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Permutation groups with few orbits on the power set. II

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Abstract. We continue the study of permutation groups acting on the power set $\mathscr{P}(\{1, 2, \ldots, n\})$. Permutation groups must have a minimum of $n + 1$ set-orbits. Previously in [3], the authors of that paper used GAP to classify permutation groups with a low number of orbits for permutation groups having $n + r$ set-orbits for some given $2 \leq r \leq 15$. We develop improvements to their theory and algorithms in GAP to classify further cases, from $16 \leq r \leq 33$.

1. Introduction

We begin with some definitions and notations we will use throughout this paper. Let $\Omega = \{1, 2, \ldots, n\}$, where $n \geq 2$ is some positive integer. Let G be a subgroup of S_n, where G acts on Ω with the action $G \times \Omega \to \Omega$ where $(g, x) \mapsto gx$. The permutation group G has degree n, and the action of G on Ω naturally induces an action of G on $\mathscr{P}(\Omega)$. This is defined by the mapping $G \times \mathscr{P}(\Omega) \to \mathscr{P}(\Omega)$, where G acts on subsets of Ω and $(g, X) \mapsto gX = \{gx : x \in X\}$. It is clear from the definition that G takes a t-element subset of Ω and maps it to another (perhaps identical) t-element subset, and these orbits under the action shall be named *set-orbits*. We will denote the number of t-element set-orbits for some given

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$0 \leq t \leq n$ as $s_t(G)$. Since Ω has at least $n+1$ different sizes of subsets, it follows that the total number of set-orbits, which we denote as $s(G)$, will be at least $n+1$. Furthermore, we can calculate $s(G)$ by finding the total number of different *t-set-orbits*, which is any set-orbit with sets of size t. Thus $s(G) = \sum_{t=0}^{n} s_t(G)$.

If a group G is transitive on Ω, then for any $x, y \in \Omega$, there exists $g \in G$ such that $gx = y$. Thus this motivates the following definition.

Definition 1.1 (Beaumont and Peterson [2]). Given an integer $0 \leq t \leq n$, a permutation group G on n letters is called *t-set-transitive* if for all t-element subsets $S, T \subseteq \Omega$, there exists $g \in G$ such that $gS = T$.

The minimum number of set-orbits that can be achieved is $n+1$, when G is t-set-transitive for every possible value of t. Hence we present the next definition.

Definition 1.2 (Beaumont and Peterson [2]). A permutation group G on n letters is called *set-transitive* if G is t-set-transitive for all integers $0 \leq t \leq n$.

With some preliminary definitions out of the way, we now discuss past progress on this topic and our goal in this paper. We consider the following classification question:
Given some integer $r \geq 1$, what are all permutation groups G such that $s(G) = n + r$, where n is the degree of the permutation group?

Set-transitive groups were studied as early as 1944 by Neumann and Morgenstern [6]. The case for set-transitive groups when $r = 1$ is answered in full by Beaumont and Peterson in [2]. Their main result was that a group G that does not contain the alternating group A_n cannot be $\lfloor \frac{n}{2} \rfloor$ set-transitive, and thus not set-transitive, with exceptions only when $n = 5, 6, 9$, and proceeded to classify all such exceptions. In [3], an algorithm using GAP [4] was developed to fully classify the cases for $2 \leq r \leq 15$, which will be discussed in greater detail later in this paper. The issues preventing further classification included the computational challenges of calculating subgroups of S_n, and determining the number of set-orbits for a large group. The first issue can be addressed using improved theory, while the second can be addressed with improved algorithms in GAP. In this paper, we successfully classify all the cases for $16 \leq r \leq 33$.

2. Lemmas

The following lemmas appeared in [3]. As they were also used in this classification, we list them here for easy reference.

Lemma 2.1. *Let G be a permutation group on n letters, and let $0 \leq t \leq n$ be an integer. Then $s_t(G) = s_{n-t}(G)$.*

Lemma 2.2. *Let G be a permutation group on n letters, and suppose $s(G) = n + r$.*

(1) *If r is even, then n is even.*

(2) *If r is odd and n is even, then $s_{n/2}(G)$ is odd.*

Lemma 2.3. *Given a permutation group G on n letters and an integer $1 \leq t \leq \frac{n}{2}$, we have $s_{t-1}(G) \leq s_t(G)$.*

The following results are in [1].

Lemma 2.4. *If $L \leq G \leq \mathrm{Sym}(\Omega)$, then $s(G) \leq s(L) \leq s(G) \cdot |G : L|$.*

Lemma 2.5. *Assume G is intransitive on Ω and has orbits $\Omega_1, \ldots, \Omega_m$. Let G_i be the restriction of G to Ω_i. Then*

$$s(G) \geq s(G_1) \times \cdots \times s(G_m).$$

Lemma 2.6. *Let G be a transitive permutation group acting on a set Ω, where $|\Omega| = n$. Let $(\Omega_1, \ldots, \Omega_m)$ denote a system of imprimitivity of G with block-size b ($1 \leq b < n$; $b = 1$ if and only if G is primitive; $bm = n$). Let $G_i = \mathrm{Stab}_G(\Omega_i)$, and denote $s = s(G_1)$. Then*

$$s(G) \geq \binom{s+m-1}{s-1} \geq \binom{b+m}{b}.$$

It should be noted that Lemma 2.2 provides conditions that allow us to avoid calculating the set-orbits of groups of particular degrees. Lemmas 2.1 and 2.3 can be used to provide lower bounds for the number of set-orbits a groups has. Lemmas 2.5 and 2.6 provide conditions under which we may eliminate certain cases for transitive and intransitive groups. Lemma 2.5 is used frequently to consider intransitive cases for a group G on n letters with $n \geq 12$.

Using Lemmas 2.1 and 2.3, it is simple to show that if G acts on n letters and $r < n$, then G must be transitive. We can show that G must be primitive rather than just transitive. This greatly reduces the computation necessary to classify groups on large values of n as there are relatively few primitive groups.

Proposition 2.7. *Let $ab = cd = n$, where $a \leq b$ and $c \leq d$ and $a < c$ and $a, b, c, d, n \in \mathbb{N}$. Then $\binom{a+b}{a} \leq \binom{c+d}{c}$.*

Proof. Assume the hypothesis. First, note that by Vandermonde's Identity we have

$$\binom{a+b}{a} = \sum_{i=0}^{a} \binom{a}{a-i}\binom{b}{i}, \qquad \binom{c+d}{c} = \sum_{k=0}^{c} \binom{c}{c-k}\binom{d}{i}.$$

We proceed by induction on the terms of the sums, up to a, since $a < c$ (we show that the desired inequality holds for each term in the sum). We note that

$$\binom{a}{a-0}\binom{b}{0} = 1 \leq 1 = \binom{c}{c-0}\binom{d}{0}.$$

Assume the result holds for $\ell < a$. That is, $\binom{a}{a-\ell}\binom{b}{\ell} \leq \binom{c}{c-\ell}\binom{d}{\ell}$. We show the result holds for $\ell + 1 \leq a$. First note that since $a + b \geq c + d$, we have

$$-\ell a - \ell b \leq -\ell c - \ell d, \qquad ab - \ell a - \ell b + \ell^2 \leq cd - \ell c - \ell d + \ell^2,$$

$$(a-\ell)(b-\ell) \leq (c-\ell)(d-\ell), \qquad 1 \leq \frac{(c-\ell)(d-\ell)}{(a-\ell)(b-\ell)}.$$

Therefore, we have

$$\begin{aligned}
\binom{a}{a-(\ell+1)}\binom{b}{\ell+1} &= \frac{a!b!}{(a-(\ell+1))!(\ell+1)!(\ell+1)!(b-(\ell+1))!} \\
&\leq \frac{a!b!}{(a-(\ell+1))!(\ell+1)!(\ell+1)!(b-(\ell+1))!} \cdot \frac{(\ell+1)^2(c-\ell)(d-\ell)}{(\ell+1)^2(a-\ell)(b-\ell)} \\
&= \frac{a!b!}{(a-\ell)![(\ell)!]^2(b-\ell)!} \cdot \frac{(c-\ell)(d-\ell)}{(\ell+1)^2} = \binom{a}{a-\ell}\binom{b}{\ell} \cdot \frac{(c-\ell)(d-\ell)}{(\ell+1)^2} \\
&\leq \binom{c}{c-\ell}\binom{d}{\ell} \cdot \frac{(c-\ell)(d-\ell)}{(\ell+1)^2} \qquad \text{(by the inductive hypothesis)} \\
&= \frac{c!d!}{(c-\ell)![\ell!]^2(d-\ell)!} \cdot \frac{(c-\ell)(d-\ell)}{(\ell+1)^2} = \frac{c!d!}{(c-(\ell+1))![(\ell+1)!]^2(d-(\ell+1))!} \\
&= \binom{c}{c-(\ell+1)}\binom{d}{\ell+1}.
\end{aligned}$$

Thus, the result holds. □

Proposition 2.8. *For all groups G acting on n elements with $s(G) = n + r$ and $n > r \geq 11$, G must be a primitive group.*

PROOF. First of all, if G is intransitive, then $s_1(G) \geq 2$, and using Lemma 2.1 and 2.3 together, we have that $s(G) \geq 2 + 2(n-1) = 2n > n + r$. Thus G must be transitive, and now we show G must also be primitive.

Suppose that G is imprimitive, and let $(\Omega_1, \Omega_2, \ldots, \Omega_m)$ denote a non-trivial system of imprimitivity of G with block size j such that $j \geq 2$ and $m \geq 2$. We know that n cannot be a prime, or it would be impossible that $n = mj$. In Lemma 2.6, we have that $s(G) \geq \binom{j+m}{j}$, and we seek to show that $\binom{j+m}{j} \geq 2n > n + r$. Let $a = \min(j, m)$ and $b = \max(j, m)$. Then for even n, we only need to consider when $a = 2$, since Proposition 2.7 guarantees that any other non-trivial factorization of n yields larger lower bounds. We prove the even case inductively. For $n = 12 = 2 \times 6$, we see that $\binom{2+6}{2} = 28 > 24 = 2n$. We have our base case and so suppose the proposition holds for some even $n \geq 12$. That is, $\binom{a+b}{2} \geq 2n$ where $a = 2$ and $b = \frac{n}{2}$. Then $n + 2 = a(b+1)$. We have that:

$$\begin{aligned}\binom{a+b+1}{2} &= \frac{1}{2}(a+b+1)(a+b) = \frac{1}{2}(a+b)(a+b-1) + (a+b) \\ &= \binom{a+b}{2} + a + b \geq 2n + 4 = 2(n+2).\end{aligned}$$

Next, we consider when n is odd, and so $n = ab$ means that $b \geq a \geq 3$, and that $b^2 \geq ab$. Since the binomial coefficients $\binom{n}{k}$ form an increasing sequence for $k \leq n/2$, we have that:

$$\binom{j+m}{j} \geq \binom{j+m}{3} = \binom{a+b}{3} = \frac{(a+b)(a+b-1)(a+b-2)}{6}.$$

Thus, we may prove that $6\binom{a+b}{3} \geq 6(2n) = 12ab$.

$$\begin{aligned}6\binom{a+b}{3} &= (a+b)(a+b-1)(a+b-2) \\ &= a^3 + 3a^2b - 3a^2 + 3ab^2 - 6ab + 2a + b^3 - 3b^2 + 2b \\ &\geq 3a^2 + 9ab - 3a^2 + 9b^2 - 6ab + 2a + 3b^2 - 3b^2 + 2b \\ &= 3ab + 9b^2 + 2a + 2b > 12ab.\end{aligned}$$

□

3. Methods

We give a brief outline of the method used to classify groups with $n + r$ set-orbits in [3]. The first step consists of bounding the value of n by finding the largest value k_0 such that $48 - \frac{n+1}{2} \leq k_0 \leq \frac{5}{54}n - \frac{1}{2}$. Then, [3, Theorem 2.7] implies that the group is not $\left(\lfloor \frac{n}{2} \rfloor + k\right)$-set-transitive for all $k \leq k_0$. This provides an upper bound on the value of n. Using other results (see [3, Lemma 2.2 and Corollary 2.6]), several of the remaining values of n can be eliminated. Noting that if $r < n - 4$, the group is primitive, and if $r < n - 2$, the group is transitive, GAP can be used to search the subgroups of S_n to find all groups with $n + r$ set-orbits.

One of the difficulties with the previous method is that it becomes computationally challenging for GAP to compute and check all subgroups of S_n when $n \geq 12$. Using the GAP libraries, the transitive and primitive groups of a certain degree can be easily checked. In the intransitive cases, we can consider the action of G on the orbits, which means, in the case where there are two orbits, we have $G = \mathsf{S}_k \times \mathsf{S}_{n-k}$ for some positive integer $k < n$. When $k = 1$, we can check the previously calculated tables to determine if there is a subgroup of S_{n-1} that will give the correct number of orbits. For larger values of k, Lemma 2.5 applies to determine which of these cases is possible; this way, we can avoid calculating the most undesirable cases when k is large.

The second major difficulty involved the need to calculate the number of set-orbits of a group. This was originally done by calculating $s_t(G)$ for $0 \leq t \leq n$. Extreme values of t can be calculated quickly, because the number of subsets of Ω of size t is given by $\binom{n}{t}$, but this grows rapidly when t becomes closer to $\lfloor \frac{n}{2} \rfloor$. We were able to circumvent this obstacle by making estimations for $s(G)$ after every $s_t(G)$ that gets calculated, thus usually allowing GAP to bypass the most expensive computations. For example, let us consider the 91-st transitive group of degree $n = 24$ listed in GAP, which is $G = \text{TransitiveGroup}(24, 91)$, and see how many set-orbits G can have. Trivially, $s_0(G) = s_n(G) = 1$, and it turns out that $s_1(G) = s_{n-1}(G) = 1$ as well. However, $s_2(G) = s_{n-2}(G) = 8$, and using Lemmas 2.1 and 2.3, this already gives us the bound that $s(G) \geq 2+2+8(24-4) = 164$, which immediately exceeds the sizes of r that we look at. These calculations take mere milliseconds, while trying to even calculate $s_6(G) = 1722$ takes nearly half a minute on the same hardware; this is very expensive considering the number of groups we must do this calculation for.

Our method improves the bound for when a permutation group must be primitive. For $n > r$, Proposition 2.8 shows that we only need to consider the primitive groups. This saves a lot of computation, since the primitive groups of degree up to 4096 are stored in the GAP library and can be checked easily. The other issue that we need to address was how to determine intransitive groups with two orbits which are subgroups of $\mathsf{S}_k \times \mathsf{S}_{n-k}$ with $k > 1$. The previous method brute force searched through all conjugacy classes of S_n and calculation time became unfeasible for $n > 12$.

We can resolve this problem easily when $k = 2$. To do this, we can use the GAP library of transitive groups to form all direct products $\mathsf{S}_2 \times T$ where T is a transitive group on $n - 2$ letters. We can then find all subgroups of $\mathsf{S}_2 \times T$ of index 1 or 2. In this way, we obtain all groups that project transitively onto the orbit of $n - 2$ points. Then GAP can be used to check each possibility. Using these methods, we have been able to calculate groups with $n + r$ set-orbits for $16 \leq r \leq 33$.

4. Examples

We explain in detail the calculation of the cases $r = 25$ and $r = 31$.

For $r = 25$, we first use GAP to calculate all groups of degree $n \leq 11$ by a relatively quick brute force search. After applying the first few steps of the algorithm, the remaining values of n are $12, 13, 14, \ldots, 30, 37$, and 38.

Next, we determine the possible transitive groups. For $n = 12 = 2 \times 6$ and $n = 14 = 2 \times 7$, Lemma 2.6 gives us bounds for $s(G)$ to be 28 and 36, respectively, and these are not sufficient to eliminate those imprimitive cases. Thus, we have to use the GAP libraries to calculate all transitive groups of degree 12 and 14. This only results in one transitive group of degree 12, listed below. Since $n = 13$ is a prime, any transitive groups of degree 13 must also be primitive, and we may check by hand that all transitive groups of degree $15 \leq n \leq 25$ must also be primitive if they are to have no more than $n + r$ set-orbits. After searching through all such primitive groups, we find that M_{24} acting on 24 points gives $24 + 25 = 49$ set-orbits.

All that remains is to check the intransitive cases of $12 \leq n \leq 25$. Take the case when $n = 13$, in which case we are looking for 38 set-orbits. Applying Lemma 2.5 to all possibilities for an action with 3 orbits on Ω gives a contradiction. If we assume that G has two orbits on Ω, then G will be a subgroup of $\mathsf{S}_1 \times \mathsf{S}_{12}$ or $\mathsf{S}_2 \times \mathsf{S}_{11}$. Using GAP to calculate the subgroups of $\mathsf{S}_2 \times \mathsf{S}_{11}$ as outlined in Section 3 yields no new groups. If we have a subgroup of $\mathsf{S}_1 \times \mathsf{S}_{12}$, we must have a subgroup of S_{12} acting on 13 points that acts on 12 points with $19 = 12 + 7$ set-orbits. By [3], we see that M_{11} will satisfy these conditions. We use GAP to check that M_{11} acting on 13 points does indeed give 38 set-orbits. We can apply this same process to the remaining values of n. GAP finds three new subgroups of $\mathsf{S}_2 \times \mathsf{S}_{12}$ for $n = 14$ only when the group splits intransitively into 2 orbits.

The last two groups found are in $n = 25$, which are S_{24} and A_{24} acting transitively on 24 elements and trivially on 1 element. This finishes the calculation for $r = 25$.

The calculations for checking higher values of r are similar, except for when finding subgroups of intransitive subgroups of lower values of n. For example, when $r = 31$ and $n = 13$, subgroups of $\mathsf{S}_3 \times \mathsf{S}_{10}$ need to be checked, and this is done by searching for subgroups of index $1, 2, 3, 6$. In general, a subgroup of $\mathsf{S}_k \times \mathsf{S}_{n-k}$ must have index dividing $k!$. This is still computationally feasible for lower values of $k \leq 4$, and the classification can continue until at least $r = 33$. However, for larger values of k, this is equivalent to brute force searching all conjugacy classes of subgroups of S_k.

We list our results for $r = 25, 31$, and a few other select values of r below. The remaining results may be obtained through [5]. One thing to note is that two non-isomorphic groups may have the same structure description, while two isomorphic groups may have different descriptions (although we have taken care to eliminate isomorphic redundancies in our listed results). As such, these groups should really be reconstructed in GAP using their generators, or selected from the output of our list functions in our GAP code, if one wishes to work with a particular group.

We first list our notation for the GAP code of groups:

- nPi denotes the i-th primitive group of degree n in GAP libraries.
- nTi denotes the i-th transitive group of degree n in GAP libraries.
- nSi denotes the i-th conjugacy class in the list generated by ConjugacyClassesSubgroups(SymmetricGroup(n)).
- $nL_f i$ denotes the i-th group in the list generated by IntransitivePartition(n,f).

	Groups with $n + 16$ set-orbits		
n	G	**Order**	**GAP ID**
6	C_4	4	6S9
8	$C_7 \rtimes C_3$	21	8S137
8	$(C_8 \rtimes C_2) \rtimes C_2$	32	8S173
8	$C_8 \rtimes (C_2 \times C_2)$	32	8S179
8	$C_{15} \rtimes C_4$	60	8S220
8	$C_3 \times (C_5 \rtimes C_4)$	60	8S221
8	$((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$	64	8S224
8	$C_2 \rtimes \mathsf{A}_5$	120	8S255

Table 1 – *Continued on next page*

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	Groups with $n+16$ set-orbits		
n	G	**Order**	**GAP ID**
8	$\mathsf{S}_3 \times (C_5 \rtimes C_4)$	120	8S258
8	GL$(2,4)$	180	8S266
8	$C_3 \rtimes \mathsf{S}_5$	360	8S279
8	$C_3 \times \mathsf{S}_5$	360	8S280
8	$\mathsf{S}_3 \times \mathsf{A}_5$	360	8S281
8	$\mathsf{S}_5 \times \mathsf{S}_3$	720	8S289
12	$(\mathsf{A}_5 \times \mathsf{A}_5) \rtimes C_4$	14400	12P278
12	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_6) \rtimes C_2$	23040	12P286
12	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_6) \rtimes C_2$	23040	12P287
12	$(\mathsf{A}_5 \times \mathsf{A}_5) \rtimes D_8$	28800	12P288
12	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_6) \rtimes (C_2 \times C_2)$	46080	12P293
12	$(\mathsf{A}_6 \times \mathsf{A}_6) \rtimes C_2$	259200	12P296
12	$(\mathsf{A}_6 \times \mathsf{A}_6) \rtimes (C_2 \times C_2)$	518400	12P297
12	$(\mathsf{A}_6 \times \mathsf{A}_6) \rtimes C_4$	518400	12P298
12	$(\mathsf{A}_6 \times \mathsf{A}_6) \rtimes D_8$	1036800	12P299
12	M_{11}	7920	
16	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_8$	322560	16P11
16	A_{15}	15!/2	
16	S_{15}	15!	

Table 1

	Groups with $n+17$ set-orbits		
n	G	**Order**	**GAP ID**
7	D_{10}	10	7S32
7	C_{10}	10	7S34
7	A_4	12	7S37
7	A_4	12	7S40
7	C_{12}	12	7S41
7	$C_3 \rtimes C_4$	12	7S44
7	D_{20}	20	7S54

Table 2 – *Continued on next page*

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	Groups with $n+17$ set-orbits		
n	G	**Order**	**GAP ID**
7	$C_5 \rtimes C_4$	20	7S55
7	$(C_6 \times C_2) \rtimes C_2$	24	7S59
7	$C_3 \times D_8$	24	7S64
7	$(C_6 \times C_2) \rtimes C_2$	24	7S65
7	$C_4 \times \mathsf{S}_3$	24	7S66
7	D_{24}	24	7S69
7	$D_8 \times \mathsf{S}_3$	48	7S79
7	S_5	60	7S80
7	S_5	120	7S86
8	$(C_2 \times C_2 \times C_2) \rtimes C_4$	32	8S171
8	$\mathsf{A}_4 \times \mathsf{A}_4$	144	8S260
8	$(\mathsf{A}_4 \times \mathsf{A}_4) \rtimes C_2$	288	8S274
8	$\mathsf{S}_4 \times \mathsf{A}_4$	288	8S275
8	$\mathsf{S}_4 \times \mathsf{S}_4$	576	8S284
9	$(\mathsf{S}_3 \times \mathsf{S}_3) \rtimes C_2$	72	9S388
10	$C_2 \times \mathsf{A}_8$	40320	10S1586
10	S_8	40320	10S1587
10	$C_2 \times \mathsf{S}_8$	80640	10S1589
11	$\mathsf{A}_6 \rtimes C_2$	720	11S2795
11	$(\mathsf{A}_6.C_2) \rtimes C_2$	1440	11S2913
12	$C_2.((C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_5)$	3840	12P256
12	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{S}_5) \rtimes C_2$	7680	12P270
12	$(C_2 \times C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_6$	11520	12P277
12	$(\mathsf{A}_5 \times \mathsf{A}_5) \rtimes (C_2 \times C_2)$	14400	12P279
12	$(C_2 \times C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{S}_6$	23040	12P285
13	$\mathrm{PSL}(3,3)$	5616	13P7
17	A_{16}	16!/2	
17	S_{16}	16!	

Table 2

	Groups with $n+18$ set-orbits		
n	G	**Order**	**GAP ID**
6	C_3	3	6S6
6	C_4	4	6S11
6	$C_2 \times C_2$	4	6S13
6	S_3	6	6S18
6	C_6	6	6S20
6	D_8	8	6S25
6	D_{12}	12	6S32
8	$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$	96	8S248
10	$((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$	216	10S1326
10	$(((C_3 \times C_3) \rtimes Q_8) \rtimes C_3) \rtimes C_2$	432	10S1432
12	$(C_2 \times C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{S}_5$	3840	12T257
18	A_{17}	$17!/2$	
18	S_{17}	$17!$	

Table 3

	Groups with $n+19$ set-orbits		
n	G	**Order**	**GAP ID**
5	C_2	2	5S2
7	D_{12}	12	7S47
8	$C_8 \rtimes C_2$	16	8S111
8	QD_{16}	16	8S122
8	$(C_2 \times C_2 \times C_2) \rtimes C_4$	32	8S174
8	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3$	48	8S197
8	$(C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2) \rtimes C_2$	64	8S223
9	$C_9 \rtimes C_3$	27	9S249
9	$(C_3 \times C_3) \rtimes C_4$	36	9S281
9	$((C_3 \times C_3) \rtimes C_3) \rtimes C_2$	54	9S349
9	$(C_3 \times C_3) \rtimes C_6$	54	9S353
9	$C_3 \rtimes \mathsf{S}_5$	360	9S501
9	$C_3 \times \mathsf{S}_5$	360	9S502
9	$\mathsf{S}_5 \times \mathsf{S}_3$	720	9S529

Table 4 – *Continued on next page*

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	Groups with $n+19$ set-orbits		
n	G	**Order**	**GAP ID**
9	$C_3 \times \mathsf{A}_6$	1080	9S532
9	$\mathsf{S}_3 \times \mathsf{A}_6$	2160	9S541
9	$C_3 \times \mathsf{S}_6$	2160	9S542
9	$\mathsf{A}_6 \rtimes \mathsf{S}_3$	2160	9S543
9	$\mathsf{S}_6 \times \mathsf{S}_3$	4320	9S546
11	$C_{11} \rtimes C_{10}$	110	11S1913
11	$\mathsf{A}_6.C_2$	720	11S2790
11	$C_2 \times \mathrm{PSL}(2,8)$	1008	11S2836
11	$C_2 \times (\mathrm{PSL}(2,8) \rtimes C_3)$	3024	11S2981
11	S_9	362880	11S3088
11	$C_2 \times \mathsf{A}_9$	362880	11S3089
11	$C_2 \times \mathsf{S}_9$	725760	11S3090
19	A_{18}	$18!/2$	
19	S_{18}	$18!$	

Table 4

	Groups with $n+20$ set-orbits		
n	G	**Order**	**GAP ID**
8	$(C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)$	32	8S166
8	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$	32	8S167
8	S_5	120	8S254
8	A_5	360	8S278
8	S_6	720	8S288
10	$C_2 \times ((C_2 \times C_2 \times C_2) \rtimes C_7)$	112	10S1132
10	$C_2 \times ((C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3))$	336	10S1394
10	$C_2 \times (\mathrm{PSL}(3,2) \rtimes C_2)$	672	10S1476
10	$C_2 \times ((C_2 \times C_2 \times C_2) \rtimes \mathrm{PSL}(3,2)$	2688	10S1394
20	S_{19}	$19!/2$	
20	A_{19}	$19!$	

Table 5

	Groups with $n+21$ set-orbits		
n	G	**Order**	**GAP ID**
6	$C_2 \times C_2 \times C_2$	8	6S21
7	C_6	6	7S21
7	$C_6 \times C_2$	12	7S35
7	D_{12}	12	7S39
7	$C_2 \times C_2 \times \mathsf{S}_3$	24	7S58
9	$C_2 \times (C_7 \rtimes C_6)$	84	9S402
9	$(((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3) \rtimes C_2$	192	9S465
9	$(((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_3$	192	9S467
9	$\mathsf{A}_4 \times (C_5 \rtimes C_4)$	240	9S488
9	$C_5 \rtimes (\mathsf{A}_4 \rtimes C_4)$	240	9S489
9	$(((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2) \rtimes C_3$	288	9S493
9	$C_2 \times \mathrm{PSL}(3,2)$	336	9S498
9	$((((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3) \rtimes C_2) \rtimes C_2$	384	9S507
9	$\mathsf{S}_4 \times (C_5 \rtimes C_4)$	480	9S517
9	$(\mathsf{A}_4 \times \mathsf{A}_4) \rtimes C_4$	576	9S519
9	$((((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2) \rtimes C_3) \rtimes C_2$	576	9S521
9	$\mathsf{A}_4 \times \mathsf{A}_5$	720	9S526
9	$(\mathsf{S}_4 \times \mathsf{S}_4) \rtimes C_2$	1152	9S533
9	$\mathsf{S}_5 \times \mathsf{A}_4$	1440	9S536
9	$\mathsf{S}_4 \times \mathsf{A}_5$	1440	9S537
9	$\mathsf{A}_5 \rtimes \mathsf{S}_4$	1440	9S538
9	$\mathsf{S}_5 \times \mathsf{S}_4$	2880	9S545
10	$\mathrm{PSL}(3,2) \rtimes C_2$	336	10S1391
12	S_{10}	3628800	12S10716
12	$C_2 \times \mathsf{A}_{10}$	362880	12S10718
12	$C_2 \times \mathsf{S}_{10}$	7257600	12S10719
14	$\mathrm{PSL}(2,13) \rtimes C_2$	2184	14T39
21	A_{20}	$20!/2$	
21	S_{20}	$20!$	

Table 6

	Groups with $n+25$ set-orbits		
n	G	**Order**	**GAP ID**
7	C_5	5	7S14
7	S_3	6	7S18
7	$C_3 \times C_3$	9	7S31
7	D_{10}	10	7S33
7	$(C_3 \times C_3) \rtimes C_2$	18	7S50
7	$C_3 \times \mathsf{S}_3$	18	7S52
7	$\mathsf{S}_3 \times \mathsf{S}_3$	36	7S72
8	$C_2 \times \mathsf{S}_4$	48	8S206
8	$C_2 \times C_2 \times \mathsf{A}_4$	48	8S208
9	$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$	96	9S403
10	$(C_2 \times C_2) \rtimes (\mathsf{A}_5 \rtimes \mathsf{S}_3)$	1440	10S1533
10	$\mathsf{A}_4 \times \mathsf{S}_5$	1440	10S1536
10	$\mathsf{S}_4 \times \mathsf{S}_5$	2880	10S1559
10	$\mathsf{A}_6 \times \mathsf{A}_4$	4320	10S1562
10	$(C_2 \times C_2) \rtimes (\mathsf{A}_6 \rtimes \mathsf{S}_3)$	8640	10S1572
10	$\mathsf{S}_6 \times \mathsf{A}_4$	8640	10S1573
10	$\mathsf{A}_6 \times \mathsf{S}_4$	8640	10S1574
10	$\mathsf{S}_6 \times \mathsf{S}_4$	17280	10S1582
11	$C_3 \times \mathsf{A}_8$	60480	11S3075
11	$C_3 \times \mathsf{S}_8$	120960	11S3081
11	$C_3 \rtimes \mathsf{S}_8$	120960	11S3082
11	$\mathsf{S}_3 \times \mathsf{A}_8$	120960	11S3083
11	$\mathsf{S}_8 \times \mathsf{S}_3$	241920	11S3086
12	$(C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2) \rtimes ((C_3 \times C_3) \rtimes C_3)$	3456	12T252
13	M_{11}	7920	$13\text{L}_1 162$
14	$C_2 \times \mathsf{S}_{12}$	958003200	$14\text{L}_2 2$
14	$C_2 \times \mathsf{A}_{12}$	479001600	$14\text{L}_2 46$
14	S_{12}	479001600	$14\text{L}_2 47$
24	M_{24}	244823040	24P1
25	A_{24}	24!/2	
25	S_{24}	24!	

Table 7

	Groups with $n+31$ set-orbits		
n	G	**Order**	**GAP ID**
8	D_8	8	7S29
8	$(C_4 \times C_2) : C_2$	16	8S107
8	$C_2 \times C_2 \times \mathsf{S}_3$	24	8S162
8	$(C_2 \times C_2 \times C_2 \times C_2) : C_2$	32	8S177
9	$C_7 : C_6$	42	9S309
9	$C_3 \times C_3 \times \mathsf{S}_3$	54	9S350
9	$C_3 \times ((C_3 \times C_3) : C_2)$	54	9S351
9	$C_5 \times \mathsf{A}_4$	60	9S358
9	$(C_3 \times \mathsf{A}_4) : C_2$	72	9S371
9	$C_3 \times \mathsf{S}_4$	72	9S372
9	$(C_3 \times C_3 \times C_3) : C_4$	108	9S418
9	$(C_3 \times C_3 \times C_3) : (C_2 \times C_2)$	108	9S420
9	$C_3 \times ((C_3 \times C_3) : C_4)$	108	9S421
9	$C_3 \times \mathsf{S}_3 \times \mathsf{S}_3$	108	9S422
9	$C_3 \times \mathsf{S}_3 \times \mathsf{S}_3$	108	9S423
9	$((C_3 \times C_3) : C_2) \times \mathsf{S}_3$	108	9S424
9	$C_5 : \mathsf{S}_4$	120	9S430
9	$C_5 \times \mathsf{S}_4$	120	9S431
9	$\mathsf{A}_4 \times D_{10}$	120	9S432
9	$\mathsf{S}_3 \times \mathsf{S}_4$	144	9S441
9	$C_2 \times ((C_3 \times \mathsf{A}_4) : C_2)$	144	9S442
9	$\mathsf{S}_4 \times \mathsf{S}_3$	144	9S445
9	$C_6 \times \mathsf{S}_4$	144	9S446
9	$\mathsf{S}_3 \times \mathsf{S}_4$	144	9S447
9	$\mathrm{PSL}(3,2)$	168	9S461
9	$(C_3 \times C_3 \times C_3) : D_8$	216	9S471
9	$C_3 \times ((\mathsf{S}_3 \times \mathsf{S}_3) : C_2)$	216	9S472
9	$C_3 \times ((C_3 \times C_3) : C_4)$	216	9S473
9	$(C_3 \times C_3 \times C_3) : D_8$	216	9S474
9	$(C_3 \times C_3 \times C_3) : D_8$	216	9S475
9	$\mathsf{S}_3 \times \mathsf{S}_3 \times \mathsf{S}_3$	216	9S476
9	$\mathsf{S}_4 \times D_{10}$	240	9S485
9	$C_2 \times \mathsf{S}_4 \times \mathsf{S}_3$	288	9S494
9	$((S_3 \times \mathsf{S}_3) : C_2) \times \mathsf{S}_3$	432	9S509

Table 8 – *Continued on next page*

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	Groups with $n+31$ set-orbits		
n	G	**Order**	**GAP ID**
11	$(((C_3 \times C_3) : Q_8) : C_3) : C_2$	432	11S2613
11	$C_2 \times (((C_3 \times C_3) : Q_8) : C_3)$	432	11S2622
11	$(C_5 \times C_5) : ((C_4 \times C_4) : C_2)$	800	11S2800
11	$C_2 \times ((((C_3 \times C_3) : Q_8) : C_3) : C_2)$	864	11S2816
11	$(C_2 \times C_2 \times C_2 \times C_2) : S_5$	1920	11S2928
11	$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) : \mathsf{A}_5)$	1920	11S2930
11	$\mathsf{S}_5 \times (C_5 : C_4)$	2400	11S2955
11	$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) : \mathsf{S}_5)$	3840	11S2994
11	$\mathsf{A}_5 : \mathsf{S}_5$	7200	11S3022
11	$(\mathsf{A}_5 \times \mathsf{A}_5) : C2$	7200	11S3023
11	$\mathsf{S}_5 \times \mathsf{A}_5$	7200	11S3025
11	$\mathsf{A}_6 \times (C_5 : C_4)$	7200	11S3028
11	$\mathsf{A}_6 : (C_5 : C_4)$	7200	11S3029
11	$(\mathsf{A}_5 \times \mathsf{A}_5) : C_4$	14400	11S3047
11	$(\mathsf{A}_5 \times \mathsf{A}_5) : (C_2 \times C_2)$	14400	11S3049
11	$\mathsf{S}_5 \times \mathsf{S}_5$	14400	11S3050
11	$\mathsf{S}_6 \times (C_5 : C_4)$	14400	11S3051
11	$\mathsf{A}_6 \times \mathsf{A}_5$	21600	11S3064
11	$(\mathsf{A}_5 \times \mathsf{A}_5) : D_8$	28800	11S3065
11	$\mathsf{A}_5 : \mathsf{S}_6$	43200	11S3072
11	$\mathsf{S}_6 \times \mathsf{A}_5$	43200	11S3073
11	$\mathsf{A}_6 \times \mathsf{S}_5$	43200	11S3074
11	$\mathsf{S}_6 \times \mathsf{S}_5$	86400	11S3080
12	$\mathsf{S}_6 : C_2$	1440	$12\mathrm{L}_2100$
13	$\mathrm{PSL}(2, 11)$	660	12T179
13	$\mathsf{S}_{10} \times \mathsf{S}_3$	10!×3!	$13\mathrm{L}_33$
13	$C_3 \times \mathsf{S}_{10}$	10!×3!/2	$13\mathrm{L}_34$
13	$\mathsf{A}_{10} \times \mathsf{S}_3$	10!×3!/2	$13\mathrm{L}_37$
13	$\mathsf{A}_{10} : \mathsf{S}_3$	10!×3!/2	$13\mathrm{L}_39$
13	$C_3 \times \mathsf{A}_{10}$	10!×3!/4	$13\mathrm{L}_310$
15	A_8	20160	15P4
17	$C_2 \times \mathsf{S}_{15}$	15!×2	$17\mathrm{L}_22$
17	$C_2 \times \mathsf{A}_{15}$	15!	$17\mathrm{L}_24$

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	Groups with $n+31$ set-orbits		
n	G	**Order**	**GAP ID**
17	S_{15}	15!	$17L_25$
17	$\mathrm{PSL}(2,16):C_4$	16320	17P8
31	A_{30}	30!/2	
31	S_{30}	30!	

Table 8

5. Conclusion

While studying the previous method and trying to extend it, we made some observations regarding the results. When $11 \le r \le 33$, we noticed that a permutation group had degree at most r. Specifically, A_{r-1} and S_{r-1} acting transitively on $r-1$ points and trivially on 1 point seemed to be the highest degree groups that gave $n+r$ set-orbits. This leads us to conjecture that for $r \ge 11$, there is no permutation group of degree $n > r$ that will give $n+r$ set-orbits. While we were unable to prove this, we know from Proposition 2.8 that if any group existed, it would have to be primitive.

There were a few computational challenges that caused difficulty in the classification. Calculating $s(G)$ is expensive when the size of the group is large, but has a relatively low number of set-orbits. This is because $s_t(G)$ has relatively small values until t grows close to $\lfloor \frac{n}{2} \rfloor$ and making estimations for $s(G)$ becomes ineffective. Specifically, the M_{24} group takes an unreasonable amount of time to calculate. Additionally, calculating all subgroups of $\mathsf{S}_k \times \mathsf{S}_{n-k}$ is still infeasible for larger k.

If one wants to extend the classification beyond $r = 33$, the computational limitations would be the greatest challenge. A more theoretical limitation would be determining an efficient method of finding all permutation groups whose orbits split into different cases.

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