Stabilizing a system with an unbounded random gain using only a finite number of bits

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Abstract

We study the stabilization of an unpredictable linear control system where the controller must act based on a rate-limited observation of the state. More precisely, we consider the system $X_{n+1} = A_n X_n + W_n - U_n$, where the A_n's are drawn independently at random at each time n from a known distribution with unbounded support, and where the controller receives at most R bits about the system state at each time from an encoder. We provide a time-varying achievable strategy to stabilize the system in a second-moment sense with fixed, finite R.

While our previous result provided a strategy to stabilize this system using a variable-rate code, this work provides an achievable strategy using a fixed-rate code. The strategy we employ to achieve this is time-varying and takes different actions depending on the value of the state. It proceeds in two modes: a normal mode (or zoom-in), where the realization of A_n is typical, and an emergency mode (or zoom-out), where the realization of A_n is exceptionally large.

1. Introduction

System design for decentralized control over communication networks requires an understanding of the informational bottlenecks that affect our ability to stabilize the system. This paper focuses on a system that grows unpredictably and is observed over a rate-limited channel. We consider a modification of the classical data-rate theorems [1], [2], [3], [4], where the system growth is unpredictable and random, and provide a strategy to control such a system over finite and fixed-rate channel. This builds on previous work [5], which considered the control of a system with unpredictable growth with unbounded support over a channel using a variable-rate code.

Specifically, we consider the control of the following system:

$$X_{n+1} = A_n X_n + W_n - U_n. \tag{1}$$

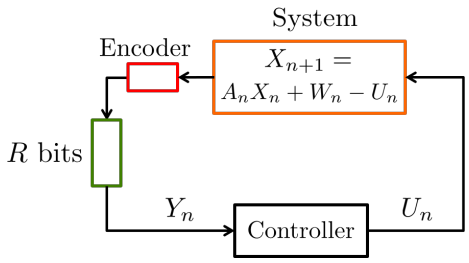

Fig. 1. Rate-limited control: The control U_n can be any function of the observations Y_0^n. The encoder chooses exactly R bits to transmit to the controller at every time-step through Y_n, i.e. $Y_n \in \{1, 2, \dots 2^R\}$. The system gains $\{A_n\}_{n\geq 0}$ are i.i.d. random variables, and so are the additive disturbances $\{W_n\}_{n\geq 0}$.

Here X_n is the (scalar) state of the system at time n. The system gains $\{A_n\}_{n\geq 0}$ are drawn i.i.d. from a known distribution and model the uncertainty the controller has about the system. The additive disturbances $\{W_n\}_{n\geq 0}$ are also i.i.d. drawn from a known distribution. The controller chooses the control U_n causally based on R-bit observations $Y_0^n := \{Y_0, \dots, Y_n\}$, where $Y_n \in \{1, 2, \dots 2^R\}$, which are transmitted by an encoder co-located with the system. Our goal is to stabilize the system in a second-moment sense, which thus requires the co-design of an encoder-controller pair.

It is known that without a rate-limit on the channel, the system in (1) is second-moment stabilizable if and only if $\sigma_A^2 < 1$, where σ_A^2 is the variance of A_n [6]. Hence, we focus on the case $\sigma_A^2 < 1$. The goal is to guarantee second-moment stability of the system. Our main theorem is stated below.

Theorem 1.1. *Assume that A_n and W_n have finite $(4+\epsilon)$-th moments. Then, for some $R \in \mathbb{N}$, there exists a rate-R-limited control strategy (see Fig. 1) that achieves second-moment stability, i.e.,*

$$\limsup_{n\to\infty} \mathbb{E}[X_n^2] < \infty. \tag{2}$$

A. Strategy overview

Previous works (e.g. [7], [8], [5]) have investigated variable-rate coding strategies for stabilizing the second-moment of the system (1), as well as other related systems. However, variable-rate strategies fundamentally require that the encoder send an unbounded number of bits in the case when the realization of A_n is large, and cannot be directly adapted for use in a fixed-rate regime.

In contrast to the variable-rate strategies, we employ a time-varying strategy in this work. It works in two different "modes," depending on whether the realization of A_n was typical or exceptionally large. We will describe the strategy precisely later, but provide an intuitive description here.

At every time-step, the controller maintains an interval that estimates system state. When the state remains in the interval that was estimated by the controller, the strategy proceeds

in "normal" (zoom-in) mode. If the state escapes the predicted interval, usually due to the realization of an exceptionally large A_n, the strategy enters "emergency" (zoom-out) mode. There, the controller grows it's guess for the interval at every time-step till we are able to locate X_n, following which we step back into normal mode. This time-varying strategy fundamentally works because we can guarantee some average decrease during normal mode, while we can bound the blow-ups from emergency mode (since we assume that the 4th moments of A and W are bounded).

B. Related work

Fundamentally, our problem setup grows out of the uncertainty threshold principle [6] and the extensive work to understand data-rate theorems [1], [2], [3], [4], [9]. However, the work on the uncertainty threshold principle [6] does not consider rate limits on the observations, and the classical data-rate theorems assume that the system model and growth rate are always perfectly known to the controller.

Martins et al. [7] were the first to consider the rate-limited control of a system with uncertain growth. Their achievable scheme is quantization-based and assumes that the uncertainty on the system growth is bounded above. Phat et al. [10] also consider rate-limited control with uncertain parameters from a robust control perspective. Their setup differs from ours in that it is not stochastic, and only considers bounded support for the uncertainty on the parameters. They provide a uniform quantization scheme that can stabilize their system.

Okano and Ishii have made significant progress on understanding rate-limited control of systems with unpredictable growth from a worst-case perspective [11], [12], [8]. However, they also consider systems where the uncertain parameters have bounded support and do not consider additive noise in their model. The achievable scheme in [8] proposes a non-uniform optimal quantizer for their problem that uses bins with logarithmically decreasing lengths, with the bins closest to zero being the largest. However, this cannot work in the setting where both A_n and X_n can have unbounded support, as is the case in our work.

We also build on a series of investigations around model unpredictability and multiplicative noise in control systems [13], [14], [15]. In this work, we take a stochastic control approach, which complements the investigations around parameter uncertainty in the robust control literature [16], [17].

In the context of linear systems with known parameters and unbounded additive disturbances, the necessity of adaptive quantization for stabilization has been long recognized. Nair and Evans [18] proved that time-invariant fixed-rate quantizers are unable to attain bounded cost if the noise is unbounded [18], regardless of their rate. The reason is that since the noise is unbounded, over time, a large magnitude noise realization will inevitably be encountered, and the dynamic range of the quantizer will be exceeded by a large margin, not permitting recovery. Adaptive quantizers of zooming type were originally proposed by Brockett and Liberzon [19], and further studied in [18], [20], [21]. Such quantizers "zoom out" (i.e. expand their quantization intervals; this corresponds to our "emergency mode") when the system is far from the target and "zoom in" when the system is close to the target (this corresponds to our "normal mode").

This paper extends our recent analysis [21] of adaptive quantization for known linear systems to unpredictably varying ones.

2. Setup

We consider the system in (1). Here W_n, A_n are i.i.d. variable noises drawn from the laws W and A. These are independent of the system state X_n and the control term U_n. We use n as the time index. Without loss of generality, we assume our initial condition $X_0 = 0$. The *encoder* and *controller* will work together to stabilize the sequence (X_n) in the second-moment sense, i.e., (2). The encoder observes X_n perfectly, and transmits a symbol from the set $\{1, 2, \ldots, 2^R\}$ to the controller at each time-step. R is a finite universal constant that we will choose later. The controller can choose U_n as a function of these R bits and any previous history. Let $\mu_A, \sigma_A, \mu_W, \sigma_W$ be the means and standard deviations of A, W. Without loss of generality, set $\mu_W = 0$.

The goal of second-moment stability cannot be accomplished for arbitrary distributions A, W. Using the independence of A_n, W_n and U_n, we see from (1) that [6]:

$$\mathbb{E}[X_{n+1}^2] = \sigma_A^2 \mathbb{E}[X_n^2] + \mathbb{E}[(\mu_A X_n - U_n)^2] + \sigma_W^2. \tag{3}$$

It is clear from (3) that the assumption $\sigma_A < 1$ is required for second-moment stabilizability (except possibly in the case $\sigma_A = 1, \sigma_W = 0$ which we ignore.) Our result will provide a strategy to achieve it. We require the mild assumption that A and W have finite $4 + \varepsilon$-moment for some $\varepsilon > 0$.

3. Description of the Strategy

At the nth time-step, three things happen in the following order:

1) The encoder transmits a codeword from a codebook of size R bits to the controller based on the observation of X_n. The exact value of R is described below.
2) The controller chooses U_n based on the transmission from the encoder and any past history.
3) The noises A_n, W_n are generated and the new state X_{n+1} is determined, based on A_n, W_n and U_n. This value of X_{n+1} is observed by the encoder.

We proceed in "rounds," which are blocks of time-steps. Most rounds will consist of a single time-step, in those cases where X_n falls in an interval predicted by the encoder-controller pair. In those cases where X_n falls outside those bounds, a round might consist of multiple time-steps, until an appropriate bounding interval is found for the state.

Throughout, we will maintain positive numbers $I_n \leq M_n$, that will be defined below. Roughly speaking, M_n will represent an estimate of the maximum value of $|X_n|$, and I_n will capture the quantization error in the controller's estimate of X_n.

We fix M_0 (a lower bound for the M_n's), a large constant P, and a small δ to be chosen later, such that $\frac{1}{\delta} \in \mathbb{Z}$. The constant P captures the controller's guess on how the M_n's will grow at each time-step. The constant δ specifies the size of the codebook to $\frac{2}{\delta} + 1$, and the transmission rate is $R = \lceil \log(\frac{2}{\delta} + 1) \rceil$.

We will now describe how the M_n's will evolve with time along with our strategy. We first initialize $M_1 = I_1 = M_0$. At each time-step, we check how X_n compares to the guess of the controller. If $|X_n| \leq PM_{n-1}$, we operate in normal mode. In this case, the encoder transmits a quantized version of the state to the controller. If $|X_n| > PM_{n-1}$ we operate in emergency mode, and the encoder simply sends a special codeword to indicate this emergency mode. A *round* of time-steps will continue until we exit emergency mode at a time ℓ such that $X_\ell \leq PM_{\ell-1}$.

Case 1: $X_n \leq PM_{n-1}$.

In this case, X_n has landed within the bounds we expected it to land in, and we are starting the round in *normal mode*. The encoder partitions the interval $[-PM_{n-1}, PM_{n-1}]$ into $\frac{2}{\delta}$ subintervals of length δPM_{n-1} and sends a codeword to represent the index of the interval (a_n, b_n) containing X_n. Set

$$M_n = \max(M_0, |a_n|, |b_n|), \tag{4}$$

$$I_n = \max\left(M_0, \frac{|a_n - b_n|}{2}\right), \tag{5}$$

$$\rho_{n+1} = \text{sgn}(a_n) = \text{sgn}(b_n), \tag{6}$$

where equality in (6) holds because $\frac{1}{\delta}$ is an integer, and so the point 0 is always an endpoint of a quantization interval. The values M_n, I_n, ρ_n are common knowledge between the encoder and controller; they are uniquely determined by the bits that have been sent by the encoder. Consequently, in normal mode, the controller knows which interval of the form

$$\rho_n[M_n - 2I_n, M_n] \tag{7}$$

contains X_n, where $\rho_n \in \{\pm 1\}$.

Now the controller will set

$$U_n = \rho_n \mu_A (M_n - I_n). \tag{8}$$

The random variables A_n, W_n are generated, and X_{n+1} is determined. Hence, we will have that:

$$|\mu_A X_n - U_n| \leq \mu_A I_n. \tag{9}$$

If $|X_{n+1}| \leq PM_n$, the *round* ends at time-step $n+1$. Else, it will proceed for at least one more time-step.

Case 2: $|X_n| > PM_{n-1}$.

If this is the case, the encoder simply uses the special ($\frac{2}{\delta} + 1$-th) codeword to indicate that we are in *emergency mode* and the state has escaped the predicted interval. The *round* will continue until we exit emergency mode.

Throughout emergency mode, we will take

$$M_{n+k+1} = PM_{n+k}$$

until exiting. More specifically, if at the start of time-step $n+k$ we are in emergency mode, the following steps happen:

1) The encoder uses the special codeword to indicate that we are in emergency mode.
2) The controller sets $U_{n+k} = 0$. Since the controller does not have a good estimate of the state, it does nothing.
3) A_{n+k}, W_{n+k} are generated, and X_{n+k+1} is determined. If $X_{n+k+1} < PM_{n+k}$, we exit emergency mode and the round ends. Else, we continue to the next time-step in emergency mode.

Remark 3.1. *Note that the state always belongs to the interval indicated by (7) at the end of a round. We also see that $M_n, I_n \geq M_0$ at all times.*

4. Preparations for Analysis

A. A Modified Sequence

The idea is to show that $|X_n|^2$ decreases on average when it is large. However this is not true during emergency mode, so we will define a dominating sequence (N_n) which does have the desired property. The analysis will also use a trick of modifying the sequence (X_n) slightly. We will freeze the value of X_n at a time of interest n_0, by setting the subsequent A_n's to 1 and W_n's to 0. This modification is an important step required to obtain the dominating sequence though Proposition 4.1 as explained in Remark 4.2. If we continued to let the state X_n evolve, then we would have to account for the probability that X_{n_0} was large but A_{n_0} was small.

First, we make a few important definitions. Fix n_0, where $\mathbb{E}[X_{n_0}^2]$ is what we would like to bound. We define a modified sequence as follows:

$$\tilde{X}_n = \begin{cases} X_n & \text{if } n < n_0, \\ X_{n_0} & \text{if } n \geq n_0. \end{cases} \tag{10}$$

which satisfies the recurrence

$$\tilde{X}_{n+1} = \tilde{A}_n \tilde{X}_n + \tilde{W}_n - \tilde{U}_n \tag{11}$$

for

$$(\tilde{A}_n, \tilde{W}_n, \tilde{U}_n) = \begin{cases} (A_n, W_n, U_n) & \text{if } n < n_0 \\ (1, 0, 0) & \text{if } n \geq n_0 \end{cases} \tag{12}$$

Define a sequence $\tilde{M}_n$:

$$\tilde{M}_n = \begin{cases} M_n & \text{if } n \leq n_0 \\ P\tilde{M}_{n-1} & \text{if } n > n_0 \text{ and } |\tilde{X}_{n_0}| > \tilde{M}_{n-1} \\ \tilde{M}_{n-1} & \text{if } n > n_0 \text{ and } |\tilde{X}_{n_0}| \leq \tilde{M}_{n-1}. \end{cases} \tag{13}$$

Note that $\tilde{M}_n$ might still be growing for $n > n_0$, even though the sequence $\tilde{X}_n$ it aims to bound is not growing for $n > n_0$. Finally define:

$$\tilde{I}_n = \begin{cases} I_n & \text{if } n \leq n_0 \\ I_{n_0} & \text{if } n > n_0. \end{cases} \tag{14}$$

Now we define a counter for the length of a round for the modified sequence $\tilde{X}_n$.

Definition 1. Define $\tau(n)$ by

$$\tau(n) = \min\left\{m \geq n : |\tilde{X}_m| \leq P\tilde{M}_{m-1}\right\}. \tag{15}$$

Assuming time-step $n < n_0$ does not involve an emergency which lasts until time n_0, the value $\tau(n)$ is simply the first time-step n or larger which starts in normal mode. Hence for a time-step n which exits in normal mode, $\tau(n) = n$. For a time-step n which starts in emergency mode, suppose that time-steps $n, ..., n+k < n_0$ exit in emergency mode and $n+k+1$ does not. Then

$$\tau(n) = \tau(n+1) = ... = \tau(n+k) = n+k+1. \tag{16}$$

Note that we always have $\tau(\tau(n)) = \tau(n) \geq n$, usually with equality.

Definition 2. Let K be a large constant. Define

$$Q_n = \sqrt{\tilde{M}_n^2 + K\tilde{I}_n^2}. \tag{17}$$

The strange-seeming Q_n is essentially M_n but with better expected-decrease guarantees that will be relevant later.

Finally, we define the dominating sequence which we will show is decreasing on average for large n. With this definition of N_n we "front-load" the cost of an emergency mode. Thus, N_n is decaying exponentially during the emergency mode, even though X_n is not, which helps us obtain the required bound.

Definition 3. For a time-step n, let

$$N_n = Q_{\tau(n)} 2^{\tau(n)-n}. \tag{18}$$

This sequence essentially measures the time until the end of the round from any n through the exponent $\tau(n) - n$. The idea is to show that N_n decreases on average when it is large, and to conclude from this that $\mathbb{E}[N_n^2]$ is uniformly bounded. We show in Proposition 4.1 below that N_{n_0} bounds X_{n_0}, allowing us to conclude that $\mathbb{E}[X_{n_0}^2]$ is bounded, for all n_0.

For technical reasons explained just below, we will work with the sequence $\tilde{X}_n$ when estimating $\mathbb{E}[X_{n_0}^2]$. We think of this modified sequence as evolving normally until time n_0. At time n_0, we freeze the value of X_n to be X_{n_0}. We finish the current round if we are in the middle of emergency mode, and then stop permanently.

Proposition 4.1. *We always have*

$$|X_{n_0}| \leq N_{n_0}. \tag{19}$$

Proof. Since $X_{n_0} = \tilde{X}_{n_0}$ and the value of $\tilde{X}_n$ is frozen to be $\tilde{X}_{n_0}$ for all $n \geq n_0$ and $\tau(n_0) \geq n_0$, we have that

$$|\tilde{X}_{n_0}| = |\tilde{X}_{\tau(n_0)}| \leq M_{\tau(n_0)} \leq Q_{\tau(n_0)} \leq N_{n_0}. \tag{20}$$

□

Remark 4.2. *If we did not switch to working with the sequence $(\tilde{X}_n)$, then the inequalities above would hold with X in place of $\tilde{X}$ but the first equality would not hold. The trick of freezing the value of $\tilde{X}_n$ at X_{n_0} for $n \geq n_0$ allows us to focus exactly on bounding the quantity of interest, which is $\mathbb{E}[X_{n_0}^2]$, without worrying about fluctuations in the sequence due to A_n's for $n \geq n_0$.*

5. Analysis of the Algorithm

Now we state our main theorem showing that N_n decreases on average when large. We need to specify a filtration; we will include in the σ-algebra $\mathcal{F}_n$ all steps of the algorithm which have happened so far.

Definition 4. Let $\mathcal{F}_n$ be the σ-algebra generated by $(A_k)_{k<n}$, $(W_k)_{k<n}$ and all bits sent by the encoder in time-steps up through $n-1$.

Theorem 5.1. *Let n_0 be any fixed time. Then, for any $n < n_0$, positive constant $c < \min\left(1-\sigma_A^2, \frac{3}{4}\right)$ that does not depend on n_0, sufficiently large P, M_0, K, and sufficiently small $\delta = \delta(c, P, M_0, K)$, there exists a constant $C(c, P, M_0, K, \sigma_W, \sigma_A)$ such that*

$$\mathbb{E}[N_n^2] \leq C \tag{21}$$

Our main result, Theorem 1.1 follows immediately as a corollary.

Proof of Theorem 1.1. For any fixed n_0, modifying the sequence at n_0 results in a sequence N_n. We know from Prop 4.1 that $|X_{n_0}| \leq N_{n_0}$. Theorem 5.1 implies inductively that

$$\sup_n(\mathbb{E}[N_n^2]) = \sup_{n_0}\left(\sup_{n<n_0} \mathbb{E}[N_n^2]\right) < \infty, \tag{22}$$

which gives the result. □

A. Proof of Theorem 5.1

For a time-step $n < n_0$ that is in normal mode, we have that $\tau(n) = n$. We write:

$$N_{n+1}^2 = N_{n+1}^2 \mathbf{1}_{\tau(n+1)=n+1} + N_{n+1}^2 \mathbf{1}_{\tau(n+1)>n+1}. \tag{23}$$

We estimate the conditional expectation of each term separately. For the first term we establish a second-moment averaged decrease from M_n to M_{n+1} when we stay in normal mode, while for the second we show that the contributions from emergency mode are small. The first step is elementary but with slightly involved algebra, while the second requires us to control the tails

for the length of a round (and thus requires a bounded $4+\epsilon$-th moment). We split this work into the following pair of lemmas.

Lemma 5.2. *Suppose that time-steps n and $n+1$ are both $< n_0$ and start in normal mode, so that $\tau(n) = n$ and $\tau(n+1) = n+1$. Then for any positive constant $c < 1 - \sigma_A^2$, sufficiently large K, arbitrary P, M_0, and sufficiently small $\delta = \delta(c, P, M_0, K)$ we have*

$$\mathbb{E}[N_{n+1}^2 \mathbf{1}_{\tau(n+1)=n+1} \mid \mathcal{F}_n] \leq (1-c)N_n^2 + 2\sigma_W^2 + KM_0^2 \text{ a.s.}. \tag{24}$$

Lemma 5.3. *Suppose that A and W have finite $4+\epsilon$-th moments. Further, suppose that time-step n starts in normal mode, so that $\tau(n) = n$. However, $X_{n+1} > PM_n$, so that $\tau(n+1) > n+1$. Then,*

$$\mathbb{E}[N_{n+1}^2 \mathbf{1}_{\tau(n+1)>n+1} \mid \mathcal{F}_n] \leq \varepsilon N_n^2 \text{ a.s.}, \tag{25}$$

where $\varepsilon(P, M_0)$ may be made made arbitrarily small by choosing M_0 sufficiently large and then P sufficiently large.

We now briefly explain how to derive Theorem 5.1 from Lemmas 5.2 and 5.3.

Proof of Theorem 5.1, using Lemmas 5.2 and 5.3. First, we consider the case when we are in normal mode and $\tau(n) = n$, and use the representation in (23). For c as in the theorem statement, we first pick P, M_0 large enough such that, in the language of Lemma 5.3, $\varepsilon(P, M_0)$ satisfies

$$c + \varepsilon < \min\left(1 - \sigma_A^2, \frac{3}{4}\right). \tag{26}$$

Lemma 5.3 then gives:

$$\mathbb{E}[N_{n+1}^2 \mathbf{1}_{\tau(n+1)>n+1} \mid \mathcal{F}_{\tau(n)}] \leq \varepsilon N_n^2. \tag{27}$$

Then substituting $c + \varepsilon$ for c in Lemma 5.2, pick K large and then δ small to make

$$\mathbb{E}[N_{n+1}^2 \mathbf{1}_{\tau(n+1)=n+1} \mid \mathcal{F}_{\tau(n)}] \leq (1-c-\varepsilon)N_n^2 + 2\sigma_W^2 + KM_0^2 \tag{28}$$

hold whenever $\tau(n) = n$. Combining (27) and (28) using (23), we have

$$\mathbb{E}[N_{n+1}^2 \mid \mathcal{F}_{\tau(n)}] \leq (1-c)N_n^2 + 2\sigma_W^2 + KM_0^2. \tag{29}$$

in this case.

Now consider the case that we are in emergency mode and $\tau(n) > n$. Here, due to (18) we automatically have $N_{n+1} = \frac{N_n}{2}$. Since $c < \frac{3}{4}$ and $\mathcal{F}_{\tau(n)}$ includes N_n, it follows that:

$$\mathbb{E}[N_{n+1}^2 \mid \mathcal{F}_{\tau(n)}] = \frac{1}{4}N_n^2 \leq (1-c)N_n^2. \tag{30}$$

Thus (29) also holds when $\tau(n) > n$.
Let

$$D = 2\sigma_W^2 + (1+K)M_0^2. \tag{31}$$

and observe that (29) implies

$$\mathbb{E}[N_{n+1}^2] \leq (1-c)\mathbb{E}[N_n^2] + 2\sigma_W^2 + KM_0^2. \tag{32}$$

Now we can proceed using induction. Assume $\mathbb{E}[N_n^2] \leq \frac{D}{c} = C$. Then, we have from (32) that

$$\mathbb{E}[N_{n+1}^2] \leq (1-c)\frac{D}{c} + D - M_0^2 \tag{33}$$

$$< \frac{D}{c}. \tag{34}$$

It remains to verify the base case. Since $X_0 = 0$, we know that $M_0 = I_0$, the initial $\tau(0) = 0$ and $Q_0 = (1+K)M_0^2$. Hence $N_0 = Q_0 \cdot 2^0 = Q_0$ and $\mathbb{E}[N_0^2] \leq D \leq \frac{D}{c}$, since $c < 1$.

This proves Theorem 5.1 assuming the lemmas. □

B. Proofs of Lemmas

Finally, we prove the two key lemmas below.

Proof of Lemma 5.2. In the case considered here, we have $\tau(n) = n$, and this implies $\mathcal{F}_{\tau(n)} = \mathcal{F}_n$ and $N_n = Q_n$. Further, we have $\tau(n+1) = n+1$, and so $N_{n+1} = Q_{n+1}$. Hence, for the remainder of this proof we will refer to the Q's instead of the N's. Similarly, since $n < n_0$ we will refer to the X's instead of the $\tilde{X}$'s, the A's instead of $\tilde{A}$'s, etc; these modifications are only important for Lemma 5.3 and not Lemma 5.2.

Note that from the definition (17) we have $Q_n = \sqrt{\tilde{M}_n^2 + K\tilde{I}_n^2} = \sqrt{\tilde{M}_n^2 + K\tilde{I}_n^2}$. We will obtain the bound (24) by bounding both M_n and I_n. We first start with obtaining a bound on M_n

From (3) we have

$$\mathbb{E}[X_{n+1}^2|\mathcal{F}_n] = \sigma_A^2 X_n^2 + (\mu_A X_n - U_n)^2 + \sigma_W^2. \tag{35}$$

In addition, since we are in normal mode we have from (9): that

$$|\mu_A X_n - U_n| \leq \mu_A I_n.$$

This implies

$$\mathbb{E}[X_{n+1}^2 \mathbf{1}_{\tau(n+1)=n+1}|\mathcal{F}_n] \leq E[X_{n+1}^2|\mathcal{F}_n] \leq \sigma_A^2 M_n^2 + \mu_A I_n^2 + \sigma_W^2. \tag{36}$$

Furthermore, we assume this round lasts exactly one time-step and $\tau(n+1) = n+1$. Hence, we also have that either

$$|M_{n+1} - X_{n+1}| \leq P\delta M_n, \tag{37}$$

or the state X_{n+1} has gotten below the lower limit M_0 and we have that $M_{n+1} = M_0$:

$$|X_{n+1}| \leq M_{n+1} = M_0. \tag{38}$$

From this we obtain the error estimate

$$\mathbb{E}[(M_{n+1}^2 - X_{n+1}^2)\mathbf{1}_{\tau(n+1)=n+1} \mid \mathcal{F}_n] \leq P\delta M_n \left(2\mathbb{E}[|X_{n+1}|\mathbf{1}_{\tau(n+1)=n+1} \mid \mathcal{F}_n] + P\delta M_n\right) + M_0^2. \tag{39}$$

This follows because we have that $(a^2 - b^2) = (a+b)(a-b)$, and further we observe that $|X + M| \leq |2X| + |M - X|$. The last term M_0^2 comes from the case where the state hits the lower limit, and follows from (38).

Now using the definition of X_{n+1} in (1), the choice of U_n in (8) and the simple estimate $\mathbb{E}[|Z|] \leq \mu_Z + \sigma_Z$ for any real random variable Z, we obtain:

$$\mathbb{E}[|X_{n+1}| \mid \mathcal{F}_n] \leq \mathbb{E}[|A_n|]|X_n| + \mathbb{E}[|W_n|] + |\mu_A M_n| \leq (2\mu_A + \sigma_A)M_n + \sigma_W. \tag{40}$$

Adding (36) and (39), and then using the bound in (40) for the conditional expectation of $|X_{n+1}|$ we get that:

$$\mathbb{E}[M_{n+1}^2 \mathbf{1}_{\tau(n+1)=n+1} \mid \mathcal{F}_n] \leq \left(\sigma_A^2 + (2\mu_A + \sigma_A)(2P\delta) + P^2\delta^2\right) M_n^2 + 2P\delta\sigma_W M_n + \mu_A I_n^2 + \sigma_W^2 + M_0^2. \tag{41}$$

Since $2P\delta\sigma_W M_n \leq P^2\delta^2 M_n^2 + \sigma_W^2$ we can eliminate the term that is linear in M_n, obtaining the slightly cleaner

$$\mathbb{E}[M_{n+1}^2 \mathbf{1}_{\tau(n+1)=n+1} | \mathcal{F}_n] \leq \left(\sigma_A^2 + (2\mu_A + \sigma_A)(2P\delta) + 2P^2\delta^2\right) M_n^2 + \mu_A I_n^2 + 2\sigma_W^2 + M_0^2. \tag{42}$$

Estimating I_{n+1}^2 is much simpler; since we know from (5) that $|I_{n+1}| \leq \max(P\delta M_n, M_0)$ whenever $\tau(n) = n$ we obviously have

$$\mathbb{E}[I_{n+1}^2 \mathbf{1}_{\tau(n)=n} | \mathcal{F}_n] \leq P^2\delta^2 M_n^2 + M_0^2. \tag{43}$$

Now, recall we defined $Q_n = \sqrt{\tilde{M}_n^2 + K\tilde{I}_n^2}$. Adding (42) and (43) we get:

$$\mathbb{E}[Q_{n+1}^2 \mathbf{1}_{\tau(n+1)=n+1} | \mathcal{F}_n] \leq \left(\sigma_A^2 + (2\mu_A + \sigma_A)(2P\delta) + (2+K)P^2\delta^2\right) M_n^2 + \mu_A I_n^2 + 2\sigma_W^2 + (K)M_0^2. \tag{44}$$

Now, we choose parameters K, δ so that we can bound the coefficients of M_n^2 and I_n^2 in (44), i.e, so that the following two inequalities hold:

$$\sigma_A^2 + (2\mu_A + \sigma_A)(2P\delta) + (2+K)P^2\delta^2 \leq 1 - c, \text{ and} \tag{45}$$

$$\mu_A \leq (1-c)K. \tag{46}$$

These can be satisfied for any positive $c < 1 - \sigma_A^2$ if we take P arbitrary, choose $K \geq \frac{\mu_A}{1-c}$, and finally take $\delta = \delta(c, K, P)$ sufficiently small. This gives:

$$\begin{aligned}\mathbb{E}[Q_{n+1}^2 \mathbf{1}_{\tau(n+1)=n+1} | \mathcal{F}_n] &\leq (1-c)M_n^2 + (1-c)KI_n^2 + 2\sigma_W^2 + (K)M_0^2 \\ &= (1-c)Q_n^2 + 2\sigma_W^2 + (K)M_0^2.\end{aligned}$$

□

Remark 5.4. *Making the choice $c < 1 - \sigma_A^2$ is technically the only place in the proof where we use the necessary assumption $\sigma_A^2 < 1$. The value for P was essentially irrelevant in the above calculations but picking P appropriately will be important for Lemma 5.3. In particular, it is important that an arbitrarily large value of P is acceptable if we take δ very small to compensate. The value M_0 did not come into play but will play a minor role in proving Lemma 5.3.*

For some intuition on the definition of Q_n, note that if $M_n = I_n$, i.e. when we hit the lower limit of M_0, then we cannot guarantee an averaged squared decrease of $M_n \to M_{n+1}$. However, in this case we expect $I_{n+1} = P\delta I_n$ to hold. In the other extreme case when we know $X_n = M_n$, we expect $M_n \to M_{n+1}$ to result in a decrease, and so picking δ small enough ensures that the regularization term KI_n^2 doesn't hurt us too much. Hence in both cases we expect a squared decrease from $Q_n \to Q_{n+1}$ when parameters are chosen as appropriately, e.g. as above.

Proof of Lemma 5.3. Again, here note that $\tau(n) = n$, and we estimate $\mathbb{E}[N_{n+1}^2 \mathbf{1}_{\tau(n+1)>n+1}]$ under this assumption. Note that if $\tau(n+1) = n+k+1$ (for $k \geq 1$) then we have

$$\begin{aligned} N_{n+1}^2 &\leq 2^{2k}\left(\tilde{M}_{n+k+1}^2 + \tilde{I}_{n+k+1}^2\right) \\ &\leq 2^{2k+1}\tilde{M}_{n+k+1}^2 = 2^{2k+1}P^{2k+2}\tilde{M}_n^2 \\ &\leq 2^{2k+1}P^{2k+2}N_n^2. \end{aligned} \tag{47}$$

Hence qualitatively, it will suffice to show $\tau(n+1)-(n+1)$ has very fast decaying tails. This is what we will do.

We have $\tilde{X}_{n+1} = \tilde{A}_n\tilde{X}_n + \tilde{W}_n - \tilde{U}_n$. For later emergency rounds, we have $\tilde{U}_{n+j} = 0$ and so

$$\tilde{X}_{n+j+1} = \tilde{A}_{n+j}\tilde{X}_{n+j} + \tilde{W}_{n+j}.$$

Hence again taking $\tau(n+1) = n+k+1$, for each $k \geq h \geq 0$ we may write

$$\tilde{X}_{n+h+1} = (\tilde{A}_{n+1}\tilde{A}_{n+2}...\tilde{A}_{n+h})(\tilde{A}_n\tilde{X}_n - \tilde{U}_n) + \sum_{i=0}^{h}\left(\tilde{W}_i \prod_{j=i+1}^{h}\tilde{A}_{n+j}\right).$$

Since the above equation only holds when $\tau(n+1) \geq n+h+1$, for each $h \geq 0$ we define

$$Z_{n+h+1} := (\tilde{A}_{n+1}\tilde{A}_{n+2}...\tilde{A}_{n+h})(\tilde{A}_n\tilde{X}_n - \tilde{U}_n) + \sum_{i=0}^{h}\left(\tilde{W}_i \prod_{j=i+1}^{h}\tilde{A}_{n+j}\right). \tag{48}$$

Since $|\tilde{U}_n| \leq |\mu_A|\tilde{M}_n$ we have

$$|Z_{n+h+1}| \leq 2\tilde{M}_n(|\tilde{A}_n| + |\mu_A|)(|\tilde{A}_{n+1}\tilde{A}_{n+2}...\tilde{A}_{n+h}|) + \sum_{i=0}^{h}\left(|\tilde{W}_i| \prod_{j=i+1}^{h}|\tilde{A}_{n+j}|\right). \tag{49}$$

To control the probability that $\tau(n+1) = n+k+1$, we will estimate the α-moments of Z_{n+h+1}. The point is that $\tau(n+1) = n+k+1$ implies $|Z_{n+k}| \geq P^k\tilde{M}_n$ which is an abnormally large value. To do this we need to control the moments for each term. For convenience, define

$$m_\alpha = \max\left(2, \mathbb{E}\left[(|A| + |\mu_A|)^\alpha\right]\right), \tag{50}$$

and

$$\ell_\alpha = \mathbb{E}[|W|^\alpha]. \tag{51}$$

Some of the $\tilde{A}_{n+j}$ terms may have the law of A, while others may be almost surely 1. However, the value m_α will serve to estimate both cases uniformly; similarly W might be identically 0, and this will also be fine. We have the simple estimate for the first term in (49), which follows from (50), the independence of $\tilde{M}_n$ and all of the $\tilde{A}_k$'s, which are i.i.d..

$$\mathbb{E}\left[\left|2\tilde{M}_n(|\tilde{A}_n| + |\mu_A|)(\tilde{A}_{n+1}\tilde{A}_{n+2}...\tilde{A}_{n+h})\right|^\alpha \middle| \mathcal{F}_n\right] \leq \tilde{M}_n^\alpha 2^\alpha m_\alpha^{h+1}. \tag{52}$$

For the individual terms in the summation in (49), we similarly have from (50), (51):

$$\mathbb{E}\left[\left(|\tilde{W}_i| \prod_{j=i+1}^{h} |\tilde{A}_{n+j}|\right)^\alpha \middle| \mathcal{F}_n\right] \leq \ell_\alpha m_\alpha^{h-i}.$$

Let $B_i = \left(|\tilde{W}_i| \prod_{j=i+1}^{h} |\tilde{A}_{n+j}|\right)$. Then, the above inequality can be written as $||B_i||_\alpha^\alpha \leq \ell_\alpha m_\alpha^{h-i}$, where the L_α-norm is taken with respect to $\mathbb{E}(\cdot \mid \mathcal{F}_n)$. The subadditivity of the L_α-norm implies that

$$||\sum_i B_i||_\alpha^\alpha \leq \left(\sum_i ||B_i||_\alpha\right)^\alpha.$$

This gives us the following bound for some constant $C(\alpha)$, where we also use $m_\alpha^{h-1} \leq m_\alpha^h$:

$$\mathbb{E}\left[\left(\sum_{i=0}^{h} |\tilde{W}_i| \prod_{j=i+1}^{h} |\tilde{A}_{n+j}|\right)^\alpha \middle| \mathcal{F}_n\right] \leq C(\alpha)\ell_\alpha m_\alpha^h. \tag{53}$$

To combine the two estimates (52) and (53) we simply note that

$$|x+y|^\alpha \leq 2^\alpha(|x|^\alpha + |y|^\alpha)$$

for all reals x, y. Hence we obtain the following, where $C_1(\alpha)$ and $C_2(\alpha)$ are also constants:

$$\mathbb{E}\left[|Z_{n+h+1}|^\alpha \mid \mathcal{F}_n\right] \leq C_1(\alpha)(\tilde{M}_n^\alpha m_\alpha^{h+1} + \ell_\alpha m_\alpha^h) \leq C_2(\alpha) m_\alpha^h(\tilde{M}_n^\alpha + 1).$$

Because we have $|\tilde{M}_n| \geq M_0$ for all n, we may simply say for a constant $C_3(\alpha)$:

$$\mathbb{E}\left[|Z_{n+h+1}|^\alpha \mid \mathcal{F}_n\right] \leq C_3(\alpha) m_\alpha^h \tilde{M}_n^\alpha. \tag{54}$$

Now in the event that $\tau(n) = n+k+1$ we must have

$$|Z_{n+k}| \geq P^k \tilde{M}_n.$$

Therefore by the Markov inequality,

$$\begin{aligned}\mathbb{P}\left[|Z_{n+k}| \geq P^k \tilde{M}_n \mid \mathcal{F}_n\right] &\leq P^{-k\alpha} \tilde{M}_n^{-k\alpha} \mathbb{E}\left[|Z_{n+k}|^\alpha \mid \mathcal{F}_n\right] \\ &\leq C_3(\alpha) P^{-k\alpha} \tilde{M}_n^{\alpha - k\alpha} m_\alpha^k. \end{aligned} \tag{55}$$

Combining our work, we have

$$\mathbb{E}[|N_{n+1}^2 \tau(n+1) > n+1] \leq \sum_{k\geq 1} \left(2^{2k+1} P^{2k+2} N_n^2 \ \mathbb{P}\left[|Z_{n+k}| \geq P^k \tilde{M}_n\right]\right) \tag{56}$$

$$\leq C_4(\alpha) N_n^2 \sum_{k\geq 1} \left(2^{2k-2} P^{2k+2-k\alpha} \tilde{M}_n^{\tau(1-k)} m_\alpha^k\right), \tag{57}$$

for a fourth constant $C_4(\alpha)$. The first inequality (56) follows from (55), and the second (57) follows from (47). This sum is a geometric series with first term $P^{4-\alpha} m_\alpha$, and ratio, $4P^{2-\alpha}\tilde{M}_n^{-\alpha} m_\alpha$. Hence we obtain

$$\mathbb{E}[N_{n+1}^2 \mathbf{1}_{\tau(n+1)>n+1}] \leq C_\alpha^{(4)} N_n^2 \left(\frac{P^{4-\alpha} m_\alpha}{1 - 4P^{2-\alpha}\tilde{M}_n^{-\alpha} m_\alpha}\right).$$

Since everything so far has been uniform in P, we see that for any $\alpha > 4$, taking P sufficiently large gives the upper bound

$$\mathbb{E}[N_{n+1}^2 \mathbf{1}_{\tau(n+1)>n+1}] \leq \varepsilon N_n^2$$

as desired. (Recall A, W have bounded α-moment for some $\alpha > 4$.)

□

6. Conclusion and Future work

Our paper considered the problem of stabilizing a system that was growing unpredictably using observations over a rate-limited channel. We provide a time-varying strategy that is able to stabilize the system. The controller takes different actions based on whether the value of the system state is in a predicted interval or not — we believe such a time-varying strategy is essential for this problem. In future work, we aim to close the gap between this strategy and the converse bound in [5]. Extensions to the vector case are also interesting, since the differential growth rate along different directions must be taken into account.

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