Two Phase Free Boundary Problem for Poisson Kernels

S. BORTZ, M. ENGELSTEIN, M. GOERING, T. TORO, & Z. ZHAO

ABSTRACT. We provide a potential theoretic characterization of vanishing chord-arc domains under mild assumptions. In particular we show that, if a domain has Ahlfors regular boundary, the oscillation of the logarithm of the interior and exterior Poisson kernels yields a great deal of geometric information about the domain. We use techniques from classical calculus of variations, potential theory, and quantitative geometric measure theory to accomplish this. One feature of this work, compared to [KT06] and [BH16], is that *a priori* we only require that the domains in question are connected.

1. INTRODUCTION

Questions concerning the connections between the geometry of a domain and the regularity of its boundary with the potential theoretic properties of the domain, the behavior of singular integrals on the boundary, and the boundary regularity to solutions of elliptic PDEs have generated a flurry of activity in the area of non-smooth analysis (see [Tor97] and [Tor19] for a brief recent history and references). In this paper we focus on the potential theoretic properties of a domain and its complement and explore their ties to the geometry of the domain. In particular, we show that if $\Omega := \Omega^+ \subset \mathbb{R}^n$ and the interior of its complement Ω^- are connected, have a shared boundary which is Ahlfors regular (see Definition 2.6), and the logarithm of the Poisson kernel of each domain is in $\mathrm{VMO}_{\mathrm{loc}}$, then the unit normal is also in $\mathrm{VMO}_{\mathrm{loc}}$ and the domain is vanishing Reifenberg flat (see Definitions 2.18 and 2.10). We contrast our result with those in the literature in order to

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emphasize the wealth of geometric information (thus far overlooked) encoded in the assumption concerning the oscillation of the logarithm of the Poisson kernels.

In [KT06] the authors established the following: suppose that $\Omega^\pm$ are chord-arc domains (i.e., NTA domains with Ahlfors regular boundary), and that $k^\pm$ are the Poisson kernels of $\Omega^\pm$ with poles $X^\pm \in \Omega^\pm$. If $\log k^\pm \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$ then the unit normal vector $\nu \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$ where $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$ (see Definition 2.15). In particular, the assumption that $\Omega^\pm$ are chord-arc domains ensures that $\partial\Omega^\pm$ are uniformly rectifiable (see Definition 2.9). In [BH16] the authors relax the geometric conditions: to be more precise, via a novel approach using layer potentials rather than blow ups, they prove that if $\Omega^\pm \subset \mathbb{R}^n$ are domains, whose common boundary is uniformly rectifiable, then $\log k^\pm \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$ implies that $\nu \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$. We also mention the recent work Prats-Tolsa [PT19], where the authors studied a different but closely related problem arising in Kenig-Toro [KT06]. They study the kernel between harmonic measures $\omega^\pm$ of $\Omega^\pm$, and show that for Reifenberg flat NTA domains, small oscillation for the logarithm of that kernel is also closely linked to small oscillation for the unit normal ν.

In this paper, we further loosen the *a priori* assumption in [KT06] and instead deduce as much geometric information as possible from the regularity of $\log k^\pm$. Furthermore, using classical tools from the calculus of variations, we establish that in this context the oscillation of the unit normal controls the flatness of the boundary. More precisely, when $\pm\nu$ are outward pointing normal vectors to $\partial\Omega^\pm$, $\sigma \mathcal{H}^{n-1} \llcorner \partial\Omega$, and $\omega_\pm^{X^\pm} = \omega^\pm$ is the harmonic measure for $\Omega^\pm$ with pole at $X^\pm$, we show the following result.

Theorem 1.1. *Let $n \geq 3$ and suppose $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$ are domains satisfying $\partial\Omega := \partial\Omega^+ = \partial\Omega^-$, and that $\partial\Omega$ is $(n-1)$-Ahlfors regular. Then, the following are equivalent:*

(i) *$\Omega^\pm$ are both vanishing chord-arc domains with $\nu \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$ (see Definition 2.19).*

(ii) *There exist $X^+ \in \Omega^+$ and $X^- \in \Omega^-$ such that $k^+ = \mathrm{d}\omega_+^{X^+}/\mathrm{d}\sigma$ and $k^- = \mathrm{d}\omega_-^{X^-}/\mathrm{d}\sigma$ exist and $\log k^\pm \in \mathrm{VMO}_{\mathrm{loc}}(\mathrm{d}\sigma)$.*

Further, we obtain corresponding quantitative results (see Theorems 4.12 and 4.14).

Remark 1.2. There is some redundancy in condition (i) of Theorem 1.1, which we leave in for the sake of emphasis. In particular, under the conditions of Theorem 1.1, Ω^+ is a vanishing chord-arc domain if and only if Ω^- is. Additionally, it is a consequence of our work in this paper (see Corollary 3.11) that $\nu \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$ is equivalent to (under the hypothesis of Theorem 1.1) $\Omega^\pm$ being vanishing chord-arc domains.

In this paper, techniques from potential theory and geometric measure theory come together, allowing one to deduce geometric properties of domains. In Section 2, basic definitions from both areas are presented. In Section 3 we apply classical tools of geometric measure theory dating back to De Giorgi's original work on sets of locally finite perimeter. (See [Mag12] for references and an approach motivating the one presented here.) The novelty is that we extend these tools from *perimeter minimizers* to sets of locally finite perimeter with Ahlfors regular boundaries[1 2], which allows us to reduce and better understand topological hypothesis from previous works concerning potential theory in "rough" domains (cf. [KT06, BE17] and the discussion in the last paragraph of Section 2). The general approach we take is analogous to [Mag12], but new ideas are also implemented in various places to extend the proof to a more general class of sets. In particular, Corollary 3.10, which is analogous to a well-known result that plays a fundamental role in the proof of regularity of perimeter minimizers, shows that control on the oscillation of the unit normal provides both local control on the flatness of the boundary as well as local separation properties (see Definition 2.18). In addition to the proofs of these separation properties, in Appendix A we also prove that if the unit normal has small oscillation in a ball centered on the perimeter, then a large portion of the perimeter inside a slightly smaller concentric ball is contained in the graph of a Lipschitz function. Within the smaller ball both the Lipschitz norm of this function and the symmetric difference of this graph and the boundary are quantitatively controlled by the oscillation inside the larger ball.

These results should be contrasted with those found in [Sem91a], [Sem91b], [KT99], [HMT10], [Mer16a], and [Mer16b]. In particular, in [Sem91a] and [Sem91b], Semmes introduced the notion of chord-arc surfaces with small constant. (His definition is similar to ours in Definition 2.19, except that he works on C^2 connected embedded hypersurfaces, whereas we assume Ahlfors regularity.) He focused on characterizing such surfaces through the behavior of singular integral operators on them. One crucial tool in Semmes's study is the "Semmes decomposition theorem" which allows one to write a large portion of the chord-arc surface with small constant as the graph of a Lipschitz function (this is in the same vein as our aforementioned Lipschitz covering in Appendix A of this paper). To obtain this decomposition, Semmes needed to assume that the surface was C^2 (though his estimates did not depend on the C^2-norm). The decomposition was later obtained in the more general context of Reifenberg flat domains by [KT97, KT99],[3] and then in the even more general context of domains with the two-sided local John condition in [HMT10]. Given the Semmes decomposition one can immediately use the oscillation of the unit normal to control the Reifenberg flatness of the chord-arc surface. Our key result along these lines, Corollary 3.12, also implies that the oscillation of the unit normal controls the Reifenberg flatness of the chord-arc surface. However, our condition (which is implied by a

[1] Rather, a representative whose boundary agrees with the support of the Gauss-Green measure. See (3.1) and Remark 3.1.

[2] The class of perimeter minimizers is a subclass of the sets we consider in Section 3, as defined in (3.1). See [Mag12, Section 16.2].

[3] We thank the referee for pointing out that no one has explicitly written the proof that a chord-arc domain with small constant in the sense of [KT97, KT99] satisfies the small γ-condition of [Sem91a]. Although the proof is straightforward, we include it in Remark 2.21 to patch this gap in the literature.

local two-sided corkscrew condition), is weaker than two-sided local John. Furthermore, our approach does not need a Semmes-type decomposition (though, as mentioned above, it does yield such a decomposition).

In addition to his geometric study of chord-arc surfaces with small constant, Semmes also expressed interest in obtaining potential theoretic characterizations. These characterizations were investigated by Kenig and Toro, with the *a priori* assumption of Reifenberg flatness in [KT97], [KT99] and [KT03]. As a consequence of results herein, we show that the flatness hypothesis is redundant[4], this in turn, allows one to remove the *a priori* topological assumption of Reifenberg flatness (or, more generally, two-sided local John) from some theorems in the aforementioned works of Kenig and Toro. In Section 4 we focus on the local two phase free boundary problem for the Poisson kernels. In Section 4.1 we show that local doubling properties of $\omega^{\pm}$ combined with the Ahlfors regularity of the boundary yield the existence of corkscrew balls on both sides (locally) and therefore imply local uniform rectifiability of the boundary (see Lemma 4.3 and Corollary 4.4). In Section 4.2 we show that in our setting, the assumption $\log k^{\pm} \in \mathrm{VMO}_{\mathrm{loc}}(\mathrm{d}\sigma)$ yields information about the doubling properties of $\omega^{\pm}$ and the local optimal behavior of $k^{\pm}$ (see Lemma 4.11). Combining the results in Sections 4.1 and 4.2, we *almost* recover the hypothesis in [BH16]. The proof of Theorem 4.12 follows the general scheme of the proof in [BH16] with an additional domain approximation scheme (see Appendix B), and special attention given to the constants in order to prove a quantitative result.

2. PRELIMINARIES

In the sequel, n is a natural number with $n \geq 3$. We typically use E to denote a set of locally finite perimeter in $\mathbb{R}^n$, that is, a Lebesgue measurable set such that for every compact set $K \subset \mathbb{R}^n$,

$$\sup\left\{\int_E \operatorname{div} T(x)\,\mathrm{d}x : T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} T \subset K,\ \sup_{\mathbb{R}^n}|T| \leq 1\right\} < \infty.$$

We use Ω to denote a domain, that is, an open and connected set, in $\mathbb{R}^n$. Oftentimes the domain Ω will also be a set of locally finite perimeter, for example, if we assume $\partial\Omega$ is upper Ahlfors regular (see [EG92, Section 5.11]). We recall a few results.

[4]As mentioned above, to show that a domain is δ-chord-arc from the oscillation of the unit normal, one can use [HMT10, Theorem 4.19] (which does not require Reifenberg flatness) instead of [KT99, Theorems 4.2 and 4.4] in the presence of the two-sided local John condition. Corollary 3.12 allows one to remove the two-sided John condition from [HMT10, Theorem 4.19]. Then, one can state the hypotheses of some theorems in [KT97], [KT99], and [KT03] in terms of the oscillation of the unit normal alone, that is, without assuming *a priori* Reifenberg flatness (or two-sided local John). See, e.g., [HMT10, Theorem 4.21].

Proposition 2.1 ([Mag12, Proposition 12.1]). *If E is a Lebesgue measurable set in $\mathbb{R}^n$, then E is a set of locally finite perimeter if and only if there exists an $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ such that*

$$\int_E \operatorname{div} T(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} T \cdot \mathrm{d}\mu_E, \quad \forall\, T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

The measure μ_E is called the Gauss-Green measure of E.

For a vector-valued Radon measure μ on $\mathbb{R}^n$, the total variation of μ is denoted by $|\mu|$. We recall (see [Mag12, Chapter 4]) that $|\mu|$ is a non-negative Radon measure that has the following characterization on open sets $V \subset \mathbb{R}^n$:

$$|\mu|(V) = \sup\left\{ \int_{\mathbb{R}^n} T \cdot \mathrm{d}\mu : T \in C_c^1(V;\mathbb{R}^n),\ |T| \le 1 \right\}. \tag{2.1}$$

If E is a set of locally finite perimeter, and μ_E the associated Gauss-Green measure, recall (see [Mag12, Chapter 15]) the reduced boundary of E, denoted $\partial^* E$, is defined by

$$\partial^* E = \left\{ x \in \operatorname{spt}\mu_E : \lim_{r\downarrow 0} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} = \nu_E(x) \in \mathbb{S}^{n-1} \right\}. \tag{2.2}$$

In fact, $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ defined by the limit in (2.2) is a Borel function called the measure-theoretic outward pointing unit normal. Moreover, the following is a version of De Giorgi's structure theorem.

Theorem 2.2 (***De Giorgi's structure theorem,*** [Mag12, Theorem 15.9]). *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter, then*

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E \quad \text{and} \quad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

Remark 2.3. For a set of locally finite perimeter $E \subset \mathbb{R}^n$ there are several notions of boundary: the reduced boundary $\partial^* E$, the measure-theoretic boundary $\partial_* E$, the support of the Gauss-Green measure, and the topological boundary (see [EG92] or [Mag12] for relevant definitions). The following relationships between different notions of the boundary hold:

$$\partial^* E \subset \partial_* E \subset \operatorname{spt}\mu_E \subset \partial E \quad \text{and} \quad \mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0.$$

In particular, $\partial^* E = \partial E$ implies $\partial^* E = \partial_* E = \operatorname{spt}\mu_E = \partial E$.

The next two propositions can be found in [Mag12, Propositions 4.29, 4.30].

Proposition 2.4 (Lower semi-continuity of weak$*$ convergence). *If μ_k and μ are vector-valued Radon measures with $\mu_k \rightharpoonup \mu$, that is, for every $\varphi \in C_c(\mathbb{R}^n,\mathbb{R}^n)$,*

$$\int \varphi \cdot \mathrm{d}\mu_k \to \int \varphi \cdot \mathrm{d}\mu,$$

then for every open set $A \subset \mathbb{R}^n$ we have $|\mu|(A) \le \liminf_{k\to\infty} |\mu_k|(A)$.

Proposition 2.5. *Let μ_k be vector-valued Radon measures on $\mathbb{R}^n$ so that $\mu_k \rightharpoonup \mu$ for some μ, a vector-valued Radon measure on $\mathbb{R}^n$. The following hold:*

(1) *If additionally $|\mu_k| \rightharpoonup \nu$ for some ν a non-negative Radon measure on $\mathbb{R}^n$, then, for every Borel set $F \subset \mathbb{R}^n$,*

$$|\mu|(F) \le \nu(F). \tag{2.3}$$

Furthermore, if $F \subset \mathbb{R}^n$ is a bounded Borel set with $\nu(\partial F) = 0$, then

$$\mu(F) = \lim_{k\to\infty} \mu_k(F). \tag{2.4}$$

(2) *If $|\mu_k|(\mathbb{R}^n) \to |\mu|(\mathbb{R}^n)$, and $|\mu|(\mathbb{R}^n) < \infty$, then $|\mu_k| \rightharpoonup |\mu|$.*

Definition 2.6 (Ahlfors regularity). A Borel measure μ on $\mathbb{R}^n$ is said to be *d-Ahlfors regular* if there exists a positive finite constant C_A such that

$$C_A^{-1} r^d \le \mu(B(x,r)) \le C_A r^d \tag{2.5}$$

for all $x \in \operatorname{spt}\mu$ and all $0 < r < \operatorname{diam}\operatorname{spt}\mu$. More generally, we say that a measure μ is *d-Ahlfors regular up to scale r_0* if (2.5) holds for all $0 < r < r_0$. In either case, the constant C_A is called the *Ahlfors regularity constant for μ.*

Let $F \subset \mathbb{R}^n$ be a closed set. If (2.5) holds for the measure $\mu = \mathcal{H}^d \llcorner F$ and some $0 < d \le n$, then F is said to be (d-)Ahlfors regular up to scale r_0. When d is understood from context, we simply say F is Ahlfors regular up to scale r_0.

Definition 2.7 (Uniformly Rectifiable (UR) sets). Let $A \subset \mathbb{R}^n$ be a closed set that is d-Ahlfors regular. It is said to be uniformly rectifiable (UR) if it contains "Big Pieces of Lipschitz Images." This means there is a pair of constants $\theta, \Lambda > 0$ such that for all $x \in A$ and all $0 < r \le \operatorname{diam}(A)$ there is a Lipschitz mapping $g : B(0,r) \subset \mathbb{R}^d \to \mathbb{R}^n$ with $\operatorname{Lip}(g) \le \Lambda$ such that $\mathcal{H}^d(E \cap g(B(0,r))) \ge \theta r^d$.

One reason uniformly rectifiable sets are ubiquitous is that they are spaces on which one can develop a rich Calderón-Zygmund theory. An example of this, to be used (implicitly) later, is the following characterization of uniformly rectifiable sets in co-dimension 1.

Theorem 2.8 ([Dav91], [MMV96], [NTV14]). *Let $F \subset \mathbb{R}^n$ be a closed and $(n-1)$-Ahlfors regular set with the associated measure $\sigma := \mathcal{H}^{n-1} \llcorner F$. Then, F is uniformly rectifiable if and only if the Riesz transform operator* (*see Definition 4.5*), *$\mathcal{R}$ is L^2 bounded with respect to σ, in the sense that its truncation $\mathcal{R}_\varepsilon$ satisfies*

$$\sup_{\varepsilon>0} \|\mathcal{R}_\varepsilon f\|_{L^2(F,\sigma)} \le C\|f\|_{L^2(F,\sigma)} \quad \forall\, f \in L^2(F,\sigma),$$

with a $C > 0$ uniform in $f \in L^2(F,\sigma)$.

Definition 2.9 (***UR domain, see*** [HMT10]). We say that an open set Ω is a UR domain if $\partial\Omega$ is UR, and the measure-theoretic boundary $\partial_*\Omega$ (see Chapter 5 in [EG92]) satisfies $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$.

We comment that in the above definition, Ω is not required to be connected; we use the term "UR domain" nonetheless following the convention set by Definition 3.7 in [HMT10] and also to distinguish them from UR sets (of Definition 2.7).

Definition 2.10 (BMO ***and*** VMO). Let $F \subset \mathbb{R}^n$ be $(n-1)$-Ahlfors regular up to scale r_0[5]. Then, for all $0 < r < r_0$, $x \in F$, and $f \in L^2_{\mathrm{loc}}(\mathcal{H}^{n-1} \llcorner F)$, define

$$\|f\|_*(x,r) = \sup_{0<s<r} \left(\fint_{B(x,s)\cap F} \left| f(y) - \fint_{B(x,s)\cap F} f(z)\, \mathrm{d}\mathcal{H}^{n-1}(z) \right|^2 \mathrm{d}\mathcal{H}^{n-1}(y) \right)^{1/2}.$$

We say the following:

(1) $f \in \mathrm{BMO}_{\mathrm{loc}}(\mathcal{H}^{n-1} \llcorner F)$ if for every compact set $K \subset \mathbb{R}^n$, there exist $R_K > 0$ and $C_K > 0$ such that

$$\sup_{0<r<R_K} \sup_{x \in F\cap K} \|f\|_*(x,r) \le C_K.$$

(2) $f \in \mathrm{BMO}_{\mathrm{loc}}(\mathcal{H}^{n-1} \llcorner F)$ with constant $\kappa > 0$ if for every compact set $K \subset \mathbb{R}^n$, there exists $R_K > 0$ such that

$$\sup_{0<r<R_K} \sup_{x \in F\cap K} \|f\|_*(x,r) \le \kappa.$$

(3) $f \in \mathrm{VMO}_{\mathrm{loc}}(\mathcal{H}^{n-1} \llcorner F)$ if for every compact set $K \subset \mathbb{R}^n$,

$$\lim_{r\to 0} \sup_{x \in \partial E\cap K} \|f\|_*(x,r) = 0.$$

Remark 2.11. It is clear that the local conditions in the definition above are equivalent to replacing arbitrary compact sets by balls centered on the boundary with radius less than, say, $\frac{1}{4}\operatorname{diam}(F)$. This is obvious if F is unbounded, and if F is bounded we can cover F by a finite collection of such balls.

Definition 2.12 (Corkscrew Condition). An open set $E \subset \mathbb{R}^n$ satisfies the (M, R_0) interior corkscrew condition if for every $x \in \partial E$ and $r \in (0, R_0)$ there is a point x_1 called the interior corkscrew point so that $B(x_1, r/M) \subset E \cap B(x,r)$.

Definition 2.13 (Two-sided Corkscrew Condition). An open set $E \subset \mathbb{R}^n$ satisfies the (M, R_0) two-sided corkscrew condition if for every $x \in \partial E$ and $r \in (0, R_0)$ there are two points $x_1 \in E$ and $x_2 \in \mathbb{R}^n \setminus E$ such that $B(x_1, r/M) \subset E$ and $B(x_2, r/M) \subset \mathbb{R}^n \setminus E$. We call x_1 and x_2 the interior and exterior corkscrew points, respectively.

[5]Of course, this notion can be defined for d-Ahlfors regular subsets of $\mathbb{R}^n$, but we are only concerned with the case $d = n - 1$

Definition 2.14 (Harnack Chain Condition). Following [JK82], we say that a domain Ω satisfies the (C,R)-Harnack Chain condition if, for every $0 < \rho \le R$, $\Lambda \ge 1$, and every pair of points $X, X' \in \Omega$ with $\delta(X), \delta(X') \ge \rho$ and with $|X - X'| < \Lambda\rho$, there is a chain of balls $B_1, \dots, B_N \subset E$ with $N \le C\log_2 \Lambda + 1$, and $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \varnothing$ for all $k = 1, \dots, N-1$ and $C^{-1}\operatorname{diam}(B_k) \le \operatorname{dist}(B_k, \partial\Omega) \le C\operatorname{diam}(B_k)$ for all $k = 1, \dots, N$. The chain of balls is called a "Harnack Chain."

Definition 2.15 (NTA and Chord-Arc Domain). We say that $\Omega \subset \mathbb{R}^n$ is a Non-Tangentially Accessible Domain (NTA) with constants (M, R_0), if it satisfies the (M, R_0)-Harnack chain condition and the (M, R_0) two-sided corkscrew condition. If Ω is unbounded, we require that $\mathbb{R}^n \setminus \partial\Omega$ consist of two, non-empty, connected components. Note that if Ω is unbounded, then $R_0 = \infty$ is allowed.

Finally, if Ω is an NTA domain whose boundary is Ahlfors regular, we say that Ω is a chord-arc domain.

Remark 2.16. Sometimes in the definition of unbounded NTA domains, it is required that $R_0 = \infty$ (see, e.g., [KT97], [KT06]). In particular, this allows one to obtain estimates on harmonic measure/functions at arbitrarily large scales. Since we are only interested in local geometric properties of Ω, we allow $R_0 < \infty$ even for unbounded domains Ω.

Also note that if Ω is an open set with an Ahlfors regular boundary and satisfies the two-sided corkscrew condition with $R_0 \approx \operatorname{diam}(\partial\Omega)$, then it is a UR domain (see [DJ90, Theorem 1] and also Badger [Bad12][6]). In addition, having interior and exterior corkscrews at arbitrarily small scales forces $\partial_*\Omega = \partial\Omega$.

Let $\Sigma \subset \mathbb{R}^n$ be a closed set. For any $x \in \Sigma$ and $r > 0$, we define

$$\Theta(x,r) = \inf_L \left\{ \frac{1}{r} D[\Sigma \cap \overline{B(x,r)}, L \cap \overline{B(x,r)}] \right\},$$

where the infimum is taken over all $(n-1)$-planes containing x. Here, D denotes the Hausdorff distance; that is, for non-empty sets $A, B \subset \mathbb{R}^n$, we have $D[A,B] := \sup\{d(a,B) : a \in A\} + \sup\{d(b,A) : b \in B\}$. With this in hand, we can define flatness as in Reifenberg [Rei60].

Definition 2.17 (Reifenberg Flat and Vanishing Reifenberg Flat sets). We say a closed set $\Sigma \subset \mathbb{R}^n$ is δ-Reifenberg flat for some $\delta > 0$ if for each compact set $K \subset \mathbb{R}^n$ there exists $R_K > 0$ such that

$$\sup_{r \in (0, R_K]} \sup_{x \in K \cap \Sigma} \Theta(x,r) < \delta. \tag{2.6}$$

We say Σ is a vanishing Reifenberg flat set if for every compact set $K \subset \mathbb{R}^n$

$$\lim_{r \to 0} \sup_{x \in \Sigma \cap K} \Theta(x,r) = 0.$$

[6] In fact, Badger shows that upper Ahlfors regularity is not necessary for the quantitative interior approximation by Lipschitz domains shown in [DJ90].

Definition 2.18 (Reifenberg Flat and Vanishing Reifenberg Flat domains). Let $\delta \in (0, \delta_n)$ where δ_n is chosen appropriately (see Remark 2.20) and depends only on the dimension n. We say that a domain $\Omega \subset \mathbb{R}^n$ is a δ-Reifenberg flat domain (or vanishing Reifenberg flat domain) if $\partial\Omega$ is δ-Reifenberg flat (respectively, vanishing Reifenberg flat) and Ω satisfies the *separation property*: for every compact set $K \subset \mathbb{R}^n$ there exists $R_K > 0$ such that for any $y \in \partial\Omega \cap K$ and $0 < r < R_K$ there exists a $\nu \in \mathbb{S}^{n-1}$ so that if $x \in B(y, r)$ and $\langle x - y, \nu \rangle > \delta r$, then $x \in \Omega^c$, and if $\langle x - y, \nu \rangle < -\delta r$ then $x \in \Omega$.

Additionally, if Ω is unbounded it is further required that $\mathbb{R}^n \setminus \partial\Omega$ consist of two connected components, and that $\delta \leq \delta_n$.[7]

Definition 2.19 (Chord-arc domains with small constants and vanishing chordarc domains). Let $\delta \in (0, \delta_n)$ (where δ_n is from Definition 2.18; see the remark below). A set of locally finite perimeter $\Omega \subset \mathbb{R}^n$ is said to be a δ-chord-arc domain (or chord-arc domain with small constant) if Ω is a δ-Reifenberg flat domain, $\partial\Omega$ is Ahlfors regular, and for each compact set $K \subset \mathbb{R}^n$ there exists some $R > 0$ such that

$$\sup_{x \in \partial\Omega \cap K} \|\nu_\Omega\|_*(x, R) < \delta. \tag{2.7}$$

We say a domain Ω is a chord-arc domain with vanishing constant if it is a chord-arc domain with small constant, and for each compact set $K \subset \mathbb{R}^n$,

$$\lim_{r \to 0} \sup_{x \in \partial\Omega \cap K} \|\nu_\Omega\|_*(x, r) = 0,$$

that is, if $\nu_\Omega \in \mathrm{VMO}_{\mathrm{loc}}(\mathcal{H}^{n-1} \llcorner \partial\Omega)$.

Remark 2.20. We recall from [KT97, Theorem 3.1] that there exists a $\delta_n > 0$ such that if $\Omega \subset \mathbb{R}^n$ is a δ-Reifenberg flat domain for some $\delta < \delta_n$, then Ω is (locally) an NTA domain. If $\partial\Omega$ is also assumed to be Ahlfors regular, then Ω is a chord-arc domain (as in Definition 2.15). This justifies the name δ-chord-arc domain (or chord-arc domain with vanishing constant).

The reader may wonder whether the smallness in (2.7) implies the smallness in (2.6), for example, when $\partial\Omega$ is smooth. In the planar case ($n = 2$) one can show that $\sup_{x,r} \Theta(x, r) \lesssim \|\nu_\Omega\|_*$; but in higher dimensions this estimate holds only if we know the smallness of both parameters *a priori*; otherwise, $\partial\Omega$ might have small *handles*. (See the discussions and main theorem in [Sem91c].) However, when $\partial\Omega$ is assumed to be Ahlfors regular (plus some weak topological assumptions), we will show in Section 3 how to bound $\Theta(x, r)$ by $\|\nu_\Omega\|_*$.

Remark 2.21. Here, we record a straightforward argument that chord arc domains with small constant in the sense of [KT97, KT99] satisfy the quantitative conditions in the definition of a chord-arc surface from [Sem91a, Sem91b].

[7]Note that the definition above is slightly different from the one in [KT03, Definition 1.6] as we do not require flatness at large scales.

By Definitions 2.17 and 2.18, an η-Reifenberg flat domain E satisfies the following flatness condition: for any $q \in \partial E$, $r > 0$ there exists some unit vector $n_{q,r}$ so that

$$|\langle n_{q,r}, y - q\rangle| \leq \eta r \quad \forall\, y \in \overline{B(q,r)} \cap \partial E. \tag{2.8}$$

In [Sem91a, Sem91b] it was assumed not only that chord-arc surfaces with small constant had small BMO norm, but also that they satisfied a flatness condition like (2.8) where $n_{q,r}$ is replaced with the specific vector

$$\nu_{q,r} = \fint_{B(q,r)\cap\partial^* E} \nu_E \,\mathrm{d}\mathcal{H}^{n-1}.$$

This height bound is an unsurprising consequence of being both Reifenberg flat and having a small BMO norm[8].

More precisely, we have the following result.

Claim 2.22. *If E has Ahlfors regular boundary, is an η-Reifenberg flat domain, and satisfies $\|\nu_E\|_*(q,r) \leq \delta$ for some $\delta \leq \frac{1}{2}$, then*

$$|\langle y - q, \nu_{q,r}\rangle| \leq Cr\sqrt{\eta + \delta} \quad \forall\, y \in B(q,r) \cap \partial E. \tag{2.9}$$

Proof. Let $\sigma = \mathcal{H}^{n-1} \llcorner \partial^* E$ and $n_{q,r}$ be the direction from the η-Reifenberg flat condition. We claim it suffices to show (2.10)–(2.12),

$$\left| \int_{B(q,r)} n_{q,r} \cdot \nu_E \,\mathrm{d}\sigma - \omega_{n-1} r^{n-1} \right| \leq C r^{n-1} \eta, \tag{2.10}$$

$$\big| |\nu_{q,r}| - 1 \big| \leq \delta, \tag{2.11}$$

$$(1 - n\eta^2)\omega_{n-1} r^{n-1} \leq \sigma(B(q,r)) \leq (1 + 2\delta)\omega_{n-1} r^{n-1}. \tag{2.12}$$

Indeed, (2.10) and (2.12) together ensure

$$|1 - \nu_{q,r} \cdot n_{q,r}| = \left| 1 - \fint_{B(q,r)} n_{q,r} \cdot \nu_E \,\mathrm{d}\sigma \right| \leq C(\eta + \delta).$$

Combining (2.11) with the preceding inequality, we deduce

$$\begin{aligned}\left| n_{q,r} \cdot \frac{\nu_{q,r}}{|\nu_{q,r}|} - 1 \right| &\leq \left| n_{q,r} \cdot \frac{\nu_{q,r}}{|\nu_{q,r}|} - \frac{1}{|\nu_{q,r}|} \right| + \left| \frac{1}{|\nu_{q,r}|} - 1 \right| \\ &\leq \frac{C}{1-\delta}[\eta + \delta] = C(\eta + \delta).\end{aligned}$$

[8]Being Reifenberg flat is not necessary, *a priori*, as seen by Corollary 3.10.

This in turn implies

$$\left| n_{q,r} - \frac{\nu_{q,r}}{|\nu_{q,r}|} \right|^2 \le C(\eta + \delta). \tag{2.13}$$

Consequently, for $y \in B(q,r) \cap \partial E$,

$$\begin{aligned} |\langle y - q, \nu_{q,r} \rangle| &\le (1+\delta) \left| \left\langle y - q, \frac{\nu_{q,r}}{|\nu_{q,r}|} \right\rangle \right| \\ &\le (1+\delta) \left\{ \left| \left\langle y - q, n_{q,r} \right\rangle \right| + \left| \left\langle y - q, \frac{\nu_{q,r}}{|\nu_{q,r}|} - n_{q,r} \right\rangle \right| \right\} \\ &\le (1+\delta)\{\eta r + Cr\sqrt{\eta + \delta}\}, \end{aligned}$$

where the first inequality used (2.11) and the final inequality follows from (2.13) and the fact that ∂E is η-Reifenberg flat. Since δ is small, this verifies (2.9). Hence, it remains to check (2.10)–(2.12).

We compare $E \cap B(q,r)$ to $B(q,r) \cap \{\langle y - q, n_{q,r} \rangle \le 0\}$ to verify (2.10). Indeed, for any constant vector e it follows that

$$\begin{aligned} 0 &= \int_{B(q,r)\cap E} \operatorname{div} e \\ &= \int_{B(q,r)\cap \partial E} e \cdot \nu_E \, \mathrm{d}\sigma + \int_{\partial B(q,r)\cap E} e \cdot \frac{y-q}{|y-q|} \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{2.14}$$

Plugging in $e = n_{q,r}$, we get

$$\int_{\partial B(q,r)\cap E} n_{q,r} \cdot \frac{y-q}{|y-q|} \, \mathrm{d}\mathcal{H}^{n-1} = - \int_{B(q,r)\cap \partial E} n_{q,r} \cdot \nu_E \, \mathrm{d}\sigma. \tag{2.15}$$

Since $n_{q,r}$ comes from the η-Reifenberg flat condition,

$$\begin{cases} E \cap \overline{B(q,r)} \subset \{\langle y - q, n_{q,r} \rangle \le \eta r\} \cap \overline{B(q,r)}, \\ \{\langle y - q, \nu_{q,r} \rangle \le -\eta r\} \cap \overline{B(q,r)} \subset E \cap \overline{B(q,r)}. \end{cases}$$

By using the divergence theorem as it was used in (2.14), it follows that

$$\begin{aligned} &\int_{\partial B(q,r)\cap\{\langle y-q, n_{q,r}\rangle \le 0\}} n_{q,r} \cdot \frac{y-q}{|y-q|} \, \mathrm{d}\mathcal{H}^{n-1} \\ &\qquad = - \int_{B(q,r)\cap\{\langle y-q, n_{q,r}\rangle = 0\}} n_{q,r} \cdot n_{q,r} \, \mathrm{d}\sigma = -\omega_{n-1} r^{n-1}, \end{aligned}$$

and a very generous estimate ensures that

$$\begin{aligned}(2.16)\qquad \Bigg| \int_{\partial B(q,r)\cap E} n_{q,r} \cdot \frac{y-q}{|y-q|}\, \mathrm{d}\mathcal{H}^{n-1} \\ - \int_{\partial B(q,r)\cap\{\langle y-q, n_{q,r}\rangle \le 0\}} n_{q,r} \cdot \frac{y-q}{|y-q|}\, \mathrm{d}\mathcal{H}^{n-1} \Bigg| \le Cr^{n-1}\eta.\end{aligned}$$

Combining (2.15)–(2.16) confirms (2.10). Equation (2.11) then follows from $\|\nu\|_*(B(q,r)) \le \delta$. Details are included when the same statement is verified in (3.4).

It only remains to show (2.12). The lower bound follows immediately from ∂E being η-Reifenberg flat and the separation property, since then

$$\sigma(B(q,r)) \ge \omega_{n-1}(r\sqrt{1-\eta^2})^{n-1} \ge \omega_{n-1}\left(1 - \frac{n-1}{2}\eta^2\right) r^{n-1}.$$

For the upper bound, we use (2.14) with $e = \nu_{q,r}$ to obtain the estimate

$$\begin{aligned}\sigma(B(q,r))|\nu_{q,r}|^2 &= \left| \int_{B(q,r)} \nu_{q,r} \cdot \nu_E \, \mathrm{d}\sigma \right| = \left| \int_{\partial B(q,r)\cap E} \nu_{q,r} \cdot \frac{y-q}{|y-q|}\, \mathrm{d}\mathcal{H}^{n-1} \right| \\ &\le |\nu_{q,r}|\omega_{n-1} r^{n-1}.\end{aligned}$$

Therefore, by (2.11),

$$\sigma(B(q,r)) \le \frac{1}{|\nu_{q,r}|}\omega_{n-1} r^{n-1} \le (1+2\delta)\omega_{n-1} r^{n-1}. \qquad \square$$

3. FLATNESS FROM CONTROL ON OSCILLATION

In this section we introduce a class of well-behaved sets $\mathcal{A}(C_A, r_0)$, and prove our key geometric result, Corollary 3.10. Specifically, in the class, $\mathcal{A}(C_A, r_0)$, the oscillation of the unit normal controls the flatness (in the sense of Reifenberg) of the boundary. One key tool is the "excess" of a set of locally finite perimeter, first introduced by De Giorgi in [DG61] and ubiquitous in the calculus of variations. Because of Lemma 3.4, all of our arguments could also be written in terms of the mean oscillation of the unit normal. Given $r_0 \in (0,\infty)$ and $C_A \in [1,\infty)$, we define a class of sets

$$\begin{aligned}(3.1)\quad \mathcal{A}(C_A, r_0) \\ = \Big\{ E \subset \mathbb{R}^n \mid E \text{ is a set of locally finite perimeter satisfying} \\ \partial E = \operatorname{spt}\mu_E \text{ and its perimeter measure } |\mu_E| \text{ is} \\ (n-1)\text{-Ahlfors regular up to scale } r_0 \text{ with constant } C_A \Big\}.\end{aligned}$$

Uniformly rectifiable domains (up to choosing a representative from the equivalence class; see Remark 3.1) with Ahlfors regularity constant C_A form a subset of $\mathcal{A}(C_A, r_0)$. The complement of a quasiminimal surface of codimension 1 is the disjoint union of two open domains of $\mathbb{R}^n$ (see [DS98]), and each of these domains would fall within the class $\mathcal{A}(C_A, r_0)$.

Remark 3.1. The condition that $\partial E = \operatorname{spt} \mu_E$ corresponds to choosing a representative for our set amongst the *equivalence class* of sets of locally finite perimeter (see [Mag12, Proposition 12.19, Remark 16.11]): for any set of finite perimeter E, we can find a Borel set F such that $|E \Delta F| = 0$, for $\partial F = \operatorname{spt} \mu_F = \operatorname{spt} \mu_E$. This choice is necessary since we want to deduce information on the topological boundary from information on the measure-theoretic unit outer normal, which is merely defined on the reduced boundary $\partial^* E$ (see, e.g., Lemma 3.8 and Theorem 3.9).

A particularly useful property of $\mathcal{A}(C_A, r_0)$ is that if $E \in \mathcal{A}(C_A, r_0)$ then $\mathbb{R}^n \setminus E \in \mathcal{A}(C_A, r_0)$. This follows since $\mu_E = -\mu_{\mathbb{R}^n \setminus E}$ and $\partial E = \partial(\mathbb{R}^n \setminus E)$.

Remark 3.2. If $E \in \mathcal{A}(C_A, r_0)$, then ∂E is $(n-1)$-Alhfors regular since $\partial E = \operatorname{spt} \mu_E$ and $\mathcal{H}^{n-1}(\partial E \setminus \partial^* E) = 0$ (see [Mat95, Theorem 6.9]). Thus,

$$|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E = \mathcal{H}^{n-1} \llcorner \partial E.$$

Definition 3.3 (***Cylinders and excess: c.f.*** [Mag12, Chapter 22]). For $r > 0$, $x \in \mathbb{R}^n$, and some $\nu \in \mathbb{S}^{n-1}$, we let

$$C(x, r, \nu) = \{y : |\langle x - y, \nu \rangle| < r,\ |x - y - \langle x - y, \nu \rangle \nu| < r\}.$$

Note that $C(x, r, \nu)$ is a cylinder with center x, radius and height r, and axial direction ν. For a set of locally finite perimeter E, $x \in \partial E$, $r > 0$, and $\nu \in \mathbb{S}^{n-1}$ we define the *cylindrical excess*

$$e(E, x, r, \nu) = \frac{1}{r^{n-1}} \int_{C(x,r,\nu) \cap \partial^* E} \frac{|\nu_E - \nu|^2}{2} \, d\mathcal{H}^{n-1}.$$

The following lemma elucidates the relationship between oscillation of the unit normal and excess.

Lemma 3.4. *Let $E \in \mathcal{A}(C_A, r_0)$ and let $Q \in \partial E$ and $0 < r < r_0$. There exists some constant $0 < C < \infty$ (which depends only on C_A and the dimension) such that*

$$\fint_{B(Q,r) \cap \partial^* E} |\nu_E - (\nu_E)_{Q,r}|^2 \, d\mathcal{H}^{n-1} \leq C e(E, Q, r, \nu) \tag{3.2}$$

for any $\nu \in \mathbb{S}^{n-1}$, where $(\nu_E)_{Q,r}$ represents the integral average of ν_E with respect to $\mathcal{H}^{n-1}$ on $B(Q, r) \cap \partial^* E$. Furthermore, as long as $|(\nu_E)_{Q,r}| \neq 0$, we have that

$$e\left(E, Q, \frac{r}{\sqrt{2}}, \frac{(\nu_E)_{Q,r}}{|(\nu_E)_{Q,r}|}\right) \leq C \fint_{B(Q,r) \cap \partial^* E} |\nu_E - (\nu_E)_{Q,r}|^2 \, d\mathcal{H}^{n-1}. \tag{3.3}$$

Proof. We first prove (3.2). Note that $B(Q,r)\cap\partial^*E \subset C(Q,r,\nu)\cap\partial^*E$ for any $Q\in\partial E$ and $r>0$. Thus,

$$e(E,Q,r,\nu) \geq c \fint_{B(Q,r)\cap\partial^*E} \frac{|\nu_E-\nu|^2}{2}\,\mathrm{d}\mathcal{H}^{n-1},$$

where c is a constant depending only on the Ahlfors regularity of E. We compute

$$\begin{aligned}
&\fint_{B(Q,r)\cap\partial^*E} |\nu_E(x)-(\nu_E)_{Q,r}|^2\,\mathrm{d}\mathcal{H}^{n-1}\\
&\qquad\leq 2\fint_{B(Q,r)\cap\partial^*E} |\nu_E(x)-\nu|^2\,\mathrm{d}\mathcal{H}^{n-1}\\
&\qquad\qquad+2\fint_{B(Q,r)\cap\partial^*E} |\nu-(\nu_E)_{Q,r}|^2\,\mathrm{d}\mathcal{H}^{n-1}\\
&\qquad\leq 4\fint_{B(Q,r)\cap\partial^*E} |\nu_E(x)-\nu|^2\,\mathrm{d}\mathcal{H}^{n-1}\leq Ce(E,Q,r,\nu),
\end{aligned}$$

where the second inequality follows from the triangle inequality and Jensen's inequality. This is exactly (3.2).

To prove (3.3) it suffices to consider

$$\fint_{B(Q,r)\cap\partial^*E} |\nu_E-(\nu_E)_{Q,r}|^2\,\mathrm{d}\mathcal{H}^{n-1}=\varepsilon<1.$$

We first estimate $|(\nu_E)_{Q,r}|$; note that

$$\begin{aligned}
(|1-|(\nu_E)_{Q,r}|)^2 &= \fint_{B(Q,r)\cap\partial^*E}(|\nu_E|-|(\nu_E)_{Q,r}|)^2\,\mathrm{d}\mathcal{H}^{n-1}\\
&\leq \fint_{B(Q,r)\cap\partial^*E}|\nu_E-(\nu_E)_{Q,r}|^2\,\mathrm{d}\mathcal{H}^{n-1}=\varepsilon
\end{aligned}\tag{3.4}$$

and

$$|(\nu_E)_{Q,r}| = \left|\fint_{B(Q,r)\cap\partial^*E}\nu_E\,\mathrm{d}\mathcal{H}^{n-1}\right| \leq \fint_{B(Q,r)\cap\partial^*E}|\nu_E|\,\mathrm{d}\mathcal{H}^{n-1}=1.\tag{3.5}$$

Now, combining (3.4) with (3.5) ensures that we have $1-\sqrt{\varepsilon}\leq|(\nu_E)_{Q,r}|\leq 1$. Let $\nu_0\equiv(\nu_E)_{Q,r}/|(\nu_E)_{Q,r}|$ and compute

$$\begin{aligned}
|\nu_E-\nu_0| &\leq |\nu_E-(\nu_E)_{Q,r}|+|(\nu_E)_{Q,r}|\left|1-\frac{1}{|(\nu_E)_{Q,r}|}\right|\\
&\leq |\nu_E-(\nu_E)_{Q,r}|+|1-(\nu_E)_{Q,r}|\\
&\leq |\nu_E-(\nu_E)_{Q,r}|+\varepsilon^{1/2},
\end{aligned}$$

so that

$$|\nu_E - \nu_0|^2 \le 2|\nu_E - (\nu_E)_{Q,r}|^2 + 2\varepsilon. \tag{3.6}$$

Notably, (3.6) and $C(Q, r/\sqrt{2}, \nu_0) \subset B(Q,r)$ imply

$$\begin{aligned}
e&\left(E, Q, \frac{r}{\sqrt{2}}, \nu_0\right) \\
&= \frac{2^{(n-1)/2}}{r^{n-1}} \int_{C(Q,r/\sqrt{2},\nu_0)\cap\partial^* E} \frac{|\nu_E - \nu_0|^2}{2} \,\mathrm{d}\mathcal{H}^{n-1} \\
&\le \frac{2^{(n-1)/2}}{r^{n-1}} \int_{C(Q,r/\sqrt{2},\nu_0)\cap\partial^* E} |\nu_E - (\nu_E)_{Q,r}|^2 \,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad + \frac{2^{(n-1)/2}\mathcal{H}^{n-1}(C(Q, r/\sqrt{2}, \nu_0) \cap \partial^* E)}{r^{n-1}}\varepsilon \\
&\le 2^{(n-1)/2} \frac{\mathcal{H}^{n-1}(B(Q,r) \cap \partial^* E)}{r^{n-1}} \\
&\qquad \times \left(\fint_{B(Q,r)\cap\partial^* E} |\nu_E - (\nu_E)_{Q,r}|^2 \,\mathrm{d}\mathcal{H}^{n-1} + \varepsilon \right) \\
&= 2^{(n+1)/2} \frac{\mathcal{H}^{n-1}(B(x,r) \cap \partial^* E)}{r^{n-1}} \varepsilon \le C_n \cdot C_A \varepsilon.
\end{aligned}$$

□

Remark 3.5. We recall some basic properties of the cylindrical excess (see, e.g., [Mag12, Chapter 22] for more details). The cylindrical excess is invariant under translation and scaling in the sense that if $E_{x,r} = (E - x)/r$, then

$$e(E_{x,r}, 0, 1, \nu) = e(E, x, r, \nu). \tag{3.7}$$

Furthermore, if $r < s$, the non-negativity of the integrand ensures

$$\begin{aligned}
&\frac{1}{r^{n-1}} \int_{C(x,r,\nu)\cap\partial^* E} \frac{|\nu_E - \nu|^2}{2} \,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad \le \left(\frac{s}{r}\right)^{n-1} \frac{1}{s^{n-1}} \int_{C(x,s,\nu)\cap\partial^* E} \frac{|\nu_E - \nu|^2}{2} \,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned}$$

that is,

$$e(E, x, r, \nu) \le \left(\frac{s}{r}\right)^{n-1} e(E, x, s, \nu). \tag{3.8}$$

Finally, since ν, ν_E are each of unit length, $|\nu_E - \nu|^2/2 = 1 - \langle \nu_E, \nu \rangle$ so that

$$e(E, x, r, \nu) = \frac{1}{r^{n-1}} \int_{C(x,r,\nu)\cap\partial^* E} (1 - \langle \nu_E, \nu \rangle) \,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.9}$$

Given a sequence of sets of locally finite perimeter $\{E_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$, we say that $\{E_k\}$ converges to E in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and write

$$E_k \xrightarrow{L^1_{\mathrm{loc}}(\mathbb{R}^n)} E$$

if $\lim_{k\to\infty} \mathcal{H}^n(E\Delta E_k) = 0$. The following compactness theorem is the key tool used in proving the flatness result.

Theorem 3.6. *If $\{E_k\}_{k\in\mathbb{N}} \subset \mathcal{A}(C_A, r_0)$ with $0 \in \partial E_k$ for all $k \geq 1$, there exist a subsequence $\{E_{k_j}\}_{j\in\mathbb{N}}$, a set E of locally finite perimeter, and a non-negative Radon measure, μ, such that as j approaches infinity,*

$$E_{k_j} \xrightarrow{L^1_{\mathrm{loc}}(\mathbb{R}^n)} E, \quad \mu_{E_{k_j}} \rightharpoonup \mu_E \quad \text{and} \quad |\mu_{E_{k_j}}| \rightharpoonup \mu. \tag{3.10}$$

Additionally, $\partial E = \operatorname{spt} \mu_E$ and μ is $(n-1)$-Ahlfors regular up to scale r_0 with constant C_A. Furthermore, $|\mu_E| \leq \mu$ and we have the following:

(1) *If $x \in \partial E$, then for all $j \in \mathbb{N}$ there exist $x_{k_j} \in \partial E_{k_j}$ such that $\lim_{j\to\infty} x_{k_j} = x$.*
(2) *If $x \in \operatorname{spt}\mu$, then for all $j \in \mathbb{N}$ there exist $x_{k_j} \in \partial E_{k_j}$ so that $\lim_{j\to\infty} x_{k_j} = x$.*
(3) *If for all $j \in \mathbb{N}$, $x_{k_j} \in \partial E_{k_j}$ and $\lim_{j\to\infty} x_{k_j} = x$, then $x \in \operatorname{spt}\mu$.*

Remark 3.7.

- We note that (2) and (3) in Theorem 3.6 combine to say that $x \in \operatorname{spt}\mu$ if and only if there exists $x_{k_j} \in \partial E_{k_j}$ such that $x_{k_j} \to x$. However, without additional hypotheses, all that is known is that $\operatorname{spt}\mu_E \subseteq \operatorname{spt}\mu$.
- Unlike in the analogous theorem [Mag12, Theorem 21.14] for perimeter minimizers, here in general we do not have $\mu = |\mu_E|$ because of possible cancellations for sets of finite perimeter. However, with further information on the excess, we will be able to conclude $\mu = |\mu_E|$ (see, e.g., Lemma 3.8).

Proof. Standard techniques and a diagonalization argument (see, e.g., [Mag12, Sections 12.4, 21.5]) verify that sets whose boundary are uniformly Ahlfors-regular (i.e., Ahlfors regular with constants independent of the element in the sequence) are pre-compact in the space of sets of locally finite perimeter. That is to say, there exists some set of locally finite perimeter $E \subset \mathbb{R}^n$ so that $\chi_{E_{k_j}} \to \chi_E$ in L^1_{loc} and $\mu_{E_{k_j}} \rightharpoonup \mu_E$ in a weak star sense. Without loss of generality (see Remark 3.1) we may assume that $\operatorname{spt}\mu_E = \partial E$. Finally, note the $|\mu_{E_{k_j}}|$ are uniformly Ahlfors regular (see Remark 3.2) and hence precompact. Without explicitly relabeling the new subsequence, there exists a Radon measure μ on $\mathbb{R}^n$ so that $|\mu_{E_{k_j}}| \rightharpoonup \mu$ in the weak star sense. Thus, (3.10) holds.

The fact that $|\mu_E| \le \mu$ follows from (2.3). This ensures that $\operatorname{spt}\mu_E \subset \operatorname{spt}\mu$, so (2) (which is a standard fact) implies (1). Moreover, (2), and the uniform upper regularity of $\{|\mu_{E_{k_j}}|\}$ imply the upper Ahlfors regularity of μ.

We show (3) and lower Ahlfors regularity of μ simultaneously. For each $j \in \mathbb{N}$ suppose $x_{k_j} \in \partial E_{k_j} = \operatorname{spt}|\mu_{E_{k_j}}|$ such that $x_{k_j} \to x$.

Fix $0 < s < r_0$ and fix $\varepsilon \in (0,1)$. Note that for k_j large enough,

$$B(x_{k_j}, s(1-\varepsilon)) \subset B(x, s(1-\varepsilon/2)).$$

Since $E_{k_j} \in \mathcal{A}(C_A, r_0)$ it follows that

$$\begin{aligned} C_A^{-1}(s(1-\varepsilon))^{n-1} &\le |\mu_{E_{k_j}}|(B(x_{k_j}, s(1-\varepsilon))) \\ &\le |\mu_{E_{k_j}}|(\overline{B(x, s(1-\varepsilon/2))}), \end{aligned}$$

so that by weak$*$ convergence of $|\mu_{E_{k_j}}|$ to μ,

$$\begin{aligned} C_A^{-1}(s(1-\varepsilon))^{n-1} &\le \limsup_j |\mu_{E_{k_j}}|(\overline{B(x, s(1-\varepsilon/2))}) \\ &\le \mu(\overline{B(x, s(1-\varepsilon/2))}), \end{aligned}$$

and taking $\varepsilon \to 0$ results in $C_A^{-1}s^{n-1} \le \mu(B(x,s))$ for all $s \in (0, r_0)$; in particular $x \in \operatorname{spt}\mu$, verifying (3). On the other hand, since (2) and (3) combine to show that $x \in \operatorname{spt}\mu$ if and only if there exists $x_{k_j} \in \partial E_{k_j}$ such that $x_{k_j} \to x$, this demonstrates μ is $(n-1)$-lower Ahlfors regular up to scale r_0 with constant C_A. $\square$

We now prove that small excess implies local measure theoretic separation. To simplify notation, define $e_n(E, x, r) = e(E, x, r, e_n)$.

Lemma 3.8 (***Separation lemma (compare with*** [Mag12, Lemma 22.10])). *Given $C_A \ge 1$, $t_0 \in (0,1)$, there is $\omega(n, t_0, C_A) \in (0, \infty)$ such that if $E \in \mathcal{A}(C_A, 2r)$ for some $r > 0$ and if there exist $x_0 \in \partial E$ and $\nu \in \mathbb{S}^{n-1}$ with*

$$e(E, x_0, 2r, \nu) \le \omega(n, t_0, C_A),$$

then

$$|\langle x - x_0, \nu\rangle| < t_0 r \quad \forall\, x \in C(x_0, r, \nu) \cap \partial E, \tag{3.11}$$

$$|\{x \in C(x_0, r, \nu) \cap E \mid \langle x - x_0, \nu\rangle > t_0 r\}| = 0, \tag{3.12}$$

and

$$|\{x \in C(x_0, r, \nu) \cap E^c \mid \langle x - x_0, \nu\rangle < -t_0 r\}| = 0. \tag{3.13}$$

(Note, here and below for any Lebesgue measurable set $\mathcal{O} \subset \mathbb{R}^n$ we write $|\mathcal{O}|$ to denote the Lesbesgue measure of $\mathcal{O}$).

Proof. This follows by a compactness-contradiction argument. If Lemma 3.8 does not hold, there exist $C_A > 1$, $t_0 \in (0,1)$, a sequence of sets $\{F_k\}_{k\in\mathbb{N}}$ and radii $r_k > 0$ such that $F_k \in \mathcal{A}(C_A, 2r_k)$, a sequence of points $x_k \in \partial F_k$, and a sequence of directions $\nu_k \in \mathbb{S}^{n-1}$, with

$$e(F_k, x_k, 2r_k, \nu_k) \le 2^{-k},$$

such that at least one of the following conditions holds for infinitely many k:

$$\{x \in C(x_k, r_k, \nu_k) \cap \partial F_k : |q_k(x)| > t_0 r_k\} \neq \emptyset, \tag{3.14}$$

$$|\{x \in C(x_k, r_k, \nu_k) \cap F_k : q_k(x) > t_0 r_k\}| > 0, \tag{3.15}$$

or

$$|\{x \in C(x_k, r_k, \nu_k) \cap F_k^c : q_k(x) < -t_0 r_k\}| > 0, \tag{3.16}$$

where $q_k(x) = \langle x - x_k, \nu_k\rangle$.

By rescaling, recentering, and rotating (see Remark 3.5) we may assume that $\nu_k \equiv e_n$, $x_k \equiv 0$, and $r_k \equiv 1$. Note that the transformed domains are now in $\mathcal{A}(C_A, 2)$. Abusing notation, we call these new sets F_k. Note that

$$e_n(F_k, 0, 2) \le 2^{-k} \quad \forall\, k \ge 1. \tag{3.17}$$

Writing $C_r = C(0, r, e_n)$ and $q(x) = \langle x, e_n\rangle$ we rewrite (3.14)–(3.16) as

$$\{x \in C_1 \cap \partial F_k \mid t_0 \le |q(x)|\} \neq \emptyset, \tag{3.18}$$

$$|\{x \in C_1 \cap F_k \mid q(x) > t_0\}| > 0, \tag{3.19}$$

or

$$|\{x \in C_1 \setminus F_k \mid q(x) < -t_0\}| > 0. \tag{3.20}$$

By Theorem 3.6, there is a set of finite perimeter $F \subset C_{5/3}$ with $0 \in \partial F = \operatorname{spt}|\mu_F|$ and a Radon measure μ such that, by passing to a subsequence we do not explicitly relabel, $F_k \cap C_{5/3} \to F$ in $L^1(\mathbb{R}^n)$, $\mu_{F_k\cap C_{5/3}} \rightharpoonup \mu_F$, and $|\mu_{F_k\cap C_{5/3}}| \rightharpoonup \mu$ with $|\mu_F| \le \mu$.

Consider an open set U such that $\bar{U} \subset C_{5/3}$. Then, (3.9) implies

$$\begin{aligned} \left(\frac{5}{3}\right)^{n-1} e_n\left(F_k, 0, \frac{5}{3}\right) &\ge \int_{U\cap\partial^* F_k} (1 - e_n \cdot \nu_{F_k})\, \mathrm{d}\mathcal{H}^{n-1} \\ &= |\mu_{F_k}|(U) - e_n \cdot \mu_{F_k}(U) \ge 0, \end{aligned} \tag{3.21}$$

where the final inequality follows since

$$\mathrm{d}\mu_{F_k} = \nu_{F_k}|\mathrm{d}\mu_{F_k}| \text{ and } |\nu_{F_k}| = 1 \quad |\mu_{F_k}|\text{-almost everywhere.} \tag{3.22}$$

Then, (3.8) and (3.17) ensure that as k tends to infinity,

$$e_n\left(F_k, 0, \frac{5}{3}\right) \leq \left(\frac{6}{5}\right)^{n-1} e_n(F_k, 0, 2) \to 0.$$

This combined with (3.21) yields

$$0 \leq \lim_{k\to\infty} \{|\mu_{F_k}|(U) - e_n \cdot \mu_{F_k}(U)\} \leq C_n \lim_{k\to\infty} e_n\left(F_k, 0, \frac{5}{3}\right) = 0.$$

Since (2.3) says $|\mu_F| \leq \mu$ we can apply (2.4) to both $|\mu_{F_k}|$ and μ_{F_k} to learn

$$\mu(U) = e_n \cdot \mu_F(U) \quad \text{for any open set } U \Subset C_{5/3}, \text{ with } \mu(\partial U) = 0. \tag{3.23}$$

Note that by Theorem 3.6, μ is Ahlfors regular with constant C_A up to scale 2 in the cylinder $C_{5/3}$. Hence, in particular for any $x \in C_{4/3} \cap \operatorname{spt}\mu$ and almost every $r \in (0, \frac{1}{3})$, $\mu(\partial B(x,r)) = 0$ and by (3.23) $\mu(B(x,r)) = e_n \cdot \mu_F(B(x,r))$. Consequently, for all $x \in \operatorname{spt}|\mu_F| \cap C_{4/3}$,

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{|\mu_F|(B(x,r))} = e_n \cdot \limsup_{r\to 0} \frac{\mu_F(B(x,r))}{|\mu_F|(B(x,r))} \leq 1,$$

where the final inequality uses the property (3.22) for the set F. Therefore, in $C_{4/3}$ we have shown $\mu \leq |\mu_F| \leq \mu$, which implies $\mu = |\mu_F| = \mathcal{H}^{n-1} \llcorner \partial^* F$. But then, (3.23) says $|\mu_F| = e_n \cdot \mu_F$ so that $\nu_F(x) = e_n$ at $\mathcal{H}^{n-1}$-almost every $x \in \partial^* F$. In particular, $e_n(F, 0, \frac{4}{3}) = 0$, at which point [Mag12, Proposition 22.2] asserts that $F \cap C_{4/3}$ is equivalent (in the sense of sets of locally finite perimeter) to $C_{4/3} \cap \{q(x) < 0\}$ or $C_{4/3} \cap \{q(x) > 0\}$. Since $|\mu_F| = e_n \cdot \mu_F$ it follows that $F \cap C_{4/3}$ is equivalent to $C_{4/3} \cap \{q(x) < 0\}$. We write this as

$$C_{4/3} \cap F \sim \{q(x) < 0\} \cap C_{4/3}. \tag{3.24}$$

We assumed that one of (3.18)–(3.20) holds for infinitely many k. First, suppose (3.18) holds for infinitely many k. By passing to a subsequence, we may assume that (3.18) holds for all $k \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$, there exists $x_k \in \partial F_k \cap C_1$ such that $t_0 \leq |q(x_k)|$. By passing to a subsequence, $x_k \to x_\infty$ for some $x_\infty \in \overline{C_1}$ and $|q(x_\infty)| \geq t_0$. By Theorem 3.6 (3), $x_\infty \in \operatorname{spt}\mu = \operatorname{spt}\mu_F = \partial F$. Hence (see [Mag12, Proposition 12.19]),

$$0 < |B(x_\infty, s) \cap F| < \omega_n s^n \quad \forall\, s > 0. \tag{3.25}$$

However, because $|q(x_\infty)| \geq t_0$, (3.24) implies that

$$|B(x_\infty, s) \cap F| = \begin{cases} \omega_n s^n & \text{if } q(x_\infty) < 0, \\ 0 & \text{if } q(x_\infty) > 0, \end{cases}$$

for any $s \le \min\{\frac{1}{8}, |q(x_\infty)|/2\}$, which contradicts (3.25). This shows that (3.18) cannot hold for infinitely many k.

Arguing as above and invoking Theorem 3.6 (3), we conclude there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$,

$$\{x \in C_{5/4} \cap \partial F_k \mid t_0 < |q(x)| \le 1\} = \emptyset. \tag{3.26}$$

However, by [Mag12, Equation 16.7], for all $r \in (1, \frac{5}{4})$,

$$|\mu_{F_k \cap C_r}| = |\mu_{C_r}| \llcorner F_k^{(1)} + |\mu_{F_k}| \llcorner (C_r \cup \{\nu_{F_k} = \nu_{C_r}\}).$$

For almost every $r \in (1, \frac{5}{4})$ we know $|\mu_{F_k}|(\partial C_r) = 0$ for all k. Then, for any such r (3.26) demonstrates

$$|\mu_{F_k \cap C_r}|(\{x \in C_r \mid t_0 < |q(x)| < 1\}) = 0 \quad \forall\, k \ge k_0. \tag{3.27}$$

We claim (3.27) implies that for almost every $r \in (1, \frac{5}{4})$, $\chi_{C_r \cap F_k}$ is constant on each connected component of $\{t_0 < |q(x)| < 1\} \cap C_r$, which implies $\chi_{C_1 \cap F_k}$ is constant on connected components of $\{t_0 < |q(x)| < 1\} \cap C_1$. Indeed, choose $r \in (1, \frac{5}{4})$ so that $|\mu_{F_k}|(\partial C_r) = 0$ for all k. Consider the sets

$$U_\pm := \{t_0 < \pm q(x) < 1\} \cap C_r,$$

which are both open and connected. The definition (2.1) and (3.27) guarantee, for all $k \ge k_0$,

$$\int_{\mathbb{R}^n} T \cdot \mathrm{d}\mu_{F_k} = 0 \quad \text{for all } T \in C_c^1(U_\pm; \mathbb{R}^n).$$

(If the integral is nonzero, we can flip the sign of T and get a contradiction with (3.27).) Thus, by Proposition 2.1,

$$\int_{\mathbb{R}^n} \chi_{F_k} \operatorname{div} T \,\mathrm{d}x = \int_{\mathbb{R}^n} T \cdot \mathrm{d}\mu_{F_k} = 0 \quad \text{for all } T \in C_c^1(U_\pm; \mathbb{R}^n),$$

that is, in the weak sense, $\nabla \chi_{F_k} = 0$ on U_+ and U_-. This implies χ_{F_k} is almost everywhere constant on each $U_\pm$ (e.g., see [Mag12, Lemma 7.5]). Combining (3.24) with χ_{F_k} constant on each $U_\pm$ and $F_k \cap C_{5/3} \xrightarrow{L^1(\mathbb{R}^n)} F$, it follows that for $k \ge k_0$,

$$\chi_{F_k \cap C_1} = \begin{cases} 0 & \text{for almost every } x \in C_1 \cap \{t_0 < q(x) < 1\}, \\ 1 & \text{for almost every } x \in C_1 \cap \{-1 < q(x) < t_0\}. \end{cases}$$

This shows that (3.19) and (3.20) cannot happen for infinitely many k. □

The (qualitative) separation lemma above can be further improved to a quantative "height bound" of ∂E. Since the proof is by fairly standard techniques in the theory of sets of locally finite perimeter, we include it in Appendix A (see Theorem A.2). Topological considerations then imply the following theorem.

Theorem 3.9. *Given $C_A \geq 1$ and $n \geq 2$, there exist positive constants $C_1 = C_1(n, C_A) < \infty$ and $\varepsilon_1 = \varepsilon_1(n, C_A)$ small such that if $E \in \mathcal{A}(C_A, 4r_0)$ for some $r_0 > 0$, and $x_0 \in \partial E$ satisfies $e(E, x_0, 2r, \nu) \leq \varepsilon_1$ for some $\nu \in \mathbb{S}^n$ and $0 < r < 2r_0$, then*

$$|\langle x - x_0, \nu\rangle| \leq C_1 r e(E, x_0, 2r, \nu)^{1/(2(n-1))} \quad \forall\, x \in C(x_0, r, \nu) \cap \partial E,$$

$$\{x \in C(x_0, r, \nu) \cap E \mid \langle x - x_0, \nu\rangle > C_1 r e(E, x_0, 2r, \nu)^{1/(2(n-1))}\} = \varnothing,$$

and

$$\{x \in C(x_0, r, \nu) \cap E^c \mid \langle x - x_0, \nu\rangle < -C_1 r e(E, x_0, 2r, \nu)^{1/(2(n-1))}\} = \varnothing.$$

An immediate quantitative consequence of Lemma 3.4 and Theorem 3.9 is the following result.

Corollary 3.10. *Given $n \geq 2$, and $C_A \geq 1$ there exist constants $\varepsilon_2 = \varepsilon_2(n, C_A)$ and $C_2 = C(n, C_A)$ (both positive and finite), such that if $E \in \mathcal{A}(C_A, r_0)$ (for some $r_0 > 0$) satisfies*

$$\sup_{r<r_0} \left(\fint_{B(x,r)\cap\partial^* E} |\nu_E - (\nu_E)_{x,r}|^2 \,\mathrm{d}\mathcal{H}^{n-1} \right)^{1/2} \leq \varepsilon_2$$

for some $x \in \partial E$, then

$$\sup_{\rho < r_0/8} \Theta(x, \rho) \leq C_2 \varepsilon_2^{1/(n-1)}.$$

In particular, if $\Omega \subset \mathbb{R}^n$ is a domain such that $\partial_*\Omega = \partial\Omega$, $\partial\Omega$ is $(n-1)$-Ahlfors regular, and ν_Ω satisfies

$$\sup_{r<r_0} \sup_{x\in\partial\Omega} \left(\fint_{B(x,r)\cap\partial\Omega} |\nu_\Omega - (\nu_\Omega)_{x,r}|^2 \,\mathrm{d}\mathcal{H}^{n-1} \right)^{1/2} \leq \varepsilon_2,$$

then Ω is a $C_2\varepsilon_2^{1/(n-1)}$-Reifenberg flat domain.

Proof. As in Remark 2.3, $\partial\Omega = \partial_*\Omega$ and $\partial\Omega$ is Ahlfors regular imply

$$\partial\Omega = \operatorname{spt}\mu_\Omega, \quad |\mu_\Omega| \text{ is Ahlfors regular.}$$

That is, $\Omega \in \mathcal{A}(C_A, r_0)$ for some constants C_A, and all r_0. Therefore, the corollary is a consequence of Theorem 3.9. ◻

An immediate qualitative consequence of Lemma 3.10 and Theorem 3.9 is the following result.

Corollary 3.11. *If $\Omega \subset \mathbb{R}^n$ is a domain such that $\partial_* \Omega = \partial\Omega$, $\partial\Omega$ is $(n-1)$-Ahlfors regular, and $\nu_E \in \mathrm{VMO}_{\mathrm{loc}}(\mathcal{H}^{n-1} \llcorner \partial\Omega)$, then $\partial\Omega$ is a vanishing Reifenberg flat set.*

Corollary 3.10 also has the following quantitative consequence for δ-CADs (see Definition 2.19).

Corollary 3.12. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial_*\Omega = \partial\Omega$ and with $(n-1)$-Ahlfors regular boundary with constant C_A. Further assume, if Ω is unbounded, that $\mathbb{R}^n \setminus \partial\Omega$ consists of two nonempty connected components. Then, there exists a $\delta_n > 0$ such that for $\delta \in (0, \delta_n]$, there exists $\varepsilon_\delta < \varepsilon_2$ (where $\varepsilon_2 > 0$ is as in Corollary 3.10) such that if for every compact set $K \Subset \mathbb{R}^n$ there exists an $R_K > 0$ such that $\sup_{x \in \partial\Omega \cap K} \|\nu\|(x, R_K) < \varepsilon_\delta$, then Ω is a δ-chord-arc domain.*

4. AN APPLICATION TO A TWO-PHASE PROBLEM FOR HARMONIC MEASURE

In this section, we consider a two-phase free boundary problem for harmonic measure, originally studied by Kenig-Toro in [KT06] and later by [BH16]. In particular, we complete the proof of Theorem 1.1, and prove a quantitative version of it (Theorem 4.14).

4.1. The existence of corkscrews. The goal of this subsection is to show that the doubling of harmonic measure implies interior corkscrews (Lemma 4.3). Later, we will show that control on the oscillation of the logarithm of the Poisson kernel implies doubling. This is an important step in proving Theorem 4.12 as it will allow us use the theory of UR domains (by way of Appendix B). First, we recall what it means for harmonic measure to be doubling.

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be a domain with harmonic measure ω. We say that ω is locally doubling with constant C if, for every compact set K, there exists $r_K > 0$ such that

$$\omega(B(x, 2r)) < C\omega(B(x, r)).$$

for all $x \in \partial\Omega \cap K$ and all $r \in (0, r_K)$. We also refer to r_K as the (local) doubling condition radius.

Remark 4.2. We often assume r_K is sufficiently small compared to the distance from the pole of ω to the boundary $\partial\Omega$. This allows us to focus on local regions away from the pole, so that we can use preliminary estimates on the harmonic measure with ease.

To prove estimates that are uniform on compacta, it is important to keep track of what the value of each constant depends on, and in particular, whether or not it depends on the choice of compact set. For simplicity, we may say the value depends on allowable constants, if it depends only on the dimension n and the Ahlfors regularity constant, and not on the compact set. The following

Lemma 4.3, which might be considered folklore, shows the existence of interior corkscrews given the doubling of harmonic measure. This is an essential step, as it allows us to gain topological information on Ω from the regularity of the Poisson kernel. We sketch the proof here, which is a small modification of the proof of [HM15, Lemma 3.14] (see also [HLMN17, Lemma 4.24]).

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain whose boundary is Ahlfors regular with constant C_A. Fix $X_0 \in \Omega$. Suppose ω^{X_0} is locally doubling with constant C_0. There exists an $\eta = \eta(n, C_A) > 0$ such that, for every closed ball K, if $r_K \ll \delta(X_0)$ is the doubling radius of ω^{X_0} in K, then Ω admits an interior corkscrew ball at every $x \in \partial\Omega \cap K$ up to radius $s_K := \eta r_K$ with constant $C_1 = C(n, C_A, C_0)$.*

Proof. Fix the closed ball K and recall that r_K is the local doubling radius. The proof of this lemma requires a slight modification of the argument in Lemma 3.14 of [HM15]. Recall the following relationship between the Green function and the harmonic measure. For $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\begin{aligned}(4.1)\qquad &\int_{\partial\Omega} \Phi(y)\,\mathrm{d}\omega^X(y) - \Phi(X)\\ &\qquad = -\iint_\Omega \nabla\mathcal{G}(X,Y)\nabla\Phi(Y)\,\mathrm{d}Y, \quad \text{for almost every } X \in \omega,\end{aligned}$$

where $\omega := \omega^X$ and $\mathcal{G}(Y) := \mathcal{G}(X,Y)$ are the harmonic measure and Green's function for Ω with pole at X.

It was proven in [HM15, Lemma 2.40] that there exists $\kappa_0 > 2$ depending only on dimension and the Ahlfors regularity constant such that for all $x \in \partial\Omega$ and $0 < r < \min\{\delta(X)/\kappa_0, \operatorname{diam}(\partial\Omega)\}$, for $B = B(x,r)$,

$$(4.2)\qquad \sup_{(1/2)B} \mathcal{G}(Y) \lesssim \frac{1}{|B|}\iint_B \mathcal{G}(Y)\,\mathrm{d}Y \lesssim r\frac{\omega(CB)}{\sigma(CB)},$$

where all implicit (and explicit) constants depend only on dimension and the Ahlfors regularity constant.

Now, let $x \in \partial\Omega$ and

$$0 < r < \min\{\delta(X_0)/\kappa_0, 10^{-3}\operatorname{diam}(\partial\Omega), 10^{-3} r_K/C\},$$

where r_K is the doubling condition radius for ω and C is as in (4.2). Without loss of generality we may assume

$$r_K \ll \min\{\delta(X), \operatorname{diam}(\partial\Omega)\},$$

so that the above minimum equals $10^{-3} r_K/C$. Set $B := B(x,r)$ and $\Phi \in C_c^\infty(\frac{1}{2}B)$ such that $0 \le \Phi \le 1$, $\Phi \equiv 1$ on $\frac{1}{100}B$ and $|\nabla\Phi| \lesssim 8/r$. Using (4.1) with $X = X_0$[9],

[9]We may move X_0 slightly using the Harnack inequality.

we obtain

$$
\begin{aligned}
(4.3)\qquad r\omega\left(\frac{1}{100}B\right) &\le r\int_{\partial\Omega\cap(1/100)B}\Phi(y)\,\mathrm{d}\omega(y)\\
&= -r\iint_{\Omega}\nabla\mathcal{G}(Y)\nabla\Phi(Y)\,\mathrm{d}Y\\
&\le 8\iint_{\Omega\cap(1/2)B}|\nabla\mathcal{G}(Y)|\,\mathrm{d}Y\\
&\le 8\iint_{((1/2)B\cap\Omega)\setminus\Sigma_\rho(r)}|\nabla\mathcal{G}(Y)|\,\mathrm{d}Y\\
&\qquad + 8\iint_{(1/2)B\cap\Sigma_\rho(r)}|\nabla\mathcal{G}(Y)|\,\mathrm{d}Y\\
&= \mathcal{A}+\mathcal{B},
\end{aligned}
$$

where $\Sigma_\rho(r)$ is the "boundary strip," $\Sigma_\rho(r) := \{Y \in \Omega : \delta(Y) \le \rho r\}$, and $\rho > 0$ is a small number to be chosen momentarily. Let $\mathcal{W} = \{I\}$ be a Whitney decomposition of Ω, and let $\mathcal{I} := \{I \in \mathcal{W} : I \cap \frac{1}{2}B \cap \Sigma_\rho(r) \ne \varnothing\}$. Then, by using standard interior estimates (the Caccioppoli inequality and the Moser estimate), we have

$$
(4.4)\qquad \mathcal{B} \le 8\sum_{I\in\mathcal{I}}\iint_I |\nabla\mathcal{G}(Y)|\,\mathrm{d}Y \le C'\sum_{I\in\mathcal{I}}\ell(I)^{n-1}|\mathcal{G}(Y_I)|,
$$

where Y_I is the center of the Whitney cube I and $\ell(I)$ is the side length of I. For each $I \in \mathcal{I}$ we use the Hölder continuity at the boundary of the Green function (which only depends on dimension and the Ahlfors regularity constant), in conjunction with (4.2), to obtain the estimate

$$
\mathcal{G}(Y_I) \lesssim \left(\frac{\ell(I)}{r}\right)^{\alpha}\frac{1}{|2B|}\iint_{2B\cap\Omega}\mathcal{G}(Y)\,\mathrm{d}Y \lesssim \left(\frac{\ell(I)}{r}\right)^{\alpha} r\frac{\omega(CB)}{\sigma(CB)}.
$$

Summing over $I \in \mathcal{I}$, and using an elementary geometric argument, whose proof we temporarily postpone, we have that

$$
(4.5)\qquad \mathcal{B} \lesssim \rho^\alpha r\omega(CB) \lesssim \rho^\alpha r\omega\left(\frac{1}{100}B\right),
$$

where we used that the harmonic measure is doubling up to r_K.

Then, there exists $\rho > 0$ depending on C_0, n, and C (which depended additionally on C_A), small enough so that the upper bound in (4.5) can be absorbed in the lefthand side of (4.3), at which point we have

$$
\mathcal{A} = 8\iint_{((1/2)B\cap\Omega)\setminus\Sigma_\rho(r)}|\nabla\mathcal{G}(Y)|\,\mathrm{d}Y \ge \frac{1}{2}r\omega\left(\frac{1}{100}B\right) > 0.
$$

Since $\mathcal{A} > 0$, there exists a point $Y_B \in \frac{1}{2}B \cap \Omega$ such that $\delta(Y_B) > \rho r$, which shows that Ω satisfies the $(1/\rho, R_0)$-interior corkscrew condition, where

$$R_0 = \min\{\delta(X)/\kappa_0, 10^{-3}\operatorname{diam}(\partial\Omega), 10^{-3}r_K/C\} = 10^{-3}r_K/C =: s_K.$$

Hence, we finish the proof of the lemma with constant $\eta := 10^{-3}/C$.

Now, we shall sketch the "elementary geometric argument," that is, how we used the estimate on $\mathcal{G}(Y_I)$ and (4.4) in order to obtain (4.5). If we first let $\tilde{\mathcal{I}} := \left\{I \in \mathcal{W} : I \cap \frac{1}{2}B \neq \varnothing\right\}$, then we observe that the Whitney property of each $I \in \tilde{\mathcal{I}}$ ensures that $\ell(I) \lesssim r$, and for each $I \in \tilde{\mathcal{I}}$ there exists $\hat{x}_I$ in $B(x, Cr) \cap \partial\Omega$ such that $\ell(I) \approx \operatorname{dist}(I, \partial\Omega) \approx \operatorname{dist}(\hat{x}_I, Y)$ for all $Y \in I$.

Now, fix k such that $2^{-k} \lesssim \rho r$, denote $\tilde{\mathcal{I}}_k := \{I \in \tilde{\mathcal{I}} : \ell(I) = 2^{-k}\}$, and cover $B(x, Cr) \cap \partial\Omega$ by balls $\{B_{k,j}\}_j = \{B(x_{k,j}, 2^{-k})\}$ with $x_{k,j} \in \partial\Omega$ such that $\{\frac{1}{5}B_{k,j}\}_j$ are disjoint. Using Ahlfors regularity to compare surface areas, we see that, for each fixed k, we have $\#\{B_{k,j}\}_j \approx r^{n-1}2^{k(n-1)}$. Now, with each $I \in \tilde{\mathcal{I}}_k$ we associate an index j such that $x_I \in B_{k,j}$, and notice we have $\operatorname{dist}(Y, x_{k,j}) \lesssim 2^{-k}$ for all $Y \in I$. Since the $I \in \tilde{\mathcal{I}}_k$ are disjoint, comparing volumes, we have that for fixed j, we have that $\#\{I \in \tilde{\mathcal{I}}_k : I \text{ is associated to } j\} \leq C$, where C depends on dimension. It follows from our bound on $\#\{B_{k,j}\}_j$ that $\#\tilde{\mathcal{I}}_k \lesssim r^{n-1}2^{k(n-1)}$. Now, breaking the sum over k in (4.4) and using our bound for $\mathcal{G}(Y_I)$, we obtain

$$\begin{aligned}
\mathcal{B} &\lesssim \omega(CB)r^{2-n-\alpha} \sum_{k \gtrsim -\log_2(\rho r)} \sum_{I \in \tilde{\mathcal{I}}_k} 2^{-k(n-1+\alpha)} \\
&\lesssim \omega(CB)r^{2-n-\alpha} \sum_{k \gtrsim -\log_2(\rho r)} r^{n-1}2^{k(n-1)}2^{-k(n-1+\alpha)} \\
&\lesssim \rho^\alpha r\,\omega(CB)
\end{aligned}$$

as desired, where we used $\sigma(CB) \approx r^{n-1}$ in the first line. ◻

One immediate corollary is that domains with Ahlfors regular boundaries have uniformly rectifiable boundaries whenever the interior and exterior harmonic measures are doubling.

Corollary 4.4. *Suppose $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$ are domains with common topological boundary $\partial\Omega := \partial\Omega^+ = \partial\Omega^-$ and* $\operatorname{diam}(\partial\Omega^+) < \infty$, *which has the additional property that $\partial\Omega$ is $(n-1)$-Ahlfors regular. Suppose further there exists $X^+ \in \Omega^+$ and $X^- \in \Omega^-$ such that the harmonic measures $\omega_\pm^{X^\pm}$ are doubling. Then, $\partial\Omega$ is uniformly rectifiable and $\partial\Omega = \partial_*\Omega$. In particular, $\Omega^\pm$ are* UR *domains.*

4.2. A localization result. The major technical result of this section is Theorem 4.12, which, roughly, states that the local oscillation of the Poisson kernel controls the local oscillation of the unit normal. Perhaps, contrary to the spirit of a "localized result," the *scale* at which we get control of the oscillation of the unit normal depends on the compact set; however, the quantitative control does not (see (4.32) and (4.33)).

Our main tool in the proof of Theorem 4.12 is the single layer potential; we recall its definition now.

Definition 4.5 (Riesz transform and the single layer potential). Let $F \subset \mathbb{R}^n$ be a closed $(n-1)$-Ahlfors regular set with surface measure $\sigma = \mathcal{H}^{n-1} \llcorner F$. We define the (vector-valued) Riesz kernel as

$$\mathcal{K}(X) = \tilde{c}_n \frac{X}{|X|^n},$$

where $\tilde{c}_n$ is chosen so that

$$\mathcal{K}(X) = \nabla \frac{c_n}{|X|^{n-2}}$$

and c_n is such that $-\Delta c_n/|X|^{n-2} = \delta_0$ (here, δ_0 is the Dirac mass at the origin).

Let $f \in L^p(\mathrm{d}\sigma)$ for some $p \in [1, n-1)$. We define the Riesz transform of f (relative to F) to be

$$\mathcal{R}f(X) := \mathcal{K} * (f\sigma)(X) = \int_F \mathcal{K}(X-y) f(y)\,\mathrm{d}\sigma(y) \quad X \in \mathbb{R}^n \setminus F,$$

as well as the truncated Riesz transforms for $X \in F$ to be

$$\mathcal{R}_\varepsilon f(X) := \int_{F \cap \{y : |X-y| > \varepsilon\}} \mathcal{K}(X-y) f(y)\,\mathrm{d}\sigma(y), \quad \varepsilon > 0.$$

We define S the (harmonic) single layer potential of f relative to F to be

$$Sf(X) := \int_F \mathcal{E}(X-y) f(y)\,\mathrm{d}\sigma(y),$$

where $\mathcal{E}(X) = c_n |X|^{2-n}$.

Remark 4.6. For f as above we have that $Sf(X)$ makes sense as an absolutely convergent integral for $X \notin F$. To see this, we may use the upper Ahlfors regularity to break the boundary up into dyadic annuli centered at $x_0 \in F$ such that $\operatorname{dist}(X, F) = |X - x_0|$, and see that $\mathcal{E}(X-y)$ is in $L^{p'}(\mathrm{d}\sigma)$ for all $p \in [1, n-1)$, where p' is the Hölder conjugate exponent to p (albeit with bounds depending on X). Notice also that for such f, $\nabla Sf(X) = \mathcal{R}f(X)$ for $X \notin F$ and $\mathcal{R}f(X)$ makes sense as an absolutely convergent integral for $X \notin F$ (here, we use the same argument as for $\mathcal{E}$ to show that $\mathcal{K}(X-y) \in L^{p'}(\mathrm{d}\sigma)$ for $p \in [1, \infty)$). To see $\nabla Sf(X) = \mathcal{R}f(X)$ for $X \notin F$ we form the difference quotients for Sf and use the dominated convergence theorem. Every function we apply the single layer potential to in the proof of Theorem 4.12 is in the space $L^1(\mathrm{d}\sigma)$.

The singular layer potential is useful in that it generates solutions to the Neumann problem (see, e.g., [HMT10, Section 5.5]). However, in order to make sense of boundary data in a rough domain we need to introduce the concept of non-tangential regions.

Definition 4.7 (Nontangential approach region and maximal function). Fix $\alpha > 0$ and let Ω be a domain; then, for $x \in \partial\Omega$ we define the nontangential approach region (or "cone"):

$$\Gamma(x) = \Gamma_\alpha(x) = \{Y \in \Omega : |Y - x| < (1+\alpha)\delta(Y)\}.$$

We also define the nontangential maximal function for $u : \Omega \to \mathbb{R}$:

$$\mathcal{N}u(x) = \mathcal{N}_\alpha u(x) = \sup_{Y \in \Gamma_\alpha(x)} |u(Y)|, \quad x \in \partial\Omega.$$

We make the convention that $\mathcal{N}u(x) = 0$ when $\Gamma_\alpha(x) = \varnothing$[10] and that $\alpha = 1$ when no subscript appears in Γ.

The relationship between the two definitions above is made clear in the following two lemmas.

Lemma 4.8 ([HMT10]). *Suppose Ω is a* UR *domain (recall Definition 2.9) and $f \in L^q(\mathrm{d}\sigma)$ for some $q \in [1, n-1)$. For all $p \in (1,\infty)$, we have*

$$\|\mathcal{N}(\nabla \mathcal{S}f)\|_{L^p(\mathrm{d}\sigma)} \leq C\|f\|_{L^p(\mathrm{d}\sigma)}, \tag{4.6}$$

where C depends on the UR *character of $\partial\Omega$, dimension, p, and the aperture of the cones defining $\mathcal{N}$.*

The bound for the non-tangential maximal function of $\nabla\mathcal{S}f$ follows from uniform bounds for the truncated singular integrals [Dav91], plus a Cotlar lemma argument; the details may be found in [HMT10, Proposition 3.20].

In addition, we have the following result proved in [HMT10].

Lemma 4.9 ([HMT10] ***Proposition 3.30***). *If Ω is a* UR *domain, whose measure theoretic and topological boundary agree up to a set of $\mathcal{H}^{n-1}$ measure zero, then for almost every $x \in \partial\Omega$, and for all $f \in L^p(\mathrm{d}\sigma)$, $1 \leq p < n-1$,*

$$\lim_{Z \to x,\, Z \in \Gamma^-(x)} \nabla\mathcal{S}f(Z) = -\frac{1}{2}\nu(x)f(x) + \mathcal{T}f(x), \tag{4.7}$$

and

$$\lim_{Z \to x,\, Z \in \Gamma^+(x)} \nabla\mathcal{S}f(Z) = \frac{1}{2}\nu(x)f(x) + \mathcal{T}f(x). \tag{4.8}$$

[10] In the settings that are treated here, this is always a set of $\mathcal{H}^{n-1}$ measure zero [HMT10, Proposition 2.9].

where $\Gamma^+(x)$ *is the cone at* x *relative to* Ω, $\Gamma^-(x)$ *is the cone at* x *relative to* Ω_{ext}, ν *is the unit outer normal to* Ω, *and* $\mathcal{T}$ *is a (vector-valued) principal value singular integral operator:*

$$\mathcal{T}f(x) = \lim_{\varepsilon\to 0+}\int_{y\in\partial\Omega\setminus B(x,\varepsilon)} \nabla\mathcal{E}(x-y)f(y)\,\mathrm{d}\sigma(y).$$

Remark 4.10. As in [BH16], we have taken our fundamental solution to be positive, so for that reason there are some changes in sign in both (4.7) and (4.8) as compared to the formulation in [HMT10].

Next, we show that if $\log k$ has *small* BMO norm, the measure $\omega = k\,\mathrm{d}\sigma$ is doubling. The proof uses the fact that σ is doubling. We comment that in general, the fact that $\|\log k\|_{\mathrm{BMO}} < \infty$ or that k satisfies a reverse Hölder inequality does not ensure that $\omega = k\,\mathrm{d}\sigma$ is doubling (see the discussions and example in [ST89, Chapter I]).

Lemma 4.11. *Let* σ *be a doubling measure on* $\mathbb{R}^n$ *and* $\omega = k\,\mathrm{d}\sigma$ *be another Radon measure with* $0 \le k \in L^1_{\mathrm{loc}}(\mathrm{d}\sigma)$. *There exists* $\tau_0 > 0$ *depending on the doubling constant of* σ, *such that if*

$$\|\log k\|_*(B(x_0,4r_0)) < \tau \le \tau_0 \quad \textit{for some } x_0 \in \operatorname{spt}\sigma \textit{ and } r_0 > 0, \tag{4.9}$$

then the following holds for $B \subset B(x_0,2r_0)$ *with* B *a ball centered on* $\operatorname{spt}\sigma$*:*

(1) *There is a constant* C *depending on* n *such that*

$$\frac{1}{1+C\tau}\fint_B k\,\mathrm{d}\sigma \le e^{\fint_B \log k\,\mathrm{d}\sigma} \le \fint_B k\,\mathrm{d}\sigma = \frac{\omega(B)}{\sigma(B)}. \tag{4.10}$$

(2) *Given* $p > 1$, *there exists* $\tau(p) \le \tau_0$ *such that if* (4.9) *holds with* $\tau \le \tau(p)$, *then for any Borel set* $E \subset B$, *where* B *is as before,*

$$\frac{\omega(E)}{\omega(B)} \ge c(p,\tau)\left(\frac{\sigma(E)}{\sigma(B)}\right)^p. \tag{4.11}$$

Here, the constant $c(p,\tau) \to 1$ *as* $\tau \to 0$.

(3) *In particular, for* $x \in \operatorname{spt}\sigma$ *such that* $B(x,2r) \subset B(x_0,2r_0)$,

$$\omega(B(x,2r)) \le C\omega(B(x,r)), \tag{4.12}$$

where the constant C *depends on* n *and the doubling constant of* σ.

(4) *Given* $r > 1$, *there exists* $\bar\tau(r) \le \tau_0$ *such that if* (4.9) *holds with* $\tau \le \bar\tau(r)$, *then the weight* k *satisfies the reverse Hölder inequality for* r, *that is,*

$$\left(\fint_B k^r\,\mathrm{d}\sigma\right)^{1/r} \le C(r,\tau)\fint_B k\,\mathrm{d}\sigma. \tag{4.13}$$

Here, the constant $C(r,\tau) \to 1$ *as* $\tau \to 0$.

Proof. By the local version of John-Nirenberg inequality for doubling measures (see [ABKY11, Theorem 5.2]), we have

$$\sigma(\{x \in B : |\log k(x) - (\log k)_B| > \lambda\}) \leq C_1 e^{-C_2\lambda/\tau}\sigma(B)$$

for all $\lambda > 0$, where the constants C_1 and C_2 depend on the doubling constant for σ. Therefore,

$$\begin{aligned}
&\fint_B e^{|\log k - (\log k)_B|}\,\mathrm{d}\sigma \\
&\quad= \frac{1}{\sigma(B)}\int_0^\infty \sigma(\{x \in B : e^{|\log k(x) - (\log k)_B|} > s\})\,\mathrm{d}s \\
&\quad\leq \frac{1}{\sigma(B)}\int_0^1 \sigma(B)\,\mathrm{d}s \\
&\qquad\quad+ \frac{1}{\sigma(B)}\int_0^\infty \sigma(\{x \in B : |\log k(x) - (\log k)_B| > \lambda\})e^{\lambda}\,\mathrm{d}\lambda \\
&\quad\leq 1 + C_1\int_0^\infty e^{-(C_2/\tau)\lambda + \lambda}\,\mathrm{d}\lambda \\
&\quad\leq 1 + C\tau,
\end{aligned}$$

if τ is sufficiently small (depending on the constant C_2). Then, (4.10) follows immediately.

Similarly, provided τ is small enough depending on p, we also have

$$\fint_B e^{(1/(p-1))|\log k - (\log k)_B|}\,\mathrm{d}\sigma \leq 1 + C_p\tau. \tag{4.14}$$

Henceforth, $\tau_0 > 0$ is chosen so that (4.14) holds with $p = 2$ and $\tau \leq \tau_0$. Let $q = p/(p-1)$ be the Hölder conjugate of p. It follows that

$$\begin{aligned}
&\fint_B k\,\mathrm{d}\sigma \cdot \left(\fint_B k^{-q/p}\,\mathrm{d}\sigma\right)^{p/q} \\
&\qquad= \fint_B e^{\log k}\,\mathrm{d}\sigma \cdot \left(\fint_B e^{-(1/(p-1))\log k}\,\mathrm{d}\sigma\right)^{p-1} \\
&\qquad= \fint_B e^{\log k - (\log k)_B}\,\mathrm{d}\sigma \cdot \left(\fint_B e^{-(1/(p-1))(\log k - (\log k)_B)}\,\mathrm{d}\sigma\right)^{p-1} \\
&\qquad\leq \fint_B e^{|\log k - (\log k)_B|}\,\mathrm{d}\sigma \cdot \left(\fint_B e^{(1/(p-1))|\log k - (\log k)_B|}\,\mathrm{d}\sigma\right)^{p-1} \\
&\qquad\leq (1 + C_p\tau)^p,
\end{aligned}$$

that is, $k \in A_p(\sigma)$, where A_p is the Muckenhaupt class with power $p > 1$.

Let $g \geq 0$ be an arbitrary measurable function on B. We have

$$\begin{aligned}\int_B g \, \mathrm{d}\sigma &\leq \left(\int_B g^p k \, \mathrm{d}\sigma\right)^{1/p} \left(\int_B k^{-q/p} \, \mathrm{d}\sigma\right)^{1/q} \\ &\leq (1 + C_p \tau)\sigma(B) \left(\int_B k \, \mathrm{d}\sigma\right)^{-1/p} \left(\int_B g^p k \, \mathrm{d}\sigma\right)^{1/p}.\end{aligned}$$

In particular, for any Borel set $E \subset B$, by plugging in the above inequality $g = \chi_E$, we get

$$\frac{\sigma(E)}{\sigma(B)} \leq (1 + C_p \tau) \left(\frac{\omega(E)}{\omega(B)}\right)^{1/p},$$

or equivalently,

$$\frac{\omega(E)}{\omega(B)} \geq c(p, \tau) \left(\frac{\sigma(E)}{\sigma(B)}\right)^p$$

with $c(p, \tau) = 1/(1 + C_p \tau)^p$. The doubling property (4.12) follows by taking $E = \frac{1}{2}B$, $p = 2$, and $\tau = \tau_0$.

Let $r > 1$; then, (4.14) applied to $p = 1 + 1/r$ implies that for τ small enough depending on r we have

$$\fint_B k^r \, \mathrm{d}\sigma \leq (1 + C_r \tau) e^{r(\log k)_B}.$$

Taking r-th root on both sides of the inequality and using (4.10), we get

$$\left(\fint_B k^r \, \mathrm{d}\sigma\right)^{1/r} \leq (1 + C_r \tau)^{1/r} e^{(\log k)_B} \leq (1 + C_r \tau)^{1/r} \fint_B k \, \mathrm{d}\sigma,$$

that is, $k \in RH_r(\sigma)$, where RH_r denotes weight that satisfies the reverse Hölder inequality with power $r > 1$. ◻

After we establish the reverse Hölder inequality (4.13), one can show

$$\left(\fint_B \left|1 - \frac{k}{a}\right|^2 \mathrm{d}\sigma\right)^{1/2} \leq C(\|\log k\|_*(4B))^{1/8} \leq C\tau^{1/8}, \tag{4.15}$$

where $a = e^{\fint_B \log k \, \mathrm{d}\sigma}$. For details of the proof we refer interested readers to [BH16, Lemma 1.33].

The following result states that control on the oscillation of the logarithm of the interior and exterior Poisson kernels provides control on the oscillation of the unit normal.

Theorem 4.12. *Let $\Omega^+ \subset \mathbb{R}^n$, $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$ be domains with common (topological) boundary, $\partial\Omega^+ = \partial\Omega^- \equiv \partial\Omega$. Assume that $\partial\Omega$ is $(n-1)$-Ahlfors regular and let $X^\pm \in \Omega^\pm$ be such that $k^\pm = \mathrm{d}\omega^\pm/\mathrm{d}\sigma$ exist. Here, $\omega^\pm = \omega_\pm^{X^\pm}$, where $\omega_\pm^{X^\pm}$ is the harmonic measure for $\Omega^\pm$ with pole at $X^\pm$. Given $\varepsilon > 0$, there exists $\kappa_1 > 0$ depending on $\delta(X^\pm)$, ε, n, and the Ahlfors regularity constant C_A such that if $\log k^\pm \in \mathrm{BMO}_{\mathrm{loc}}(\sigma)$ with constant $0 < \kappa \leq \kappa_1$, then $\nu \in \mathrm{BMO}_{\mathrm{loc}}(\sigma)$ with constant at most ε. In particular, if $\log k^\pm \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$, then $\nu \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$.*

Remark 4.13. The proof of the above theorem yields a quantitative estimate (see (4.32) and (4.33)).

Proof. Let $A > 2$ be a constant depending on dimension and the Ahlfors regularity constant[11] such that if $x_0 \in \partial\Omega$ and $r_0 \in (0, \operatorname{diam}\partial\Omega)$, then there exists[12] a dyadic cube Q as in Lemma B.2 such that

$$\Delta(x_0, r_0/A) \subset Q \subset \Delta(x_0, r_0).$$

Let $\tau(p)$ be as in Lemma 4.11 such that (4.11) holds with power $p = 1 + 1/(2(n-1))$. Suppose that $\log k^\pm \in \mathrm{BMO}_{\mathrm{loc}}(\sigma)$ with $\mathrm{BMO}_{\mathrm{loc}}$ semi-norm κ satisfying $\kappa \in (0, \kappa_1)$, where $\kappa_1 \leq \tau(p)$ will be determined after (4.31). Notice that in the case when $\log k^\pm \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$, this holds for every $\kappa > 0$. Fix $B^* B(y_0, 4R)$ for some $y_0 \in \partial\Omega$ and $R \in (0, \operatorname{diam}(\partial\Omega)/4)$, and set $\tilde{B} = \frac{1}{4}\overline{B^*}$. Since $\log k^\pm \in \mathrm{BMO}_{\mathrm{loc}}(\sigma)$ with constant κ, there exists a radius $r_0 = r_0(\tau(p), B^*) < c\min\{R, \delta(X^\pm)\}$ (with $c > 0$ depending on dimension and Ahlfors dimension and Ahlfors regularity) such that

$$\|\log k\|_*(B(z_0, 2r_0)) < \kappa, \quad \forall\, z_0 \in B^* \cap \partial\Omega.$$

The proof of Lemma 4.11 establishes that $\omega^\pm$ are doubling[13] up to radius r_0 on balls centered on $B^* \cap \partial\Omega$, with a doubling constant depending on n and C_A. Moreover, by choice of c and Lemma 4.3, the domains $\Omega^\pm$ both admit an interior corkscrew ball for every $x \in B^* \cap \partial\Omega$ up to radius r_0. Thus, we record for later use that, in the language of Appendix B, Ω satisfies the (x_0, M_0, r_0)-DLTSCS[14] for all $x_0 \in \tilde{B}$.

Henceforth, x_0 will denote an arbitrary point in $\tilde{B} \cap \partial\Omega$. Let $1 < M < \infty$ and $\theta \in (0,1)$ be determined later. For $x \in B(x_0, r_0/(20A)) \cap \partial\Omega$, let $r \in (0, \theta r_0)$ be such that $\Delta := \Delta(x, r) \subset \Delta^* := \Delta(x, Mr) \subset B(x_0, r_0/(5A))$.

For any $y, z \in \Delta$, we let y^* and z^* denote arbitrary points in the non-tangential approach regions in Ω^-, $\Gamma^-(y) \cap B(y, r/2)$ and $\Gamma^-(z) \cap B(z, r/2)$,

[11] We use A to simplify notation. In fact, we take $A = C_3$ as in Lemma B.2 and as used in Lemma B.4.

[12] See Remark B.3.

[13] Here, we have uniform control on the doubling constant by Lemma 4.11 and the choice of κ_1.

[14] This is a local two-sided corkscrew condition; see Definition B.1.

respectively. Following [BH16, Theorem 1.1], we first show

$$
\begin{aligned}
(4.16)\quad &\left(\fint_{\Delta}\left|\nabla S 1_{\Delta^*}(z^*)-\fint_{\Delta}\nabla S 1_{\Delta^*}(y^*)\,\mathrm{d}\sigma(y)\right|^2\mathrm{d}\sigma(z)\right)^{1/2}\\
&\qquad\le \frac{C_1}{\omega(B(x_0,r_0/(5A)))}\cdot\left(\frac{r}{r_0}\right)^{1/2}\cdot\frac{1}{\sqrt{M}}+C_2M^{(n-1)/2}\kappa^{1/8}+\frac{C_3}{M},
\end{aligned}
$$

where ω is the harmonic measure of Ω^+ with pole X^+, and where the constants $C_1, C_2, C_3 > 0$ depend only on n, the Ahlfors regularity constant C_A, and $\delta(X^\pm)$. In particular, $\omega = k^+\,\mathrm{d}\sigma$. We decompose 1_{Δ^*} as

$$
1_{\Delta^*}=\left[\left(1-\frac{k^+}{a}\right)1_{\Delta^*}\right]+\left[\frac{k^+}{a}\right]-\left[\left(\frac{k^+}{a}\right)1_{(\Delta^*)^c}\right],
$$

where $a = a_{x,Mr} = e^{\fint_{\Delta^*}\log k^\pm\,\mathrm{d}\sigma}$. We want to estimate the lefthand side of (4.16) by using this decomposition and the triangle inequality. This gives three terms, which we denote as I, II, and III:

$$
\begin{aligned}
I=\Bigg(\fint_{\Delta}\Bigg|&\nabla S\left[\left(1-\frac{k}{a}\right)1_{\Delta^*}\right](z^*)\\
&-\fint_{\Delta}\nabla S\left[\left(1-\frac{k}{a}\right)1_{\Delta^*}\right](y^*)\,\mathrm{d}\sigma(y)\Bigg|^2\mathrm{d}\sigma(z)\Bigg)^{1/2},\\
II=\Bigg(\fint_{\Delta}\Bigg|&\nabla S\left[\frac{k}{a}\right](z^*)-\fint_{\Delta}\nabla S\left[\frac{k}{a}\right](y^*)\,\mathrm{d}\sigma(y)\Bigg|^2\mathrm{d}\sigma(z)\Bigg)^{1/2},
\end{aligned}
$$

and

$$
\begin{aligned}
III=\Bigg(\fint_{\Delta}\Bigg|&\nabla S\left[\left(\frac{k}{a}\right)1_{(\Delta^*)^c}\right](z^*)\\
&-\fint_{\Delta}\nabla S\left[\left(\frac{k}{a}\right)1_{(\Delta^*)^c}\right](y^*)\,\mathrm{d}\sigma(y)\Bigg|^2\mathrm{d}\sigma(z)\Bigg)^{1/2}.
\end{aligned}
$$

For simplicity, we drop the super-index and write $k = k^+$. We will leave the estimate of I for last, as it requires the use of the localization Lemma B.4.

For II, we recall that $k = k^+$ is the Poisson kernel for Ω with pole at X^+. Moreover, $\mathcal{E}(\cdot - z^*)$ and $\mathcal{E}(\cdot - y^*)$ are harmonic in Ω since $z^*, y^* \in \Omega^-$, and decay to 0 at infinity, and are therefore equal to their respective Poisson integrals in Ω. Consequently,

$$
(4.17)\quad II\le\frac{1}{a}\left(\fint_{\Delta}\fint_{\Delta}\left|\nabla\mathcal{E}(X^+-z^*)-\nabla\mathcal{E}(X^+-y^*)\,\mathrm{d}\sigma(y)\right|^2\mathrm{d}\sigma(z)\right)^{1/2}.
$$

Note that, since $y^*, z^* \in B(x, 2r)$ and $|X^+ - x| > r_0$,

$$
|\nabla\mathcal{E}(X^+-z^*)-\nabla\mathcal{E}(X^+-y^*)|\lesssim\frac{r}{r_0^n}.
$$

Then, continuing (4.17), we have, using (4.11) with power $p = 1 + \frac{1}{2(n-1)}$,

$$
\begin{aligned}
(4.18)\quad II &\lesssim \frac{1}{ar_0^n} r \approx \frac{\sigma(\Delta^*)}{r_0^n \omega(\Delta^*)} r = \frac{\sigma(\Delta^*)}{\omega(B(x_0, r_0/(5A)))} \frac{\omega(B(x_0, r_0/(5A)))}{r_0^n \omega(\Delta^*)} r \\
&\le \frac{C}{\omega(B(x_0, r_0/(5A)))} \left(\frac{Mr}{r_0}\right)^{n-1} \left(\frac{r_0}{Mr}\right)^{n-1/2} \frac{r}{r_0} \\
&\le \frac{C}{\omega(B(x_0, r_0/(5A)))} \left(\frac{r}{r_0}\right)^{1/2} \cdot \frac{1}{\sqrt{M}},
\end{aligned}
$$

where $C > 0$ depends on n and the Ahlfors regularity constant.

For III, we use basic Calderón-Zygmund type estimates as follows. Let

$$\Delta_j := \Delta(x, 2^j r), \quad A_j := \Delta_j \setminus \Delta_{j-1},$$

so that

$$
\begin{aligned}
(4.19)\quad III &= \left(\fint_\Delta \left| \fint_\Delta \left(\nabla S\left[\left(\frac{k}{a}\right) 1_{(\Delta^*)^c} \right](z^*) \right.\right.\right. \\
&\qquad \left.\left.\left. - \nabla S\left[\left(\frac{k}{a}\right) 1_{(\Delta^*)^c} \right](y^*) \right) d\sigma(y) \right|^2 d\sigma(z) \right)^{1/2} \\
&= \left(\fint_\Delta \left| \fint_\Delta \int_{\partial\Omega\setminus\Delta^*} \left[\nabla \mathcal{E}(z^* - w) - \nabla \mathcal{E}(y^* - w) \right] \right.\right. \\
&\qquad \left.\left. \times \frac{k(w)}{a} \, d\sigma(w)\, d\sigma(y) \right|^2 d\sigma(z) \right)^{1/2} \\
&\le \sum_{\{j | 2^j \ge M\}} \left(\fint_\Delta \left[\fint_\Delta \int_{A_j} |\nabla \mathcal{E}(z^* - w) - \nabla \mathcal{E}(y^* - w)| \right.\right. \\
&\qquad \left.\left. \times \frac{k(w)}{a} \, d\sigma(w)\, d\sigma(y) \right]^2 d\sigma(z) \right)^{1/2} \\
&\lesssim \sum_{\{j | 2^j \ge M\}} \left(\fint_\Delta \left[\fint_\Delta \int_{A_j} \frac{r}{(2^j r)^n} \frac{k(w)}{a} \, d\sigma(w)\, d\sigma(y) \right]^2 d\sigma(z) \right)^{1/2},
\end{aligned}
$$

where we understand that, if $\operatorname{diam}(\partial\Omega) < \infty$, the sums are finite and terminate for $2^j r \ge \operatorname{diam}(\partial\Omega)$.

$$
\begin{aligned}
(4.20)\quad III &\le \sum_{\{j | 2^j \ge M\}} \left(\fint_\Delta \left[\fint_\Delta \int_{A_j} \frac{r}{(2^j r)^n} \frac{k(w)}{a} \, d\sigma(w)\, d\sigma(y) \right]^2 d\sigma(z) \right)^{1/2} \\
&\lesssim \sum_{\{j | M \le 2^j \le r_0/(2r)\}} \frac{r\,\omega(A_j)}{(2^j r)^n a} + \sum_{\{j | 2^j \ge r_0/(2r)\}} \frac{r\,\omega(A_j)}{(2^j r)^n a} \\
&= III_a + III_b.
\end{aligned}
$$

To estimate III_a and III_b we use (4.10), the fact that $A_j \subset \Delta_j$ (in III_a), that ω is a probability measure (in III_b), and (4.11) again with $p = 1 + 1/(2(n-1))$:

$$\begin{aligned}
III_a &= \sum_{\{j \mid M \le 2^j \le r_0/(2r)\}} \frac{r\omega(A_j)}{(2^j r)^n a} \\
&\lesssim \sum_{\{j \mid M \le 2^j \le r_0/(2r)\}} \frac{r\omega(A_j)}{(2^j r)^n} \cdot \frac{\sigma(\Delta^*)}{\omega(\Delta^*)} \\
&\lesssim \sum_{\{j \mid M \le 2^j \le r_0/(2r)\}} \frac{r\sigma(\Delta^*)}{(2^j r)^n} \cdot \frac{\omega(\Delta_j)}{\omega(\Delta^*)} \\
&\lesssim \sum_{\{j \mid M \le 2^j \le r_0/(2r)\}} \frac{1}{M} \cdot \frac{(Mr)^n}{(2^j r)^n} \cdot \left(\frac{2^j r}{Mr}\right)^{n-1/2} \\
&\lesssim \frac{1}{\sqrt{M}} \sum_{\{j \mid M \le 2^j \le r_0/(2r)\}} 2^{-j/2} = \frac{C}{M},
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
III_b &= \sum_{\{j \mid 2^j \ge r_0/(2r)\}} \frac{r\omega(A_j)}{(2^j r)^n a} \\
&\lesssim \sum_{\{j \mid 2^j \ge r_0/(2r)\}} \frac{r\omega(A_j)}{(2^j r)^n} \cdot \frac{\sigma(\Delta^*)}{\omega(\Delta^*)} \\
&\lesssim \sum_{\{j \mid 2^j \ge r_0/(2r)\}} \frac{r}{(2^j r)^n} \cdot \frac{\sigma(\Delta^*)}{\omega(\Delta^*)} \\
&\lesssim \frac{r}{r_0^n} \cdot \frac{\sigma(\Delta^*)}{\omega(B(x_0, r_0/(5A)))} \cdot \frac{\omega(B(x_0, r_0/(5A)))}{\omega(\Delta^*)} \\
&\lesssim \frac{1}{M} \cdot \left(\frac{Mr}{r_0}\right)^n \cdot \frac{1}{\omega(B(x_0, r_0/(5A)))} \left(\frac{r_0}{Mr}\right)^{n-1/2} \\
&\le \frac{C}{\omega(B(x_0, r_0/(5A)))} \cdot \left(\frac{r}{r_0}\right)^{1/2} \cdot \frac{1}{\sqrt{M}}.
\end{aligned} \tag{4.22}$$

As before, the constant $C > 0$ in (4.21) and (4.22) depends only on n and the Ahlfors regularity constant. Combining (4.19), (4.20), (4.21), and (4.22), we conclude that

$$III \le \frac{C(n, C_A)}{M} + \frac{C(n, C_A)}{\omega(B(x_0, r_0/(5A)))} \cdot \left(\frac{r}{r_0}\right)^{1/2} \cdot \frac{1}{\sqrt{M}}. \tag{4.23}$$

The idea in estimating I is to approximate Ω, locally, by UR domains, so that we may exploit Lemmas 4.8 and 4.9 on those approximate domains. Using the fact that the (x_0, M_0, r_0)-DLTSCS holds, we may invoke Lemma B.4 to construct two UR "domains" $T_Q^\pm \subseteq \Omega^\pm$, where Q is a dyadic cube such that we have

$\Delta(x_0, r_0/(4A)) \subset Q \subset \Delta(x_0, r_0/4)$, where the definition of A above allows us to find such a cube. In particular,

$$\partial T_Q^{\pm} \cap \Delta(x_0, r_0/(4A)) = \Delta(x_0, r_0/(4A)),$$

and for $\mathcal{H}^{n-1}$-almost every $x \in \Delta(x_0, r_0/(4A))$, the unit outer normals $\nu_{T_Q^{\pm}}(x)$ exist and satisfy

$$\nu_{T_Q^{\pm}}(x) = \pm \nu_{\Omega^+}(x). \tag{4.24}$$

For any open set U with Ahlfors regular boundary, define

$$S_U f(X) := \int_{\partial U} \mathcal{E}(X - y) f(y) \, \mathrm{d}\sigma(y).$$

In our context, U is either $\Omega^{\pm}$ or $T_Q^{\pm}$. The coincidence of $\partial T_Q^{\pm} \cap \Delta(x_0, r_0/(4A))$ and $\Delta(x_0, r_0/(4A))$ allows us to conclude, for

$$f \in L^2(\Delta(x_0, r_0/(4A))) \quad \text{with } \operatorname{spt} f \subseteq \Delta(x_0, r_0/(4A)),$$

$$S_{\Omega^+} f(X) = S_{\Omega^-} f(X) = S_{T_Q^{\pm}} f(X), \tag{4.25}$$

for all $X \notin \Delta(x_0, r_0/(4A))$.

Recall

$$I = \left(\fint_{\Delta} \left| \nabla S\left[\left(1 - \frac{k}{a}\right) 1_{\Delta^*}\right](z^*) - \fint_{\Delta} \nabla S\left[\left(1 - \frac{k}{a}\right) 1_{\Delta^*}\right](y^*) \, \mathrm{d}\sigma(y) \right|^2 \mathrm{d}\sigma(z) \right)^{1/2},$$

where z^* and y^* are in non-tangential regions in Ω^- over $y, z \in \partial\Omega$. We want to dominate $\nabla S[(1 - k/a) 1_{\Delta^*}](z^*)$ by a non-tangential maximal function in T_Q^-. To this end, we make the observation that if r/r_0 is sufficiently small (which we may ensure by adjusting the value of θ), then for any $y \in \Delta$, the non-tangential cone $\Gamma^-(y) \cap B(y, r/2) \subset T_Q^-$, provided we take the constant K in the definition of $T_Q^{\pm}$ large enough depending on dimension and the Ahlfors regularity of $\partial\Omega$[15]. To see this, one needs to inspect the definition of $\mathcal{W}_Q$ (see Appendix B), and note that if $Z \in \Gamma^-(y) \cap B(y, 2r)$ then $\delta(Z) \sim |Z - y| < 2r$, and therefore Z is inside a Whitney cube I for Ω^- with

$$\operatorname{dist}(I, y) \sim \ell(I) \sim \delta(Z) < 2r \lesssim \ell(Q).$$

[15] This does not affect the validity of Lemma B.4.

By choosing K sufficiently large, depending on allowable parameters, we can guarantee the existence of a cube $Q' \subset Q$ containing $y \in Q'$ with length $\ell(Q') \approx_K \ell(I)$. Hence, $Z \in U^-_{Q'} \subset T^-_Q$. Moreover, in the construction of the Whitney region $U_{Q'}$, $\operatorname{int} I^* \subset U_{Q'}$ where $I^* = (1+\tau)I$ for some (small) parameter $\tau > 0$ (see Appendix B, and note this τ is unrelated to $\tau(p)$ above). This forces $\operatorname{dist}(Z, \partial T^-_Q) \gtrsim_\tau \ell(I) \sim |Z - y|$, and therefore,

$$Z \in \Gamma_{\beta, T^-_Q}(y) := \{Y \in T^-_Q : |Y - y| < (1+\beta)\operatorname{dist}(Y, \partial T^-_Q)\},$$

where $\beta = \beta(n, C_A, \theta) \gg_\tau 1$. We conclude that

$$\Gamma^-(y) \cap B(y, r/2) \subset \Gamma_{\beta, T^-_Q}(y) \cap B(y, r/2).$$

With these observations in hand, we can estimate I. By (4.6) and (4.15),

$$\begin{aligned} I &\le 2\Big(\fint_\Delta \Big| \tilde{\mathcal{N}}\Big(\nabla S_{T^-_Q}\Big[\Big(1 - \frac{k}{a}\Big) 1_{\Delta^*}\Big]\Big)\Big|^2 \,\mathrm{d}\sigma\Big)^{1/2} \\ &\le C\Big(\frac{\sigma(\Delta^*)}{\sigma(\Delta)}\Big)^{1/2} \Big(\fint_{\Delta^*} \Big|1 - \frac{k}{a}\Big|^2 \,\mathrm{d}\sigma\Big)^{1/2} \\ &\le CM^{(n-1)/2}(\|\log k\|_*(B(x_0, r_0)))^{1/8} \le CM^{(n-1)/2}\kappa^{1/8}, \end{aligned} \tag{4.26}$$

where $\tilde{\mathcal{N}}$ is the non-tangential maximal function in T^-_Q with aperture β (which dominates $S_{T^-_Q}[(1-k/a)1_{\Delta^*}](y^*)$ by the arguments in the preceding paragraph). Note that $C > 0$ above depends only on $\beta > 0$, n, C_A, and the UR constants of $\partial\Omega$, which in turn depend only on n, C_A, and $\delta(X^\pm)$.

Putting (4.18), (4.23), and (4.26) together, we finally obtain (4.16). The estimate analogous to (4.16) when y^* and z^* are in $\Gamma^+(y) \cap B(y, r/2)$ and $\Gamma^+(z) \cap B(z, r/2)$ is also true by symmetry. It remains to use the jump relations to get an estimate on the oscillation of unit outer normal. Here, we again use the approximations $T^\pm_Q$. Applying the jump relation in Lemma 4.9 to $T^\pm_Q$, and using (4.25), (4.24), and the containment $\Gamma^\pm(y) \cap B(y, r/2) \subset \Gamma_{\beta, T^\pm_Q}(y) \cap B(y, r/2)$, we obtain for $\mathcal{H}^{n-1}$ almost every $y \in \Delta(x_0, r_0/(4A))$

$$\nu_{\Omega^+}(y) 1_{\Delta^*}(y) = \lim_{\substack{Z \to y \\ Z \in \Gamma^+(y)}} \nabla S 1_{\Delta^*}(Z) - \lim_{\substack{Z \to y \\ Z \in \Gamma^-(y)}} \nabla S 1_{\Delta^*}(Z). \tag{4.27}$$

Here, we need to make the further observation that the principal value singular integral operators $\mathcal{T}_{T^\pm_Q}$[16] in (4.7) and (4.8) have the property that $\mathcal{T}_{T^+_Q} f = \mathcal{T}_{T^-_Q} f$ whenever $f \in L^2(\Delta(x_0, r_0/(4A)))$ with $\operatorname{spt} f \subseteq \Delta(x_0, r_0/(4A))$. This is a consequence of the definition of $\mathcal{T}$ and that

$$\partial T^+_Q \cap B(x_0, r_0/(4A)) = \partial T^-_Q \cap B(x_0, r_0/(4A)).$$

[16]The operator $\mathcal{T}_U$ is defined in the same way as S_U.

Taking nontangential limits[17] in (4.16), and using (4.27), we obtain

$$\left(\fint_{B(x,r)} \left| \nu_{\Omega^+}(y) - \fint_{B(x,r)} \nu_{\Omega^+}(z)\, d\sigma(z) \right|^2 d\sigma(y)\right)^{1/2} \tag{4.28}$$
$$\leq \frac{C_1}{\omega(B(x_0, r_0/(4A)))} \cdot \left(\frac{r}{r_0}\right)^{1/2} \cdot \frac{1}{\sqrt{M}} + C_2 M^{(n-1)/2} \kappa^{1/8} + \frac{C_3}{M},$$

for $x \in \partial\Omega \cap B(x_0, r_0/(20A))$ and $0 < r \leq \theta r_0$. Here, as above, the constants $C_1, C_3 > 0$ depend on n and C_A, and C_2 depends on n, C_A, and $\delta(X^\pm)$. Notice that we may apply the same argument to Ω^- and $\log k^-$ to get an analogous estimate to (4.28).

We define a constant

$$C_4 = \frac{C_1}{\inf\limits_{x_0 \in \tilde{B} \cap \partial\Omega} \omega^\pm(B(x_0, r_0/(5A)))}. \tag{4.29}$$

In fact, for each $x_0 \in \tilde{B} \cap \partial\Omega$, the harmonic measure $\omega^\pm(B(x_0, r_0/(5A))) > 0$ since $\sigma \ll \omega^\pm$. Consider an arbitrary pair $x_0, x_0' \in \tilde{B} \cap \partial\Omega$ such that $|x_0 - x_0'| < r_0/(5A)$. By the doubling property of $\omega^\pm$ (up to radius r_0), we have

$$\omega^\pm(B(x_0, r_0/(5A))) \leq \omega^\pm(B(x_0', r_0)) \leq C\omega^\pm(B(x_0', r_0/(5A))).$$

Since $\tilde{B} \cap \partial\Omega$ is compact, it can be covered by finitely many balls centered on $\tilde{B} \cap \partial\Omega$ with radii $r_0/(5A)$. In particular, the denominator in (4.29) is a strictly positive constant depending on the domains $\Omega^\pm$ and $\tilde{B}$, and thus the constant C_4 is well defined. Notice that the same argument applied to $\log k^-$ combined with (4.28) and (4.29) yields

$$\left(\fint_{B(x,r)} \left| \nu_{\Omega^\pm}(y) - \fint_{B(x,r)} \nu_{\Omega^\pm}(z)\, d\sigma(z) \right|^2 d\sigma(y)\right)^{1/2} \tag{4.30}$$
$$\leq C_4 \left(\frac{r}{r_0}\right)^{1/2} \cdot \frac{1}{\sqrt{M}} + C_2 M^{(n-1)/2} \kappa^{1/8} + \frac{C_3}{M},$$

where $C_4 = C_4(n, C_A, \tilde{B}, \Omega^\pm)$. For $\varepsilon > 0$ sufficiently small (satisfying $C_3 \varepsilon \leq 4$), we choose the constant M such that $1/\sqrt{M} = \varepsilon/4$ and $C_3/\sqrt{M} \leq 1$; we also choose the constant θ such that $M\theta < 1/(10A)$ and $C_4 \theta^{1/2} \leq 1$. Then, (4.30) becomes

$$\left(\fint_{B(x,r)} \left| \nu_{\Omega^\pm}(y) - \fint_{B(x,r)} \nu_{\Omega^\pm}(z)\, d\sigma(z) \right|^2 d\sigma(y)\right)^{1/2} \tag{4.31}$$
$$\leq \frac{\varepsilon}{2} + C_5 \varepsilon^{-(n-1)} \kappa^{1/8},$$

[17] This is justified by Lemma 4.8 and the dominated convergence theorem.

where C_5 depends on n and C_A. Note that in the above estimate, only θ depends on $\tilde{B}$. Thus, perhaps further shrinking κ_1 (depending on ε, n, C_A, and $\delta(X^\pm)$, and independent of $\tilde{B}$), (4.31) becomes

$$\begin{aligned}\left(\fint_{B(x,r)} \left| \nu_{\Omega^\pm}(y) - \fint_{B(x,r)} \nu_{\Omega^\pm}(z)\, \mathrm{d}\sigma(z) \right|^2 \mathrm{d}\sigma(y) \right)^{1/2} \\ \leq \frac{\varepsilon}{2} + C_5(n, C_A)\varepsilon^{-(n-1)}\kappa_1^{1/8} \leq \varepsilon.\end{aligned}$$

To sum up, we have shown that, given $\varepsilon > 0$ there exists a small constant κ_1 depending on ε, n, C_A, and $\delta(X^\pm)$ such that the following holds: for every ball B^* centered on the boundary with radius less than $\frac{1}{4}\operatorname{diam}(\partial\Omega)$, if there is a radius $r_0 = r_0(B^*)$ such that

$$\sup_{x_0 \in B^* \cap \partial\Omega} \|\log k^\pm\|_*(B(x_0, r_0)) \leq \kappa \leq \kappa_1, \tag{4.32}$$

then we can find $\theta \in (0,1)$ depending on n, C_A, the domains $\Omega^\pm$, and $\tilde{B} := \frac{1}{4}\overline{B^*}$ so that

$$\sup_{x_0 \in \tilde{B} \cap \in \partial\Omega} \|\nu\|_*(B(x_0, \theta r_0)) \leq \varepsilon. \tag{4.33}$$

Thus, $\nu \in \mathrm{BMO}_{\mathrm{loc}}(\sigma)$ with constant at most ε (see Remark 2.11). This concludes the proof of Theorem 4.12. ∎

4.3. Free boundary results. In this section we combine Theorem 4.12 with Corollaries 3.10 and 3.11 to obtain information about the local geometry of a domain (with minimal hypothesis) from the local oscillation of the logarithm of the interior and exterior Poisson kernels.

Theorem 1.1. *Let $n \geq 3$ and suppose $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$ are domains satisfying $\partial\Omega := \partial\Omega^+ = \partial\Omega^-$, and that $\partial\Omega$ is $(n-1)$-Ahlfors regular. Then, the following are equivalent:*

(i) *$\Omega^\pm$ are both vanishing chord-arc domains with $\nu \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$ (see Definition 2.19).*

(ii) *There exist $X^+ \in \Omega^+$ and $X^- \in \Omega^-$ such that $k^+ = \mathrm{d}\omega_+^{X^+}/\mathrm{d}\sigma$ and $k^- = \mathrm{d}\omega_-^{X^-}/\mathrm{d}\sigma$ exist and $\log k^\pm \in \mathrm{VMO}_{\mathrm{loc}}(\mathrm{d}\sigma)$.*

Proof. (i) implies (ii) is the main theorem in [KT03]. That (ii) implies (i) follows from Theorem 4.12. Indeed, by Corollary 3.11, to show that $\Omega^\pm$ are vanishing chord-arc domains it suffices to prove that $\nu \in \mathrm{VMO}_{\mathrm{loc}}(\mathrm{d}\sigma)$. Theorem 4.12 asserts this is the case when $\log k^\pm \in \mathrm{VMO}_{\mathrm{loc}}(d\sigma)$. ∎

The following is a quantified version of Theorem 1.1 which results from the remark at the end of the proof of Theorem 4.12.

Theorem 4.14 (Quantified version of Theorem 1.1). *Let $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$ be domains with common (topological) boundary $\partial\Omega = \partial\Omega^+ = \partial\Omega^-$. Assume that $\partial\Omega$ is $(n-1)$-Ahlfors regular, and let $X^\pm \in \Omega^\pm$ be such that $k^\pm = \mathrm{d}\omega_\pm^{X^\pm}/\mathrm{d}\sigma$ exist. Given $\delta > 0$ there exists $\kappa = \kappa(\delta, n, C_A, \delta(X^\pm)) > 0$ such that if $\log k^\pm \in \mathrm{BMO}_{\mathrm{loc}}(\sigma)$ with constant less than κ, then Ω^+ and Ω^- are δ-chord-arc domains.*

Conversely, for every $\kappa > 0$ there exists $\delta = \delta(\eta, n, C_A) > 0$ if $\nu \in \mathrm{BMO}_{\mathrm{loc}}(\sigma)$ with constant less than δ; then, $\log k^\pm \in \mathrm{BMO}_{\mathrm{loc}}(\sigma)$ with constant less than κ.

Proof. This is a combination of Theorem 4.12, Corollary 3.10, and the work in [KT99]. □

Appendix A. Proof of Theorem 3.9

In this section we prove Theorem 3.9. Recall, roughly speaking, that Theorem 3.9 says small excess implies flatness in the sense of Reifenberg. We will show Theorem 3.9 as a corollary of the height bound, Theorem A.2. Many of the techniques, included for completeness, are standard. Another consequence of Theorem A.2 is a Lipschitz approximation theorem, Theorem A.4, which is proven at the end of this section. It is of independent interest and is not used in this paper.

The next lemma is contained in [Mag12, Lemma 22.11]. Here, we recall some notation introduced in other sections. We define $q(x) = \langle x, e_n\rangle$, $p(x) = x - q(x)e_n$, $C_r = \{|q(x)| < r\} \cap \{|p(x)| < r\}$, $D_r = p(C_r)$ and $D = p(C_1)$. We consider D, D_r to be subsets of $\mathbb{R}^{n-1}$. Finally, when the set E is clear from context, recall $e_n(x,r) = e(E, x, r, e_n)$ and if $x = 0$, $e_n(r) = e(E, 0, r, e_n)$.

Lemma A.1 (Excess Measure). *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in $\mathbb{R}^n$ with $0 \in \partial E$, such that for some $t_0 \in (0,1)$, (3.11), (3.12), and (3.13) are each satisfied with $r = 1$ and $\nu = e_n$, then writing $M = C_1 \cap \partial^* E$ it follows that, for any Borel $G \subset D$,*

$$\mathcal{H}^{n-1}(G) = \int_{M \cap p^{-1}(G)} \langle \nu_E, e_n\rangle \, \mathrm{d}\mathcal{H}^{n-1}.$$

Moreover, for every $\varphi \in C_c^0(D)$ and $t \in (-1,1)$,

$$\int_D \varphi \, \mathrm{d}x = \int_M \varphi(p(x))\langle \nu_E(x), e_n\rangle \, \mathrm{d}\mathcal{H}^{n-1}$$

and

$$\int_{E_t \cap D} \varphi \, \mathrm{d}x = \int_{M \cap \{q(x) > t\}} \varphi(p(x))\langle \nu_E(x), e_n\rangle \, \mathrm{d}\mathcal{H}^{n-1} \qquad \forall\, t \in (-1,1)$$

where $E_t = \{z \in \mathbb{R}^{n-1} \mid (z,t) \in E\}$. In fact, the set function

$$\zeta(G) = \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G)$$

defines a Radon measure in D, and is called the excess measure of E over D since $\zeta(D) = e(E, 0, 1, e_n)$.

Theorem A.2 (***Height bound: cf.*** [Mag12, Theorem 22.8]). *Given $C_A \geq 1$, $r_0 > 0$ and $n \geq 2$, there exist constants $\varepsilon_1 = \varepsilon(n, C_A) > 0$ and $C_1 = C(n, C_A) \geq 1$ such that if $E \subset \mathbb{R}^n$ is Ahlfors regular with constant C_A up to scale $4r_0$ and $x_0 \in \partial E$ satisfies*

$$e_n(x_0, 4r_0) \leq \varepsilon_1,$$

then

$$\frac{1}{r_0} \sup\{|q(x_0) - q(y)| : y \in C(x_0, r_0, e_n) \cap \partial E\} \leq C_1 e_n(x_0, 4r_0)^{1/(2(n-1))}.$$

Proof. By Remark 3.5, we let $x_0 = 0$ and $2r_0 = 1$. We then want to show that $|q(x)| \leq c_0(n) e_n(2)^{1/(2(n-1))}$ whenever $x \in C_{1/2} \cap \partial E$.

First, assume $\varepsilon_1 \leq \min\{\omega(n, \frac{1}{4}, C_A), 2^{-n}\mathcal{H}^{n-1}(D)\}$, with $\omega(n, \frac{1}{4}, C_A)$ from Lemma 3.8. Then, by Lemma 3.8, $|q(x)| \leq \frac{1}{4}$ whenever $x \in C_1 \cap \partial^* E =: M$, and moreover E satisfies the hypotheses of Lemma A.1 with $t_0 = \frac{1}{4}$. Therefore,

$$0 \leq \mathcal{H}^{n-1}(M) - \mathcal{H}^{n-1}(D) \leq e_n(1) \leq 2^{n-1} e_n(2) \tag{A.1}$$

and

$$0 \leq \mathcal{H}^{n-1}(M \cap \{q(x) > t\}) - \mathcal{H}^{n-1}(E_t \cap D) \leq 2^{n-1} e_n(2) \quad \forall\, t \in (-1, 1). \tag{A.2}$$

Now, we consider $f : (-1, 1) \to [0, \mathcal{H}^{n-1}(M)]$ defined by

$$f(t) = \mathcal{H}^{n-1}(M \cap \{q(x) > t\}).$$

By Lemma 3.8

$$f(t) = \begin{cases} \mathcal{H}^{n-1}(M) & -1 < t < -\dfrac{1}{4}, \\ 0 & \dfrac{1}{4} < t < 1. \end{cases}$$

Since f is decreasing and right continuous, there exists $|t_0| < \frac{1}{4}$ such that

$$\begin{cases} f(t) \leq \dfrac{\mathcal{H}^{n-1}(M)}{2} & t \geq t_0, \\ f(t) > \dfrac{\mathcal{H}^{n-1}(M)}{2} & t < t_0. \end{cases} \tag{A.3}$$

Claim A.3. *If $x \in C_{1/2} \cap \partial E$, then $|q(x) - t_0| \leq c(n) e_n(2)^{1/(2(n-1))}$. In particular, since $0 \in \partial E$, this ensures $|t_0| \leq c(n) e_n(2)^{1/(2(n-1))}$.*

The claim will be verified by showing that $q(x) - t_0 \le c(n)e_n(2)^{1/(2(n-1))}$, then considering $\mathbb{R}^n \setminus E$ to get $|q(x) - t_0| \le c(n)e_n(2)^{1/(2(n-1))}$. Since $\partial E = \operatorname{spt} \mu_E = \overline{\partial^* E}$ and the projection function q is continuous, it suffices to prove the estimate for $x \in C_{1/2} \cap \partial^* E$. To bound $q(x) - t_0$, we first show there exists t_1 with $q(x) - t_1 \le c(n)e_n(2)^{1/(2(n-1))}$ and then that $t_1 - t_0$ satisfies a similar upper-bound.

By choice of ε_1,

$$\sqrt{e_n(2)} < \frac{1}{2C_A} \le \frac{\mathcal{H}^{n-1}(M)}{2}.$$

So, we choose $t_1 \in (t_0, \frac{1}{4})$ such that

$$\begin{cases} f(t) \le \sqrt{e_n(2)} & \forall\, t \ge t_1, \\ f(t) > \sqrt{e_n(2)} & \forall\, t < t_1. \end{cases} \tag{A.4}$$

To see $q(x) - t_1 \le c(n)e_n(2)^{1/(2(n-1))}$ for all $x \in C_{1/2} \cap \partial^* E$, note if $y \in C_{1/2} \cap \partial^* E$ and $q(y) > t_1$, then $q(y) - t_1 < \frac{1}{2}$ since $t_1 \in (t_0, \frac{1}{4})$ and $|q(y)| < \frac{1}{4}$. In particular, $(q(y) - t_1)$ is a small enough scale for Ahlfors-regularity to hold. Hence,

$$C_A^{-1}(q(y) - t_1)^{n-1} \le |\mu_E|(B(y, q(y) - t_1)). \tag{A.5}$$

Since $x \in B(y, q(y) - t_1)$ implies $q(y) - q(x) \le |x - y| < q(y) - t_1$ and since $y \in C_{1/2}$ with $q(y) - t_1 < \frac{1}{2}$,

$$B(y, q(y) - t_1) \subset \{x \in C_1 \mid q(x) > t_1\}. \tag{A.6}$$

Thus, $B(y, q(y) - t_1) \cap \partial^* E \subset M \cap \{q > t_1\}$. So, (A.5) and (A.6) imply

$$\begin{aligned} C_A^{-1}(q(y) - t_1)^{n-1} &\le |\mu_E|(C_1 \cap \{q(x) > t_1\}) \\ &= \mathcal{H}^{n-1}(M \cap \{q(x) > t_1\}) = f(t_1). \end{aligned}$$

By the choice of t_1 in (A.4), under the standing assumption $q(y) - t_1 > 0$ we have

$$q(y) - t_1 \le c(n, C_A)e_n(2)^{1/(2(n-1))}, \tag{A.7}$$

as desired. Note, (A.7) is trivially true when $q(y) \le t_1$.

Next, we show that $t_1 - t_0 \le c_n e_n(2)^{1/(2(n-1))}$, which verifies Claim A.3. We will use a slicing result (see [Mag12, Theorem 18.11]) which ensures that for almost every $t \in (-1, 1)$,

$$\mathcal{H}^{n-2}((\partial^* E_t) \Delta (\partial^* E)_t) = 0,$$

where

$$(\partial^* E)_t = \{z \in \mathbb{R}^{n-1} : (z,t) \in \partial^* E\} \subset \mathbb{R}^{n-1}$$
$$E_t = \{z \in \mathbb{R}^{n-1} \mid (z,t) \in E\} \subset \mathbb{R}^{n-1}.$$

Furthermore, the co-area formula ensures that for any $g : \mathbb{R}^n \to [0,\infty]$ a non-negative Borel function,

$$\int_{\partial^* E} g\sqrt{1 - \langle \nu_E, e_n\rangle^2}\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{R}} \Big(\int_{(\partial^* E)_t} g\,\mathrm{d}\mathcal{H}^{n-2}\Big)\,\mathrm{d}t.$$

In particular, realizing the square-root term on the left is just the Jacobian of the projection p, and choosing the function $g = \chi_{C_1}$, recalling that $C_1 \cap \partial^* E \supset M$ is Ahlfors regular up to scale 2,

$$\begin{aligned}
\int_{-1}^{1} &\mathcal{H}^{n-2}\Big((\partial^* E)_t \cap D\Big)\,\mathrm{d}t \\
&= \int_M \sqrt{1 - \langle \nu_E, e_n\rangle^2}\,\mathrm{d}\mathcal{H}^{n-1} \\
&\le (2\mathcal{H}^{n-1}(M))^{1/2}\Big(\int_M (1 - \langle \nu_E, e_n\rangle)\,\mathrm{d}\mathcal{H}^{n-1}\Big)^{1/2} \\
&\le c(n, C_A)\sqrt{e_n(2)}.
\end{aligned}$$

We extract from the above that

$$\int_{t_0}^{1} \mathcal{H}^{n-2}(\partial^* E_t \cap D)\,\mathrm{d}t \le \int_{-1}^{1} \mathcal{H}^{n-2}(\partial^* E_t \cap D)\,\mathrm{d}t \le c(n)\sqrt{e_n(2)}. \tag{A.8}$$

For almost all $t \in [t_0, 1)$ it follows from

$$\mathcal{H}^{n-1}(E_t \cap D) \le \mathcal{H}^{n-1}(M \cap \{q(x) > t\}),$$

(A.1), (A.2), and (A.3) that

$$\mathcal{H}^{n-1}(E_t \cap D) \le \frac{\mathcal{H}^{n-1}(M)}{2} \le \frac{\mathcal{H}^{n-1}(D)}{2} + 2^{n-2}e_n(2) \le \frac{3}{4}\mathcal{H}^{n-1}(D),$$

where we used that $e_n(2) \le 2^{-n}\mathcal{H}^{n-1}(D)$.

Applying the relative isoperimetric inequality (see [Mag12, (12.45)]) in $\mathbb{R}^{n-1}$ to the set $E_t \cap D$, we have

$$\mathcal{H}^{n-2}(D \cap \partial^* E_t) \ge c(n)\mathcal{H}^{n-1}(E_t \cap D)^{(n-2)/(n-1)} \quad \text{for almost every } t \in [t_0, 1). \tag{A.9}$$

Then, (A.8) and (A.9) together imply (where the constant $c(n)$ can change in every instance, but only depends on n)

$$\text{(A.10)} \qquad \int_{t_0}^{t_1} \mathcal{H}^{n-1}(E_t \cap D)^{(n-2)/(n-1)}\,\mathrm{d}t \\ \leq c(n)\int_{t_0}^{1} \mathcal{H}^{n-1}(E_t \cap D)^{(n-2)/(n-1)}\,\mathrm{d}t \leq c(n)\sqrt{e_n(2)}.$$

Finally, (A.2) and (A.4) yield, for $t < t_1$,

$$\mathcal{H}^{n-1}(E_t \cap D) \geq \mathcal{H}^{n-1}(M \cap \{q(x) > t\}) - 2^{n-1}e_n(2) \\ \geq \sqrt{e_n(2)} - 2^{n-1}e_n(2) \geq c(n)\sqrt{e_n(2)},$$

which combined with (A.10) ensures

$$(t_1 - t_0)e_n(2)^{(n-1)/(2(n-1))-1/(2(n-1))} \\ = (t_1 - t_0)\sqrt{e_n(2)}^{(n-2)/(n-1)} \leq c(n)\sqrt{e_n(2)},$$

so that $t_1 - t_0 \leq c(n)e_n(2)^{1/(2(n-1))}$ as desired. ∎

We are now ready to prove Theorem 3.9 which first appears in Section 3 above. We restate it here for convenience.

Theorem. *Fix $C_A \geq 1$, $r_0 > 0$, and $n \geq 2$. Let $\varepsilon_1 = \varepsilon(C_A, n) > 0$ be as in Theorem A.2. If $E \in \mathcal{A}(C_A, 4r_0)$ and $x_0 \in \partial E$ satisfies*

$$e(E, x_0, 2r, \nu) \leq \varepsilon_1$$

for some $\nu \in \mathbb{S}^n$ and $0 < r < 2r_0$, then

$$\text{(A.11)} \qquad \Big\{x \in C(x_0, r, \nu) \cap E \mid \\ \langle x - x_0, \nu\rangle > rC_1 e(E, x_0, 2r, \nu)^{1/(2(n-1))}\Big\} = \varnothing$$

and

$$\text{(A.12)} \qquad \Big\{x \in C(x_0, r, \nu) \cap E^c \mid \\ \langle x - x_0, \nu\rangle < -rC_1 e(E, x_0, 2r, \nu)^{1/(2(n-1))}\Big\} = \varnothing.$$

Proof of Theorem 3.9. We will verify (A.11), and (A.12) follows similarly. By translation and rotation, without loss of generality we suppose $x_0 = 0$ and $\nu = e_n$.

Suppose (A.11) fails. Then, there exists

$$x \in C_r \cap E \quad \text{with } q(x) > rC_1 e_n(2r)^{1/(2(n-1))}.$$

However, $\varepsilon_1 \le \omega(n, \frac{1}{4}, C_A)$ guarantees that (3.12) holds with $t_0 = \frac{1}{4}$. However, (3.12) guarantees that there exists some $y \in C_r \cap E^c$ with $q(x) < q(y) < r$. But then, there exists $z \in \partial E$ which lies on the line segment connecting x and y. In particular, $q(z) > q(x) > rC_1 e_n(2r)^{1/(2(n-1))}$ contradicting Theorem A.2. ◻

The following theorem is another consequence of the height bound, Theorem A.2. Hereafter, ∇' denotes the gradient in $\mathbb{R}^{n-1}$.

Theorem A.4 (***Lipschitz function approximation: cf. Theorem 23.7 in*** [Mag12]). *There exist positive $C_3 = C(n, C_A)$, $\varepsilon_3 = \varepsilon(n, C_A)$, $\delta_0 = \delta(n, C_A)$, and $L = L(n, C_A) < 1$ with the following properties. If $E \in \mathcal{A}(C_A, 13r)$ and $e_n(x_0, 13r) \le \varepsilon_3$ with $x_0 \in \partial E$, then for $M = C(x_0, r) \cap \partial E$ and for $M_0 = \{y \in M \mid \sup_{0<s<8r} e_n(y, s) < \delta_0\}$ there is $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with* $\mathrm{Lip}(u) \le L$ *and*

$$\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r} \le C_3 e_n(x_0, 13r)^{1/(2(n-1))}$$

such that $M_0 \subset M \cap \Gamma$ where $\Gamma = x_0 + \{(z, u(z)) \mid z \in D_r\}$. Furthermore,

$$\frac{\mathcal{H}^{n-1}(M \Delta \Gamma)}{r^{n-1}} \le C_3 e_n(x_0, 13r),$$
$$\frac{1}{r^{n-1}} \int_{D_r} |\nabla' u|^2 \le C_3 e_n(x_0, 13r),$$

and

$$\text{(A.13)} \qquad \begin{aligned} &\mathrm{dist}(x, (p(x), u(p(x)))) \\ &\quad = |q(x) - u(p(x))| \le 2L\, \mathrm{dist}(p(x), p(M_0)) \quad \forall\, x \in M. \end{aligned}$$

In fact, (A.13) *ensures there exist Lipschitz functions $u_\pm$ defined by*

$$\text{(A.14)} \qquad u_+(x) = \begin{cases} u(x) & x \in p(M_0), \\ \inf_{y \in p(M_0)} u(y) + L|x - y| & x \in D \setminus p(M_0), \end{cases}$$

$$\text{(A.15)} \qquad u_-(x) = \begin{cases} u(x) & x \in p(M_0), \\ \sup_{y \in p(M_0)} u(y) - L|x - y| & x \in D \setminus p(M_0), \end{cases}$$

with the property that

$$\text{(A.16)} \qquad u_-(p(x)) \le q(x) \le u_+(p(x)) \quad \forall\, x \in M.$$

Proof. Step 1: Up to replacing E with $E_{x_0,r}$ and correspondingly replacing u with $u_r(z) = r^{-1}u(rz)$, we can reduce to proving that if $E \in \mathcal{A}(C_A, 13)$ with $0 \in \partial E$, if

$$\text{(A.17)} \qquad M = C \cap \partial E, \quad M_0 = \{y \in M \mid \sup_{0<s<8} e_n(y, s) < \delta_0(n, C_A)\},$$

and if $e_n(0,13) \le \varepsilon_3$, then there exists a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\mathrm{Lip}(u) \le L < 1$ such that

$$\sup_{\mathbb{R}^{n-1}} |u| \le C_3 e_n(0,13)^{1/(2(n-1))}, \tag{A.18}$$

such that $M_0 \subset M \cap \Gamma$ where

$$\Gamma = \{(z, u(z)) \mid z \in D\}. \tag{A.19}$$

Furthermore,

$$\mathcal{H}^{n-1}(M \Delta \Gamma) \le C_3 e_n(0,13) \tag{A.20}$$

and

$$\int_D |\nabla' u|^2 \le C_3 e_n(0,13). \tag{A.21}$$

By Theorem A.2 it follows that

$$\sup\{|q(x)| : x \in C_2 \cap \partial E\} \le C_1 e_n(0,13)^{1/(2(n-1))}. \tag{A.22}$$

By choosing $\varepsilon_3 \le \varepsilon_1 \le \omega(n, \frac{1}{4}, C_A)$, E satisfies the hypotheses of Lemma 3.8. Consequently, Lemma A.1 and (3.8) imply

$$0 \le \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G) \le e_n(0,1) \le 13^{n-1} e_n(0,13),$$

for every Borel set $G \subset D$. Meanwhile, Theorem 3.9 ensures

$$\left\{x \in C_2 \mid q(x) < -\frac{1}{4}\right\} \subset C_2 \cap E \subset \left\{x \in C_2 \mid q(x) < \frac{1}{4}\right\}.$$

Step 2: We show that M_0 is contained in the graph of a Lipschitz function u, satisfying (A.18) and (A.20). In order to create the Lipschitz function, we first need to know M_0 is non-empty. This follows from a covering argument done later in more detail in (A.27).

Define $\|\cdot\| = \max\{|p(\cdot)|, |q(\cdot)|\}$. Then, $C(y,s) = \{z \in \mathbb{R}^n \mid \|z - y\| < s\}$. For fixed $y \in M_0$ and $x \in M$, consider $F = E_{y, \|x-y\|}$. Notably, $\|x - y\| < 2$. Since $y \in M_0$ and $4\|x - y\| < 8$ it follows from (3.7) and (A.17) that

$$e_n(F, 0, 4) = e_n(E, y, 4\|x - y\|) \le \delta_0.$$

Thus, choosing $\delta_0 \le \varepsilon_1$ allows us to apply Theorem A.2 to $F \in \mathcal{A}(C_A, 4)$ and conclude that

$$\sup\{|q(w)| : w \in C \cap \partial F\} \le C_1 e_n(F, 0, 4)^{1/(2(n-1))} \le C_1 \delta_0^{1/(2(n-1))}.$$

Applying this height bound to the specific point $w = (x - y)/\|x - y\|$ we find

$$|q(x) - q(y)| \le C_0(n)\delta_0^{1/(2(n-1))}\|y - x\|. \tag{A.23}$$

If we now define $L = C_1\delta_0^{1/(2(n-1))}$ and choose δ_0 so small that $L < 1$, it follows from (A.23) that

$$|q(x) - q(y)| < \|x - y\|,$$

which ensures that $\|x - y\| = |p(x) - p(y)|$, and hence (A.23) can be written

$$|q(x) - q(y)| \le L|p(x) - p(y)|, \quad \forall y \in M_0,\ x \in M, \tag{A.24}$$

which implies that $p|_{M_0}$ is invertible. Define $u : p(M_0) \to \mathbb{R}$ such that $u(p(x)) = q(x)$ for every $x \in M_0$. Evidently, (A.24) ensures u satisfies

$$|u(p(x)) - u(p(y))| \le L|p(x) - p(y)|, \quad \forall x, y \in M_0.$$

Since $M_0 \subset M$, it follows from (A.22) that

$$|u(p(x))| = |q(x)| \le C_1 e_n(0, 13)^{1/(2(n-1))}, \quad \forall x \in M_0. \tag{A.25}$$

Via Kirzbraun's theorem and truncation we extend u from $p(M_0)$ to $\mathbb{R}^{n-1}$ with Lipschitz constant $L < 1$ such that the L^∞-bound from (A.25) holds on all of $\mathbb{R}^{n-1}$, which verifies (A.18). The definition of u on $p(M_0)$ guarantees $M_0 \subset M \cap \Gamma$ where Γ is as in (A.19).

Next, we show (A.20). By definition of M_0, for every $y \in M \setminus M_0$ there exists $s_y \in (0, 8)$ with

$$\delta_0 s_y^{n-1} < \int_{C(y,s_y)\cap\partial E} \frac{|\nu_E - e_n|^2}{2}\, \mathrm{d}\mathcal{H}^{n-1}. \tag{A.26}$$

Let $\mathcal{F}$ be the set of all balls $B(y_k, \sqrt{2}s_k)$ centered on $M \setminus M_0$ satisfying (A.26) of radius at most $8\sqrt{2}$. Each ball is contained in $C_{1+8\sqrt{2}} \subset C_{13}$. By Besicovitch's covering theorem (see [EG92, Theorem 2, Section 1.5.2]) we partition $\mathcal{F}$ into N_n disjoint families of balls $\mathcal{G}_j$. Then, there exists j such that

$$\begin{aligned}
\mathcal{H}^{n-1}(M \setminus M_0) &\le N_n \sum_{B(y_k,s_k)\in\mathcal{G}_j} \mathcal{H}^{n-1}((M \setminus M_0) \cap B(y_k, \sqrt{2}s_k)) \\
&\le N_n \sum_{k\in\mathbb{N}} \mathcal{H}^{n-1}(M \cap B(y_k, \sqrt{2}s_k)) \\
&\le N_n C_A 2^{(n-1)/2} \sum_{k\in\mathbb{N}} s_k^{n-1}.
\end{aligned}$$

Since $C(y_k, s_k, e_n) \subset B(y_k, \sqrt{2}s_k)$, the family of cylinders are also mutually disjoint. Thus, (A.26) combined with the preceding computation yields

$$\begin{aligned}
\mathcal{H}^{n-1}(M \setminus M_0) &\le C \sum_{k\in\mathbb{N}} s_k^{n-1} \\
&\le \frac{C}{\delta_0} \sum_k \int_{C(y_k,s_k)} \frac{|\nu_E - e_n|^2}{2} \, \mathrm{d}\mathcal{H}^{n-1} \\
&\le \frac{C}{\delta_0} e_n(0, 13).
\end{aligned} \tag{A.27}$$

Keeping in mind that $\delta_0 < \min\{C_1^{-2(n-1)}, \varepsilon_1\}$, if ε_3 is small enough that $\delta_0 \ge C\varepsilon_3/\mathcal{H}^{n-1}(D)$ it follows that M_0 is non-empty. This also adds an additional constraint on ε_3. A consequence of (A.27) and $M \setminus \Gamma \subset M \setminus M_0$ is

$$\mathcal{H}^{n-1}(M \setminus \Gamma) \le Ce_n(0, 13). \tag{A.28}$$

To finish verifying (A.20) it remains to bound $\mathcal{H}^{n-1}(\Gamma \setminus M)$.

Indeed, $\mathrm{Lip}(u) \le 1$ and $M_0 \subset \Gamma$ together ensure

$$\begin{aligned}
\mathcal{H}^{n-1}(\Gamma \setminus M) &\le \sqrt{1 + |\nabla' u|^2}\mathcal{H}^{n-1}(p(\Gamma \setminus M)) \\
&\le \sqrt{2}\mathcal{H}^{n-1}(M \cap p^{-1}(p(\Gamma \setminus M))).
\end{aligned}$$

But, $M \cap p^{-1}(p(\Gamma \setminus M)) \subset M\setminus\Gamma$, so by the bound in (A.28), we have the necessary bound on $\mathcal{H}^{n-1}(\Gamma \setminus M)$, verifying (A.20) with a constant we denote as C_3.

Step 3: We verify (A.21). The first necessary observation is to note that for almost every $x \in M \cap \Gamma$,

$$\nu_E(x) = \lambda(x) \frac{(-\nabla' u(p(x)), 1)}{\sqrt{1 + |\nabla' u(p(x))|^2}} \tag{A.29}$$

where $\lambda(x) \in \{-1, 1\}$. Since $|\nu_E - e_n|^2 = |p(\nu_E)|^2$, (A.29) implies

$$\begin{aligned}
e_n(0, 1) &\ge \frac{1}{2} \int_{M\cap\Gamma} |p(\nu_E)|^2 \, \mathrm{d}\mathcal{H}^{n-1} \\
&= \frac{1}{2} \int_{M\cap\Gamma} \frac{|\nabla' u(p(x))|^2}{1 + |\nabla' u(p(x))|^2} \, \mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{1}{2} \int_{p(M\cap\Gamma)} \frac{|\nabla' u(z)|^2}{\sqrt{1 + |\nabla' u(z)|^2}} \, \mathrm{d}\mathcal{H}^{n-1}(z).
\end{aligned}$$

Since $\mathrm{Lip}(u) < 1$ it follows that

$$\int_{p(M\cap\Gamma)} |\nabla' u(z)|^2 \le 2^{3/2} e_n(0, 1). \tag{A.30}$$

On the other hand, $\mathrm{Lip}(u) < 1$ and (A.20) imply

$$\int_{p(M\Delta\Gamma)} |\nabla' u|^2 \le \mathcal{H}^{n-1}(p(M\Delta\Gamma)) \le \mathcal{H}^{n-1}(M\Delta\Gamma) \le C_3 e_n(0,13). \tag{A.31}$$

As $e_n(0,1) \le 13^{n-1} e_n(0,13)$, (A.30) and (A.31) together guarantee (A.21).

Step 4: Note that (A.24) and the definition of $u_\pm$ in (A.14) and (A.15) ensure (A.16) holds. Thus, we conclude by showing (A.13). In fact, if M_0 were closed, then (A.24) would immediately verify (A.13).

In case M_0 is not closed, fix $\varepsilon > 0$ small. For $x \in M \setminus M_0$ choose $y \in M_0$ such that $\mathrm{dist}(p(x), p(y)) \le \mathrm{dist}(x, p(M_0)) + \varepsilon$. Then,

$$\begin{aligned} |q(x) - u(p(x))| &\le u_+(p(x)) - u_-(p(x)) \\ &\le (u(p(y)) + L|p(x) - p(y)|) - (u(p(y)) \\ &\qquad - L|p(x) - p(y)|) \\ &\le 2L|p(x) - p(y)| \\ &\le 2L\,\mathrm{dist}(x, p(M_0)) + 2L\varepsilon. \end{aligned}$$

Taking $\varepsilon \to 0$ verifies (A.13). ∎

APPENDIX B. APPROXIMATION OF UR DOMAINS WITH DOUBLY LOCAL TWO-SIDED CORKSCREWS

In this appendix we will build UR domains[18] which (locally) approximate open sets satisfying a (doubly) local two-sided corkscrew (DLTSCS) condition with Ahlfors regular boundary. This will allow us to directly use the work of [HMT10] on singular integrals on UR domains.

Definition B.1 (Doubly local two-sided corkscrew condition). We let $R_0 \in (0,\infty)$, $M_0 \ge 2$, and $x_0 \in \mathbb{R}^n$. We say an open set $\Omega \subset \mathbb{R}^n$ with $x_0 \in \partial\Omega$ satisfies the (x_0, M_0, R_0)-doubly local two-sided corkscrew condition or (x_0, M_0, R_0)-*DLTSCS condition*, if for every $x \in B(x_0, R_0) \cap \partial\Omega$ and $r \in (0, R_0)$ there are two points X_1, X_2 so that $B(X_1, r/M_0) \subset B(x,r) \cap \Omega$ and $B(X_2, r/M_0) \subset B(x,r) \setminus \bar{\Omega}$.

The first step in the construction is to introduce the appropriate notion of boundary "cubes" for sets with $(n-1)$-dimensional Ahlfors regular boundary. These constructions were introduced in the work of David [Dav88] and were refined by Christ [Chr90]. The dyadic "families" built later by Hytönen and Kairema in [HK12] are better adapted to our needs; thus, we describe them below.

Lemma B.2 (***Dyadic cubes*** [Dav88, Chr90, HK12]). *Suppose $E \subset \mathbb{R}^n$ is an $(n-1)$-dimensional, closed Ahlfors regular set. Then, there exist N, a_0, γ, C_2, and*

[18] Recall that in [HMT10], the authors use the word domain to mean an open set; we have adopted this convention only in the context of "UR domains".

C_3 depending on n and the Ahlfors regularity constant such that the following holds. For each $t \in \{1, \dots, N\}$ there exists a collection of Borel sets ("cubes")

$$\mathbb{D}_k^t(E) := \mathbb{D}_k^t := \{Q_j^k \subset E \mid j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying the following:

(i) $E = \bigcup_j Q_j^k$ *for each* $k \in \mathbb{Z}$.

(ii) *If* $m \geq k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \emptyset$.

(iii) *For each* (j, k) *and each* $m < k$, *there is a unique* i *such that* $Q_j^k \subset Q_i^m$.

(iv) $\operatorname{diam}(Q_j^k) \leq C_2 2^{-k}$.

(v) *Each* Q_j^k *contains some "surface ball"*

$$\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E.$$

(vi) $\mathcal{H}^{n-1}(\{x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \leq \varrho 2^{-k}\}) \leq C_2 \varrho^\gamma \mathcal{H}^{n-1}(Q_j^k)$, *for all* k, j *and for all* $\varrho \in (0, a_0)$.

(vii) *For every surface ball* $\Delta(x, r) = B(x, r) \cap E$, $x \in E$, *and* $r \in (0, \operatorname{diam} E)$ *there exists* t *and* $Q \in \mathbb{D}^t := \bigcup_k \mathbb{D}_k^t$ *with* $B \subset Q$ *and* $\operatorname{diam}(Q) \leq C_3 r$.

If $Q \in \mathbb{D}_k^t$ *for some* $t \in \{1, \dots, N\}$ *and* $k \in \mathbb{Z}$ *we set* $\ell(Q) = 2^{-k}$. *Evidently,* $\operatorname{diam}(Q) \approx \ell(Q)$, *provided* $2^{-k} \lesssim \operatorname{diam}(E)$[19], *and we refer to* $\ell(Q)$ *as the "side length" of* Q.

Remark B.3. When we use these dyadic cubes we always start by knowing that the DLTSCS condition holds on some ball $B(x_0, R_0)$. The flexibility of the families (the index t above) allows us to use property (vii) to find a cube Q such that $B(x_0, C_3^{-1} R_0) \cap \partial\Omega \subset Q \subset B(x_0, R_0) \cap \partial\Omega$.

From this point onward, we work with $E \subset \mathbb{R}^n$, an $(n-1)$-dimensional Ahlfors regular set (E will eventually be the boundary of an open set) and a particular dyadic grid $\mathbb{D} := \mathbb{D}^t$ for some t to be chosen when needed to ensure the existence of a cube as in Remark B.3. There will be no constants that depend on t.

For $E \subset \mathbb{R}^n$ an $(n-1)$-dimensional Ahlfors regular set, we denote by $\mathcal{W} = \mathcal{W}(E^c)$ the collection of (closed) n-dimensional dyadic Whitney cubes of $\mathbb{R}^n \setminus E$; that is, the collection $\mathcal{W} = \{I\}$ form a pairwise non-overlapping (their boundaries may intersect) covering of $\mathbb{R}^n \setminus E$ with the property that

$$4 \operatorname{diam}(I) \leq \operatorname{dist}(4I, E) \leq \operatorname{dist}(I, E) \leq 40 \operatorname{diam}(I)$$

(see [Ste70, Chapter VI]). Moreover, whenever $I_1, I_2 \in \mathcal{W}$ with $I_1 \cap I_2 \neq \emptyset$, we have $\operatorname{diam}(I_1) \approx \operatorname{diam}(I_2)$. For $I \in \mathcal{W}$ we let $\ell(I)$ denote the side length of I.

[19] We ignore the cubes for which $2^{-k} \gg \operatorname{diam}(E)$, because (v) implies that eventually $\mathbb{D}_k^t$ consists of a single cube if $\operatorname{diam}(E) < \infty$ and k is sufficiently large.

Now we relate these two notions of cubes to form Carleson and Whitney-type regions associated with each boundary cube Q. These are almost exactly as in [HM14][20].

We let $K \gg 1$ be a large parameter, and for $Q \in \mathbb{D}(E)$ we define

$$\begin{aligned}\mathcal{W}_Q &:= \mathcal{W}_Q(K)\\ &:= \{I \in \mathcal{W}(E^c) \mid K^{-1}\ell(Q) \le \ell(I) \le K\ell(Q),\ \operatorname{dist}(I,Q) \le K\ell(Q)\}.\end{aligned}$$

Since E is Ahlfors regular, one can show that $\mathcal{W}_Q$ is non-empty provided K is chosen large enough. We do not fix K at this point because we will eventually set $E = \partial\Omega$ and want to choose K to take advantage of the existence of the (local) corkscrew points afforded by the DLTSCS condition.

Next, we fix τ a small parameter depending on dimension so that the $(1+\tau)$-dilates of $I \in \mathcal{W}$, $I^* := I^*(\tau) = (1+\tau)I$ maintain the Whitney property

$$\ell(I) \approx \ell(I^*) \approx \operatorname{dist}(I^*, E) \approx \operatorname{dist}(I, E)$$

and I^* meets J^* if and only if $I \cap J \neq \emptyset$. We also may ensure (by choice of τ small) that if $I \cap J \neq \emptyset$ and $I \neq J$ then $I^* \cap (\frac{3}{4}J) = \emptyset$.

Finally, we define the *Whitney regions relative to Q*

$$U_Q(K) := \bigcup_{I \in \mathcal{W}_Q(K)} I^* \tag{B.1}$$

and the *Carleson boxes relative to Q*

$$T_Q(K) := \operatorname{int}\Big(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}(K)\Big), \tag{B.2}$$

where $\mathbb{D}_{\mathbb{Q}} := \{Q' \in \mathbb{D} : Q' \subseteq Q\}$.

Now we are ready to state our approximation lemma.

Lemma B.4. *Let $M_0 \ge 2$ and $R_0 > 0$. If $\Omega \subset \mathbb{R}^n$ is an open set with $(n-1)$-dimensional Ahlfors regular boundary $\partial\Omega$ satisfying $\partial_*\Omega = \partial\Omega$ with $x_0 \in \partial\Omega$ such that Ω satisfies the $(x_0, M_0, 2R_0)$-DLTSCS condition, then there exist $K \gg 1$ and $M_0' \ge M_0$ depending on n, R_0, M_0, and the Ahlfors regularity constant such that the following holds.*

Let $E = \partial\Omega$, $\mathbb{D}(E)$, $\mathcal{W} = \mathcal{W}(E^c)$, and so on be as above. Suppose $Q \in \mathbb{D}^t$ for some t such that $B(x_0, C_3^{-1}R_0) \cap \partial\Omega \subseteq Q \subseteq B(x_0, R_0)$[21]. Then, the sets

$$T_Q^+ := T_Q^+(K) := T_Q(K) \cap \Omega \qquad \text{and} \qquad T_Q^- := T_Q^-(K) := T_Q(K) \cap (\bar{\Omega})^c$$

[20]The difference here is that the regions are not "augmented" by exploiting connectivity which was present in [HM14].

[21]See Remark B.3.

are non-empty. Also, they satisfy the $(M_0', \ell(Q))$*-two sided corkscrew condition* (*see Definition 2.13*) *and* $\partial T_Q^{\pm}$ *are* $(n-1)$*-Ahlfors regular with constant depending on* M_0*,* R_0*, and the Ahlfors regularity constant for* $\partial\Omega$*. In particular,* $T_Q^{\pm}$ *are* UR *domains with constants depending on* n*,* R_0*,* M_0*, and the Ahlfors regularity constant for* $\partial\Omega$[22]*, and* $\partial T_Q^{\pm} \cap Q = Q$*. Moreover, for* $\mathcal{H}^{n-1}$*-almost every* $x \in Q$ *the measure theoretic outer normals to* $T_Q^{\pm}$*, denoted by* $\nu_{T_Q^{\pm}}(x)$*, exist and satisfy* $\nu_{T_Q^{\pm}}(x) = \pm\nu_{\Omega}(x)$*.*

Proof. Fix $Q \subseteq B(x_0, R_0)$. Choose K big enough to ensure that for $Q' \in \mathbb{D}_Q$ with $Q' \subseteq B(x_0, R_0)$ the sets

$$U_{Q'}^{+} := U_{Q'}^{+}(K) := U_{Q'}(K) \cap \Omega \quad \text{and} \quad U_{Q'}^{-} := U_{Q'}^{-}(K) := U_{Q'}(K) \cap (\bar{\Omega})^c$$

are non-empty. To see that such a choice (depending on M_0, R_0, and the Ahlfors regularity constant for $\partial\Omega$) exists, we note that if $x \in Q' \subseteq B(x_0, R_0)$, then necessarily $\ell(Q') \le CR_0$ and the ball $B(x_{Q'}, (1/C)\ell(Q'))$ contains two corkscrew points, one for Ω and one for $(\bar{\Omega})^c$. Choosing $K^{-1} \ll 1/(CM_0)$ ensures that these points are contained in $U_Q(K)$.

We also have that $\partial T_Q^{\pm}$ are both Ahlfors regular by the work of [HM14] (see the Appendix therein). It is also easy to see that $\partial T_Q^{\pm} \cap Q = Q$, since for every $x \in Q$, $x \in Q_j \in \mathbb{D}_Q$ with $\ell(Q_j) \to 0$ as $j \to \infty$. Using that $U_{Q_j}^{\pm}$ are non-empty, we see there exist $X_j \in U_{Q_j} \to x$ as $j \to \infty$, and hence $x \in \partial T_Q^{\pm}$ (see (B.1) and (B.2)).

Next, we show that $T_Q^{\pm}$ both satisfy the $(M_0', \ell(Q))$-two sided corkscrew condition. The hypotheses are symmetric so we may just show that T_Q^{+} satisfies the $(M_0', \ell(Q))$-two sided corkscrew condition. To this end, let $x \in \partial T_Q^{+}$ and $r \in (0, \ell(Q))$, and fix A_0 to be chosen[23]. We break into cases, following closely [HM14, HMM16].

Case 1. $r < A_0\delta(x)$, where $\delta(x) := \operatorname{dist}(x, \partial\Omega)$. In this case, $\delta(x) > 0$ and x is "far" from $\partial\Omega$. Necessarily (since $\delta(x) > 0$), $x \in \partial I^*$ for some "fat" Whitney cube I^* with $\operatorname{int}(I^*) \subset T_Q^{+}$ and also $x \in J$ for some $J \in \mathcal{W} \setminus (\bigcup_{Q' \in \mathbb{D}_Q} \mathcal{W}_{Q'})$. The Whitney property of I^* and J yields $\ell(I^*) \approx \ell(J) \approx \delta(x) \gtrsim r/A_0$. It follows (from our choice of τ) that J contains an exterior corkscrew point and I^* contains an interior corkscrew point for T_Q^{+} at x at scale r, with constants depending on A_0, for now.

Case 2: $r \ge A_0\delta(x)$. In this case, we are close enough to the boundary so that we may exploit the (M_0, R_0)-DLTSCS condition for Ω. We break into further cases.

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[22]See the discussion following Definition 2.9 and note that since $\operatorname{diam}(T_Q) \approx_K \ell(Q)$, T_Q satisfies the two-sided corkscrew condition.

[23]Note that the choice of A_0 depends on K, which is now fixed.

Case 2a: $\delta(x) > 0$. In this case $x \in \partial I^*$ for some I as in *Case 1*. Let $\hat{x} \in \bar{Q}$ be such that $\delta(x) \approx |x - \hat{x}|$, where the implicit constants depend on K (which we have fixed). Note that the existence of $\hat{x}$ is afforded by the Whitney property of I^*. Moreover, $I \in \mathcal{W}_{Q'}$ for some $Q' \subset Q$. Since

$$|x - \hat{x}| \leq C_K \delta(x) \leq C_K \frac{r}{A_0} < C_K \frac{\ell(Q)}{A_0},$$

choosing A_0 large enough we may find Q^* whose closure contains $\hat{x}$, $Q^* \subset Q$ and $\ell(Q^*) \approx \dfrac{r}{A_0}$, where the implicit constants depend on n, the Ahlfors regularity constant, and K. Note that by the $(x_0, M_0, 2R_0)$-DLTSCS condition of Ω, and choice of K, $U^{\pm}_{Q^*}$ are both non-empty, and we may find two points $X^{\pm}_{Q^*} \in U^{\pm}_{Q^*}$ with

$$\operatorname{dist}(X^{\pm}_{Q^*}, \partial T^{\pm}_{Q}) \geq C_K \ell(Q^*) \approx \frac{r}{A_0}.$$

Here, one may take each $X^{\pm}_{Q^*}$ to be the center of a Whitney cube in $\mathcal{W}_{Q^*}$. We then choose $A_0 \gg 2$ such that

$$|x - X^{\pm}_{Q^*}| \leq |x - \hat{x}| + |\hat{x} - X^{\pm}_{Q^*}| \lesssim \frac{r}{A_0} < \frac{r}{2}.$$

Having fixed such an A_0, depending on the allowable parameters, we have

$$\operatorname{dist}(X^{\pm}_{Q^*}, \partial T^{\pm}_{Q}) \geq C_K \ell(Q^*) \gtrsim r$$

so that $X^{\pm}_{Q^*}$ may serve as interior and exterior corkscrews (respectively) for $T^{\pm}_{Q}$ at x at scale r.

Case 2b: $\delta(x) = 0$. In this case, things are easier than *Case 2a*:, provided we can show $x \in \bar{Q}$. Indeed, we may forgo the step of finding $\hat{x}$ above, by setting $\hat{x} = x$ and repeating the above argument verbatim. To show $x \in \bar{Q}$, we use that $\delta(x) = 0$ and $x \in \partial T^{+}_{Q}$, so there exists a sequence of points $X_i \in U^{+}_{Q_i}$ with $Q_i \subset Q$ and $\ell(Q_i) \to 0$, $|X_i - x| \to 0$ as $i \to \infty$. Here, we used $\delta(X_i) \approx \ell(Q_i)$ by the Whitney property of cubes in $\mathcal{W}_{Q_i}$ and that $\delta(\cdot)$ is continuous. Moreover, for each i there exists $\widehat{X_i} \in Q_i$ with $|\widehat{X_i} - X_i| \lesssim \ell(Q_i)$ so that

$$|x - \widehat{X_i}| \leq |x - X_i| + |X_i - \widehat{X_i}| \to 0 \quad \text{as } i \to \infty.$$

Since $\widehat{X_i} \in Q$ this shows $x \in \bar{Q}$, and we can proceed as in *Case 2a*:.

Again, by [DJ90, Theorem 1], an open set with Ahlfors regular boundary that satisfies a two-sided corkscrew condition on scales up to its diameter is a UR domain. Thus, the only thing left to do is show that the measure theoretic unit normals for $T^{\pm}_{Q}$ agree with the unit normal of Ω up to a sign. Again, the symmetry of the hypotheses in the theorem and the fact that $\partial_* \Omega = \partial\Omega$ allow us to only consider T_Q^+.

Since T_Q^+ has $(n-1)$-Ahlfors regular boundary and satisfies the two-sided corkscrew condition, Federer's criteria ensures that T_Q^+ is a set of locally finite perimeter [EG92, Theorem 1, Section 5.11]. The structure theorem for sets of locally finite perimeter ensures that the measure theoretic unit normal to ∂T_Q^+ exists $\mathcal{H}^{n-1}$-almost everywhere [EG92, Theorem 2, Section 5.7.3]. Since $Q \subset \partial\Omega$ and $\partial T_Q^+ \cap Q = Q$ the measure theoretic tangents to ∂T_Q^+ and $\partial\Omega$ must agree $\mathcal{H}^{n-1}$-almost everywhere in Q. Thus, the measure theoretic outer unit normal for T_Q^+ and Ω must agree up to a sign for $\mathcal{H}^{n-1}$-almost every $x \in Q$.

To show that $\nu_{T_Q^+}(x) = \nu_\Omega(x)$ $\mathcal{H}^{n-1}$-almost everywhere in Q, assume that $x \in \partial^* T_Q \cap Q$; then, $\nu_{T_Q^+}(x) = \pm\nu_\Omega(x)$. Suppose, for the sake of obtaining a contradiction, that $\nu_{T_Q^+}(x) = -\nu_\Omega(x)$, and set

$$H^+ := \{y \in \mathbb{R}^n \mid (y - x) \cdot \nu_\Omega(x) \geq 0\}.$$

This is a half-space through x, perpendicular to $\nu_\Omega(x)$. The blow-up of the reduced boundary [EG92, Section 5.7, Corollary 1] gives

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap \Omega \cap H^+)}{\mathcal{L}^n(B(x,r))} = 0,$$

which of course implies

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap T_Q^+ \cap H^+)}{\mathcal{L}^n(B(x,r))} = 0.$$

On the other hand, using $\nu_{T_Q^+}(x) = -\nu_\Omega(x)$, and applying Corollary 1 in Section 5.7 of [EG92] to the set T_Q^+ give

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap T_Q^+ \cap H^+)}{\mathcal{L}^n(B(x,r))} = \frac{1}{2},$$

which is impossible. Thus, $\nu_{T_Q^+}(x) = \nu_\Omega(x)$, and we have proved the lemma. ◻

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S. BORTZ:
Department of Mathematics
University of Alabama
Tuscaloosa, AL, 35487, USA
E-MAIL: sbortz@ua.edu

M. ENGELSTEIN:
Department of Mathematics
University of Minnesota
Minneapolis, MN, 55455, USA
E-MAIL: mengelst@umn.edu

M. GOERING:
Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstr. 22
04103 Leipzig, Germany
E-MAIL: goering@mis.mpg.de

T. TORO:
Department of Mathematics
University of Washington
Seattle, WA 98195, USA
E-MAIL: toro@uw.edu

Z. ZHAO:
Department of Mathematics
University of Chicago
Chicago, IL 60637, USA
E-MAIL: zhaozh@uchicago.edu

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