Selecta Mathematica (2022) 28:71
https://doi.org/10.1007/s00029-022-00785-z

Selecta Mathematica
New Series

Two-dimensional topological theories, rational functions and their tensor envelopes

Mikhail Khovanov[1] · Victor Ostrik[2,3] · Yakov Kononov[1,4,5]

Accepted: 19 May 2022 / Published online: 8 July 2022

Abstract

We study generalized Deligne categories and related tensor envelopes for the universal two-dimensional cobordism theories described by rational functions, recently defined by Sazdanovic and one of the authors.

Mathematics Subject Classification 18M10 · 57R56 · 18M25 · 17B10 · 05A18 · 81R05

Contents

✉ Yakov Kononov
ya.kononoff@gmail.com

Mikhail Khovanov
khovanov@math.columbia.edu

Victor Ostrik
ostrik@uoregon.edu

1 Department of Mathematics, Columbia University, New York, NY 10027, USA

2 Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

3 Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow, Russia

4 Department of Mathematics, Yale University, New Haven, CT 06511, USA

5 Kharkevich Institute for Information Transmission Problems, Moscow 127994, Russia

1 Introduction

Throughout the paper we work over a field $\mathbf{k}$, occasionally specializing to a characteristic zero field. We will often consider $\mathbf{k}-$linear symmetric monoidal categories and call them *tensor categories*. Note that this is different from the convention in [16], where tensor categories are assumed to be abelian.

The Deligne category $\mathrm{Rep}(S_t)$, where t ranges over elements of $\mathbf{k}$, interpolates between categories of finite-dimensional representations $\mathrm{Rep}(S_n)$ of the symmetric group S_n over $\mathbf{k}$, for various n [8, 10]. It is a rigid symmetric monoidal Karoubi-closed $\mathbf{k}$-linear category that depends on a parameter $t \in \mathbf{k}$. The Deligne category is equipped with a natural symmetric trace form tr that allows to form the ideal $J_t \in \mathrm{Rep}(S_t)$ of negligible morphisms. A morphism $x : a \to b$ is negligible if for any $y : b \to a$ the trace $\mathrm{tr}(yx) = 0$. The ideal J_t is non-trivial only when $t = n$ is a non-negative integer, and the quotient category

$$\underline{\mathrm{Rep}}(S_n) \ := \ \mathrm{Rep}(S_n)/J_n \tag{1}$$

is equivalent to the category of symmetric group representations in characteristic 0, with some modifications needed in characteristic p [20]. For the other values of t the ideal J_t is trivial and $\underline{\mathrm{Rep}}(S_t)$ is equivalent to $\mathrm{Rep}(S_t)$.

As observed by Comes [7], there is a functor from the category Cob_2 of oriented two-dimensional cobordisms between one-manifolds to the Deligne category. Modifications of this functor, coupled with the universal construction of two-dimensional topological theories [3, 22], lead to generalizations of the Deligne category $\mathrm{Rep}(S_t)$ and of its quotient $\underline{\mathrm{Rep}}(S_t)$ by the negligible ideal [29].

Objects in Cob_2 are non-negative integers $n \in \mathbb{Z}_+$ and morphisms from n to m are oriented two-dimensional cobordisms from the union of n circles to the union of m circles, up to rel boundary diffeomorphisms [29].

Working over $\mathbf{k}$, choose a rational function and its power series expansion (the series of α)

$$Z_\alpha(T) = \frac{P(T)}{Q(T)} = \sum_{n\geq 0} \alpha_n T^n, \ \alpha_n \in \mathbf{k}, \ \alpha = (\alpha_0, \alpha_1, \ldots), \tag{2}$$

where polynomials $P(T), Q(T) \in \mathbf{k}[T]$ are relatively prime and

$$N = \deg P(T), \ M = \deg Q(T), \ K := \max(N+1, M). \tag{3}$$

We may also write $P_\alpha(T)$ and $Q_\alpha(T)$ in lieu of $P(T)$ and $Q(T)$ to emphasize dependence on α, and normalize so that

$$Z_\alpha(T) = \frac{P_\alpha(T)}{Q_\alpha(T)} = \sum_{n\geq 0} \alpha_n T^n, \quad (P_\alpha(T), Q_\alpha(T)) = 1, \quad Q_\alpha(0) = 1. \tag{4}$$

With this normalization, $P_\alpha(T)$ and $Q_\alpha(T)$ are uniquely determined by α. We may refer to α as a (rational) sequence and to $Z_\alpha(T)$ as its associated series. To keep track of α, we may also denote

$$N_\alpha = \deg P_\alpha(T), \ M_\alpha = \deg Q_\alpha(T), \ K_\alpha := \max(N_\alpha + 1, M_\alpha) \tag{5}$$

in place of (3).

One can "linearize" the category Cob_2 of two-dimensional cobordisms using the sequence α. To do that, first allow linear combinations of cobordisms with coefficients in $\mathbf{k}$ and also evaluate a closed connected oriented surface of genus g, when it is a component of a cobordism, to $\alpha_g \in \mathbf{k}$. This results in the $\mathbf{k}$-linear tensor category VCob_α, which has the same objects $n \in \mathbb{Z}_+$ as Cob_2 (this category is denoted Cob'_α in [29]).

In the notation VCob_α letter V stands for *viewable* or *visible*. A 2D cobordism x is called *viewable* or *visible* if it has no closed components, that is, any connected component of x has non-empty boundary. The above evaluation of closed components allows to reduce a morphism from n to m to a linear combination of viewable cobordisms. Thus, morphisms in VCob_α are $\mathbf{k}$-linear combinations of viewable cobordisms, with the composition of morphisms given by composition of cobordisms and the above α-*evaluation* applied to all closed components of the composition. Note that hom spaces in VCob_α are infinite-dimensional, since a component of a cobordism can have any number of handles.

A further reduction in the size of the category is given by considering the ideal $J_\alpha \subset \mathrm{VCob}_\alpha$ of negligible morphisms, relative to the trace form tr_α associated with α and forming the quotient category

$$\mathrm{Cob}_\alpha := \mathrm{VCob}_\alpha / J_\alpha. \tag{6}$$

In this paper we call Cob_α the *gligible quotient* of VCob_α. We choose this terminology over more cumbersome *non-negligible quotient* and over *radical quotient*, for the latter may be somewhat ambiguous.

The trace form is given on a cobordism y from n to n by closing it via n annuli connecting n top and n bottom circles of the boundary of y into a closed oriented surface $\widehat{y}$ and applying α,

$$\mathrm{tr}_\alpha(y) := \alpha(\widehat{y}), \tag{7}$$

see formula (11) in [29]. It is shown in [29] that the hom spaces in Cob_α are finite-dimensional over $\mathbf{k}$ iff the generating function $Z_\alpha(T)$ is rational, see formula (2), although the category Cob_α is defined for any sequence α.

Notation

$$A_\alpha(n) := \mathrm{Hom}_{\mathrm{Cob}_\alpha}(0, n) \tag{8}$$

is used in [22] to denote the *state space* of n circles in the theory associated to α. Vector space $A_\alpha(n)$ can be described as the quotient of the $\mathbf{k}$-vector space with a basis $\{[S]\}_S$ of viewable cobordisms S with n boundary circles modulo skein relations that hold in the evaluation α for any closure of these n circles by a cobordism T on the other size, so that TS are closed surfaces and evaluations $\alpha(TS)$ make sense as elements of $\mathbf{k}$. These state spaces are finite-dimensional for rational sequences α.

Category Cob_α is a symmetric monoidal $\mathbf{k}$-linear category with objects $n \in \mathbb{Z}_+$ and finite-dimensional hom spaces (recall the assumption that α is rational). One can form the additive Karoubi closure

$$\underline{\mathrm{DCob}}_\alpha := \mathrm{Kar}(\mathrm{Cob}_\alpha^\oplus) \tag{9}$$

by first allowing formal finite direct sums of objects in Cob_α and extending morphisms correspondingly to get the finite additive closure $\mathrm{Cob}_\alpha^\oplus$, then adding idempotents to get a Karoubi-closed category.

Category $\underline{\mathrm{DCob}}_\alpha$ is the analogue of the gligible quotient of the Deligne category. It is a $\mathbf{k}$-linear additive idempotent-complete rigid symmetric monoidal category with finite-dimensional hom spaces. It carries nondegenerate bilinear pairings on its hom spaces

$$\mathrm{Hom}(a, b) \otimes \mathrm{Hom}(b, a) \longrightarrow \mathbf{k}.$$

This category is the analogue of the category $\underline{\mathrm{Rep}}(S_t)$ above given by modding out the Deligne category $\mathrm{Rep}(S_t)$ by the ideal of negligible morphisms.

To recover the analogue of the Deligne category itself, one needs to insert an intermediate category SCob_α (*skein cobordisms*) into the chain of categories and functors below, in between categories VCob_α and Cob_α:

$$\mathrm{Cob}_2 \longrightarrow \mathrm{VCob}_\alpha \longrightarrow \mathrm{SCob}_\alpha \longrightarrow \mathrm{Cob}_\alpha \longrightarrow \underline{\mathrm{DCob}}_\alpha. \tag{10}$$

In this additional intermediate step, instead of modding out by all negligible morphisms, one can first form a skein relation (the *handle relation*) quotient of VCob_α

which reduces a component with K handles (K is given in formula (3)) to a linear combination of components with fewer handles, using the relation

$$x^K - b_1\, x^{K-1} + b_2\, x^{K-2} - \ldots + (-1)^M b_M\, x^{K-M} = 0, \tag{11}$$

where x denotes addition of a handle to a component, and x^m stands for adding m handles.

Polynomial in the left hand side of (11) is the minimal polynomial of the handle endomorphism x of the circle in the category Cob_α. We call it *the handle polynomial* of α and denote

$$U_\alpha(x) \ := \ x^K - b_1\, x^{K-1} + b_2\, x^{K-2} - \ldots + (-1)^M b_M\, x^{K-M}. \tag{12}$$

The handle polynomial is monic. We use x to denote the handle operator, use formal variable T in the power series, and switch between x and T as convenient throughout the paper.

Coefficients b_i of the handle polynomial are the coefficients of the polynomial in the denominator of (2) in the reverse order:

$$Q(T) = 1 - b_1 T + b_2 T^2 + \ldots + (-1)^M b_M T^M, \ b_i \in \mathbf{k}, \tag{13}$$

also see [22, Sect. 2.4]. We have

$$U_\alpha(x) = x^K Q(1/x) = x^{K-M}(x^M - b_1\, x^{M-1} + b_2\, x^{K-2} - \ldots + (-1)^M b_M). \tag{14}$$

Recall that denominator $Q(T)$ is normalized to have constant term 1, so that $Q(0) = 1$. Since $Q(T)$ is a denominator of the power series $Z_\alpha(T)$, it has a non-zero constant term, which is rescaled to 1. Note that changing $Z_\alpha(T)$ to $\lambda Z_\alpha(T)$ for $\lambda \in \mathbf{k}^*$ does not change the above skein relation, but does change the evaluation and the resulting categories VCob_α, Cob_α, and $\underline{\mathrm{DCob}}_\alpha$.

When $P(T)/Q(T)$ in (2) is a *proper* fraction, that is $\deg P(T) < \deg Q(T)$, then $K = M$ and $U_\alpha(0) = (-1)^M b_M \neq 0$.

We denote by SCob_α the quotient of VCob_α by the handle relation (11). This category is denoted PCob_α in [29]. The quotient of SCob_α by the ideal of negligible morphisms (the gligible quotient) is naturally isomorphic to Cob_α (isomorphic and not only equivalent, since objects of these categories are non-negative integers).

We see that SCob_α has a place in (10) as an intermediate category between the two categories in the middle. The additive Karoubi envelope of SCob_α is denoted by DCob_α. It is the analogue of the Deligne category. There is a natural equivalence between the category obtained from SCob_α by first forming the additive Karoubi closure and then modding out by negligible morphisms (two consecutive right arrows then the down arrow in the square below) and the category obtained from SCob_α by first forming the quotient by negligible morphisms and then passing to the additive Karoubi envelope (down arrow followed by two right arrows in the square below).

We summarize the resulting collection of categories and functors between them in the following diagram, with the square commutative.

$$
\begin{array}{ccccccccc}
\mathrm{Cob}_2 & \longrightarrow & \mathrm{VCob}_\alpha & \longrightarrow & \mathrm{SCob}_\alpha & \longrightarrow & \mathrm{SCob}_\alpha^\oplus & \longrightarrow & \mathrm{DCob}_\alpha \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \mathrm{Cob}_\alpha & \longrightarrow & \mathrm{Cob}_\alpha^\oplus & \longrightarrow & \underline{\mathrm{DCob}}_\alpha
\end{array}
\tag{15}
$$

The four rightmost categories are additive, the three categories to the left of them are **k**-linear and pre-additive (category Cob_2 is neither pre-additive nor **k**-linear). All eight categories are rigid symmetric monoidal. The six categories on the right each have finite-dimensional hom spaces. The bottom three categories are gligible quotients of the respective categories above them (that is, quotients by the ideals of negligible morphisms), and their hom spaces carry non-degenerate bilinear forms. The table below provides brief summaries for most of these categories.

Notation	Category
Cob_2	Oriented 2D cobordisms
VCob_α	Viewable cobordisms; evaluate closed components via α
SCob_α	"Skein" category; quotient of VCob_α by the handle relation (11)
Cob_α	"Gligible" quotient of SCob_α by the kernels of trace forms (equivalently, by the negligible ideal)
DCob_α	Deligne category; additive Karoubi completion of the skein category SCob_α
$\underline{\mathrm{DCob}}_\alpha$	Gligible quotient of the Deligne category; equivalent to the additive Karoubi completion of Cob_α

Category DCob_α is the analogue of the Deligne category $\mathrm{Rep}(S_t)$ and specializes to it when the sequence α is constant,

$$
\alpha(t) = (t, t, \ldots), \quad Z_{\alpha(t)} = \frac{t}{1-T}, \ t \in \mathbf{k}. \tag{16}
$$

Category $\underline{\mathrm{DCob}}_\alpha$ is the analogue of the quotient $\underline{\mathrm{Rep}}(S_t)$ of $\mathrm{Rep}(S_t)$ by negligible morphisms. It specializes to $\underline{\mathrm{Rep}}(S_t)$ when α is the constant sequence $\alpha(t)$.

In this paper we study generalized Deligne categories DCob_α, their quotients $\underline{\mathrm{DCob}}_\alpha$ as well as categories SCob_α and Cob_α for other rational series α. We refer to these categories as *tensor envelopes* of α.

For particular key rational generating functions $Z_\alpha(T)$ we establish or recall the connection between tensor envelopes of α and the known representation categories:

- Generating function $\beta/(1-\lambda T)$ relates to the Deligne category of representations of symmetric group $\mathrm{Rep}(S_t)$, $t = \beta\gamma$, see [29] and Sect. 6.1. For these series α the category DCob_α is equivalent to $\mathrm{Rep}(S_t)$, inducing an equivalence of gligible quotient categories as well, $\underline{\mathrm{DCob}}_\alpha \cong \underline{\mathrm{Rep}}(S_t)$.

- Tensor envelopes for the constant generating function $Z(T) = \beta$, $\beta \in \mathbf{k}$, relate to the representation category of the Lie algebra $osp(1|2)$, see Sect. 5 and Theorem 5.5.
- Categories for the linear generating function $\beta_0+\beta_1 T$ relate to the Deligne category $\mathrm{Rep}(O_t)$ for the orthogonal group, also known as the unoriented Brauer category, and to its gligible quotients, Sect. 7.1.

Below is a brief summary, section by section, of the constructions and results in the paper.

- In Sect. 2 we discuss basic properties of tensor envelopes.
  - Section 2.1 points out that the scaling $Z(T) \longrightarrow \lambda^{-1}Z(\lambda T)$ for an invertible $\lambda = \mu^2$ does not change the categories we consider.
  - In Sect. 2.2 we explain that any commutative Frobenius algebra object in a pre-additive tensor category gives rise to a power series α with coefficients in the commutative ring $\mathrm{End}(\mathbf{1})$ of endomorphisms of the unit object.
  - In Sect. 2.3 we recall the universal property of Cob_α and Proposition 2.4.
  - Section 2.4 studies direct sum decompositions of commutative Frobenius algebra objects mirroring partial fraction decompositions of their rational generating series.

- Section 3 contains key semisimplicity and abelian realization criteria for the tensor envelopes of α, including Theorems 3.2 and 3.7. In particular, we classify series α with the semisimple category $\underline{\mathrm{DCob}}_\alpha$.
- Section 4 reviews properties of the endomorphism ring of the one-circle object in categories SCob_α and Cob_α.
- Section 5 describes the structure of the gligible quotient category Cob_α for the constant function (series $\alpha = (\beta, 0, 0, \dots)$). Theorem 5.1 states that the dimension of the state space $A(n)$ of n circles for this function is the Catalan number, for $\mathbf{k}$ of characteristic 0. A monoidal equivalence between the Karoubi envelope $\underline{\mathrm{DCob}}_\alpha$ of Cob_α and a suitable category of finite-dimensional representations of the Lie superalgebra $osp(1|2)$ is established in Sect. 5.5.
- Section 6.1 studies Gram determinants of a natural spanning set of surfaces for the function $\beta/(1-\gamma T)$, where tensor envelopes correspond to the Deligne category. These are rank one theories. Determinant computations for various rank two theories are given in Sect. 6.2.
- Section 7 considers the case of a polynomial generating function, beyond the constant function case studied in Sect. 5. When the function is linear, associated tensor envelopes can be expressed via the unoriented Brauer category and its gligible quotient, due to the presence of a commutative Frobenius object in the Brauer category with a linear generating function, see Sect. 7.1. Section 7.2 provides numerical data for the Gram determinants in categories when the generating function is a polynomial of degree two or three. Section 7.3 considers arbitrary degree polynomials. A conjectural basis in the state space of n circles for the theory is proposed there, and some properties of the Gram determinant for that set of vectors is established.
- In Sect. 8 we explain how to enrich category Cob_2 of two-dimensional oriented cobordisms by adding codimension two defects (dots). Presence of the handle

cobordism allows one to add relations intertwining the handle cobordism with dot decorations. Going from less general to more general examples, dots may be viewed as fractional handles, elements of a commutative monoid, or elements of a commutative algebra. Theory developed in the rest of this paper should extend to least some of these generalizations.

We'd like to mention related papers [18, 33] that came out after this paper appeared on arXiv.

2 Properties of α-theories

2.1 Scaling by invertible elements

Consider a theory α over $\mathbf{k}$ with generating function $Z_\alpha(T)$ and state spaces $A_\alpha(k)$ of k circles. Choose an invertible element $\mu \in \mathbf{k}^*$, denote $\lambda = \mu^2$, and change the sequence $\alpha = (\alpha_0, \alpha_1, \dots)$ to

$$\alpha' = (\lambda^{-1}\alpha_0, \alpha_1, \lambda\alpha_2, \lambda^2\alpha_3, \dots), \tag{17}$$

that is, $(\alpha')_n = \lambda^{n-1}\alpha_n$. The generating function for α' is

$$Z_{\alpha'}(T) = \lambda^{-1}Z_\alpha(\lambda T) = \lambda^{-1}\alpha_0 + \alpha_1 T + \lambda\alpha_2 T^2 + \lambda^2\alpha_3 T^3 + \dots \tag{18}$$

Note that $Z_\alpha(T)$ and $Z_{\alpha'}(T)$ have the same linear term α_1.

Consider the $\mathbf{k}$-vector space $\underline{\mathrm{Fr}}(k)$ with a basis $\{[S]\}_S$ given by surfaces S without closed components and with $\partial S \cong \sqcup_k \mathbb{S}^1$, one for each diffeomorphism class rel boundary of such surfaces. Sequence α determines a $\mathbf{k}$-bilinear symmetric form on $\underline{\mathrm{Fr}}(k)$ with the pairing $(,)_\alpha$ given on generators by

$$([S_1], [S_2])_\alpha = \alpha((-S_1) \sqcup S_2) \tag{19}$$

and extended by linearity, where $(-S_1) \sqcup S_2$ is the closed surface given by gluing S_1 and S_2 along the common boundary. Recall that the state space

$$A_\alpha(k) \ := \ \underline{\mathrm{Fr}}(k)/\ker((,)_\alpha) \tag{20}$$

is the quotient of the free module by the kernel of this form.

Alternatively, consider the bilinear form on $\underline{\mathrm{Fr}}(k)$ given by α'. The quotient of $\underline{\mathrm{Fr}}(k)$ by the kernel of this form is the state space for α':

$$A_{\alpha'}(k) \ := \ \underline{\mathrm{Fr}}(k)/\ker((,)_{\alpha'}). \tag{21}$$

Introduce the $\mathbf{k}$-linear map

$$\phi : \underline{\mathrm{Fr}}(k) \longrightarrow \underline{\mathrm{Fr}}(k), \quad \phi([S]) = \mu^{-\chi(S)}[S]. \tag{22}$$

This map scales $[S]$ by $\mu^{-\chi(S)}$, where $\chi(S)$ is the Euler characteristic of S. It intertwines the bilinear forms on these spaces

$$([S_1],[S_2])_{\alpha'} = (\phi([S_1]), \phi([S_2]))_{\alpha} \tag{23}$$

and induces an isomorphism of vector spaces

$$\phi \ : \ A_{\alpha'}(k) \longrightarrow A_{\alpha}(k), \tag{24}$$

also denoted ϕ. This isomorphism intertwines nondegenerate R-valued bilinear forms $(,)_{\alpha'}$ and $(,)_{\alpha}$ on these spaces and shows that α and α' define equivalent topological theories as defined in [22].

On the level of categories, the scaling map ϕ in (22) induces $\mathbf{k}$-linear isomorphisms between the morphism spaces in VCob_{α} and $\mathrm{VCob}_{\alpha'}$

$$\mathrm{Hom}_{\mathrm{VCob}_{\alpha'}}(n,m) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{VCob}_{\alpha}}(n,m)$$

compatible with the composition in these categories and leading to an isomorphism of categories $\mathrm{VCob}_{\alpha'} \cong \mathrm{VCob}_{\alpha}$. This isomorphism is compatible with the various quotient and Karoubi envelope categories that follow and leads to isomorphisms or equivalences of the corresponding categories, including isomorphisms $\mathrm{SCob}_{\alpha'} \cong \mathrm{SCob}_{\alpha}$, $\mathrm{Cob}_{\alpha'} \cong \mathrm{Cob}_{\alpha}$ and equivalences $\mathrm{DCob}_{\alpha'} \cong \mathrm{DCob}_{\alpha}$, $\underline{\mathrm{DCob}}_{\alpha'} \cong \underline{\mathrm{DCob}}_{\alpha}$. We see that scaling by $\lambda = \mu^2$, $\mu \in \mathbf{k}^*$, gives isomorphic theories and isomorphic or equivalent associated categories. This scaling changes the handle relation by rescaling the handle.

2.2 Commutative Frobenius algebra objects in symmetric monoidal categories

Let $\mathcal{C}$ be a symmetric monoidal category. Let $A = (A, m, \iota) \in \mathcal{C}$ be a commutative algebra object in $\mathcal{C}$, i.e. an object $A \in \mathcal{C}$ equipped with associative and commutative multiplication $m : A \otimes A \to A$ such that $\iota : \mathbf{1} \to A$ satisfies the unit axiom, see e.g [16, Sections 7.8.1 and 8.8.1]. We say that A is a *commutative Frobenius monoid* in $\mathcal{C}$ if it is equipped with a morphism $\epsilon : A \to \mathbf{1}$ such that the composition $b : A \otimes A \xrightarrow{m} A \xrightarrow{\epsilon} \mathbf{1}$ is a non-degenerate pairing, i.e. there exists a morphism $c : \mathbf{1} \to A \otimes A$ such that the morphisms b and c satisfy the axioms of evaluation and coevaluation maps, see e.g. [16, 2.10.1]. Equivalently, the object A^* exists and the morphism $A \to A^*$ which is the image of b under the natural isomorphism $\mathrm{Hom}(A \otimes A, \mathbf{1}) \simeq \mathrm{Hom}(A, A^*)$ is an isomorphism. We will often identify A and A^* using this morphism. For example we define the comultiplication morphism $\Delta : A \to A \otimes A$ as dual to the multiplication morphism. Clearly Δ is coassociative. It is easy to see that Δ is a morphism of $A \times A^{op}$-objects. One shows that the morphism c equals to the composition of ι and Δ.

Given a Frobenius monoid $A \in \mathcal{C}$ and $n \in \mathbb{Z}_{\geq 0}$ we get a morphism $a_n : \mathbf{1} \xrightarrow{\iota} A \xrightarrow{\Delta_n} A^{\otimes n} \xrightarrow{m_n} A \xrightarrow{\epsilon} \mathbf{1}$ where $\Delta_n : A \to A^{\otimes n}$ is n-fold comultiplication and $m_n : A^{\otimes n} \to A$ is n-fold multiplication. Thus $a_0 = \epsilon\,\iota$ is the composition of ϵ and ι, and $a_1 = \epsilon\, m\, \Delta\, \iota = c\, b$ is the composition of c and b, that is the dimension of A.

Equivalently, consider the *handle* endomorphism

$$x \ : \ A \xrightarrow{\Delta} A \otimes A \xrightarrow{m} A, \tag{25}$$

which has a topological interpretation as a tube with a handle on it. Iterating this morphism yields x^n, a tube with n handles. The map a_n can be written as

$$a_n = \epsilon \, x^n \, \iota. \tag{26}$$

Elements a_n are endomorphisms of the unit object $\mathbf{1}$ of $\mathcal{C}$.

We distinguish between the *handle endomorphism* x above and the *handle morphism*. The latter is the morphism $\mathbf{1} \longrightarrow A$ given by $x\iota$. Handle morphism corresponds to a one-punctured torus, with the puncture circle on the target on the morphism, while handle endomorphism corresponds to the twice-punctured torus, with one circle on both the source and target cobordisms.

From here on we assume that $\mathcal{C}$ is a $\mathbf{k}-$linear symmetric monoidal category and the canonical map $\mathbf{k} \to \mathrm{End}(\mathbf{1})$ is an isomorphism. Then $a_n = \alpha_n \mathrm{id}_{\mathbf{1}}$ for some $\alpha_n \in \mathbf{k}$. The sequence $\alpha = (\alpha_n)_{n \in \mathbb{Z}_{\geq 0}}$ will be called $\alpha-$*evaluation* or just the *evaluation* of A.

Example 2.1 Let $\mathcal{C}$ be an additive $\mathbf{k}-$linear symmetric monoidal category and let $V \in \mathcal{C}$ be an object equipped with a non-degenerate symmetric pairing $V \otimes V \to \mathbf{1}$. We define a $\mathbb{Z}-$graded commutative Frobenius algebra $A = A(V)$ as follows: $A = A_0 \oplus A_1 \oplus A_2$ where $A_0 = A_1 = \mathbf{1}$ and $A_1 = V$; A_0 is the image of the unit morphism, the multiplication $A_1 \otimes A_1 \to A_2$ is given by the symmetric pairing, and the linear form $\epsilon : A \to \mathbf{1}$ factors through the projection $A \to A_0 \oplus A_2$ and is nonzero when restricted to A_2. It is easy to verify that the $\alpha-$evaluation of the algebra $A(V)$ has $\alpha_i = 0$ for $i > 1$; also $\alpha_1 = \dim(A) = 2 + \dim(V)$. Parameter α_0 is the composition of ϵ and the unit morphism and can be chosen to be any element of $\mathbf{k}$. The generating function is then $Z_\alpha(T) = \alpha_0 + (2 + \dim(V))T$. Possible dimensions $\dim(V)$ of such objects V depend on $\mathcal{C}$. For instance, when $\mathcal{C}$ is the category of $\mathbf{k}$-vector spaces, these dimensions belong to the image of $\mathbb{Z}_+ \in \mathbf{k}$. When $\mathcal{C}$ is the unoriented Brauer category $\mathrm{Rep}(O_t)$ with the parameter $t \in \mathbf{k}$, $\dim(V) = t$ for the standard generator V of $\mathcal{C}$, see Sect. 7.1, for instance, and references there.

Remark One can informally compare this setup with the problem of reconstructing or understanding a system from observable data on it. Here one can imagine that the system consists of an object $A \in \mathcal{C}$, handle endomorphism x of A (and, more generally, endomorphisms of A associated to arbitrary cobordisms from a circle to itself). Object A is unknown to us, but we can observe values of closed cobordisms, which are α_n for a connected genus n cobordism. Then the universal pairing construction of [22, 29] in dimension two (and its counterpart [3] in three dimensions) consists of recovering a minimal model for x and $\mathcal{C}$ from the closed cobordism data. This toy example in two dimensions can be compared to more complicated reconstructions in control theory. We probe category $\mathcal{C}$ via evaluations of closed cobordisms, which allow us to fully reconstruct it, in the universal pairing setup.

Example 2.2 Given two commutative Frobenius algebra objects A_1, A_2 in $\mathcal{C}$, their sum $A_1 \oplus A_2$ is naturally a commutative Frobenius algebra in $\mathcal{C}$. If sequences α and β are evaluations of A_1 and A_2, respectively, the evaluation of $A_1 \oplus A_2$ is the sequence $\alpha + \beta = (\alpha_n + \beta_n)_{n \in \mathbb{Z}_+}$.

Example 2.3 Hadamard product of power series α and β is the series $\alpha\beta$ with $(\alpha\beta)_n = \alpha_n \beta_n$, that is, we multiply the two series term-wise. Hadamard product of rational power series is rational [31]. The tensor product $A_1 \otimes A_2$ of commutative Frobenius algebras in $\mathcal{C}$ is naturally a commutative Frobenius algebra in $\mathcal{C}$. The evaluation of $A_1 \otimes A_2$ is the Hadamard product of evaluations of A_1 and A_2.

If char $\mathbf{k} = p$, the p-th tensor power $A^{\otimes p}$ of a commutative Frobenius algebra A has evaluation α^p equal to the application of the Frobenius endomorphism of $\mathbf{k}$ to each term of α.

2.3 Universal property

It is well known (see e.g. [40, Theorem 0.1]) that the category Cob_2 has the following universal property: for a symmetric monoidal category $\mathcal{C}$ an evaluation of symmetric monoidal functors on the circle object gives an equivalence of categories

$$\{\text{tensor functors } \mathrm{Cob}_2 \to \mathcal{C}\} \to \{\text{commutative Frobenius algebras in } \mathcal{C}\}.$$

One deduces easily a similar universal property of $\mathbf{k}\mathrm{Cob}_2$ where the categories $\mathcal{C}$ and functors are assumed to be $\mathbf{k}-$linear. Likewise,category VCob_α has the following universal property: for an $\mathbf{k}-$linear symmetric category $\mathcal{C}$ an evaluation at the circle object gives an equivalence of categories:

$$\{\mathbf{k} - \text{linear tensor functors } \mathrm{VCob}_\alpha \to \mathcal{C}\} \to \left\{ \begin{array}{c} \text{commutative Frobenius algebras in } \mathcal{C} \\ \text{with evaluation } \alpha \end{array} \right\}$$

We pick an inverse equivalence of categories and for a commutative Frobenius algebra $A \in \mathcal{C}$ we will denote by F_A the corresponding tensor functor,

$$F_A \colon \mathrm{VCob}_\alpha \to \mathcal{C}. \tag{27}$$

A sequence α is called *linearly recurrent* or *homogeneously linearly recurrent* if $\alpha_{k+n+1} = a_n \alpha_{k+n} + a_{n-1}\alpha_{k+n-1} \cdots + a_1 \alpha_{k+1}$ for all $k \geq N$ for some N and fixed $a_1, \ldots, a_n$, see [17]. In this paper we refer to such α as recurrent sequences.

Assume that the sequence α is recurrent. Functor F_A factors through the category SCob_α if and only if F_A annihilates the handle polynomial in (12). If the category $\mathcal{C}$ is Karoubian the functor F_A extends uniquely to the category DCob_α.

We will often use the following result, see [6, Lemma 2.6], specialized to $\underline{\mathrm{DCob}}_\alpha$:

Proposition 2.4 *Assume that the category $\mathcal{C}$ is a* $\mathbf{k}$*-linear additive Karoubian nondegenerate symmetric monoidal category with finite-dimensional* hom *spaces and the functor F_A satisfies the following properties:*

(1) *Any indecomposable object of* $\mathcal{C}$ *is a direct summand of* $F_A(n)$ *for some object* n *in* VCob_α.
(2) *The functor* F_A *is full (i.e. surjective on* Hom*'s).*

Then the functor F_A *induces an equivalence* $\underline{\mathrm{DCob}}_\alpha \simeq \mathcal{C}$.

Here we say that $\mathcal{C}$ is *nondegenerate* if any negligible morphism is the zero morphism between some objects.

2.4 Direct sums decompositions

Let $A \in \mathcal{C}$ be a commutative Frobenius algebra object in a $\mathbf{k}-$linear symmetric monoidal category $\mathcal{C}$. Then the multiplication in A induces a commutative algebra structure on the vector space $A_1 := \mathrm{Hom}(\mathbf{1}, A)$; this algebra acts on A via left (equivalently, right) multiplications, so we get a natural injective homomorphism

$$\phi : A_1 \to \mathrm{End}_{\mathcal{C}}(A). \tag{28}$$

Let $x_0 \in A_1$ be the handle morphism; its image $\phi(x_0)$ is the handle endomorphism of A. Let $A_0 \subset A_1$ be the unital subalgebra generated by x_0.

We assume that A_0 is finite dimensional. Thus x_0 is annihilated by a nonzero polynomial. We let $U(T) \in \mathbf{k}[T]$ be the minimal polynomial of x_0, which is assumed monic. It factors

$$U(T) = T^a \underline{U}(T), \tag{29}$$

with $\underline{U}(0) \neq 0$ and $a \geq 0$.

We recall that the idempotents $e \in A_0$ are naturally labeled by factorizations $U(T) = U_1(T)U_2(T)$ such that the factors $U_1(T)$ and $U_2(T)$ are relatively prime. Namely given such a factorization we can find $a(T), b(T) \in \mathbf{k}[T]$ such that

$$a(T)U_1(T) + b(T)U_2(T) = 1,$$

and then

$$e \ = \ a(x_0)U_1(x_0) \ \in A_0 \tag{30}$$

is an idempotent. Conversely for an idempotent $e \in A_0$ let us choose a polynomial $s(T)$ such that $e = s(x_0)$; then setting $U_1(T) = \mathrm{gcd}(U(T), s(T))$ and $U_2(T) = \mathrm{gcd}(U(T), 1 - s(T))$ we get a factorization as above.

We furthermore assume that the category $\mathcal{C}$ is Karoubian. Let $e \in A_0$ be an idempotent; then it is easy to see that the image of $\phi(e)$, see (28), is a Frobenius subalgebra of A in $\mathcal{C}$; moreover there is a decomposition

$$A = \phi(e)A \oplus \phi(1 - e)A$$

of A as a direct sum of its Frobenius subalgebras (direct sum as objects in $\mathcal{C}$ and direct product as algebras). Note that the unit elements of these subalgebras become idempotents in A.

Let us compute $\alpha-$invariants of subalgebras $\phi(e)A$ and $\phi(1-e)A$ in terms of the $\alpha-$invariant of A. Recall that the generating function of A can be written as a rational function

$$Z(T) = \frac{P(T)}{Q(T)} \tag{31}$$

with $Q(0) = 1$, where $Q(T)$ is the polynomial given by

$$Q(T) = T^d\, U(T^{-1}), \quad d = \deg(U(T)). \tag{32}$$

Here d is the degree of $U(T)$, and $Q(T)$ is the reverse polynomial of $\underline{U}(T)$. The orders of the coefficients of $Q(T)$ and $U(T)$ are reversed. Note that $Q(0) = 1$ since $U(T)$ is monic.

Example 2.5 Let $U(T) = T^5 + 9T^4 - 6T^3$ then $d = 5$, $\underline{U}(T) = T^2 + 9T - 6$ and $Q(T) = -6T^2 + 9T + 1$.

In particular, any factorization

$$U(T) = U_1(T)U_2(T) \tag{33}$$

for $U(T)$ as above into two relatively prime monic polynomials induces a factorization

$$Q(T) = Q_1(T)Q_2(T), \tag{34}$$

where $Q_1(T)$ and $Q_2(T)$ are determined from $U_1(T)$ and $U_2(T)$, respectively, in the same way as $Q(T)$ is determined by $U(T)$, via relation (32).

Since polynomials $U_1(T)$ and $U_2(T)$ are relatively prime, at most one of them is divisible by T. Thus we can and will assume that $U_2(T)$ is not divisible by T. Polynomial $Q(T)$ is divisible by T iff $a \neq 0$ in formula (29).

There is a unique partial fraction decomposition

$$Z(T) = \frac{v_1(T)}{Q_1(T)} + \frac{v_2(T)}{Q_2(T)}, \tag{35}$$

where $\deg(v_1) < \deg(Q_1)$. Denote the terms on the right hand side by $Z_1(T)$ and $Z_2(T)$, respectively, and write

$$Z(T) = Z_1(T) + Z_2(T)$$

where $v_1(T) = Z_1(T)Q_1(T)$ and $v_2(T) = Z_2(T)Q_2(T)$ are polynomials and

$$\deg v_1(T) = \deg(Z_1(T)Q_1(T)) < \deg Q_1(T).$$

Thus, $Z_1(T)$ is a proper fraction, but $Z_2(T)$ may not be proper. Recall idempotent $e \in$ defined by (30).

Proposition 2.6 *In the notations above, the generating functions of commutative Frobenius algebra objects $\phi(e)A$ and $\phi(1-e)A$ are $Z_1(T)$ and $Z_2(T)$ respectively.*

Proof Let $'x_0$ and $''x_0$ be the handle endomorphisms of the algebras $\phi(e)A$ and $\phi(1-e)A$, so that $'x_0 = ex_0$ and $''x_0 = (1-e)x_0$. The evaluation series α' of $\phi(e)A$ has n-th coefficient

$$\alpha'_n = \epsilon(('x_0)^n) = \epsilon(ex_0^n) = \epsilon(s(x_0)x_0^n), \tag{36}$$

where $s(T)$ is a polynomial such that $e = s(x_0)$. This shows that the evaluation series β and γ of $\phi(e)A$ and $\phi(1-e)A$ are uniquely determined by the factorization (33) of $U(T)$. The expression (36) is somewhat implicit since we don't write down a formula for $s(T)$.

Pick commutative Frobenius algebras $A' \in \mathcal{C}'$ and $A'' \in \mathcal{C}''$ with generating functions $Z_1(T)$ and $Z_2(T)$, and such that their handle endomorphisms x'_0 and x''_0 have minimal polynomials $U_1(T)$ and $U_2(T)$ respectively. Here $\mathcal{C}'$ and $\mathcal{C}''$ are $\mathbf{k}$-linear symmetric monoidal categories. Such algebras and categories exists in view of [22, 29].

Consider commutative Frobenius algebra

$$A' \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes A'' \in \mathcal{C}' \boxtimes \mathcal{C}'', \tag{37}$$

where $\mathcal{C}' \boxtimes \mathcal{C}''$ is the external tensor product of $\mathcal{C}'$ and $\mathcal{C}''$, see [37, Sect. 2.2]. This is the *naive* tensor product of $\mathbf{k}$-linear monoidal categories.

A more sophisticated exterior tensor product was defined by Deligne for abelian monoidal categories, subject to additional assumptions, see [16] and references therein, but it is not used here.

Objects of the naive tensor product $\mathcal{C}' \boxtimes \mathcal{C}''$ are finite direct sums of external tensor products $V' \boxtimes V''$ of objects V' and V'' of $\mathcal{C}'$ and $\mathcal{C}''$. The tensor product $\mathcal{C}' \boxtimes \mathcal{C}''$ is additive but $\mathcal{C}', \mathcal{C}''$ do not have to be additive, only $\mathbf{k}$-linear.

The generating function of this algebra is $Z_1(T) + Z_2(T) = Z(T)$ and its handle endomorphism $\tilde{x}_0$ is $x'_0 \oplus x''_0$, where x'_0 and x''_0 are handle endomorphisms of A' and A'', respectively. Hence, for any polynomial $a(T)$, we have $a(\tilde{x}_0) = a(x'_0) \oplus a(x''_0)$. Since the polynomials $U_1(T)$ and $U_2(T)$ are relatively prime, the minimal polynomial of $\tilde{x}_0$ is $U_1(T)U_2(T) = U(T)$. Moreover it is clear that the idempotents determined by the factorization $U(T) = U_1(T)U_2(T)$ are precisely the unit elements of $A' \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes A''$. Thus, the $\alpha-$invariants of the algebras $A' \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes A''$ can be computed via formula (36) applied to them. The result follows. □

Example 2.7 Assume that char $\mathbf{k} \neq 2$ and the generating function of A is

$$Z_\alpha(T) = \frac{T^3 + 1}{1 - 3T + 2T^2}. \tag{38}$$

Then the handle polynomial $U(T) = U_\alpha(T) = T^2(T^2-3T+2) = T^2(T-1)(T-2)$. Note that $T^2 - 3T + 2$ is the reciprocal if $1 - 3T + 2T^2$. The degree of $U(T)$ equals $K_\alpha = \max(\deg P_\alpha + 1, \deg Q_\alpha) = \max(3+1, 2) = 4$, see (5).

Consider the factorization (33) with $Q_1(T) = T^2(T-1)$ and $Q_2(T) = T-2$. Then, see formula (30), $e = \frac{1}{4}(x_0^3 - x_0^2)$ and

$$Z_1(T) = \frac{9/4}{1-2T}, \quad Z_2(T) = \frac{-T^2/2 - T/4 - 5/4}{1-T}.$$

Rational series β and γ give rise to commutative Frobenius objects A_β and A_γ in the skein categories SCob_β and SCob_γ, respectively. Consider the tensor product category

$$\mathrm{SCob}_{\beta,\gamma} \ := \ \mathrm{SCob}_\beta \boxtimes \mathrm{SCob}\gamma \tag{39}$$

with the Frobenius object

$$A_{\beta,\gamma} \ := \ A_\beta \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes A_\gamma, \tag{40}$$

see also (37). Let

$$Z_\beta(T) = \frac{P_\beta(T)}{Q_\beta(T)}, \quad Z_\gamma(T) = \frac{P_\gamma(T)}{Q_\gamma(T)} \tag{41}$$

be the standard presentations of rational series for β and γ, see formulas (4), (13), with $Q_\beta(0) = Q_\gamma(0) = 1$ and co-prime numerators and denominators in each of the two fractions. Polynomials $U_\beta(x)$ and $U_\gamma(x)$ describe handle skein relations for series β and γ, respectively. They are reciprocal polynomials of $Q_\beta(x)$ and $Q_\gamma(x)$, respectively, scaled by suitable powers of x when the fractions are not proper.

Lemma 2.8 *The handle polynomial of the Frobenius object $A_{\beta,\gamma}$ in $\mathrm{SCob}_{\beta,\gamma}$ is*

$$U_{\beta,\gamma}(x) := \mathrm{lcm}(U_\beta(x), U_\gamma(x)), \tag{42}$$

the least common multiple of $U_\beta(x)$ and $U_\gamma(x)$.

Proof The handle endomorphism of $A_{\beta,\gamma}$ is the sum of handle endomorphisms of its direct summands $A_\beta \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes A_\gamma$. □

To understand the handle polynomial for $\beta + \gamma$, we convert the series for β and γ into sums of proper fractions and polynomial terms:

$$Z_\beta(T) = \frac{\overline{P}_\beta(T)}{Q_\beta(T)} + R_\beta(T), \quad \deg \overline{P}_\beta(T) < \deg Q_\beta(T),$$
$$Z_\gamma(T) = \frac{\overline{P}_\gamma(T)}{Q_\gamma(T)} + R_\gamma(T), \quad \deg \overline{P}_\gamma(T) < \deg Q_\gamma(T).$$

The handle polynomials for β and γ are the reciprocals of $Q_\beta(T)$ and $Q_\gamma(T)$ multiplied by T to the exponent the degree of $R_\beta(T)$ and $R_\gamma(T)$, respectively.

From the corresponding decomposition for the series of $\beta+\gamma$,

$$Z_{\beta+\gamma}(T) \;=\; Z_\beta(T) + Z_\gamma(T) \;=\; \frac{\overline{P}_{\beta+\gamma}(T)}{Q_{\beta+\gamma}(T)} + R_\beta(T) + R_\gamma(T), \tag{43}$$

with the reduced fraction

$$\frac{\overline{P}_{\beta+\gamma}(T)}{Q_{\beta+\gamma}(T)} \;=\; \frac{\overline{P}_\beta(T)Q_\gamma(T) + \overline{P}_\gamma(T)Q_\beta(T)}{Q_\beta(T)Q_\gamma(T)}, \tag{44}$$

one sees that $Q_{\beta+\gamma}(T)$ is a divisor of $\mathrm{lcm}(Q_\beta(T), Q_\gamma(T))$ and $R_{\beta+\gamma}(T) = R_\beta(T) + R_\gamma(T)$. Consequently, the handle polynomial $U_{\beta+\gamma}(x)$ is the reciprocal of a divisor of $\mathrm{lcm}(Q_\beta(x), Q_\gamma(x))$ times a power of x of degree $\deg(R_\beta + R_\gamma) \le \max(\deg R_\beta, \deg R_\gamma)$.

Corollary 1 *The handle polynomial $U_{\beta+\gamma}(x)$ of $\beta+\gamma$ is a divisor of the polynomial $U_{\beta,\gamma}(x)$ in* (42).

Definition 2.9 A pair (β,γ) of rational sequences is called *regular* if $U_{\beta+\gamma}(x) = U_{\beta,\gamma}(x)$.

Proposition 2.10 *(β,γ) is regular (and $U_{\beta,\gamma}(x) = U_{\beta+\gamma}(x)$) iff there is a functor*

$$F^S_{\beta,\gamma} \;:\; \mathrm{SCob}_{\beta+\gamma} \longrightarrow \mathrm{SCob}_\beta \boxtimes \mathrm{SCob}_\gamma \tag{45}$$

taking the circle object $A_{\beta+\gamma}$ of $\mathrm{SCob}_{\beta+\gamma}$ to the object $A_{\beta,\gamma}$, see (40)*, and the Frobenius structure of $A_{\beta+\gamma}$ to that of $A_{\beta,\gamma}$. In particular, the handle endomorphism of $A_{\beta+\gamma}$ must go to that of $A_{\beta,\gamma}$.*

Proof The handle polynomial of $A_{\beta+\gamma}$ is $U_{\beta+\gamma}(x)$, while that of $A_{\beta,\gamma}$ is $U_{\beta,\gamma}(x)$. For the functor to exist, one needs $U_{\beta,\gamma}(x_0) = 0$, where x_0 is the handle endomorphism of $A_{\beta+\gamma}$. □

If $U_{\beta+\gamma}(x)$ is a proper divisor of $U_{\beta,\gamma}(x)$, then the handle endomorphism of $A_{\beta+\gamma}$ satisfies a stronger relation than that of the handle endomorphism of $A_{\beta,\gamma}$, and such a functor cannot be set up.

Note that the category $\mathrm{SCob}_{\beta+\gamma}$ in (45) is not additive, while the target category is additive. To remedy that, one can first pass to finite additive closures of these categories to get an additive functor

$$F^{\oplus}_{\beta,\gamma} \;:\; \mathrm{SCob}^{\oplus}_{\beta+\gamma} \longrightarrow \mathrm{SCob}^{\oplus}_\beta \boxtimes \mathrm{SCob}^{\oplus}_\gamma \cong \mathrm{SCob}_\beta \boxtimes \mathrm{SCob}_\gamma. \tag{46}$$

Proposition 2.11 *If at least one of the fractions in* (41) *is proper and $Q_\beta(T), Q_\gamma(T)$ are relatively prime then the pair (β,γ) is regular, so that $U_{\beta+\gamma}(x) = U_{\beta,\gamma}(x)$.*

Proof A fraction $P(T)/Q(T)$ is proper if $\deg P(T) < \deg Q(T)$. This is equivalent to the condition that the handle polynomial $U(x)$ for this rational series is not divisible by x, that is, $U(0) \neq 0$. Proposition follows by considering partial fraction decompositions for series $Z_\beta(T)$ and $Z_\gamma(T)$. Then

$$U_{\beta,\gamma}(x) = \mathrm{lcm}(U_\beta(x), U_\gamma(x)) = U_\beta(x)U_\gamma(x) = U_{\beta+\gamma}(x). \tag{47}$$

□

Remark 2.12 The implication in the above proposition goes only one way, as one can see by taking $\beta = \gamma$ when $\mathrm{char}\,\mathbf{k} \neq 2$. The pair (β, β) is regular then.

Proposition 2.11 gives a sufficient condition for the functor $F^S_{\beta,\gamma}$ in (45) to exist.

Now we will show that for any regular pair (β, γ) functors $F^S_{\beta,\gamma}$ and $F^{\oplus}_{\beta,\gamma}$ are fully faithful.

Proposition 2.13 *For a regular (β, γ), functor $F^S_{\beta,\gamma}$, see (45), induces a fully faithful functor*

$$\mathrm{Cob}_{\beta+\gamma} \longrightarrow \mathrm{Cob}_\beta \boxtimes \mathrm{Cob}_\gamma \tag{48}$$

Proof Observe that the category $\mathrm{Cob}_\beta \boxtimes \mathrm{Cob}_\gamma$ is nondegenerate. Indeed let $X \boxtimes Y$ and $Z \boxtimes T$ be some objects of $\mathrm{Cob}_\beta \boxtimes \mathrm{Cob}_\gamma$. The trace form $\mathrm{Hom}(X\boxtimes Y, Z\boxtimes T)\times \mathrm{Hom}(Z\boxtimes T, X\boxtimes Y) \to \mathbf{k}$ is the tensor product of the trace forms $\mathrm{Hom}(X, Z)\times\mathrm{Hom}(Z, X) \to \mathbf{k}$ and $\mathrm{Hom}(Y, T) \times \mathrm{Hom}(T, Y) \to \mathbf{k}$. Since the tensor product of non-degenerate pairings is non-degenerate, we see that $\mathrm{Hom}(X \boxtimes Y, Z \boxtimes T)$ has no nonzero negligible morphisms and the result follows.

The functor $F^S_{\beta,\gamma}$ induces a full functor $\mathrm{SCob}_{\beta+\gamma} \longrightarrow \mathrm{Cob}_\beta \boxtimes \mathrm{Cob}_\gamma$. Since the category $\mathrm{Cob}_\beta \boxtimes \mathrm{Cob}_\gamma$ is nondegenerate, this functor factors through $\mathrm{Cob}_{\beta+\gamma}$ and gives rise to the fully faithful functor in (48) . □

Passing to additive Karoubi envelopes results in an equivalence of categories:

Proposition 2.14 *For a regular pair (β, γ) and upon passing to additive Karoubi envelopes, functor $F^S_{\beta,\gamma}$ induces an equivalence of tensor categories*

$$F^D_{\beta,\gamma} \ : \ \mathrm{DCob}_{\beta+\gamma} \simeq \mathrm{DCob}_\beta \boxtimes \mathrm{DCob}_\gamma. \tag{49}$$

Further passage to gligible quotients produces a tensor equivalence

$$\underline{F}_{\beta,\gamma} \ : \ \underline{\mathrm{DCob}}_{\beta+\gamma} \simeq \underline{\mathrm{DCob}}_\beta \boxtimes \underline{\mathrm{DCob}}_\gamma. \tag{50}$$

Note that the tensor products $\boxtimes$ above are still the naive tensor products of additive $\mathbf{k}$-linear tensor categories.

Proof The category $\mathrm{SCob}_\beta \boxtimes \mathrm{SCob}_\gamma$ has a commutative Frobenius algebra $A_{\beta,\gamma}$, see (40), so there is a tensor functor $\mathrm{SCob}_\alpha \to \mathrm{SCob}_\beta \boxtimes \mathrm{SCob}_\gamma$ sending A_α to $A_{\beta+\gamma}$ by the universal property from Sect. 2.3, with $\alpha = \beta + \gamma$.

Let $U(T)$ be the polynomial representing the handle skein relation in the category SCob_α. We have a factorization $U(T) = U_1(T)U_2(T)$ corresponding to factorization of the denominator of $Z(T)$ into product of the denominators of $Z_1(T)$ and $Z_2(T)$ (in particular we assume that $U_1(T)$ is not divisible by T). Thus we have a corresponding idempotent $e \in \mathrm{Hom}(\mathbf{1}, A_\alpha)$ and decomposition $A_\alpha = \phi(e)A_\alpha \oplus \phi(1-e)A_\alpha$ where the generating functions of the algebras $\phi(e)A_\alpha$ and $\phi(1-e)A_\alpha$ are precisely $Z_1(T)$ and $Z_2(T)$.

By the universal property there are tensor functors $\mathrm{SCob}_\beta \to \mathrm{DCob}_\alpha$ and $\mathrm{SCob}_\gamma \to \mathrm{DCob}_\alpha$ from the skein categories to the Deligne category (additive Karoubi closure of the skein category) for α sending A_β to $\phi(e)A_\alpha$ and A_γ to $\phi(1-e)A_\alpha$. Thus by the universal property of the external tensor product, see e.g. [37, 2.2], there is a tensor functor $\mathrm{SCob}_\beta \boxtimes \mathrm{SCob}_\gamma \to \mathrm{DCob}_\alpha$ sending $A_\beta \boxtimes \mathbf{1}$ to $\phi(e)A_\alpha$ and $\mathbf{1} \boxtimes A_\gamma$ to $\phi(1-e)A_\alpha$.

Passing to the additive Karoubi closure of the source category gives a tensor functor

$$F \ : \ \mathrm{DCob}_\beta \boxtimes \mathrm{DCob}_\gamma \longrightarrow \mathrm{DCob}_\alpha.$$

The composition of the above tensor functors

$$\mathrm{DCob}_\alpha \xrightarrow{F^D_{\beta,\gamma}} \mathrm{DCob}_\beta \boxtimes \mathrm{DCob}_\gamma \xrightarrow{F} \mathrm{DCob}_\alpha$$

sends A_α to itself and thus is isomorphic to the identity functor. Similarly, the composition

$$\mathrm{DCob}_\beta \boxtimes \mathrm{DCob}_\gamma \xrightarrow{F} \mathrm{DCob}_\alpha \xrightarrow{F^D_{\beta,\gamma}} \mathrm{DCob}_\beta \boxtimes \mathrm{DCob}_\gamma$$

sends $A_\beta \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes A_\gamma$ to themselves and thus is also isomorphic to the identity functor.

These functors intertwine the Frobenius structures of A_α and $A_{\beta,\gamma}$, so the isomorphisms are that of tensor functors. This completes the proof. □

Remark 2.15 A similar argument can be applied in a slightly more general situation where the polynomial relation $U(x) = 0$ is replaced by the polynomial relation $\widetilde{U}(x) = 0$, with $U(t)$ a factor of the polynomial $\widetilde{U}(t)$. This allows to generalize the skein category SCob_α to a category $\widetilde{S}\mathrm{Cob}_\alpha$ that maps onto SCob_α. Notice that such a skein relation is still compatible with evaluation α.

Considering these categories $\widetilde{S}\mathrm{Cob}_\alpha$ with handle skein relations of a fixed degree n gives a family of tensor categories that depend on $2n$ parameters, that is, the coefficients of the polynomial $\widetilde{U}(T) = T^n + \text{l.o.t}$ and the evaluations $\alpha_0 = \alpha(1), \ldots, \alpha_{n-1} = \alpha(x^{n-1})$. This is a flat family of tensor categories, in a suitable sense.

Starting with $Z_\alpha(T)$ as in (4), let us extract the polynomial term by writing

$$Z_\alpha(T) = \frac{P(T)}{Q(T)} = \frac{\overline{P}(T)}{Q(T)} + R(T), \quad R(T) \in \mathbf{k}[T], \quad \deg \overline{P}(T) < \deg Q(T), \tag{51}$$

so that $\overline{P}(T)/Q(T)$ is a proper fraction. Factor $Q(T)$ over $\mathbf{k}$ into

$$Q(T) = Q_1(T)\dots Q_\ell(T), \tag{52}$$

where each factor is a power of an irreducible polynomial over $\mathbf{k}$, the factors are mutually coprime, $(Q_i(T), Q_j(T)) = 1$ for $i \neq j$ and $Q_i(0) = 1$ for all i. Each $Q_i(T)$ is a power of an irreducible polynomial over $\mathbf{k}$, with distinct polynomials for different i. Now form the partial fraction decomposition

$$Z_\alpha(T) = \sum_{i=1}^{\ell} \frac{P_i(T)}{Q_i(T)} + R(T), \quad \deg P_i(T) < \deg Q_i(T), \tag{53}$$

with $Q_i(T)$ and $R(T)$ as above. Denote by $\alpha[i]$ the sequence associated to the rational function $P_i(T)/Q_i(T)$, so that $Z_{\alpha[i]}(T) = P_i(T)/Q_i(T)$, and by $\alpha[0]$ the sequence of coefficients of $R(T)$, so that $Z_{\alpha[0]}(T) = R(T)$. There is a functor

$$F_\alpha^S \ : \ \mathrm{SCob}_\alpha \longrightarrow \mathrm{SCob}_{\alpha[0]} \boxtimes \mathrm{SCob}_{\alpha[1]} \boxtimes \cdots \boxtimes \mathrm{SCob}_{\alpha[\ell]} = \boxtimes_{i=0}^{\ell} \mathrm{SCob}_{\alpha[i]} \tag{54}$$

taking the circle object A_α to the direct sum of objects $\mathbf{1}^{\otimes(i-1)} \otimes A_{\alpha[i]} \otimes \mathbf{1}^{\otimes(\ell-i)}$, for $i = 0, \dots, \ell$.

This functor induces an additive functor with the same target category from the additive closure of the source category:

$$F_\alpha^{\oplus} \ : \ \mathrm{SCob}_\alpha \longrightarrow \boxtimes_{i=0}^{\ell} \mathrm{SCob}_{\alpha[i]}^{\oplus} \cong \boxtimes_{i=0}^{\ell} \mathrm{SCob}_{\alpha[i]}, \tag{55}$$

as well as functors

$$F_\alpha^D : \mathrm{DCob}_\alpha \longrightarrow \boxtimes_{i=0}^{\ell} \mathrm{DCob}_{\alpha[i]} \tag{56}$$

$$\underline{F}_\alpha : \underline{\mathrm{DCob}}_\alpha \longrightarrow \boxtimes_{i=0}^{\ell} \underline{\mathrm{DCob}}_{\alpha[i]} \tag{57}$$

Proposition 2.16 *Functors F_α^D and $\underline{F}_\alpha$ are equivalences of categories for any rational α over a field* $\mathbf{k}$.

Proof This follows by iteratively applying the previous proposition. □

3 Abelian realizations

Let $\alpha = \{\alpha_i, i \in \mathbb{Z}_{\geq 0}\}$ be a sequence of elements of $\mathbf{k}$. We say that a Frobenius algebra A in a symmetric monoidal category $\mathcal{C}$ defined over some field extension L of $\mathbf{k}$ (not necessarily a finite extension) is a *realization* of α if the evaluation of A is α, see [22]. Category $\mathcal{C}$ is then an L-linear category. We say that the realization of α is *finite* if the Hom spaces in $\mathcal{C}$ are finite dimensional over L. Sequence α is called *recognizable* if it admits a finite realization. The following result closely mirrors the one in [22].

Theorem 3.1 *A sequence α admits a finite realization if and only if it is recurrent.*

Proof Let $x = m \circ \Delta$ be the handle endomorphism of A. Note that $\mathrm{tr}(x^n) = \alpha_n \in \mathbf{k}$ for $n \in \mathbb{Z}_{\geq 0}$. If $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is finite dimensional over L then there exists a nonzero polynomial $U(X) \in L[X]$ such that $U(x) = 0$. This implies $x^i U(x) = 0$ for any $i \in \mathbb{Z}_{\geq 0}$. Computing the traces of all terms in this relation we get a recurrent relation with constant coefficients in L satisfied by α_i for $i \gg 0$. Eventual recurrence property can be written as vanishing of suitable Hankel determinants, see references in [22], and computing a determinant made of α_i's gives the same answer in L and $\mathbf{k}$. Thus, the sequence α is recurrent over $\mathbf{k}$.

Conversely, assume α is recurrent. Then the category $\mathcal{C} = \mathrm{SCob}_\alpha$ or its additive Karoubi closure DCob_α, see diagram (15), with Frobenius object A given by one circle is a finite realization of α. □

We say that a realization of α is *abelian* if $\mathcal{C}$ is a symmetric abelian tensor category in the sense of [16, 4.1.1] (the hom spaces are finite-dimensional, all objects have finite length, $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = L$ and $\mathcal{C}$ is rigid).

In particular, an abelian realization is finite in the above sense.

Theorem 3.2 *A sequence α admits an abelian realization if and only if the category $\underline{\mathrm{DCob}}_\alpha$ is semisimple.*

Proof Assume that the category $\underline{\mathrm{DCob}}_\alpha$ is semisimple. Then the object $A \in \underline{\mathrm{DCob}}_\alpha$ gives an abelian realization of α. Conversely, the existence of an abelian realization implies that the quotient of DCob_α by the negligible morphisms is semisimple, see [1, Theorem 1]. □

Remark 3.3 (i) The above theorem shows that if a sequence α admits an abelian realization over a field extension $L \supset \mathbf{k}$ then it also admits an abelian realization over $\mathbf{k}$.

(ii) Assume that a sequence α admits an abelian realization over field $\mathbf{k}$ and let $L \supset \mathbf{k}$ be a finite separable extension of $\mathbf{k}$. Then the sequence α admits an abelian realization over L. This follows from the construction of scalars extension of a tensor category, see [11, 5.3]. It is not clear, though, what can happen when $L \supset \mathbf{k}$ is inseparable.

The following result gives necessary conditions for a sequence α to admit an abelian realizations in terms of its generating function $Z(T) = Z_\alpha(T)$, see (2). Below, in Theorem 3.6, it is shown that these conditions are also sufficient.

Theorem 3.4 *Assume that a sequence α admits an abelian realization. Then*

(1) *The generating function $Z(T)$ is rational, so $Z(T) = \frac{P(T)}{Q(T)}$ where $P(T), Q(T) \in \mathbf{k}[T]$ are relatively prime.*
(2) *The denominator $Q(T)$ is separable, i.e., it has no multiple roots in an algebraic closure of $\mathbf{k}$.*
(3) $\deg P(T) \leq \deg Q(T) + 1$.
(4) *Assume that* $\mathrm{char}\, \mathbf{k} = p > 0$. *Then all the residues of the form $Z(T)\frac{dT}{T^2}$ (computed over the algebraic closure $\overline{\mathbf{k}}$) lie in the prime subfield $\mathbb{F}_p \subset \mathbf{k}$.*

Note that conditions (2) and (3) say that the form $Z(T)\frac{dT}{T^2}$ has no poles of order ≥ 2 (including at $T = \infty$) except, possibly, the point $T = 0$.

Proof Statement (1) is implied by Theorem 3.1 as any abelian realization is finite.

To prove (2) let us consider the morphism $x = m \circ \Delta \in \mathrm{End}_{\mathcal{C}}(A)$ as in the Proof of Theorem 3.1. Let $p \in \mathbf{k}[X]$ be a nonzero polynomial such that $p(x) = 0$ (this polynomial exists since the Hom spaces in the category $\mathcal{C}$ are finite dimensional). Let p_0 be the product of all irreducible factors of p, each appearing with multiplicity 1. Then for any $i \in \mathbb{Z}_{\geq 0}$ the endomorphism $x^i p_0(x)$ is a nilpotent element of $\mathrm{End}_{\mathcal{C}}(A)$ as some power of p_0 is divisible by p. Thus we have $\mathrm{tr}(x^i p_0(x)) = 0$ which gives a linear recurrent relation with constant coefficients for α_n with n sufficiently large. This relation implies that the generating function $Z(T)$ can be written as a fraction with denominator $p_0(T)$. Thus we proved that the factorization of $Q(T)$ into irreducible factors is square free.

A related property is that in a rigid abelian **k**-linear tensor category the trace of a nilpotent endomorphism is zero, see e.g. [10, Corollaire 3.6].

We still have to show that each irreducible factor of $Q(T)$ is separable in the case $\mathrm{char}\,\mathbf{k} = p > 0$. Observe that the tensor power $A^{\otimes p} \in \mathcal{C}$ is a commutative Frobenius algebra object that gives an abelian realization of the sequence $\mathrm{Fr}(\alpha) = \{\alpha_0^p, \alpha_1^p, \ldots\}$, see Example 2.3. The generating function of the sequence $\mathrm{Fr}(\alpha)$ is $\mathrm{Fr}(Z(T))$ where $\mathrm{Fr}(Z(T))$ is obtained from $Z(T)$ by applying the Frobenius endomorphism $\lambda \mapsto \lambda^p$ to all the coefficients of $Z(T)$. Now for the sake of contradiction assume that one of the irreducible factors of $Q(T)$ is not separable. We recall that a nonseparable irreducible polynomial is of the form $r(T) = \sum_{i=0}^k c_i T^{pi}$. Thus, one of the factors of $\mathrm{Fr}(Q(T))$ is $\mathrm{Fr}(r(T)) = \sum_{i=0}^k c_i^p T^{pi} = (\sum_{i=0}^k c_i T^i)^p$ and $\mathrm{Fr}(Q(T))$ is not square free. This is a contradiction (note that the polynomials $\mathrm{Fr}(P(T))$ and $\mathrm{Fr}(Q(T))$ are relatively prime). Thus (2) is proved.

Recall that the relation $\mathrm{tr}(x^i p_0(x)) = 0$ holds for any $i \in \mathbb{Z}_{\geq 0}$, where the polynomial $p_0 = p_0(T)$ is square free. In particular the multiplicity of factor T in $p_0(T)$ is ≤ 1. It follows that the sequence $\alpha_2, \alpha_3, \ldots$ satisfies a linear recurrent relation with constant coefficients, which implies (3).

Let us prove (4). We can assume that the denominator $Q(T) = \prod_{i=1}^r (1 - \gamma_i T)$ for distinct nonzero constants $\gamma_i \in L$, where L is a finite separable field extension of **k**. Let $Z(T) = \frac{\beta_i}{1-\gamma_i T} + Z'$ where $\beta_i \in L$ and Z' has no poles at $T = {\gamma_i}^{-1}$, $1 \leq i \leq r$. Let us consider an abelian realization of α over L, see Remark 3.3(2). For a suitable idempotent e the algebra $\phi(e)A$ will have the generating function $\frac{\beta_i}{1-\gamma_i T}$, see Sect. 2.4. Thus $\beta_i \gamma_i = \dim(\phi(e)A)$ must be an element of the prime subfield $\mathbb{F}_p \subset L$, see [14, Lemma 2.2]. Observe that $-\beta_i \gamma_i$ is precisely the residue of the 1-form $Z(T)\frac{dT}{T^2}$ at $T = {\gamma_i}^{-1}$. Thus the statement (4) is proved for all finite nonzero poles of $Z(T)$. The residue at $T = 0$ is $\alpha_1 = \dim(A)$, and we can apply [14, Lemma 2.2] again. Finally in the remaining case $T = \infty$ we use the Residue Theorem, which holds in characteristic p as well [19, Corollary 2.5.4]. □

Example 3.5 Sequence $\alpha = (1, 2, 3, 4, 5, \ldots)$ describing the function $Z(T) = 1/(1 - T)^2$ does not admit an abelian realization over any field, see condition (2) of the above theorem. More explicitly, the handle endomorphism x satisfies $(x - 1)^2 = 0$ in Cob_α

for this α. However, $\mathrm{tr}(x-1) = \alpha_2 - \alpha_1 = 3 - 2 = 1 \neq 0$. This is a contradiction: in any abelian category trace of a nilpotent endomorphism is zero [16, Proposition 4.7.5].

If $\mathbf{k}$ is algebraically closed, decomposition (53) can be refined to

$$Z_\alpha(T) = \sum_{i=1}^{\ell} \frac{P_i(T)}{(1-\gamma_i T)^{m_i}} + R(T), \quad \deg P_i(T) < m_i, \tag{58}$$

with distinct $\gamma_1, \ldots, \gamma_\ell \in \mathbf{k}$ and polynomial $R(T)$. Conditions (2)-(4) of Theorem 3.4 translate to

- $m_i = 1$ for all i, $1 \leq i \leq \ell$.
- $\deg R(T) \leq 1$, that is, the polynomial $R(T)$ is at most linear, $R(T) = r_0 + r_1 T$, and $r_1 \in \mathbb{F}_p$ if $\mathrm{char}\,\mathbf{k} = p$.
- Due to $m_i = 1$ we restrict to a constant polynomial $P_i(T) = p_i \in \mathbf{k}$, simplifying the residue to

$$\mathrm{res}_{\gamma_i^{-1}} \left(\frac{p_i dT}{(1-\gamma_i T)T^2} \right) = -p_i \gamma_i. \tag{59}$$

Thus, condition (4) can be rewritten as $p_i \gamma_i \in \mathbb{F}_p$.

Summarizing, over an algebraically closed $\mathbf{k}$, a rational function $Z(T)$ admits an abelian realization iff

$$Z_\alpha(T) = \sum_{i=1}^{\ell} \frac{p_i}{1-\gamma_i T} + r_0 + r_1 T, \quad p_i\gamma_i \in \mathbb{F}_p, \; r_1 \in \mathbb{F}_p, \tag{60}$$

for distinct $\gamma_1, \ldots, \gamma_\ell$.

In characteristic 0, one can use Remark 3.3(i) to pass from $\mathbf{k}$ to its algebraic closure. In characteristic p, irreducible inseparable factors in the denominator also constitute an obstruction to existence of an abelian realization, by condition (2) of Theorem 3.4.

Theorem 3.6 *Assume that a sequence α satisfies conditions (1)–(4) from Theorem 3.4. Then α admits an abelian realization.*

Proof By Remark 3.3 (i) we can and will assume that the field $\mathbf{k}$ is algebraically closed. We start by giving abelian realizations for some special sequences.

(1) Assume $Z(T) = \alpha_0 + \alpha_1 T$ and $\mathrm{char}\,\mathbf{k} = p > 0$ with $\alpha_1 \in \mathbb{F}_p$. Then we can choose $\mathcal{C} = \mathrm{Vec}_{\mathbf{k}}$ and use Example 2.1 with vector space V of suitable dimension.
(2) Assume $Z(T) = \alpha_0 + \alpha_1 T$ and $\mathrm{char}\,\mathbf{k} = 0$. Again we use Example 2.1; however in all cases when $\dim(V) \notin \mathbb{Z}_{\geq 0}$ we use the abelian specialization of the Deligne category $\mathcal{C} = \mathrm{Rep}(O_t)$ (see e.g. [10, 9]) with $t = \dim(V)$.
(3) Assume $Z(T) = \frac{\beta}{1-\gamma T}$ with $\beta\gamma = 1$. We choose $\mathcal{C} = \mathrm{Vec}_{\mathbf{k}}$ and $A = \mathbf{k}$ such that $\epsilon(1) = \beta$.

(4) Assume $Z(T) = \frac{\beta}{1-\gamma T}$ and $\operatorname{char} \mathbf{k} = p > 0$, with $\beta\gamma \in \mathbb{F}_p \setminus \{0\}$. We choose $\mathcal{C} = \mathrm{Vec}_{\mathbf{k}}$ and take A to be a direct sum of several copies of the algebra from (3).

(5) Assume $Z(T) = \frac{\beta}{1-\gamma T}$ and $\operatorname{char} \mathbf{k} = 0$, with $t = \beta\gamma \neq 0$. We take A to be the standard Frobenius algebra in the semisimple quotient of the Deligne category $\mathcal{C} = \mathrm{Rep}(S_t)$, see [10, Théorèmes 2.18, 6.2].

Any sequence α satisfying the conditions (1)–(4) from Theorem 3.4 is a sum of sequences considered in (1), (2), (4), (5) above. Thus the following result completes the Proof of the Theorem. □

The last two theorems together are equivalent to the following result.

Theorem 3.7 *A sequence α over a field $\mathbf{k}$ admits an abelian realization if and only if it satisfies conditions (1)–(4) in Theorem 3.4.*

Lemma 3.8 *Assume that sequences α' and α'' admit abelian realizations over an algebraically closed field $\mathbf{k}$. Then the sequence $\alpha' + \alpha''$ also admits an abelian realization over $\mathbf{k}$.*

Proof By Theorem 3.2 there are semisimple categories $\mathcal{C}'$ and $\mathcal{C}''$ with Frobenius algebras $A' \in \mathcal{C}'$ and $A'' \in \mathcal{C}''$ giving the realizations of α' and α''. Then the algebra

$$A' \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes A'' \in \mathcal{C}' \boxtimes \mathcal{C}''$$

gives a realization of α in a semisimple (and hence abelian) category $\mathcal{C}' \boxtimes \mathcal{C}''$. □

Remark 3.9 The Proof of Theorem 3.6 shows that in the case of algebraically closed field $\mathbf{k}$ of positive characteristic the sequence α admits an abelian realization if and only if it admits a realization with $\mathcal{C} = \mathrm{Vec}_{\mathbf{k}}$.

Example 3.10 (a) Let $\alpha = (1, 1, 2, 3, 5, 8, \dots)$ be the Fibonacci sequence, with the generating function $Z(T) = 1/(1 - T - T^2)$. Then α admits an abelian realization in characteristic zero. It admits an abelian realization in positive characteristic p if $p \neq 5$ and 5 is a quadratic residue modulo p, i.e., $p = 2$ or $p \equiv \pm 1 (\mathrm{mod}\ 5)$. Indeed, in characteristic 5 the denominator is $(1 + 2T)^2$, hence has a multiple root and is not separable. If 5 is not a quadratic residue modulo p, the differential form residue of condition (4) in either of the two roots of the denominator does not lie in the prime subfield.

(b) Let $\beta = (-1, 2, 1, 3, 4, 7, 11, \dots)$ be the (shifted) Lucas sequence. It satisfies the Fibonacci relation $\beta_{n+2} = \beta_{n+1} + \beta_n$ for $n \geq 0$ but with a different initial condition. The generating function $Z(T) = \frac{\phi^{-1}}{1-\phi T} + \frac{\bar{\phi}^{-1}}{1-\bar{\phi} T}$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ its Galois conjugate. Both residues of the one-form in Theorem 3.4 (4) equal 1 in this case. Thus, β admits an abelian realization in any characteristic.

4 Endomorphisms of object 1

4.1 Algebras B_S and B

Define the *rank* K of a rational theory α by formula (3). Rank is the maximum of the degree of the numerator $P(T)$ of $Z(T)$ plus one and the degree of the denominator $Q(T)$. For a theory of rank K, elements $1, x, \dots, x^{K-1}$ of $A_\alpha(1)$ are linearly independent and there is a linear relation (11), reproduced below

$$U_\alpha(x) := x^K - b_1\, x^{K-1} + b_2\, x^{K-2} - \dots + (-1)^M b_M\, x^{K-M} = 0, \qquad (61)$$

where b_i's are the coefficients of the denominator $Q(T)$, see (13), normalized so that $Q(0) = 1$, and $p_\alpha(x) = x^K Q(x^{-1})$.

Recall that category Cob_α is the quotient of the skein category SCob_α by the ideal of negligible morphisms. In the category SCob_α we evaluate closed components via α and reduce K handles on a connected component via (12). In Cob_α we further mod out by all negligible morphisms.

Consider the endomorphism algebras

$$B_S := \mathrm{End}_{\mathrm{SCob}_\alpha}(1) \cong \mathrm{Hom}_{\mathrm{SCob}_\alpha}(0, 2), \qquad (62)$$

$$B = \mathrm{End}_{\mathrm{Cob}_\alpha}(1) \cong A_\alpha(2). \qquad (63)$$

Both algebras are commutative unital $\mathbf{k}$-algebras, under the pants cobordism multiplication. Isomorphisms on the right are those of $\mathbf{k}$-vector spaces, given by moving the bottom circle of a $(1, 1)$-cobordism to the top. Algebra B is the quotient of B_S by the two-sided ideal J_{neg} of negligible endomorphisms,

$$B \cong B_S / J_{neg}. \qquad (64)$$

Elements u, x in Fig. 1 generate the algebra B_S.

It is easy to write down a basis in each hom space of the category SCob_α, see [29]. A basis in $B = \mathrm{End}_{\mathrm{SCob}_\alpha}(1)$ is given by the set of tube cobordisms with at most $K-1$ dots on them and the cup-cap cobordisms u decorated by at most $K-1$ dots on each connected component.

Proposition 4.1 *Elements*

$$x^n,\ 0 \le n < K,\quad x^n u x^k,\ 0 \le n, k < K \qquad (65)$$

constitute a basis of B_S*, and* $\dim(B_S) = K^2 + K$.

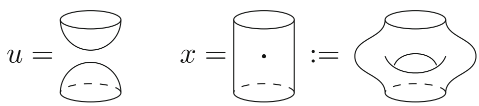


Fig. 1 Generators u and x of B_S

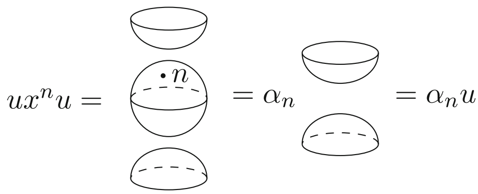


Fig. 2 One of the defining relations in B_S

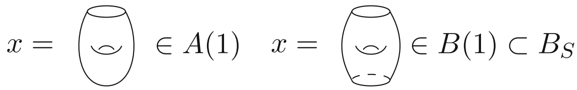


Fig. 3 Element x of $A(1)$ on the left is a one-holed torus (or a handle with one hole). Element x of $B(1) \subset B_S$ on the right is a two-holed torus (a handle with two holes). Both have the same defining relation $U_\alpha(x) = 0$ in $A(1)$ and $B(1)$, respectively. Left x is obtained from the right x by capping off the bottom circle with a disk

Proposition 4.2 *The following is a set of defining relations in B_S on generators u, x:*

$$ux^n u = \alpha_n u, \quad n \geq 0, \tag{66}$$

$$U_\alpha(x) = 0. \tag{67}$$

Defining relations of the first type are shown in Fig. 2.

Algebra B_S has the bar anti-involution $a \longmapsto \overline{a}$ given by the identity on the generators, $\overline{x} = x, \overline{u} = u$, and $\overline{x^n u x^k} = x^k u x^n$.

The trace form on B_S is defined by closing up a (1, 1)-cobordism and evaluating it via α. The trace is given in the above basis by

$$\mathrm{tr}(x^n) = \alpha_{n+1}, \quad \mathrm{tr}(x^n u x^k) = \alpha_{n+k}.$$

Recall the hom space

$$A(1) = \mathrm{Hom}_{\mathrm{Cob}_\alpha}(0, 1) \cong \mathrm{Hom}_{\mathrm{SCob}_\alpha}(0, 1) \cong \mathbf{k}[x]/(U_\alpha(x)) \tag{68}$$

of dimension K, with the commutative algebra structure given by the pants cobordism.

Denote by $B(1) \subset B_S$ the subalgebra of B_S generated by the handle endomorphism. There is an algebra isomorphism

$$B(1) \cong A(1) \tag{69}$$

given by taking the handle endomorphism in $B(1)$ to the corresponding handle element of $A(1)$, see Fig. 3. The skein relation on powers of the handle holds on linear combinations of powers of handle on a disk as well as on an annulus. For this reason the algebras are isomorphic. The geometric definitions of multiplications in the two algebras are slightly different: in $A(1)$ it is given by the pants cobordism, while in $B(1)$ and B_S it is the composition of (1, 1)-cobordisms.

Relatedly, we maintain a slight abuse of notation, also shown in Fig. 3, where x is used to denote handle cobordisms with either one or two boundary components, respectively, generating algebras $A(1)$ and $B(1) \subset B_S$.

The two-sided ideal (u) of B_S is

$$(u) = BuB = B(1)uB(1) \cong B(1) \otimes_{\mathbf{k}} B(1)^{op}, \tag{70}$$

where the second isomorphism is that of $B(1)$-bimodules. It is spanned by cobordisms with two connected viewable components, and there is a split exact sequence of this 2-sided ideal of B_S and the quotient ring

$$0 \longrightarrow B(1)uB(1) \longrightarrow B_S \rightleftarrows B(1) \longrightarrow 0. \tag{71}$$

The quotient by the 2-sided ideal is spanned by powers of x and is naturally isomorphic to the subring $B(1)$ spanning by cobordisms with one connected component, via the inclusion $B(1) \subset B_S$, which is a section of the surjection above.

Recall that B is the quotient of B_S given by (64). B_S acts on the space $A(1)$ by left multiplication by cobordisms, see isomorphisms (68) for equivalent descriptions of that space. The action takes a cobordism from 0 to 1 (that can be assumed to be connected and of genus less than K) and composes with a cobordism from 1 to 1. Closed components that may result are removed via α-evaluation and the genus is reduced to at most $K - 1$. The action factors through that of B, since negligible endomorphisms act by 0.

Passing to gligible quotients results in a short exact sequence

$$0 \longrightarrow B(1)uB(1) \xrightarrow{\phi} B \longrightarrow B/\mathrm{im}(\phi) \longrightarrow 0, \tag{72}$$

of a two-sided ideal, algebra, and the quotient algebra. Map ϕ is injective, and the quotient $B/\mathrm{im}(\phi)$ is trivial iff α is multiplicative, that is, if the product map $A(1) \otimes A(1) \longrightarrow A(2)$, which corresponds to ϕ, is an isomorphism, also see [22].

4.2 Examples

Example: $K = 1$. In this case a handle on a component reduces to a multiple of the component without the handle and $Z(T)$ is either the constant function, $Z(T) = \alpha_0$, or

$$Z(T) = \frac{\alpha_0}{1 - \gamma T} = \alpha_0 + \alpha_0 \gamma T + \alpha_0 \gamma^2 T^2 + \dots \tag{73}$$

The ring B_S has a basis $\{1, u\}$ with $u^2 = \alpha_0 u$. The trace on B_S is $\mathrm{tr}(1) = \alpha_0 \gamma$, $\mathrm{tr}(u) = \alpha_0$.

The only possible functions in this case are $Z_\alpha = \alpha_0, \alpha_0 \neq 0$ and $Z_\alpha = \frac{\alpha_0}{1-\gamma T}, \alpha_0, \gamma \neq 0$.

The quotient map $B_S \longrightarrow B$ is an isomorphism iff the Gram matrix

$$\begin{pmatrix} \alpha_0^2 & \alpha_0 \\ \alpha_0 & \alpha_0\gamma \end{pmatrix} \tag{74}$$

of the basis $\{1, u\}$ of B_S is nondegenerate. It has determinant $\alpha_0^2(\alpha_0\gamma - 1)$. We see that the quotient map $B_S \longrightarrow B$ is an isomorphism (and the theory is not multiplicative) iff $\gamma \neq \alpha_0^{-1}$.

Example: Linear function. Let $Z(T) = \beta_0 + \beta_1 T$, $\beta_1 \neq 0$. Then B_S has a spanning set $\{1, x, u, xu, ux\}$, which is a basis iff $\beta_1 \neq 2$, see Sect. 7.1 and [22]. Let us assume the latter case. The multiplication rules in B_S follow from the relations

$$uxu = \beta_1 u, \quad xux = \beta_1 x, \quad u^2 = \beta_0 u, \quad x^2 = 0,$$

and the quotient of B_S by the 2-sided ideal BuB is one-dimensional. The action of B on the left ideal $Bu = \mathbf{k}u \oplus \mathbf{k}xu \cong B(1)$ surjects B onto the matrix algebra $\mathrm{Mat}_2(\mathbf{k})$ and leads to the direct product decomposition

$$B \cong \mathrm{Mat}_2(\mathbf{k}) \times \mathbf{k}. \tag{75}$$

It is given explicitly as follows. Let $z = xu + ux - \beta_0 x - \beta_1$. Then $z^2 = -\beta_1 z$ and $-\beta_1^{-1} z$ is a central idempotent splitting off $\mathbf{k}$ from B. The complementary factor is given by the (non-unital) homomorphism $\mathrm{Mat}_2(\mathbf{k}) \longrightarrow B$,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \beta_1^{-1} ux, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \beta_1^{-1}(u - \beta_0\beta_1^{-1} ux),$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longrightarrow x, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \beta_1^{-1}(xu - \beta_0 x).$$

On the other hand, if $\beta_1 = 2$, the map ϕ in (72) is an isomorphism, the theory is multiplicative ($A(2) \cong A(1)^{\otimes 2}$), and B is isomorphic to the matrix algebra of size 2 over $\mathbf{k}$, see [22].

5 Constant generating function β

5.1 State spaces, partitions, and Catalan numbers

Consider the evaluation corresponding to the series

$$Z(T) = \beta, \;\; \beta \in \mathbf{k}^*, \tag{76}$$

which is just the constant function, so the associated sequence $\alpha = (\beta, 0, 0, \ldots)$. Scaling invariance explained in Sect. 2.1 allows us to set $\beta = 1$ without "changing" any categories. We keep β arbitrary, but this is just a matter of preference.

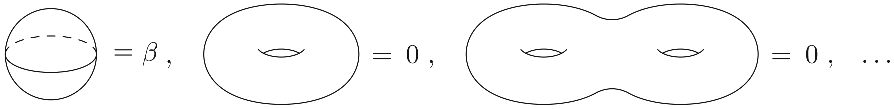


Fig. 4 Constant series evaluations

For α describing a constant function, see Fig. 4, any closed surface S which has a component of genus greater than zero evaluates to 0. Otherwise, S (which is then necessarily the disjoint union of 2-spheres) evaluates to

$$S \mapsto \alpha(S) = \beta^{\dim H_0(S,\mathbf{k})},$$

that is, β to the power the number of components of S.

Define the genus of a connected surface with boundary as the genus of a surface obtained by attaching 2-disks to all boundary components.

Consider the space state of n circles $A(n) = A_\alpha(n)$ in this theory α as discussed in [22, Sect. 2.7]. For an arbitrary α the state space is defined by the formula (20), also see [22]. Recall from the latter reference that the circles are ordered and numbered by $1, 2, \ldots, n$; their union is denoted $\sqcup_n \mathbb{S}^1$. All surfaces with a component of genus greater than zero are in the kernel of the bilinear form, so that $A(n)$ is spanned by diffeomorphism classes of viewable surfaces S, with each component of genus 0, and the boundary diffeomorphic to the disjoint union of n circles.

Such surfaces can be canonically identified with partitions of the set $\{1, 2, \ldots, n\}$. Denote the set of partitions by $D(n)$. Cardinality of $D(n)$ is known as the Bell number B_n and it has the following generating function:

$$\sum_{n\geq 0} \frac{B_n}{n!} t^n = \exp(\exp(t) - 1).$$

To a partition $\lambda \in D(n)$ there is associated a viewable surface S_λ as above, with each component of genus 0. Recall that by *viewable surface* we mean a surface without closed components.

Consider the vector space $\mathbf{k}^{D(n)}$ with a basis of vectors v_λ, over all partitions $\lambda \in D(n)$. Form the linear map

$$\mathbf{k}^{D(n)} \longrightarrow A(n), \quad v_\lambda \mapsto [S_\lambda] \tag{77}$$

into the state space of n circles which takes basis vectors to corresponding surfaces S_λ. This map is surjective, as follows from the discussion above.

For $n \leq 3$ the map (77) is an isomorphism, that is, the induced bilinear form is nondegenerate on $\mathbf{k}^{D(n)}$, see [22, Sect. 2.7].

However, starting from $n = 4$ map (77) has a nontrivial kernel, and the dimension of the state space (if char $\mathbf{k} = 0$) is equal to the Catalan number

$$c_n = \frac{1}{2n+1}\binom{2n}{n}. \tag{78}$$

n	0	1	2	3	4	5	6	7
B_n	1	1	2	5	15	52	203	877
$c_n = \dim A(n)$	1	1	2	5	14	42	132	429

Theorem 5.1 *Over a field* $\mathbf{k}$ *of characteristic zero the state space* $A(n)$ *for the theory with the constant generating function* (76) *has dimension equal to the Catalan number* c_n*:*

$$\dim A(n) = c_n.$$

It has a basis $\mathbb{PS}^n$ *of crossingless surfaces, as described below in Sect. 5.2.*

One proof of this theorem is given starting here and through Sect. 5.3. Another proof is contained in Sect. 5.5, via a connection to representations of osp(1|2).

Recall notations from [22], where y_{ij} denotes a surface that consists of a tube connecting circles i and j and $n-2$ disks that cap off the remaining circles. More generally, for $1 \le i_1 < \dots i_r \le n$ denote by $y_J = y_{i_1,i_2,\dots,i_r}$ the surface that consists of a 2-sphere with r holes bounding circles $i_1, i_2, \dots, i_r$ and $n-r$ disks capping off the remaining $n-r$ circles. Here $J = \{i_1, \dots, i_r\}$. Figure 5 shows examples of these surfaces for $n = 3$ and Fig. 6 shows examples for larger n. Note also that $A(n)$ is naturally a commutative associative unital algebra under the multiplication given by composing two diagrams via the disjoint union of n pants cobordisms.

Fifteen elements of this spanning set for $A(4)$ can be separated into five types, as follows and see Fig. 7:

(1) Unit cobordism 1.
(2) Six cobordisms y_{ij}, $i < j$.
(3) Three cobordisms $y_{ij}y_{kl}$ with i, j, k, l distinct: $y_{12}y_{34}$, $y_{13}y_{24}$, $y_{14}y_{23}$, each a disjoint union of two tubes.
(4) Four cobordisms y_{ijk}, each a union of a 3-holed sphere and a disk.
(5) Cobordism y_{1234}, which is a 4-holed sphere.

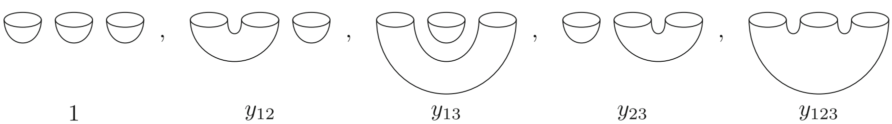


Fig. 5 A spanning set (in fact, a basis) of $A(3)$: the unit element, tubes y_{ij}, and connected surface $y_{123} = y_{12}y_{13}$

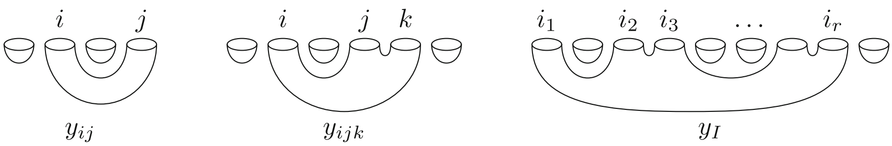


Fig. 6 Examples of surfaces y_J

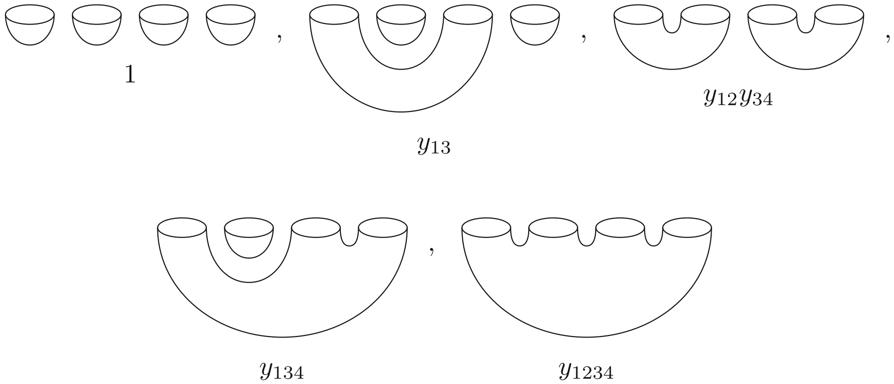
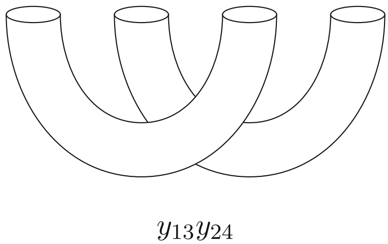


Fig. 7 Elements 1, y_{1234} and examples of elements of types y_{ij}, $y_{ij}y_{kl}$, y_{ijk} in $A(4)$

Fig. 8 Two tubes in the diagram of $y_{13}y_{24}$ overlap



Recall from [22, Sect. 2.7] that there is an S_4-invariant skein relation on the eight vectors of the last three types:

$$(y_{12}y_{34} + y_{13}y_{24} + y_{14}y_{23}) - (y_{123} + y_{124} + y_{134} + y_{234}) + \beta y_{1234} = 0 \quad (79)$$

Among these eight vectors, only $y_{13}y_{24}$ has a diagram with an "intersection" of its surfaces, see Fig. 8. In fact, the remaining fourteen elements of this spanning set all have "planar" diagrams without overlapping components, see examples in Fig. 7.

Consequently, element $y_{13}y_{24}$ of the spanning set for $A(4)$ can be written as a linear combination of "crossingless" cobordisms. Informally, we call a surface with n boundary circles *crossingless* if it can be drawn such that the components do not interlace.

5.2 Planar partitions, crossingless matchings, and crossingless surfaces

Planar partitions and crossingless matchings. Recall that $\mathbb{B}^n$ denotes the set of crossingless matchings of $2n$ points on a horizontal line. It has cardinality c_n, the n-th Catalan number, see formula (78).

Consider the set $\mathbb{PD}^n$ of planar partitions of an n-element set. These are decompositions of $\{1, \ldots, n\}$ into non-empty subsets such that the configuration of these subsets can be drawn in the lower half-plane by connecting points in each subset by arcs and without arcs from different subsets intersecting. Equivalently, there should

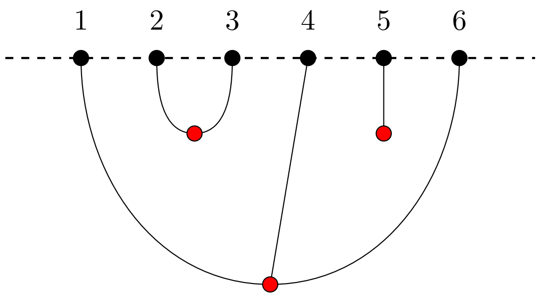


Fig. 9 Diagram of the planar partition $(\{1, 4, 6\}, \{2, 3\}, \{5\})$ in $\mathbb{PD}^6$. Points in the lower half-plane where the arcs end are shown in red

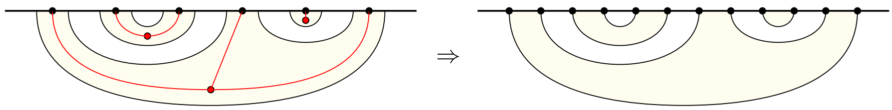


Fig. 10 Left: non-crossing partition λ as in Fig. 9, here shown in red, and its regular neighbourhood $N(\lambda)$, shaded in yellow. Right: Arcs on the boundary of $N(\lambda)$ constitute a crossingless matching $\Phi_0(\lambda)$

exist no quadruple of numbers $1 < i_1 < i_2 < i_3 < i_4 \leq n$ with i_1, i_3 in one subset and i_2, i_4 in another.

Given a planar partition $\lambda \in \mathbb{PD}^n$, it can be depicted by connecting points in each m-element subset (these points lies on the horizontal line with n marked points $p_1, \dots, p_n$) by m arcs to a central point somewhere in the lower half-plane. In this configuration there are n non-intersecting arcs connecting n points on the horizontal line to k points in the lower half-plane, where k is the number of sets in the planar partition. Figure 9 shows an example of the diagram for the planar partition $(\{1, 4, 6\}, \{2, 3\}, \{5\})$ in $\mathbb{PD}^6$.

To a planar partition $\lambda \in \mathbb{PD}^n$, also called a *non-crossing partition*, we assign a crossingless matching $\Phi_0(\lambda) \in \mathbb{B}^n$ as follows. Take a planar diagram of λ and form a standard retract closed neighbourhood in $\mathbb{R}^2_-$ of the configuration of n arcs and k inner points. This neighbourhood $N(\lambda)$ consists of k connected components. Each component deformation retracts onto the corresponding tree of the diagram of λ. The intersection of $N(\lambda)$ with the horizontal line $\mathbb{R}$ consists of n closed intervals, one for each point $p_1, \dots, p_n$.

The boundary of these intervals constitute $2n$ points $p'_1, \dots, p'_{2n}$, with points p'_{2i-1}, p'_{2i} being the boundaries of the interval that contains the point p_i. Boundary of $N(\lambda)$ consists of these n intervals together with n arcs that lie in the lower half-plane and constitute a crossingless matching of points $p'_1, \dots, p'_{2n}$. Denote this matching by $\Phi_0(\lambda)$. This map

$$\Phi_0 : \mathbb{PD}^n \longrightarrow \mathbb{B}^n \tag{80}$$

is a bijection between planar partitions and crossingless matchings.

An example of the bijection Φ_0 is depicted in Fig. 10. To construct the inverse bijection Φ_0^{-1}, start with a crossingless matching b. The matching decomposes the lower half-plane $\mathbb{R}^2_-$ into $n+1$ connected regions. Label these regions by colors 1 and 2 in a checkerboard fashion so that the outer region (the unique unbounded region)

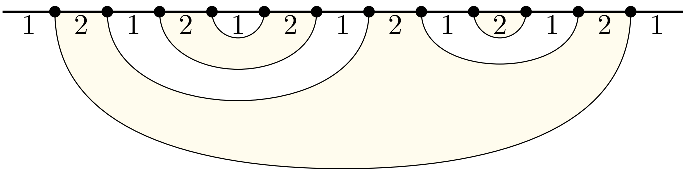


Fig. 11 Checkerboard coloring in the complement of a crossingless matching

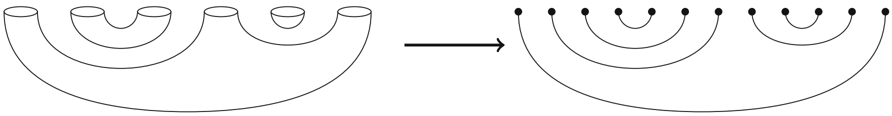


Fig. 12 Bijection between (isotopy classes of) crossingless surfaces and matchings

is labelled 1, see Fig. 11. Each region colored 2 has a boundary that is a union of horizontal intervals and inner arcs in $\mathbb{R}^2_-$. Given a region that contains m horizontal intervals, choose a point inside each interval, a point u inside the region, and connect the m points to the point u by m non-intersecting arcs. The region deformation retracts onto the union of these arcs. Taking these unions over all regions of b colored 2 gives a diagram of planar partition. This is the planar partition $\Phi_0^{-1}(b)$.

Crossingless surfaces and crossingless matchings. For an accurate definition, position n circles, each of diameter 1, on the xy-plane $\mathbb{R}^2$ so that their centers are located on the x-axis, at points with the x-coordinate $2, 4, \dots, 2n$, respectively, and denote this collection of circles C_n. The circles intersect the x-axis at $2n$ points $2 \pm \frac{1}{2}, \dots, 2n \pm \frac{1}{2}$.

Place $\mathbb{R}^2$ in $\mathbb{R}^3$ in the usual way, by adding the third coordinate z and taking $\mathbb{R}^2$ to be the plane $z = 0$. This plane splits $\mathbb{R}^3$ into half-spaces $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$. We consider surfaces S properly embedded in $\mathbb{R}^3_-$ with the boundary C_n. Form the intersection $S \cap \mathbb{R}^2_{xz}$ of S with the xz-coordinate plane. Upon a slight deformation of S while keeping its boundary on the xy-plane fixed we can assume that the intersection $S \cap \mathbb{R}^2_{xz}$ is a one-manifold which is the union of circles and n arcs with boundary the above $2n$ points on the x-axis: $2 \pm \frac{1}{2}, 4 \pm \frac{1}{2}, \dots, 2n \pm \frac{1}{2}$.

These n arcs in the lower half-plane $\mathbb{R}^2_{xz,-}$ constitute a crossingless matching $b \in \mathbb{B}^n$ of $2n$ points.

Vice versa, to a crossingless matching $b \in \mathbb{B}^n$ we can associate a 2-manifold $S(b)$. First, color the regions of the lower half-plane $\mathbb{R}^2_- = \mathbb{R}^2_{xz,-}$ which contains the matching by colors 1 and 2 in a checkerboard fashion so that the outer color is 1. Regions of $\mathbb{R}^2_-$ colored 2 are bounded. Each one has a boundary that is a union of horizontal intervals and inner arcs in $\mathbb{R}^2_-$. Horizontal intervals connect points $2i \pm \frac{1}{2}$ for various i, $1 \leq i \leq n$. To associate a surface $S(b)$ to b we thicken each region V colored 2 into a 3-dimensional region in $\mathbb{R}^3_- = \mathbb{R}^2_- \times \mathbb{R}$. One way to do that is by forming $V \times [0, 1] \subset \mathbb{R}^3_-$ and then smoothing its corner arcs to get a region bounding a smooth surface in $\mathbb{R}^3_-$. Each horizontal interval $[2i - \frac{1}{2}, 2i + \frac{1}{2}]$ in V first gets multiplied by $I = [0, 1]$ and then smoothed out to a circle of diameter one. In this way n pairs of consecutive points on the boundary of the matching b turn into n circles on the plane $\mathbb{R}^2 = \partial\mathbb{R}^3_-$. Each region V of b colored 2 turns into a 3-dimensional region that bounds the union of disks, each of diameter 1, in $\mathbb{R}^2$ and a surface $S(V)$ in $\mathbb{R}^3_-$. Now to b assign the union $S(b)$ of surfaces $S(V)$ over all regions V labelled 2. The boundary of $S(b)$ consists of the union C_n of n circles.

We refer to $S(b)$ as a *crossingless surface* associated to the matching b. Denote by $\mathbb{PS}^n$ the set of crossingless surfaces associated to matchings $b \in \mathbb{B}^n$ and by Φ_1 the corresponding bijection

$$\Phi_1 \ : \ \mathbb{B}^n \longrightarrow \mathbb{PS}^n. \tag{81}$$

This assignment is inverse to the map depicted in Fig. 12.

Isotopy classes of surfaces $S(b) = \Psi_1(b)$ that result from this construction are in a bijection with planar partitions of n. These surfaces are determined by their intersection with the lower half-plane $\mathbb{R}^2_- \subset \mathbb{R}^3_-$. This intersection is a crossingless matching b.

Composing the two bijections above results in the bijection Φ from the set $\mathbb{PD}^n$ of non-crossing partitions to the set of crossingless surfaces:

$$\Phi \ : \ \mathbb{PD}^n \xrightarrow{\Phi_0} \mathbb{B}^n \xrightarrow{\Phi_1} \mathbb{PS}^n \tag{82}$$

Any oriented surface S with a component of genus greater than 0 and $\partial S \cong \sqcup_n \mathbb{S}^1$ evaluates to the zero vector, $[S] = 0$ in $A(n)$ for our series α. Relation (79) allows to reduce $y_{13}y_{24}$, which can be viewed as a crossing, to a linear combination of crossingless diagrams. Inductive application of this relation shows that $A(n)$ is spanned by vectors $[S(b)]$ associated to crossingless surfaces $S(b)$, for $b \in \mathbb{B}^n$. In particular,

$$\dim A(n) \leq c_n \tag{83}$$

for any field $\mathbf{k}$ and $\beta \in \mathbf{k}^*$.

5.3 Meander determinants and size of $A(n)$

To prove the opposite inequality to (83) when $\mathbf{k}$ has zero characteristic, it is enough to show that the bilinear form is nondegenerate on the subspace with a basis $\{[S(b)]\}$, $b \in \mathbb{B}^n$.

It turns out that the matrix of this bilinear form in the spanning set of crossingless matchings is the same as one of the auxiliary matrices appearing in [25], namely, the matrix for the deformed *meander determinant*.

Consider two matchings $a, b \in \mathbb{B}^n$, their surfaces $S(a)$, $S(b)$ and the closed surface $S(a, b) := \overline{S}(b)S(a)$ given by reflecting the surface $S(b)$ about the horizontal plane $\mathbb{R}^2$ and composing with $S(a)$ along the common n circles.

To matchings a, b there is also associated a collection $\overline{b}a$ of circles in the plane, which has a unique checkerboard coloring of its connected components (regions) by $\{1, 2\}$ with the outer component colored 1. Define $h_1(a, b)$, $h_2(a, b)$ as the number of connected components of colors 1 and 2 respectively.(Fig. 13).

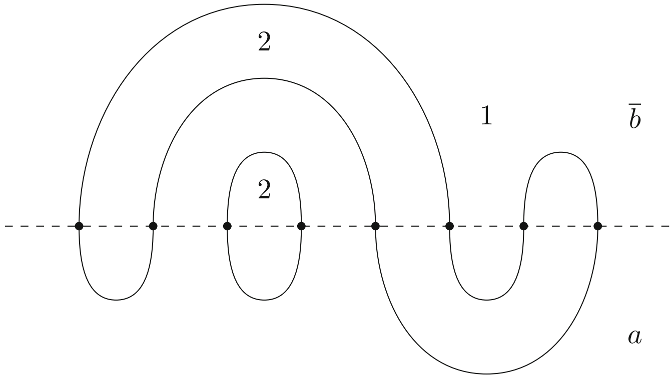


Fig. 13 Diagram $\overline{b}a$ with $h_1(a, b) = 1$. Note that regions colored 2 have no "holes", that is, each one is homeomorphic to a disk

A component of the closed surface $S(a, b)$ has genus 0 if and only if the corresponding region of the planar diagram $\overline{b}a$ colored 2 is a disk. All components of $S(a, b)$ have genus 0 iff no region of $\overline{b}a$ of color 2 contains a region of color 1 inside. Equivalently, all color 2 regions are non-nested disks and there is a unique region of color 1 (the outer region). This happens iff $h_1(a, b) = 1$.

The number of color 2 components is $h_2(a, b)$. The bilinear form on $A(n)$ for the constant series $Z_\alpha(T) = \beta$ in the spanning set of crossingless surfaces $[S(b)]$ is given by

$$([S(a)], [S(b)]) \;=\; \delta_{1,h_1(a,b)}\beta^{h_2(a,b)}, \quad a, b \in \mathbb{B}^n, \tag{84}$$

where

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Denote by $D_n(\beta)$ this matrix of size $\mathbb{B}^n \times \mathbb{B}^n$.

In [25] the authors study the determinant of $\mathbb{B}^n \times \mathbb{B}^n$ matrix $M(y_1, y_2)$ with the (a, b)-entry

$$y_1^{h_1(a,b)} y_2^{h_2(a,b)},$$

where y_1, y_2 are formal variables, and show that it can be expressed in terms of Chebyshev polynomials of the second kind. Namely, let

$$M_n(y) = \prod_{h=1}^{n} U_h(y)^{c_{n,h} - c_{n,h+1}}, \quad \text{where}$$
$$c_{n,h} = \binom{2n}{n-h} - \binom{2n}{n-h-1}, \quad U_h(2\cos\theta) = \frac{\sin(h+1)\theta}{\sin\theta}.$$

Chebyshev polynomial $U_h(y)$ of the second kind is a polynomial of degree h in y with integer coefficients. Then (see [25])

$$\det(M(y_1, y_2)) = (y_1 y_2)^{|\mathbb{B}^n|/2} \cdot M_n(\sqrt{y_1 y_2}). \tag{85}$$

For any a, b

$$h_1(a, b), h_2(a, b) \geq 1,$$

so that

$$D_n(\beta) = \lim_{y_1 \to 0} \frac{1}{y_1} M(y_1, \beta). \tag{86}$$

We need to prove that the following limit is nonzero, when the ground field has characteristic 0:

$$\lim_{y_1 \to 0} \frac{\det(M(y_1, \beta))}{y_1^{|\mathbb{B}^n|}}, \tag{87}$$

so that the determinant below does not vanish:

$$\det(D_n(\beta)) = \lim_{y_1 \to 0} \left(\frac{\beta}{y_1}\right)^{|\mathbb{B}^n|/2} M_n(\sqrt{y_1 \beta}). \tag{88}$$

At the point $y = 0$ even-indexed Chebyshev polynomials $U_{2h}(y)$ do not vanish, while odd-indexed polynomials $U_{2h+1}(y)$ have simple poles, when char $\mathbf{k} = 0$:

$$y \to 0 \Rightarrow U_{2h}(y) \sim 1, \quad U_{2h+1}(y) \sim y.$$

The order of vanishing of $M_n(y)$ is

$$\sum_{i=1,odd}^{n} (c_{n,i} - c_{n,i+1}) = \binom{2n}{n-1} - 2\binom{2n}{n-2} + 2\binom{2n}{n-3} + \ldots + (-1)^n 2\binom{2n}{0}.$$

Lemma 5.2

$$\binom{2n}{n} = 2\sum_{i=1}^{n} (-1)^{i-1} \binom{2n}{n-i},$$

Proof The lemma follows from the identity

$$(1+t)^{2n} = \binom{2n}{n} t^n + \sum_{i=1}^{n} \binom{2n}{n-k} \left(t^{n-k} + t^{n+k}\right),$$

evaluated at $t = -1$.

It follows from the lemma that the order of vanishing can be rewritten as

$$\binom{2n}{n} - \binom{2n}{n-1},$$

which equals the Catalan number c_n. □

This completes the Proof of Theorem 5.1. Note that, over a field **k** of characteristic p, $\dim A(n) = c_n$ if p does not divide the corrresponding product of the values of even Chebyshev polynomials $\mathsf{U}_{2h}(0)$ for $h \leq n/2$ and derivatives of odd Chebyshev polynomials $\mathsf{U}'_{2h+1}(0)$ for $h \leq (n-1)/2$. Otherwise, $\dim A(n) < c_n$. □

5.4 Graded case and the Narayana numbers

5.4.1 Non-crossing partitions and Narayana numbers

Consider points $1, 2, \ldots, n$ placed in this order on a circle. A partition of them into a disjoint union

$$\{1, 2, \ldots, n\} = \{i_1, \ldots, i_t\} \bigsqcup \{j_1, \ldots, j_s\} \bigsqcup \ldots$$

is called non-crossing (or planar, see earlier) if the parts avoid interlaps, when drawn via trees in the disk. Instead of the circle and the disk it bounds one can use the x-axis and the lower half-plane, see earlier.

The number of non-crossing partitions of n points on a circle with exactly k parts, $1 \leq k \leq n$, is called the Narayana number [35]:

$$N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Narayana numbers provide a distinguished refinement of Catalan numbers

$$c_n = \sum_{k=1}^{n} N(n, k)$$

and have the following generating function

$$\sum_{n \geq 0, 1 \leq k \leq n} N(n, k) z^n t^{k-1} = \frac{1 - z(t+1) - \sqrt{1 - 2z(t+1) + z^2(t-1)^2}}{2tz}.$$

5.4.2 Graded dimensions of $A(n)$

The vector space $A(n)$ is spanned by diffeomorphism classes of viewable surfaces (elements of $\mathbb{PS}^n$, see formula (81)) with the boundary diffeomorphic to the disjoint union of n disks and each component of genus 0. Assume that char $\mathbf{k} = 0$. Then surfaces in $\mathbb{PS}^n$ constitute a basis of $A(n)$ and carry a natural degree. The degree of $[S] \in A(n)$, for $S \in \mathbb{PS}^n$, is

$$\deg[S] = n - \chi(S), \tag{89}$$

where $\chi(S)$ is the Euler characteristic of S. In this way $A(n)$ becomes a $2\mathbb{Z}_+$-graded vector space and even a $2\mathbb{Z}_+$-graded commutative associative algebra. The unit element of $A(n)$, viewed as an algebra, is given by the union of n disks and has degree 0. Other basis elements $[S]$ of $A(n)$ have positive even degrees.

This can be written via the action of the grading operator

$$q^{n-\chi} : A(n) \to A(n) \otimes \mathbf{k}[q^{\pm 1}], \quad q^{n-\chi}[S] = q^{n-\chi(S)}[S].$$

It is natural to consider the trace of $q^{n-\chi}$ as a Laurent polynomial in q. Since the homogeneous summands have non-negative gradings, the trace of $q^{n-\chi}$ is a genuine polynomial in q^2 rather than a Laurent polynomial. This polynomial depends on n and describes the graded dimension of $A(n)$ with the above grading.

Proposition 5.3 *If* char $\mathbf{k} = 0$, *the space $A(n)$ is naturally graded. Dimensions of its homogeneous components (i.e. the coefficients of the polynomial* $\mathrm{tr}_{A(n)} q^{n-\chi}$*) are Narayana numbers:*

$$\mathrm{tr}_{A(n)} q^{n-\chi} = \sum_{k=0}^{n-1} N(n, n-k) q^{2k}.$$

Proof Crossingless surfaces correspond to non-crossing partitions. The connected components of a crossingless surface provide a partition of the set of the boundary components. From the discussion in earlier subsections, crossingless surfaces form a basis of the set $A(n)$. A crossingless surface $S \in \mathbb{PS}^n$ with k components gives a decomposition

$$n = i_1 + \ldots + i_k.$$

The Euler characteristic of a component with i boundary circles is $2 - i$. Thus,

$$n - \chi(S) = n - \sum_{j=1}^{k} (2 - i_j) = 2(n-k).$$

□

Remark 5.4 Commutative algebra $A(n)$ is naturally a Frobenius algebra, via the trace map of capping off a surface S by n disks and evaluating the resulting closed surface. The trace map does not respect the grading. To make the trace map homogeneous, one can make β a formal variable with $\deg(\beta) = -2$ and change from a field $\mathbf{k}$ to a polynomial ring $\mathbf{k}[\beta]$. The resulting pairing on $A(n)$, defined over $\mathbf{k}[\beta]$, is not perfect, though.

5.5 A commutative Frobenius algebra in Rep($osp(1|2)$)

In this Section we give an alternative derivation of Theorem 5.1 and relate the category $\underline{\mathrm{DCob}}_\alpha$ for the constant series α with the representation category of Lie superalgebra $osp(1|2)$.

Start with the category $\mathcal{C}'$ of finite-dimensional representations of $osp(1|2)$, viewed as a Lie superalgebra over a field $\mathbf{k}$ of characteristic 0, see [4, 5, 13, 30, 38, 41] and [15, Theorem A.3]. An object of $\mathcal{C}'$ is a $\mathbb{Z}/2$-graded representation of $U(osp(1|2))$. The defining representation $V \cong \mathbf{k}^{1|2}$ generates a Karoubi-closed tensor subcategory $\mathcal{C}$ of $\mathcal{C}'$. One can think of $\mathcal{C}$ as "one-half" of the category $\mathcal{C}'$. An irreducible object of $\mathcal{C}'$ is either isomorphic to an irreducible object of $\mathcal{C}$ or to such an object tensored with the odd one-dimensional representation of $osp(1|2)$. Both $\mathcal{C}$ and $\mathcal{C}'$ are semisimple $\mathbf{k}$-linear categories.

Category $\mathcal{C}$ is similar to the category of representations of the Lie group $SO(3)$. Namely, it has one irreducible representation V_{2n} in each odd dimension $2n+1$, $n = 0, 1, \ldots,$ just like $SO(3)$, and the tensor product decomposition

$$V_{2n} \otimes V_{2m} \cong V_{2|n-m|} \oplus V_{2|n-m|+2} \oplus \cdots \oplus V_{2(n+m)}$$

has the same multiplicities as for the corresponding representations of $SO(3)$. In particular, taking $W = V_0 \oplus V_2$, the dimension of the space of invariants $\mathrm{Hom}_{\mathcal{C}}(V_0, W^{\otimes n})$ equals to the corresponding multiplicity for representations of $SO(3)$. In the latter case, the analogue of W is the four-dimensional representation $\widetilde{W}$ of $SO(3)$ isomorphic, as a representation of $sl(2)$, to $\widetilde{V}_1 \otimes \widetilde{V}_1 \cong \widetilde{V}_0 \oplus \widetilde{V}_2$, where $\widetilde{V}_1$ is the fundamental representation of $sl(2)$, and $\widetilde{V}_n$ is the irreducible representation of $sl(2)$ of dimension $n+1$. Multiplicities for these representations are the same in the categories of $SO(3)$ and $sl(2)$ representations. The identity representations $\mathbf{1}$ in these categories are isomorphic to V_0 and $\widetilde{V}_0$, respectively. One obtains that

$$\dim \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W^{\otimes n}) = \dim \mathrm{Hom}_{SO(3)}(\mathbf{1}, \widetilde{W}^{\otimes n}) = \dim \mathrm{Hom}_{sl(2)}(\mathbf{1}, \widetilde{V}_1^{\otimes 2n}) = c_n, \tag{90}$$

where c_n is the n-th Catalan number.

Let E be a 2-dimensional $\mathbf{k}$-vector space with the basis $\{a, b\}$. Consider the exterior algebra

$$A = \Lambda^* E,$$

thus, $a^2 = b^2 = 0$ and $ab = -ba$ in A. Algebra A has the following super-derivations:

$$x = (ab+1)\partial_a, \;\; y = (ab+1)\partial_b.$$

These super-derivations act on the left in the basis $\{1, a, b, ab + 1\}$ of A as follows:

	x	y	$[x, x] = 2x^2$	$[x, y] = xy + yx$	$[y, y] = 2y^2$
1	0	0	0	0	0
a	$ab{+}1$	0	$2b$	$-a$	0
b	0	$ab{+}1$	0	b	$-2a$
$ab{+}1$	b	$-a$	0	0	0

Thus, x and y generate an action of the Lie super-algebra $osp(1|2)$ on A. As an $osp(1|2)$-module, $A \cong \mathbf{1} \oplus V_2$, where $\mathbf{1}$ is the trivial module spanned by $1 \in A$ and $V_2 \cong \mathbf{k}^{1|2}$ is irreducible of dimension $(1|2)$ spanned by $a, b, ab + 1$. In particular, A is an object of $\mathcal{C}$.

The multiplication $A \otimes A \longrightarrow A$ is a map of $osp(1|2)$-modules, since $osp(1|2)$ acts by super-derivations. The unit map $\iota : \mathbf{1} \longrightarrow A$ is an $osp(1|2)$-module map as well. Being the exterior algebra, the algebra A is a commutative algebra object in the category of super-vector spaces and, consequently, in the category $\mathcal{C}$.

The algebra object $A \in \mathcal{C}$ is Frobenius with respect to the linear form $A \to \mathbf{1}$ sending $1 \in A$ to a non-zero constant $\beta \in \mathbf{1}$ and $a, b, ab + 1$ to zero.

Assume $\beta = 1$. Then the Frobenius comultiplication $A \to A \otimes A$ is given by

$$\begin{aligned} 1 &\mapsto 1 \otimes ab - a \otimes b + b \otimes a + ab \otimes (1 + ab) \\ a &\mapsto a \otimes ab + ab \otimes a \\ b &\mapsto b \otimes ab + ab \otimes a \\ ab &\mapsto ab \otimes ab \end{aligned} \tag{91}$$

Composing with the multiplication, we see that the composition $m\Delta : A \longrightarrow A$, which is the handle endomorphism, is the zero map. One computes immediately that $\alpha_0 = \beta = 1$, $\alpha_i = 0$ for $i > 0$, so that the generating function for A is $Z(T) = 1$. For arbitrary invertible β one should insert β^{-1} after the arrow in each line in the map (91) above. Then $\alpha_0 = \beta$ and $\alpha_i = 0$ for $i > 0$, with $Z(T) = \beta$.

Consider the skein category SCob_α for the constant series $Z_\alpha(T) = \beta$, its gligible quotient Cob_α, and the Karoubi envelope $\underline{\mathrm{DCob}}_\alpha$ of the latter. The skein category has the relations that the handle is equal to 0 while the 2-sphere evaluates to β.

There is a functor $F_A : \mathrm{SCob}_\alpha \longrightarrow \mathcal{C}$ taking the circle object 1 to A as explained in Sect. 2.3, see (27). To apply Proposition 2.4, note that $\mathcal{C}$ is semisimple and that any object of $\mathcal{C}$ is a direct summand of $A^{\otimes n}$ for some n. To show that F_A is surjective on homomorphisms, it suffices to check that the maps

$$\mathrm{Hom}(n, m) \xrightarrow{F_A} \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n}, A^{\otimes m})$$

induced by F_A are surjective. Furthermore, by duality, it is enough to establish surjectivity of maps

$$\mathrm{Hom}(0, n) \xrightarrow{F_A} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A^{\otimes n}). \tag{92}$$

The image $F_A(\mathrm{Hom}(0, n))$ is a subspace of dimension bounded from below by the rank of the matrix $D_n(\beta)$ from Sect. 5.3, see (84),

$$\dim(F_A(\mathrm{Hom}(0, n))) \geq \mathrm{rk}(D_n(\beta)).$$

We proved in that section that $D_n(\beta)$ has nonzero determinant over a characteristic zero field and has rank equal to the Catalan number c_n. Therefore,

$$\dim(F_A(\mathrm{Hom}(0, n))) \geq c_n. \tag{93}$$

From Eq. (90) we know that dimension of $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A^{\otimes n})$ is also c_n. Consequently, the inequality in (93) is equality and the map (92) is an isomorphism. We conclude that F_A is surjective on homomorphisms. By Proposition 2.4 it induces an equivalence from the Karoubi completion of the negligible quotient Cob_α of SCob_α to $\mathcal{C}$.

Theorem 5.5 *Over a field* $\mathbf{k}$ *of characteristic zero, the category* $\underline{\mathrm{DCob}}_\alpha$ *is equivalent, as a symmetric monoidal category, to the above category* $\mathcal{C}$ *of representations of* $osp(1|2)$*:*

$$\underline{\mathrm{DCob}}_\alpha \cong \mathcal{C}. \tag{94}$$

This gives an alternative proof that $\dim A(n) = c_n$ over a characteristic zero field, which is part of Theorem 5.1 .

Remark 5.6 (*Vera Serganova*) Let G be a supergroup and $\mathfrak{g} = \mathfrak{g}(G) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_1$ is the odd part of the corresponding Lie superalgebra. Assume that the top exterior power $\Lambda^{top}\mathfrak{g}_1 = \Lambda^{\dim(\mathfrak{g}_1)}\mathfrak{g}_1$ is the trivial representation V_0 of $\mathfrak{g}_0$. Then the ring of functions $A = \mathrm{Spec}(G/G_0)$ on G/G_0 is a super-commutative Frobenius algebra isomorphic to the exterior algebra $\Lambda^*\mathfrak{g}_1$. A is a commutative Frobenius algebra object in the category of G-modules (the category of super vector spaces with an action of G).

As a G-module, A is isomorphic to the induced representation $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V_0)$,

$$A \cong \Lambda^*\mathfrak{g}_1 \cong \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V_0). \tag{95}$$

The $G-$invariant trace map $\epsilon : A \longrightarrow \mathbf{k}$ comes from the Frobenius reciprocity via the identity map of $\mathfrak{g}_0$-modules from the trivial representation V_0 to itself.

For our case of $G = OSp(1|2)$ and $G_0 \cong SL(2)$ this gives an alternative construction of the commutative Frobenius algebra as considered in this section.

6 Gram determinants for theories of rank one and two

Recall that the rank $K = \dim A(1)$ of the theory with $Z(T) = P(T)/Q(T)$ is $\max(\deg P + 1, \deg Q)$.

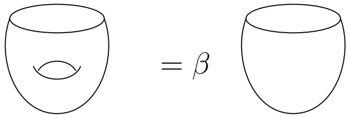


Fig. 14 Handle relation for the generating function in (98)

6.1 Generating function $\beta/(1-\gamma T)$ and the Deligne category

Generating function of a rank one theory has the form

$$Z_\alpha(T) = \frac{\beta}{1-\gamma T} = \beta + \beta\gamma T + \beta\gamma^2 T^2 + \ldots \tag{96}$$

with $\beta \in \mathbf{k}^*$ and $\gamma \in \mathbf{k}$. When $\gamma = 0$, the generating function is constant and the theory is studied in Sect. 5.

Assume now $\gamma \neq 0$. Rescaling $T \longmapsto \lambda T$ and $Z(T) \longmapsto \lambda^{-1} Z(\lambda T)$ by invertible λ leads to isomorphic theories, see Sect. 2.1. Rescaling T to $\gamma^{-1}T$ and $Z(T)$ to $\gamma Z(\gamma^{-1}T)$ reduces the theory to that for the generating function

$$Z(T) = \frac{\beta\gamma}{1-T} \tag{97}$$

and allows us to restrict to the case $\gamma = 1$ and generating function

$$Z(T) = \frac{\beta}{1-T} = \beta + \beta T + \beta T^2 + \ldots, \tag{98}$$

with the handle relation in this case shown in Fig. 14 (note that the rescaling above changes the handle relation, in general, by rescaling x).

The skein category SCob_α for this sequence $\alpha = (\beta, \beta, \beta, \ldots)$ is equivalent to the partition category Pa_β, the category DCob_α to the Deligne category $\mathrm{Rep}(S_\beta)$, and the gligible quotient $\underline{\mathrm{DCob}}_\alpha$ is equivalent to the gligible quotient $\underline{\mathrm{Rep}}(S_\beta)$.

Going back to the rational function in (96) with $\gamma \neq 0$, we obtain three equivalences of categories and a commutative diagram

$$\begin{array}{ccccc} \mathrm{SCob}_\alpha & \longrightarrow & \mathrm{DCob}_\alpha & \longrightarrow & \underline{\mathrm{DCob}}_\alpha \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathrm{Pa}_{\beta\gamma} & \longrightarrow & \mathrm{Rep}(S_{\beta\gamma}) & \longrightarrow & \underline{\mathrm{Rep}}(S_{\beta\gamma}) \end{array} \tag{99}$$

where in the bottom row appear the partition category, the Deligne category, and its gligible quotient, respectively, going from left to right, for the parameter $t = \beta\gamma$, and $\alpha = (\beta, \beta\gamma, \beta\gamma^2, \ldots)$.

When $\mathrm{char}\,\mathbf{k} = 0$ and $\beta\gamma \notin \mathbb{Z}_+ \subset \mathbf{k}$, the negligible ideal is zero, the quotient does not change the category, and there are equivalences

$$\mathrm{DCob}_\alpha \cong \underline{\mathrm{DCob}}_\alpha \cong \underline{\mathrm{Rep}}(S_{\beta\gamma}) \cong \mathrm{Rep}(S_{\beta\gamma}), \tag{100}$$

Table 1 Determinants of the bilinear form on $A(n)$ for the generating function $Z(T) = \frac{\beta}{1-\gamma T}$

n	B_n	det
1	1	β
2	2	$\beta^2\,(\beta\gamma-1)$
3	5	$\beta^5\,(\beta\gamma-1)^4\,(\beta\gamma-2)$
4	15	$\beta^{15}\,\gamma\,(\beta\gamma-1)^{14}\,(\beta\gamma-2)^7\,(\beta\gamma-3)$
5	52	$\beta^{52}\gamma^{10}\,(\beta\gamma-1)^{51}\,(\beta\gamma-2)^{36}\,(\beta\gamma-3)^{11}\,(\beta\gamma-4)$
6	203	$\beta^{203}\gamma^{73}\,(\beta\gamma-1)^{202}\,(\beta\gamma-2)^{171}\,(\beta\gamma-3)^{81}\,(\beta\gamma-4)^{16}\,(\beta\gamma-5)$
7	877	$\beta^{877}\gamma^{490}\,(\beta\gamma-1)^{876}\,(\beta\gamma-2)^{813}\,(\beta\gamma-3)^{512}\,(\beta\gamma-4)^{162}\,(\beta\gamma-5)^{22}\,(\beta\gamma-6)$

between the four categories in the middle and on the right of the diagram (99), see [10, Théorème 2.8].

When $\beta\gamma = n$ and char $\mathbf{k} = 0$, the category $\underline{\mathrm{DCob}}_\alpha$ is equivalent to the category of finite-dimensional representations of the symmetric group S_n over $\mathbf{k}$. If char $\mathbf{k} = p$, one should replace S_n by S_m, where m is the remainder after the division of n by p.

For the generating function in (96) the handle relation is obtained from that in Fig. 14 by replacing β by γ.

The n-circle state space $A_\alpha(n)$ of that theory is spanned by visible cobordisms with every connected component of genus zero. Diffeomorphism classes of these cobordisms are in a bijection with set-theoretic partitions of n. Properties of the Deligne category imply that for $\gamma \neq 0$ and $\beta\gamma$ not the image of an integer in $\mathbf{k}$ the set of these genus zero surfaces is a basis of $A_\alpha(n)$, so that in this case

$$\dim A_\alpha(n) \;=\; B_n, \tag{101}$$

where B_n is the Bell number, see Sect. 5.1. Recall from Sect. 5 that for $\gamma = 0$ genus zero surfaces span $A_\alpha(n)$ and constitute a basis of the latter when char $\mathbf{k} = 0$ and $\beta \neq 0$, leading to the formula (101) in this case as well.

Bilinear form data. We next compute the bilinear form on this spanning set, keeping β, γ as formal variables, for small values of n. Table 1 below describes determinants of Gram matrices G_n of size $B_n \times B_n$, with rows and columns enumerated by set partitions λ and μ of n. To each set partition λ we assign a surface $S(\lambda)$ that matches n boundary circles into genus zero connected components via parts of the partition. These surfaces span $A_\alpha(n)$ for all values of $\beta, \gamma \in \mathbf{k}$ and constitute a basis in the generic case.

Two surfaces $S(\lambda), \underline{S(\mu)}$ share the common boundary of n circles and can be glued into a closed surface $\overline{S(\mu)}S(\lambda)$. Evaluating this surface via the generating function $Z_\alpha(T)$ in (96) gives a monomial in β and γ that we put as the (λ, μ)-entry of the matrix G_n and then compute its determinant.

For instance, for $n = 2$ there are two partitions, and the Gram matrix is $\begin{pmatrix} \beta^2 & \beta \\ \beta & \beta\gamma \end{pmatrix}$, with the determinant $\beta^2(\beta\gamma - 1)$.

Exponents of γ. The main surprising feature of the above table is given by terms that are powers of γ. Nonzero powers of γ in the above determinant for $\beta/(1-\gamma T)$ start on line four. Adding zero as the power of γ on line three gets us the sequence $(0, 1, 10, 73, 490)$ matching the Sloan sequence A200580, see https://oeis.org/A200580. It relates to the supercharacters of the finite group of unipotent upper-triangular matrices over the 2-element field [2, 9] and can also be expressed as the combination $-2B_{n+2}+(n+4)B_{n+1}$ of two Bell numbers.

Exponents of determinant factors $\beta\gamma - k$. The exponents of factors of det G_n are related to representation theory of the symmetric group. Namely, for n shown in the table, the difference between the Bell number B_n (the dimension of $A(n)$ in the generic case) and the exponent $b_{k,n}$ of $\beta\gamma - k$ in det G_n, for $k \geq 1$, is equal to the dimension of invariants in $V^{\otimes n}$, where V is the natural k-dimensional representation of S_k:

$$B_n - b_{k,n} = \dim \left(V^{\otimes n}\right)^{\text{inv}} . \tag{102}$$

We expect this pattern to hold for all n, see Fig. 2.

The dimension of invariants can be calculated via characters. Conjugacy classes in S_k are parametrized by cycle types of permutations, which are in a bijective correspondence with Young diagrams with k boxes. The diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$ corresponds to the conjugacy class of permutations with cycle lengths given by λ. The trace of such permutation on V equals to $n - l(\lambda)$. Hence, the multiplicity of the trivial representation in $V^{\otimes n}$ is

$$\dim \left(V^{\otimes n}\right)^{\text{inv}} = \sum_{|\lambda|=k} \frac{1}{z_\lambda} \left(m_1(\lambda)\right)^n ,$$

where

$$\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \ldots).$$

For $k = 1$ representation V of S_1 is trivial, and (102) specializes to

$$B_n - b_{1,n} = \dim(V^{\otimes n})^{\text{inv}} = 1, \quad n > 0.$$

Exponents of β. Denote by $b_{0,n}$ the exponent of β in det G_n, see Table 1. The sequence $(1, 2, 5, 15, 52, 203, 877)$ of exponents of β in that table is the Bell numbers sequence A000110, see https://oeis.org/A000110, and we expect this pattern to hold for all n, so that $b_{0,n} = B_n$. Furthermore,

$$b_{0,n} - b_{1,n} = 1, \quad n > 0.$$

which matches the data in Table 1 (difference in exponents of β and $\beta\gamma - 1$ is 1), so we expect

$$b_{0,n} = b_{1,n} + 1 = B_n, \quad n > 0.$$

Table 2 Prediction for the exponents of linear factors are given in the last three columns, for $n = 8, 9, 10$

factor	Group	$A(0)$	$A(1)$	$A(2)$	$A(3)$	$A(4)$	$A(5)$	$A(6)$	$A(7)$	$A(8)$	$A(9)$	$A(10)$
$\beta\gamma - 1$	S_1	0	0	1	4	14	51	202	876	4139	21146	115974
$\beta\gamma - 2$	S_2	0	0	0	1	7	36	171	813	4012	20891	115463
$\beta\gamma - 3$	S_3	0	0	0	0	1	11	81	512	3046	17866	106133
$\beta\gamma - 4$	S_4	0	0	0	0	0	1	16	162	1345	10096	72028
$\beta\gamma - 5$	S_5	0	0	0	0	0	0	1	22	295	3145	29503
$\beta\gamma - 6$	S_6	0	0	0	0	0	0	0	1	29	499	6676
$\beta\gamma - 7$	S_7	0	0	0	0	0	0	0	0	1	37	796
$\beta\gamma - 8$	S_8	0	0	0	0	0	0	0	0	0	1	46
$\beta\gamma - 9$	S_9	0	0	0	0	0	0	0	0	0	0	1
$\beta\gamma - 10$	S_{10}	0	0	0	0	0	0	0	0	0	0	0

Table 3 Two-colored Bell numbers and Gram determinants for the function $Z(T) = \beta/(1-\gamma T)^2$

n	$B_n^{(2)}$	det
1	2	$-\beta^2$
2	6	$-\beta^{10}\gamma^{12}$
3	22	$-\beta^{50}\gamma^{66}$
4	94	$-\beta^{266}\gamma^{376}$
5	454	$-\beta^{1522}\gamma^{2270}$

6.2 Gram determinants for rank two theories

1. Consider the generating function

$$Z(T) = \frac{\beta}{(1-\gamma T)^2}, \quad \beta, \gamma \in \mathbf{k}^*. \tag{103}$$

This theory has $K = 2$ and the handle relation $(x-\gamma)^2 = 0$. Sequence α for this theory has no abelian realizations (Table 2).

The space $A(n)$ has a spanning set consisting of viewable surfaces with $\sqcup_n \mathbb{S}^1$ as the boundary and each component of genus at most one. Elements of the spanning set are set partitions of n carrying labels 0, 1 (the genus of a component), and their count is the generalized (two-colored) Bell number $B_n^{(2)}$, see [32, 36], with the generating exponential function

$$\sum_{n\geq 0} B_n^{(2)} \frac{t^n}{n!} = \exp\left(2(\exp(x) - 1)\right). \tag{104}$$

The first few values of $B_n^{(2)}$ are listed in Tables 3 and 7.

Table 4 Dimensions and determinants for the function $Z(T) = (\beta_0 + \beta_1 T)/(1-\gamma T)^2$. The difference with the previous table is $\beta_0\gamma + \beta_1$ taking place of β

n	dim	det
1	2	$-(\beta_0\gamma + \beta_1)^2$
2	6	$-\gamma^2(\beta_0\gamma + \beta_1)^{10}$
3	22	$-\gamma^{16}(\beta_0\gamma + \beta_1)^{50}$
4	94	$-\gamma^{110}(\beta_0\gamma + \beta_1)^{266}$
5	454	$-\gamma^{748}(\beta_0\gamma + \beta_1)^{1522}$

Values of the determinant of the Gram matrix for this spanning set, from computer computations, are shown in Table 3. For each $n \leq 5$ the determinant is the negative product of powers of β and γ.

The following guess works for the powers of β in the table: it is the total number of components of all elements of the spanning set, that is, of two-colored partitions of n. More explicitly, the exponent of β matches the sequence $B^{(2)}_{n+1} - 2B^{(2)}_n$.

Consider a sequence with the terms given by half of the power of β in the n-th Gram determinant plus $B^{(2)}_n$, see Table 3. The sequence of exponents has the form $(1, 3, 11, 47, 227, 1215, \ldots)$, with the first few terms matching the sequence A035009 in the Sloan encyclopedia, see https://oeis.org/A035009. Multiplying terms of the latter sequence by 2 recovers the sequence $(B^{(2)}_n)_n$, with the index shifted by one.

The degree of γ in the table matches the product of numbers in the first two columns of the table, that is, $nB^{(2)}_n$.

2. Table 4 below shows the Gram determinants of the same spanning set of cobordisms with components of genus at most one for the generating function

$$Z(T) = \frac{\beta_0 + \beta_1 T}{(1-\gamma T)^2} \tag{105}$$

that deforms the function in (103) without changing the parameter $K = 2 = \dim A(1)$ of the theory, that is, its rank.

3. Gram determinants for the generating function

$$Z(T) = \frac{\beta}{(1-T)(1-\gamma T)} \tag{106}$$

are given in Table 5.

4. Consider the most general generating function, over an algebraically closed $\mathbf{k}$, for a theory of rank two $(K = 2)$:

$$Z(T) = \frac{\beta_0 + \beta_1 T}{(1-\gamma_1 T)(1-\gamma_2 T)}. \tag{107}$$

Its partial fraction decomposition is given by

$$Z(T) = \frac{\beta_0\gamma_1 + \beta_1}{\gamma_1 - \gamma_2}\frac{1}{1-\gamma_1 T} + \frac{\beta_0\gamma_2 + \beta_1}{\gamma_2 - \gamma_1}\frac{1}{1-\gamma_2 T}.$$

Table 5 Determinants of the bilinear form on $A(n)$ for the generating function $Z(T) = \frac{\beta}{(1-T)(1-\gamma T)}$

n	det
1	$-\gamma\,\beta^2$
2	$-\gamma^4\beta^8\,(\gamma+\beta-1)\left(\beta\,\gamma^2-\gamma+1\right)$
3	$-\gamma^{17}\beta^{34}\,(\gamma+\beta-1)^7\left(\beta\,\gamma^2-\gamma+1\right)^7(\beta+2\,\gamma-2)\left(\beta\,\gamma^2-2\,\gamma+2\right)$
4	$-\gamma^{80}\beta^{158}\,(\gamma+\beta-1)^{42}\left(\beta\,\gamma^2-\gamma+1\right)^{42}(\beta+2\,\gamma-2)^{11}\left(\beta\,\gamma^2-2\,\gamma+2\right)^{11}\cdot$ $\cdot\,(\beta+3\,\gamma-3)\left(\beta\,\gamma^2-3\,\gamma+3\right)$
5	$-\gamma^{417}\beta^{804}\,(\gamma+\beta-1)^{251}\left(\beta\,\gamma^2-\gamma+1\right)^{251}(\beta+2\,\gamma-2)^{91}\left(\beta\,\gamma^2-2\,\gamma+2\right)^{91}\cdot$ $(\beta+3\,\gamma-3)^{16}\left(\beta\,\gamma^2-3\,\gamma+3\right)^{16}(\beta+4\,\gamma-4)\left(\beta\,\gamma^2-4\,\gamma+4\right)$

Table 6 Dimensions and determinants for the function $Z(T) = (\beta_0+\beta_1 T)/((1-\gamma_1 T)(1-\gamma_2 T))$

n	dim	det
1	2	$-\left(\beta_0\gamma_1+\beta_1\right)\left(\beta_0\gamma_2+\beta_1\right)$
2	6	$-\left(\beta_0\gamma_1+\beta_1\right)^4\left(\beta_0\gamma_1^2+\beta_1\gamma_1-\gamma_1+\gamma_2\right)\left(\beta_0\gamma_2+\beta_1\right)^4\left(\beta_0\gamma_2^2+\beta_1\gamma_2-\gamma_2+\gamma_1\right)$
3	22	$-\left(\beta_0\gamma_1+\beta_1\right)^{17}\left(\beta_0\gamma_1^2+\beta_1\gamma_1-\gamma_1+\gamma_2\right)^7\left(\beta_0\gamma_1^2+\beta_1\gamma_1-2\gamma_1+2\gamma_2\right)$ $\left(\beta_0\gamma_2+\beta_1\right)^{17}\left(\beta_0\gamma_2^2+\beta_1\gamma_2-\gamma_2+\gamma_1\right)^7\left(\beta_0\gamma_2^2+\beta_1\gamma_2-2\gamma_2+2\gamma_1\right)$
4	94	$-\gamma_1\gamma_2\left(\beta_0\gamma_1+\beta_1\right)^{79}\left(\beta_0\gamma_1^2+\beta_1\gamma_1-\gamma_1+\gamma_2\right)^{42}\left(\beta_0\gamma_1^2+\beta_1\gamma_1-2\gamma_1+2\gamma_2\right)^{11}$ $\left(\beta_0\gamma_1^2+\beta_1\gamma_1-3\gamma_1+3\gamma_2\right)\left(\beta_0\gamma_2+\beta_1\right)^{79}\left(\beta_0\gamma_2^2+\beta_1\gamma_2-\gamma_2+\gamma_1\right)^{42}$ $\left(\beta_0\gamma_2^2+\beta_1\gamma_2-2\gamma_2+2\gamma_1\right)^{11}\left(\beta_0\gamma_2^2+\beta_1\gamma_2-3\gamma_2+3\gamma_1\right)$
5	454	$-\gamma_1^{15}\gamma_2^{15}\left(\beta_0\gamma_1+\beta_1\right)^{402}\left(\beta_0\gamma_1^2+\beta_1\gamma_1-\gamma_1+\gamma_2\right)^{251}\left(\beta_0\gamma_1^2+\beta_1\gamma_1-2\gamma_1+2\gamma_2\right)^{91}$ $\left(\beta_0\gamma_1^2+\beta_1\gamma_1-3\gamma_1+3\gamma_2\right)^{16}\left(\beta_0\gamma_1^2+\beta_1\gamma_1-4\gamma_1+4\gamma_2\right)$ $\left(\beta_0\gamma_2+\beta_1\right)^{402}\left(\beta_0\gamma_2^2+\beta_1\gamma_2-\gamma_2+\gamma_1\right)^{251}\left(\beta_0\gamma_2^2+\beta_1\gamma_2-2\gamma_2+2\gamma_1\right)^{91}$ $\left(\beta_0\gamma_2^2+\beta_1\gamma_2-3\gamma_2+3\gamma_1\right)^{16}\left(\beta_0\gamma_2^2+\beta_1\gamma_2-4\gamma_2+4\gamma_1\right)$

Table 6 shows values of the Gram determinant for the same spanning set of $A(n)$.

Powers of $\beta_0\gamma_i+\beta_1$, $i=1,2$ are given by the sequence (1, 4, 17, 79, 402), which matches the beginning of the Sloan sequence $A289924$, see https://oeis.org/A289924. The differences between total dimensions, shown in the middle column, and these exponents, for the simplest factors $\beta_0\gamma_i+\beta_1$, give the sequence (1, 2, 5, 15, 52), matching Bell numbers.

For the factor $\left(\beta_0\gamma_1^2+\beta_1\gamma_1-\gamma_1+\gamma_2\right)$ and its image under the index transposition $\gamma_1 \leftrightarrow \gamma_2$ the differences are given by the sequence (2, 5, 15, 52, 203) matching the shifted sequence of Bell numbers.

For the factor $\left(\beta_0\gamma_1^2+\beta_1\gamma_1-2\gamma_1+2\gamma_2\right)$ and its transposition under $\gamma_1 \leftrightarrow \gamma_2$, the sequence of differences is (2, 6, 21, 83, 363), which can be represented as $\frac{1}{2}(B_{n+2}-B_{n+1}+B_n)$.

We expect that various patterns observed above hold for all n.

7 Polynomial generating functions

In this section the categories $\underline{\mathrm{DCob}}_\alpha$ are investigated for polynomial generating functions $Z(T) = Z_\alpha(T)$, i.e. $\alpha_i = 0$ for $i \gg 0$. The case of the constant function $Z(T) = \beta$ was studied in Sect. 5, so we will assume that the degree of the polynomial $Z(T)$ is at least one. There are two main cases:

- Function $Z(T)$ is linear. Its tensor envelopes are closely related to the unoriented Brauer category $\mathrm{Rep}(O_t)$.
- Function $Z(T)$ has degree at least two. Its series α has no abelian realizations, see condition (3) in Theorem 3.4. Experimental data, discussed below, indicates that $\dim A_\alpha(n)$ depends only on the degree of the polynomial $Z(T)$.

Note that a theory of degree K has a skein relation that reduces the K-th power of a handle to a linear combination of lower degree powers. Consequently, the state space $A(n)$ has a spanning set given by partitions with an integer between 0 and $K-1$ (inclusive) assigned to each component.

Generalized Bell numbers $B_n^{(k)}$ count set partition of n together with an assignment of an integer between 0 and $k-1$ (inclusive) to each part of the partition [32, 36]. Elements of the latter set are in a bijection with diffeomorphism classes of viewable surfaces with n fixed boundary components and at most $k-1$ handles on each component.

For a given k, generalized Bell numbers have the following exponential generating function:

$$\exp(k(\exp(t)-1)) = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}. \tag{108}$$

When $Z(T)$ is a polynomial $P(T)$, rank K of the theory is $1 + \deg P$.

7.1 Linear generating function and the unoriented Brauer category

Here we consider the case

$$Z(T) = \beta_0 + \beta_1 T \tag{109}$$

of a linear generating function, with $\beta_0, \beta_1 \in \mathbf{k}$ and $\beta_1 \neq 0$. Evaluations of connected surfaces for this α are shown in Fig. 15. Alternatively, one can treat this theory as defined over a ring that contains $\mathbf{k}[\beta_0, \beta_1]$, in which case β_1 may not be invertible.

Scaling by $\lambda = \mu^2$ as in Sect. 2.1 changes $Z(T)$ to $\lambda^{-1}\beta_0 + \beta_1 T$. Consequently, if every invertible element of $\mathbf{k}$ is a square, we can reduce to one of the two cases:

$$(1)\ \ Z_0(T) = \beta_1 T, \qquad (2)\ \ Z_1(T) = 1 + \beta_1 T.$$

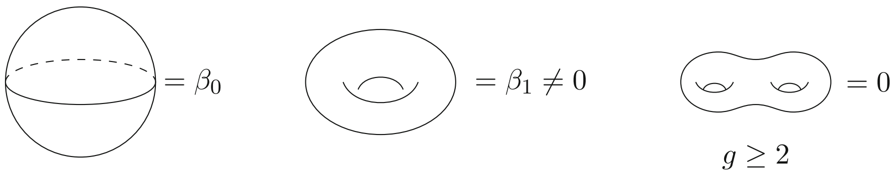


Fig. 15 Evaluation is zero beyond genus one, $\beta_1 \neq 0$

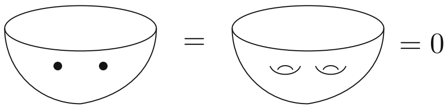


Fig. 16 Two or more dots on a connected component evaluate to zero, so for a spanning set one can reduce to each connected component carrying at most one dot. Handles in diagrams can be converted to dots for convenience

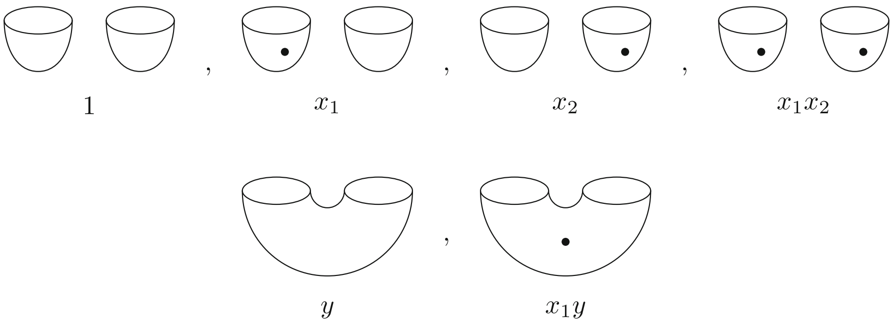


Fig. 17 Six vectors that span $A_\alpha(2)$. Notation x_i denotes a dot on the i-th cup, while y stands for the tube

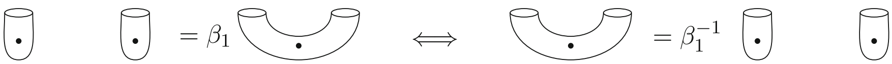


Fig. 18 Skein relation on diagrams with two boundary circles. One can either exclude two cups with dots or exclude a tube with dots. If not working over a field, only transformation on the left may be allowed

Skein relations. For now, consider the general case, with both $\beta_0, \beta_1 \in \mathbf{k}$, $\beta_1 \neq 0$. Recall that a dot on a connected component is a shorthand for a handle. Due to the particular evaluation we are considering, two or more dots on a component evaluate the entire diagram to zero, see Fig. 16.

The state space of two circles $A(2) = A_\alpha(2)$ for this theory was considered in [22, Sect. 6.2]. It has a spanning set of six vectors, shown in Fig. 17: a pair of disks, each with no dots or a single dot, and a tube, either dotless or with a dot.

One skein relation on these six vectors, shown in Fig. 18, holds for all values of the parameters and allows to exclude the vector x_1x_2 from the list. To verify this relation observe that the pairing of the both sides with any vector from Fig. 17 except for the vector 1 is zero; and it is easy to see that the pairings of both sides with vector 1 are β_1^2. Thus the difference of the left hand side and the right hand side is a negligible morphism, so it vanishes in categories Cob_α and $\underline{\mathrm{DCob}}_\alpha$.

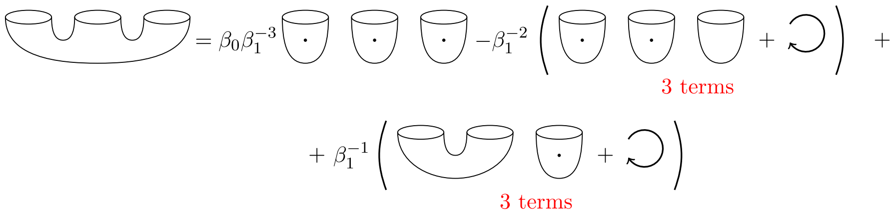


Fig. 19 Sphere with 3 holes as a linear combination of seven terms. The expression on the RHS is invariant under the permutation action of S_3 and terms permuted under the action are grouped together into sets of three. Terms in the first group of 3 differ in dot placement

The Gram determinant for the remaining five vectors is $\beta_1^6(\beta_1 - 2)$. In particular, if $\beta_1 \neq 2$, these five vectors constitute a basis of $A(2)$ and $\dim A(2) = 5$.

The case $\beta_1 = 2$ gives a multiplicative theory discussed at the end of [22, Sect. 2.5]. Note that when $\beta_1 = 2$, we have char $\mathbf{k} \neq 2$ since $\beta_1 \neq 0$.

Let us now assume that $\beta_1 \neq 2$. Figure 18 relation allows to reduce a component of genus one with more than one boundary circle to components of genus one with just one boundary circle each.

Consider the state space $A(3) = A_\alpha(3)$ for three circles. These reductions give a spanning set of cobordisms with each component of genus at most one (equivalently, of genus 0 with at most one dot), and genus one components with only one boundary circle each. Furthermore, a direct computation shows that a genus zero component with three boundary circles is a linear combination of other cobordisms from this spanning set, see Fig. 19.

Relation in Fig. 19 implies the relation in Fig. 18 by capping off one of the three boundary circles with a one-holed torus. Inductive application of these relations, together with the one in Fig. 16, allows to reduce any connected component with three or more boundary circles to a linear combination of surfaces where

- each component has at most one boundary circle,
- all components have genus zero or one,
- each genus one component bounds one circle.

Proposition 7.1 *The space $A(n)$ has a spanning set $\mathscr{A}(n)$ that consists of viewable cobordisms with n boundary circles, with each connected component having one or two boundary circles, all genus one components with one boundary circle and no components of genus two or higher. The cardinality a_n of the above set $\mathscr{A}(n)$ of cobordisms satisfies the following recurrent relation*

$$a_n = 2a_{n-1} + (n-1)a_{n-2}, \tag{110}$$

and has the following generating function

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n = \exp\left(2t + \frac{t^2}{2}\right)$$

Table 7 Determinants of the bilinear form on $A(n)$ for the generating function $Z(T) = \beta_0 + \beta_1 T$. Notice the appearance of the term $\beta_1 + 2$ in the last two lines

n	$B_n^{(2)}$	dim $A(n)$	det
1	2	2	$-\beta_1^2$
2	6	5	$(\beta_1 - 2)\,\beta_1^8$
3	22	14	$-(\beta_1 - 2)^6 \beta_1^{30}$
4	94	43	$(\beta_1 - 3)^2 (\beta_1 - 2)^{27} \beta_1^{113}$
5	454	142	$-(\beta_1 - 3)^{20} (\beta_1 - 2)^{110} \beta_1^{440}$
6	2430	499	$(\beta_1 - 4)^5 (\beta_1 - 3)^{134} (\beta_1 - 2)^{435} \beta_1^{1774} (\beta_1 + 2)$
7	14214	1850	$-(\beta_1 - 4)^{70} (\beta_1 - 3)^{756} (\beta_1 - 2)^{1722} \beta_1^{7406} (\beta_1 + 2)^{14}$

Proof Using the relations in Figs. 18 and 19 we can express any cobordism as a linear combination of the elements of $\mathscr{A}(n)$. Let us show that the relation (110) holds. Consider the first boundary circle of a cobordism from $\mathscr{A}(n)$. If it is the only boundary circle of its connected component C then the cobordism is a union of this component of genus zero or one and an element of $\mathscr{A}(n-1)$, giving $2a_{n-1}$ possibilities. Otherwise, C has genus zero and two boundary circles. There are $n-1$ options for the second circle, giving $(n-1)a_{n-2}$ possibilities in this case and proving (110). The derivation of the generating function from the recurrence relation is standard. □

Corollary 2 dim $A(n) \leq a_n$ *for* a_n *as above, for any field* $\mathbf{k}$ *and* $\beta_0 \in \mathbf{k}$, $\beta_1 \in \mathbf{k}^*$.

Remark 7.2 The sequence $(a_n)_{n\geq 0}$ is the Sloan sequence A005425, see https://oeis.org/A005425,

$$(a_0, a_1, a_2, \dots) = (1, 1, 2, 5, 14, 43, 142, 499, 1850, 7193, \dots).$$

Numerical data for dim $A(n)$ for generic values of β_0, β_1 and the Gram determinant for the spanning set $\mathscr{A}(n)$ with $n \leq 7$ is given in Table 7. The third column shows dim $A(n)$ for generic values of β_0, β_1. The last column shows the values of the Gram determinant for the set of vectors in the above spanning set $\mathscr{A}(n)$. Observe that the determinants do not depend on β_0 and can vanish only when β_1 is in the image of $\mathbb{Z}$ in $\mathbf{k}$. Note that non-vanishing of the determinants implies that dim $A(n) = a_n$ for $n \leq 7$ and generic β's. We are going to show that the same is true for any n.

Consider the Deligne orthogonal category Rep(O_t), $t \in \mathbf{k}$, see e.g. [10, Sect. 9]. Let $V \in \mathrm{Rep}(O_t)$ be the generating object corresponding to one element set in [10, Definition 9.2]. By definition, we have

$$\begin{aligned} \mathrm{inv}\left(V^{\otimes 2n}\right) &:= \dim \mathrm{Hom}_{\mathrm{Rep}(O_t)}(\mathbf{1}, V^{\otimes 2n}) = (2n-1)!!, \\ \mathrm{inv}\left(V^{\otimes 2n+1}\right) &= \dim \mathrm{Hom}_{\mathrm{Rep}(O_t)}(\mathbf{1}, V^{\otimes 2n+1}) = 0, \end{aligned}$$

where $(2n-1)!! = (2n-1)\dots 3 \cdot 1$ is the odd factorial.

Proposition 7.3 *a_n is the dimension of invariants of the n-th tensor power of the object $\mathbf{1} \oplus \mathbf{1} \oplus V \in \mathrm{Rep}(O_t)$,*

$$a_n \;=\; \dim \mathrm{Hom}_{\mathrm{Rep}(O_t)}(\mathbf{1}, (\mathbf{1}^2 \oplus V)^{\otimes n}). \tag{111}$$

Proof Let us compute the exponential generating function of dimensions of invariants

$$\begin{aligned}
&\sum_{n\geq 0} \dim\left(\mathrm{inv}\left((\mathbf{1}^2 \oplus V)^{\otimes n}\right)\right)\frac{u^n}{n!} = \dim(\mathrm{inv}\exp(2u + Vu))\\
&\quad = \dim(\mathrm{inv}\exp(2u))\exp(Vu)\\
&\quad = \exp(2u)\sum_{n\geq 0}\frac{u^n}{n!}\dim(\mathrm{inv}\ V^{\otimes n}) = \exp(2u)\sum_{n\geq 0}\frac{u^{2n}}{2n!}(2n-1)!!\\
&\quad = \exp\left(2u + \frac{u^2}{2}\right).
\end{aligned} \tag{112}$$

Here for a G-representation V we denote by

$$\exp(Vu) = \sum_{n\geq 0}\frac{u^n}{n!}V^{\otimes n} \in K(G)[[u]],$$

where $K(G)$ is the representation ring (tensored with $\mathbb{Q}$) of the group G. □

Proposition 7.3 motivates the following construction. Let $A \in \mathrm{Rep}(O_t)$ be the commutative Frobenius algebra obtained from a symmetrically self-dual object $V \in \mathrm{Rep}(O_t)$ by the construction in Example 2.1 in Sect. 2.2. The generating function of the α−invariant of algebra A is $\alpha_0 + \alpha_1 T$ where α_0 can be chosen arbitrarily and $\alpha_1 = t + 2$. It is also easy to see that the skein relation $x^2 = 0$ holds for the handle morphism of the algebra A. Thus by the universal property from Sect. 2.3 there is a symmetric tensor functor $F_\alpha : \mathrm{DCob}_\alpha \to \mathrm{Rep}(O_{\beta_1 - 2})$ sending the circle object to $A = \mathbf{1}^2 \oplus V$ (we assume here that $\beta_1 \neq 0$ since the skein relation is different in the case $\beta_1 = 0$). The following simple result is crucial:

Proposition 7.4 *Assume $\beta_1 \neq 0$. Then the functor F_α is full and essentially surjective.*

Proof By definition, the image of the functor F_α contains $A = \mathbf{1}^2 \oplus V$. This implies the essential surjectivity, as any object of $\mathrm{Rep}(O_t)$ is a direct summand of a direct sum of tensor powers of V.

Let us show that the functor F_α is full. Since the object A is self-dual it is sufficient to show that any morphism from $\mathrm{Hom}(\mathbf{1}, A^{\otimes n})$ is in the image of the functor F_α. Using the decomposition

$$A^{\otimes n} = (\mathbf{1}^2 \oplus V)^{\otimes n} = \bigoplus_{S\subset[1,\ldots,n]}\bigotimes_{i=1}^{n} X_i^S$$

where $X_i^S = \mathbf{1}^2$ if $i \in S$ and $X_i^S = V$ if $i \notin S$, we see that the space $\mathrm{Hom}(\mathbf{1}, A^{\otimes n})$ is spanned by the tensor products of morphisms from $\mathrm{Hom}(\mathbf{1}, A)$ and the pairing $\mathbf{1} \to V \otimes V \to A \otimes A$. Thus it is sufficient to check that F_α is surjective on $\mathrm{Hom}(\mathbf{1}, A^{\otimes n})$ for $n = 1, 2$. This is clear for $n = 1$ since the space $\mathrm{Hom}(\mathbf{1}, A)$ is two dimensional and the image of the functor F_α is at least two dimensional (by the first row of Table 7), as F_α does not annihilate non-negligible morphisms. The same argument (based on the second row of Table 7) works for $n = 2$ provided that $\beta_1 \neq 2$. Finally in the case $n = 2$, $\beta_1 = 2$ and $\mathrm{char}\, \mathbf{k} \neq 2$ one verifies by an explicit computation that the unique up to scaling negligible morphism in $\mathrm{Hom}(\mathbf{1}, A^{\otimes 2})$ is not annihilated by F_α. □

Remark 7.5 One verifies that the functor F_α annihilates relations in Figs. 18 and 19. Let $\mathrm{DCob}_\alpha^\bullet$ be the quotient of DCob_α by these relations. It is clear that the inequality from Corollary 2 holds in the category $\mathrm{DCob}_\alpha^\bullet$. Thus Proposition 7.4 implies that the functor $\mathrm{DCob}_\alpha^\bullet \to \mathrm{Rep}(O_{\beta_1-2})$ is also faithful. Hence, there is an equivalence of tensor categories $\mathrm{DCob}_\alpha^\bullet \simeq \mathrm{Rep}(O_{\beta_1-2})$.

Combining Propositions 7.4 and 2.4 results the following:

Theorem 7.6 *Assume $Z_\alpha = \beta_0 + \beta_1 T$ with $\beta_1 \neq 0$. The functor F_α induces an equivalence of tensor categories $\underline{\mathrm{DCob}}_\alpha \simeq \underline{\mathrm{Rep}}(O_{\beta_1-2})$, where $\underline{\mathrm{Rep}}(O_{\beta_1-2})$ is the gligible quotient of the Deligne category* $\mathrm{Rep}(O_{\beta_1-2})$.

Here is a special case. Assume that $\mathrm{char}\, \mathbf{k} = 0$. By a theorem of H. Wenzl (see e.g. [10, Théorèm 9.7]) we have $\mathrm{Rep}(O_{\beta_1-2}) = \underline{\mathrm{Rep}}(O_{\beta_1-2})$ when $\beta_1 \notin \mathbb{Z}$. It follows from Proposition 7.3 that in this case $\dim A(n) = a_n$, and the set $\mathscr{A}(n)$ is linearly independent. We have the following implications for the determinant $\det_n$ of the bilinear form on $A(n)$:

Proposition 7.7 *The polynomial* $\det_n$ *is nonzero and depends only on β_1 (and not on β_0); moreover its irreducible factors are of the form $\beta_1 - s$, $s \in \mathbb{Z}$.*

Proof It is clear that $\det_n$ is a polynomial in variables β_0 and β_1 with integer coefficients. As explained above this polynomial can vanish only when $\beta_1 \in \mathbb{Z}$, so that $\det_n$ does not depend on β_0, by elementary algebraic geometry. □

In the case $\mathrm{char}\, \mathbf{k} = 0$ and $t \in \mathbb{Z}$, the gligible quotients $\underline{\mathrm{Rep}}(O_t)$ are computed in [10, Théorèm 9.6]. Recall that

$$\underline{\mathrm{Rep}}(O_t) \cong \mathrm{Rep}(G, \varepsilon),$$

where G is one of the super groups $O(n)$ (if $t = n \geq 0$), $Sp(2m)$ (if $t = -2m$ is negative and even), $OSp(1, 2m)$ (if $t = 1 - 2m$ is negative and odd) and $\varepsilon \in G$ is a suitable involution. Thus we get the following examples illustrating Theorem 7.6:

Example 7.8 ($\mathrm{char}\, \mathbf{k} = 0$)

(1) Assume $Z_\alpha = \beta_0 + 2T$. Then $\underline{\mathrm{DCob}}_\alpha$ is the category Vec and the circle object corresponds to the Frobenius algebra $\mathbf{k}[x]/(x^2)$ with $\epsilon(1) = \beta_0$ and $\epsilon(x) = 1$. Note that in this case $\dim A(n) = 2^n$.

(2) Assume $Z_\alpha = \beta_0+3T$. Then $\underline{\mathrm{DCob}}_\alpha$ is the category $\mathrm{Rep}(\mathbb{Z}/2)$ and the circle object corresponds to the Frobenius algebra $A = \mathbf{k}[x]/(x^3)$ with $\epsilon(1) = \beta_0$, $\epsilon(x) = 0$, $\epsilon(x^2) = 1$, and the group $\mathbb{Z}/2$ acting on A via $x \mapsto -x$. Thus the character of the $\mathbb{Z}/2-$representation A takes values 3 and 1 on the elements $0, 1 \in \mathbb{Z}/2$, and $\dim A(n) = \frac{3^n+1}{2}$.

(3) Assume $Z_\alpha = \beta_0 - 2T$. Then $\underline{\mathrm{DCob}}_\alpha$ is the category $\mathrm{Rep}(Sp(4))$ (with the modified commutativity constraint), and the circle object corresponds to the Frobenius algebra $H^*(\Sigma_2, \mathbf{k})$, where Σ_2 is a oriented closed connected surface of genus two. Here $Sp(4)$ acts trivially on $H^{even}(\Sigma_2, \mathbf{k})$ and via the natural representation on $H^1(\Sigma_2, \mathbf{k})$. The commutativity constraint in $\mathrm{Rep}(Sp(4), \varepsilon)$ is modified in a way making the natural representation into an odd vector space, so the algebra $H^*(\Sigma_2, \mathbf{k})$ is commutative in the category $\mathrm{Rep}(Sp(4), \varepsilon)$.

We will see later (Proposition 7.12) that the leading coefficient of the polynomial $\det_n$ is ± 1. It follows that $\det_n$ is nonzero even if $\mathrm{char}\, \mathbf{k} > 0$; moreover its roots lie in the prime subfield of $\mathbf{k}$. Thus there is an equivalence $\underline{\mathrm{DCob}}_\alpha \cong \mathrm{Rep}(O_{\beta_1-2})$ provided that β_1 is not an element of the prime subfield. Note that in this case the category $\mathrm{Rep}(O_{\beta_1-2})$ is not semisimple by [14, Lemma 2.2].

Corollary 3 *Assume* $\mathrm{char}\, \mathbf{k} > 0$ *and* t *is not in the prime subfield of* $\mathbf{k}$. *Then the category* $\mathrm{Rep}(O_t)$ *is non-degenerate (i.e. has no nonzero negligible morphisms) and non-semisimple.*

In the case $\mathrm{char}\, \mathbf{k} > 0$ and β_1 is in the prime subfield we expect that the categories $\underline{\mathrm{DCob}}_\alpha$ are equivalent to the fusion categories associated with super groups (G, ε) as above (i.e. gligible quotients of suitable tilting modules categories). In particular, the categories $\underline{\mathrm{DCob}}_\alpha$ should have finitely many simple objects up to isomorphism.

We discuss now the multiplicities of the roots of polynomials $\det_n$. Here are some patterns that can be observed in Table 7:

- The differences $\dim A(n) - u_n$, where u_n is the exponent of $\beta_1 - 2$, are given by powers of two: (2, 4, 8, 16, 32, 64).
- The differences $\dim A(n) - w_n$, where w_n is the exponent of $\beta_1 - 3$, are given by (2, 5, 14, 41, 122, 365). These exponents match the sequence $(3^n + 1)/2$.

Comparing this patterns with Example 7.8 (1) and (2) we arrive at the following

Conjecture 7.9 *Let* $s \neq 0$ *be an integer. The exponent of the factor* $\beta_1 - s$ *in the polynomial* $\det_n$ *is given by*

$$a_n - \dim A_{\alpha(s)}(n)$$

where $\alpha(s) = (\beta_0, s, 0, 0, \dots)$, *so that the generating function* $Z_{\alpha(s)}(T) = \beta_0 + sT$ *(for arbitrary* β_0 *and* $\mathrm{char}\, \mathbf{k} = 0$*).*

Remark 7.10 By definition, the bilinear form on the space $\mathbf{k}\mathscr{A}(n)$ has a null space of dimension $a_n - \dim A_{\alpha(s)}(n)$. Thus, a standard argument (see e.g. [21, Lemma 8.4]) implies that the exponent of the factor $\beta_1 - s$ in $\det_n$ is greater or equal to $a_n - \dim A_{\alpha(s)}(n)$.

Note that, according to Theorem 7.6, the dimensions dim $A_{\alpha(s)}(n)$ are given by the dimensions of invariants of the (super) groups $G = O(k)$, $Sp(2k)$, $OSp(1|2k)$ in the representation $(\mathbf{1}^2 \oplus V)^{\otimes n}$ where V is the defining representation of G and $s = k+2$, $2-2k$, $3-2k$, respectively. We tabulated the exponents predicted by Conjecture 7.9 in Table 8. Here are some observations about Tables 8 and 7:

- The exponents for $\beta_1 - 1$ and $\beta_1 - 5$ coincide. The same applies to the exponents for β_1+1 and β_1-7 and to the exponents for β_1+3 and β_1-9 etc. This is explained by the coincidence of the multiplicities for tensor products for $OSp(1, 2k)$ and $O(2k+1)$, see [38].
- The irreducible factors of $\det_{2n}$ and $\det_{2n+1}$ coincide for any $n \geq 0$.
- The irreducible factors which appear in $\det_{2n}$ and do not appear at $\det_{2n-1}$ are

$$\beta_1 - n - 1 \text{ (for } n \geq 1), \beta_1 + 2n - 4 \text{ (for } n \geq 3), \text{ and } \beta_1 + n - 5 \text{ (for even } n \geq 4).$$

Conjecture 7.9 does not predict the exponent of the factor β_1 in $\det_n$. We propose the following

Conjecture 7.11 *The exponent of the factor β_1 in* $\det_n$ *is given by*

$$2na_{n-1} + a_n - c_{n+1}$$

where c_n is the Catalan number.

It would be interesting to find a categorical interpretation of this conjecture. The numerical data for it are given in the last row of Table 8. Term na_{n-1} in the above conjecture is the total number of connected components of genus one in the set of cobordisms $\mathscr{A}(n)$.

Table 7 suggests that the leading coefficient of the polynomial $\det_n$ is $(-1)^n$. Using this together with Conjectures 7.9 and 7.11 we can predict the polynomials $\det_n$. For example, the prediction for $\det_8$ is

$$(\beta_1-5)^{14}(\beta_1-4)^{630}(\beta_1-3)^{3912}(\beta_1-2)^{6937}(\beta_1-1)^{14}\beta_1^{31931}(\beta_1+2)^{133}(\beta_1+4).$$

One verifies that the degree of this polynomial agrees with Corollary 5 below.

7.2 Polynomials of degree two and three

Consider a polynomial generating function of degree two,

$$Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2 \tag{113}$$

The dimension of $A(n)$ is bounded from above by $B_n^{(3)}$, since all surfaces with a component of genera at least 3 are in the kernel of the bilinear form. However the computation shows that the actual dimension is strictly less than $B_n^{(3)}$ starting from $n = 2$, see the data in Table 9 for the quadratic $Z(T)$.

Table 10 shows the determinants for a generic polynomial of degree three.

Table 8 Prediction for the exponents of linear factors. Column for $A(0)$ is not shown, it contains zeros only. For a naive interpolation, the question mark in the bottom row on the left may be replaced by $Sp(2)$, but Conjecture 7.11 gives more complicated rules for the entries of this row than Conjecture 7.9 (if extended to $s = 0$) that should govern the other rows of the table

factor	Group	$A(1)$	$A(2)$	$A(3)$	$A(4)$	$A(5)$	$A(6)$	$A(7)$	$A(8)$	$A(9)$	$A(10)$	$A(11)$	$A(12)$
$\beta_1 - 2$	$O(0)$	0	1	6	27	110	435	1722	6937	28674	122085	536030	2426259
$\beta_1 - 3$	$O(1)$	0	0	0	2	20	134	756	3912	19344	93584	449504	2164634
$\beta_1 - 4$	$O(2)$	0	0	0	0	0	5	70	630	4620	30219	184338	1076229
$\beta_1 - 5$	$O(3)$	0	0	0	0	0	0	0	14	252	2862	26004	207350
$\beta_1 - 6$	$O(4)$	0	0	0	0	0	0	0	0	0	42	924	12705
$\beta_1 - 7$	$O(5)$	0	0	0	0	0	0	0	0	0	0	0	132
$\beta_1 - 8$	$O(6)$	0	0	0	0	0	0	0	0	0	0	0	0
$\beta_1 + 2$	$Sp(4)$	0	0	0	0	0	1	14	133	1050	7491	50226	323796
$\beta_1 + 4$	$Sp(6)$	0	0	0	0	0	0	0	1	18	216	2112	18370
$\beta_1 + 6$	$Sp(8)$	0	0	0	0	0	0	0	0	0	1	22	319
$\beta_1 + 8$	$Sp(10)$	0	0	0	0	0	0	0	0	0	0	0	1
$\beta_1 - 1$	$OSp(1,2)$	0	0	0	0	0	0	0	14	252	2862	26004	207350
$\beta_1 + 1$	$OSp(1,4)$	0	0	0	0	0	0	0	0	0	0	0	132
$\beta_1 + 3$	$OSp(1,6)$	0	0	0	0	0	0	0	0	0	0	0	0
β_1	?	2	8	30	113	440	1774	7406	31931	141864	648043	3038464	14601327

Table 9 Computation of dimensions and the determinant for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2$

n	$B_n^{(3)}$	dim	det
0	1	1	1
1	3	3	$-\beta_2^3$
2	12	11	$-\beta_2^{20}$
3	57	46	β_2^{118}
4	309	213	β_2^{696}
5	1866	1073	$-\beta_2^{4225}$

Table 10 Computation of dimensions and the determinant for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2 + \beta_3 T^3$

n	$B_n^{(4)}$	dim	det
0	1	1	1
1	4	4	β_3^4
2	20	19	$-\beta_3^{35}$
3	116	102	$-\beta_3^{266}$
4	756	604	β_3^{2007}
5	5428	3884	β_3^{15540}

7.3 Polynomials of arbitrary degree

Now consider the case of an arbitrary polynomial generating function:

$$Z = \beta_0 + \beta_1 T + \ldots + \beta_m T^m, \quad m \geq 1.$$

Let $\mathscr{A}^m(n)$ be the set of viewable cobordisms with n boundary circles such that for each component S of genus g with ℓ boundary circles the following inequality holds:

$$g + \ell \leq m + 1. \tag{114}$$

Note that $\mathscr{A}^1(n)$ is precisely the set $\mathscr{A}(n)$ from Proposition 7.1. Let us consider the matrix of the bilinear form on the space $A(n)$ computed at the elements of the set $\mathscr{A}^m(n)$, and let $\det_n^{(m)}$ denote its determinant. It is clear that $\det_n^{(m)}$ is a polynomial in variables $\beta_0, \beta_1, \ldots, \beta_m$. In the next Proposition we are going to compute the leading term of this polynomial. Let $d_n^{(m)}$ be the total number of connected components of all elements of the set $\mathscr{A}^m(n)$.

Proposition 7.12 *The polynomial* $\det_n^{(m)}$ *is of the form*

$$\pm(\beta_m)^{d_n^{(m)}} + \textit{lower terms}$$

where each lower term monomial has either less than $d_n^{(m)}$ factors or precisely $d_n^{(m)}$ factors but involves some β_i with $i < m$.

Proof The expansion of the determinant $\det_n^{(m)}$ is a sum over all permutations π of the set $\mathscr{A}^m(n)$ of terms

$$t_\pi = \pm \prod_{a \in \mathscr{A}^m(n)} b_{a,\pi(a)},$$

where $b_{a,\pi(a)}$ is the pairing of a and $\pi(a)$, hence some monomial in β_i's. The number of factors in the monomial $b_{a,\pi(a)}$ is precisely the number of connected components of the surface obtained from a and $\pi(a)$ by gluing along the boundary. Thus it is clear that the number of factors is less or equal to the number of connected components of a. Moreover, we have equality only if the partition of the boundary circles determined by the connected components of $\pi(a)$ is a refinement of the partition determined by a.

Thus, the total number of factors in t_π is less or equal than the total number of connected components of all elements $a \in \mathscr{A}^m(n)$, and every monomial in the polynomial $\det_n^{(m)}$ has $\leq d_n^{(m)}$ factors. The term t_π has precisely $d_n^{(m)}$ factors if and only if the permutation π has the following property:

(*) for any a the partition of the boundary circles determined by the connected components of $\pi(a)$ is a refinement of the partition determined by a.

Note that there exists $r > 0$ such that $\pi^r(a) = a$. Thus the condition (*) is equivalent to the following property:

(**) for any a the partition of the boundary circles determined by the connected components of $\pi(a)$ coincides with the partition determined by a.

Now let π_0 be the following permutation:

$\pi_0(a)$ is obtained from a by replacing each connected component of genus g with l boundary circles by the connected component of genus $g' = m + 1 - g - l$ with the same boundary circles. This transformation preserves inequality (114) and defines an involution π_0 on $\mathscr{A}^m(n)$.

Then every connected component of the closed surface $\overline{a}\pi_0(a)$ given by gluing a and $\pi_0(a)$ along the boundary has genus $g + g' + l - 1 = m$, and the term $t_{\pi_0} = \pm(\beta_m)^{d_n^{(m)}}$. It is also clear that for any other π satisfying (**) the term t_π will be either zero (if one of the components of $\overline{a}\pi_0(a)$ has genus greater than m) or will involve β_i with $i < m$ (if one of the components of the gluing has genus $< m$). This completes the proof of the proposition. □

Remark 7.13 The sign of the leading term is the sign of the permutation π_0; since π_0 is an involution, the sign can be computed from the number of fixed points.

Corollary 4 *The set $\mathscr{A}^m(n)$ is linearly independent in $A(n)$, for generic values of β_i's.*

Using the standard methods one computes the exponential generating functions for the sizes of the sets $\mathscr{A}^m(n)$ and for the sequence $d_n^{(m)}$:

$$\sum_{n\geq 0}\frac{|\mathscr{A}^m(n)|}{n!}t^n = \exp\left(\sum_{g=0}^{m}\frac{t^{m+1-g}}{(m+1-g)!}(g+1)\right),$$

$$\sum_{n\geq 0}\frac{d_n^{(m)}}{n!}t^n = \left(\sum_{g=0}^{m}\frac{t^{m+1-g}}{(m+1-g)!}(g+1)\right)\exp\left(\sum_{g=0}^{m}\frac{t^{m+1-g}}{(m+1-g)!}(g+1)\right).$$

In particular, for $m=1$ we get

Corollary 5 *The degree $d_n = d_n^{(1)}$ of the polynomial $\det_n = \det_n^{(1)}$ satisfies*

$$\sum_{n\geq 0}\frac{d_n}{n!}t^n = \left(2t+\frac{t^2}{2}\right)\exp\left(2t+\frac{t^2}{2}\right).$$

Equivalently, $d_n = \frac{1}{2}n(a_n + 2a_{n-1})$.

Conjecture 7.14 *$\mathscr{A}^m(n)$ spans $A(n)$.*

Proposition 7.15 *Assume that Conjecture 7.14 holds for some $m>1$. Then*

$$\det_n^{(m)} = \pm(\beta_m)^{d_n^{(m)}}.$$

Thus $\mathscr{A}^m(n)$ is a basis of $A(n)$ for any $\beta_0, \beta_1, \ldots, \beta_m$ with $\beta_m \neq 0$ and in any characteristic.

Proof The set of zeroes of $\det_n^{(m)}$ should be invariant under the scaling $(\beta_0, \beta_1, \beta_2, \ldots, \beta_m) \mapsto (\lambda^{-1}\beta_0, \beta_1, \lambda\beta_2 \ldots, \lambda^{m-1}\beta_m)$(see Sect. 2.1). Now the result follows from Proposition 7.12 since the leading term is multiplied by $(\lambda)^{d_n^{(m)}(m-1)}$ under the scaling and the potential lower terms are multiplied by lower power of λ. □

Remark 7.16 The argument above does not work for $m=1$ since β_1 does not change under the scaling. However it gives an alternative proof to the known fact that $\det_n = \det_n^{(1)}$ does not depend on β_0.

Theorem 7.17 *The Conjecture 7.14 holds for $m \leq 2$.*

Proof For $m=1$ it was established earlier. To prove it for $m=2$, let us first introduce the following notation.

The symmetric group S_n acts on $A(n)$ via the permutation cobordisms that permute n circles. Suppose given a cobordism y which is stabilized by a parabolic subgroup $S_\lambda \subset S_n$, for a decomposition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, so that $\sigma y = y$ for $y \in S_\lambda$. To y and λ assign the element $\sum_\lambda y$ of $A(n)$ given by

$$\sum_\lambda y \;:=\; \sum_{\sigma\in S_n/S_\lambda}\sigma y. \qquad (115)$$

That is, pick a representative τ in each coset S_n/S_λ, form τy and sum over cosets.

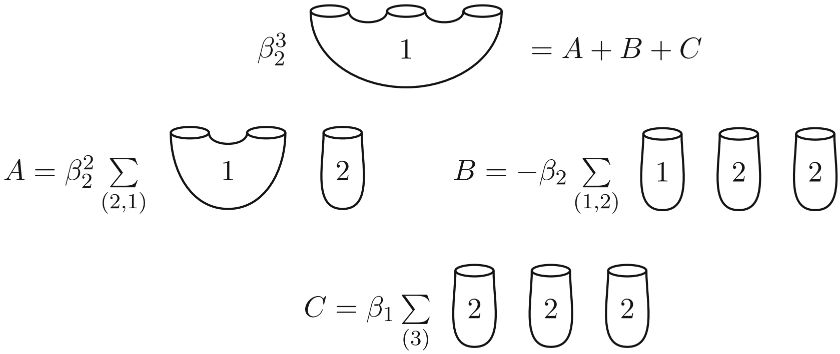


Fig. 20 Relation in $A(3)$ for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2$. Numbers 1 and 2 show the number of handles (dots) on the component. Summation means symmetrization with respect to permutations of boundary components parametrized by cosets of the stabilizer of the surface in S_3. Sums A, B, C have 3, 3, 1 terms respectively (7 terms in the right hand side in total)

For $m = 2$, the following relations hold in $A(3)$ and $A(4)$, see Figs. 20 and 21. Figure 20 relation reduces a 3-holed torus to a linear combination of other cobordisms, with each summand A, B, C also invariant under the permutation action of S_3. Each of these three terms is associated to a surface that has an obvious S_λ-invariance, for the decomposition λ shown under the sum sign. The term is the sum over surfaces in its orbit, as described above in (115).

Figure 21 relation has a similar presentation. For term B there the stabiliser of the surface is the dihedral group $D_{(4)} \subset S_4$ generated by the permutations (12), (34), and (13)(24). The corresponding subgroup is denoted $(2, 2)'$, it contains $S_{(2,2)}$ as an index two subgroup. This relation implies Fig. 20 relation by capping off a circle by a handle. Capping off by a disk results in a trivial relation.

We can exclude components with four boundary circles ($\ell = 4$) using Fig. 21 relation. To obtain the relations that simplify a genus i surface with $4 - i$ boundary components for $i = 1, 2$, see inequality (114), cap i boundary components by handles (one-holed tori) in Fig. 21 relation, resulting in Figs. 20 and 22 left relations. For $i = 3$, there is also the relation that a one-holed connected surface of genus three is 0 in $A(1)$, see Fig. 22 right.

These relations show that any connected component of genus g with ℓ boundary circles and $g + \ell > 2 + 1$ (since $m = 2$) simplifies to a linear combination of surfaces in the set $\mathscr{A}^2(n)$. Consequently, this set spans $A(n)$, establishing Conjecture 7.14 for $m = 2$. □

Remark 7.18 Originally, relation (21) was computed in Sage by finding the kernel of the $(|\mathscr{A}^2(4)| + 1) \times (|\mathscr{A}^2(4)| + 1)$-matrix of the quadratic form restricted to the elements of $\mathscr{A}^2(4)$ and the four-holed sphere.

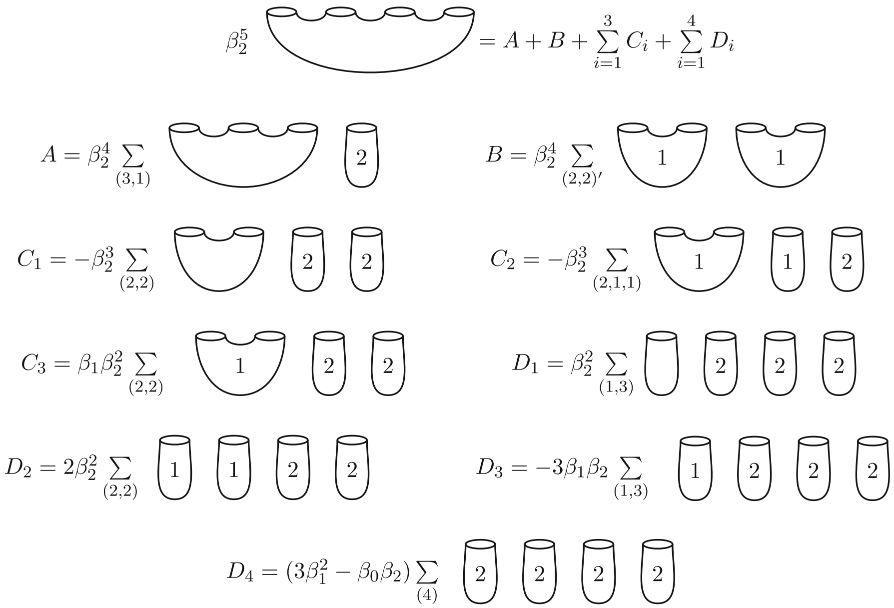


Fig. 21 Relation in $A(4)$ for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2$. Numbers 1 and 2 show the number of handles (dots) on the component. Summation means symmetrization with respect to permuting the boundary components, as described in the proof. Sums $A, B, C_1, C_2, C_3, D_1, D_2, D_3, D_4$ have 4, 3, 6, 12, 6, 4, 6, 4, 1 terms respectively (46 terms in the right hand side in total)

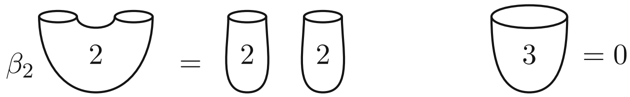


Fig. 22 Relations in $A(2)$ and $A(1)$ for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2$. Numbers 2 and 3 show the number of handles (dots) on the component

8 Cobordisms of fractional genus and other decorations

Fractional genus. Recall the defining relations in Proposition 4.2 on generators x and u of the algebra B_S. If $\alpha_n \neq 0$ for some even $n \geq 0$, then the element

$$\alpha_n^{-1} x^{n/2} u x^{n/2} \tag{116}$$

is an idempotent in B_S. This is an obvious way to get an idempotent in B_S unless the power series $Z(T)$ has nontrivial coefficients only at odd powers of T. With a minor effort, a version of the above idempotent can be produced in the latter case as well. Namely, for odd n and with $\alpha_n \neq 0$, we can try to make sense of the expression (116). For that one needs "cobordism" $x^{1/2}$, which should be a "genus 1/2" surface, with some boundary components. Let us consider an even more general case of a "genus $1/\ell$" surface for some $\ell > 1$. We simply introduce a fractional dot $x^{1/\ell}$ with the relation that its ℓ-th power is the handle, see Figs. 23 and 24.

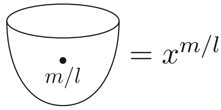


Fig. 23 Dot of fractional order m/ℓ

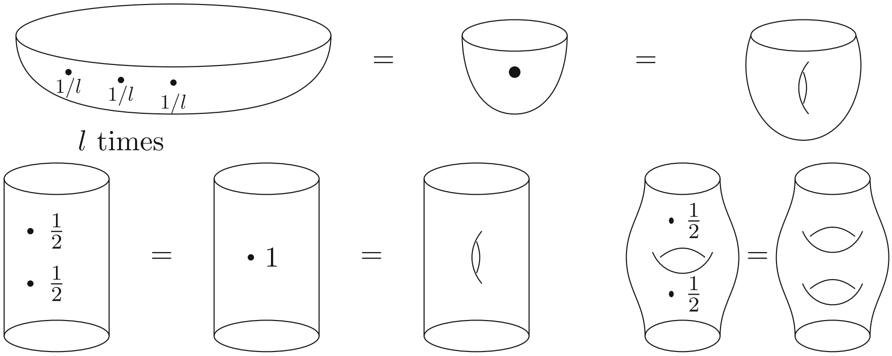


Fig. 24 Top: Fractional dot $x^{1/\ell}$ can freely float along a connected component. Its ℓ-th power is the handle. Bottom: examples of relations on dots and handles

Fractional dots a/ℓ and b/ℓ, $a, b \in \mathbb{Z}_+$, floating on the same component, can merge into the fractional dot $(a+b)/\ell$. Vice versa, the latter can split into a/ℓ and b/ℓ dots. Dot $\ell/\ell = 1$ converts into x and equals a handle on the component.

Formally, one can introduce the category of fractional cobordisms Cob_2^{ℓ}. Its objects are non-negative integers $n \geq 0$ and morphisms from n to m are the diffeomorphism classes rel boundary of oriented cobordisms from n to m circles with dots floating on the components and labelled by elements of the commutative semigroup $H = \frac{1}{\ell}\mathbb{Z}_+ = \{0, 1/\ell, 2/\ell, \dots\}$. Dots can merge, adding their labels, and the label 1 dot equals the handle. Dot 0 can be erased. A connected closed cobordism of genus g with a dot $\frac{m}{\ell}$ reduces to a 2-sphere decorated by the dot $\frac{m+g\ell}{\ell} \in H$.

Universal constructions for 2-dimensional cobordisms, as described here and in [22, 29], extend in a straightforward way to Cob_2^{ℓ} for any $\ell \geq 2$ (the original theory corresponds to $\ell = 1$). Parameters of the theory are $\alpha_{n/\ell} \in \mathbf{k}$, over all $n \geq 0$, encapsulated by the power series in $T^{1/\ell}$,

$$Z_\alpha(T^{1/\ell}) = \sum_{n \geq 0} \alpha_{n/\ell} T^{n/\ell}. \tag{117}$$

State spaces $A_\alpha(k)$ of k circles are defined as in [22], and the rationality result is proved in the same way.

Proposition 8.1 *Vector spaces $A_\alpha(k)$ are finite-dimensional for all $k \geq 0$ iff $A_\alpha(1)$ is finite-dimensional iff $Z_\alpha(T^{1/\ell})$ is a rational function,*

$$Z_\alpha(T^{1/\ell}) = \frac{P(T^{1/\ell})}{Q(T^{1/\ell})}, \tag{118}$$

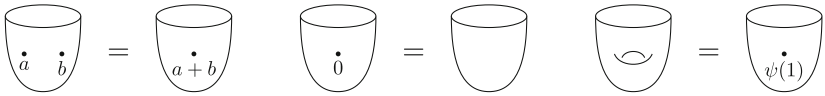


Fig. 25 Dots a and b merge into $a+b$ dot; dot 0 can be erased; handle equals the dot labelled $\psi(1)$

for coprime polynomials P and Q, with $Q(0) \neq 0$.

Generalizations of the Deligne category extend to this case as well, and each rational function as in (118) gives rise to several categories by direct analogy with [29]. These categories have finite-dimensional hom spaces and include the analogue of the partition category, the Deligne category, and their quotients by ideals of negligible morphisms.

We should warn the reader that cobordisms of fractional genus, as above, are simply decorated surfaces that are morphisms of Cob_2^{ℓ}. They don't carry any of the rich structure associated with the usual surfaces, such as the mapping class group, moduli spaces of complex structures, and so on.

Commutative monoid decorations. More generally, one can take any commutative monoid H (with the binary operation written additively as $+$), together with a monoid homomorphism $\psi : \mathbb{Z}_+ \longrightarrow H$. To ψ one can assign the category Cob_2^{ψ} of oriented decorated 2D cobordisms. As before, objects of this category are non-negative integers $n \in \mathbb{Z}_+$, while the morphisms are oriented 2D cobordisms (modulo rel boundary diffeomorphisms) decorated by dots labelled by elements of H. Dots labelled $a, b \in H$ floating on the same component can merge into a dot labelled $a+b$, see Fig. 25. Dot labelled $0 \in H$ can be erased. A handle on a component equals a dot on that component labelled by $\psi(1)$.

Closed ψ-cobordisms are disjoint unions of their connected components, classified by their genus, which is an element of H. The analogue of evaluation is a map of sets $\alpha : H \longrightarrow \mathbf{k}$ which can be written via formal power series

$$Z_\alpha = \sum_{h \in H} \alpha_h \, h \tag{119}$$

and viewed as an element of the dual vector space $(\mathbf{k}H)^*$.

In Cob_2^{ψ} connected cobordisms from 0 to 1 are parametrized by elements $h \in H$ and correspond to a 2-disk with a dot labelled H. Consequently, the space $A_\alpha(1)$ is the $\mathbf{k}H$-submodule of $(\mathbf{k}H)^*$ generated by the functional Z_α. It is finite-dimensional iff α is a representative function on H, see [22].

Notice that ψ does not have to be injective. However, one can specialize to the case when H is a free monoid and ψ is injective, and then get any desired defining relations on generators of H by restricting to suitable subspaces of $(\mathbf{k}H)^*$. Taking large H, however, may move the emphasis from 2D cobordisms and representative functions on them to, for the most part, studying representative functions on $\mathbf{k}H$, with only a meagre input from cobordisms.

Another potentially interesting specialization is to the periodic genus. For that specialization genus does not need to be fractional. Consider the quotient of the cobordism category by the relation that the M-th power of the handle is identity, that is, can be removed. This corresponds to working with the monoid and map

$$H = \mathbb{Z}/M\mathbb{Z}, \quad \psi ; \mathbb{Z}_+ \longrightarrow H, \ \psi(1) = 1, \tag{120}$$

that is, modding out $\mathbb{Z}_+$ by $M\mathbb{Z}_+$. In the language of α-evaluations, one is looking at "M-periodic" power series, that is, $\alpha_{M+n} = \alpha_n$ for all $n \geq 0$. Equivalently, the power series

$$Z_\alpha(T) = \frac{P(T)}{1 - T^M}, \quad \deg(P(T)) < M, \tag{121}$$

is determined by the coefficients of $P(T)$. Such evaluations necessarily extend to the category of cobordisms with integral genus, see the next remark. Fractional version of (121) also makes sense, with the power series

$$Z_\alpha(T^{1/\ell}) = \frac{P(T^{1/\ell})}{1 - T^{M/\ell}}, \quad \deg(P(x)) < M, \tag{122}$$

and not necessarily integral M/ℓ. In this theory $M/\gcd(M, \ell)$ handles on a component can be erased.

Remark Paper [28] discusses several rank two Frobenius extensions $R_* \subset A_*$ used in various flavors of $SL(2)$ link homology. Here R_* is a ground commutative ring and A_* is a Frobenius R_*-algebra, which is, in particular, free of rank two over R_*. Extension $(R_\mathcal{D}, A_\mathcal{D})$ considered in [28] makes use of the *anti-handle*, a formal inverse of the handle cobordism, denoted by $\star^{-1}$ in that paper. This extension essentially describes Lee's homology theory and also gives a monoidal functor from the category Cob_2^ψ to the category of free $R_\mathcal{D}$-modules, where

$$H = \mathbb{Z}, \quad \psi : \mathbb{Z}_+ \longrightarrow \mathbb{Z} \tag{123}$$

is the usual homomorphism from the monoid of non-negative integers (under addition) to integers. The functor assigns $A_\mathcal{D}^{\otimes n}$ to the union of n circles and the structure maps of that Frobenius algebra (unit, counit, multiplication, comultiplication) to the basic cobordisms: cup, cap, pants, copants. Multiplication by the handle endomorphism of $A_\mathcal{D}$ is invertible and allows to introduce the antihandle (dot labelled -1) as the inverse of the handle endomorphism.

A similar localization appears in [27] in the context of evaluations of unoriented $SL(3)$ foams, where one can invert the discriminant and work with suitable decorations on foams.

Allowing connected sums of cobordisms in Cob_2 with $\mathbb{RP}^2$ (which results in unorientable cobordisms) corresponds to working with the monoid and the map

$$H = \langle 1, b \rangle / (3b = b + 1), \quad \psi : \mathbb{Z}_+ \longrightarrow H, \psi(1) = 1, \tag{124}$$

with dot labelled b corresponding to the connected sum with $\mathbb{RP}^2$. In this monoid there is no cancellation, and $b + b \neq 1$ although $b + b + b = b + 1$. Topologically,

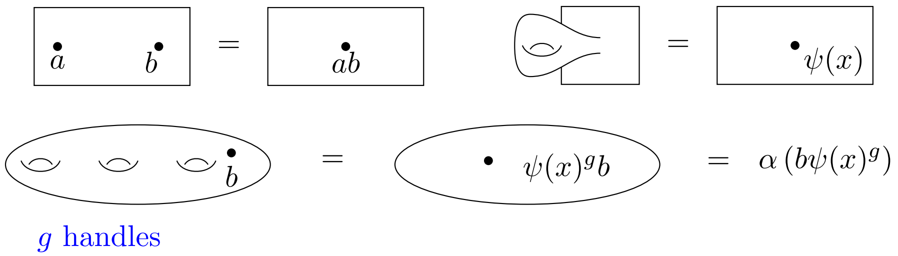


Fig. 26 Skein relations in category Cob_2^{ψ}. Note that a, b are elements of an algebra B rather than of a monoid H (where the binary operation is denoted $+$), which leads to the addition in Fig. 25 becoming multiplication here

connected sum with three copies of $\mathbb{RP}^2$ is diffeomorphic to the connected sum with one $\mathbb{RP}^2$ and the torus, but connected sum with two copies of $\mathbb{RP}^2$ (equivalently, with the Klein bottle) is not diffeomorphic to adding a handle, when applied to an orientable connected component. Monoid H surjects onto $\frac{1}{2}\mathbb{Z}_+$ by sending b to $\frac{1}{2}$, intertwining homomorphisms ψ for these monoids.

Extending to commutative algebras. Decorations of two-dimensional cobordisms by elements of a commutative monoid can be further generalized. Observe that a map ψ as above from the "genus" semigroup $\mathbb{Z}_+$ into a commutative monoid H, together with the "trace" or evaluation $\alpha : H \longrightarrow \mathbf{k}$ gives rise to two maps, that we still denote ψ and α,

$$\mathbf{k}[x] \cong \mathbf{k}\mathbb{Z}_+ \xrightarrow{\psi} \mathbf{k}H \xrightarrow{\alpha} \mathbf{k}. \tag{125}$$

The first map is a homomorphism of commutative $\mathbf{k}$-algebras, the second (evaluation) map is $\mathbf{k}$-linear. One can generalize from $\mathbf{k}H$ to a commutative $\mathbf{k}$-algebra B together with a homomorphism ψ and a $\mathbf{k}$-linear trace map

$$\mathbf{k}\mathbb{Z}_+ \cong \mathbf{k}[x] \xrightarrow{\psi} B \xrightarrow{\alpha} \mathbf{k}. \tag{126}$$

Instead of the set-theoretic category Cob_2 which does not have a linear structure one starts with the $\mathbf{k}$-linear category $\mathbf{k}\mathrm{Cob}_2$ with the same objects as Cob_2 and morphisms – finite $\mathbf{k}$-linear combinations of morphisms in Cob_2. One then modifies $\mathbf{k}\mathrm{Cob}_2$ to the category Cob_2^{ψ} as follows. Category Cob_2^{ψ} has objects $n \in \mathbb{Z}_+$. Morphisms are $\mathbf{k}$-linear combinations of oriented 2D cobordisms as before with dots labelled by elements of B floating on components and the following relations, see also Fig. 26:

- Dots are subject to the obvious addition and product rules for elements of B,
- A handle on a cobordism can be replaced by a dot labelled $\psi(x)$,
- A closed surface of genus g with dot labelled b evaluates to $\alpha(b\psi(x)^g) \in \mathbf{k}$.

In this way, a surface of genus g with dots labelled $a_1, \ldots, a_k$ floating on it reduces to a genus zero surface with the same boundary and a single dot labelled $\psi(x)^g a_1 \ldots a_k$. A closed connected component reduces to a 2-sphere with a dot b. It then evaluates to $\alpha(b) \in \mathbf{k}$. Thus, any closed component evaluates to an element of $\mathbf{k}$. In this way a dotted cobordism reduces to a viewable cobordism with all components of genus zero and at most a single dot on each connected component, labelled by some element of B.

Pick a basis $\{b\}_{b\in C}$ of B that contains $1 \in B$. A morphism from n to m in Cob_2^{ψ} reduces to a linear combination of genus zero viewable cobordisms with dots on each components labelled by elements of the basis C of B. In fact, such dotted cobordisms constitute a basis in the hom space $\mathrm{Hom}_{\mathrm{Cob}_2^{\psi}}(n, m)$. Vice versa, category Cob_2^{ψ} can be defined via these bases and multiplication rules that come from composition of cobordisms, converting a handle to $\psi(x)$, multiplication in the basis C of B and evaluation map $\alpha \in B^*$. Basis elements are parametrized by a choice of a set-theoretical partition $\lambda \in D_n^m$ of $n+m$ boundary circles together with an assignment of an element of the basis C of B to each component.

Next, assume that $\alpha \in B^* = \mathrm{Hom}_{\mathbf{k}}(B, \mathbf{k})$ is a representative function and exclude the trivial case $\alpha = 0$. This means that the hyperplane $\ker(\alpha) \subset B$ contains an ideal $I \subset B$ of finite codimension, $\dim(B/I) < \infty$, see [34], for instance. Assume that I is the largest such ideal. Element α generates B-subrepresentation $B\alpha \subset B^*$ that factors through the action of B/I and is isomorphic to a free rank one B/I-module.

Remark Algebra $B' = B/I$ is a commutative Frobenius $\mathbf{k}$-algebra, with the nondegenerate trace α and a preferred element, which is the image of $\psi(x)$ under the quotient map $B \longrightarrow B/I$. The constructions that follow can alternatively be done with such a commutative algebra B' (necessarily finite-dimensional over $\mathbf{k}$), equipped with a nondegenerate trace and a preferred element.

Next, we quotient Cob_2^{ψ} by the relations that a dot labelled by $z \in I$ is zero. Such a dot can be expanded as a linear combination in the basis C. A possible convenient basis can be formed by choosing a basis a basis C_I of I and extending it to a basis of B that contains 1 (the latter condition is also for convenience). Let us denote such a basis by $C = C_I \sqcup C'$, with $1 \in C'$ and C' descending to a basis of the quotient B/I. Set C' is finite.

Denote the quotient category by $\mathrm{SCob}_{\alpha}^{\psi}$. It is the analogue of the category PCob_{α} from [29]. One can check that a basis of $\mathrm{Hom}_{\mathrm{SCob}_{\alpha}^{\psi}}(n, m)$ is given by choosing a set-theoretic partition $\lambda \in D_n^m$ for $n+m$ boundary circles and assigning an element of C' to each component of the partition. In particular, hom spaces in the category $\mathrm{SCob}_{\alpha}^{\psi}$ are finite-dimensional. The space of homomorphisms

$$A_{\alpha}^{\psi}(1) := \mathrm{Hom}_{\mathrm{SCob}_{\alpha}^{\psi}}(0, 1) \tag{127}$$

is a commutative algebra under the pants cobordism, naturally isomorphic to the Frobenius algebra B/I above. Now form the additive Karoubi closure

$$\mathrm{DCob}_{\alpha}^{\psi} \ := \ \mathrm{Kar}(\mathrm{SCob}_{\alpha}^{\psi}) \tag{128}$$

to get a $\mathbf{k}$-linear idempotent-complete rigid symmetric monoidal category with finite-dimensional hom spaces. This is the analogue of the Deligne category for the data

(B, ψ, α) as in (126), with a representative functional α (trivial case $\alpha = 0$ gives the zero category).

Categories Cob_2^{ψ}, $\mathrm{SCob}_{\alpha}^{\psi}$, and $\mathrm{DCob}_{\alpha}^{\psi}$ have a trace map given on a decorated (n, n)-cobordism x representing an element in $\mathrm{Hom}(n, n)$ by closing x via n annuli into a closed cobordism $\widehat{x}$ and evaluating via α:

$$\mathrm{tr}(x) = \alpha(\widehat{x}). \tag{129}$$

Denote by J_{α}^{ψ} the two-sided ideal in $\mathrm{SCob}_{\alpha}^{\psi}$ of *negligible morphisms* for the trace tr. Let

$$\mathrm{Cob}_{\alpha}^{\psi} = \mathrm{SCob}_{\alpha}^{\psi} / J_{\alpha}^{\psi} \tag{130}$$

be the quotient category by this ideal. Likewise, let $\underline{\mathrm{DCob}}_{\alpha}^{\psi}$ be the quotient of the Deligne category $\mathrm{DCob}_{\alpha}^{\psi}$ by the negligible ideal for tr.

For a representative α, as before, these categories can be organized into the following diagram of categories and functors, with a commutative square on the right.

$$\begin{array}{ccccccccc} \mathrm{Cob}_2 & \longrightarrow & \mathbf{k}\mathrm{Cob}_2 & \longrightarrow & \mathrm{Cob}_2^{\psi} & \longrightarrow & \mathrm{SCob}_{\alpha}^{\psi} & \longrightarrow & \mathrm{DCob}_{\alpha}^{\psi} \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & \mathrm{Cob}_{\alpha}^{\psi} & \longrightarrow & \underline{\mathrm{DCob}}_{\alpha}^{\psi} \end{array} \tag{131}$$

This diagrams of categories is fully analogous to the ones described in (15) above and in [23, 26, 29]. The four categories in the commutative square have finite-dimensional hom spaces. The two categories on the far right are idempotent complete.

Remark The construction above is likely to be more interesting when the algebra B is not very large. One may, for instance, take $B = \mathbf{k}[x, y]/(g(x, y))$, the quotient of the ring of polynomials in two variables by a polynomial that depends nontrivially on both x and y, and define $\psi : \mathbf{k}[x] \longrightarrow B$ by $\psi(x) = x$.

Remark The category of thin flat 2-dimensional cobordisms in [26] has commuting *hole* and *handle* cobordisms. Similar to the discussion above, dot-decorated version of that category can be introduced, with elements of a commutative monoid H floating on components of cobordism. One fixes two elements of H, to equate to a handle and a hole, respectively. Equivalently, a homomorphism $\psi : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow H$ can be fixed for that.

If, instead, elements of a commutative $\mathbf{k}$-algebra B are made to float on cobordism's components, one should choose two elements of B, to equate to the handle and the hole, respectively. In the version of the thin cobordism category [26] where side boundaries are colored by colors $\{1, \ldots, r\}$, there are r different holes, one for each color of its boundary. Then to combine B with the handles and holes, one chooses $r + 1$ elements in B.

Acknowledgements The authors are grateful to Pavel Etingof for a valuable email discussion. The authors would like to thank Vera Serganova and Noah Snyder for discussions of the Frobenius algebra described in Sect. 5.5 and the referee for many valuable comments and suggestions. M.K. was partially supported by NSF grant DMS-1807425 while working on this paper. Y.K. was partially supported by NSF FRG grant DMS-1564497. The work of V. O. was partially supported by the HSE University Basic Research Program, Russian Academic Excellence Project '5-100' and by NSF grant DMS-1702251.

References

1. André, Y., Kahn, B.: Erratum: Nilpotence, radicaux et structures monoidales. Rend. Sem. Univ. Padova **113**, 125–128 (2005)
2. Bergeron, N., Thiem, N.: A supercharacter table decomposition via power-sum symmetric functions. Int. J. Algebra. Comput. (IJAC) **23–4**, 763–778 (2013). arXiv:1112.4901
3. Blanchet, C., Habegger, N., Masbaum, G., Vogel, P.: Topological quantum field theories derived from the Kauffman bracket. Topology **34**(4), 883–927 (1995)
4. Blumen, S.C.: Quantum Superalgebras at Roots Of Unity and Topological Invariants of Three-manifolds. PhD Thesis, University of Sydney, Australia (2005)
5. Blumen, S.C.: On the $U_q(osp(1|2n))$ and $U_{-q}(so(2n+1))$ uncolored quantum link invariants. J. Knot Theory Ramif. **19**(3), 335–353 (2010)
6. Brundan, J., Entova-Aizenbud, I., Etingof, P., Ostrik, V.: Semisimplification of the category of tilting modules for GL_n. Adv. Math. **375**, 1 (2020). arXiv:2002.01900
7. Comes, J.: Jellyfish partition categories. Algebras and Rep. Theory **23**, 327–347 (2020). arXiv:1612.05182
8. Comes, J., Ostrik, V.: On blocks of Deligne's category $Rep(S_t)$. Adv. in Math. **226**, 1331–1377 (2011). arXiv:0910.5695
9. De Stavola, D.: A Plancherel measure associated to set partitions and its limit. Adv. Appl. Math. **92**, 73–98 (2018). arXiv:1612.03061
10. Deligne, P.: La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas en entier naturel, In: Proceedings of the Int. Colloquium on Alg. Groups and Homogeneous Spaces, Tata Inst. Fund. Res. Studies Math. Mumbai, 209-273 (2007)
11. Deligne, P.: Semi-simplicité de produits tensoriels en charactéristique p. Inv. Math. **197**, 587–611 (2014)
12. Di Francesco, P.: Meander determinants. Comm. Math. Phys. **191**, 543–583 (1998). arXiv:hep-th/9612026
13. Entova-Aizenbud, I., Serganova, V.: Jacobson–Morozov lemma for algebraic supergroups. Adv. Math. **398**, 108240 (2022). arXiv:2007.08731
14. Etingof, P., Harman, N., Ostrik, V.: p-adic dimensions in symmetric tensor categories in characteristic p, Quantum. Topology **9**, 119–140 (2018)
15. Etingof, P., Ostrik, V.: On semisimplification of tensor categories, arXiv:1801.04409
16. Etingof, P., Gelaki, S., Nikshych, D., Ostrik, V.: Tensor categories, Math. Surveys and Monographs **205**, AMS (2015)
17. Everest, G., van der Poorten, A., Shparlinski, I., Ward, T.: Recurrence Sequences, Math. Surv. Monographs **104**, AMS (2003)
18. Flake, J., Laugwitz, R., Posur, S.: Indecomposable objects in Khovanov-Sazdanovic's generalizations of Deligne's interpolation categories, arXiv:2106.05798
19. Goldschmidt, D.M.: Algebraic Functions and Projective Curves. In: Grad. Texts in Math, vol. 215. Springer, Berlin (2003)
20. Harman, N.: Deligne categories as limits in rank and characteristic, arXiv:1601.03426
21. Kac, V. G., Raina, A. K., Rozhkovskaya, N.: Bombay lectures on highest weight representations of infinite dimensional Lie algebras. Second edition. Advanced Series in Mathematical Physics, 29. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, xii+237 pp (2013)
22. Khovanov, M.: Universal construction of topological theories in two dimensions, arXiv:2007.03361
23. Khovanov, M.: Decorated one-dimensional cobordisms and tensor envelopes of noncommutative recognizable power series, arXiv:2010.05730

24. Khovanov, M., Kitchloo, N.: A deformation of Robert-Wagner foam evaluation and link homology, arXiv:2004.14197
25. Khovanov, M., Kitchloo, N., Kononov, Ya.: In progress
26. Khovanov, M., Qi, Y., Rozansky, L.: Evaluating thin flat surfaces. Commun. Math. Phys. **385**(3), 1835–1870 (2021). arXiv:2009.01384
27. Khovanov, M., Robert, L.-H.: Foam evaluation and Kronheimer-Mrowka theories, to appear in Advances in Mathematics. Adv. Math. **376**, 107433 (2021). arXiv:1808.09662
28. Khovanov, M., Robert, L.-H.: Link homology and Frobenius extensions II. Fundamenta Mathematicae **256**, 1–46 (2022). arXiv:2005.08048
29. Khovanov, M., Sazdanovic, R.: Bilinear pairings on two-dimensional cobordisms and generalizations of the Deligne category, arXiv:2007.11640
30. Kulish, P.P., Reshetikhin, NYu.: Universal R-matrix of the quantum superalgebra $osp(2|1)$. Lett. Math. Phys. **18**, 143–149 (1989)
31. Larson, R.G., Taft, E.J.: The algebraic structure of linearly recursive sequences under Hadamard product. Israel J. Math. **72**(1–2), 118–132 (1990)
32. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, Oxford, England (1995)
33. Meir, Y.: Interpolations of monoidal categories and algebraic structures by invariant theory, arXiv:2105.04622
34. Montgomery, S.: Hopf algebras and their actions on rings, CBMS Regional Conf. Ser. Math. **82**, AMS, (1993)
35. Petersen, T.K.: Eulerian Numbers. Birkhäuser Adv, Texts, Basel (2015)
36. Stanley, R.P.: Enumerative Combinatorics, vol. 1. Cambridge University Press, Cambridge (1997)
37. Ostrik, V.: On symmetric fusion categories in positive characteristic. Selecta Mathematica **26**(3), 1–19 (2020). arXiv:1503.01492
38. Rittenberg, V., Scheunert, M.: A remarkable connection between the representations of the Lie superalgebras $osp(1, 2n)$ and the Lie algebras of $o(2n + 1)$. Comm. Math. Phys. **83**, 1–9 (1982)
39. Robert, L.-H., Wagner, E.: A closed formula for the evaluation of $\mathfrak{sl}_N$-foams, to appear in Quantum Topology. Quan. Topol. **11**(3), 411–487 (2020). arXiv:1702.04140
40. Schommer-Pries, C.: The classification of two-dimensional extended topological field theories, arXiv:1112.1000
41. Zhang, R.B.: Finite-dimensional representations of $U_q(\mathrm{osp}(1/2n))$ and its connection with quantum $so(2n + 1)$. Letters in Math. Phys. **25**, 317–325 (1992)