LINE TRANSVERSALS IN FAMILIES OF CONNECTED SETS IN THE PLANE

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Abstract. We prove that if a family of compact connected sets in the plane has the property that every three members of it are intersected by a line, then there are three lines intersecting all the sets in the family. This answers a question of Eckhoff from 1993 [3], who proved that, under the same condition, there are four lines intersecting all the sets. In fact, we prove a colorful version of this result, under weakened conditions on the sets. A triple of sets A, B, C in the plane is said to be a *tight* if $\mathrm{conv}(A\cup B)\cap\mathrm{conv}(A\cup C)\cap\mathrm{conv}(B\cap C)\neq\emptyset$. This notion was first introduced by Holmsen in [6], where he showed that if $\mathcal{F}$ is a family of compact convex sets in the plane in which every three sets form a tight triple, then there is a line intersecting at least $\frac{1}{8}|\mathcal{F}|$ members of $\mathcal{F}$. Here we prove that if $\mathcal{F}_1,\dots,\mathcal{F}_6$ are families of compact connected sets in the plane such that every three sets, chosen from three distinct families $\mathcal{F}_i$, form a tight triple, then there exists $1\leq j\leq 6$ and three lines intersecting every member of $\mathcal{F}_j$. In particular, this improves $\frac{1}{8}$ to $\frac{1}{3}$ in Holmsen's result.

1. Introduction

Let $\mathcal{F}$ be a family of sets in the plane. We say that $\mathcal{F}$ has *property* $T(r)$ if every r or fewer sets in $\mathcal{F}$ admit a *line transversal*, that is, there exists a line intersecting these sets. We say that $\mathcal{F}$ is *pierced* by k lines if there are k lines in the plane whose union intersects all the sets in $\mathcal{F}$. The *line-piercing number* of the family is the minimum k so that $\mathcal{F}$ is pierced by k lines.

The problem of bounding the line-piercing numbers of families of compact convex sets in the plane with the $T(r)$ property has been investigated since the 1960's. In 1964 Eckhoff [1] proved that if a family of compact convex sets satisfies the $T(4)$ property then it can be pierced by two lines. In 1974 he gave an example of a family of compact convex sets satisfying the $T(3)$ property that is not pierced by two lines [2]. Various upper bounds on the line-piercing numbers were proved when further restrictions on the sets are imposed (see [3] for more details).

For a while it was not clear whether the $T(3)$ property implies a finite universal upper bound on the line-piercing number. This question was eventually resolved in 1975 by Kramer [9], who showed that a family of compact convex sets in $\mathbb{R}^2$ with the $T(3)$ property is pierced by 5 lines. Finally, in 1993 Eckhoff [3] proved that such families are pierced by 4 lines, and asked whether this bound can be improved to 3.

Quantitative versions have been studied too. In 1980, Katchalski and Liu [7] showed the existence of a constant $0<\alpha(r)<1$, so that every finite family $\mathcal{F}$ of

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convex sets in the plane with the $T(r)$ property admits a line intersecting $\alpha(r)|\mathcal{F}|$ of its members. In 2010 Holmsen [5] showed that $\left(\frac{2}{r(r-1)}\right)^{\frac{1}{r-2}} \leq \alpha(r) \leq \frac{r-2}{r-1}$, and in particular, $\frac{1}{3} \leq \alpha(3) \leq \frac{1}{2}$.

In [6], Holmsen introduced the notion of tight triples. Three compact, connected sets in the plane A, B, C are said to be a *tight triple* if

$$\mathrm{conv}(A \cup B) \cap \mathrm{conv}(A \cup C) \cap \mathrm{conv}(B \cap C) \neq \emptyset.$$

We will call a family of sets in the plane a *family of tight triples* if every three sets in the family form a tight triple. Note that if A, B, C has a line transversal, then it is a tight triple. Holmsen [6] proved that if $\mathcal{F}$ is a family of tight triples in which every set is compact and convex, then there is a line intersecting at least $\frac{1}{8}|\mathcal{F}|$ members of $\mathcal{F}$.

The sets investigated in all the above results are assumed to be convex, but the results apply also for families of connected sets. This follows from the fact that if S is a connected set in $\mathbb{R}^2$ and ℓ is a line intersecting $\mathrm{conv}(S)$ then ℓ must intersect S. Similarly, three connected sets A, B, C form a tight triple if and only if $\mathrm{conv}(A), \mathrm{conv}(B), \mathrm{conv}(C)$ form a tight triple.

In this paper we show that the line-piercing number of a family of compact connected tight triples is at most 3. This improves the $\frac{1}{8}$ in Holmsen's result to $\frac{1}{3}$.

In fact, we show a colorful version of this fact.

Theorem 1.1. *Let $\mathcal{F}_1, \dots, \mathcal{F}_6$ be families of compact connected sets in $\mathbb{R}^2$. If every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, $1 \leq i_1 < i_2 < i_3 \leq 6$, form a tight triple, then there exists $i \in [6]$ such that the line-piercing number of $\mathcal{F}_i$ is at most* 3.

In particular, if $\mathcal{F}$ is a family with the $T(3)$ property and we take $\mathcal{F}_i = \mathcal{F}$ for $1 \leq i \leq 6$, then Theorem 1.1 implies that $\mathcal{F}$ has line-piercing number at most three. This gives an affirmative answer to Eckhoff's question. This also gives another proof to Holmsen's result $\alpha(3) \geq 1/3$.

Our main tool is the colorful version of the topological KKM theorem [8] due to Gale [4]. Let $\Delta^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ denote the $(n-1)$-dimensional simplex in $\mathbb{R}^n$, whose vertices are the canonical basis vectors $e_1, \dots, e_n$. Let S_n be the group of permutations on $[n]$.

Theorem 1.2 (The colorful KKM theorem [4])**.** *Let $A^i_1, \dots, A^i_n$, $i \in [n]$, be open sets of Δ^{n-1}, such that for every $i \in [n]$ and for every face σ of Δ^{n-1} we have $\sigma \subset \bigcup_{e_j \in \sigma} A^i_j$. Then there exists a permutation $\pi \in S_n$ such that $\cap_{i=1}^n A_i^{\pi(i)} \neq \emptyset$.*

2. Proof of Theorem 1.1

Throughout the proof, addition in integers is taken modulo 6. For $a, b \in \mathbb{R}^2$, let $[a, b] = \mathrm{conv}\{a, b\}$ be the line segment connecting a, b.

As is explained in [3], the compactness of the sets in each $\mathcal{F}_j$ allows us to assume that $\mathcal{F}_j$ is finite. Thus we may scale the plane so that every set in $\mathcal{F}_j$ is contained in the unit disk D for each j. Denote by U the unit circle. Let $f(t)$ be a parameterization of U defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$.

A point $x = (x_1, \dots, x_6) \in \Delta^5$ corresponds to 6 points on U given by $f_i(x) = f(\sum_{j=1}^i x_j)$ for $1 \leq i \leq 6$. Let $l_1(x) = l_4(x) = [f_1(x), f_4(x)]$, $l_2(x) = l_5(x) = [f_2(x), f_5(x)]$, and $l_3(x) = l_6(x) = [f_3(x), f_6(x)]$.

For $i = 1, \dots, 6$ let R^i_x be the interior of the region bounded by $l_{i-1}(x)$, $l_i(x)$ and the arc on U connecting $f_{i-1}(x)$ and $f_i(x)$ (see Figure 1). Notice that $R^i_x = \emptyset$ when $x_i = 0$. Also, it is possible that some of the regions R^i_x intersect.

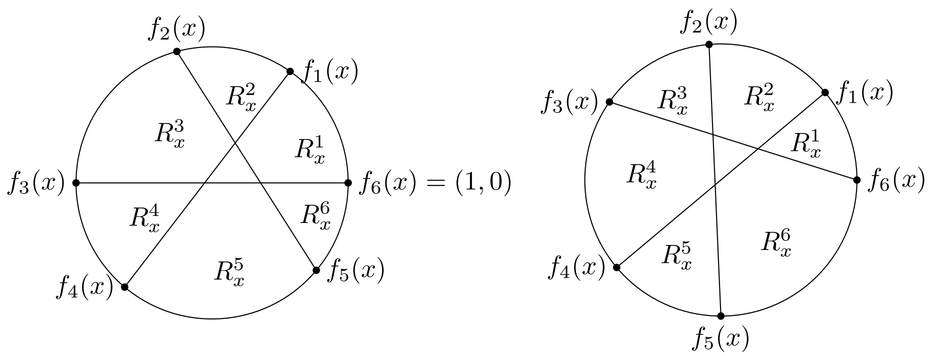

FIGURE 1. A point $x \in \Delta^5$ corresponds to six regions R^i_x. The regions R^1_p, R^3_p, R^5_p are pairwise disjoint (on the right) or the regions R^2_p, R^4_p, R^6_p are pairwise disjoint (on the left), depending on the orientation of the triangle bounded by the lines $l_1(x), l_2(x), l_3(x)$.

Set $1 \le j \le 6$ and let A^j_i be the set of points $x \in \Delta^5$ so that R^i_x contains a set $F \in \mathcal{F}_j$. Since the sets $F \in \mathcal{F}_j$ are closed, A^j_i is open. If there is some $x \in \Delta^5$ for which $x \notin \bigcup_{i=1}^6 A^j_i$, then since the sets in $\mathcal{F}_j$ are connected, every set in $\mathcal{F}_j$ must intersect $\bigcup_{i=1}^3 l_i(x)$, and we are done. So we assume for contradiction that $\Delta^5 = \bigcup_{i=1}^6 A^j_i$ for all j. Observe that if $x \in \operatorname{conv}\{e_i : i \in I\}$ for some $I \subset [6]$ then $R^k_x = \emptyset$ for $k \notin I$, and therefore, $x \in \bigcup_{i \in I} A^j_i$ for all j. This shows that the conditions of Theorem 1.2 hold.

Thus, by Theorem 1.2, there exists some permutation $\pi \in S_6$ and a point $p = (p_1, \dots, p_6) \in \bigcap_{i=1}^6 A^{\pi(i)}_i$. Therefore, each of the open regions R^i_p contains a set $S_i \in \mathcal{F}_{\pi(i)}$, $i = 1, \dots, 6$, and in particular $R^i_p \neq \emptyset$ and thus $p_i \neq 0$ for all i. We claim that at least one of the triples $\{S_1, S_3, S_5\}$ or $\{S_2, S_4, S_6\}$ is not a tight triple. To see this, note that the regions R^1_p, R^3_p, R^5_p are pairwise disjoint or the regions R^2_p, R^4_p, R^6_p are pairwise disjoint (depending on the orientation of the triangle bounded by the lines l_1, l_2, l_3, see Figure 1). Without loss of generality, we assume R^1_p, R^3_p, R^5_p are pairwise disjoint, and in this case, the three sets S_1, S_3, S_5 is not a tight triple. This is a contradiction.

3. Concluding remarks

The proof of Theorem 1.1 implies a slightly stronger result: when each $\mathcal{F}_i$ is finite, one can fix a point lying on one of the three piercing lines of $\mathcal{F}_i$, as long as this point is outside $\operatorname{conv}(\cup_i \mathcal{F}_i)$.

A similar proof can be used to prove a colorful version of Eckhoff's result that $T(4)$ families are pierced by two lines.

Theorem 3.1. *Let $\mathcal{F}_1, \dots, \mathcal{F}_4$ be families of compact, connected sets in the plane such that any collection of four sets, one from each $\mathcal{F}_i$, has a line transversal. Then for some $i \in [4]$, $\mathcal{F}_i$ has line piercing number at most 2.*

This can be proved by associating a point in Δ^3 with two lines and applying a similar argument as in the proof of Theorem 1.1

When each of the families $\mathcal{F}_i$ is finite, one may drop the condition that the sets are compact. This is because we may replace each set $S \in \mathcal{F}_j$ with a compact, convex set $S' \subset \mathrm{conv}(S)$ such that the resulting family is still a family of tight triples.

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