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Title: Mind-Bending Geometry: Children's and adults' intuitions about linearity on spheres

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Data Availability Statement
This work was preregistered on the Open Science Framework (OSF), and the protocol, data, and analysis code are accessible at: https://osf.io/thbpv/. This work's umbrella project on the OSF also includes a full report of two pilot studies that were preregistered. These pilot studies are accessible through the same link.

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Abstract

Humans appear to intuitively grasp definitions foundational to formal geometry, like definitions that describe points as infinitely small and lines as infinitely long. Nevertheless, previous studies exploring human's intuitive natural geometry have consistently focused on geometric principles in planar Euclidean contexts and thus may not comprehensively characterize humans' capacity for geometric reasoning. The present study explores whether children and adults can reason about linearity in spherical contexts. We showed 48 6- to 8-year-old children and 48 adults from the Northeast United States two different paths between the same two points on pictures of spheres and asked them to judge which path was the most efficient for an actor to get from a starting point to a goal object. In one kind of trial, both paths looked curved in the pictures, and in another kind of trial, the correct curved-looking path was paired with an incorrect straight-looking path. Adults were successful on both kinds of trials, and although children often chose the incorrect straight-looking path, they were surprisingly successful at identifying the efficient path when comparing two that were curved. Children thus may build on a natural geometry that gives us humans intuitions that are not limited to the formal axioms of Euclidean geometry or even to the Euclidean plane.

Public Significance Statement: Children and adults succeed in judgements of spherical linearity, i.e., identifying a "line" on a sphere as the most efficient path between two points. Children's seemingly advanced judgments about spherical geometry suggest the possibility of effective geometry pedagogies that go beyond planar contexts.

Introduction

Before surfaces, there are lines, at least according to Euclid's *Elements*. Definitions 2 and 4 of the *Elements*, historically among the most important texts in all of formal mathematics, introduce a line as infinitely thin and a straight line as lying evenly with its points (Euclid, 2007/300BCE). (Earlier definitions—1 and 3—introduce points, and later definitions—5 through 7—introduce surfaces.) This definition of a straight line is innovative and curious upon reflection (Trudeau, 2001), but we intuitively grasp it as picking out the shortest, most efficient path between two points. For example, imagine taking a string by its ends: It does not form a straight line until you pull it taut so that it lies evenly with its ends. Euclid may have thus intended to exclude any curve, and this is the meaning of Definition 4 on a plane. But is our identification of straight lines, like their definition, prior to and perhaps not limited to any particular kind of geometric surface? On a sphere, for example, a taut string becomes a curved geodesic. Are our intuitions strictly Euclidean or are they more flexible, allowing us to identify such lines on surfaces that are not planar, like spherical surfaces?

Prior work investigating humans' ability to identify and reason about such foundational definitions, principles, and figures of formal geometry has consistently focused on geometric intuitions that align with planar Euclidean geometry. For example, recent research relying on cross-species, cross-cultural, developmental, and computational approaches suggests that from childhood and regardless of formal schooling, humans, but not non-human primates, are spontaneously attuned to foundational principles of planar Euclidean geometry (e.g., lines, length, parallelism, perpendicularity, and symmetry) such that humans uniquely are able to mentally compose these Euclidean principles with an algorithmic-like "language of thought" for geometry (Amalric et al., 2017; Sablé-Meyer et al., 2021). Other research relying on these same broad approaches suggests that separate cognitive systems for geometry inherited by humans through evolution—one system that prioritizes distance and directional information to support navigation and one system that prioritizes length and angle information to support visual form recognition—provide complementary geometric sensitivities that get productively combined through human development to form an intuitive natural geometry consistent with planar Euclidean geometry (Dillon et al., 2013; Dillon & Spelke, 2018; Spelke et al., 2010). Even studies that have probed humans' both planar and spherical geometric intuitions have nevertheless emphasized that intuitive geometry reflects planar Euclidean principles. For example, Izard et al. (2011) investigated the geometric intuitions of adults and children from the Unites States, France, and from an Amazonian village, in which there is no formal schooling in geometry. Participants were asked to reason about the properties of points and lines described in the context of a planar surface and, for a subset of questions, a spherical surface (e.g., "Can two lines never intersect?"). Older children and adults across cultures performed well on the questions presented in the planar context and changed their answers as needed when the same questions were presented in the spherical context. Younger children showed less sensitivity to context but also produced fewer correct planar responses. Izard et al. (2011) concluded that humans are cognitively prepared to reason about planar surfaces and that planar geometric intuitions spontaneously develop in all humans and are reinforced by everyday experiences (e.g., navigation) and formal education. Because of the small sample sizes (an inherent limitation to testing the Amazonian population) and because of the relatively few questions presented in the spherical context, this study nevertheless could not support any strong developmental conclusions about participants' spherical intuitions.

The present work does not adjudicate among the cognitive theories outlining human's intuitive natural geometry described above or refute the idea that humans may develop geometric intuitions that support reasoning about planar Euclidean geometry. Rather, the present work suggests that in its sole focus on planar contexts, prior work falls short in comprehensively describing human's intuitive geometry as both a central cognitive achievement of the human mind and as a foundation for humans' capacity to understand formal geometries more generally, both Euclidean and non-Euclidean. While different formal geometries describe different surfaces, they nevertheless adopt the same principle of a line as the shortest, most efficient path between two points. In investigating children's and adults' intuitions about lines on spherical surfaces in the present study, we thus explore the possibility that children and adults have geometric intuitions that go beyond planar contexts, allowing us to provide a more complete picture of human intuitive natural geometry.

Methods

Participants

Sample sizes and exclusion criteria were specified in advance of data collection and were preregistered on the Open Science Framework (OSF; osf.io/thbpv/). Forty-eight 6- to 8-year-old English-speaking children (*M*age 7y7m; *range* 6y1m–8y11m; 27 girls) participated. One additional child was excluded because they had participated in a pilot version of the study. Participants were recruited from the National Museum of Mathematics in New York City and New York University's child participant database and were given a small thank-you gift. Forty-eight English-speaking adults (*M*age 19y; *range* 18-22y; 34 women) also participated. An additional 6 adults were excluded because of: failure to follow task directions (3); choosing the response on one side of the screen over 90% of the time (1); or experimenter error (2). Adults were recruited from New York University's participant study pool and were given course credit or payment. The use of human participants for this study was approved by the Institutional Review Board at New York University.

Materials, Design, and Procedure

The stimuli consisted of 90 2D pictures of 3D spheres generated by custom code in Mathematica (version 11). Each sphere depicted a purple point and an orange point, connected by a thin black path. The path could either be the shortest path between the points, i.e., it could be a *geodesic* (which, if the path continued around the whole sphere, would be a great-circle and cut the sphere in half; **Fig. 1**), or it could *not* be the shortest path between the two points, i.e., it could be an *arc* (which, if the path continued around the whole sphere, would *not* cut the sphere in half). Because each picture captured the sphere from only one point of view, the 2D geometric properties of how the paths looked varied. In particular, both geodesics and arcs could look *straight* or *curved* in the picture. By varying the point of view at which the spheres were presented, we could therefore vary both spherical linearity (i.e., whether the path was a *geodesic* or *arc*) and planar linearity (i.e., whether the path looked *straight* or *curved* on the 2D picture plane).

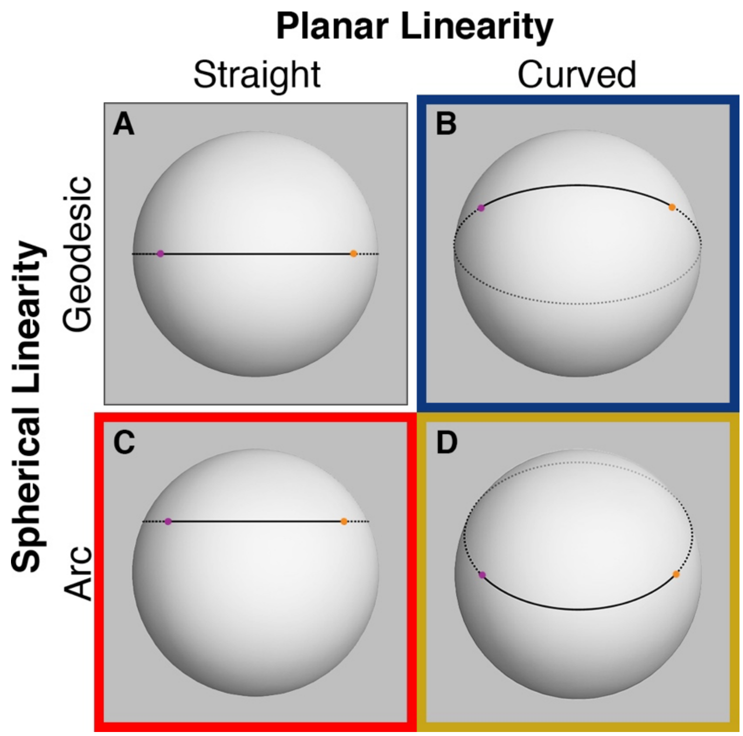

Fig. 1. Examples of spherical and planar linearity. Geodesics only look straight in a picture when they circumscribe the sphere's equatorial plane or are rotated only around the front-back axis (**A**); they look curved when shown from another point of view (**B**). Arcs, in contrast, can look either straight (**C**) or curved when shown at a point of view other than one intersecting the equator; they look curved when they intersect the equator (**D**). Participants in this experiment compared curved geodesics (**B**) to straight arcs (**C**) and curved arcs (**D**). For illustrative purposes, we show straight geodesics (**A**) here, but these paths were not included in the experiment. We also show here (but not in the experiment) the continuation of the depicted paths with dotted lines beyond the purple and orange end points to illustrate that, while geodesics will cut spheres in half, arcs will not.

On each trial, participants saw a pair of sphere pictures (**Fig. 2**) presented using PsychoPy (version 1.90.3) on a 13" laptop screen by an experimenter in a quiet testing room. The distance between the two depicted points on the spheres and their heights on the spheres were always matched across pictures in the same trial but varied across trials, with 5 possible distances and 3 possible heights above or below the equator. Curved paths varied in curvature in a semi-continuous way based on a geodesic's true curvature. Paths farther from the equator appeared more curved than paths nearer to the equator since geodesics appear straight at the equator (see **Fig. 1**); and, for paths at the same height on the sphere, those whose endpoints were farther apart versus closer together appeared more curved since paths with more distant points cover more of the sphere's curved surface.

In 2 blocks of 30 trials each, participants were asked to evaluate which of the two depicted paths was the "easiest," most efficient path from one point to the other on the sphere (**Fig. 2A**). In one trial type (the *curved arc condition*), participants compared curved geodesics to curved arcs (**Fig. 2B**). The curved arcs were generated in Photoshop (CC 2015.5 version 17.0.0) by reflecting the curved geodesics across the principal axis between the two points. For these trials, participants thus compared two paths that matched in their depicted length and curvature. In the other trial type (the *straight arc condition*), participants compared curved geodesics of the same length and curvature as those in the curved arc condition to straight arcs (**Fig. 2C**). The straight arcs depicted what looked like a straight path in the picture between the two points, and so for these trials, participants compared two paths that did not match either in their depicted length or in their curvature. Curved geodesics were the correct responses in both conditions.

Trial types were mixed within each block, trial order within each block was randomized across participants, and curved geodesics appeared 50% of the time on each side of the screen. To protect against any effects of path orientation on performance, paths were not presented within 10° of the horizontal. They also varied in orientation across trials and across participants (whole-degree values: 10°-170°; 190°-350°) but were matched across the two pictures in each trial.

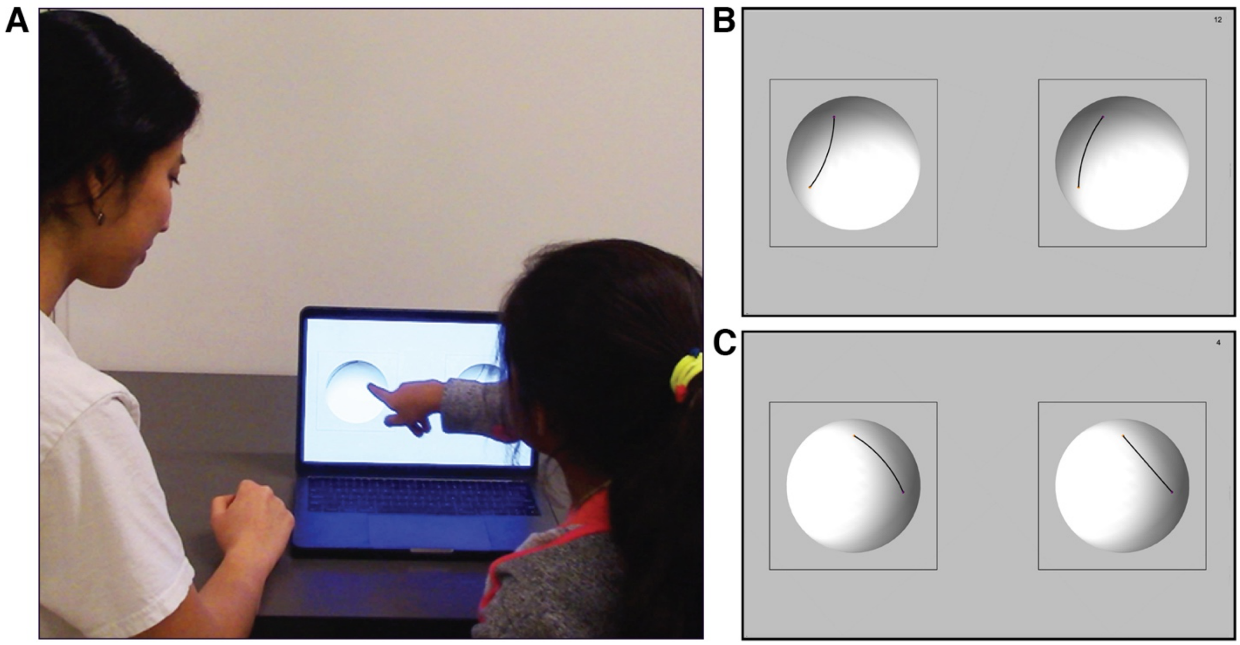

Fig. 2. A child participant (**A**) and screen shots of example trials. In the curved arc condition (**B**), participants compared curved arcs (incorrect; left) to curved geodesics (correct; right). In the straight arc condition (**C**), participants compared curved geodesics (correct; left) of the same length and curvature as those in the curved arc condition to straight arcs (incorrect; right). The curved geodesics were presented at different orientations across conditions.

The task was designed to probe participants' intuitions of a line as the most efficient path between two points on a sphere without requiring their knowledge of any formal definitions. Prior to completing the test trials, participants completed practice trials, in which they were introduced to a "very lazy" purple snail, who always took the most efficient path from a starting point to an orange mushroom, a favorite food. Across five practice trials, participants were asked to judge, e.g., whether the snail would push two blocks (correct) or three blocks (incorrect) out of the way to get to the mushroom. Participants received corrective feedback on the practice trials. Participants were then shown a picture of a purple point and an orange point on an otherwise blank screen and were told that the snail would look like the purple point, and the mushroom would look like the orange point. Finally, they were shown a large picture of sphere (with no points or paths) and were told that the snail and mushroom would be on a "perfectly round land" shaped like a "really big ball." For each test trial, participants saw two pictures of spheres, one on each side of the screen and each presenting a path. The experimenter asked which path was the *easiest* path the snail could take to the mushroom and recorded with a button press to which picture participants pointed. Participants received no corrective feedback on the test trials.

Results

Results with 6- to 8-year-old children

Children's responses are presented in **Fig. 3**. All analyses were specified prior to data collection and preregistered on the OSF (osf.io/thbpv/). We focused on the accuracy and consistency of participants' responses. A binomial mixed-model logistic regression found that children performed significantly below chance overall (P = 0.455, 95% CI = [0.426, 0.485], p = .003). An additional regression with accuracy as the dependent variable, condition as a fixed effect, and random intercepts for participants revealed a main effect of condition (Wald Test, $\chi^2(1) = 568.44$, $p < .001$), with children performing above chance in the condition comparing curved geodesics to curved arcs (P = 0.695, 95% CI = [0.658, 0.730], $p < .001$) but below chance in the condition comparing curved geodesics to straight arcs (P = 0.216, 95% CI = [0.187, 0.248], $p < .001$). A third regression that added curvature as a fixed effect revealed an effect of condition (Wald Test, $\chi^2(1) = 133.11$, $p < .001$), curvature (Wald Test, $\chi^2(1) = 24.99$, $p < .001$), and an interaction between condition and curvature (Wald Test, $\chi^2(1) = 35.34$, $p < .001$). Curvature had a significant effect on accuracy in both conditions (curved arc condition: $P = 0.888$, 95% CI = [0.779, 0.947], $p < .001$; straight arc condition: $P = 0.167$, 95% CI = [.076, 0.330], $p < .001$): In the curved arc condition, children performed better with greater curvature, but in the straight arc condition, children performed worse with greater curvature.

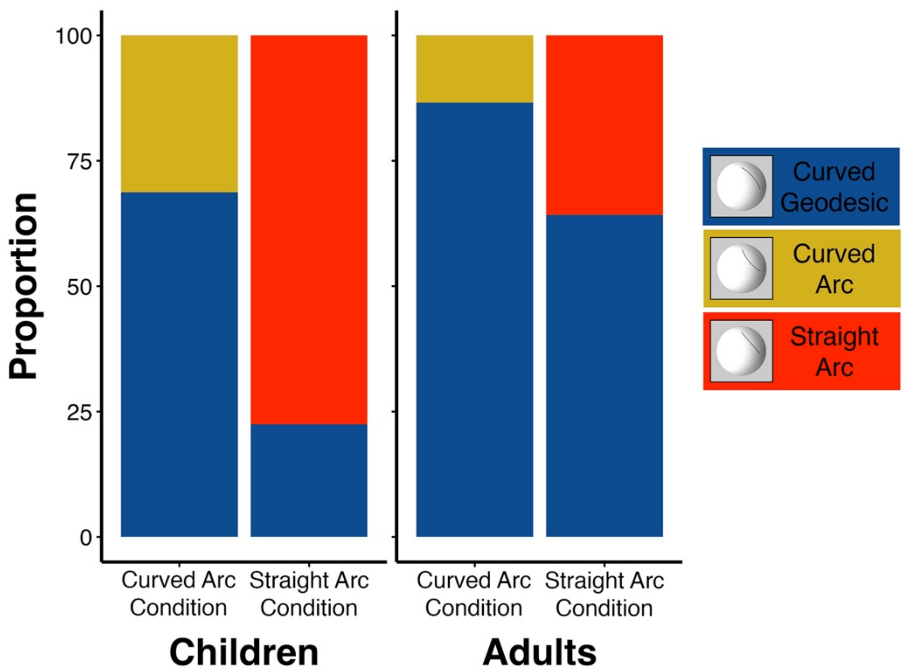

Fig. 3. The proportion of geodesic and arc responses across all trials and participants in the curved arc and straight arc conditions for both children and adults. The curved geodesics were always the correct response, and chance responding was 50%; see text for statistical analyses.

We next examined the consistency of children's responses by evaluating whether an individual's correct response to a geodesic curve in the straight arc condition predicted their correct response to a geodesic curve of the same length and curvature in the curved arc condition. A binomial mixed-model logistic regression with responses to curved geodesics in the curved arc condition as the dependent variable, responses to curved geodesics in the straight arc condition and curvature as fixed effects, and random intercepts for the particular geodesic queried and for participants revealed that children's responses in the straight arc condition did not predict their responses in the curved arc condition (Wald Test, $\chi^2(1) = 0.27$, $p = .604$). There was a main effect of curvature (Wald Test, $\chi^2(1) = 19.54$, $p < .001$), and there was no effect of the interaction term (Wald Test, $\chi^2(1)= 0.51$, $p = .476$).

Results with adults

Adults' responses are presented in **Fig. 3**. All analyses were specified prior to data collection, preregistered on the OSF (osf.io/thbpv/), and were identical to those run on the children's data. Adults performed above chance overall ($P = 0.870$, 95% CI = [0.789, 0.923], $p < .001$), and while their performance differed by condition (Wald Test, $\chi^2(1) = 253.95$, $p < .001$), it was nevertheless above chance in both conditions (curved arc condition: $P = 0.959$, 95% CI = [0.923, 0.979], $p < .001$; straight arc condition: $P = 0.763$, 95% CI = [0.644, 0.881], $p < .001$). In the model with curvature as an additional fixed effect, there was a main effect of condition (Wald Test, $\chi^2(1) = 87.95$, $p < .001$), curvature (Wald Test, $\chi^2(1) = 9.90$, $p = .002$), and an interaction between condition and curvature (Wald Test, $\chi^2(1) = 6.24$, $p = .012$): Adults performed better when the paths were more curved in the curved arc condition ($P = 0.883$, 95% CI = [0.681, 0.963], $p = .002$), but curvature did not affect their accuracy in the straight arc condition ($P = 0.509$, 95% CI = [0.293, 0.721], $p = .940$).

Finally, adults' responses in the straight arc condition predicted their responses in the curved arc condition (Wald Test, $\chi^2(1) = 4.15$, $p = .042$). In this regression, there was no main effect of curvature (Wald Test, $\chi^2(1) = .92$, $p = .337$), and there was no effect of the interaction term (Wald Test, $\chi^2(1) = 2.70$, $p = .101$).

Exploratory Results

An unplanned analysis investigating the effects of age (treated as a continuous variable) and condition on accuracy in the child sample found a main effect of condition (Wald Test, $\chi^2(1) = 566.54$, $p = < .001$), with better performance in the curved arc condition, a main effect of age (Wald Test, $\chi^2(1) = 7.21$, $p = .007$), with older children performing better than younger children, and an interaction between condition and age (Wald Test, $\chi^2(1) = 12.67$, $p < .001$). Older children performed better than younger children in the curved arc condition ($P = 0.500$, 95% CI = [0.500, 0.500], $p = .007$), but not in the straight arc condition ($P = 0.500$, 95% CI = [0.500, 0.500], $p = .442$). Younger children (median split) nevertheless still performed above chance in the curved arc condition ($P = 0.640$, 95% CI = [0.559, 0.731], $p < .001$).

Additional unplanned analyses investigated the effects of age group (treated as a categorical variable) on accuracy and consistency across children and adults. The accuracy analysis revealed a main effect of condition (Wald Test, $\chi^2(1) = 246.59$, $p < .001$), with better performance in the curved arc condition, and a main effect of age group (Wald Test, $\chi^2(1) = 42.54$, $p < .001$), with adults performing better than children. The interaction term further characterized these results (Wald Test, $\chi^2(1) = 2.85$, $p = .091$). The consistency analysis revealed a main effect of age (Wald Test, $\chi^2(1) = 22.62$, $p < .001$), with adults performing better than children, but responses in the straight arc condition did not predict responses in the curved arc condition (Wald Test, $\chi^2(1) = .08$, $p = .774$), and the interaction was not significant (Wald Test, $\chi^2(1) = .33$, $p = .566$).

Discussion

Children and adults were shown paths between two points on 2D pictures of 3D spheres and were asked to judge which paths were the most efficient for an actor to get from a starting point to a goal object. Six- to 8-year-old children answered below chance when comparing curved geodesics to straight arcs, but they answered above chance when comparing curved geodesics to curved arcs. Like children, adults performed better when curved geodesics were compared to curved versus straight arcs, but unlike children, they succeeded in both conditions. Moreover, adults' responses across the two conditions showed some internal consistency: Those adults who responded correctly to curved geodesics in the straight arc condition were also more likely to respond correctly to curved geodesics in the curved arc condition. Finally, from 6 to 8 years, children improve in their identification of curved geodesics versus curved arcs.

The difference between children's and adults' performance when comparing curved geodesics with curved arcs versus straight arcs and children's worse performance with more-curved geodesics compared with straight arcs suggests that both children and adults are biased to judge the most efficient path between two points based on planar linearity, consistent with prior work (Izard et al., 2011). Strikingly, however, our results also show that adults recognize spherical linearity (i.e., geodesics) despite this bias and that both children and adults succeed in identifying spherical linearity when there is no conflicting planar linearity.

Children and adults' success in identifying curved geodesics in pictures of spheres is particularly surprising given that even adults are rarely taught the principles of spherical geometry (Lénárt, 2003; Sinclair et al., 2017) and prior work had suggested a strong and *growing* planar bias in children's and adults' geometric reasoning about spheres across development, especially in children and adults from formally educated societies (Izard et al., 2011). Human intuitions about the shortest paths between two points in space in general may thus be flexible beyond the Euclidean plane to include spherical surfaces. In addition, children's seemingly advanced judgments about spherical geometry suggest the possibility of effective geometry pedagogies that go beyond planar contexts.

The present study may even underestimate this ability. For example, in the straight arc condition, the locations of the start and end points of the paths were matched between curved geodesics and straight arcs. Controlling for these start- and end-point locations meant that the depicted curved geodesics were longer in the picture than the depicted straight arcs, which may have interfered with participants' judgements instead of, or in addition to, the interference from planar linearity. Future studies might investigate how matching the depicted path lengths by moving the start and end points closer together for the curved geodesics might affect performance. Second, participants saw 2D pictures of 3D surfaces, as they might see them in a formal geometry textbook. But, using 2D pictures may have made any intuitions about 3D geometry harder to access, especially intuitions about straight arcs, which appear straight from only one viewpoint of the sphere. Paths presented on real 3D objects or on real or animated 3D objects, in which an actor's movement along paths unfolds over time, might facilitate performance (e.g., Hart et al., 2022; Joh et al., 2011; Smith et al., 2018). Future studies could thus evaluate how the dimensionality and dynamics of experimental displays might differentially engage participants' intuitions about 3D geometry and explore whether simulation versus rule-based reasoning supports accurate judgments about spherical linearity.

Our design also relied on eliciting participants' judgments in contexts that may have enhanced their performance. In particular, questions about spherical linearity were posed in the context of judgments about an agent's navigation and efficient action. Evidence from studies with humans and non-human animals suggest flexibility with geometry for navigation, including use of slopes and curvature (Jeffery et al., 2013; Nardi et al., 2011; Widdowson & Wang, 2022), although the specific geometric representations underlying these abilities are still debated. Recognition of the shortest path between two points on a non-planar surface might thus be present in human judgements about navigation. In addition, a large body of research on infants' expectations about the goal-directed actions of others has found that infants expect others to take the most efficient paths to their goals (e.g., Gergely et al., 1995; Liu & Spelke, 2017). While these studies have strictly relied on planar surfaces, infants' expectations may extend to curved surfaces. Future studies might evaluate what sensitivities to surfaces with different geometries underlie infants' and children's judgments of navigation and efficient action and whether such sensitivities are elicited and accessible outside of the domains of place or action understanding.

Conclusion

Philosophers throughout history to the nineteenth century debated the alignment between the natural geometry in our minds and that of the world (Kant, 1998/1781; Plato, 1949/385 BCE), but always within the context of what would become formalized as Euclidean geometry. The history of mathematics then showed us that we humans are not limited to the formal system of planar Euclidean geometry when describing the world or the formal system of geometry itself (Trudeau, 2001). Previous work focusing on the origins of humans' unique capacity for understanding geometry has nevertheless continued to emphasize only where our natural geometric intuitions align with planar Euclidean geometry (e.g., Dillon et al., 2013; Dillon & Spelke, 2018; Izard et al., 2011; Sablé-Meyer et al., 2021; Spelke et al., 2010). The present findings instead emphasize the development of those geometric intuitions that are not Euclidean, insisting that a comprehensive understanding of humans' geometric cognition, including its readiness for learning formal geometry, requires looking beyond planar Euclidean contexts. The present work thus also contributes to growing evidence that our explicit reasoning about simple geometric figures is not comprehensively explained solely by Euclidean principles (e.g., Hart et al., 2022).

Both natural and Euclidean geometry have sets of principles, and the results of previous research indicate that within natural geometry are principles that allow for an intuitive grasp Euclidean geometry. Our present work suggests that Euclidean geometry does not exhaust natural geometry or vice versa. We found that intuitions about a foundational principle in all formal geometries—linearity—are at least present in judgments about an agent's efficient navigation, even if that navigation is happening on a complex surface in terms of its formal description. Children may not naturally develop into "little Euclids"; rather, they may develop a natural geometry that gives us humans intuitions not limited to the formal axioms of Euclidean geometry or even to the Euclidean plane.

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