $(Mr^2)^{-1}$, so we should take r such that $(Mr^2)^{-1} = \mu$. Notice that $r \le R_0$ by our condition on R_0. We get equal probabilities for integer translations of that ball, as usual, by the translation invariance of our setting. Now each cube $Q \in \{Q\}_{\kappa\,\mu^{-1/2}}$ can be covered by less than C integer translations of $B_{r/3}(0)$ (taken from a fixed subgrid), and for each one the probability that $\frac{1}{u} \le \mu$ somewhere on the ball is estimated as in (3.49). Therefore

$$N_u^E(\mu) \le C(\kappa\mu^{-1/2})^{-d}\gamma_1 F(c_P r^{-2})^{\gamma_2 r^d} \le \gamma_3 \mu^{d/2} F(Mc_P\mu)^{\gamma_4 \mu^{-d/2}},$$

as announced. □

We now give a lower bound for $N_u^E(\mu)$.

Lemma 3.52. *Let Ω, L, and V be as in Theorem 3.1 and in the previous lemmas. There exist constants $m, \widetilde{c}_P, \gamma_5, \gamma_6$, depending on the dimension only, such that*

$$N_u^E(\mu) \ge \gamma_5\, \mu^{d/2}\, F(\widetilde{c}_P\, \mu)^{\,\gamma_6\, \mu^{-d/2}}, \tag{3.53}$$

whenever $\mu \le 1$ and $R_0 > (\mu m)^{-1/2}$.

Proof. Much as above, we start observing that

$$N_u^E(\mu) = \frac{1}{|\Omega|} \times \mathbb{E}\left\{\text{the number of cubes } Q \in \{Q\}_{\kappa\,\mu^{-1/2}} \text{ such that } \min_Q \frac{1}{u} \le \mu\right\} \tag{3.54}$$

$$\ge \frac{1}{|\Omega|} \times \sum_{Q\in\{Q\}_{\kappa\,\mu^{-1/2}}} \mathbb{P}\left\{\min_Q \frac{1}{u} \le \mu\right\}. \tag{3.55}$$

Now we recall again from [1], Lemma 4.1 (or (1.3)), that

$$\int_\Omega |\nabla f|^2 + V\, f^2\, dx \ge \int_\Omega \frac{1}{u}\, f^2\, dx,$$

for all f in the space of periodic functions in $W^{1,2}(\Omega)$, and in particular for $f \in C_0^\infty(\Omega)$. We will choose f to be a standard cut-off on $4C_1Q$, $C_1 \ge 1$; that is, $f \in C_0^\infty(4C_1Q)$, $f = 1$ on C_1Q and $|\nabla f| \le (C_1 l(Q))^{-1}$. We will need that $4C_1Q \subset \Omega$, i.e., Ω should be large enough to accommodate this. This is ensured by the condition $R_0 > (\mu m)^{-1/2}$ if m is small enough. It follows that

$$\min_{C_1Q} \frac{1}{u} \le \frac{1}{|C_1Q|}\left(\int_{C_1Q} \frac{1}{u} f^2\right) \le \frac{1}{|C_1Q|}\left(\int_\Omega |\nabla f|^2 + V f^2\right)$$

$$\leq \frac{1}{|C_1 Q|}\left(\int_{4C_1Q} (C_1 l(Q))^{-2} + V\right) \leq 4^d\Big((C_1 l(Q))^{-2} + \fint_{4C_1Q} V\Big).$$

We choose C_1 such that $4^d C_1^{-2} \leq 1/2$; then $4^d (C_1 l(Q))^{-2} \leq l(Q)^{-2}/2 = \kappa^{-1}\mu/2 \leq \mu/2$, and now $\min_{C_1Q} \frac{1}{u} \leq \mu/2 + 4^d \fint_{4C_1Q} V$. Therefore

$$\mathbb{P}\left\{\min_{C_1Q} \frac{1}{u} \leq \mu\right\} \geq \mathbb{P}\left\{4^d \fint_{4C_1Q} V\, dx \leq \mu/2\right\} \geq \mathbb{P}\left\{\max_{4C_1Q} V \leq 4^{-d}\mu/2\right\}.$$

Note that

$$\mathbb{P}\left\{\min_{C_1Q} \frac{1}{u} \leq \mu\right\} \leq \sum_{Q' \in C_1Q \bigcap \{Q\}_{\kappa\mu^{-1/2}}} \mathbb{P}\left\{\min_{Q'} \frac{1}{u} \leq \mu\right\}.$$

Therefore,

$$\begin{aligned}\sum_{Q\in\{Q\}_{\kappa\mu^{-1/2}}} \mathbb{P}\left\{\min_{C_1Q} \frac{1}{u} \leq \mu\right\} &\leq \sum_{Q\in\{Q\}_{\kappa\mu^{-1/2}}} \sum_{Q' \in C_1Q \bigcap \{Q\}_{\kappa\mu^{-1/2}}} \mathbb{P}\left\{\min_{Q'} \frac{1}{u} \leq \mu\right\} \\ &\leq C_1^d \sum_{Q\in\{Q\}_{\kappa\mu^{-1/2}}} \mathbb{P}\left\{\min_{Q} \frac{1}{u} \leq \mu\right\}.\end{aligned}$$

Combining all of the above and using the independence of the ω_j, we conclude that

$$\begin{aligned}\sum_{Q\in\{Q\}_{\kappa\mu^{-1/2}}} \mathbb{P}\left\{\min_{Q} \frac{1}{u} \leq \mu\right\} &\geq C_1^{-d} \sum_{Q\in\{Q\}_{\kappa\mu^{-1/2}}} \mathbb{P}\left\{\omega_j \leq 4^{-d}\mu/2 \;\forall\, j \in 5C_1Q \cap \mathbb{Z}^d\right\} \\ &= C_1^{-d} \frac{|\Omega|}{|(\kappa\mu^{-1/2})^d|} F\left(4^{-d}\mu/2\right)^{\mathrm{Card}\,\{5C_1Q\cap\mathbb{Z}^d\}},\end{aligned}$$

which yields the desired conclusion. □

We are now finished with the proof of Theorem 3.1, which is a combination of Corollary 3.50 and Lemma 3.52. We just renamed the four γ_j, and also renamed Mc_P from Lemma 3.52 as c_P, but both of these constants depend only on d and the expectation of the ω_j.

We shall now see how Theorem 3.1 provides the desired estimates on the expectation of the density of states.

Theorem 3.56. *Let Ω, L, and V be as in Theorems 1.13 and 3.1. Then there exist constants $C_5, C_6 > 0$, depending on the dimension and the expectation of the random variables ω_j, only and a constant $C_4 > 0$, depending on the dimension only, such that*

$$C_5 N_u^E(C_6\,\mu) \le N^E(\mu) \le N_u^E(C_4\mu), \tag{3.57}$$

for every $\mu > 0$.

In particular, there exist constants $\mu^*, m_1, c_P, \gamma_1, \gamma_2$, *depending on the dimension and the expectation of the random variable only, and constants* $\widetilde{c}_P, \gamma_3, \gamma_4$, *depending on the dimension only, such that*

$$\gamma_3\,\mu^{d/2} F(\widetilde{c}_P\mu)^{\gamma_4\mu^{-d/2}} \le N^E(\mu) \le \gamma_1\,\mu^{d/2} F(c_P\mu)^{\gamma_2\mu^{-d/2}}, \tag{3.58}$$

whenever $\mu < \mu^*$ *and* $R_0 > (\mu m_1)^{-1/2}$.

Notice that Theorem 3.56 is a combination of Theorem 1.13 and the statement (3.6) in Theorem 3.5. Since the other part of Theorem 3.5, (3.7), was proved in Theorem 3.1, both Theorems 1.13 and 3.5 will follow as soon as we prove Theorem 3.56.

Proof. The right-hand side inequality in (3.57) is the right-hand side inequality in (1.8), hence it has been proved in Theorem 1.7. The proof of the left-hand side of (1.8) will be split into two parts, where $\mu > \mu^\sharp$ and $\mu \le \mu^\sharp$ for some suitable $\mu^\sharp$.

For the values of $\mu > \mu^\sharp$ we are going to proceed as for the proof of (1.11) in Theorem 1.7, and prove that for any given μ_0,

$$N_u(\mu) \le N(C'\mu) \quad \text{for all } \mu > \mu_0, \tag{3.59}$$

where $C' = (d, \mu_0)$ depends only on μ_0 and the dimension. We will essentially use the fact that the function u^2 is a doubling weight. Indeed, given that $\|V\|_{L^\infty(\Omega)} \le 1$, the Harnack inequality (see, [11], Theorem 8.17 and 8.18) guarantees that

$$\sup_{Q_{2s}} u \le C(s)\left(\inf_{Q_s} u + s^2\right).$$

Here the constant $C(s)$ depends on s; specifically, the examination of the proof shows that $C(s) \le C_0^s$ for some dimensional constant C_0 (see the comment right after the statement of Theorem 8.20 in [11] to this effect or simply use the Harnack inequality at scale 1 roughly s times to treat larger s). Hence, if s is bounded from above by some constant depending on d and some $\mu_0 > 0$, we have

$$\sup_{Q_{2s}} u \le C(d, \mu_0)\left(\inf_{Q_s} u + s^2\right).$$

Going further, we recall that $u \ge 1$ on Ω (see [1], Proposition 3.2), so that possibly further adjusting $C(d, \mu_0)$ we have

$$\sup_{Q_{2s}} u \le C(d, \mu_0) \inf_{Q_s} u,$$

again assuming that s is bounded from above by some constant depending on d and μ_0. We now follow the argument in (2.2)–(2.7), except that this time we take $C_2 = 1$. Then the sidelength of the cube under consideration is $\kappa\mu^{-1/2} \leq 2(\mu_0)^{-1/2}$, and we will be using doubling on cubes of the size at most $16\,(\mu_0)^{-1/2}$ (in fact, we even use smaller κ). The argument follows the same path, only arriving at the bound by some constant $C'(d,\mu_0)\,\mu$ in place of $C_{d,5}C_2\mu$ on the right-hand side of (2.7). Thus (3.59) holds: $N_u(\mu) \leq N(C'(d,\mu_0)\mu)$, for all $\mu > \mu_0$. We can write an upper bound on $C'(d,\mu_0) \leq e^{\widetilde{C}\mu_0^{-1/2}}$ explicitly, for a suitable dimensional constant $\widetilde{C}$. Note that $\mu_0 e^{\widetilde{C}\mu_0^{-1/2}} \to \infty$ either as $\mu_0 \to 0$ or as $\mu_0 \to \infty$. Therefore, we choose

$$\mu^\sharp = \min_{\mu_0>0} \mu_0 e^{\widetilde{C}\mu_0^{-1/2}} \tag{3.60}$$

and choose μ_0 to attain the minimum. In other words,

$$N_u(C'(d,\mu_0)^{-1}\,\mu) \leq N(\mu),$$

for all $\mu > \mu^\sharp = \mu_0\,C'(d,\mu_0)$.

Now recall the first inequality in (1.8) of Theorem 1.7 and fix the constants C_1, C_2, C_3 (depending on dimension only) from this inequality. For the $\mu^\sharp$ given as above, we claim that for a suitable choice of $\alpha < 2^{-4}$, depending on dimension and the expectation of the ω_j, and also depending on $\mu^\sharp$,

$$C_3 N_u^E(C_2\alpha^{d+4}\mu) \leq \frac{1}{2}\,C_1\alpha^d N_u^E(C_2\alpha^{d+2}\mu), \tag{3.61}$$

whenever $\mu < \mu^\sharp$ and $R_0 > (\mu m_1)^{-1/2}$ (for some $m_1 > 0$, that depends on the dimension and the expectation of the ω_j only). As we shall see, this is basically a consequence of the fact that according to Theorem 3.1, $N_u^E(\mu)$ is exponentially small for small μ, far beating the polynomial increase of $\alpha^{-d/2}$. Indeed, Theorem 3.1 says that

$$N_u^E(C_2\alpha^{d+4}\mu) \leq \gamma_1\,(C_2\alpha^{d+4}\mu)^{d/2} F(c_P C_2\alpha^{d+4}\mu)^{\gamma_2(C_2\alpha^{d+4}\mu)^{-d/2}}, \tag{3.62}$$

provided that $(C_2\alpha^{d+4}\mu) < \mu^*$ and $R_0 > (C_2\alpha^{d+4}\mu m)^{-1/2}$. These last conditions are ensured if we take $C_2\alpha^{d+4}\mu^\sharp \leq \mu^*$ and $m_1 \leq C_2\alpha^{d+4}m$. Theorem 3.1 also says that

$$N_u^E(C_2\alpha^{d+2}\mu) \geq \gamma_3\,(C_2\alpha^{d+2}\mu)^{d/2} F(\widetilde{c}_P C_2\alpha^{d+2}\mu)^{\gamma_4(C_2\alpha^{d+2}\mu)^{-d/2}} \tag{3.63}$$

provided that $(C_2\alpha^{d+2}\mu) < \mu^*$ and $R_0 > (C_2\alpha^{d+2}\mu m)^{-1/2}$, which will hold if we take $C_2\alpha^{d+2}\mu^\sharp \leq \mu^*$ and $m_1 \leq C_2\alpha^{d+2}m$.

Set $F_2 = F(\widetilde{c}_P C_2\alpha^{d+2}\mu)$ and $F_4 = F(c_P C_2\alpha^{d+4}\mu)$; if we want to prove our claim (3.61), it is enough to prove that

$$C_3\gamma_1\,(C_2\alpha^{d+4}\mu)^{d/2} F_4^{\gamma_2(C_2\alpha^{d+4}\mu)^{-d/2}} \leq \frac{1}{2}C_1\alpha^d\gamma_3\,(C_2\alpha^{d+2}\mu)^{d/2} F_2^{\gamma_4(C_2\alpha^{d+2}\mu)^{-d/2}}. \tag{3.64}$$

Take α so small that $c_P C_2 \alpha^{d+4} < \widetilde{c}_P C_2 \alpha^{d+2}$; thus α depends also on the expectation of ω, through c_P. Then $F_4 \le F_2$. Also choose α so small that $\widetilde{c}_P C_2 \alpha^{d+2} \mu^\sharp < \delta_0$, with $\delta_0 = \mathbb{E}(\omega)/2$. This way, if ν denotes the probability measure defined by F, $\mathbb{E}(\omega) = \int_{[0,1]} \delta d\nu(\delta) \le \delta_0 + \nu((\delta_0, 1]) = \delta_0 + 1 - F(\delta_0)$, so $F(\delta_0) \le 1 - \mathbb{E}(\omega)/2 < 1$. Therefore $F_4 \le F_2 \le 1 - \mathbb{E}(\omega)/2$ in the estimates above; now

$$\frac{F_4^{\gamma_2 (C_2 \alpha^{d+4} \mu)^{-d/2}}}{F_2^{\gamma_4 (C_2 \alpha^{d+2} \mu)^{-d/2}}} \le F_2^{a \mu^{-d/2}}, \tag{3.65}$$

with $a = \gamma_2 (C_2 \alpha^{d+4})^{-d/2} - \gamma_4 (C_2 \alpha^{d+2})^{-d/2} \ge \frac{1}{2} \gamma_2 (C_2 \alpha^{d+4})^{-d/2}$ if $\alpha \le (\gamma_4/\gamma_2)^{1/2}$. Thus the right-hand side of (3.65) is exponentially decreasing when α tends to 0. The powers of μ in (3.64) are the same, and the rest is polynomial in α; thus (3.64) holds for α small, and (3.61) follows.

Now we average (1.8) and use (3.61); we get that

$$\frac{C_1}{2} \alpha^d N_u^E (C_2 \alpha^{d+2} \mu) \le N^E(\mu) \le N_u^E (C_4 \mu), \tag{3.66}$$

which is the same as (3.57) (recall that we are allowed to let C_5 and C_6 depend on α, which is now chosen depending on $\mathbb{P}(\omega)$ and d), except that we have to assume that $\mu < \mu^\sharp$ and $R_0 > (m_1 \mu)^{-1/2}$, and

$$\alpha = \min \left\{ \left(\frac{\mu^*}{C_2 \mu^\sharp} \right)^{\frac{1}{d+2}}, \left(\frac{\delta_0}{\widetilde{c}_P C_2 \mu^\sharp} \right)^{\frac{1}{d+2}}, \left(\frac{\gamma_4}{\gamma_2} \right)^{\frac{1}{2}} \right\}. \tag{3.67}$$

Taken along with Theorem 3.1, this also automatically gives (3.58). As usual, we silently redefine the constants, still depending on the same parameters. □

References

[1] D. Arnold, G. David, M. Filoche, D. Jerison, S. Mayboroda, Localization of eigenfunctions via an effective potential, Commun. Partial Differ. Equ. 44 (11) (2019) 1186–1216, https://doi.org/10.1080/03605302.2019.1626420.

[2] D. Arnold, G. David, M. Filoche, D. Jerison, S. Mayboroda, Computing spectra without solving eigenvalue problems, SIAM J. Sci. Comput. 41 (1) (2019) B69–B92, https://doi.org/10.1137/17M1156721.

[3] D. Arnold, G. David, D. Jerison, S. Mayboroda, M. Filoche, Effective confining potential of quantum states in disordered media, Phys. Rev. Lett. 116 (2016) 056602, https://doi.org/10.1103/PhysRevLett.116.056602.

[4] D. Chaudhuri, J.C. Kelleher, M.R. O'Brien, E.P. O'Reilly, S. Schulz, Electronic structure of semiconductor nanostructures: a modified localization landscape theory, Phys. Rev. B 101 (2020) 035430, https://doi.org/10.1103/PhysRevB.101.035430.

[5] A. Comtet, C. Texier, Comment on "Effective confining potential of quantum states in disordered media", Phys. Rev. Lett. 124 (2020) 219701, https://doi.org/10.1103/PhysRevLett.124.219701.

[6] P. Desforges, S. Mayboroda, S. Zhang, G. David, D. Arnold, W. Wang, M. Filoche, Sharp estimates for the integrated density of states in Anderson tight-binding models, arXiv:2010.09287 [math-ph], 2020.

[7] C. Fefferman, The uncertainty principle, Bull. Am. Math. Soc. (N.S.) 9 (1983) 129–206, https://doi.org/10.1090/S0273-0979-1983-15154-6.
[8] M. Filoche, D. Arnold, G. David, D. Jerison, S. Mayboroda, Filoche et al. Reply, Phys. Rev. Lett. 124 (2020) 219702, https://doi.org/10.1103/PhysRevLett.124.219702.
[9] M. Filoche, S. Mayboroda, Universal mechanism for Anderson and weak localization, Proc. Natl. Acad. Sci. USA 109 (37) (2012) 14761–14766, https://doi.org/10.1073/pnas.1120432109.
[10] M. Filoche, M. Piccardo, Y.R. Wu, C.-K. Li, C. Weisbuch, S. Mayboroda, Localization landscape theory of disorder in semiconductors I: theory and modeling, Phys. Rev. B 95 (2017) 144204, https://doi.org/10.1103/PhysRevB.95.144204.
[11] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer Verlag, Berlin, Heidelberg, 1983.
[12] Q. Han, F. Lin, Elliptic Partial Differential Equations, second edition, Courant Lecture Notes in Mathematics, vol. 1, Courant Institute of Mathematical Sciences/American Mathematical Society, New York/Providence, RI, 2011.
[13] W. Kirsch, An Invitation to Random Schrödinger Operators, Soc. Math. France, Paris, 2008, pp. 1–119, With an appendix by Frédéric Klopp. Panor. Synthèses, 25, Random Schrödinger operators.
[14] W. Kirsch, B. Metzger, The integrated density of states for random Schrödinger operators, in: Spectral Theory and Mathematical Physics: a Festschrift in Honor of Barry Simon's 60th Birthday, in: Proc. Sympos. Pure Math., vol. 76, Amer. Math. Soc., Providence, RI, 2007, pp. 649–696, Part 2.
[15] W. König, The Parabolic Anderson Model. Random Walk in Random Potential, Pathways in Mathematics, Birkhäuser/Springer, Cham, 2016.
[16] K. Kurata, On doubling properties for non-negative weak solutions of elliptic and parabolic PDE, Isr. J. Math. 115 (2000) 285–302, https://doi.org/10.1007/BF02810591.
[17] J.M. Luck, Th.M. Nieuwenhuizen, Lifshitz tails and long-time decay in random systems with arbitrary disorder, J. Stat. Phys. 52 (1988) 1–22, https://doi.org/10.1007/BF01016401.
[18] G. Lemut, M.J. Pacholski, O. Ovdat, A. Grabsch, J. Tworzydło, C.W.J. Beenakker, Localization landscape for Dirac fermions, Phys. Rev. B 101 (2020), 081405(R), https://doi.org/10.1103/PhysRevB.101.081405.
[19] B. Metzger, Asymptotische Eigenschaften im Wechselspiel von Diffusion und Wellenausbreitung in zufälligen Medien, Ph.D. thesis, TU Chemnitz, 2005.
[20] L. Pastur, A. Figotin, Spectra of Random and Almost-Periodic Operators, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 297, Springer-Verlag, Berlin, 1992.
[21] M. Piccardo, C.-K. Li, Y.-R. Wu, J. Speck, B. Bonef, R. Farrell, M. Filoche, L. Martinelli, J. Peretti, C. Weisbuch, Localization landscape theory of disorder in semiconductors. II. Urbach tails of disordered quantum well layers, Phys. Rev. B 95 (2017) 144205, https://doi.org/10.1103/PhysRevB.95.144205.
[22] A. Polti, T. Schneider, Corrections to the Lifshitz tail and the long-time behaviour of the trapping problem, Europhys. Lett. 5 (1988) 715, https://doi.org/10.1209/0295-5075/5/8/009.
[23] Z. Shen, Eigenvalue asymptotics and exponential decay of eigenfunctions for Schrödinger operators with magnetic fields, Trans. Am. Math. Soc. 348 (11) (1996) 4465–4488.
[24] Z. Shen, On bounds of $N(\lambda)$ for a magnetic Schrödinger operator, Duke Math. J. 94 (3) (1998) 479–507, https://doi.org/10.1215/S0012-7094-98-09420-0.
[25] B. Simon, Functional Integration and Quantum Physics, Pure and Applied Mathematics, vol. 86, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1979.
[26] B. Simon, Lifschitz tails for the Anderson model, J. Stat. Phys. 38 (1985) 65–76, https://doi.org/10.1007/BF01017848.
[27] T.Y. Tsai, K. Michalczewski, P. Martyniuk, C.-H. Wu, Y.-R. Wu, Application of localization landscape theory and the k · p model for direct modeling of carrier transport in a type II superlattice InAs/InAsSb photoconductor system, J. Appl. Phys. 127 (2020) 033104, https://doi.org/10.1063/1.5131470.