

THE OVERFULLNESS OF GRAPHS WITH SMALL MINIMUM DEGREE AND LARGE MAXIMUM DEGREE*

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Abstract. Given a simple graph G , denote by $\Delta(G)$, $\delta(G)$, and $\chi'(G)$ the maximum degree, the minimum degree, and the chromatic index of G , respectively. We say G is Δ -critical if $\chi'(G) = \Delta(G) + 1$ and $\chi'(H) \leq \Delta(G)$ for every proper subgraph H of G , and G is *overfull* if $|E(G)| > \Delta(G)\lfloor |V(G)|/2 \rfloor$. Since a maximum matching in G can have size at most $\lfloor |V(G)|/2 \rfloor$, it follows that $\chi'(G) = \Delta(G) + 1$ if G is overfull. Conversely, let G be a Δ -critical graph. The well known overfull conjecture of Chetwynd and Hilton asserts that G is overfull provided $\Delta(G) > |V(G)|/3$. In this paper, we show that any Δ -critical graph G is overfull if $\Delta(G) - 7\delta(G)/4 \geq (3|V(G)| - 17)/4$.

Key words. overfull conjecture, overfull graph, multifan, Kierstead path

MSC code. 05C15

DOI. 10.1137/21M1432776

1. Introduction. We will mainly use the notation from the book [14]. Graphs in this paper are simple. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We denote by $n(G)$, $\Delta(G)$, and $\delta(G)$ the order, maximum degree, and minimum degree of G , respectively. When there is no risk of confusion, we simply use n , Δ , and δ . For two integers p, q , let $[p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$.

An *edge k -coloring* or simply a *k -coloring* of a graph G is a mapping φ from $E(G)$ to the set of integers $[1, k]$, called *colors*, such that no two adjacent edges receive the same color with respect to φ . The *chromatic index* of G , denoted $\chi'(G)$, is defined to be the smallest integer k so that G has an edge k -coloring. We denote by $\mathcal{C}^k(G)$ the set of all edge k -colorings of G . A graph G with $\Delta(G) = \Delta$ is Δ -critical if $\chi'(G) = \Delta + 1$ and $\chi'(H) \leq \Delta$ for every proper subgraph H of G . In the 1960s, Vizing [15] and, independently, Gupta [10] proved that $\Delta \leq \chi'(G) \leq \Delta + 1$. This leads to a natural classification of graphs. Following Fiorini and Wilson [8], we say a graph G is of *class 1* if $\chi'(G) = \Delta$ and of *class 2* if $\chi'(G) = \Delta + 1$. Holyer [11] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. Nevertheless, if a graph G has too many edges compared to its maximum degree, i.e., $|E(G)| > \Delta\lfloor n/2 \rfloor$, then we have to color $E(G)$ using exactly $\Delta + 1$ colors. Such graphs are *overfull*. Although the converse is not true in general, finding sufficient conditions for a class 2 graph to be overfull has been a major focus in graph edge chromatic theory.

Applying Edmonds' matching polytope theorem, Seymour [13] showed that whether a graph G contains an overfull subgraph of maximum degree $\Delta(G)$ can be determined

*Received by the editors July 12, 2021; accepted for publication (in revised form) July 5, 2022; published electronically September 13, 2022.

<https://doi.org/10.1137/21M1432776>

Funding: The second author was supported by National Science Foundation grants DMS-1855716 and DMS-2154331; the third and fourth authors were supported by National Science Foundation grants DMS-2001130, and DMS-2153938, respectively.

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in polynomial time. A number of long-standing conjectures listed in *Twenty Pretty Edge Coloring Conjectures* in [14] lie in deciding when a Δ -critical graph is overfull. Chetwynd and Hilton [5, 6], in 1986, proposed the following conjecture.

CONJECTURE 1.1 (overfull conjecture). *If G is a class 2 graph with $\Delta(G) > \frac{n}{3}$, then G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.*

The degree condition of $\Delta > \frac{n}{3}$ in the conjecture above is best possible, as seen by the graph obtained from the Petersen graph by deleting one vertex. If the overfull conjecture is true, then the NP-complete problem of determining the chromatic index becomes polynomial-time solvable for graphs G with $\Delta > \frac{n}{3}$. Furthermore, the overfull conjecture implies several other conjectures in edge colorings such as the 1-factorization conjecture. Despite its importance, very little is known about its truth. It was confirmed only for graphs with $\Delta \geq n - 3$ by Chetwynd and Hilton [6] in 1989. By restricting the minimum degree, Plantholt [12] in 2004 showed that the overfull conjecture is affirmative for graphs G with even order n and minimum degree $\delta \geq \sqrt{7}n/3 \approx 0.8819n$. The 1-factorization conjecture is a special case of the overfull conjecture (when G is regular with $\Delta \geq n/2$), which in 2013 was confirmed for large n by Csaba et al. [7]. The overfull conjecture is still wide open in general, and it seems extremely difficult even for graphs G with $\Delta = n - 4$.

Since every class 2 graph of maximum degree Δ contains a Δ -critical subgraph, the overfull conjecture is equivalent to saying that every Δ -critical graph with $\Delta(G) > \frac{n}{3}$ is overfull. In this paper, we obtain the following result.

THEOREM 1.2. *Let G be a Δ -critical graph of order n . If $\Delta(G) - \frac{7\delta(G)}{4} \geq \frac{3n-17}{4}$, then G is overfull.*

The remainder of this paper is organized as follows. In the next section, we introduce some preliminaries on edge colorings such as multifans and Kierstead paths. In section 3, we prove Theorem 1.2. In the last section, we prove one of the main lemmas.

2. Preliminaries. In this section, we introduce the fundamental tools such as multifans and Kierstead paths in edge colorings and list some new developments built on these tools. Let G be a graph and $e \in E(G)$. Denote by $G - e$ the graph obtained from G by deleting the edge e . A vertex is called a k -vertex if its degree is k , and a neighbor of a vertex v is a k -neighbor if it is a k -vertex in G .

An edge $e \in E(G)$ is a *critical edge* of G if $\chi'(G - e) < \chi'(G)$. It is not hard to see that a connected graph is Δ -critical if and only if every of its edge is critical. Critical graphs are useful since they have many more graph structural properties than general class 2 graphs do. For example, Vizing's adjacency lemma (VAL) from 1965 [15] is a useful tool in counting the number of Δ -neighbors of a vertex.

LEMMA 2.1 (VAL). *Let G be a class 2 graph. If xy is a critical edge of G , then x has at least $\Delta - d(y) + 1$ Δ -neighbors from $V(G) \setminus \{y\}$.*

Let G be a graph, $e \in E(G)$, and $\varphi \in \mathcal{C}^k(G - e)$ for some integer $k \geq 0$. For any $v \in V(G)$, the set of colors *present* at v is $\varphi(v) = \{\varphi(f) : f \in E(G) \text{ is incident to } v\}$, and the set of colors *missing* at v is $\bar{\varphi}(v) = [1, k] \setminus \varphi(v)$. If $|\bar{\varphi}(v)| = 1$, we will also use $\bar{\varphi}(v)$ to denote the color that is missing at v . For a vertex set $X \subseteq V(G)$, define $\bar{\varphi}(X) = \bigcup_{v \in X} \bar{\varphi}(v)$. The set X is called *elementary* with respect to φ or simply φ -*elementary* if $\bar{\varphi}(u) \cap \bar{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. Moreover, we sometimes just say that X is elementary if the edge coloring is understood. For two distinct colors $\alpha, \beta \in [1, \Delta]$, the components of the subgraph induced by edges with

colors α or β are called (α, β) -chains. Clearly, each (α, β) -chain is either a path or an even cycle. For an (α, β) -chain P , if it is a path with an endvertex x , we also denote it by $P_x(\alpha, \beta, \varphi)$ to stress the endvertex x . If we interchange the colors α and β on an (α, β) -chain C of G , we get a new edge k -coloring of G , which is denoted by φ/C . This operation is a *Kempe change*. If C is a path with one endvertex x , we may also refer this Kempe change as “do an (α, β) -swap at x .”

Let $x, y \in V(G)$. If x and y are contained in the same (α, β) -chain of G with respect to φ , we say x and y are (α, β) -linked with respect to φ . Otherwise, x and y are (α, β) -unlinked with respect to φ . Without specifying φ , when we just say x and y are (α, β) -linked or x and y are (α, β) -unlinked, we mean they are linked or unlinked with respect to the current edge coloring.

A *multifan* at x with respect to edge $e = xy \in E(G)$ and coloring $\varphi \in \mathcal{C}^k(G - e)$ for some integer $k \geq 0$ is a sequence $F = (x, e_1, y_1, \dots, e_p, y_p)$ with $p \geq 1$ consisting of edges $e_1, e_2, \dots, e_p$ and distinct vertices $x, y_1, y_2, \dots, y_p$ satisfying the following two conditions:

- $e_1 = e$ and $e_i = xy_i$ for $i = 1, \dots, p$;
- for every edge e_i with $i \in [2, p]$, there is $j \in [1, i - 1]$ such that $\varphi(e_i) \in \overline{\varphi}(y_j)$.

Denote by $V(F)$ the set of vertices contained in F .

Notice that a multifan is slightly more general than a Vizing-fan which requires $j = i - 1$ in the second condition. The following lemma shows that the vertex set of a multifan in a Δ -critical graph is elementary. The proof can be found in the book [14].

LEMMA 2.2 (Stiebitz et al. [14]). *Let G be a class 2 graph with maximum degree Δ , $e_1 \in E(G)$, $\varphi \in \mathcal{C}^\Delta(G - e_1)$, and let $F = (x, e_1, y_1, \dots, e_p, y_p)$ be a multifan at x with respect to e_1 and φ . Then the following statements hold:*

- $\{x, y_1, y_2, \dots, y_p\}$ is φ -elementary.
- For any $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y_i)$ for some $i \in [1, p]$, it holds that $P_x(\alpha, \beta, \varphi) = P_{y_i}(\alpha, \beta, \varphi)$.

Let multifan $F = (x, e_1, y_1, \dots, e_p, y_p)$ be at x with respect to e_1 and φ and $y_{\ell_1}, \dots, y_{\ell_k}$ be a subsequence of $y_1, \dots, y_p$. We call $y_{\ell_1}, y_{\ell_2}, \dots, y_{\ell_k}$ an α -sequence with respect to φ and F if the following holds:

$$\varphi(xy_{\ell_1}) = \alpha \in \overline{\varphi}(y_1), \quad \varphi(xy_{\ell_i}) \in \overline{\varphi}(y_{\ell_{i-1}}), \quad i \in [2, k].$$

A vertex in an α -sequence is called an α -inducing vertex with respect to φ and F , and a missing color at an α -inducing vertex is called an α -inducing color. For convenience, α itself is also an α -inducing color. We say a color β is *induced by* α if β is α -inducing. By Lemma 2.2(a) and the definition of a multifan, each color in $\overline{\varphi}(V(F))$ is induced by a unique color in $\overline{\varphi}(y_1)$. Also if α_1, α_2 are two distinct colors in $\overline{\varphi}(y_1)$, then an α_1 -sequence is disjoint with an α_2 -sequence. For two distinct α -inducing colors β and γ , we write $\gamma \prec \beta$ if there exists an α -sequence $y_{\ell_1}, y_{\ell_2}, \dots, y_{\ell_k}$ such that $\gamma \in \overline{\varphi}(y_{\ell_i})$, $\beta \in \overline{\varphi}(y_{\ell_j})$, and $i < j$. For convenience, $\alpha \prec \beta$ for any α -inducing color $\beta \neq \alpha$. As a consequence of Lemma 2.2(a), we have the following properties for a multifan. A proof of the result can be found in [2, Lemma 3.2].

LEMMA 2.3. *Let G be a class 2 graph and $F = (x, e_1, y_1, \dots, e_p, y_p)$ be a multifan with respect to a critical edge $e_1 = xy_1$ and a coloring $\varphi \in \mathcal{C}^\Delta(G - e_1)$. For two colors $\gamma \in \overline{\varphi}(y_i)$ and $\lambda \in \overline{\varphi}(y_j)$ with $i, j \in [1, p]$ and $i \neq j$, the following statements hold:*

- If γ and λ are induced by different colors, then y_i and y_j are (γ, λ) -linked with respect to φ .
- If γ and λ are induced by the same color, $\gamma \prec \lambda$, and y_i and y_j are (γ, λ) -unlinked with respect to φ , then $x \in P_{y_j}(\gamma, \lambda, \varphi)$.

Let G be a graph, $e = v_0v_1 \in E(G)$, and $\varphi \in \mathcal{C}^k(G - e)$ for some integer $k \geq 0$. A *Kierstead path* with respect to e and φ is a sequence $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, \dots, v_{p-1}, v_{p-1}v_p, v_p)$ with $p \geq 1$ consisting of distinct vertices $v_0, v_1, \dots, v_p$ and distinct edges $v_0v_1, v_1v_2, \dots, v_{p-1}v_p$ satisfying the following condition:

- (K1) For every edge $v_{i-1}v_i$ with $i \in [2, p]$, there exists $j \in [0, i - 2]$ such that $\varphi(v_{i-1}v_i) \in \overline{\varphi}(v_j)$.

Clearly, a Kierstead path with at most 3 vertices is a multifan, and so the set of its vertices is elementary for the case of Δ -critical graphs. The following result shows that the sets of the vertices of Kierstead paths with 4 vertices are “nearly” elementary. (See Theorem 3.3 from [14].)

LEMMA 2.4. *Let G be a class 2 graph, $e = v_0v_1 \in E(G)$ be a critical edge, and $\varphi \in \mathcal{C}^\Delta(G - e)$. If $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, v_2v_3, v_3)$ is a Kierstead path with respect to e and φ , then the following statements hold:*

- (a) *If $\min\{d(v_1), d(v_2)\} < \Delta$, then $V(K)$ is φ -elementary.*
 (a) $|\overline{\varphi}(v_3) \cap (\overline{\varphi}(v_0) \cup \overline{\varphi}(v_1))| \leq 1$.

LEMMA 2.5. *Let G be an n -vertex Δ -critical graph, and let $a \in V(G)$. If $d(a) \leq \frac{2\Delta - n + 2}{3}$, then for each $v \in V(G) \setminus \{a\}$, either $d(v) \geq \Delta - d(a) + 1$ or $d(v) \leq n - \Delta + 2d(a) - 6$. Furthermore, if $d(v) \geq \Delta - d(a) + 1$, then for any $b \in N(a)$ with $d(b) = \Delta$ and $\varphi \in \mathcal{C}^\Delta(G - ab)$, $|\overline{\varphi}(v) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \leq 1$.*

Proof. Let $k = d(a)$. Assume to the contrary that there exists $v \in V(G) \setminus \{a\}$ such that $n - \Delta + 2k - 5 \leq d(v) \leq \Delta - k$. By Lemma 2.1 (VAL), we may take $b \in N(a)$ with $d(b) = \Delta$ and $\varphi \in \mathcal{C}^\Delta(G - ab)$. As $d(b) = \Delta$, we have $|N(b) \setminus \{a\}| = \Delta - 1$. Since $d(v) \leq \Delta - k$, by VAL, $a \notin N(v)$. As $b, v \notin N(b) \cap N(v)$ and $d(v) \geq n - \Delta + 2k - 5$, it follows that

$$\begin{aligned} |N(v) \cap (N(b) \setminus \{a\})| &= |((N(v) \cap (N(b) \setminus \{a\})) \setminus \{b, v\})| \\ &\geq |N(v)| + |N(b) \setminus \{a\}| - (n - 3) - |N(b) \cap \{v\}| - |N(v) \cap \{b\}| \\ &\geq 2k - 3 - |N(b) \cap \{v\}| - |N(v) \cap \{b\}|. \end{aligned}$$

Note that $|[1, \Delta] \setminus (\overline{\varphi}(a) \cup \overline{\varphi}(b))| = k - 2$. If $bv \notin E(G)$, we can find a vertex $u \in N(v) \cap (N(b) \setminus \{a\})$ such that $\varphi(bu), \varphi(vu) \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$. If $bv \in E(G)$, then as $d(v) \leq \Delta - k$, by Lemma 2.2(a), we must have $\varphi(bv) \notin \overline{\varphi}(a)$ and therefore $\varphi(bv) \notin \overline{\varphi}(a) \cup \overline{\varphi}(b)$. Thus there are at most $2(k - 3)$ edges between $\{b, v\}$ and $N(v) \cap (N(b) \setminus \{a\})$ colored by colors from $[1, \Delta] \setminus (\overline{\varphi}(a) \cup \overline{\varphi}(b))$. Thus again we can find a vertex $u \in N(v) \cap (N(b) \setminus \{a\})$ such that $\varphi(bu), \varphi(vu) \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$. Therefore, $K = (a, ab, b, bu, u, uv, v)$ is a Kierstead path. Since $d(v) \leq \Delta - k$, $|\overline{\varphi}(v)| \geq k$. This implies that

$$|\overline{\varphi}(v) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \geq 2,$$

showing a contradiction to Lemma 2.4(b).

For the second part, since $|N(b) \setminus \{a\}| = \Delta - 1$ and $d(v) \geq \Delta - k + 1$,

$$|N(v) \cap (N(b) \setminus \{a\})| \geq 2\Delta - n - k \geq 2k - 2,$$

as $k = d(a) \leq \frac{2\Delta - n + 2}{3}$. The rest of the proof then follows the same argument where the second inequality above is implied by for the first part, and so we omit it. $\square$

Lemmas 2.6 to 2.8 below aim to study the “elementary properties” of small tree-like structures that contain Kierstead paths as substructures.

Let G be a Δ -critical graph, $ab \in E(G)$, and $\varphi \in \mathcal{C}^\Delta(G - ab)$. A *fork* H with respect to φ is a graph with

$$V(H) = \{a, b, u, s_1, s_2, t_1, t_2\} \quad \text{and} \quad E(H) = \{ab, bu, us_1, us_2, s_1t_1, s_2t_2\}$$

such that $\varphi(bu) \in \overline{\varphi}(a)$, $\varphi(us_1), \varphi(us_2) \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$, and $\varphi(s_1t_1) \in (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \cap \overline{\varphi}(t_2)$ and $\varphi(s_2t_2) \in (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \cap \overline{\varphi}(t_1)$. Fork was defined in [3], and it was shown in [3, Proposition B] that a fork cannot exist in a Δ -critical graph if the degree sum of a , t_1 , and t_2 is small.

LEMMA 2.6. *Let G be a Δ -critical graph, $ab \in E(G)$, and $\{u, s_1, s_2, t_1, t_2\} \subseteq V(G)$. If $\Delta \geq d(a) + d(t_1) + d(t_2) + 1$, then for any $\varphi \in \mathcal{C}^\Delta(G - ab)$, G does not contain the fork on $\{a, b, u, s_1, s_2, t_1, t_2\}$ with respect to φ .*

A *short-kite* H is a graph with

$$V(H) = \{a, b, c, u, x, y\} \quad \text{and} \quad E(H) = \{ab, ac, bu, cu, ux, uy\}.$$

The lemma below was proved in [4].

LEMMA 2.7. *Let G be a class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in \mathcal{C}^\Delta(G - ab)$. Suppose*

$$K = (a, ab, b, bu, u, ux, x) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, uy, y)$$

are two Kierstead paths with respect to ab and φ . If $\overline{\varphi}(x) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$ and $\overline{\varphi}(y) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$, then $\max\{d(x), d(y)\} = \Delta$.

Note that in [4, Lemma 5], the condition used was “ $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$ ” instead of “ $\overline{\varphi}(x) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$ and $\overline{\varphi}(y) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$.” However, the two conditions play an equivalent role in the proof of the lemma. If we assume to the contrary that $d(x) \leq \Delta - 1$ and $d(y) \leq \Delta - 1$, then the proof in [4] (without using the condition $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$) implies that $d(x) = d(y) = \Delta - 1$. Thus $\overline{\varphi}(x) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$ and $\overline{\varphi}(y) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$ imply that $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$.

A *kite* H is a graph with

$$V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\} \quad \text{and} \quad E(H) = \{ab, ac, bu, cu, us_1, us_2, s_1t_1, s_2t_2\}.$$

LEMMA 2.8. *Let G be a class 2 graph, $H \subseteq G$ be a kite with $V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\}$, and $\varphi \in \mathcal{C}^\Delta(G - ab)$. Suppose*

$$K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, us_2, s_2, s_2t_2, t_2)$$

are two Kierstead paths with respect to ab and φ . If $\varphi(s_1t_1) = \varphi(s_2t_2)$, then $|\overline{\varphi}(t_1) \cap \overline{\varphi}(t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \leq 4$.

The proof of Lemma 2.8 will be given in the last section.

3. Proof of Theorem 1.2. Since all vertices not missing a given color α are saturated by the matching that consists of all edges colored by α in G , we have the parity lemma below, which has appeared in many papers; for example, see [9, Lemma 2.1].

LEMMA 3.1 (parity lemma). *Let G be an n -vertex multigraph and $\varphi \in \mathcal{C}^k(G)$ for some integer $k \geq \Delta(G)$. Then for any color $\alpha \in [1, \Delta]$, $|\{v \in V(G) : \alpha \in \overline{\varphi}(v)\}| \equiv n \pmod{2}$.*

Proof of Theorem 1.2. Since the overfull conjecture is true for $\Delta \geq n - 3$ by Chetwynd and Hilton [6], we may assume that $\Delta \leq n - 4$. Thus $\Delta(G) - \frac{7\delta(G)}{4} \geq \frac{3n-17}{4}$ implies that $\delta(G) \leq \frac{n+1}{7}$. As $\delta(G) \geq 2$, it follows that $n \geq 13$ and $\Delta(G) \geq \frac{3n-3}{4}$.

Choose $a \in V(G)$ such that $d(a) = \delta(G) =: k$. As $\Delta \geq \frac{3n-3}{4}$, $\frac{2\Delta-n+2}{3} \geq \frac{n+1}{6} > k$. By Lemma 2.5, for every $v \in V(G) \setminus \{a\}$, we have

$$\text{either } d(v) \geq \Delta - k + 1 \quad \text{or} \quad d(v) \leq (n - \Delta) + 2k - 6.$$

Let $b \in N(a)$ with $d(b) = \Delta$ and $\varphi \in \mathcal{C}^\Delta(G - ab)$. We consider two cases below to finish the proof.

Case 1. There exist two distinct vertices $t_1, t_2 \in V(G) \setminus \{a\}$ such that $d(t_1), d(t_2) \leq (n - \Delta) + 2k - 6$.

Let $s_1 \in N(t_1)$ and $s_2 \in N(t_2)$ be any two distinct vertices such that $\varphi(s_1t_1), \varphi(s_2t_2) \in \bar{\varphi}(a) \cup \bar{\varphi}(b)$. Such vertices s_1, s_2 exist since $d(a) = k = \delta(G)$ and $|[1, \Delta] \setminus (\bar{\varphi}(a) \cup \bar{\varphi}(b))| = k - 2$; there are at least two edges incident to each of t_1 and t_2 colored by colors from $\bar{\varphi}(a) \cup \bar{\varphi}(b)$. Since G is Δ -critical and $s_i \in N(t_i)$ for $i \in [1, 2]$, VAL implies $d(s_i) \geq \Delta + 2 - (n - \Delta + 2k - 6) > n - \Delta + 2k - 6$, where the last inequality follows as $\Delta - \frac{7k}{4} \geq \frac{3n-17}{4}$ and $n \geq 13$.

By Lemma 2.5, $d(s_i) \geq \Delta - k + 1$ for each $i \in [1, 2]$. In fact we can assume $d(s_i) \geq \Delta - k + 2$ for each $i \in [1, 2]$. Since, say, if $d(s_1) = \Delta - k + 1$, then t_1 has at least k neighbors of degree Δ by VAL. Note that $k > k - 2 = |[1, \Delta] \setminus (\bar{\varphi}(a) \cup \bar{\varphi}(b))|$. Then we can choose s_1 such that $\varphi(s_1t_1) \in \bar{\varphi}(a) \cup \bar{\varphi}(b)$ with $d(s_1) = \Delta \geq \Delta - k + 2$ and $s_1 \neq s_2$. Thus we assume $d(s_i) \geq \Delta - k + 2$ for each $i \in [1, 2]$. Let $c \in N(a)$ such that $\varphi(ac) \in \bar{\varphi}(b)$. By VAL, $d(c) \geq \Delta - k + 2$. Thus, since $b, c \notin N(b) \cap N(c)$ and $s_1, s_2 \notin N(s_1) \cap N(s_2)$,

$$\begin{aligned} (1) \quad & |N(b) \cap N(c) \cap N(s_1) \cap N(s_2)| \\ &= |(N(b) \cap N(c) \cap N(s_1) \cap N(s_2)) \setminus \{b, c, s_1, s_2\}| \\ &\geq |N(s_1) \cap N(s_2)| - |N(s_1) \cap N(s_2) \cap \{b, c\}| \\ &\quad + |N(b) \cap N(c)| - |N(b) \cap N(c) \cap \{s_1, s_2\}| - (n - 4) \\ &\geq (2\Delta - n - 2k + 4) - |N(s_1) \cap N(s_2) \cap \{b, c\}| \\ &\quad + (2\Delta - n - k + 2) - |N(b) \cap N(c) \cap \{s_1, s_2\}| - n + 4 \\ &= 4\Delta - 3n - 3k + 10 - |N(s_1) \cap N(s_2) \cap \{b, c\}| - |N(b) \cap N(c) \cap \{s_1, s_2\}| \\ &\geq (k - 2 - |N(s_1) \cap \{b, c\}|) + (k - 2 - |N(s_2) \cap \{b, c\}|) + \\ &\quad (k - 2 - |N(b) \cap \{s_1, s_2\}|) + (k - 2 - |N(c) \cap \{s_1, s_2\}|) + 1, \end{aligned}$$

where the last inequality follows from $\Delta \geq 3n/4 + (7k - 17)/4$. Note that $|(\bar{\varphi}(a) \cup \bar{\varphi}(b)) \cap \bar{\varphi}(t_i)| \geq 2$ under any Δ -coloring of $G - ab$ since $d(t_i) \leq (n - \Delta) + 2k - 6 \leq \Delta - k$ and that $s_i \neq a$ as $d(s_i) \geq \Delta - k + 2 > k = d(a)$. Therefore if $s_ib \in E(G)$, then $\varphi(s_ib) \notin \bar{\varphi}(a)$ by Lemma 2.4(b), giving $\varphi(s_ib) \notin \bar{\varphi}(a) \cup \bar{\varphi}(b)$. By considering the coloring φ' obtained from φ by coloring ab with $\varphi(ac)$ and uncoloring ac , we also have that if $s_ic \in E(G)$, then $\varphi(s_ic) \notin \bar{\varphi}(a) \cup \bar{\varphi}(b)$. Since $|[1, \Delta] \setminus (\bar{\varphi}(a) \cup \bar{\varphi}(b))| = k - 2$, (1) implies that there exists $u \in N(b) \cap N(c) \cap N(s_1) \cap N(s_2)$ such that $\varphi(ub), \varphi(uc), \varphi(us_1), \varphi(us_2) \in \bar{\varphi}(a) \cup \bar{\varphi}(b)$. As both $\{a, b, t_i\}$ and $\{a, b, s_i, t_i\}$ are not elementary, it follows that $u \notin \{a, t_1, t_2\}$ by $\varphi(s_it_i) \in \bar{\varphi}(a) \cup \bar{\varphi}(b)$ for $i \in [1, 2]$, Lemma 2.2, and Lemma 2.4. Thus H with $V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\}$ is a kite, and both

$K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1)$ and $K^* = (b, ab, a, ac, c, cu, u, us_2, s_2, s_2t_2, t_2)$ are Kierstead paths with respect to ab and φ .

If there exist $s_1 \in N(t_1)$ and $s_2 \in N(t_2)$ such that $\varphi(s_1t_1), \varphi(s_2t_2) \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$, and $\varphi(s_1t_1) = \varphi(s_2t_2)$, then let $\Gamma = \overline{\varphi}(t_1) \cap \overline{\varphi}(t_2)$. Since $|[1, \Delta] \setminus (\overline{\varphi}(a) \cup \overline{\varphi}(b))| = k - 2$, $|\Gamma \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \geq |\Gamma| - k + 2$. By the assumption of this case, $|\Gamma| \geq \Delta - 2(n - \Delta + 2k - 6) = 3\Delta - 2n - 4k + 12$. As $\Delta \geq \frac{3n+7k-17}{4}$, $n \geq 13$ and $k \geq 2$, we get

$$|\Gamma \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \geq |\Gamma| - k + 2 \geq 3\Delta - 2n - 5k + 14 \geq \frac{n+k+5}{4} \geq 5 > 4,$$

contradicting Lemma 2.8.

Thus for any $s_1 \in N(t_1)$ and $s_2 \in N(t_2)$ such that $\varphi(s_1t_1), \varphi(s_2t_2) \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$, it follows that $\varphi(s_1t_1) \neq \varphi(s_2t_2)$. Thus $\varphi(s_1t_1) \in \overline{\varphi}(t_2)$ and $\varphi(s_2t_2) \in \overline{\varphi}(t_1)$. Hence H^* with $V(H^*) = \{a, b, u, s_1, s_2, t_1, t_2\}$ is a fork. However, $d(a) + d(t_1) + d(t_2) \leq k + 2(n - \Delta + 2k - 6) = 2n - 2\Delta + 5k - 12 < \Delta$, as $n \geq 13$ and $\Delta \geq 3n/4 + (7k - 17)/4$, contradicting Lemma 2.6.

Case 2. There exists at most one vertex $t \in V(G) \setminus \{a\}$ such that $d(t) \leq (n - \Delta) + 2k - 6$.

Assume that G is not overfull. We show that we can always find two distinct vertices $x, y \in V(G) \setminus \{a\}$ with

$$(*) \quad \Delta - k + 1 \leq d(x), d(y) < \Delta, \quad \overline{\varphi}(x) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset, \quad \overline{\varphi}(y) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset.$$

Then we will apply Lemma 2.7 to reach a contradiction. To see this, let $c \in N(a)$ such that $\varphi(ac) \in \overline{\varphi}(b)$. By VAL, $d(c) \geq \Delta - k + 2$. We claim that we can find $u \in N(b) \cap N(c) \cap N(x) \cap N(y)$ such that $u \notin \{a, x, y\}$ and $\varphi(ub), \varphi(uc), \varphi(ux), \varphi(uy) \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$. Since with the existence of the vertex u , H with $V(H) = \{a, b, c, u, x, y\}$ is a short-kite, and both

$$K = (a, ab, b, bu, u, ux, x) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, uy, y)$$

are Kierstead paths with respect to ab and φ . As $\max\{d(x), d(y)\} < \Delta$, we achieve a contradiction to Lemma 2.7.

To show the existence of u , note that, for any $z \in \{x, y\}$, if $bz \in E(G)$, then $\varphi(bz) \notin \overline{\varphi}(a) \cup \overline{\varphi}(b)$ by the assumption $\overline{\varphi}(x) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$. Similarly, by considering the coloring φ' obtained from φ by coloring ab with $\varphi(ac)$ and uncoloring ac , it holds also that if $cz \in E(G)$, then $\varphi(cz) \notin \overline{\varphi}(a) \cup \overline{\varphi}(b)$. Thus if $d(x), d(y) \geq \Delta - k + 2$, by exactly the same argument as in (1), we can find $u \in N(b) \cap N(c) \cap N(x) \cap N(y)$ such that $u \notin \{a, x, y\}$ and $\varphi(ub), \varphi(uc), \varphi(ux), \varphi(uy) \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$. If, say $d(x) = \Delta - k + 1$ and $d(y) \geq \Delta - k + 3$, then we can still get similar inequalities as in (1) for showing the existence of u . Thus we assume that $\min\{d(x), d(y)\} = \Delta - k + 1$ and $\max\{d(x), d(y)\} \leq \Delta - k + 2$.

When $k = 2$, then $\overline{\varphi}(a) \cup \overline{\varphi}(b) = [1, \Delta]$. As $\overline{\varphi}(x) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$ and $\overline{\varphi}(y) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \neq \emptyset$, it follows that $bx, by \notin E(G)$. By considering the coloring φ' obtained from φ by coloring ab with $\varphi(ac)$ and uncoloring ac , it holds also that $cx, cy \notin E(G)$. Then we have $|N(x) \cap N(y)| \geq 2(\Delta - k + 1) - (n - 2) = 2\Delta - n - 2k + 4$ in the place of $|N(s_1) \cap N(s_2)|$ in (1), and so we can find the existence of the vertex u too. Thus we further assume that $k \geq 3$.

Now we have $\min\{d(x), d(y)\} = \Delta - k + 1$ and $\max\{d(x), d(y)\} \leq \Delta - k + 2$, and $k \geq 3$. With the existence of x and y , we claim that if a vertex $z \in N(a)$ satisfies $d(z) \geq \Delta - k + 1$, then $d(z) = \Delta$. Assume otherwise that $d(z) \leq \Delta - 1$.

Let $\varphi' \in \mathcal{C}^\Delta(G - az)$. Assume, without loss of generality (w.l.o.g.), that $y \neq z$. Then as $|N(y) \cap N(z) \setminus \{a\}| \geq 2(\Delta - k + 1) - n - 1 > 2(k - 3)$, there exists $w \in (N(y) \cap N(z)) \setminus \{a\}$ such that $\varphi'(wy), \varphi'(wz) \in \overline{\varphi}'(a) \cup \overline{\varphi}'(z)$. Since $d(z) < \Delta$, $\{a, z, w, y\}$ is φ' -elementary by Lemma 2.4(a). However, $|\overline{\varphi}'(a) \cup \overline{\varphi}'(z)| \geq \Delta - k + 3$ and $|\overline{\varphi}'(y)| \geq k - 2$ (recall $\max\{d(x), d(y)\} \leq \Delta - k + 2$), a contradiction. Therefore, the statement that if a vertex $z \in N(a)$ satisfies $d(z) \geq \Delta - k + 1$, then $d(z) = \Delta$ is true. This particularly implies that $d(c) = \Delta$ and $a \notin N(x) \cup N(y)$. The latter implies $|N(x) \cap N(y)| \geq 2(\Delta - k + 1) - (n - 1)$.

Thus, since $b, c \notin N(b) \cap N(c)$ and $x, y \notin N(x) \cap N(y)$, we get

$$\begin{aligned} & |N(b) \cap N(c) \cap N(x) \cap N(y)| \\ &= |(N(b) \cap N(c) \cap N(x) \cap N(y)) \setminus \{b, c, x, y\}| \\ &\geq |N(x) \cap N(y)| - |N(x) \cap N(y) \cap \{b, c\}| \\ &\quad + |N(b) \cap N(c)| - |N(b) \cap N(c) \cap \{x, y\}| - (n - 4) \\ &\geq (2\Delta - (n - 1) - 2k + 2) - |N(x) \cap N(y) \cap \{b, c\}| \\ &\quad + (2\Delta - n) - |N(b) \cap N(c) \cap \{x, y\}| - n + 4 \\ &= 4\Delta - 3n - 2k + 7 - |N(x) \cap N(y) \cap \{b, c\}| - |N(b) \cap N(c) \cap \{x, y\}| \\ &\geq (k - 2 - |N(x) \cap \{b, c\}|) + (k - 2 - |N(y) \cap \{b, c\}|) + \\ &\quad (k - 2 - |N(b) \cap \{x, y\}|) + (k - 2 - |N(c) \cap \{x, y\}|) + k - 2 \\ &\geq (k - 2 - |N(x) \cap \{b, c\}|) + (k - 2 - |N(y) \cap \{b, c\}|) + \\ &\quad (k - 2 - |N(b) \cap \{x, y\}|) + (k - 2 - |N(c) \cap \{x, y\}|) + 1, \end{aligned}$$

where the last inequality follows since $\Delta \geq 3n/4 + (7k - 17)/4$ and $k \geq 3$. By the same argument as in Case 1, the inequality above implies the existence of a desired vertex u .

Below we only show the existence of the vertices x and y . Assume first that n is odd. As G is not overfull, some color in $[1, \Delta]$ is missing at least three distinct vertices of G by the parity lemma. If there exists exactly one vertex $t \in V(G) \setminus \{a\}$ such that $d(t) \leq (n - \Delta) + 2k - 6$, then each color in $\overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$ is missing at another vertex from $V(G) \setminus \{a, b, t\}$. As $|\overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \geq 2\Delta - n - 2k + 6 - (k - 2) = 2\Delta - n - 3k + 8 > k$ and $d(v) \geq \Delta - k + 1$ for every $v \in V(G) \setminus \{a, b, t\}$, we can find $x, y \in V(G) \setminus \{a\}$ with the desired property as in (*). Thus for every vertex $v \in V(G) \setminus \{a, b\}$, $d(v) \geq \Delta - k + 1$. Let $\beta \in [1, \Delta]$ such that β is missing at at least three vertices, say u, v, w , from $V(G)$. If $\beta \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$, as $\overline{\varphi}(a) \cup \overline{\varphi}(b)$ is φ -elementary, then letting $x, y \in \{u, v, w\} \setminus \{a, b\}$ will give us a desired choice. So we assume $\beta \notin \overline{\varphi}(a) \cup \overline{\varphi}(b)$. Let $\alpha \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$, say, w.l.o.g., that $\alpha \in \overline{\varphi}(a)$. As at most one of u, v, w , say w is (α, β) -linked with a , we do an (α, β) -swap at the other two vertices u and v . Call the new coloring still φ . Now u and v can play the role of x and y .

Assume now that n is even. As G is not overfull, each color in $[1, \Delta]$ is missing at an even number of vertices of G by the parity lemma. In particular, we have fact (a): each color in $\overline{\varphi}(a) \cup \overline{\varphi}(b)$ is missing at a vertex from $V(G) \setminus \{a, b\}$. By the second part of Lemma 2.5, we have fact (b): for every $v \in V(G) \setminus \{a, b\}$, if $\Delta - k + 1 \leq d(v) \leq \Delta - 1$, then $|\overline{\varphi}(v) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \leq 1$. Let $t \in V(G) \setminus \{a, b\}$ such that $d(t)$ is the smallest among the degrees of vertices from $V(G) \setminus \{a, b\}$. As $|\overline{\varphi}(a) \cup \overline{\varphi}(b)| = \Delta - k + 2$ and $|\overline{\varphi}(t)| \leq \Delta - k$, there exist at least two distinct colors $\alpha, \beta \in (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \setminus \overline{\varphi}(t)$. By the assumption of this case and Lemma 2.5, we know that every vertex from $V(G) \setminus \{a, b, t\}$ has degree at least $\Delta - k + 1$. Thus by facts (a) and (b), we can find

$x, y \in V(G) \setminus \{a, b, t\}$ such that $\alpha \in \bar{\varphi}(x)$ and $y \in \bar{\varphi}(y)$. Clearly, x and y satisfy the property in (*). This completes the proof. $\square$

4. Proof of Lemma 2.8. In this section, we prove Lemma 2.8. We start with some notations. Let G be a graph and $\varphi \in \mathcal{C}^k(G - e)$ for some edge $e \in E(G)$ and some integer $k \geq 0$. For all the concepts below, when we use them later on, if we omit φ , we mean the concept is defined with respect to the current edge coloring.

Let $x, y \in V(G)$ and $\alpha, \beta, \gamma \in [1, k]$ be three colors. Let P be an (α, β) -chain of G with respect to φ that contains both x and y . If P is a path, denote by $P_{[x,y]}(\alpha, \beta, \varphi)$ the subchain of P that has endvertices x and y . By *swapping colors* along $P_{[x,y]}(\alpha, \beta, \varphi)$, we mean exchanging the two colors α and β on the path $P_{[x,y]}(\alpha, \beta, \varphi)$.

Define $P_x(\alpha, \beta, \varphi)$ to be an (α, β) -chain or an (α, β) -subchain of G with respect to φ that starts at x and ends at a different vertex missing exactly one of α and β . If x is an endvertex of the (α, β) -chain that contains x , then $P_x(\alpha, \beta, \varphi)$ is unique. Otherwise, we take one segment of the whole chain to be $P_x(\alpha, \beta, \varphi)$. We will specify the segment when it is used.

If u is a vertex on $P_x(\alpha, \beta, \varphi)$, we write $u \in P_x(\alpha, \beta, \varphi)$, and if uv is an edge on $P_x(\alpha, \beta, \varphi)$, we write $uv \in P_x(\alpha, \beta, \varphi)$. If $u, v \in P_x(\alpha, \beta, \varphi)$ such that u lies between x and v on the path, then we say that $P_x(\alpha, \beta, \varphi)$ *meets u before v* . Suppose the current color of an edge uv of G is α ; the notation $uv : \alpha \rightarrow \beta$ means to recolor the edge uv using the color β . Suppose that $\alpha \in \bar{\varphi}(x)$ and $\beta, \gamma \in \varphi(x)$. An $(\alpha, \beta) - (\beta, \gamma)$ -swap at x consists of two operations: first swap colors on $P_x(\alpha, \beta, \varphi)$ to get an edge k -coloring φ' , and then swap colors on $P_x(\beta, \gamma, \varphi')$. By convention, an (α, α) -swap at x does nothing at x .

Let $\alpha, \beta, \gamma, \tau, \eta \in [1, k]$. We will use a matrix with two rows to denote a sequence of operations taken on φ . Each entry in the first row represents a path, and each entry in the second row indicates the action taken on the object above it. We require the operations to be taken to follow the “left to right” order as they appear in the matrix. For example, the matrix below indicates three sequential operations taken on the graph based on the coloring from the previous step:

$$\begin{bmatrix} P_{[a,b]}(\alpha, \beta) & rs & ab \\ \alpha/\beta & \gamma \rightarrow \tau & \eta \end{bmatrix}.$$

Step 1. Swap colors on the (α, β) -subchain $P_{[a,b]}(\alpha, \beta, \varphi)$.

Step 2. Do $rs : \gamma \rightarrow \tau$.

Step 3. Color the edge ab using color η .

We will need the following result on Kierstead paths with 5 vertices in proving Lemma 2.8. General properties on Kierstead paths with 5 vertices were proven by the first author of this paper [1]. Here we stress only one of the cases.

LEMMA 4.1. *Let G be a class 2 graph, $ab \in E(G)$ be a critical edge, $\varphi \in \mathcal{C}^\Delta(G - ab)$, and $K = (a, ab, b, bu, u, us, s, st, t)$ be a Kierstead path with respect to ab and φ . If $|\bar{\varphi}(t) \cap (\bar{\varphi}(a) \cup \bar{\varphi}(b))| \geq 3$, then the following holds.*

(a) *There exists $\varphi^* \in \mathcal{C}^\Delta(G - ab)$ satisfying the following properties:*

- (i) $\varphi^*(bu) \in \bar{\varphi}^*(a) \cap \bar{\varphi}^*(t)$,
- (ii) $\varphi^*(us) \in \bar{\varphi}^*(b) \cap \bar{\varphi}^*(t)$, and
- (iii) $\varphi^*(st) \in \bar{\varphi}^*(a)$.

(b) $d(b) = d(u) = \Delta$.

Figure 1 shows a Kierstead path with the properties described in (a), where the dashed lines represent missing colors.

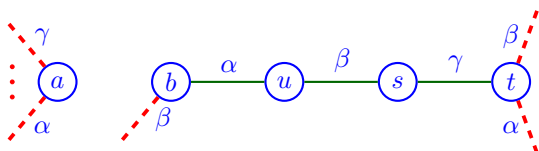


FIG. 1. Colors on a Kierstead path of 5 vertices.

Proof. We prove (a). By Lemma 2.2, for every $\varphi' \in \mathcal{C}^\Delta(G - ab)$, $\{a, b\}$ is φ' -elementary, and for every $i \in \overline{\varphi}'(a)$ and $j \in \overline{\varphi}'(b)$, a and b are (i, j) -linked with respect to φ' .

Let $\Gamma = \overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$ and $\alpha, \beta \in \Gamma$ be distinct. If $\alpha, \beta \in \overline{\varphi}(a)$, then we take $\lambda \in \overline{\varphi}(b)$ and do a (β, λ) -swap at b . If $\alpha, \beta \in \overline{\varphi}(b)$, then we take $\lambda \in \overline{\varphi}(a)$ and do a (β, λ) -swap at a . Therefore, we may assume that

$$\alpha \in \overline{\varphi}(a) \quad \text{and} \quad \beta \in \overline{\varphi}(b).$$

If $\varphi(bu) = \tau \neq \alpha$, which implies $\tau \in \overline{\varphi}(a)$, then we do a (β, τ) -swap and then an (α, β) -swap at t and rename the color τ as α and vice versa. Thus we may assume

$$\varphi(bu) = \alpha.$$

Assume first that $\varphi(us) \in \overline{\varphi}(b)$, and let $\varphi(us) = \tau$. If $us \notin P_t(\beta, \tau)$, we do a (β, τ) -swap at t and still call the resulting coloring φ ; we see that $\tau \in \overline{\varphi}(b) \cap \overline{\varphi}(t)$. By exchanging the colors β and τ , we have $\varphi(us) = \beta$. If $us \in P_t(\beta, \tau)$, we do a (β, τ) -swap at t and further do an (α, β) -swap and then an (α, τ) -swap at t , which gives a new coloring, still calling it φ , such that $\varphi(us) = \beta$. Let $\varphi(st) = \gamma$. Since $\alpha, \beta \in \overline{\varphi}(t)$, $\gamma \neq \alpha, \beta$. If $\gamma \in \overline{\varphi}(a)$, we are done. So we assume $\gamma \in \overline{\varphi}(b) \cup \overline{\varphi}(u)$. We color ab by α and uncolor bu . Denote this resulting coloring by φ' . Then $K' = (b, bu, u, us, s, st, t)$ is a Kierstead path with respect to bu and φ' . However, $\alpha, \beta \in \overline{\varphi}'(t) \cap (\overline{\varphi}'(b) \cup \overline{\varphi}'(u))$, showing a contradiction to Lemma 2.4(b).

Thus we let $\varphi(us) = \tau \in \overline{\varphi}(a)$. Then $\tau \neq \beta$ by Lemma 2.2(a). Let $\varphi(st) = \gamma$. Clearly, $\gamma \neq \alpha, \beta, \tau$. We have either $\gamma \in \overline{\varphi}(a)$ or $\gamma \in \overline{\varphi}(b) \cup \overline{\varphi}(u)$. We consider three cases below.

Case 1: $\gamma \in \overline{\varphi}(b)$. If $u \in P_a(\beta, \tau) = P_b(\beta, \tau)$, we do a (β, τ) -swap at t . Since a and b are (γ, τ) -linked and $u \in P_t(\gamma, \tau)$, we do a (γ, τ) -swap at a . By renaming colors, this gives a desired coloring φ^* .

If $u \notin P_a(\beta, \tau) = P_b(\beta, \tau)$, we first do a (β, τ) -swap at a and then a (β, γ) -swap at a . Again this gives a desired coloring φ^* after renaming colors.

Case 2: $\gamma \in \overline{\varphi}(u)$. If $\tau \in \Gamma$, since b and u are (β, γ) -linked by Lemma 2.2(b), we do (β, γ) -swap at t . Now $u \in P_t(\beta, \tau)$, we do a (β, τ) -swap at a . This gives a desired coloring φ^* . Thus we assume $\tau \notin \Gamma$. Since b and u are (β, γ) -linked by Lemma 2.2(b) and a and u are (γ, τ) -linked by Lemma 2.3(a), we do $(\beta, \gamma) - (\gamma, \tau)$ -swaps at t . Finally, since $u \in P_t(\beta, \tau)$, we do a (β, τ) -swap at a . This gives a desired coloring φ^* .

Case 3: $\gamma \in \overline{\varphi}(a)$. If $\tau \in \Gamma$, we do a (β, γ) -swap at t and then a (β, τ) -swap at a to get a desired coloring φ^* . Thus we assume $\tau \notin \Gamma$. Let $\lambda \in \Gamma \setminus \{\alpha, \beta\}$. If $\lambda \in \overline{\varphi}(u)$, since a and u are (τ, λ) -linked by Lemma 2.3(a), we do a (τ, λ) -swap at t . This gives back to the previous case that $\tau \in \Gamma$. Next we assume $\lambda \in \overline{\varphi}(b)$. It is clear that $u \in P_a(\tau, \lambda) = P_b(\tau, \lambda)$, as otherwise, a (τ, λ) -swap at a gives a desired coloring. Thus we do a (τ, λ) -swap at t , giving back to the previous case that $\tau \in \Gamma$.

Now we assume $\lambda \in \overline{\varphi}(a)$. If $u \notin P_a(\beta, \tau)$, we do a (β, τ) -swap at a . Since a and b are (α, τ) -linked and $u \in P_a(\alpha, \tau)$, we do an (α, τ) -swap at t . Now since $u \in P_t(\gamma, \tau)$, we do a (γ, τ) -swap at a and do $(\beta, \gamma) - (\gamma, \alpha)$ -swaps at t . Since a and b are (γ, λ) -linked, we do a (γ, λ) -swap at t and then a (β, γ) -swap at a . Now since $u \in P_t(\beta, \tau)$, we do a (β, τ) -swap at a . This gives a desired coloring. Thus, we assume $u \in P_a(\beta, \tau)$. We do a (β, τ) -swap at t and then a (β, λ) -swap at t . Next we do a (β, γ) -swap at a and then a (γ, τ) -swap at a . This gives a desired coloring.

For statement (b), let $\varphi^* \in \mathcal{C}^\Delta(G - ab)$ satisfying (i)–(iii). Let $\alpha, \gamma \in \overline{\varphi}^*(a)$, $\beta \in \overline{\varphi}^*(b)$ with $\alpha, \beta \in \overline{\varphi}(t)$ such that

$$\varphi^*(bu) = \alpha, \quad \varphi^*(us) = \beta, \quad \text{and} \quad \varphi^*(st) = \gamma.$$

Let $\tau \in (\overline{\varphi}^*(t) \cap (\overline{\varphi}^*(a) \cup \overline{\varphi}^*(b))) \setminus \{\alpha, \beta\}$. Suppose to the contrary first that $d(b) \leq \Delta - 1$. Let $\lambda \in \overline{\varphi}^*(b) \setminus \{\beta\}$. We do $(\tau, \lambda) - (\lambda, \gamma)$ -swaps at t . Now we color ab by α and uncolor bu to get a coloring φ' . Then $K' = (b, bu, u, us, s, st, t)$ is a Kierstead path with respect to bu and φ' . However, $\alpha, \beta \in \overline{\varphi}'(t) \cap (\overline{\varphi}'(b) \cup \overline{\varphi}'(u))$, contradicting Lemma 2.4(b).

Assume then that $d(b) = \Delta$ and $d(u) \leq \Delta - 1$. Let $\lambda \in \overline{\varphi}^*(u)$. Since (a, ab, b, bu, u) is a multifan, $\lambda \notin \{\alpha, \beta, \gamma\}$. Since u and b are (β, λ) -linked and u and a are (γ, λ) -linked by Lemma 2.3(b), we do $(\beta, \lambda) - (\lambda, \gamma)$ -swap(s) at t . Now we color ab by α and uncolor bu to get a coloring φ' . Then $K' = (b, bu, u, us, s, st, t)$ is a Kierstead path with respect to bu and φ' . However, $\alpha \in \overline{\varphi}'(t) \cap \overline{\varphi}'(u)$, which contradicts Lemma 2.4(a) since $d(u) < \Delta$. $\square$

We are now ready to prove Lemma 2.8.

LEMMA 2.8. *Let G be a class 2 graph, $H \subseteq G$ be a kite with $V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\}$, and $\varphi \in \mathcal{C}^\Delta(G - ab)$. Suppose*

$$K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, us_2, s_2, s_2t_2, t_2)$$

are two Kierstead paths with respect to ab and φ . If $\varphi(s_1t_1) = \varphi(s_2t_2)$, then $|\overline{\varphi}(t_1) \cap \overline{\varphi}(t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \leq 4$.

Proof. Let $\Gamma = \overline{\varphi}(t_1) \cap \overline{\varphi}(t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$. Assume to the contrary that $|\Gamma| \geq 5$. By considering K and applying Lemma 4.1, we conclude that $d(b) = d(u) = \Delta$. We show that there exists $\varphi^* \in \mathcal{C}^\Delta(G - ab)$ satisfying the following properties:

- (i) $\varphi^*(bu), \varphi^*(cu), \varphi^*(us_2) \in \overline{\varphi}^*(a) \cap \overline{\varphi}^*(t_1) \cap \varphi^*(t_2)$,
- (ii) $\varphi^*(us_1) \in \overline{\varphi}^*(b) \cap \overline{\varphi}^*(t_1) \cap \varphi^*(t_2)$, and
- (iii) $\varphi^*(s_1t_1) = \varphi^*(s_2t_2) \in \overline{\varphi}^*(a)$.

See Figure 2 for a depiction of the colors described above.

Let $\alpha, \beta, \tau, \eta \in \Gamma$ be distinct, and let $\varphi(s_1t_1) = \varphi(s_2t_2) = \gamma$. We may assume that $\alpha \in \overline{\varphi}(a)$ and $\beta \in \overline{\varphi}(b)$. Otherwise, since $d(b) = \Delta$, we have $\alpha, \beta \in \overline{\varphi}(a)$. Let $\lambda \in \overline{\varphi}(b)$. As a and b are (β, λ) -linked, we do a (β, λ) -swap at b . Note that this operation may change some colors of the edges of K and K^* , but they are still Kierstead paths with respect to ab and the current coloring. Note that $\varphi(ac) = \beta$ since $\overline{\varphi}(b) = \{\beta\}$ and K^* is a Kierstead path.

Since $d(b) = d(u) = \Delta$ and $\beta \in \overline{\varphi}(b) \cap \overline{\varphi}(t_1)$, we know that $\gamma \in \overline{\varphi}(a)$, as K_1 is a Kierstead path. Next, we may assume that $\varphi(bu) = \alpha$. If not, let $\varphi(bu) = \alpha'$. Since a and b are (α, β) -linked, we do an (α, β) -swap at b . Now a and b are (α, α') -linked, we do an (α, α') -swap at b . Finally, we do an (α', β) -swap at b . All these swaps do not change the colors in Γ , so now we get the color on bu to be α .

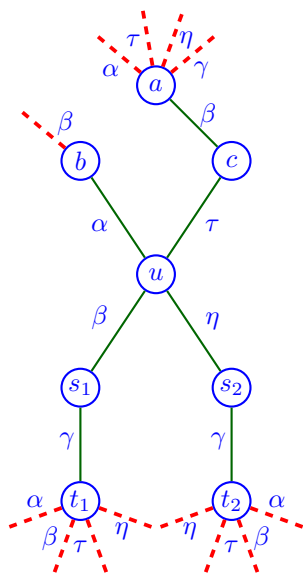


FIG. 2. Colors on the edges of a kite.

We may now assume that $\varphi(cu) = \tau$. If not, let $\varphi(cu) = \tau'$. Then $\tau' \in \overline{\varphi}(a)$. Since a and b are (β, τ) -linked, we do a (β, τ) -swap at b . Then do $(\tau, \tau') - (\tau', \beta)$ -swaps at b .

Finally, we show that we can modify φ to get φ' such that $\varphi'(us_1) = \beta$ and $\varphi'(us_2) = \eta$. Let $\lambda \in \Gamma \setminus \{\alpha, \beta, \tau, \eta\}$. Assume firstly that $\varphi(us_1) = \beta' \neq \beta$. Then $\beta' \in \overline{\varphi}(a)$ as $\overline{\varphi}(b) = \{\beta\}$ and $\overline{\varphi}(u) = \emptyset$. If $\beta' \in \Gamma$, we do $(\beta, \gamma) - (\gamma, \beta')$ -swaps at b . Thus, we assume $\beta' \notin \Gamma$. If $u \notin P_a(\beta, \beta') = P_b(\beta, \beta')$, we simply do a (β, β') -swap at b . Thus, we assume $u \in P_a(\beta, \beta') = P_b(\beta, \beta')$. We do a (β, β') -swap at both t_1 and t_2 . Since a and b are (β, λ) -linked, we do a (β, λ) -swap at both t_1 and t_2 . Now we do $(\beta, \gamma) - (\gamma, \beta')$ -swaps at b . By switching the role of β and β' , we have $\varphi(us_1) = \beta$.

Lastly, we show that $\varphi(us_2) = \eta$. Assume otherwise that $\varphi(us_2) = \eta'$. By coloring ab with the color β and uncoloring ac , we have $d(c) = \Delta$ by Lemma 4.1. Thus $\eta' \in \overline{\varphi}(a)$. Note that $bu \in P_{t_1}(\alpha, \gamma)$. Otherwise, let φ' be obtained from φ by doing an (α, γ) -swap at t_1 . Then $P_b(\alpha, \beta) = bus_1t_1$, showing a contradiction to the fact that a and b are (α, β) -linked with respect to φ' . Thus, $bu \in P_{t_1}(\alpha, \gamma)$. Next, we claim that $P_{t_1}(\alpha, \gamma)$ meets u before b . As otherwise, we do the following operations to get a Δ -coloring of G :

$$\begin{bmatrix} s_1t_1 & P_{[s_1, b]}(\alpha, \gamma) & us_1 & bu & ab \\ \gamma \rightarrow \beta & \alpha/\gamma & \beta \rightarrow \alpha & \alpha \rightarrow \beta & \gamma \end{bmatrix}.$$

This gives a contradiction to the assumption that G is Δ -critical. Thus, we have that $P_{t_1}(\alpha, \gamma)$ meets u before b . This implies that it is not the case that $P_{t_2}(\alpha, \gamma)$ meets u before b . In turn, this implies that $u \in P_a(\beta, \eta') = P_b(\beta, \eta')$. As otherwise, we get a Δ -coloring of G by doing a (β, η') -swap along the (β, η') -chain containing u and then by doing the same operation as above with t_2 playing the role of t_1 .

Since $u \in P_a(\beta, \eta') = P_b(\beta, \eta')$, we do a (β, η') -swap at both t_1 and t_2 . As $u \in P_a(\beta, \tau) = P_b(\beta, \tau)$, we do a (β, τ) -swap at both t_1 and t_2 . Since $us_1 \in P_{t_1}(\beta, \gamma)$, we do a (β, γ) -swap at b , then a (γ, λ) -swap at b . Since a and b are (τ, λ) -linked, we do

a (τ, λ) -swap at both t_1 and t_2 . Now $(\lambda, \eta) - (\eta, \gamma) - (\gamma, \beta)$ -swaps at b give a desired coloring.

Still, by the same arguments as above, we have that $P_{t_1}(\alpha, \gamma)$ meets u before b , and $u \in P_a(\beta, \eta) = P_b(\beta, \eta)$. Let $P_u(\beta, \eta)$ be the (β, η) -chain starting at u not including the edge us_2 . It is clear that $P_u(\beta, \eta)$ ends at either a or b . We may assume that $P_u(\beta, \eta)$ ends at a . Otherwise, we color ab by β , uncolor ac , and let τ play the role of α . Let $P_u(\alpha, \gamma)$ be the (α, γ) -chain starting at u not including the edge bu , which ends at t_1 by our earlier argument. We do the following operations to get a Δ -coloring of G :

$$\begin{bmatrix} P_u(\alpha, \gamma) & bu & P_u(\beta, \eta) & us_2t_2 & ab \\ \alpha/\gamma & \alpha \rightarrow \beta & \beta/\eta & \eta/\gamma & \alpha \end{bmatrix}.$$

This gives a contradiction to the assumption that G is Δ -critical. The proof is now finished. $\square$

Acknowledgments. The authors are extremely grateful to the two anonymous referees for their careful reading and valuable comments.

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