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Equilibrium analysis of game on heterogeneous networks with coupled activities

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***Abstract*— We study a game where agents interacting over a network engage in two coupled activities and have to strategically decide their production for each of these activities. Agent interactions involve local and global network effects, as well as a coupling between activities. We consider the general case where the network effects are heterogeneous across activities, i.e., the underlying graph structures of the two activities differ and/or the parameters of the network effects are not equal. In particular, we apply this game in the context of palm oil tree cultivation and timber harvesting, where network structures are defined by spatial boundaries of concessions. We first derive a sufficient condition for the existence and uniqueness of a Nash equilibrium. This condition can be derived using the potential game property of our game or by employing variational inequality framework. We show that the equilibrium can be expressed as a linear combination of two Bonacich centrality vectors.**

I. INTRODUCTION

We study interactions between economic agents who are simultaneously engaged in the production of multiple goods and compete in a market to sell these goods. In many situations, such trade relationships are described by a network structure that captures how the aggregate production of each good is influenced by the manner in which each agent is connected with other agents. When such *network connections are heterogeneous* across goods, their impact on the agents' utility need to be modeled separately. Often, the production levels of goods (henceforth, referred as activities) are *coupled* because of the underlying complementarity or substituability effects.

For example, palm oil tree cultivation and timber harvesting from forest concessions in the tropical regions of Southeast Asia are inherently coupled activities [1, 2, 3]. Here, the incentives of individual agents (palm oil and logging companies) are not only shaped by the spatial distribution (i.e., network structure) of timber and logging concessions, but also depend on how these activities are coupled. One can argue that in this example the coupling depends on the extent to which the activities can be carried in a synergistic manner (e.g., by using similar means of production and transport of harvested goods [4]) or compete with each other in terms of resources (e.g., water, sunlight, and soil nutrients and/or labor and capital [5, 6, 7]). The competition for resources and economic outlet arises also at the global level between agents, giving rise to a global network effect besides a local network effect.

In this paper, we study a network game in which the activities (i.e., production decisions) of each agent is influenced by her interactions with other agents in the network, as well as the coupling between these activities. Importantly, the network interactions corresponding to each activity can be heterogeneous and coupling vary across agents. In this sense, our game-theoretic approach extends the well-known network games with single activity [8, 9, 10] and multiple activities [11, 12].

In [8, 9, 10, 13], the agent utility is a linear-quadratic function given by $u(y_i, y_{-i}, \mathbf{G}) = py_i - \frac{1}{2}cy_i^2 - \mu\sum_{j=1}^n y_iy_j + \delta\sum_{j=1}^n G_{ij}y_iy_j$, where $\mathbf{G}$ is the adjacency matrix of the graph $\mathcal{G}(\mathcal{N}, \mathcal{E})$ underlying the network structure, where the set of nodes of the graph $\mathcal{N}$, with $|\mathcal{N}| = n$, models agents and the set of edges $\mathcal{E}$ represent their interactions. For any node i, the neighborhood of i is the set of nodes j connected to i by an edge, i.e., $G_{ij} = 1$. Furthermore, y_i is the production of agent i, y_{-i} is the production of all agents except i, p is the price of the commodity, c is the cost of production, μ is the parameter quantifying the global network effect due to the competition of agents in a market to sell their productions, δ is the parameter quantifying the local network effect that arises from the interaction of agents with their neighbors. While [8] considered this model in the context of criminal network, [9, 10, 13] applied it to education, R&D and financial risk. Our focus is on production networks and our context pertains to harvesting and trade of coupled forest products (e.g., timber and palm oil) where network effects arise from spatial connections between

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forest regions and concessions, and coupling arises from other aspects such as availability of natural resources, production technology and transportation routes [4, 5].

Previous works on network games with single activity [8, 10, 13] have shown that provided the local network effect is small enough compared to the parameter for own concavity c, namely $\delta\lambda_{max}(\mathbf{G}) < c$, where λ_{max} is the largest eigenvalue of $\mathbf{G}$, then a unique Nash equilibrium exists and the production of individual agents at the equilibrium can be expressed in terms of the vector of Bonacich centralities, a canonical centrality measure on network.

Definition 1. For a graph with adjacency matrix $\mathbf{G}$ and scalar $\delta > 0$, let $\mathbf{M}(\mathbf{G}, \delta) \equiv (\mathbf{I} - \delta\mathbf{G})^{-1} = \sum_{k=0}^{+\infty} \delta^k \mathbf{G}^k$. Given a vector of weights $\mathbf{w} \in \mathbb{R}_+^n$, the vector of weighted Bonacich centralities for the network $\mathbf{G}$ is defined as

$$\mathbf{B}_{\mathbf{w}}(\mathbf{G}, \delta) = \mathbf{M}(\mathbf{G}, \delta)\mathbf{w} = (\mathbf{I} - \delta\mathbf{G})^{-1}\mathbf{w} = \sum_{k=0}^{+\infty} \delta^k \mathbf{G}^k \mathbf{w}.$$

When $\mathbf{w} = \mathbf{1}$, we simply denote $\mathbf{B_1} = \mathbf{B}$. Then the unweighted Bonacich centrality of node i is given by $\mathbf{B}_i(\mathbf{G}, \delta) = \sum_{j=1}^n \mathbf{M}_{ij}(\mathbf{G}, \delta)$, where $\mathbf{M}_{ij}$ is the (i, j)-th entry of matrix $\mathbf{M}$. As $\mathbf{G}_{ij}^k$ counts the number of paths of length k from node i to node j, the Bonacich centrality of node i counts the total number of walks that start at node i in the graph with adjacency matrix $\mathbf{G}$. Each walk is exponentially discounted by δ, i.e., longer walks have a lower weight in the centrality measure than shorter walks. In the context of trade, this discounting captures reduced impact of nodes (agents) that are connected via longer trading routes - with distances counted as number of hops.

We can express the (unweighted) Bonacich centrality as follows:

$$\mathbf{B}_i(\mathbf{G}, \delta) = \mathbf{M}_{ii}(\mathbf{G}, \delta) + \sum_{i \neq j} \mathbf{M}_{ij}(\mathbf{G}, \delta),$$

where $\mathbf{M}_{ii}(\mathbf{G}, \delta)$ counts the number of self-loops from i to itself and $\sum_{i \neq j} \mathbf{M}_{ij}(\mathbf{G}, \delta)$ counts the number of outer walks from i to any other player $j \neq i$. By definition $\mathbf{M}_{ii}(\mathbf{G}, \delta) \geq 1$, hence $\mathbf{B}_i(\mathbf{G}, \delta) \geq 1$, with equality when $\delta = 1$.

A particularly relevant work to ours is [11] in which the authors considered a game where agents engage in two coupled activities with homogeneous network effects (i.e., when both activities are subject to identical network effects). In this game, agent's utility is given by $u(y_i^A, y_{-i}^A, y_i^B, y_{-i}^B) = p_i^A y_i^A - \frac{1}{2}c(y_i^A)^2 + \delta \sum_{j=1}^n G_{ij} y_i^A y_j^A + p_i^B y_i^B - \frac{1}{2}c(y_i^B)^2 + \delta \sum_{j=1}^n G_{ij} y_i^B y_j^B + \beta y_i^A y_i^B$, where y_i^A and y_i^B are the production of agent i in activity A and B respectively, δ^A and δ^B are the local network effect for each activity, and β is the parameter for the coupling effect, measuring the interdependance between activities.

In the setting of [11], the networks for activities A and B are assumed to be the same, i.e., the adjacency matrix $\mathbf{G}$ encodes for the structure of the underlying graph. Furthermore, the local network effect encoded by the parameter δ is also identical for the two activities. The authors show that in equilibrium, production of each activity can be described by the sum of two terms, where each term is a weighted Bonacich centrality associated with the adjacency matrix $\mathbf{G}$. In each of these terms, the discount factor is determined by the level of coupling β between activities, and weights depend on the price vectors $\mathbf{p}^A = [p_1^A, \cdots, p_n^A]^\top$ and $\mathbf{p}^A = [p_1^B, \cdots, p_n^B]^\top$. Importantly, when the two activities are coupled, the condition for existence and uniqueness of an equilibrium is tighter, namely $\delta\lambda_{\max}(\mathbf{G}) < c - |\beta|$. It is easy to see that for the uncoupled case ($\beta = 0$), the equilibrium level of each activity is given by a Bonacich centrality associated with the adjacency matrix $\mathbf{G}$, with discount factor δ, and weight vectors $\mathbf{p}^A$ and $\mathbf{p}^B$ for activity A and B respectively.

In this paper, we extend the work of [11] in two directions: firstly, we consider heterogeneous network effects wherein an agent's interaction with other agents for each activity is described by a different network structure, and/or the local network effect δ is different across activities. This allows us to capture more realistic situations such as that of harvesting from timber concessions and palm oil plantations, where the spatial configurations of forest concessions and plantations are described by different network structures and the production of the two activities are coupled. In particular, we consider agent-specific coupling parameter β_i: a positive parameter β_i expresses the complementarity of activities A and B, for instance because of common technologies, shared transportation and/or supply chains. By contrast, a negative parameter β_i means that activities A and B are substitutes, for instance because of resources competition (groundwater, land, nutrients, sunlight). [1] Secondly, we also consider a global network effect for each activity (via fully connected network) to model the competition that agents face in selling the produced goods in a market (for example, global market of timber and palm oil products).

Our technical contributions are as follows: We show that the condition for the existence and uniqueness of Nash equilibrium can be derived by analyzing the potential game formulation, or by leveraging the results on variational inequality for equilibrium. In Section III, we derive this condition for our network game with coupled activities (Theorem 1) by leveraging a preliminary result (Lemma 1 in Section II). Furthermore, we show that the

[1] The coupling effect can modulate the local network effects since a large parameter β_i in absolute value will affect both the parameters δ^A and δ^B, and the structures of the networks.

equilibrium can be expressed as a linear combination of two Bonacich centralities for our general network game with heterogeneous local network effects and presence of global competition among agents (Theorem 2). To further interpret our results, we conduct numerical results in Section IV. We provide some concluding remarks in Section V.

II. Preliminaries

Agents strategically interact over a network structure and each agent's payoff thus depends on other agents' action. We denote the generic game by $\Gamma : \langle \mathcal{N}, (\mathbf{Y}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}} \rangle$, where $\mathcal{N} = \{1, \cdots, n\}$ is the set of agents, $\mathbf{Y}_i$ is the set of available actions for agent i, and u_i is the agent individual utility function. We define by $\mathbf{Y} = \mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n$ the set of all action profiles. Each agent's action is multi-dimensional and denoted by $\mathbf{y}_i \in \mathbb{R}^N$, where N is the number of activities. For simplicity, we will limit our attention to $N = 2$. We also define $\mathbf{y}_{-i}$ the action profiles for all players except i, and $\mathbf{y} = (\mathbf{y}_i, \mathbf{y}_{-i})$ the action profiles for all players. The objective of each agent is to maximize her utility. An action profile $\mathbf{y} \in \mathbb{R}^{nN}$ is called a *Nash equilibrium* (NE) if no agent has an incentive to unilaterally change her strategy.

Definition 2. A pure strategy *Nash equilibrium* is a profile of actions $\mathbf{y} \in \mathbf{Y} = \mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n$ such that for all $i \in \mathcal{N}$:

$$u_i(\mathbf{y}_i, \mathbf{y}_{-i}) \geq u_i(\tilde{\mathbf{y}}_i, \mathbf{y}_{-i});\ \forall \tilde{\mathbf{y}}_i \in \mathbf{Y}_i.$$

It turns out that our network game Γ is a *potential game* (See Section III).

Definition 3. A game Γ is a potential game if there exists a function $\Phi : \mathbf{Y} \to \mathbb{R}$ such that $\forall i \in n, \forall \mathbf{y}_{-i} \in \mathbf{Y}_{-i}, \forall \mathbf{y}_i, \tilde{\mathbf{y}}_i \in \mathbf{Y}_i$, we have

$$\Phi(\mathbf{y}_i, \mathbf{y}_{-i}) - \Phi(\tilde{\mathbf{y}}_i, \mathbf{y}_{-i}) = u_i(\mathbf{y}_i, \mathbf{y}_{-i}) - u_i(\tilde{\mathbf{y}}_i, \mathbf{y}_{-i}).$$

The function Φ is called the *potential function* of the game Γ.

By [9] and [14], a profile of action $\mathbf{y}$ is a Nash equilibrium of Γ if and only if $\mathbf{y}$ satisfies the Kuhn-Tucker conditions of the problem $\max_{\mathbf{y}} \Phi(\mathbf{y}_i, \mathbf{y}_{-i})$. Each agent chooses its production as if she wanted to maximize the potential function, given other agent's production. This maximization problem has a unique solution when there is only one solution to its first-order conditions. For $\mathbf{y}_i \geq 0, \forall i$, a sufficient condition for a unique solution is for the potential function $\Phi(\mathbf{y}_i, \mathbf{y}_{-i})$ to be strictly concave, i.e., if the negative of the Hessian matrix of $\Phi(\mathbf{y}_i, \mathbf{y}_{-i})$ is positive definite, $-\mathbf{H} \succ 0$.

As an alternative to the potential function approach, one can use the variational inequality framework [12, 15, 16] to study existence and uniqueness of network games.

Assumption 4. *For all $i \in \{1, \cdots, n\}$, set $\mathbf{Y}_i$ is non-empty, closed and convex, and the utility function $u_i(\mathbf{y}_i, \mathbf{y}_{-i})$ is continuously differentiable and convex in $\mathbf{y}_i$ and for all $\mathbf{y}_j \in \mathbf{Y}_j$, $\forall j \neq i$ in the neighborhood of i. Furthermore, the utility function is twice differentiable in $[\mathbf{y}_i, \mathbf{z}_i]^\top$, and $\nabla_{\mathbf{y}_i} u_i(\mathbf{y}_i, \mathbf{y}_{-i})$ is Lipschitz in $[\mathbf{y}_i, \mathbf{z}_i]^\top$, where $\mathbf{z}_i = [\sum_{j=1}^n G_{ij}^X y_j^X]_{X=1}^N \in \mathbb{R}^N$.*

Definition 5. A vector $\bar{\mathbf{y}} \in \mathbb{R}^{nN}$ solves the variational ineqality $\mathrm{VI}(\mathbf{Y}, V)$ with set $\mathbf{Y} = \mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n$ and operator $V : \mathbf{Y} \to \mathbb{R}^{nN}$ if and only if

$$V(\bar{\mathbf{y}})^\top (\mathbf{y} - \bar{\mathbf{y}}) \geq 0, \forall \mathbf{y} \in \mathbf{Y}. \tag{1}$$

Under Assumption 4, [15, Proposition 1.4.2] show that $\mathbf{y}$ is a Nash equilibrium for game Γ if and only if it solves the variational inequality $\mathrm{VI}(\mathbf{Y}, V)$. Furthermore, in [15, Theorem 2.3.3], the authors show that the variational inequality $\mathrm{VI}(\mathbf{Y}, V)$ (1), where V is continuous and $\mathbf{Y}$ is nonempty, closed and convex, admits a unique solution if V is *strongly monotone*.

Definition 6. An operator V is strongly monotone if there exists $\alpha > 0, \alpha \in \mathbb{R}$, such that

$$\Big(V(\mathbf{y}) - V(\tilde{\mathbf{y}})\Big)^\top (\mathbf{y} - \tilde{\mathbf{y}}) \geq \alpha \left\| \mathbf{y} - \tilde{\mathbf{y}} \right\|_2^2,$$

for all $\tilde{\mathbf{y}}, \mathbf{y} \in \mathbf{Y}$.

It follows that for $\mathbf{y}$ to be a Nash equilibrium of game Γ, it is necessary and sufficient to prove that it solves the variational inequality $\mathrm{VI}(\mathbf{Y}, V)$; moreover a sufficient condition to prove the uniqueness of equilibrium is to show that the operator V is strongly monotone.

If V is an affine map, strong monotonicity is equivalent to strict monotonicity, i.e., $\Big(V(\mathbf{y}) - V(\tilde{\mathbf{y}})\Big)^\top (\mathbf{y} - \tilde{\mathbf{y}}) \geq 0, \forall \tilde{\mathbf{y}}, \mathbf{y} \in \mathbf{Y}$. A sufficient condition for V to be strictly monotone is to show that its Jacobian $\nabla_{\mathbf{y}} V(\mathbf{y})$ is positive definite [15, Proposition 2.3.2].

In fact, [14, Lemma 4.4] show that a game is a potential game with potential function Φ if and only if $\nabla_{\mathbf{y}} \Phi(\mathbf{y}) = [\nabla_{\mathbf{y}_i} u_i(\mathbf{y}_i, \mathbf{y}_{-i})^\top]_{i=1}^n$. We define $V(\mathbf{y}) := -\nabla_{\mathbf{y}} \Phi(\mathbf{y}) = -[\nabla_{\mathbf{y}_i} u(\mathbf{y}_i, \mathbf{y}_{-i})^\top]_{i=1}^n$. Then, the first-order optimality conditions of the optimization problem $\max_{\mathbf{y}} \Phi(\mathbf{y})$ can be expressed as $\nabla_{\mathbf{y}} \Phi(\bar{\mathbf{y}})(\bar{\mathbf{y}} - \mathbf{y})^\top \geq 0$, which coincide with the variational inequality (1) $V(\bar{\mathbf{y}})^\top (\mathbf{y} - \bar{\mathbf{y}}) \geq 0, \forall \mathbf{y} \in Y$. Consequently, we can establish the equivalence between strict concavity of Φ and strong monotonicity of $V(\mathbf{y}) := -\nabla_{\mathbf{y}} \Phi(\mathbf{y})$ as condition for uniqueness of Nash equilibrium.

Lemma 1. *A game Γ with utility function $u_i(\mathbf{y}_i, \mathbf{y}_{-i}), i \in \{1, \cdots, n\}$ is a potential game with potential function $\Phi(\mathbf{y})$ if and only if*

$\nabla_{\mathbf{y}}\Phi(\mathbf{y}) = -V(\mathbf{y}) = [\nabla_{\mathbf{y}_i} u_i(\mathbf{y}_i, \mathbf{y}_i)^\top]_{i=1}^n$. *Then, there exists a unique Nash equilibrium if* Φ *is strictly concave (equivalently if* V *is strongly monotone).*

Proof. From [9], strict concavity of Φ guarantees that the Nash equilibrium is unique. By definition, Φ is strictly concave if and only if $\Big(\nabla_{\mathbf{y}}\Phi(\mathbf{y}) - \nabla_{\bar{\mathbf{y}}}\Phi(\bar{\mathbf{y}})\Big)^\top(\bar{\mathbf{y}} - \mathbf{y}) > 0, \forall \bar{\mathbf{y}}, \mathbf{y} \in \mathbf{Y}$; that is, there exists $\alpha \in \mathbb{R}, \alpha > 0$, such that $\Big(\nabla_{\mathbf{y}}\Phi(\mathbf{y}) - \nabla_{\bar{\mathbf{y}}}\Phi(\bar{\mathbf{y}})\Big)^\top(\bar{\mathbf{y}} - \mathbf{y}) \geq \alpha \|\bar{\mathbf{y}} - \mathbf{y}\|^2$. Equivalently, the Nash equilibrium of Γ is unique if and only if $-\nabla_{\mathbf{y}}\Phi(\mathbf{y})$ is strongly monotone. The strict concavity of Φ is equivalent to the strong monotonicity of V. $\square$

III. Existence, uniqueness and equilibrium characterization

In this section, we specify our network game Γ and derive a condition for the existence and uniqueness of NE, which we then characterize.

A. Model

In our model, n agents interact over a network. Each agent's decision is her levels of production of two coupled activities denoted by A and B. Let $\mathbf{G}^A$ and $\mathbf{G}^B$ denote the adjacency matrices for the network influencing activity A and B, respectively. Each agent corresponds to a single node in the graph $\mathcal{G}^A(\mathcal{N}, \mathcal{E}^A)$ and the graph $\mathcal{G}^B(\mathcal{N}, \mathcal{E}^B)$, where $\mathcal{N}$ is the set of nodes and $\mathcal{E}^A$ (resp. $\mathcal{N}^B, \mathcal{E}^B$) is the set of edges in the network A (resp. B). For any node i, the neighborhood of i in the network A (resp. B) is the set of nodes j connected to i by an edge, i.e., $G^A_{ij} = 1$ (resp. $G^B_{ij} = 1$). Each agent chooses a level of production for activities A and B, denoted by y_i^A and y_i^B respectively, when y_i^A and y_i^B are non-negative. Each agent's action is thus two-dimensional. Let us denote $\mathbf{y}_i = [y_i^A, y_i^B]^\top$, $\mathbf{y}^A = [y_1^A, \cdots, y_n^A]^\top$, $\mathbf{y}^B = [y_1^B, \cdots, y_n^B]^\top$, and $\mathbf{y} = [\mathbf{y}^A, \mathbf{y}^B]^\top$. We also define $\mathbf{y}_{-i} = [y_1^A, y_1^B, \cdots, y_{i-1}^A, y_{i-1}^B, y_{i+1}^A, y_{i+1}^B, \cdots, y_n^A, y_n^B]^\top$ the productions of agents other than i. We denote by $\mathbf{p}^A = [p_1^A, \cdots, p_n^A]^\top$ and $\mathbf{p}^B = [p_1^B, \cdots, p_n^B]^\top$ the vectors of prices.

The utility of agent i follows a linear-quadratic function:

$$
\begin{aligned}
&u_i(\mathbf{y}_i, \mathbf{y}_{-i}) = \\
&\underbrace{p_i^A y_i^A - \frac{1}{2}c_i^A (y_i^A)^2}_{\text{Proceeds from A}} + \underbrace{p_i^B y_i^B - \frac{1}{2}c_i^B (y_i^B)^2}_{\text{Proceeds from B}} \\
&- \underbrace{\mu^A \sum_{j=1}^n y_i^A y_j^A}_{\text{Global network effect in activity } A} - \underbrace{\mu^B \sum_{j=1}^n y_i^B y_j^B}_{\text{Global network effect in activity } B} \\
&+ \underbrace{\delta^A \sum_{j=1}^n G_{ij}^A y_i^A y_j^A}_{\text{Local network effect from activity } A} + \underbrace{\delta^B \sum_{j=1}^n G_{ij}^B y_i^B y_j^B}_{\text{Local network effect from activity } B} \\
&+ \underbrace{\beta_i y_i^A y_i^B}_{\text{Interaction between activity } A \text{ and activity } B}.
\end{aligned}
\tag{2}
$$

The total utility of individual agents is the sum of their net gain from trade of both products, the effect of network interaction due to global competition between agents, the local network effects corresponding to each activity and the coupling term that captures the dependence of the two activities. Notice that we make no assumption on the sign of β_i for each agent i. Thus, the activities A and B may be either complements ($\beta_i > 0$) or substitutes ($\beta_i < 0$) for some agents, and even uncoupled for others ($\beta_i = 0$). We explain model parameters in Table I.

Parameter	Description
p_i^A	Price for activity A in dollar per unit of production ($\$/m^3$)
p_i^B	Price for activity B in dollar per unit of production ($\$/m^3$)
c_i^A	Unit cost of activity A ($\$/m^3$)
c_i^B	Unit cost of activity B ($\$/m^3$)
δ^A	Marginal local network effect for activity A ($\$/m^6$)
δ^B	Marginal local network effect for activity B ($\$/m^6$)
μ^A	Marginal effect of global competition for activity A ($\$/m^6$)
μ^B	Marginal effect of global competition for activity B ($\$/m^6$)
$\mathbf{G}^A$	Adjacency matrix of the network underlying activity A
$\mathbf{G}^B$	Adjacency matrix of the network underlying activity B
β_i	Agent-specific coupling parameter between activities ($\$/m^6$)

TABLE I: Notation

Note that the agents' equilibrium activity levels can be shaped by other factors. For example, one can consider identical prices across agents ($p_i^A \equiv p^A$ and $p_i^B \equiv p^B$) or cost of production ($c_i^A \equiv c^A$ and $c_i^B \equiv c^B$). Our results (Theorem 2) can be used to evaluate the impact of such factors, although our main focus is on the impact of coupling introduced by β_i on y_i^A, y_i^B under heterogeneous network structures underlying A and B.

B. Existence and uniqueness

It is important to recall from [9] that a simpler network game with a single activity admits a potential function. We establish analogous result for our more general game of two coupled activities. We know from [14] that for a game with continuous and twice-differentiable utility function u_i, there exists a potential function if and only if $\frac{\partial u_i(\mathbf{y}_i, \mathbf{y}_{-i})}{\partial y_i \partial y_j} = \frac{\partial u_i(\mathbf{y}_i, \mathbf{y}_{-i})}{\partial y_j \partial y_i}, \forall i, j \in \{1 \cdots, n\}$. Our game Γ satisfies these conditions since $\frac{\partial u_i(\mathbf{y}_i, \mathbf{y}_{-i})}{\partial y_i^A \partial y_j^A} = \frac{\partial u_i(\mathbf{y}_i, \mathbf{y}_{-i})}{\partial y_j^A \partial y_i^A} = \delta^A \mathbf{G}_{ij}^A$, $\frac{\partial u_i(\mathbf{y}_i, \mathbf{y}_{-i})}{\partial y_i^B \partial y_j^B} =$

$\frac{\partial u_i(\mathbf{y}_i,\mathbf{y}_{-i})}{\partial y_j^B \partial y_i^B} = \delta^B \mathbf{G}_{ij}^B$, and $\frac{\partial u_i(\mathbf{y}_i,\mathbf{y}_{-i})}{\partial y_i^A \partial y_j^B} = \frac{\partial u_i(\mathbf{y}_i,\mathbf{y}_{-i})}{\partial y_j^B \partial y_i^A} = 0$, thus Γ admits a potential function. We consider the potential function of the game Γ:

$$\begin{aligned}
&\Phi(\mathbf{y}_i, \mathbf{y}_{-i}) \\
&= p_i^A \sum_{i=1}^n y_i^A - \frac{c_i^A + \mu^A}{2} \sum_{i=1}^n (y_i^A)^2 \\
&+ \frac{\delta^A}{2} \sum_{i=1}^n \sum_{j=1}^n G_{ij}^A y_i^A y_j^A - \frac{\mu^A}{2} \sum_{i=1}^n \sum_{j=1}^n y_i^A y_j^A \\
&+ p_i^B \sum_{i=1}^n y_i^B - \frac{c_i^B + \mu^B}{2} \sum_{i=1}^n (y_i^B)^2 \\
&+ \frac{\delta^B}{2} \sum_{i=1}^n \sum_{j=1}^n G_{ij}^B y_i^B y_j^B - \frac{\mu^B}{2} \sum_{i=1}^n \sum_{j=1}^n y_i^B y_j^B \\
&+ \sum_{i=1}^n \beta_i y_i^A y_i^B.
\end{aligned} \tag{6}$$

We can check that Φ, as defined in (6), satisfies Definition 3.

Its Hessian matrix is given by $\mathbf{H} = \begin{bmatrix} -\mathbf{D} & \boldsymbol{\beta} \\ \boldsymbol{\beta} & -\mathbf{Q} \end{bmatrix}$ where

$$\mathbf{D} = (c^A + \mu^A)\mathbf{I} + \mu^A \mathbf{J} - \delta^A \mathbf{G}^A,$$
$$\mathbf{Q} = (c^B + \mu^B)\mathbf{I} + \mu^B \mathbf{J} - \delta^B \mathbf{G}^B.$$

and $\mathbf{J}$ is the matrix of full ones, and $\boldsymbol{\beta}$ is the diagonal matrix such that $\boldsymbol{\beta}_{ii} = \beta_i$.

Recall from Section II, the game has a unique Nash equilibrium if the potential function is strictly concave, i.e., if $-\mathbf{H}$ is positive definite. As $-\mathbf{H}$ is a block matrix, it is positive definite if and only if both $((c^A + \mu)\mathbf{I} + \mu \mathbf{J} - \delta^A \mathbf{G}^A)$ and its Schur complement are positive definite. We simplify these conditions to show the following result.

Theorem 1. *If* $\min(c^A + \mu^A; c^B + \mu^B) - \max_i\{|\beta_i|\} > \max(\delta^A \lambda_{max}(\mathbf{G}^A), \delta^B \lambda_{max}(\mathbf{G}^B))$, *then there exists a unique Nash equilibrium to the game* Γ.

The proof can be found in Appendix A. The authors in [11] provide an analogous condition but without proof. Besides, our condition applies to the game with global network effect (non-zero μ^A and μ^B) and heterogeneous cost of two activities ($c^A \neq c^B$). Intuitively, the condition given in Theorem 1 requires that the network effect must be small enough compared to own individual concavity, i.e., quadratic terms (here costs) that only involve own agent's production. Otherwise, a large enough local network effect would compromise the positive definiteness of $-\mathbf{H}$. Notice that the global competition term does not show this effect because they are counted negatively in the utility function. In an economic context, it means that the influence of direct neighbors should not exceed the effect of one's own production cost and global competition net the coupling effect between the two activities.

In proving Theorem 1, we simply leveraged the fact that our game is a potential game and concluded existence and uniqueness of the Nash equilibrium based on positive definiteness of the negative of the Hessian of potential function of the game. On the other hand, authors of [12, 16] (building on results of [15]), offer a more general result on the equilibrium existence and uniqueness based on variational inequality framework provided Assumption 4 and strong monotonicity of operator V holds.

Indeed, for the game Γ, Theorem 1 can be also derived through the lens of the variational inequality with set $\mathbf{Y} = \mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n$ and operator $V : \mathbf{Y} \to \mathbb{R}^{n \times 2}$, $V(\mathbf{y}) := -[\nabla_{\mathbf{y}_i} u_i(\mathbf{y}, \mathbf{y}_{-i})^\top]_{i=1}^n$

C. Equilibrium characterization

For each agent i, the first-order conditions of the game are given by:

$$p_i^A - c_i^A y_i^A + \beta_i y_i^B - \mu^A y_i^A - \mu^A \sum_{j=1}^n y_j^A + \delta^A \sum_{j=1}^n G_{ij}^A y_j^A = 0,$$

$$p_i^B - c_i^B y_i^B + \beta_i y_i^A - \mu^B y_i^B - \mu^B \sum_{j=1}^n y_j^B + \delta^B \sum_{j=1}^n G_{ij}^B y_j^B = 0.$$

Following the notation in Section III-B, we can express these conditions in matrix form:

$$\begin{bmatrix} \mathbf{D} & -\boldsymbol{\beta} \\ -\boldsymbol{\beta} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{y}^A \\ \mathbf{y}^B \end{bmatrix} = \begin{bmatrix} \mathbf{p}^A \\ \mathbf{p}^B \end{bmatrix}. \tag{3}$$

If the existence and uniqueness condition in Theorem 1 holds, systems (3) is invertible (refer to [17, Lemma*]):

$$\begin{bmatrix} \mathbf{y}^A \\ \mathbf{y}^B \end{bmatrix} = \begin{bmatrix} \mathbf{D} & -\boldsymbol{\beta} \\ -\boldsymbol{\beta} & \mathbf{Q} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}^A \\ \mathbf{p}^B \end{bmatrix}. \tag{4}$$

Let us proceed with the following notation:

$$\begin{aligned}
\widetilde{\mathbf{G}}^A &= -\frac{\mu^A}{c^A + \mu^A} \mathbf{J} + \frac{\delta^A}{c^A + \mu^A} \mathbf{G}^A, \\
\widetilde{\mathbf{G}}^B &= -\frac{\mu^B}{c^B + \mu^B} \mathbf{J} + \frac{\delta^B}{c^B + \mu^B} \mathbf{G}^B, \\
\mathbf{L}_A &= \frac{1}{c^A + \mu^A} (\mathbf{I} - \widetilde{\mathbf{G}}^A)^{-1}, \\
\mathbf{L}_B &= \frac{1}{c^B + \mu^B} (\mathbf{I} - \widetilde{\mathbf{G}}^B)^{-1}.
\end{aligned}$$

Here, $(c^A + \mu^A)\mathbf{L}^A$ and $(c^B + \mu^B)\mathbf{L}^B$ are the so-called *Leontief matrices* [13]. Let us also define $\mathbf{B}_A = (c^A + \mu^A)\mathbf{L}_A \mathbf{1} = (\mathbf{I} - \widetilde{\mathbf{G}}^A)^{-1}\mathbf{1}$ and $\mathbf{B}_B = (c^B + \mu^B)\mathbf{L}_B \mathbf{1} = (\mathbf{I} - \widetilde{\mathbf{G}}^B)^{-1}\mathbf{1}$. Following Definition 1, the vectors $\mathbf{B}_A$ and $\mathbf{B}_B$ can be interpreted as Bonacich centralities for the networks with adjacency matrices $\widetilde{\mathbf{G}}^A$ and $\widetilde{\mathbf{G}}^B$ respectively.

We also define the weighted Bonacich centralities $\widehat{\mathbf{B}}_A = (c^A + \mu^A)\mathbf{L}_A \mathbf{p}^A = (\mathbf{I} - \widetilde{\mathbf{G}}^A)^{-1}\mathbf{p}^A$ and $\widehat{\mathbf{B}}_B = (c^B + \mu^B)\mathbf{L}_B \mathbf{p}^B = (\mathbf{I} - \widetilde{\mathbf{G}}^B)^{-1}\mathbf{p}^B$, where the weights are the corresponding price vectors for activities A and B, i.e., $\mathbf{p}^A = [p_1^A, \cdots, p_n^A]^\top$ and $\mathbf{p}^B = [p_1^B, \cdots, p_n^B]^\top$.

For ease of notation and simplify equations, let us adopt the following notation:

$$\bar{\mathbf{B}}_\mathbf{A} \equiv \frac{1}{c^A + \mu^A} \widehat{\mathbf{B}}_A$$
$$\bar{\mathbf{B}}_\mathbf{B} \equiv \frac{1}{c^B + \mu^B} \widehat{\mathbf{B}}_B.$$

Rewriting (4), the system to be solved is the following:

$$\begin{bmatrix} \mathbf{y}^A \\ \mathbf{y}^B \end{bmatrix} = \begin{bmatrix} \mathbf{L}_A^{-1} & -\boldsymbol{\beta} \\ -\boldsymbol{\beta} & \mathbf{L}_B^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}^A \\ \mathbf{p}^B \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{bmatrix} \begin{bmatrix} \mathbf{p}^A \\ \mathbf{p}^B \end{bmatrix},$$

where by the inversion formulae of block diagonal matrices, $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3$ and $\mathbf{Z}_4$ are given as follows:

$$\begin{aligned}
\mathbf{Z}_1 &= [\mathbf{L}_A^{-1} - \boldsymbol{\beta}\mathbf{L}_B\boldsymbol{\beta}]^{-1}, \\
&= [\mathbf{I} - \mathbf{L}_A\boldsymbol{\beta}\mathbf{L}_B\boldsymbol{\beta}]^{-1}\mathbf{L}_A \\
\mathbf{Z}_2 &= \mathbf{L}_A\boldsymbol{\beta}[\mathbf{L}_B^{-1} - \boldsymbol{\beta}\mathbf{L}_A\boldsymbol{\beta}]^{-1}, \\
&= \mathbf{L}_A\boldsymbol{\beta}[\mathbf{I} - \mathbf{L}_B\boldsymbol{\beta}\mathbf{L}_A\boldsymbol{\beta}]^{-1}\mathbf{L}_B \\
\mathbf{Z}_3 &= \mathbf{L}_B\boldsymbol{\beta}[\mathbf{L}_A^{-1} - \boldsymbol{\beta}\mathbf{L}_B\boldsymbol{\beta}]^{-1}, \\
&= \mathbf{L}_B\boldsymbol{\beta}[\mathbf{I} - \mathbf{L}_A\boldsymbol{\beta}\mathbf{L}_B\boldsymbol{\beta}]^{-1}\mathbf{L}_A \\
\mathbf{Z}_4 &= [\mathbf{L}_B^{-1} - \boldsymbol{\beta}\mathbf{L}_A\boldsymbol{\beta}]^{-1}, \\
&= [\mathbf{I} - \mathbf{L}_B\boldsymbol{\beta}\mathbf{L}_A\boldsymbol{\beta}]^{-1}\mathbf{L}_B.
\end{aligned}$$

Further, in order to simplify notation, we introduce coefficients $\mathbf{K}_1, \mathbf{K}_2, \widetilde{\mathbf{K}}_1$ and $\widetilde{\mathbf{K}}_2$, such that $\mathbf{Z}_1 = \mathbf{K}_1\mathbf{L}_A, \mathbf{Z}_2 = \widetilde{\mathbf{K}}_2\mathbf{L}_B, \mathbf{Z}_3 = \widetilde{\mathbf{K}}_1\mathbf{L}_A$ and $\mathbf{Z}_4 = \widetilde{\mathbf{K}}_2\mathbf{L}_B$, where we define

$$\begin{aligned}
\mathbf{K}_1 &= [\mathbf{I} - \mathbf{L}_A\boldsymbol{\beta}\mathbf{L}_B\boldsymbol{\beta}]^{-1}, \\
\mathbf{K}_2 &= [\mathbf{I} - \mathbf{L}_B\boldsymbol{\beta}\mathbf{L}_A\boldsymbol{\beta}]^{-1}, \\
\widetilde{\mathbf{K}}_1 &= \mathbf{L}_B\boldsymbol{\beta}[\mathbf{I} - \mathbf{L}_A\boldsymbol{\beta}\mathbf{L}_B\boldsymbol{\beta}]^{-1} = \mathbf{L}_B\boldsymbol{\beta}\mathbf{K}_1, \\
\widetilde{\mathbf{K}}_2 &= \mathbf{L}_A\boldsymbol{\beta}[\mathbf{I} - \mathbf{L}_B\boldsymbol{\beta}\mathbf{L}_A\boldsymbol{\beta}]^{-1} = \mathbf{L}_A\boldsymbol{\beta}\mathbf{K}_2.
\end{aligned}$$

We can now characterize the equilibrium of the game Γ.

Theorem 2. *Assume* $\min(c^A + \mu^A; c^B + \mu^B) - \max_i\{|\beta_i|\} > \max(\delta^A\lambda_{max}(\mathbf{G}^A), \delta^B\lambda_{max}(\mathbf{G}^B))$*, then the game* Γ *admits a unique Nash equilibrium given by:*

$$\begin{aligned}
\mathbf{y}^A &= \mathbf{K}_1\bar{\mathbf{B}}_\mathbf{A} + \widetilde{\mathbf{K}}_2\bar{\mathbf{B}}_\mathbf{B}, \\
\mathbf{y}^B &= \widetilde{\mathbf{K}}_1\bar{\mathbf{B}}_\mathbf{A} + \mathbf{K}_2\bar{\mathbf{B}}_\mathbf{B}.
\end{aligned} \tag{5}$$

For the special case of uniform prices across agents (i.e., $\mathbf{p}^A = p^A\mathbf{1}$ *and* $\mathbf{p}^B = p^B\mathbf{1}$*):*

$$\begin{aligned}
\mathbf{y}^A &= p^A\mathbf{K}_1\mathbf{B}_A + p^B\widetilde{\mathbf{K}}_2\mathbf{B}_B, \\
\mathbf{y}^B &= p^A\widetilde{\mathbf{K}}_1\mathbf{B}_A + p^B\mathbf{K}_2\mathbf{B}_B.
\end{aligned} \tag{6}$$

To interpret this result, we recall that [11] consider a simpler game in which the activities A and B have the same structure ($\mathbf{G}^A = \mathbf{G}^B$) and $\delta^A = \delta^B$. In their case, they show that, in equilibrium activities can be expressed as a sum of two Bonacich centralities. In our game, networks $\mathbf{G}^A$ and $\mathbf{G}^B$ are different and $\delta^A \neq \delta^B$. Theorem 2 provides that the equilibrium for the game Γ - if unique - exists, can be given by a linear combination of $\bar{\mathbf{B}}_\mathbf{A}$ and $\bar{\mathbf{B}}_\mathbf{B}$, i.e., the vectors of Bonacich centralities of the networks with adjacency matrices $\widetilde{\mathbf{G}}^A$ and $\widetilde{\mathbf{G}}^B$ respectively.

Furthermore, if there is no coupling between activities (i.e. $\boldsymbol{\beta} = \mathbf{0}$), then $\mathbf{y}^A = \bar{\mathbf{B}}_\mathbf{A}$ and $\mathbf{y}^B = \bar{\mathbf{B}}_\mathbf{B}$. Indeed, in such a case, the equilibrium of each activity can be analyzed independently, and we are left with the result from [8, 9, 10, 13]. When $\boldsymbol{\beta} \neq \mathbf{0}$, the vectors of Bonacich centralities $\bar{\mathbf{B}}_\mathbf{A}$ and $\bar{\mathbf{B}}_\mathbf{B}$ only depend on the corresponding network for A and B respectively and are weighted by the weight matrices $\mathbf{K}_1, \mathbf{K}_2, \widetilde{\mathbf{K}}_1$ and $\widetilde{\mathbf{K}}_2$. We can interpret these weight matrices coefficient by going back to matrices $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3$ and $\mathbf{Z}_4$.
Let us discuss the general case of agent-specific prices $\mathbf{p}^A$ and $\mathbf{p}^B$. Consider a marginal increase in the price for agent 1, that is p_1^A increases by $+1$ for agent 1. Then, by Theorem 2, Equation (5), the new equilibrium is given by

$$\mathbf{y}^A_{new} = \mathbf{Z}_1\mathbf{p}^A + \mathbf{Z}_1[1, 0, \cdots, 0]^\top + \mathbf{Z}_2\mathbf{p}^B = \mathbf{y}^A + \mathbf{Z}_1[1, 0, \cdots, 0]^\top$$

Since $\mathbf{Z}_1[1, 0, \cdots, 0]^\top$ is equal to the first column of $\mathbf{Z}_1$, we deduce that the i-th column of $\mathbf{Z}_1$ is the change in effort in activity A for each agent after the price p_i^A of product A has been raised by $+1$ for agent i.

If now the price $\mathbf{p}^A$ is uniform across agents, then by Theorem 2, Equation (6), the sum of the entries of the j-row of $\mathbf{Z}_1$ is the change in effort in activity A for agent j after the price p^A of product A has been (*globally*) raised by $+1$.

An analogous interpretation can be provided for $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$.

Moreover, the entry $(\mathbf{K}_1)_{ji}$ represents the weight of agent i, due to its engagement in activity A, on the production of agent j in activity A. If the price p_i^A of agent i is increased by $+1$, then the vector of Bonacich centralities $\bar{\mathbf{B}}_\mathbf{A}$ becomes $\bar{\mathbf{B}}_\mathbf{A}^{new} = \bar{\mathbf{B}}_\mathbf{A} + (\mathbf{L}_A)_i$, where $(\mathbf{L}_A)_i$ is the i-th column of $\mathbf{L}_A$. The new vector of productions at the equilibrium becomes $\mathbf{y}^A_{new} = \mathbf{y}^A + \mathbf{K}_1(\mathbf{L}_A)_i$. Therefore, the sum-product $\sum_{l=1}^n (\mathbf{K}_1)_{jl}(\mathbf{L}_A)_{li}$ is the change in production produced by agent j, when the price p_i^A of agent i is increased by $+1$. The same reasoning follows for $\mathbf{K}_2$, $\widetilde{\mathbf{K}}_1$ and $\widetilde{\mathbf{K}}_2$.

In the context of trade, $\mathbf{p}^A$ and $\mathbf{p}^B$ can be determined in equilibrium. For instance, assuming homogeneous prices across agents, one can consider the consumer utility function $\pi(q^A, q^B) = a^A q^A - \frac{1}{2}p^A(q^A)^2 + a^B q^B - \frac{1}{2}p^B(q^B)^2$, where $q^A = \sum_{i=1}^n y_i^A, q^B = \sum_{i=1}^n y_i^B$, and a^A, a^B are consumers' marginal monetary value for activity A and B respectively. Solving for the first-order conditions $a^A - p^A q^A = 0$ and $a^B - p^B q^B = 0$, the prices are endogenously given by $p^A = \frac{a^A}{\sum_{i=1}^n y_i^A}$ and $p^B = \frac{a^B}{\sum_{i=1}^n y_i^B}$ respectively.

IV. Numerical Study

In this section, we conduct some numerical experiments in order to analyze how the equilibrium activity level of activity A varies with the local network effects δ^A and δ^B, and with the coupling parameters between activities β_i. By symmetry, we would follow the same reasoning for $\mathbf{y}^B$. For simplicity, we henceforth assume that the coupling parameters are identical across agents, $\beta_i = \beta, \forall i$.

We first consider the effect of the parameter δ^A. We consider a network with 10 agents, $c^A = c^B = 10$, global network effect $\mu = 0.01$, coupling parameter $\beta = 0.4$ and we vary the parameter δ^A. We compute the effect of this variation for different network structures $\mathbf{G}^A$ and $\mathbf{G}^B$. The effort in activity A is displayed (in log) in Figure 1 as a function of δ^A (here we set $\delta^A = \delta^B$).

We first notice that for given network structures in A and B and a positive cross-activity parameter β, the level of effort in A at the equilibrium increases with the network parameters δ^A. For a positive network effect $\delta^A > 0$, the feedback from neighbors affect individual agents positively. In such situation, an increase in magnitude in δ^A enables to improve further the positive effect of the local network term in (2). If $\delta^A < 0$, then the negative feedback from neighbors decreases individual activity levels. Under this circumstance, an increase in the magnitude δ^A allows to mitigate the negative effect of the local network term in (2).

When the coupling parameter β is negative, then an increase in the local network effect parameter δ^A has an ambiguous effect. In such a situation, an increase in δ^A may tend to increase production through the local network term $\sum_{j=0}^n G_{ij}^B y_i^A y_j^A$. However, such an increase may be counterbalanced by the cross-activity term $\beta y_i^A y_i^B$. The effect of an increase in magnitude in δ^A will thus depend on the relative magnitude

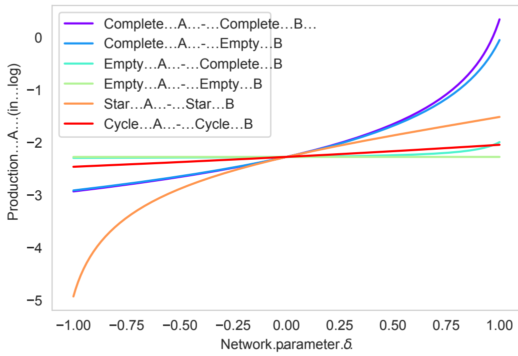

Fig. 1: Individual production (in log) of activity A as a function of the network parameter δ_A. We assume $\delta^A = \delta^B$, $\mu^A = \mu^B = 0.01$, $\beta_i = \beta = 0.4, \forall i$.

between the local network parameters delta δ^A, δ^B and the coupling parameter β.

The density of the network structures also affects the production level at the equilibrium. For positive network effects δ^A, δ^B and a positive coupling effect β, a denser network in A (e.g. complete network where all agents are connected) increases the effort made by the agents compared to a sparser network (e.g., empty network where the degree of every node is 0). This is expected as a positive network effect $\delta^A > 0$ means that the feedback from neighbors affect individual agents positively. When $\delta^A < 0$, the feedback from neighbors is negative. In such a case, a denser network in activity A will negatively affect the equilibrium activity levels.

The density of the other activity B also impacts the effort in activity A, because of the coupling β between activities. In particular, assuming $\beta > 0$, given a network structure in activity A, if the network effects δ^A and δ^B are positive, the production in A will be higher when the network structure for activity B is denser. For instance, in Figure 1, when the network structure in A is complete and $\delta^A, \delta^B > 0$ (respectively $\delta^A, \delta^B < 0$), the production in A is higher (respectively smaller) when the network structure in activity B is complete compared to an empty network.

These effects are ambiguous if β and (δ^A, δ^B) are of different signs. For instance, if $\beta < 0$ (and $\delta^A > 0, \delta^B > 0$), a denser network in B increases the utility through the local network term $\delta^B \sum_{j=1}^n \mathbf{G}_{ij}^B y_i^B y_j^B$, but this also has a negative impact because of the cross-activity term $\beta y_i^A y_i^B$. If $\beta > 0$ (and $\delta^A < 0, \delta^B < 0$), a denser network in B decreases the utility through the local network term $\delta^B \sum_{j=1}^n \mathbf{G}_{ij}^B y_i^B y_j^B$, thus pushing y_i^A downwards, while the cross-activity term $\beta y_i^A y_i^B$ tends to push it upwards.

Ceteris paribus, the analysis in the coupling parameter β is more straightforward. A negative β parameter affects the equilibrium level negatively while it has a positive effect when it is positive ($\beta > 0$). Overall, for fixed local parameters δ^A, δ^B and fixed network structures, an increase in β increases the level of effort at the equilibrium.

In summary, a trade-off arises between the local network effect (δ^A, δ^B) and the coupling parameter (β), namely for a given production level, a decrease (resp. increase) in the local network terms $\delta^A \sum_{j=1}^n \mathbf{G}_{ij}^A y_i^A y_j^A, \delta^B \sum_{j=1}^n \mathbf{G}_{ij}^B y_i^B y_j^B$ can be traded against an increase (resp. decrease) in the coupling term $\beta x_i^A x_i^B$. This trade-off is illustrated in Figure 2.

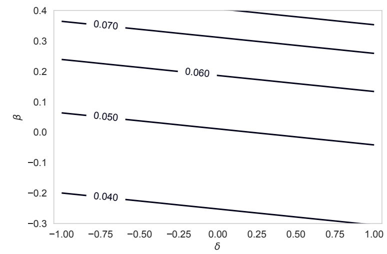

Fig. 2: Curves of iso-production in activity A for an individual agent at the equilibrium. The network structure is complete for both $\mathbf{G}^A$ and $\mathbf{G}^B$, $n = 20$, $\mu = 0.01$.

V. Conclusion

We study a linear-quadratic game with two activities and heterogeneous network structures for these activities. We find a sufficient condition for the existence and uniqueness of a Nash equilibrium and we show that this condition can be equivalently studied through the potential game property of this game or using variational inequality results. Furthermore, we prove that the equilibrium can be written as a linear combination of Bonacich centralities. As future work, an econometric analysis based on this model should be conducted in order to empirically test our theoretical model and in particular estimate the effect of the local network terms. In the context of illegal logging, such an analysis would enable us to estimate the effect of the spatial layout of concessions.

Appendix

A. Proof of Theorem 1

In order to simplify the notation, we show the proof for Theorem 1 assuming $\mu^A = \mu^B = \mu$. $((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)$ is positive definite if and only if:

$$\begin{aligned} &\mathbf{a}^\top((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)\mathbf{a} > 0, \forall \mathbf{a} \neq 0,\\ &\quad (c^A+\mu)\mathbf{a}^\top\mathbf{a}+\mu\mathbf{a}^\top\mathbf{J}\mathbf{a} > \delta^A\mathbf{a}^\top\mathbf{G}^A\mathbf{a}, \forall \mathbf{a} \neq 0,\\ &(c^A+\mu)\frac{\mathbf{a}^\top\mathbf{a}}{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}}+\mu\frac{\mathbf{a}^\top\mathbf{J}\mathbf{a}}{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}} > \delta^A, \forall \mathbf{a} \neq 0. \end{aligned} \tag{7}$$

In particular, this must be true for any $\mathbf{a} \neq 0$. We have:

$$\begin{aligned} &(c^A+\mu)\min_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{a}}{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}}+\mu\min_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{J}\mathbf{a}}{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}} > \delta^A,\\ &(c^A+\mu)\min_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{a}}{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}}+\mu\min_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{J}\mathbf{a}}{\mathbf{a}^\top\mathbf{a}}\frac{\mathbf{a}^\top\mathbf{a}}{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}} > \delta^A,\\ &(c^A+\mu)\frac{1}{\max_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}}{\mathbf{a}^\top\mathbf{a}}}+\mu\min_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{J}\mathbf{a}}{\mathbf{a}^\top\mathbf{a}}\frac{1}{\max_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{a}}{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}}} > \delta^A. \end{aligned} \tag{8}$$

By the Rayleigh-Ritz theorem, we have $\max_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}}{\mathbf{a}^\top\mathbf{a}} = \lambda_{max}(\mathbf{G}^A)$, where $\lambda_{max}(\mathbf{G}^A)$ is the largest eigenvalue of $\mathbf{G}^A$ and $\min_{\mathbf{a}\neq\mathbf{0}}\frac{\mathbf{a}^\top\mathbf{G}^A\mathbf{a}}{\mathbf{a}^\top\mathbf{a}} = \lambda_{min}(\mathbf{G}^A)$ where $\lambda_{min}(\mathbf{G}^A)$ is the smallest eigenvalue of $\mathbf{G}^A$. Hence, the condition for the positive definiteness property is,

$$\delta^A < (c^A+\mu)\lambda_{max}(\mathbf{G}^A)+\mu\frac{\lambda_{min}(\mathbf{J})}{\lambda_{max}(\mathbf{G}^A)}. \tag{9}$$

The eigenvalues of $\mathbf{J}$ are n with multiplicity 1 and 0 with multiplicity $n-1$. We thus deduce that $((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)$ is positive definite if and only if

$$c^A+\mu > \delta^A \lambda_{max}(\mathbf{G}^A). \tag{10}$$

Let us now find a sufficient condition for the positive definiteness of the Schur complement of $((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)$. The Schur complement of $((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)$ is $((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)-\boldsymbol{\beta}((c^B+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^B\mathbf{G}^B)^{-1}\boldsymbol{\beta}$. We thus want to show a condition such that

$$\begin{aligned}
&((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)\\
&-\boldsymbol{\beta}((c^B+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^B\mathbf{G}^B)^{-1}\boldsymbol{\beta} > \mathbf{0},\\
&((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)\\
&-(\max_i\{\beta_i\})^2((c^B+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^B\mathbf{G}^B)^{-1} > \mathbf{0},\\
&((c^B+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^B\mathbf{G}^B)\\
&((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A) > (\max_i\{\beta_i\})^2\mathbf{I}.
\end{aligned} \tag{11}$$

Then $\forall \mathbf{a}^\top \neq \mathbf{0}$, we have:

$$\begin{aligned}
&\max_{\mathbf{a}^\top\neq\mathbf{0}} \frac{\mathbf{a}^\top((c^B+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^B\mathbf{G}^B)((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)\mathbf{a}}{\mathbf{a}^T\mathbf{a}}\\
&> (\max_i\{|\beta_i|\})^2.
\end{aligned} \tag{12}$$

Furthermore, for A and B two positive definite matrices, we have $\lambda_{max}(AB) < \lambda_{max}(A)\lambda_{max}(B)$. Therefore, we have:

$$\begin{aligned}
&\max_{\mathbf{a}^\top\neq\mathbf{0}} \frac{\mathbf{a}^\top((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)\mathbf{a}}{\mathbf{a}^T\mathbf{a}}\\
&\max_{\mathbf{a}^\top\neq\mathbf{0}} \frac{\mathbf{a}^\top((c^B+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^B\mathbf{G}^B)\mathbf{a}}{\mathbf{a}^T\mathbf{a}} > (\max_i\{|\beta_i|\})^2,\\
&\max_{X\in\{A,B\}} \left(\max_{\mathbf{a}^\top\neq\mathbf{0}} \frac{\mathbf{a}^\top((c^A+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^A\mathbf{G}^A)\mathbf{a}}{\mathbf{a}^T\mathbf{a}}\right)^2\\
&> (\max_i\{|\beta_i|\})^2.
\end{aligned} \tag{13}$$

Quantities on both sides of the inequality are positive scalars and we can take the square root of these quantities.

$$\max_{X\in\{A,B\}} \max_{\mathbf{a}^\top\neq\mathbf{0}} \frac{\mathbf{a}^\top((c^X+\mu)\mathbf{I}+\mu\mathbf{J}-\delta^X\mathbf{G}^X)\mathbf{a}}{\mathbf{a}^\top\mathbf{a}} > \max_i\{|\beta_i|\}. \tag{14}$$

Since $\lambda_{max}(\mathbf{J}) = n$, a sufficient condition is:

$$\begin{aligned}
&\max_{X\in\{A,B\}}(c^X+\mu) - \max_{\mathbf{a}^\top\neq\mathbf{0}} \delta^X \frac{\mathbf{a}^\top\mathbf{G}^X\mathbf{a}}{\mathbf{a}^\top\mathbf{a}} > \max_i\{|\beta_i|\},\\
&\max(\delta^A\lambda_{max}(\mathbf{G}^A);\delta^B\lambda_{max}(\mathbf{G}^B))\\
&< -\max_i\{|\beta_i|\} + \max(c^A+\mu; c^A+\mu),
\end{aligned} \tag{15}$$

and we deduce that the condition for existence and uniqueness of a Nash equilibrium is

$$\begin{aligned}
&\min(c^A+\mu-\max_i\{|\beta_i|\}; c^B+\mu-\max_i\{|\beta_i|\})\\
&> \max(\delta^A\lambda_{max}(\mathbf{G}^A), \delta^B\lambda_{max}(\mathbf{G}^B)).
\end{aligned} \tag{16}$$

References

[1] K. Carlson, L. Curran, G. Asner, A. Pittman, S. Trigg, and J. Adeney, “Carbon emissions from forest conversion by kalimantan oil palm plantations,” *Nature Climate Change*, vol. 3, 10 2012.

[2] J. Busch, K. Ferretti-Gallon, J. Engelmann, M. Wright, K. G. Austin, F. Stolle, S. Turubanova, P. V. Potapov, B. Margono, M. C. Hansen, and A. Baccini, “Reductions in emissions from deforestation from indonesia’s moratorium on new oil palm, timber, and logging concessions,” *Proceedings of the National Academy of Sciences*, vol. 112, no. 5, pp. 1328–1333, 2015. [Online]. Available: https://www.pnas.org/doi/abs/10.1073/pnas.1412514112

[3] K. M. Carlson, R. Heilmayr, H. K. Gibbs, P. Noojipady, D. N. Burns, D. C. Morton, N. F. Walker, G. D. Paoli, and C. Kremen, “Effect of oil palm sustainability certification on deforestation and fire in indonesia,” *Proceedings of the National Academy of Sciences*, vol. 115, no. 1, pp. 121–126, 2018. [Online]. Available: https://www.pnas.org/doi/abs/10.1073/pnas.1704728114

[4] E. I. Agency, “Permitting crime: How palm oil expansion drives illegal logging in indonesia,” 2014.

[5] Y. Shigetomi, Y. Ishimura, and Y. Yamamoto, “Trends in global dependency on the indonesian palm oil and resultant environmental impacts,” *Scientific Reports*, vol. 10, p. 20624, 11 2020.

[6] M. Qaim, K. T. Sibhatu, H. Siregar, and I. Grass, “Environmental, economic, and social consequences of the oil palm boom,” *Annual Review of Resource Economics*, vol. 12, no. 1, pp. 321–344, 2020. [Online]. Available: https://doi.org/10.1146/annurev-resource-110119-024922

[7] D. Gaveau, B. Locatelli, M. Salim, H. Msi, T. Manurung, A. Descals, A. Angelsen, E. Meijaard, and D. Sheil, “Slowing deforestation in indonesia follows declining oil palm expansion and lower oil prices,” 01 2021.

[8] C. Ballester, A. Calvó-Armengol, and Y. Zenou, “Who’s who in networks. wanted: The key player,” *Econometrica*, vol. 74, no. 5, pp. 1403–1417, 2006. [Online]. Available: http://www.jstor.org/stable/3805930

[9] Y. Bramoullé, R. Kranton, and M. D’Amours, “Strategic interaction and networks,” *American Economic Review*, vol. 104, no. 3, pp. 898–930, March 2014. [Online]. Available: https://www.aeaweb.org/articles?id=10.1257/aer.104.3.898

[10] M. O. Jackson and Y. Zenou, “Games on networks,” in *Handbook of Game Theory with Economic Applications*. Elsevier, 2015, vol. 4, ch. 3, pp. 95–163. [Online]. Available: https://EconPapers.repec.org/RePEc:eee:gamchp:v:4:y:2015:i:c:p:95-163

[11] Y.-J. Chen, Y. Zenou, and J. Zhou, “Multiple activities in networks,” *American Economic Journal: Microeconomics*, vol. 10, no. 3, pp. 34–85, August 2018. [Online]. Available: https://www.aeaweb.org/articles?id=10.1257/mic.20160253

[12] F. Parise and A. Ozdaglar, “A variational inequality framework for network games: Existence, uniqueness, convergence and sensitivity analysis,” *Games and Economic Behavior*, vol. 114, pp. 47–82, 2019. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0899825618301891

[13] D. Acemoglu, A. Ozdaglar, and A. Tahbaz-Salehi, “Networks, shocks, and systemic risk,” National Bureau of Economic Research, Working Paper 20931, February 2015. [Online]. Available: http://www.nber.org/papers/w20931

[14] D. Monderer and L. Shapley, “Potential games,” *Games and Economic Behavior*, vol. 14, pp. 124–143, 05 1996.

[15] F. Facchinei and J. S. Pang, “Finite-dimensional variational inequalities and complementarity problems,” ser. Operations Research. Springer, 2003.

[16] E. Melo, “A variational approach to network games,” *SSRN Electronic Journal*, 05 2017.

[17] G. Debreu and I. N. Herstein, “Nonnegative square matrices,” *Econometrica*, vol. 21, no. 4, pp. 597–607, 1953. [Online]. Available: http://www.jstor.org/stable/1907925