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Independence number of edge-chromatic critical graphs

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Abstract

Let G be a simple graph with maximum degree $\Delta(G)$ and chromatic index $\chi'(G)$. A classical result of Vizing shows that either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. A simple graph G is called edge-Δ-critical if G is connected, $\chi'(G) = \Delta(G) + 1$ and $\chi'(G - e) = \Delta(G)$ for every $e \in E(G)$. Let G be an n-vertex edge-Δ-critical graph. Vizing conjectured that $\alpha(G)$, the independence number of G, is at most $\frac{n}{2}$. The current best result on this conjecture, shown by Woodall, is $\alpha(G) < \frac{3n}{5}$. We show that for any given $\varepsilon \in (0, 1)$, there exist positive constants $d_0(\varepsilon)$ and $D_0(\varepsilon)$ such that if G is an n-vertex edge-Δ-critical graph with minimum degree at least d_0 and maximum degree at least D_0, then $\alpha(G) < \left(\frac{1}{2} + \varepsilon\right)n$. In particular, we show that if G is an n-vertex edge-Δ-critical graph with minimum degree at least d and $\Delta(G) \geq (d+1)^{4.5d+11.5}$, then

$$\alpha(G) < \begin{cases} \dfrac{7n}{12} & \text{if } d = 3, \\ \dfrac{4n}{7} & \text{if } d = 4, \\ \dfrac{d + 2 + \sqrt[3]{(d-1)d}}{2d + 4 + \sqrt[3]{(d-1)d}}n < \dfrac{4n}{7} & \text{if } d \geq 19. \end{cases}$$

KEYWORDS

chromatic index, edge-chromatic critical graph, Vizing's independence number conjecture

1 | INTRODUCTION

All graphs considered in this paper are simple. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. For a vertex $v \in V(G), N_G(v)$ is the set of neighbors of v in G, and $d_G(v) = |N_G(v)|$ is the degree of vertex v in G. We simply write $N(v)$ and $d(v)$ if G is clear. For $e \in E(G)$, $G - e$ denotes the graph obtained from G by deleting the edge e. Let $\Delta(G)$ and $\delta(G)$ be the maximum and minimum degree of G, respectively. We reserve the symbol Δ for $\Delta(G)$ throughout this paper. The *independence number* of G, denoted $\alpha(G)$, is the largest size of an independent set in G.

An *edge k-coloring* of G is a mapping φ from $E(G)$ to the set of integers $[1, k] := \{1, \ldots, k\}$, called *colors*, such that no adjacent edges receive the same color under φ. The *chromatic index* of G, denoted $\chi'(G)$, is defined to be the smallest integer k so that G has an edge k-coloring. We denote by $\mathcal{C}^k(G)$ the set of all edge k-colorings of G. In 1965, Vizing [9] showed that a graph of maximum degree Δ has chromatic index either Δ or $\Delta + 1$. If $\chi'(G) = \Delta$, then G is said to be of *class 1*; otherwise, it is said to be of *class 2*. Holyer [4] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. Similar to vertex coloring, it is essential to edge-color the "core" part of a graph and then extend the coloring to the whole graph without increasing the total number of colors. This leads to the concept of *edge-chromatic criticality*. A graph G is called *edge-chromatic critical* if for any proper subgraph $H \subseteq G, \chi'(H) < \chi'(G)$. We say G is edge-Δ-critical if G is edge-chromatic critical and $\chi'(G) = \Delta + 1$. It is clear that G is edge-Δ-critical if and only if G is connected with $\chi'(G) = \Delta + 1$ and $\chi'(G - e) = \Delta$, for every $e \in E(G)$. By this definition, every class 2 graph with maximum degree Δ can be reduced to an edge-Δ-critical graph by removing edges or vertices. Vizing conjectured that edge-Δ-critical graphs have some special structural properties. In particular, he proposed the following conjectures.

Conjecture 1.1 (Vizing's Independence Number Conjecture [10]). *If G is an edge-Δ-critical graph of order n, then $\alpha(G) \leq n/2$.*

Conjecture 1.2 (Vizing's 2-factor Conjecture [8]). *If G is an edge-Δ-critical graph, then G contains a 2-factor; that is, a 2-regular subgraph H of G with $V(H) = V(G)$.*

Conjecture 1.3 (Vizing's Average Degree Conjecture [8]). *If G is an n-vertex edge-Δ-critical graph, then the average degree of G is at least $\Delta - 1 + \frac{3}{n}$.*

Partial results have been obtained for each of these conjectures. In this paper, we investigate Vizing's Independence Number Conjecture. This conjecture was confirmed for special graph classes including graphs with many edges such as overfull graphs by Grünewald and Steffen [3], and n-vertex edge-Δ-critical graphs G with $\Delta \geq \frac{n}{2}$ by Luo and Zhao [5]. Let G be an n-vertex edge-Δ-critical graph. Brinkmann et al. [1], in 2000, proved that $\alpha(G) < 2n/3$; and the upper bound is further improved when the maximum degree is between 3 and 10. Luo and Zhao [5], in 2008, by improving the result of Brinkmann et al., showed $\alpha(G) < (5\Delta - 6)n/(8\Delta - 6) < 5n/8$ when $\Delta \geq 6$. In 2009, Woodall [11] further improved the upper bound of $\alpha(G)$ to $3n/5$. In this paper, by using new adjacency lemmas, we obtain the following results.

Theorem 1.4. *For any given $\varepsilon \in (0, 1)$, there exist positive constants $d_0(\varepsilon)$ and $D_0(\varepsilon)$ such that if G is an n-vertex edge-Δ-critical graph with minimum degree at least d_0 and maximum degree at least D_0, then $\alpha(G) < \left(\frac{1}{2} + \varepsilon\right)n$.*

By choosing d such that $d \geq 19$ and $\frac{\frac{1}{2}\sqrt[3]{(d-1)d}}{2d+4+\sqrt[3]{(d-1)d}} \leq \varepsilon$, we see that Theorem 1.4 is implied by the third inequality in the following result.

Theorem 1.5. *If G is an n-vertex edge-Δ-critical graph with minimum degree at least d and $\Delta \geq (d+1)^{4.5d+11.5}$, then*

$$\alpha(G) < \begin{cases} \frac{7n}{12} & \text{if } d = 3, \\ \frac{4n}{7} & \text{if } d = 4, \\ \frac{d+2+\sqrt[3]{(d-1)d}}{2d+4+\sqrt[3]{(d-1)d}} n < \frac{4n}{7} & \text{if } d \geq 19. \end{cases}$$

When $d \geq 19$, $\frac{d+2+\sqrt[3]{(d-1)d}}{2d+4+\sqrt[3]{(d-1)d}} < \frac{4}{7}$. In fact, we suspect the following might be true.

Conjecture 1.6. *Let $d \geq 2$ be a positive integer. Then there exists a constant D_0 depending only on d such that if G is an n-vertex edge-Δ-critical graph with $\delta(G) \geq d$ and $\Delta \geq D_0$, then*

$$\alpha(G) < \frac{d+4}{2d+6} n.$$

The case for $d = 2$ was confirmed by Woodall's result [11], and the cases for $d = 3, 4$ are covered in Theorem 1.5.

It is worth mentioning that Steffen [6] showed that for every $\varepsilon > 0$, Vizing's Independence Number Conjecture is equivalent to its restriction on a specific set of edge-chromatic critical graphs that have independence ratio smaller than $\frac{1}{2} + \varepsilon$. In particular, the specific set of edge-chromatic critical graphs G has vertices only of degree either $\Delta(G) - 1$ or $\Delta(G)$. Given any $\varepsilon > 0$ and any edge-Δ-critical graph G, by the "Meredith extension" introduced in [6], one can construct another edge-Δ-critical graph H based on G with $\Delta(H) = \Delta(G)$ and $\delta(H) = \Delta(G) - 1$ such that $\frac{\alpha(H)}{|V(H)|} < \frac{1}{2} + \varepsilon$. However, the independence ratio of H does not imply any reasonable upper bound on the independence ratio of G. This is essentially different from the result in Theorem 1.4.

The remainder of the paper is organized as follows. We introduce some edge-coloring notation and technical lemmas in Section 2, and we prove Theorem 1.5 in Section 3.

2 | TECHNICAL LEMMAS

In this section, we list the classical Vizing's Adjacency Lemma (VAL) and some new developed adjacency lemmas that will be used for proving Theorem 1.5.

For an edge $xy \in E(G)$ and a given positive number q, let

$$\sigma_q(x, y) = |\{z \in N(y)\backslash\{x\} : d(z) \geq q\}|$$

be the number of neighbors of y other than x that are of degree at least q. By the definition, it is clear that $\sigma_q(x, y) \leq \Delta - 1$. The case that $q = \Delta$ gives rise to VAL.

Lemma 2.1 (Vizing's Adjacency Lemma—VAL)**.** *If G is an edge-Δ-critical graph, then $\sigma_\Delta(x, y) \geq \Delta - d(x) + 1$ for every $xy \in E(G)$.*

Let G be an edge-Δ-critical graph, $xy \in E(G)$, and $\varphi \in \mathcal{C}^\Delta(G - xy)$. For any $v \in V(G)$, the set of colors present at v is $\varphi(v) = \{\varphi(e) : e$ is incident to $v\}$, and the set of colors missing at v is $\overline{\varphi}(v) = [1, \Delta]\backslash\varphi(v)$. For a vertex-set X, let $\overline{\varphi}(X) = \bigcup_{v \in X}\overline{\varphi}(v)$. We call X *φ-elementary* if the sets $\overline{\varphi}(v)$ with $v \in X$ are pairwise disjoint.

A *multifan* at x with respect to edge $e = xy \in E(G)$ and coloring $\varphi \in \mathcal{C}^\Delta(G - e)$ is a sequence $F = (x, e_1, y_1, ..., e_p, y_p)$ with $p \geq 1$ consisting of edges $e_1, e_2, ..., e_p$ and vertices $x, y_1, y_2, ..., y_p$ satisfying the following two conditions:

- The edges $e_1, e_2, ..., e_p$ are distinct, $e_1 = e$ and $e_i = xy_i$ for $i = 1, ..., p$.
- For every edge e_i with $2 \leq i \leq p$, there is a vertex y_j with $j \in [1, i - 1]$ such that $\varphi(e_i) \in \overline{\varphi}(y_j)$.

Multifans are well defined in multigraphs, but in this paper we only discuss them in simple graphs. The following lemma shows that a multifan is elementary, and its proof can be found in [7, Theorem 2.1].

Lemma 2.2 (Stiebitz, Scheide, Toft, and Favrholdt [7])**.** *Let G be an edge-Δ-critical graph, $e_1 = xy_1 \in E(G)$, and $\varphi \in \mathcal{C}^\Delta(G - e_1)$. Let $F = (x, e_1, y_1, ..., e_p, y_p)$ be a multifan at x with respect to e_1 and φ. Then $\{x, y_1, y_2, ..., y_p\}$ is φ-elementary.*

Let G be an edge-Δ-critical graph, $xy \in E(G)$, and $\varphi \in \mathcal{C}^\Delta(G - xy)$. Note that by the edge-$\Delta$-criticality of G, $\overline{\varphi}(x) \cap \overline{\varphi}(y) = \emptyset$. Thus $\overline{\varphi}(x)$, $\overline{\varphi}(y)$, and $\varphi(x) \cap \varphi(y)$ form a partition of the color set $[1, \Delta]$. Let q be a positive integer, we partition $\varphi(x) \cap \varphi(y)$ by the following two sets

$$A_\varphi(x, y, q) = \{\alpha \in \varphi(x) : \exists\ u \in N(y) \text{ such that } \varphi(yu) = \alpha \text{ and } d(u) < q\}, \tag{1}$$

$$B_\varphi(x, y, q) = \{\beta \in \varphi(x) : \exists\ u \in N(y) \text{ such that } \varphi(yu) = \beta \text{ and } d(u) \geq q\}. \tag{2}$$

Let $M_\varphi(x, y, q) = A_\varphi(x, y, q) \cup \overline{\varphi}(x) \cup \overline{\varphi}(y)$. We simply write $A_\varphi(q), B_\varphi(q)$, and $M_\varphi(q)$ if xy is specified and clear, and we may also use A_φ, B_φ, and M_φ if both xy and q are specified and clear. It is easy to see that $M_\varphi(q) \cup B_\varphi(q) = [1, \Delta]$. We now partition $N(x)\backslash\{y\}$ into two sets and define $\varphi^{\text{bad}}(v)$ for any $v \in V(G)$ (when q is specified and clear) as the following:

$$N(x, M_\varphi(x, y, q)) = \{z \in N(x) : \varphi(xz) \in M_\varphi(x, y, q)\},$$
$$N(x, B_\varphi(x, y, q)) = \{z \in N(x) : \varphi(xz) \in B_\varphi(x, y, q)\}, \text{and}$$
$$\varphi^{\text{bad}}(v) = \overline{\varphi}(v) \cup \{\alpha \in [1, \Delta] : \exists\ v' \in N(v) \text{so that } \varphi(vv') = \alpha \text{ and } d(v') < q\}.$$

Observe that for $v \in N(x)$,

$$\sigma_q(x,v) = \begin{cases} \Delta - |\varphi^{\text{bad}}(v)| & \text{if } \varphi(xv) \in \varphi^{\text{bad}}(v), \\ \Delta - 1 - |\varphi^{\text{bad}}(v)| & \text{if } \varphi(xv) \notin \varphi^{\text{bad}}(v). \end{cases} \tag{3}$$

Figure 1 gives a depiction of $A_\varphi(x,y,q), B_\varphi(x,y,q), N(x, M_\varphi(x,y,q)), N(x, B_\varphi(x,y,q))$, and $\varphi^{\text{bad}}(y)$.

For any $\varepsilon, \lambda \in (0,1)$, define

$$c_0 = \left\lceil \frac{1-\varepsilon}{\varepsilon} \right\rceil, \tag{4}$$

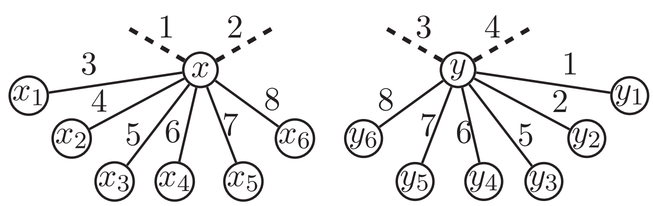
$$f(\varepsilon) = \begin{cases} \frac{1}{\varepsilon^2}\left(3c_0^4 + 12c_0^3 + 10c_0^2 + 4c_0 + 1\right) & \text{if } \varepsilon \le \frac{30}{31}, \\ \frac{31}{\varepsilon} & \text{otherwise}, \end{cases} \tag{5}$$

$$N = (c_0+1)\left(\frac{1}{\lambda}+1\right)^{3c_0+1}, \tag{6}$$

$$D_0 = \max\left\{f(\varepsilon), \frac{3c_0+1}{\lambda^2}, \frac{N+c_0}{\varepsilon^3}\right\}. \tag{7}$$

Letting $q = (1-\varepsilon)\Delta$, in the proof of Theorem 1.5, we will show that the average degree of some vertices of G is at least q. The following two lemmas analyze the neighborhood structures of vertices of degree less than q but at least $\varepsilon\Delta$ and less than $\varepsilon\Delta$, respectively.

Lemma 2.3 (Cao and Chen [2, Corollary 10]). *Let G be an edge-Δ-critical graph with $\Delta \ge D_0$, $q = (1-\varepsilon)\Delta$, and let $xy \in E(G)$ with $d(x) < q$. Then for any $\varepsilon, \lambda \in (0,1)$ and*



$$\begin{aligned} d(y_3), d(y_4) < q, &\qquad d(y_5), d(y_6) \ge q; \\ \overline{\varphi}(x) = \{1,2\}, &\qquad \overline{\varphi}(y) = \{3,4\}; \\ A_\varphi(q) = \{5,6\}, &\qquad B_\varphi(q) = \{7,8\}; \\ N(x, M_\varphi) = \{x_1,x_2,x_3,x_4\}, &\qquad N(x, B_\varphi) = \{x_5, x_6\}; \\ \varphi^{\text{bad}}(y) = \{3,4,5,6\} &\qquad \text{if assuming } d(y_1), d(y_2) \ge q. \end{aligned}$$

FIGURE 1 Edge 8-coloring: illustration of $A_\varphi(q), B_\varphi(q), N(x, M_\varphi(q)), N(x, B_\varphi(q))$, and $\varphi^{\text{bad}}(y)$

$\varphi \in \mathcal{C}^{\Delta}(G - xy)$, except at most N vertices in $N(x, M_{\varphi})$, for any other remaining vertex $x^* \in N(x, M_{\varphi})$,

$$|\varphi^{\text{bad}}(x^*) \setminus \{\varphi(xx^*)\}| < \lambda\Delta.$$

Lemma 2.4 (Cao and Chen [2, Lemmas 12 and 14])**.** *Let G be an edge-Δ-critical graph, $\varepsilon \in (0, 1)$ and $q = (1 - \varepsilon)\Delta$, and let $xy \in E(G)$ with $d(x) < \varepsilon\Delta$ and $\varphi \in \mathcal{C}^{\Delta}(G - xy)$. Then for any $x^* \in N(x, M_{\varphi})$,*

$$\varphi^{\text{bad}}(x^*) \setminus \{\varphi(xx^*)\} \subseteq B_{\varphi}(q).$$

Moreover, for any distinct $x_1, x_2 \in N(x, M_{\varphi})$,

$$\varphi^{\text{bad}}(x_1) \cap \varphi^{\text{bad}}(x_2) = \varnothing.$$

Lemma 2.5, Corollary 2.6, and Lemma 2.7 are refinements of Lemma 2.4 and are used to analyze the neighborhood of vertices of degree between 4 and 7.

Lemma 2.5 (Cao and Chen [2, Lemma 13])**.** *Let G be an edge-Δ-critical graph, $\varepsilon \in (0, 1)$ and $q = (1 - \varepsilon)\Delta$, and let $xy \in E(G)$ with $d(x) < \varepsilon\Delta$ and $\varphi \in \mathcal{C}^{\Delta}(G - xy)$. Then for $x^* \in N(x, M_{\varphi})$ and $\beta \in \varphi^{\text{bad}}(x^*)$, there exists a vertex $x' \in N(x, B_{\varphi})$ such that $\varphi(xx') = \beta$ and $\varphi^{\text{bad}}(x') \subseteq B_{\varphi}(q)$.*

If $|B_{\varphi}(q)| = 1$, then $|N(x, B_{\varphi})| = 1$. Let $x' \in N(x, B_{\varphi})$ be the vertex. Then as $\varphi(xx')$ is colored by the color in $B_{\varphi}(q)$, all other edges incident with x' are colored by colors from $M_{\varphi}(q)$ as $[1, \Delta] = M_{\varphi}(q) \cup B_{\varphi}(q)$. Thus, for any $u \in N(x') \setminus \{x\}$, $\varphi(x'u) \notin \varphi^{\text{bad}}(x')$ and so $d(u) \geq q$. So as a special case of Lemma 2.5, we get the following result.

Corollary 2.6. *Let G be an edge-Δ-critical graph, $\varepsilon \in (0, 1)$ and $q = (1 - \varepsilon)\Delta$, and let $xy \in E(G)$ with $d(x) < \varepsilon\Delta$ and $\varphi \in \mathcal{C}^{\Delta}(G - xy)$. If $|B_{\varphi}(q)| = 1$ and there exists $x^* \in N(x, M_{\varphi})$ such that $\varphi^{\text{bad}}(x^*) \cap B_{\varphi}(q) \neq \varnothing$, then for the vertex $x' \in N(x, B_{\varphi})$ with $\varphi(xx') = \beta$, we have $d(u) \geq q$ for any $u \in N(x') \setminus \{x\}$.*

Lemma 2.7. *Let G be an edge-Δ-critical graph, $\varepsilon \in (0, 1)$ and $q = (1 - \varepsilon)\Delta$, and let $xy \in E(G)$ with $d(x) < \varepsilon\Delta$ and $\varphi \in \mathcal{C}^{\Delta}(G - xy)$. Suppose $z \in N(x, M_{\varphi})$, $\beta \in \varphi^{\text{bad}}(z) \cap B_{\varphi}(x, y, q)$, and $w \in N(x, B_{\varphi})$ such that $\varphi(xw) = \beta$. Define $B_w(\varphi) = \{\varphi(ww') : w' \in N(w), \varphi(ww') \in B_{\varphi}(x, y, q) \setminus \{\beta\}$ and $d(w') < q\}$. Then for any $\beta' \in B_w(\varphi)$, there exist $z' \in N(x, B_{\varphi})$ and $u \in N(z')$ such that $\varphi(xz') = \beta'$, $\varphi(z'u) = \beta$, and $d(u) \geq q$.*

Proof. A coloring $\varphi' \in \mathcal{C}^{\Delta}(G - xy)$ is called *valid* if

$$M_{\varphi'}(x, y, q) = M_{\varphi}(x, y, q), B_{\varphi'}(x, y, q) = B_{\varphi}(x, y, q), \text{ and } \quad B_w(\varphi') = B_w(\varphi).$$

Let $z \in N(x, M_{\varphi})$ and let $\beta \in \varphi^{\text{bad}}(z) \cap B_{\varphi}(x, y, q)$.

Let $\varphi(xz) = \alpha$. Let us show that *we may assume* $\alpha \in \overline{\varphi}(y)$. Otherwise, $\alpha \in A_{\varphi}(x, y, q)$. Let $v \in N(y)$ such that $\varphi(yv) = \alpha$ and $\gamma \in \overline{\varphi}(y)$. If $\gamma \in \overline{\varphi}(v)$, we recolor the edge yv

using the color γ and let φ' be the new coloring of $G - xy$. It is clear that for any edge $e \in E(G - xy)$ with $e \neq yv, \varphi(e) = \varphi'(e)$. Furthermore, φ' is a valid coloring. However, under $\varphi', \alpha \in \overline{\varphi}'(y)$. So we assume that $\gamma \in \varphi(v)$. Since $d(x) < \varepsilon\Delta$ and $d(v) < q = (1 - \varepsilon)\Delta$, there exits a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(v)$. Note that $\gamma \in \varphi(x)$ and $\delta \in \varphi(y)$ by the edge-Δ-criticality of G. Let $P_v(\gamma, \delta), P_x(\gamma, \delta)$, and $P_y(\gamma, \delta)$ be the paths induced by the edges colored with the two colors γ and δ that start at v, x, and y, respectively. We claim that $P_x(\gamma, \delta) = P_y(\gamma, \delta)$. For otherwise, let φ'' be the new coloring of $G - xy$ obtained by switching the colors γ and δ on the path $P_x(\gamma, \delta)$. Then φ'' is an edge Δ-coloring of $G - xy$ such that $\gamma \in \overline{\varphi}''(x) \cap \overline{\varphi}''(y)$. Now coloring the edge xy using the color γ gives an edge Δ-coloring of G, showing a contradiction to the assumption that $\chi'(G) = \Delta + 1$. Thus, $P_x(\gamma, \delta) = P_y(\gamma, \delta)$. This implies that $P_v(\gamma, \delta)$ is vertex-disjoint from $P_x(\gamma, \delta)$. We let φ_1 be the new coloring of $G - xy$ obtained by switching the colors γ and δ on the path $P_v(\gamma, \delta)$. We now have that $\gamma \in \overline{\varphi}_1(v)$. Since the switching of colors on $P_v(\gamma, \delta)$ does not affect the colors on the edges incident to y, we still have $\gamma \in \overline{\varphi}_1(y)$. Let φ_2 be the new coloring of $G - xy$ obtained from φ_1 by recoloring the edge yv using the color γ. We see that $\alpha \in \overline{\varphi}_2(y)$. Because $\delta, \gamma, \alpha \in M_\varphi(x, y, q)$ and to get φ_2, we only switched the two colors γ and δ on the path $P_v(\gamma, \delta)$, and then changed the color on the edge yv from α to γ, φ_2 is a valid coloring. Furthermore, $x \notin V(P_v(\gamma, \delta))$, for any edge e that is incident to x or u with $u \in N(x)$ and $\varphi(xu) = \gamma$, we have $\varphi_2(e) = \varphi(e)$. Thus, we can use φ_2 as a coloring for $G - xy$ that satisfies $\alpha \in \overline{\varphi}_2(y)$. Figure 2 shows this sequence of Kempe changes.

We now take $z \in N(x, M_\varphi), \beta \in \varphi^{\text{bad}}(z) \cap B_\varphi(x, y, q)$ such that $\varphi(xz) = \alpha$ and $\alpha \in \overline{\varphi}(y)$. We take the color α on the edge xz out and color the edge xy using the color α, and we get a coloring φ_3 of $G - xz$. Note that $\alpha \in \overline{\varphi}_3(z)$ and $\beta \in M_{\varphi_3}(x, z, q)$ as $\beta \in \varphi^{\text{bad}}(z)$. Since $\varphi_3(e) = \varphi(e)$ for any $e \notin \{xy, xz\}$, for the specified vertex $w \in N(x, B_\varphi(x, y, q))$ such that $\varphi(xw) = \beta$, we still have that $\varphi_3(xw) = \beta$, and $B_w(\varphi) = B_w(\varphi_3)$. By the definitions of $B_w(\varphi)$ and $\varphi_3^{\text{bad}}(w)$, we have $B_w(\varphi) \subseteq \varphi_3^{\text{bad}}(w)$. Since $\beta \in M_\varphi(x, z, q)$, by Lemma 2.4 with the vertex w playing the role of the vertex x^*, we get $\varphi_3^{\text{bad}}(w) \backslash \{\varphi_3(xw)\} \subseteq B_{\varphi_3}(x, z, q)$. As $\varphi_3(xw) = \beta \notin B_w(\varphi)$ and $B_w(\varphi) \subseteq \varphi_3^{\text{bad}}(w)$, we have $B_w(\varphi) \subseteq B_{\varphi_3}(x, z, q)$. By Lemma 2.5, again with w playing the role of x^* and β' playing the role of β in the lemma, we know that for any $\beta' \in B_w(\varphi_3)$, there exists $z' \in N(x, B_{\varphi_3}(x, z, q))$ with $\varphi_3(xz') = \beta'$ and $\varphi^{\text{bad}}(z') \subseteq B_{\varphi_3}(x, z, q)$. As $\beta \in M_{\varphi_3}(x, z, q)$ and $M_{\varphi_3}(x, z, q) \cap B_{\varphi_3}(x, z, q) = \varnothing$, $\beta \notin \varphi^{\text{bad}}(z')$. Thus there exists $u \in N(z')$ such that $\varphi_3(z'u) = \beta$ and $d(u) \geq q$. Since $B_w(\varphi) = B_w(\varphi_3), B_w(\varphi) \subseteq B_\varphi(x, y, q)$, $B_w(\varphi) \subseteq B_{\varphi_3}(x, z, q)$, and $\varphi_3(xz') = \varphi(xz')$, we see that $z' \in N(x, B_{\varphi_3}(x, z, q))$ with $\varphi_3(xz') = \beta'$

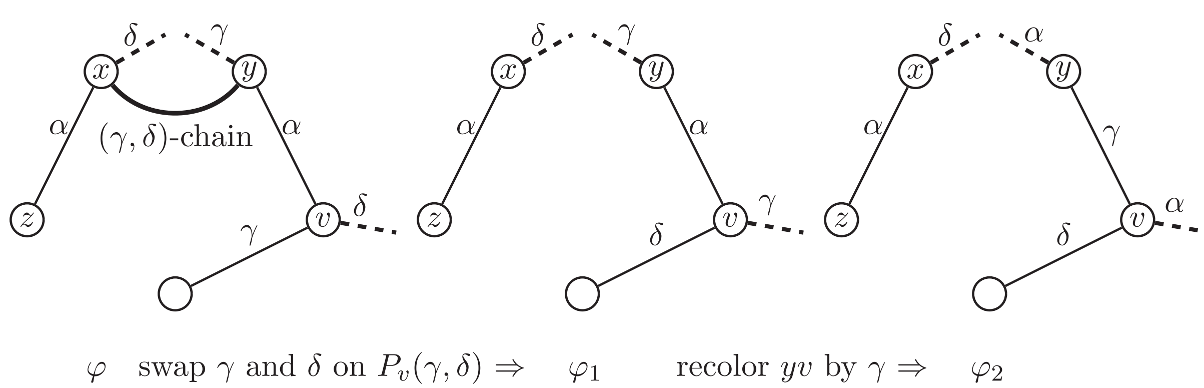


FIGURE 2 Process of changing from φ to φ_2

implies that $z' \in N(x, B_\varphi(x, y, q))$. Furthermore, since $\varphi_3(xz') = \varphi(xz')$ and $\varphi_3(z'u) = \varphi(z'u)$, we get $\varphi(xz') = \beta'$, $\varphi(z'u) = \beta$, and $d(u) \geq q$, proving Lemma 2.7. □

Corollary 2.8. *Let G be an edge-Δ-critical graph, $\varepsilon \in (0, 1)$ and $q = (1 - \varepsilon)\Delta$, and let $xy \in E(G)$ with $d(x) < \varepsilon\Delta$ and $\varphi \in \mathcal{C}^\Delta(G - xy)$. If $B_\varphi(x, y, q) \subseteq \bigcup_{z \in N(x, M_\varphi)} \varphi^{\text{bad}}(z)$ and $|B_\varphi(x, y, q)| = 2$, then there exists $z' \in N(x, B_\varphi)$ such that $d(u) \geq q$ for any $u \in N(z') \backslash \{x\}$.*

Proof. Let $B_\varphi(x, y, q) = \{\beta, \beta'\}$. Since $B_\varphi(x, y, q) \subseteq \bigcup_{z \in N(x, M_\varphi)} \varphi^{\text{bad}}(z)$, there exists $x^* \in N(x, M_\varphi)$ such that $\beta \in \varphi^{\text{bad}}(x^*)$. By Lemma 2.5, there exists $x' \in N(x, B_\varphi)$ such that $\varphi(xx') = \beta$ and $\varphi^{\text{bad}}(x') \subseteq B_\varphi(q)$. In particular, for every color $\alpha \in M_\varphi(q) = [1, \Delta] \backslash B_\varphi(q)$, there exists $u \in N(x')$ such that $\varphi(x'u) = \alpha$ and $d(u) \geq q$. If $\beta' \notin \varphi(x')$ or $\beta' \in \varphi(x')$ but $\beta' \notin \varphi^{\text{bad}}(x')$, then letting $z' = x'$ gives the desired vertex. Thus we assume that $\beta' \in \varphi(x')$ and $\beta' \in \varphi^{\text{bad}}(x')$. As $B_\varphi(q) = \{\beta, \beta'\}$, we see that $B_{x'}(\varphi) = \{\beta'\}$ as defined in Lemma 2.7. Applying Lemma 2.7, there exist $z' \in N(x, B_\varphi)$ with $\varphi(xz') = \beta'$ and $v \in N(z')$ such that $\varphi(z'v) = \beta$ and $d(v) \geq q$. As $B_\varphi(x, y, q) \subseteq \bigcup_{z \in N(x, M_\varphi)} \varphi^{\text{bad}}(z)$ and so $\beta' \in \varphi^{\text{bad}}(z)$ for some $z \in N(x, M_\varphi)$, and φ is a proper coloring, we know that z' is the vertex guaranteed by Lemma 2.5 such that $\varphi(xz') = \beta'$. Again, by Lemma 2.5, $\varphi^{\text{bad}}(z') \subseteq B_\varphi(q)$. In particular, for every color $\alpha \in M_\varphi(q) = [1, \Delta] \backslash B_\varphi(q)$, there exists $u \in N(z')$ such that $\varphi(z'u) = \alpha$ and $d(u) \geq q$. Furthermore, since $\beta \in \varphi(z')$ and for the vertex $v \in N(z')$, we have $\varphi(z'v) = \beta$ and $d(v) \geq q$, we conclude that z' is the desired vertex. □

3 | PROOF OF THEOREM 1.5

The setting up of the discharging idea in the proof below uses some of the approaches that Woodall had in [11], but the computations are much more complicated.

Proof of Theorem 1.5. Let G be an edge-Δ-critical graph of order n with minimum degree at least d and maximum degree $\Delta \geq (d + 1)^{4.5d+11.5}$. In fact, the proof only needs $\Delta \geq D_0$, where D_0 is defined in Equation (7). Define

$$\omega = \begin{cases} 2 & \text{if } d = 3, 4, \\ \sqrt[3]{(d-1)d} & \text{if } d \geq 19, \end{cases} \tag{8}$$

and let

$$\varepsilon = \frac{\omega}{d + 2}, \quad q = \frac{d + 2 - \omega}{d + 2}\Delta = (1 - \varepsilon)\Delta, \quad \text{and} \quad \lambda = \frac{\omega^3}{2(d + 2)^3} = \frac{\varepsilon^3}{2}, \tag{9}$$

where we have two different expressions for λ as $\varepsilon = \frac{\omega}{d+2}$ and so $\frac{\omega^3}{2(d+2)^3} = \frac{\varepsilon^3}{2}$. We will use both of the two expressions in computations later.

Since $\omega = \sqrt[3]{(d-1)d} = \sqrt[3]{18 \times 19} = \sqrt[3]{342} < 7$ when $d = 19$ and ε is decreasing in d, it follows that $\varepsilon \leq \max\left\{\frac{2}{5}, \frac{7}{21}\right\} = \frac{2}{5} < \frac{30}{31}$. Thus by the definition of $f(\varepsilon)$ in (5), we have

$f(\varepsilon) = \frac{1}{\varepsilon^2}(3c_0^4 + 12c_0^3 + 10c_0^2 + 4c_0 + 1)$. By Equation (4), $c_0 = \left\lceil \frac{1-\varepsilon}{\varepsilon} \right\rceil \le \frac{1}{\varepsilon} \le c_0 + 1$. Also, as $c_0 \ge \left\lceil \frac{1-2/5}{2/5} \right\rceil = 2$, we have $10c_0^4 > 12c_0^3 + 10c_0^2 + 4c_0 + 1$. Thus

$$\begin{aligned}
f(\varepsilon) &= \frac{1}{\varepsilon^2}\left(3c_0^4 + 12c_0^3 + 10c_0^2 + 4c_0 + 1\right)\\
&\le (c_0+1)^2\left(3c_0^4 + 12c_0^3 + 10c_0^2 + 4c_0 + 1\right)\\
&< 13c_0^4(c_0+1)^2,\\
\frac{3c_0+1}{\lambda^2} &= \frac{4(3c_0+1)}{\varepsilon^6} \le 4(3c_0+1)(c_0+1)^6 < 12(c_0+1)^7,\\
N &= (c_0+1)\left(\frac{1}{\lambda}+1\right)^{3c_0+1}\\
&= (c_0+1)\left(\frac{2}{\varepsilon^3}+1\right)^{3c_0+1}\\
&\ge (c_0+1)\left(2c_0^3+1\right)^7 > (c_0+1)\left(2c_0^2+2\right)^7 = 2^7(c_0+1)\left(c_0^2+1\right)^7\\
&> \max\left\{13c_0^4(c_0+1)^2, 12(c_0+1)^7\right\}.
\end{aligned}$$

Hence as

$$\begin{aligned}
c_0 &= \left\lceil \frac{1-\varepsilon}{\varepsilon} \right\rceil = \left\lceil \frac{d+2-\omega}{\omega} \right\rceil\\
&= \begin{cases} \left\lceil \frac{d}{2} \right\rceil \le \frac{d+1}{2} & \text{if } d = 3, 4,\\ \left\lceil \frac{d+2-\sqrt[3]{(d-1)d}}{\sqrt[3]{(d-1)d}} \right\rceil \le \left\lceil \frac{d}{\sqrt[3]{(d-1)d}} \right\rceil \le \frac{d+1}{2} & \text{if } d \ge 19, \end{cases}
\end{aligned}$$

we have

$$\begin{aligned}
D_0 &= \max\left\{f(\varepsilon), \frac{3c_0+1}{\lambda^2}, \frac{N+c_0}{\varepsilon^3}\right\}\\
&= \frac{N+c_0}{\varepsilon^3} \le (c_0+1)^3\left((c_0+1)(2(c_0+1)^3+1)^{3c_0+1} + c_0\right)\\
&< (c_0+1)^3(c_0+2)(2(c_0+1)^3+1)^{3c_0+1}\\
&\le \left(\frac{d+3}{2}\right)^3\left(\frac{d+5}{2}\right)\left(\frac{(d+3)^3}{4}+1\right)^{1.5d+2.5}\\
&< (d+1)^4(d+1)^{4.5d+7.5} = (d+1)^{4.5d+11.5} \le \Delta,
\end{aligned}$$

where the last inequality was obtained because $\frac{d+3}{2} \le d+1$, $\frac{d+5}{2} \le d+1$, and $\frac{d+3}{d+1} \le \frac{3}{2}$ implying $\frac{(d+3)^3}{4} + 1 \le \frac{1}{4}\left(\frac{3}{2}(d+1)\right)^3 + \frac{(d+1)^3}{64} < (d+1)^3$.

Let X be a largest independent set in G and let $Y = V(G)\backslash X$. Note that Y is not an independent set. For otherwise, G is a bipartite graph, which is of class 1. To prove Theorem 1.5, it is sufficient to show $\alpha(G) < \frac{d+2+\omega}{2d+4+\omega}n$, as $\frac{d+2+\omega}{2d+4+\omega} = \frac{7}{12}$ when $d = 3$ and $\frac{d+2+\omega}{2d+4+\omega} = \frac{4}{7}$ when $d = 4$. We now partition X as the following:

$$\begin{aligned}
X^{++} &= \{x \in X : d(x) = \Delta\}, \\
X^{+} &= \{x \in X : q \le d(x) < \Delta\}, \\
X_1^- &= \{x \in X : \varepsilon\Delta \le d(x) < q\}, \\
X_2^- &= \{x \in X : d \le d(x) < \varepsilon\Delta\} && \text{if } d \ge 19, \\
X_2^- &= \{x \in X : 3d-3 \le d(x) < \varepsilon\Delta\} && \text{if } d = 3, 4, \\
X_3^- &= \{x \in X : d \le d(x) < 3d-3\} && \text{if } d = 3, 4, \\
X^- &= \{x \in X : d \le d(x) < q\}.
\end{aligned}$$

Since G has minimum degree at least d, by the definitions above, we see that $X = X^{++} \cup X^{+} \cup X^{-}$, $X^- = X_1^- \cup X_2^-$ when $d \ge 19$, and $X^- = X_1^- \cup X_2^- \cup X_3^-$ when $d = 3, 4$.

For each positive integer k, define

$$g_1(k) = \frac{(d+2)(\Delta - k)}{k} \quad \text{and} \quad g_2(k) = \frac{\omega\Delta}{k-1}. \tag{10}$$

Clearly, $g_1(k)$ and $g_2(k)$ are both decreasing functions of k. Since $q = \frac{d+2-\omega}{d+2}\Delta = (1-\varepsilon)\Delta$, we have

$$\begin{aligned}
g_1(q) &= \frac{(d+2)(\Delta - q)}{q} \\
&= \frac{(d+2)\varepsilon\Delta}{q} = \frac{(d+2)(\omega/(d+2))\Delta}{q} = \frac{\omega\Delta}{q} \qquad (11)\\
&= \frac{\omega(d+2)}{d+2-\omega}. \qquad (12)
\end{aligned}$$

Claim 3.1. If $x \in X^+$ and $k = d(x) \ge 3$, then $g_1(k) \le g_2(k)$.

Proof. Let $g(k) = g_2(k) - g_1(k) = \frac{\omega\Delta}{k-1} - \frac{(d+2)(\Delta-k)}{k}$. The derivative of $g(k)$ is

$$g'(k) = \frac{(k-1)^2(d+2)\Delta - k^2\omega\Delta}{(k-1)^2k^2}.$$

Since $\frac{(d+2)^3}{(d-1)d} \cdot \frac{4^3}{9^3} \ge \frac{5^3}{6} \cdot \frac{4^3}{9^3} > \frac{20^3}{2^3 \cdot 9^3} > 1$, we have $\sqrt[3]{(d-1)d} \le \frac{4(d+2)}{9}$. Thus

$$\omega \le \max\{2, \sqrt[3]{(d-1)d}\} \le \frac{4(d+2)}{9} \le \frac{(k-1)^2(d+2)}{k^2}.$$

Consequently, $(k-1)^2(d+2)\Delta - k^2\omega\Delta = k^2\Delta\left(\frac{(k-1)^2(d+2)}{k^2} - \omega\right) \ge 0$ and so $g'(k) \ge 0$. Thus $g(k)$ is increasing in k. Since $k \ge q$ when $x \in X^+$, we have $g(k) \ge g(q)$. By (10) and (11),

$$\begin{aligned}
g(k) \ge g(q) &= g_2(q) - g_1(q) \\
&= \frac{\omega\Delta}{q-1} - \frac{\omega\Delta}{q} > 0.
\end{aligned}$$

Hence, $g_1(k) \le g_2(k)$. □

Define three charge functions M_0, M_1, and M_2 on $V(G)$ as follows.

$$\begin{array}{llll} M_0(x) = 0, & M_1(x) = (d+2)d(x), & M_2(x) = (d+2)\Delta, & \text{if } x \in X, \\ M_0(y) = (d+2+\omega)\Delta, & M_1(y) = \omega\Delta, & M_2(y) = 0, & \text{if } y \in Y. \end{array}$$

We redistribute the charge according to the following *Discharging Rule*:

Step 0: Starting with M_0, each vertex $y \in Y$ gives charge $d+2$ to each vertex $x \in N(y) \cap X$. Denote the resulting charge by M_0^*. It is clear that $M_0^*(v) \ge M_1(v)$ for each $v \in V(G)$. By dropping the surplus charges, we assume that the current distribution is M_1.

Step 1: Starting with M_1, each vertex $y \in Y$ gives charge $g_1(d(x))$ to each $x \in N(y) \cap X^+$. Denote the resulting charge by M_1^*.

Step 2: Stating with M_1^*, for each vertex $y \in Y$, if $M_1^*(y) > 0$, y distributes its remaining charge equally among all vertices (if any) $x \in N(y) \cap X^-$. Denote the resulting charge by M_2^*.

Claim 3.2. If $M_2^*(v) \ge M_2(v)$ for each $v \in V(G)$, then $\alpha(G) < \frac{d+2+\omega}{2d+4+\omega}n$. Consequently, Theorem 1.5 holds.

Proof. By Step 0 of Discharging Rule,

$$\begin{aligned} M_0^*(x) &= \sum_{y \in N(x)} (d+2) = (d+2)d(x) = M_1(x) && \text{for each } x \in X, \\ M_0^*(y) &= M_0(y) - \sum_{x \in N(y) \cap X} (d+2) \\ &\ge (d+2+\omega)\Delta - (d+2)\Delta = \omega\Delta = M_1(y) && \text{for each } y \in Y. \end{aligned}$$

Since G is edge-Δ-critical and so it is not bipartite, there exists $y \in Y$ so that $|N(y) \cap X| < \Delta$ and thus $M_0^*(y) > M_1(y)$. Hence,

$$\sum_{v \in V(G)} M_1(v) < \sum_{v \in V(G)} M_0^*(v) = \sum_{v \in V(G)} M_0(v) = (d+2+\omega)\Delta|Y|.$$

Since M_2^* is obtained based on M_1 by Steps 1 and 2 of Discharging Rule, if $M_2^*(v) \ge M_2(v)$ for each $v \in V(G)$, then we have

$$(d+2)\Delta|X| = \sum_{v \in V(G)} M_2(v) \le \sum_{v \in V(G)} M_2^*(v) = \sum_{v \in V(G)} M_1(v) < (d+2+\omega)\Delta|Y|.$$

The inequality above together with the fact that $|X| + |Y| = n$ implies

$$\alpha(G) = |X| < \frac{d+2+\omega}{2d+4+\omega}n,$$

as desired. □

By Claim 3.2, we only need to show that for each $v \in V(G)$, we have $M_2^*(v) \geq M_2(v)$. We show this by considering different cases according to which set v belongs to.

Claim 3.3. For each $y \in Y, M_2^*(y) \geq M_2(y) = 0$.

Proof. Let $y \in Y$. By Step 2 in Discharging Rule, to show $M_2^*(y) \geq 0 = M_2(y)$, it suffices to show that $M_1^*(y) \geq 0$. If $N(y) \cap X^+ = \emptyset$, then y did not send any of its charge to its neighbors in Step 1 of Discharging Rule. Thus $M_1(y) = M_1^*(y) = \omega\Delta > 0$. So we let $k_0 = \min\{d(x) : x \in N(y) \cap X^+\}$. By Lemma 2.1, y is adjacent to at least $\Delta - k_0 + 1$ neighbors of degree Δ. Thus y is adjacent to at most $d(y) - (\Delta - k_0 + 1) \leq k_0 - 1$ neighbors in $X^+ \cup X^-$. By Step 1 in Discharging Rule, we have

$$M_1^*(y) = M_1(y) - \sum_{x \in N(y) \cap X^+} g_1(d(x)).$$

By Claim 3.1, for $x \in X^+$, $g_1(d(x)) \leq g_2(d(x))$. Since $g_2(k)$ is decreasing in k and k_0 is the minimum value among the degrees of x in $N(y) \cap X^+$, $g_2(d(x)) \leq \frac{\omega\Delta}{k_0 - 1}$. Combining the arguments above, we get

$$\begin{aligned} M_1^*(y) &= M_1(y) - \sum_{x \in N(y) \cap X^+} g_1(d(x)) \geq M_1(y) - \sum_{x \in N(y) \cap X^+} g_2(d(x)) \\ &\geq M_1(y) - |N(y) \cap X^+| \frac{\omega\Delta}{k_0 - 1} \geq M_1(y) - (k_0 - 1)\frac{\omega\Delta}{k_0 - 1} \\ &= \omega\Delta - \omega\Delta = 0. \end{aligned}$$

□

Claim 3.4. For each $x \in X^{++} \cup X^+, M_2^*(x) \geq M_2(x) = (d+2)\Delta$.

Proof. For each $x \in X^{++}$, by Step 0, we have

$$M_0^*(x) = \sum_{y \in N(x) \cap Y} (d+2) = (d+2)\Delta,$$

where we get $|N(x) \cap Y| = \Delta$ since X is an independent set in G. The charge of $x \in X^{++}$ keeps unchanged in Steps 1 and 2, thus $M_2^*(x) = M_0^*(x) = (d+2)\Delta$. For each $x \in X^+$, by Discharging Rule,

$$\begin{aligned} M_2^*(x) = M_1^*(x) &= M_1(x) + \sum_{y \in N(x)} g_1(d(x)) \\ &= (d+2)d(x) + \sum_{y \in N(x)} \frac{(d+2)(\Delta - d(x))}{d(x)} \\ &= (d+2)d(x) + d(x)\frac{(d+2)(\Delta - d(x))}{d(x)} = (d+2)\Delta. \end{aligned}$$

The next claim will be used for showing that for each $x \in X^-, M_2^*(x) \geq M_2(x) = (d+2)\Delta$.

Claim 3.5. Let ℓ be a nonnegative integer and $y \in Y$ be a neighbor of $x \in X^-$, and $k = d(x)$. If $\sigma_q(x, y) \geq \Delta - k + 1 + \ell$, then y gives x at least

$$h(k, \ell) := \frac{1}{k-\ell-1}(\omega\Delta - \ell g_1(q)) = \frac{1}{k-\ell-1}\left(\omega\Delta - \ell\frac{(d+2)\omega}{d+2-\omega}\right)$$

charge in Step 2.

Proof. Let L^{++} be a set of $\Delta - k + 1$ neighbors of y with degree Δ (y has at least $\Delta - k + 1$ neighbors of degree Δ by VAL), and let L^+ be a set, disjoint from L^{++}, of ℓ neighbors of y with degree at least q, which exists since $\sigma_q(x, y) \geq \Delta - k + 1 + \ell$. Let $L = N(y)\backslash(L^{++} \cup L^+)$, which note may contain both vertices from X^+ and X^-. Then in Steps 1 and 2, y gives nothing to vertices in L^{++}, and in Step 1, for each vertex $x' \in N(y) \cap L^+$ and each vertex $x' \in N(y) \cap (L \cap X^+)$, y gives $g_1(d(x')) \leq g_1(q)$ to x'. In Step 2, y's remaining charge after Step 1 is divided equally among y's remaining $d(y) - (\Delta - k + 1 + \ell) - |L \cap X^+| \leq k - \ell - 1 - |L \cap X^+|$ neighbors. For x, being in X^-, receives charge of at least $\frac{\omega\Delta - \ell g_1(q)}{k-\ell-1}$ from y as seen below:

(a) For each $x' \in L \cap X^+$, we have $g_1(d(x')) \leq g_1(q) \leq g_2(q) \leq g_2(k) = \frac{\omega\Delta}{k-1}$, and thus

$$\begin{aligned}\frac{\omega\Delta - \ell g_1(q)}{k-\ell-1} &\geq \frac{\omega\Delta - \ell g_2(q)}{k-\ell-1} \geq \frac{\omega\Delta - \ell g_2(k)}{k-\ell-1}\\ &= \frac{\omega\Delta - \frac{\ell\omega\Delta}{k-1}}{k-\ell-1} = \frac{\omega\Delta(k-1-\ell)}{(k-1)(k-1-\ell)}\\ &= \frac{\omega\Delta}{k-1} = g_2(k).\end{aligned}$$

(b) By (a), $g_1(d(x')) \leq g_1(q) \leq g_2(q) \leq g_2(k) \leq \frac{\omega\Delta - \ell g_1(q)}{k-\ell-1}$, and therefore the charge that y gives to x is

$$\begin{aligned}&\frac{\omega\Delta - \ell g_1(q) - \sum_{x' \in L \cap X^+} g_1(d(x'))}{|L| - |L \cap X^+|}\\ \geq\ &\frac{\omega\Delta - \ell g_1(q) - |L \cap X^+|\frac{\omega\Delta - \ell g_1(q)}{k-\ell-1}}{|L| - |L \cap X^+|}\\ \geq\ &\frac{\omega\Delta - \ell g_1(q) - |L \cap X^+|\frac{\omega\Delta - \ell g_1(q)}{k-\ell-1}}{k-\ell-1-|L \cap X^+|}\\ =\ &\frac{(\omega\Delta - \ell g_1(q))(k-\ell-1-|L \cap X^+|)}{(k-\ell-1)(k-\ell-1-|L \cap X^+|)} = \frac{\omega\Delta - \ell g_1(q)}{k-\ell-1}.\end{aligned}$$

□

Let $x \in X^-$ and $k = d(x)$. Define

$$p = \min_{y' \in N(x)} \{\sigma_q(x, y') - (\Delta - k + 1)\}.$$

Note that $p \le k - 2$ as $\sigma_q(x, y') \le \Delta - 1$ by (3). Assume that $y \in N(x)$ achieves $\sigma_q(x, y) = (\Delta - k + 1) + p$. Let $\varphi \in \mathcal{C}^{\Delta}(G - xy)$,

$$C_y(\varphi) = \{\alpha \in \overline{\varphi}(x) : \text{there exists } y_\alpha \text{ with } \varphi(yy_\alpha) = \alpha \text{ and } d(y_\alpha) < q\} \quad \text{and}$$
$$|C_y(\varphi)| = t.$$

These two vertices x and y, and the coloring φ will be fixed in the rest of proof.

Then as each neighbor y' of y with $d(y') \ge q$ is colored by a color either from $B_\varphi(q)$ or from $\overline{\varphi}(x)$, we have $\sigma_q(x, y) = |B_\varphi(q)| + |\overline{\varphi}(x)| - t$. Thus

$$|B_\varphi(q)| + |\overline{\varphi}(x)| = \sigma_q(x, y) + t. \tag{13}$$

By the definition of $M_\varphi(q)$, which recall is defined by $A_\varphi(q) \cup \overline{\varphi}(x) \cup \overline{\varphi}(y)$ with $A_\varphi(q)$ defined in (1), we have $N(x, M_\varphi) = \overline{\varphi}(y) \cup A_\varphi(q)$. Thus

$$\begin{aligned} |N(x, M_\varphi)| &= \Delta - |\overline{\varphi}(x) \cup B_\varphi(q)| = \Delta - (|B_\varphi(q)| + |\overline{\varphi}(x)|) \\ &= \Delta - \sigma_q(x, y) - t = k - p - 1 - t. \end{aligned} \tag{14}$$

We first claim that

$$t \le c_0. \tag{15}$$

To see this, let $Y(\varphi) = \{y_\alpha : \varphi(yy_\alpha) = \alpha \in \overline{\varphi}(x)\}$. Since $\{x, y\} \cup Y(\varphi)$ is the vertex set of a multifan at y, it is φ-elementary by Lemma 2.2. Then $\Delta \ge |\bigcup_{y_\alpha \in Y(\varphi)} \overline{\varphi}(y_\alpha)| > \sum_{\alpha \in C_y(\varphi)} (\Delta - q) = |C_y(\varphi)| \varepsilon \Delta$, which gives $|C_y(\varphi)| < \frac{1}{\varepsilon}$. Thus $|C_y(\varphi)| \le \left\lceil \frac{1}{\varepsilon} \right\rceil - 1 = c_0$, as $c_0 = \left\lceil \frac{1-\varepsilon}{\varepsilon} \right\rceil$. This proves the claim.

Assume $N(x) = \{z_1, z_2, \ldots, z_k\}$. For each $i \in [1, k]$, we define

$$\ell_i = \sigma_q(x, z_i) - (\Delta - k + 1).$$

Note that by the definition of p, we have

$$\ell_i = \sigma_q(x, z_i) - (\Delta - k + 1) \ge (\Delta - k + 1) + p - (\Delta - k + 1) = p. \tag{16}$$

By Claim 3.5, in Step 2, x receives charge of at least

$$M(k, p) := \sum_{i=1}^{k} h(k, \ell_i) = \sum_{i=1}^{k} \frac{1}{k - \ell_i - 1} (\omega\Delta - \ell_i g_1(q)). \tag{17}$$

Claim 3.6. If $x \in X_1^-$, then $M_2^*(x) \geq M_2(x) = (d+2)\Delta$.

Proof. By the definition of X_1^-, $\varepsilon\Delta \leq k < q = (1-\varepsilon)\Delta$. Now by Lemma 2.3, x has at least $k-1-p-t-N$ neighbors $z \in N(x, M_\varphi)$ satisfying

$$\begin{aligned}\sigma_q(x,z) &\geq \Delta - 1 - (\lambda\Delta - 1) = \Delta - \lambda\Delta + (\Delta - k + 1) - (\Delta - k + 1)\\ &= (\Delta - k + 1) + k - \lambda\Delta - 1.\end{aligned}$$

Thus the corresponding ℓ_i value for each of those vertices is at least $k - \lambda\Delta - 1$. The remaining $p+1+t+N$ neighbors z' of x satisfy $\sigma_q(x,z') \geq \Delta - k + 1 + p$ by the minimality of p. Thus the corresponding ℓ_i value for each of those vertices is at least p. By (17), in Step 2, x receives charge of at least

$$M(k,p) \geq (k-p-1-t-N)h(k, k-\lambda\Delta-1) + (p+1+t+N)h(k,p).$$

By Step 0, $M_1(x) = (d+2)k$. Since $M_2^*(x) \geq M_1(x) + M(k,p) = (d+2)k + M(k,p)$, to show $M_2^*(x) \geq (d+2)\Delta$, it suffices to show $M(k,p) \geq (d+2)(\Delta - k)$. Since $h(k,\ell) = \frac{1}{k-\ell-1}(\omega\Delta - \ell g_1(q))$, we get

$$\begin{aligned}M(k,p) &= (k-p-1-t-N)h(k,k-\lambda\Delta-1) + (p+1+t+N)h(k,p)\\
&= \frac{k-p-1-t-N}{\lambda\Delta}(\omega\Delta - kg_1(q) + (\lambda\Delta+1)g_1(q)) + \frac{p+1+t+N}{k-p-1}\\
&\qquad (\omega\Delta - pg_1(q))\\
&= \frac{k-p-1-t-N}{\lambda\Delta}(\omega\Delta - kg_1(q) + (\lambda\Delta+1)g_1(q))\\
&\quad + \frac{p+1+t+N}{k-p-1}(\omega\Delta - kg_1(q) + (k-p)g_1(q))\\
&= \left(\frac{k-p-1-t-N}{\lambda\Delta} + \frac{p+1+t+N}{k-p-1}\right)(\omega\Delta - kg_1(q))\\
&\quad + \frac{k-p-1-t-N}{\lambda\Delta}(\lambda\Delta+1)g_1(q) + \frac{p+1+t+N}{k-p-1}(k-p)g_1(q)\\
&> \left(\frac{k-p-1-t-N}{\lambda\Delta} + \frac{p+1+t+N}{k-p-1}\right)(\omega\Delta - kg_1(q)) + kg_1(q)\\
&= \left(\frac{k-p-1}{\lambda\Delta} - \frac{t+N}{\lambda\Delta} + \frac{k+t+N}{k-p-1} - 1\right)(\omega\Delta - kg_1(q)) + kg_1(q)\\
&\geq \left(2\sqrt{\frac{k+t+N}{\lambda\Delta}} - \frac{t+N}{\lambda\Delta} - 1\right)(\omega\Delta - kg_1(q)) + kg_1(q)\\
&> \left(2\sqrt{\frac{k}{\lambda\Delta}} - \frac{t+N}{\lambda\Delta} - 1\right)(\omega\Delta - kg_1(q)) + kg_1(q).\end{aligned}$$

Let

$$\begin{aligned} f(k) &= \left(2\sqrt{\frac{k}{\lambda\Delta}} - \frac{N+t}{\lambda\Delta} - 1\right)(\omega\Delta - kg_1(q)) + kg_1(q) - (d+2)(\Delta - k) \\ &= \left(2\sqrt{\frac{k}{\lambda\Delta}} - \frac{N+t}{\lambda\Delta} - 1\right)(\omega\Delta - kg_1(q)) - \frac{d+2}{\omega}(\omega\Delta - kg_1(q)) \\ &= \left(2\sqrt{\frac{k}{\lambda\Delta}} - \frac{N+t}{\lambda\Delta} - 1 - \frac{d+2}{\omega}\right)(\omega\Delta - kg_1(q)), \end{aligned}$$

where $d+2+g_1(q) = (d+2) + \frac{\omega(d+2)}{d+2-\omega} = (d+2)(1+\frac{\omega}{d+2-\omega}) = (d+2)\frac{d+2}{d+2-\omega} = \frac{d+2}{\omega}g_1(q)$. Since $k < q$ and $g_1(q) = \frac{\omega\Delta}{q}$ from (11), we have $\omega\Delta - kg_1(q) \geq \omega\Delta - qg_1(q) = 0$ always. As $\left(2\sqrt{\frac{k}{\lambda\Delta}} - \frac{N+t}{\lambda\Delta} - 1 - \frac{d+2}{\omega}\right)$ is increasing in k and $k \geq \varepsilon\Delta$, now we know that to show $f(k) \geq 0$, it suffices to show $f(\varepsilon\Delta) \geq 0$. Recall that $\varepsilon = \frac{\omega}{d+2}$, $\lambda = \frac{\omega^3}{2(d+2)^3} = \frac{\varepsilon^3}{2}$, $t \leq c_0$ by (15), and $\Delta \geq D_0 \geq \frac{N+c_0}{\varepsilon^3}$. Thus we get

$$\begin{aligned} f(\varepsilon\Delta) &= \left(2\sqrt{\frac{\varepsilon\Delta}{\lambda\Delta}} - \frac{N+t}{\lambda\Delta} - 1 - \frac{d+2}{\omega}\right)(\omega\Delta - \varepsilon\Delta g_1(q)) \\ &\geq \left(2\sqrt{\frac{2}{\varepsilon^2}} - \frac{2(N+c_0)}{\varepsilon^3\Delta} - 1 - \frac{d+2}{\omega}\right)(\omega\Delta - \varepsilon\Delta g_1(q)) \\ &\geq \left(\frac{2\sqrt{2}(d+2)}{\omega} - 2 - 1 - \frac{d+2}{\omega}\right)(\omega\Delta - \varepsilon\Delta g_1(q)) \\ &= \left((2\sqrt{2}-1)\frac{d+2}{\omega} - 3\right)(\omega\Delta - \varepsilon\Delta g_1(q)) \geq 0, \end{aligned}$$

where note that $\frac{d+2}{\omega} > 3$ when $d \geq 19$, since $\frac{d+2}{\omega}$ is increasing in d and when $d = 19$, $\omega = \sqrt[3]{(d-1)d} < 7$. □

We assume now that $x \in X_2^- \cup X_3^-$. By the definition of X_2^- and X_3^-, we have $k < \varepsilon\Delta$. By Lemma 2.2, for any $v \in N(y)$ such that $\varphi(vy) \in \overline{\varphi}(x)$, $d(v) \geq |\overline{\varphi}(x)| = \Delta - k + 1 > q$. Therefore $|C_y(\varphi)| = 0$. By (13) and (14), we get

$$|B_\varphi(q)| = \sigma_q(x,y) - |\overline{\varphi}(x)| = \Delta - k + 1 + p - (\Delta - k + 1) = p, \tag{18}$$

$$|N(x, M_\varphi)| = \Delta - (|B_\varphi(q)| + |\overline{\varphi}(x)|) = k - p - 1. \tag{19}$$

Recall that $N(x) = \{z_1, \ldots, z_k\}$. Assume, without loss of generality, that $\{z_1, \ldots, z_{k-p-1}\} = N(x, M_\varphi)$. Let $i, j \in [1, k-p-1]$ be distinct. Define

$$b_i = |\varphi^{\text{bad}}(z_i) \setminus \{\varphi(xz_i)\}|.$$

By Lemma 2.4, $\varphi^{\text{bad}}(z_i)\backslash\{\varphi(xz_i)\} \subseteq B_\varphi(q)$ and $\varphi^{\text{bad}}(z_i) \cap \varphi^{\text{bad}}(z_j) = \emptyset$. This, together with (18), gives

$$\sum_{i=1}^{k-p-1} b_i = \left| \left(\bigcup_{z\in N(x,M_\varphi)} \varphi^{\text{bad}}(z) \right) \cap B_\varphi(q) \right| \le |B_\varphi(q)| = p. \tag{20}$$

By the definition of $\varphi^{\text{bad}}(z_i)$, we have $\sigma_q(x, z_i) + b_i = \Delta - 1$, and so $\sigma_q(x, z_i) = \Delta - b_i - 1$. Therefore, for $i \in [1, k - p - 1]$, we have

$$\begin{aligned} \ell_i &= \sigma_q(x, z_i) - (\Delta - k + 1) = k - b_i - 2, \\ h(k, \ell_i) &= \frac{1}{k - \ell_i - 1}(\omega\Delta - \ell_i g_1(q)) \\ &= \frac{1}{b_i + 1}(\omega\Delta - (k - 1 - (b_i + 1))g_1(q)) \\ &= \begin{cases} \dfrac{1}{b_i + 1}(\omega\Delta - (k - 1)g_1(q)) + g_1(q), \\ \dfrac{1}{b_i + 1}(\omega\Delta - (k - 2 - b_i)g_1(q)). \end{cases} \end{aligned} \tag{21}$$

For each $i \in [k - p - 1, k]$, we have $\ell_i \ge p$ by (16). By (17), in Step 2, x receives charge of at least

$$M(k, p) \ge \sum_{i=1}^{k-p-1} h(k, \ell_i) + (p + 1)h(k, p) \tag{22}$$

$$= \begin{cases} \displaystyle\sum_{i=1}^{k-p-1} \frac{1}{b_i + 1}(\omega\Delta - (k - 1)g_1(q)) + (k - p - 1)g_1(q) \\ \displaystyle+ \frac{p + 1}{k - p - 1}(\omega\Delta - pg_1(q)), \\ \displaystyle\sum_{i=1}^{k-p-1} \frac{1}{b_i + 1}(\omega\Delta - (k - 2 - b_i)g_1(q)) + \frac{p + 1}{k - p - 1}(\omega\Delta - pg_1(q)). \end{cases} \tag{23}$$

Claim 3.7. If $x \in X_2^-$, then $M_2^*(x) \ge M_2(x) = (d + 2)\Delta$.

Proof. By the definition of X_2^-, $6 \le k < \varepsilon\Delta$ if $d = 3$, $9 \le k < \varepsilon\Delta$ if $d = 4$, and $d \le k < \varepsilon\Delta$ if $d \ge 19$. We will apply (22) in proving Claim 3.7. We show first that $\sum_{i=1}^{k-p-1} \frac{1}{b_i+1} \ge \frac{k-p-1}{1+\frac{p}{k-p-1}} = \frac{(k-p-1)^2}{k-1}$. This follows by applying Cauchy–Schwarz inequality as below, where noticing that $\sum_{z_i\in N(x,M_\varphi)} b_i \le p$ and so $\sum_{i=1}^{k-p-1}(b_i + 1) \le p + (k - p - 1)$,

$$\begin{aligned} \sum_{i=1}^{k-p-1} \frac{1}{b_i + 1}(k - 1) = \sum_{i=1}^{k-p-1} \frac{1}{b_i + 1}(p + k - p - 1) \ge \sum_{i=1}^{k-p-1} \frac{1}{b_i + 1} \sum_{i=1}^{k-p-1} (b_i + 1) \\ \ge \left(\sum_{i=1}^{k-p-1} \sqrt{\frac{1}{b_i + 1}(b_i + 1)} \right)^2 = (k - p - 1)^2. \end{aligned}$$

Thus by (22),

$$M(k,p) \geq \frac{(k-p-1)^2}{k-1}(\omega\Delta-(k-1)g_1(q))+(k-p-1)g_1(q)+\frac{p+1}{k-p-1}(\omega\Delta-pg_1(q))$$
$$>\frac{(k-p-1)^2}{k-1}(\omega\Delta-(k-1)g_1(q))+(k-p-1)g_1(q)+\frac{p+1}{k-p-1}(\omega\Delta-kg_1(q)+(k-p-1)g_1(q))$$
$$>\frac{(k-p-1)^2}{k-1}(\omega\Delta-kg_1(q))+(k-p-1)g_1(q)+\frac{p+1}{k-p-1}(\omega\Delta-kg_1(q))+(p+1)g_1(q)$$
$$=\left(\frac{(k-p-1)^2}{k-1}+\frac{p+1}{k-p-1}\right)(\omega\Delta-kg_1(q))+kg_1(q)$$
$$=\left(\frac{(k-p-1)^2}{k-1}+\frac{k}{k-p-1}-1\right)(\omega\Delta-kg_1(q))+kg_1(q)$$
$$=\left(\frac{(k-p-1)^2}{k-1}+\frac{k}{2(k-p-1)}+\frac{k}{2(k-p-1)}-1\right)(\omega\Delta-kg_1(q))+kg_1(q)$$
$$\geq\left(3\sqrt[3]{\frac{k^2}{4(k-1)}}-1\right)(\omega\Delta-kg_1(q))+kg_1(q).$$

Since $M_2^*(x) \geq M_1(x)+M(k,p)=(d+2)k+M(k,p)$, to show $M_2^*(x)\geq(d+2)\Delta$, we only need to show $M(k,p)\geq(d+2)(\Delta-k)$. To do so, let

$$f(k)=\left(3\sqrt[3]{\frac{k^2}{4(k-1)}}-1\right)(\omega\Delta-kg_1(q))+kg_1(q)-(d+2)(\Delta-k)$$
$$=\left(3\sqrt[3]{\frac{k^2}{4(k-1)}}-1\right)(\omega\Delta-kg_1(q))-(d+2)\Delta+kg_1(q)+k\frac{\omega(d+2)(d+2-\omega)}{(d+2-\omega)\omega}$$
$$=\left(3\sqrt[3]{\frac{k^2}{4(k-1)}}-1\right)(\omega\Delta-kg_1(q))-(d+2)\Delta+kg_1(q)+kg_1(q)\frac{d+2-\omega}{\omega}$$
$$=\left(3\sqrt[3]{\frac{k^2}{4(k-1)}}-1\right)(\omega\Delta-kg_1(q))-\frac{d+2}{\omega}(\omega\Delta-kg_1(q))$$
$$=\left(3\sqrt[3]{\frac{k^2}{4(k-1)}}-1-\frac{d+2}{\omega}\right)(\omega\Delta-kg_1(q)).$$

Let $f_1(k)=\left(3\sqrt[3]{\frac{k^2}{4(k-1)}}-1-\frac{d+2}{\omega}\right)$. Since $\omega\Delta-kg_1(q)>0$ ($k<q$ and $g_1(q)=\frac{\omega\Delta}{q}$ from Equation 11) and $f_1(k)$ is increasing in k, we check that $f_1(3d-3)\geq 0$ when

$d = 3, 4$ and $f_1(d) \geq 0$ when $d \geq 19$. Recall that $\omega = 2$ when $d = 3, 4$, and $\omega = \sqrt[3]{d(d-1)} < \frac{d}{2}$ when $d \geq 19$, we get

$$f_1(6) > 3.649 - 1 - 2.5 > 0, \text{ when } d = 3,$$
$$f_1(9) > 4.08 - 1 - 3 > 0, \text{ when } d = 4,$$
$$\begin{aligned} f_1(d) &= 3\sqrt[3]{\frac{d^3}{4d(d-1)}} - 1 - \frac{d+2}{\omega} \\ &= \frac{3d\sqrt[3]{\frac{1}{4}}}{\omega} - 1 - \frac{d+2}{\omega} > \frac{1.88d}{\omega} - \frac{d+2+\omega}{\omega} > \frac{1.88d - 1.5d - 2}{\omega} > 0, \\ &\text{when } d \geq 19. \end{aligned}$$
□

Claim 3.8. If $x \in X_3^-$, then $M_2^*(x) \geq M_2(x) = (d+2)\Delta$.

Proof. By the definition of $X_3^-, 3 \leq k < 3(d-1) = 6$ if $d = 3$ and $4 \leq k < 3(d-1) = 9$ if $d = 4$. Thus $k = 3$ implies $d = 3$, and $\omega = 2$ in this case.

We use the same charge function $M(k, p)$ in (22) or (23)

$$\text{except when } d = 4 \text{ and } (k, p) \in \{(4, 1), (5, 2), (6, 3), (7, 3)\}.$$

Again, we show $M(k, p) \geq (d+2)(\Delta - k)$. We first consider the case when $p = k - 2$. In this case, $\sigma_q(x, z) \geq p + (\Delta - k + 1) = \Delta - 1$ for any $z \in N(x)$. As $\sigma_q(x, z) \leq \Delta - 1$ on the other hand, we get $\sigma_q(x, z) = \Delta - 1$. Thus $\ell_i = k - 2 = p$ for each $i \in [1, k]$. By (17), we get

$$\begin{aligned} M(k, k-2) &= k(2\Delta - (k-2)g_1(q)) \\ &= (k+2)\Delta + (k-2)(\Delta - kg_1(q)) \\ &> (d+2)(\Delta - k), \end{aligned}$$

where $\Delta - kg_1(q) > 0$ since $\Delta \geq (d+1)^{4.5d+11.5} > 36$ and $kg_1(q) = k\frac{\omega(d+2)}{d+2-\omega} \leq (3d-4)\frac{2(d+2)}{d} < 6(d+2) \leq 36$.

Since $M(k, p) - (d+2)(\Delta - k)$ is increasing in k, the minimum value of $M(k, p) - (d+2)(\Delta - k)$ is achieved at the smallest value of k. Since the b_i's are nonnegative integers, the minimum of the first summands in (22) or (23) is achieved when the values of any two b_i's differ by at most 1. Thus by (23),

$$\begin{aligned} M(k, 0) &= (k-1)(\omega\Delta - (k-2)g_1(q)) + \frac{\omega\Delta}{k-1} \quad (b_i = 0 \text{ for all } i) \\ &= (k-1)\omega\left(\Delta - (k-2)\frac{d+2}{d+2-\omega}\right) + \frac{\omega\Delta}{k-1} \quad \left(g_1(q) = \frac{(d+2)\omega}{d+2-\omega}\right) \\ &= 2(k-1)(\Delta - (k-2)(1+2/d)) + \frac{2\Delta}{k-1} \\ &\geq \begin{cases} 5\Delta - 4(1+2/d) > 5(\Delta - 3) = (d+2)(\Delta - k) & \text{if } k = 3, \\ \frac{20\Delta}{3} - 12(1+2/d) > 6(\Delta - k) = (d+2)(\Delta - k) & \text{if } k \geq 4. \end{cases} \end{aligned}$$

The following calculations are based on (22) and using $g_1(q) = \frac{(d+2)\omega}{d+2-\omega} = 2(1+2/d)$.

$$
\begin{aligned}
M(k,1) \geq & \left(k-3+\frac{1}{2}\right)(\omega\Delta-(k-1)g_1(q)) + (k-1-1)g_1(q) + \frac{2}{k-2}(\omega\Delta - g_1(q)) \\
& (\text{exactly one } b_i = 1 \text{ and all others are } 0) \\
> & \left(k-3+\frac{1}{2}\right)(\omega\Delta-(k-1)g_1(q)) + \frac{2}{k-2}(\omega\Delta - g_1(q)) \\
= & \left(k-3+\frac{1}{2}\right)(\omega\Delta-(k-1)\omega(1+2/d)) + \frac{2}{k-2}(\omega\Delta - \omega(1+2/d)) \\
= & 2\left(k-3+\frac{1}{2}\right)(\Delta-(k-1)(1+2/d)) + \frac{4}{k-2}(\Delta - (1+2/d)) \\
\geq & \begin{cases} 2\left(5-3+\frac{1}{2}\right)(\Delta-4(1+2/d)) + \frac{4}{5-2}(\Delta-(1+2/d)) & \text{when } 5 \leq k \leq 8, \\ 2\left(4-3+\frac{1}{2}\right)(\Delta-3(1+2/3)) + \frac{4}{4-2}(\Delta-(1+2/3)) & \text{when } d=3, k=4 \end{cases} \\
= & \begin{cases} \frac{19\Delta}{3} - \frac{64}{3}(1+2/d) > 6\Delta + \frac{3\Delta-320}{9} > 6\Delta & \text{when } 5 \leq k \leq 8, \\ 5\Delta - \frac{55}{3} > 5\Delta - 20 & \text{when } d=3 \text{ and } k=4 \end{cases} \\
\geq & (d+2)(\Delta-k),
\end{aligned}
$$

$$
\begin{aligned}
M(k,2) \geq & \left(k-5+\frac{1}{2}+\frac{1}{2}\right)(\omega\Delta-(k-1)g_1(q)) + (k-2-1)g_1(q) \\
& + \frac{3}{k-3}(\omega\Delta - 2g_1(q)) \\
& (\text{the minimum is attained when two of the } b_i'\text{s are } 1) \\
> & \left(k-5+\frac{1}{2}+\frac{1}{2}\right)(\omega\Delta-(k-1)g_1(q)) + \frac{3}{k-3}(\omega\Delta - 2g_1(q)) \\
= & 2(k-4)(\Delta-(k-1)(1+2/d)) + \frac{6}{k-3}(\Delta-(1+2/d)) \\
\geq & \begin{cases} 2(6-4)(\Delta-5(1+2/d)) + \frac{6}{6-3}(\Delta-(1+2/d)) & \text{if } k \geq 6, \\ 2(5-4)(\Delta-4(1+2/3)) + \frac{6}{5-3}(\Delta-(1+2/3)) & \text{if } d=3 \text{ and } k=5 \end{cases} \\
= & \begin{cases} 6\Delta - 22(1+2/d) > (d+2)(\Delta-k) & \text{if } k \geq 6 (\text{so } d=4), \\ 5\Delta - \frac{55}{3} > (d+2)(\Delta-k) & \text{if } d=3 \text{ and } k=5, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
M(k,k-3) \geq & \frac{4}{k-1}(\omega\Delta-(k-1)g_1(q)) + 2g_1(q) + \frac{k-2}{2}(\omega\Delta-(k-3)g_1(q)) \\
& \left(\text{taking } b_1 = b_2 = \frac{k-3}{2}\right) \\
> & \frac{8}{k-1}(\Delta-(k-1)(1+2/d)) + (k-2)(\Delta-(k-3)(1+2/d)) \\
\geq & \frac{8}{7-1}(\Delta-(7-1)(1+2/d)) + (7-2)(\Delta-(7-3)(1+2/d)) \quad \text{if } k \geq 7 \\
= & \frac{19\Delta}{3} - 28(1+2/d) > 6(\Delta-k),
\end{aligned}
$$

$$\begin{aligned} M(8,3) &\geq (1+1/2+1/2+1/2)(\omega\Delta-(k-1)g_1(q))+4g_1(q)+(\omega\Delta-3g_1(q)) \\ &\quad (\text{the minimum is attained when } 3 \text{ of the } b_i'\text{s are } 1 \text{ and the last one is } 0) \\ &= 7\Delta-\frac{33}{2}g_1(q)=7\Delta-49.5>6(\Delta-8)\cdot\left(d=4, g_1(q)=\frac{\omega(d+2)}{d+2-\omega}=3\right), \\ M(8,4) &\geq (1/2+1/2+1/3)(\omega\Delta-(k-1)g_1(q))+3g_1(q)+\frac{5}{3}(\omega\Delta-4g_1(q)) \\ &\quad (\text{the minimum is attained when } 2 \text{ of the } b_i'\text{s are } 1 \text{ and the last one is } 2) \\ &= 6\Delta-13g_1(q)=6\Delta-39>6(\Delta-8)\cdot(d=4, g_1(q)=3). \end{aligned}$$

When $d=3$, we have $k=3,4,5$ and $0\leq p\leq k-2$. The computations for $M(k,k-2)$, $M(k,0)$, $M(k,1)$, $M(k,2)$ above cover all the cases for $d=3$. For $d=4$, we have $k=4$, 5, 6, 7, 8 and $0\leq p\leq k-2$, and the following cases are covered by the computations above:

$$\begin{cases} M(4,0), M(4,2)=M(k,k-2) & \text{if } k=4, \\ M(5,0), M(5,1), M(5,3)=M(k,k-2) & \text{if } k=5, \\ M(6,0), M(6,1), M(6,2), M(6,4)=M(k,k-2) & \text{if } k=6, \\ M(7,0), M(7,1), M(7,2), M(7,4)=M(k,k-3), M(7,5)=M(k,k-2) & \text{if } k=7, \\ M(8,p), p\in\{0,1,2,3,4\}, M(8,5)=M(k,k-3), M(8,6)=M(k,k-2) & \text{if } k=8. \end{cases}$$

We are now left to check that when $d=4$ and $(k,p)\in\{(4,1),(5,2),(6,3),(7,3)\}$, we have $M(k,p)\geq 6(\Delta-k)$. For those cases, the desired bounds do not follow directly from (22) or (23). We will use (17) instead.

For $k=4$ and $p=1$, we have $|B_\varphi(q)|=p=1$ by (18). Recall $N(x)=\{z_1,\ldots,z_4\}$, $N(x,M_\varphi)=\{z_1,z_2\}$, $b_1+b_2\leq p=1$ by (20), $\ell_i=k-b_i-2$ by (21) for $i\in[1,2]$, and $\ell_i\geq p$ for each $i\in[1,4]$ by (16). Let $y=z_3$ and $N(x,B_\varphi)=\{z_4\}$. If $b_1=b_2=0$, then $\ell_1=\ell_2=2$. If $b_1+b_2=p=1$, we assume by symmetry that $b_1=1$ and $b_2=0$. Thus $\ell_1=1$ and $\ell_2=2$. Furthermore, by Lemma 2.4, $b_1=1$ implies that $(\varphi^{\text{bad}}(z_1)\setminus\{\varphi(xz_1)\})\cap B_\varphi(q)\neq\varnothing$. Then by Corollary 2.6, we know that for the vertex z_4, all its neighbors other than x are of degree at least q and so $\sigma_q(x,z_4)=\Delta-1$ and $\ell_4=2$. In either case, we have two ℓ_i's being at least 2 and the other two being at least 1. Therefore by (17), we have

$$\begin{aligned} M(4,1) &\geq \sum_{i=1}^{4}\frac{1}{k-\ell_i-1}(\omega\Delta-\ell_i g_1(q)) \\ &\geq 2(\omega\Delta-2g_1(q))+2\left(\frac{\omega\Delta-g_1(q)}{2}\right) \\ &= 6\Delta-5g_1(q)=6\Delta-5\times 3>6(\Delta-4), \end{aligned}$$

where $g_1(q)=\frac{\omega(d+2)}{d+2-\omega}=3$.

We then consider the case that $d(x)=k=5$ and $p=2$. In this case $|B_\varphi(q)|=p=2$, $|N(x,M_\varphi)|=k-1-p=2$. If $b_1+b_2\leq 1$, we may assume $b_1=0$. Then $\ell_1=k-b_1-2=3$. By (16), $\ell_i\geq p=2$ for $i\in[2,4]$. By (17), we get

$$M(5,2) \geq (\omega\Delta - 3g_1(q)) + 4\left(\frac{\omega\Delta - 2g_1(q)}{2}\right) = 6\Delta - 21 > 6(\Delta - 5).$$

Thus, we assume $b_1 + b_2 = p = 2$. Since $|B_\varphi(q)| = p = 2$, by Lemma 2.4 that $\varphi^{\text{bad}}(z_i)\setminus\{\varphi(xz_i)\} \subseteq B_\varphi(q)$ for $i \in [1,2]$ and $\varphi^{\text{bad}}(z_1) \cap \varphi^{\text{bad}}(z_2) = \emptyset$, we know that $B_\varphi(q) \subseteq (\varphi^{\text{bad}}(z_1) \cup \varphi^{\text{bad}}(z_2))$. Now by Corollary 2.8, there exists $z \in N(x, B_\varphi)$ such that z is adjacent to exactly one vertex of degree less than q which is x. We obtain the same charge function as above.

We next consider $(k,p) = (6,3)$ and $(k,p) = (7,3)$. Note that $|B_\varphi(q)| = p = 3$ by (18) and $|N(x, M_\varphi)| = k - p - 1$ by (19). Assume first that $|(\bigcup_{z\in N(x,M_\varphi)}\varphi^{\text{bad}}(z)) \cap B_\varphi(q)| \leq 2$. Then $\sum_{i=1}^{k-p-1} b_i \leq 2$ by (20). Using the calculation in (22) by taking two of the b_i's in $\{b_1, \ldots, b_{k-p-1}\}$ to be 1 and the rest to be 0, we have

$$M(6,3) \geq \left(\frac{1}{2} + \frac{1}{2}\right)(\omega\Delta - 5g_1(q)) + 2g_1(q) + 2(\omega\Delta - 3g_1(q)) = 6\Delta - 27 > 6(\Delta - 6),$$

$$M(7,3) \geq \left(\frac{1}{2} + \frac{1}{2} + 1\right)(\omega\Delta - 6g_1(q)) + 3g_1(q) + \frac{4}{3}(\omega\Delta - 3g_1(q)) = \frac{20}{3}\Delta - 39 > 6(\Delta - 7).$$

Thus we assume $B_\varphi(q) \subseteq \bigcup_{z\in N(x,M_\varphi)}\varphi^{\text{bad}}(z)$. Notice that when $(k,p) = (6,3)$ and so $N(x, M_\varphi) = \{z_1, z_2\}$, this assumption already leads to a contradiction as explained below. As $\varphi(xz_i) \in \varphi^{\text{bad}}(z_i)$, we have $|\varphi^{\text{bad}}(z_i)| = \Delta - \sigma_q(x, z_i)$ by (3). Thus $|\varphi^{\text{bad}}(z_i)| = \Delta - \sigma_q(x, z_i) \geq k - p - 1 = 2$. Since $\varphi(xz_i) \in \varphi^{\text{bad}}(z_i)$ and $\varphi(xz_i) \notin B_\varphi(q)$ for $i \in [1,2]$, we have

$$|(\varphi^{\text{bad}}(z_1) \cup \varphi^{\text{bad}}(z_2)) \cap B_\varphi(q)| \leq 1 + 1 = 2,$$

contradicting $|B_\varphi(q)| = 3$.

Lastly consider the case $(k,p) = (7,3)$. We first claim that there exists a vertex $z' \in N(x, B_\varphi)$ such that $\sigma_q(x, z') \geq \Delta - 2$. This is equivalent to show $|\varphi^{\text{bad}}(z')| \leq 2$ since $\varphi(z'x) \in \varphi^{\text{bad}}(z')$ implying $\sigma_q(x, z') = \Delta - |\varphi^{\text{bad}}(z')|$ by (3). Since $B_\varphi(q) \subseteq \bigcup_{z\in N(x,M_\varphi)}\varphi^{\text{bad}}(z)$, by Lemma 2.5, for each $w \in N(x, B_\varphi)$, $\varphi^{\text{bad}}(w) \subseteq B_\varphi(q)$. To prove the claim, since $|B_\varphi(q)| = p = 3$, it suffices to show that there is $z' \in N(x, B_\varphi)$ and a color $\beta \in B_\varphi(q)$ such that $\beta \notin \varphi^{\text{bad}}(z')$. Let $w \in N(x, B_\varphi)$ be any vertex. As $B_\varphi(q) \subseteq \bigcup_{z\in N(x,M_\varphi)}\varphi^{\text{bad}}(z)$, there exists $z \in N(x, M_\varphi)$ such that $\varphi(xw) \in \varphi^{\text{bad}}(z) \cap B_\varphi(q)$. Define $B_w(\varphi) = \{\varphi(ww') : w' \in N(w), \varphi(ww') \in B_\varphi(q)\setminus\{\varphi(xw)\} \text{ and } d(w') < q\}$ the same way as in Lemma 2.7. If $B_w(\varphi) = \emptyset$, then $\sigma_q(x, w) = \Delta - 1$ and so w is a vertex that can play the role of z'. Thus we assume $B_w(\varphi) \neq \emptyset$. Applying Lemma 2.7, there exists $w' \in N(x, B_\varphi)$ such that $\varphi(xw) \notin \varphi^{\text{bad}}(w')$ and so $\sigma_q(x, w') \geq \Delta - 2$. Now w' can play the role of z'.

By (19), $|N(x, M_\varphi)| = 3$. By (20), $b_1 + b_2 + b_3 \leq |B_\varphi(q)| = 3$. For the vertex z', we have $\sigma_q(x, z') - (\Delta - k + 1) \geq \Delta - 2 - (\Delta - 6) = 4$, and for the three remaining neighbors y' of x with one being y and the other two from $N(x, B_\varphi)\setminus\{z'\}$, they all satisfy $\sigma_q(x, y') - (\Delta - k + 1) \geq p = 3$ by (16). When $b_1 + b_2 + b_3 = |B_\varphi(q)| = 3$, the minimum value of $\sum_{i=1}^{3} h(k, \ell_i)$ is achieved when $b_1 = b_2 = b_3 = 1$, or equivalently when

$\ell_1 = \ell_2 = \ell_3 = k - 1 - 2 = 4$. Thus among the values $\ell_1, \ldots, \ell_7$, four of them are at least 4 and the rest three are at least 3. Applying (17), we have

$$M(7,3) \geq \frac{4}{7-4-1}(\omega\Delta - 4g_1(q)) + \frac{3}{7-3-1}(\omega\Delta - 3g_1(q))$$
$$= 6\Delta - 11g_1(q) = 6\Delta - 33 > 6(\Delta - 7). \qquad \square$$

The proof of Theorem 1.5 is now complete. $\square$

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