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Ising Model on Trees and Factors of IID

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Abstract: We study the ferromagnetic Ising model on the infinite d-regular tree under the free boundary condition. This model is known to be a factor of IID in the uniqueness regime, when the inverse temperature $\beta \geq 0$ satisfies $\tanh \beta \leq (d-1)^{-1}$. However, in the reconstruction regime ($\tanh \beta > (d-1)^{-\frac{1}{2}}$), it is not a factor of IID. We construct a factor of IID for the Ising model beyond the uniqueness regime via a strong solution to an infinite dimensional stochastic differential equation which partially answers a question of Lyons (Comb Probab Comput 2(2):285–300, 2017). The solution $\{X_t(v)\}$ of the SDE is distributed as

$$X_t(v) = t\tau_v + B_t(v),$$

where $\{\tau_v\}$ is an Ising sample and $\{B_t(v)\}$ are independent Brownian motions indexed by the vertices in the tree. Our construction holds whenever $\tanh \beta \leq c(d-1)^{-\frac{1}{2}}$, where $c > 0$ is an absolute constant.

1. Introduction

Let $\mathcal{T} = (\mathcal{T}^d, \rho)$ be the infinite d-regular tree rooted at a vertex ρ. For two measurable spaces Ω_0 and Ω_1, a map $\phi : \Omega_0^{\mathcal{T}} \to \Omega_1^{\mathcal{T}}$ is called a $\mathcal{T}$**-factor** if it is measurable in terms of the product σ-algebra and satisfies

$$\phi(\eta(\omega)) = \eta(\phi(\omega)) \quad (\eta \in \mathrm{Aut}(\mathcal{T}),\ \omega \in \Omega_0^{\mathcal{T}}),$$

where $\mathrm{Aut}(\mathcal{T})$ denotes the automorphism group of $\mathcal{T}$, that is, the collection of graph isomorphisms from $\mathcal{T}$ to itself. We are interested in the notion of *factor of IID* on $\mathcal{T}$, which is an ergodic property of measures defined as follows.

Definition 1.1 (*Factor of IID on $\mathcal{T}$*). For a measure ν on a measurable space $\Omega_1^{\mathcal{T}}$, ν is called a **factor of IID** if there exists a measurable space Ω_0, a measure μ_0 on Ω_0, and a $\mathcal{T}$-factor $\phi : \Omega_0^{\mathcal{T}} \to \Omega_1^{\mathcal{T}}$ such that ν is the ϕ-push-forward of the product measure $\mu_0^{\otimes \mathcal{T}}$.

Besides the classical works from ergodic theory [4,19–21], study of factor of IIDs has drawn extensive interests in probability theory, see e.g. [1,2,9,10,14,23,25,26,28]. In particular, a factor of IID on $\mathcal{T}$ can be interpreted as an infinite analogue of local algorithms on the random d-regular graph (whose limiting local structure is $\mathcal{T}$) [5,8,22]. For a detailed introduction to this concept and related works in probability theory, we refer to [13] and the references therein.

We study the ferromagnetic Ising model defined on $\mathcal{T}$ under the free boundary condition, and determine if it is a factor of IID. This model is also called as *binary symmetric channel on trees*, or *broadcasting problem on trees*, and has drawn substantial interest not only from probability theory and statistical mechanics, but also from information theory and theoretical computer science (see, e.g., [3,6,7,12,16–18]). When the inverse temperature β satisfies $\tanh\beta \le (d-1)^{-1}$, the Ising model on $\mathcal{T}$ has the unique Gibbs measure and is known to be a factor of IID. On the other hand, when $\tanh\beta > (d-1)^{-\frac{1}{2}}$, it is in the reconstruction regime (see Sect. 1.1 for its definition) and the second author proved that it is not a factor of IID ([13, Theorem 3.1]). However, it is not known whether the Ising model on $\mathcal{T}$ is a factor of IID when $(d-1)^{-1} < \tanh\beta \le (d-1)^{-\frac{1}{2}}$. Our main result proves that it is a factor of IID beyond the uniqueness regime to a certain extent, partially answering a question of Lyons [13].

Theorem 1. *There exist absolute constants $c, d_0 > 0$ such that for all $d \ge d_0$ and $\beta \ge 0$ with $\tanh\beta \le c(d-1)^{-\frac{1}{2}}$, the Ising model on $\mathcal{T}^d$ under the free boundary condition at inverse temperature β is a factor of IID.*

A common approach to show that a measure is a factor of IID is the *divide-and-color* method from the study of Random Cluster Model (RCM) (see [27] and references therein for details). This is particularly useful to study the Ising/Potts model, which can be viewed as RCM through their natural connections to FK percolation, and each cluster is assigned an independent color. If β is small (i.e. in the high temperature regime), the percolation clusters are a.s. finite, we can construct a factor of IID by choosing the color of each cluster in an automorphism invariant way. Unfortunately, in our case, this method only works for $\tanh\beta \le (d-1)^{-1}$, as otherwise there exist infinite clusters. Thus proving Theorem 1 requires a different idea.

One question asked by Steif and Tykesson in [27] is whether there were examples of RCM, such that there are infinite clusters but the coloring process is still a factor of IID. In [24], the second and third author verified that the stationary measures of the voter model on $\mathbb{Z}^d$ are factors of IID, which is another case where there are infinite clusters in the corresponding RCM, and a naive divide-and-color method does not work. Their approach was to utilize the *coalescing random walk*, which is a dual model whose $t \to \infty$ limit gives the stationary measure of the voter model, and is a factor of IID at all finite times by the divide-and-color method. They constructed a factor of IID at $t \to \infty$ limit by a clever coupling of the finite-time divide-and-color configurations, where all vertices change their color only finitely many times a.s. as t goes to infinity. However, a similar approach fails in our case; see Sect. 1.2 for details.

Our approach is to construct a factor of IID from the independent one-dimensional standard Brownian motions $\mathbf{W} = \{(W_t(v))_{t\ge 0}\}_{v\in\mathcal{T}}$. We construct an infinite-dimensional system of stochastic differential equations whose strong solution recovers the Ising model in the $t \to \infty$ limit. Constructing the desired strong solution is done using the approximation by the corresponding finite-dimensional system of SDEs and requires technically sophisticated computations. Although our method works for large d and up to the point where $\tanh \beta \le c(d-1)^{-\frac{1}{2}}$, we conjecture that the Ising model on $\mathcal{T}$ should be a factor of IID for all $d \ge 3$ and $\tanh \beta \le (d-1)^{-\frac{1}{2}}$.

1.1. Ising model on trees. In this subsection, we briefly review the definition of the Ising model and its basic properties. On a finite graph $\mathcal{G}$ with edge set $E(\mathcal{G})$, the *Ising model on $\mathcal{G}$ under the free boundary condition at inverse temperature β* is a probability measure $\pi_{\mathcal{G}}$ on $\{\pm 1\}^{\mathcal{G}}$ defined as follows. For $\boldsymbol{\sigma} = (\sigma_v)_{v \in \mathcal{G}}$,

$$\pi_{\mathcal{G}}(\boldsymbol{\sigma}) = \frac{1}{Z_{\mathcal{G}}} \exp\left(\beta \sum_{(u,v)\in E} \sigma_u \sigma_v \right), \tag{1.1}$$

where $Z_{\mathcal{G}}$ is the normalizing constant given by $Z_{\mathcal{G}} = \sum_{\boldsymbol{\sigma}\in\{\pm 1\}^{\mathcal{G}}} \exp\left(\beta \sum_{(u,v)\in E} \sigma_u\sigma_v\right)$. We will be interested in the ferromagnetic case where β is non-negative.

For given $d \in \mathbb{N}$, let $(\mathcal{T}, \rho) = (\mathcal{T}^d, \rho)$ denote the inifnite d-regular tree rooted at ρ, and let $\mathcal{T}_R$ be the depth-R subtree defined by

$$\mathcal{T}_R = \mathcal{T}_R^d := \{v \in \mathcal{T} : \mathrm{dist}(\rho, v) \le R\}. \tag{1.2}$$

It is well-known that $\pi_{\mathcal{T}_R}$ converges in product σ-algebra. We denote the limit as $\pi_{\mathcal{T}}$, which is called the *infinite-volume Gibbs measure on $\mathcal{T}$ with free boundary condition*, and is the main object of interest in this paper. The measure $\pi_{\mathcal{T}}$ can also be constructed by a recursive Markovian fashion as follows. Let $\theta := \tanh \beta$, and consider the 2×2 transition matrix

$$P := \begin{pmatrix} \frac{1+\theta}{2} & \frac{1-\theta}{2} \\ \frac{1-\theta}{2} & \frac{1+\theta}{2} \end{pmatrix}.$$

This defines a Markov chain on the single-spin space $\{-1, +1\}$: for instance, +1 becomes -1 with probability $\frac{1-\theta}{2}$, and stays at +1 with probability $\frac{1+\theta}{2}$. Then, it is well-known that $\pi_{\mathcal{T}}$ is equivalent to the law of $\boldsymbol{\sigma}$ generated by the following recursive scheme:

- At the root ρ, set $\sigma_\rho = \pm 1$ each with probability $\frac{1}{2}$.
- For each edge (u, v) in $\mathcal{T}$, suppose that $\mathrm{dist}(\rho, u) + 1 = \mathrm{dist}(\rho, v)$ and σ_u is determined. Then, σ_v is obtained by a single-step Markov chain from σ_u with respect to the transition matrix P.

This construction explains why the model is also called the *broadcasting problem on trees*: a vertex passes its spin correctly to a child with probability θ, and otherwise (i.e., with probability $1-\theta$) it delivers a randomized spin that is either +1 or -1 with equal probability. This procedure happens independently at each edge.

We now introduce the two thresholds for β in the Ising model on $\mathcal{T}$. The first is the *uniqueness threshold* $\beta_c := \tanh^{-1}((d-1)^{-1})$. When $\beta \le \beta_c$, it is known that the infinite-volume Gibbs measure is unique regardless of the boundary condition; when $\beta > \beta_c$, the measure $\pi_{\mathcal{T}}$ is not the unique infinite-volume Gibbs measure on $\mathcal{T}$. For detailed introduction on this subject, we refer to [12] and [15, Chapter 17]. For its relation to the FK percolation in the context of RCM, see [27, Section 1.3].

Besides the uniqueness threshold, the *reconstruction threshold* $\beta_r = \tanh^{-1}((d-1)^{-\frac{1}{2}})$ gives the phase transition for whether the reconstruction problem is solvable. For $\beta \le \beta_r$, given by $\boldsymbol{\sigma} \sim \pi_{\mathcal{T}}$, it is impossible to guess σ_ρ better than probability $\frac{1}{2}$ in the $R \to \infty$ limit from just looking at $(\sigma_v)_{v \in \mathcal{T} \setminus \mathcal{T}_R}$. However, we can beat the random guess if $\beta > \beta_r$, the *reconstruction regime*. For a detailed study on this concept, we refer to [7] and references therein.

As mentioned above, we are interested in the *intermediate regime* of $\beta_c < \beta \le \beta_r$, and this is the only case where it is unknown whether $\pi_{\mathcal{T}}$ is a factor of IID.

1.2. A Glauber dynamics approach. The first approach that one may come up with to prove Theorem 1 is using the Glauber dynamics, which is a Markov chain that converges to some Gibbs measure. Since the Glauber dynamics can be encoded by IID information assigned at each vertex, we may hope to construct a factor of IID from this Markov chain starting with an IID process. This approach is indeed possible in the uniqueness regime $\tanh \beta \le (d-1)^{-1}$. In this subsection, we briefly explain why it does not look promising beyond the uniqueness regime.

To begin with, we give a short description of the Glauber dynamics. Each vertex $v \in \mathcal{T}$ is assigned with an IID rate-one Poisson process, which defines the update times at v. When an update occurs at $v \in \mathcal{T}$ at time t, $\sigma_t(v)$ is updated with respect to the Ising measure on $\{v\}$ conditioned on its neighborhood profile $\{\sigma_{t^-}(u)\}_{u \sim v}$. Namely, it becomes +1 with probability

$$\frac{1}{2}\left(1 + \tanh\left(\beta \sum_{u \sim v} \sigma_{t^-}(u)\right)\right),$$

and transitions to -1 otherwise.

To construct the Glauber dynamics that converges to $\pi_{\mathcal{T}}$, one can start from the initial condition given by $\sigma_{t=0}(v) = \pm 1$ with probability $\frac{1}{2}$, independently for each vertex v. Assume that we have two instances of Glauber dynamics $\boldsymbol{\sigma}_t = (\sigma_t(v))_{v \in \mathcal{T}}$ and $\boldsymbol{\sigma}'_t = (\sigma'_t(v))_{v \in \mathcal{T}}$, and further suppose that at some time t we had $\sigma_t(v) = \sigma'_t(v)$ at all $v \in \mathcal{T}$ except at a single vertex $v_0 \in \mathcal{T}$. Even under an optimal coupling between $\boldsymbol{\sigma}_t$ and $\boldsymbol{\sigma}'_t$, the spin at vertex $v \sim v_0$ will be updated differently in the two instances with probability

$$\frac{1}{2}\left|\tanh\left(\beta \sum_{u \sim v} \sigma_t(u)\right) - \tanh\left(\beta \sum_{u \sim v} \sigma'_t(u)\right)\right|,$$

which is equal to $\tanh \beta$ in the worst case. Then, $\tanh \beta > (d-1)^{-1}$ implies that the *disagreement percolation* is not necessarily subcritical. Thus, the function that maps the IID update information of the Glauber dynamics to the Ising model is not measurable, since it has long-range dependence between distant vertices. This infers that in contrast to [24], we may have to look for a different approach rather than relying on a stochastic process that converges to the Ising model.

2. Factor Construction by Independent Brownian Motions

In this section, we give the construction of the factor of IID for the Ising model, and prove Theorem 1.

2.1. Notations and setup. From now on we fix an arbitrary d, assuming it being large enough.

We work on the rooted d-regular tree $(\mathcal{T}, \rho)$. For any $\rho' \in \mathcal{T}$ and $R \in \mathbb{N}$, we denote $\mathcal{T}_R(\rho')$ as the subgraph induced by

$$\{v \in \mathcal{T} : \operatorname{dist}(\rho', v) \leq R\}. \tag{2.1}$$

We also write $\mathcal{T}_R := \mathcal{T}_R(\rho)$ for ease of notations.

For a finite graph $\mathcal{G}$ with edge set $E(\mathcal{G})$, given the external field $\mathbf{x} = (x(v))_{v\in\mathcal{G}}$, and inverse temperature $\boldsymbol{\beta} = \{\beta_{u,v}\}_{(u,v)\in E(\mathcal{G})}$, let $\pi_{\mathcal{G}}^{\boldsymbol{\beta},\mathbf{x}}$ be the measure on $\{\pm 1\}^{\mathcal{G}}$, such that

$$\pi_{\mathcal{G}}^{\boldsymbol{\beta},\mathbf{x}}(\boldsymbol{\sigma}) = \frac{1}{Z_{\mathcal{G}}^{\boldsymbol{\beta},\mathbf{x}}} \exp\left(\sum_{(u,v)\in E(\mathcal{G})} \beta_{u,v}\sigma_u\sigma_v + \sum_{v\in\mathcal{G}} x(v)\sigma_v \right), \tag{2.2}$$

for any $\boldsymbol{\sigma} = (\sigma_v)_{v\in\mathcal{G}}$, where $Z_{\mathcal{G}}^{\boldsymbol{\beta},\mathbf{x}} := \sum_{\boldsymbol{\sigma}\in\{\pm1\}^{\mathcal{G}}} \exp\Big(\sum_{(u,v)\in E(\mathcal{G})} \beta_{u,v}\sigma_u\sigma_v + \sum_{v\in\mathcal{G}} x(v)\sigma_v \Big)$ is the normalizing constant. For simplicity of notations, we shall usually write $\pi_{\mathcal{G}}^{\mathbf{x}}$, $Z_{\mathcal{G}}^{\mathbf{x}}$ for $\pi_{\mathcal{G}}^{\boldsymbol{\beta},\mathbf{x}}$, $Z_{\mathcal{G}}^{\boldsymbol{\beta},\mathbf{x}}$. We shall also omit $\mathbf{x}$ when $\mathbf{x} = \mathbf{0}$, and omit $\mathcal{G}$ when it is clear which graph we are working on.

For any measure μ defined on the probability space $\{\pm1\}^{\mathcal{G}}$, and two measurable functions $f, g : \{\pm1\}^{\mathcal{G}} \to \mathbb{R}$, we write

$$\langle f \rangle_\mu := \sum_{\boldsymbol{\sigma}\in\{\pm1\}^{\mathcal{G}}} f(\boldsymbol{\sigma})\mu(\boldsymbol{\sigma}), \quad \langle f\,;\,g\rangle_\mu := \langle fg\rangle_\mu - \langle f\rangle_\mu \langle g\rangle_\mu.$$

For simplicity of notations we also write $\langle\cdot\rangle_R^{\mathbf{x}}$ for $\langle\cdot\rangle_{\mathcal{T}_R^{\mathbf{x}}}$ and $\langle\cdot\,;\,\cdot\rangle_R^{\mathbf{x}}$ for $\langle\cdot\,;\,\cdot\rangle_{\mathcal{T}_R^{\mathbf{x}}}$. We denote $\pi_{\mathcal{T}}$ as the free boundary Gibbs measure on $\mathcal{T}$ (i.e. the weak limit of $\pi_{\mathcal{T}_R}$ as $R \to \infty$).

2.2. Construction via finite systems of SDEs. In this subsection we take constant $\boldsymbol{\beta}$, i.e. $\beta_{u,v} = \beta$ for any edge (u, v), and we give the construction for the factor of IID of $\pi_{\mathcal{T}}$. Take independent one-dimensional Brownian motions $\mathbf{W} = \{(W_t(v))_{t\geq0}\}_{v\in\mathcal{T}}$, which is an i.i.d. process on $\mathcal{T}$. We shall construct $\pi_{\mathcal{T}}$ as a function of $\mathbf{W}$, invariant under automorphisms of $\mathcal{T}$.

We start from the finite tree $(\mathcal{T}_R, \rho)$, the depth-R subtree rooted at ρ, for $R > 0$. Define the function $\mathbf{F}^R : \mathbb{R}^{\mathcal{T}_R} \to \mathbb{R}^{\mathcal{T}_R}$, where for any $v \in \mathcal{T}_R$ and $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$, we let $F_v^R(\mathbf{x}) := \langle\sigma_v\rangle_R^{\mathbf{x}}$. We also let $\mathbf{X}^R = (\mathbf{X}_t^R)_{t\geq0} = \{(X_t^R(v))_{t\geq0}\}_{v\in\mathcal{T}_R}$ be the strong solution of the following finite dimensional stochastic differential equation system on $C([0,\infty);\mathbb{R})^{\mathcal{T}_R}$:

$$d\mathbf{X}_t^R = \mathbf{F}^R(\mathbf{X}_t^R)dt + d\mathbf{W}_t^R, \quad \mathbf{X}_0 = \mathbf{0}, \tag{2.3}$$

where $\mathbf{W}^R$ denotes the restriction of $\mathbf{W}$ to $\mathcal{T}_R$. We note that for any $u, v \in \mathcal{T}_R$, $\partial_{x_u} F_v^R(\mathbf{x}) = \langle\sigma_u\,;\,\sigma_v\rangle_R^{\mathbf{x}}$, and its absolute value is always bounded by 1. This implies that F_v^R is 1-Lipschitz in each coordinate. Thus, the strong solution of this system of SDEs exists and is unique.

We claim that $\mathbf{X}^R$ has the same law as the following process. Let $\boldsymbol{\tau} = (\tau_v)_{v\in\mathcal{T}}$ be sampled from $\pi_{\mathcal{T}}$, and let $\mathbf{B} = \{(B_t(v))_{t\geq 0}\}_{v\in\mathcal{T}}$ be another collection of independent one-dimensional standard Brownian motions. Consider the stochastic process $\overline{\mathbf{X}} = (\overline{\mathbf{X}}_t)_{t\geq 0} = \{(\overline{X}_t(v))_{t\geq 0}\}_{v\in\mathcal{T}}$ defined as

$$\overline{\mathbf{X}}_t := t\boldsymbol{\tau} + \mathbf{B}_t. \tag{2.4}$$

Let $\overline{\mathbf{X}}^R$ be the restriction of $\overline{\mathbf{X}}$ on $\mathcal{T}_R$. We have that $\overline{\mathbf{X}}^R$ has the same law as $\mathbf{X}^R$, by the following lemma.

Lemma 2.1. *The process $\overline{\mathbf{X}}^R$ is a weak solution to* (2.3).

Proof. For any $t \geq 0$, let $\mathcal{F}_t$ be the σ-algebra generated by $(\overline{\mathbf{X}}_s^R)_{0\leq s\leq t}$. Then $\overline{\mathbf{X}}^R$ is a weak solution to the following system of SDEs (for details see e.g., [11, Section 7.4])

$$d\overline{X}_t^R(v) = \mathbb{E}[\tau_v|\mathcal{F}_t]dt + dW_t(v),\ \forall v\in\mathcal{T}_R,\quad \overline{\mathbf{X}}_0^R = \mathbf{0}.$$

It remains to compute $\mathbb{E}[\tau_v|\mathcal{F}_t]$. Denote $\boldsymbol{\tau}^R$ as the restriction of $\boldsymbol{\tau}$ on $\mathcal{T}_R$. For any $\mathcal{F}_t$ measurable set A, and $\boldsymbol{\sigma} = (\sigma_v)_{v\in\mathcal{T}_R} \in \{\pm 1\}^{\mathcal{T}_R}$,

$$\begin{aligned}\mathbb{P}\big(\boldsymbol{\tau}^R = \boldsymbol{\sigma},\ (\overline{\mathbf{X}}_s^R)_{0\leq s\leq t}\in A\big) &= \pi_{\mathcal{T}_R}(\boldsymbol{\sigma})\mathbb{P}\big((\mathbf{B}_s + s\boldsymbol{\sigma})_{0\leq s\leq t}\in A\big)\\ &= \pi_{\mathcal{T}_R}(\boldsymbol{\sigma})\frac{\mathbb{P}\big((\mathbf{B}_s + s\boldsymbol{\sigma})_{0\leq s\leq t}\in A\big)}{\mathbb{P}\big((\mathbf{B}_s)_{0\leq s\leq t}\in A\big)}\mathbb{P}\big((\mathbf{B}_s)_{0\leq s\leq t}\in A\big).\end{aligned}$$

Thus by Girsanov theorem, we have

$$\mathbb{P}\big(\boldsymbol{\tau}^R = \boldsymbol{\sigma}|\mathcal{F}_t\big) = \pi_{\mathcal{T}_R}(\boldsymbol{\sigma})\prod_{v\in\mathcal{T}_R}\exp\left(\overline{X}_t(v)\sigma_v - \frac{t}{2}\right)f((\overline{\mathbf{X}}_s^R)_{0\leq s\leq t})$$

where f is the Radon-Nikodym derivative of the law of $(\mathbf{B}_s^R)_{0\leq s\leq t}$ over the law of $(\overline{\mathbf{X}}_s^R)_{0\leq s\leq t}$. Thus we have that conditional on $\mathcal{F}_t$, the law of $\boldsymbol{\tau}^R$ is given by the Ising model with external field $\overline{\mathbf{X}}_t^R$; then for any $v\in\mathcal{T}_R$ we have $\mathbb{E}[\tau_v|\mathcal{F}_t] = \langle\sigma_v\rangle_R^{\overline{\mathbf{X}}_t^R} = F_v^R(\overline{\mathbf{X}}_t^R)$, and the conclusion follows. □

Given this lemma, our general strategy is to show almost sure convergence of $\mathbf{X}^R$ as $R\to\infty$; and the limit is the same if one starts from a different root ρ'. The limit would have the same law as $\overline{\mathbf{X}}$, from which we could recover the Ising model by taking $\lim_{t\to\infty}\operatorname{sign}(\overline{\mathbf{X}}_t)$.

To get the desired convergence, our key step bounds the difference between $\mathbf{X}^R$ and $\mathbf{X}^{R-1}$, as follows.

Proposition 2.2. *There exist an absolute constant $c > 0$, a constant $C_d > 0$ depending only on d, and $\alpha = \alpha(d,\beta)\in(0,1)$, such that when $\beta\geq 0$ and $\tanh\beta \leq c(d-1)^{-\frac{1}{2}}$, we have*

$$\mathbb{E}\left[\left(X_t^R(u) - X_t^{R-1}(u)\right)^2\right] \leq C_d e^{C_d t}\alpha^{R-\operatorname{dist}(\rho,u)-3}(d-1)^{\operatorname{dist}(\rho,u)}, \tag{2.5}$$

for all t, $R > 0$ and $u\in\mathcal{T}_{R-1}$.

Now we study the equations starting from a different root. Take any $\rho' \in \mathcal{T}$. We define the function $\mathbf{F}^{R,\rho'} : \mathbb{R}^{\mathcal{T}_R(\rho')} \to \mathbb{R}^{\mathcal{T}_R(\rho')}$, where for any $v \in \mathcal{T}_R(\rho')$ and $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R(\rho')}$, we let $F_v^{R,\rho'}(\mathbf{x}) := \langle \sigma_v \rangle_{\pi^{\mathbf{x}}_{\mathcal{T}_R(\rho')}}$. We also let $\mathbf{X}^{R,\rho'} = (\mathbf{X}_t^{R,\rho'})_{t\geq 0} = \{(X_t^{R,\rho'}(v))_{t\geq 0}\}_{v\in\mathcal{T}_R}$ be the strong solution of the following finite dimensional stochastic differential equation system on $C([0,\infty);\mathbb{R})^{\mathcal{T}_R(\rho')}$:

$$d\mathbf{X}_t^{R,\rho'} = \mathbf{F}^{R,\rho'}(\mathbf{X}_t^{R,\rho'})dt + d\mathbf{W}_t^{R,\rho'}, \quad \mathbf{X}_0 = \mathbf{0}, \tag{2.6}$$

where $\mathbf{W}^{R,\rho'}$ denotes the restriction of $\mathbf{W}$ to $\mathcal{T}_R(\rho')$.

Proposition 2.3. *Let c, C_d, α, β be as in Proposition 2.2, and ρ' be a neighbor of the root ρ. For any $t, R > 0$ and $u \in \mathcal{T}_R(\rho) \cap \mathcal{T}_R(\rho')$, we have*

$$\mathbb{E}\left[\left(X_t^R(u) - X_t^{R,\rho'}(u)\right)^2\right] \leq C_d e^{C_d t} \alpha^{R-\mathrm{dist}(\rho,u)-4}(d-1)^{\mathrm{dist}(\rho,u)}.$$

We now deduce Theorem 1 from Propositions 2.2 and 2.3.

Proof of Theorem 1. Let c be the same as in Propositions 2.2 and 2.3. By Proposition 2.2 we have that for any $t \geq 0$ and $u \in \mathcal{T}$,

$$\mathbb{E}\left[\sum_{R=1}^{\infty}\left|X_t^R(u) - X_t^{R-1}(u)\right|\right] < \infty.$$

This means that as $R \to \infty$, $X_t^R(u)$ converges almost surely. For each $t \in \mathbb{Q}_{\geq 0}$ we define $X_t(u)$ as the limit. Then the process $(\mathbf{X}_t)_{t\in\mathbb{Q}_{\geq 0}} = \{(X_t(v))_{t\in\mathbb{Q}_{\geq 0}}\}_{v\in\mathcal{T}}$ is defined for a.s. $\mathbf{W}$, and its finite dimensional distribution is the same as that of $(\overline{\mathbf{X}}_t)_{t\in\mathbb{Q}_{\geq 0}}$.

For $n \in \mathbb{N}$ we define $\mathbf{G}_n = \{G_n(u)\}_{u\in\mathcal{T}}$ as $G_n(u) := \mathrm{sign}(X_{2^n}(u))$. Then as $n \to \infty$, $\mathbf{G}_n$ converges in law to $\pi_{\mathcal{T}}$. We now show that $\mathbf{G}_n$ converges almost surely. Indeed, we have

$$\begin{aligned}\mathbb{P}\left(G_n(u) \neq G_{n+1}(u)\right) &= \mathbb{P}\left(\mathrm{sign}(\overline{X}_{2^n}(u)) \neq \mathrm{sign}(\overline{X}_{2^{n+1}}(u))\right)\\ &\leq \mathbb{P}\left(|B_{2^n}(u)| > 2^{n-1}\right) + \mathbb{P}\left(|B_{2^{n+1}}(u) - B_{2^n}(u)| > 2^{n-1}\right).\end{aligned}$$

Thus, we can see that $\mathbb{P}(G_n(u) \neq G_{n+1}(u))$ is summable in n, which by Borel-Cantelli lemma implies that $G_n(u)$ converges a.s., as $n \to \infty$. We denote the limit by $\mathbf{G} \in \mathbb{R}^{\mathcal{T}}$, which is a measurable function of $\mathbf{W}$, defined for a.s. $\mathbf{W}$, and the law of $\mathbf{G}$ is $\pi_{\mathcal{T}}$.

Finally we show that this function $\mathbf{W} \mapsto \mathbf{G}$ is invariant under $\mathrm{Aut}(\mathcal{T})$. From our construction of $\mathbf{G}$ it suffices to prove the following: take any $\rho' \in \mathcal{T}$, then as $R \to \infty$, $X_t^{R,\rho'}(u)$ a.s. converges to $X_t(u)$, for any $t \in \mathbb{Q}_{\geq 0}$ and $u \in \mathcal{T}$. Indeed, $X_t^{R,\rho'}(u)$ a.s. converges, since the function $\mathbf{W} \mapsto X_t^{R,\rho'}(u)$ is the push-forward of $\mathbf{W} \mapsto X_t^R(u)$ under an action in $\mathrm{Aut}(\mathcal{T})$. Denote this limit as $X_t^{\rho'}(u)$. Let $\rho_0, \rho_1, \cdots, \rho_k$ be the path from $\rho_0 = \rho$ and $\rho_k = \rho'$. Applying Proposition 2.3 to the consecutive pairs (ρ_i, ρ_{i+1}) and sending $R \to \infty$, we have $\mathbb{E}\left[\left(X_t(u) - X_t^{\rho'}(u)\right)^2\right] = 0$; then $X_t(u) = X_t^{\rho'}(u)$ almost surely, and our conclusion follows. $\square$

2.3. An interpolation approach. In this subsection, we develop a framework to analyze the difference between $\mathbf{X}^{R-1}$ and $\mathbf{X}^R$, the unique strong solutions of (2.3). As a result, we reduce Propositions 2.2 and 2.3 into more tractable forms. From now on, denote $E_R := E(\mathcal{T}_R)$, and let ∂E_R denote the boundary edges of $\mathcal{T}_R$, i.e.

$$\partial E_R := E_R \setminus E_{R-1} = \{(u, v) \in E(\mathcal{T}_R) : v \in \partial \mathcal{T}_R\},$$

where $\partial \mathcal{T}_R := \{v \in \mathcal{T}_R : \mathrm{dist}(\rho, v) = R\}$.

The Ising model on $\mathcal{T}_{R-1}$ can be viewed as the Ising model on $\mathcal{T}_R$, by setting $\beta_{u,v} = 0$ for $(u, v) \in \partial E_R$. Thus, our approach to compare $\mathbf{X}^{R-1}$ and $\mathbf{X}^R$ is by interpolating the inverse temperature: we investigate the Ising model on $\mathcal{T}_R$ with inverse temperature $\boldsymbol{\beta}^\gamma := \{\beta^\gamma_{u,v}\}_{(u,v)\in E_R}$, where $\beta^\gamma_{u,v} = \beta$ for $(u, v) \in E_{R-1}$ and $\beta^\gamma_{u,v} = \gamma$ for $(u, v) \in \partial E_R$, for some $\gamma \in [0, \beta]$. Formally, for $\mathbf{x} = (x(v))_{v\in\mathcal{T}_R} \in \mathbb{R}^{\mathcal{T}_R}$, we define the probability measure $\pi^{\gamma,\mathbf{x}} := \pi^{\boldsymbol{\beta}^\gamma,\mathbf{x}}_{\mathcal{T}_R}$. Define the function $\mathbf{F}^\gamma : \mathbb{R}^{\mathcal{T}_R} \to \mathbb{R}^{\mathcal{T}_R}$, where for any $v \in \mathcal{T}_R$ and $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$, we let $F^\gamma_v(\mathbf{x}) := \langle \sigma_v \rangle_{\pi^{\gamma,\mathbf{x}}}$. We also let $\mathbf{Y}^\gamma = (\mathbf{Y}^\gamma_t)_{t\geq 0} = \{(Y^\gamma_t(v))_{t\geq 0}\}_{v\in\mathcal{T}_R}$ be the strong solution of the following finite dimensional stochastic differential equation system on $C([0,\infty); \mathbb{R})^{\mathcal{T}_R}$:

$$d\mathbf{Y}^\gamma_t = \mathbf{F}^\gamma(\mathbf{Y}^\gamma_t)dt + d\mathbf{W}^R_t, \quad \mathbf{Y}^\gamma_0 = \mathbf{0}, \tag{2.7}$$

where as before $\mathbf{W}^R$ denotes the restriction of $\mathbf{W}$ to $\mathcal{T}_R$, and are the driving Brownian motions. In particular, we have $\mathbf{Y}^0 = \mathbf{X}^{R-1}$ on $\mathcal{T}_{R-1}$, and $\mathbf{Y}^\beta = \mathbf{X}^R$. As for $\mathbf{X}^R$, the law of $\mathbf{Y}^\gamma = (\mathbf{Y}^\gamma_t)_{t\geq 0}$ is the same as that of $(t\boldsymbol{\tau} + \mathbf{B}_t)_{t\geq 0}$, where $\boldsymbol{\tau} = (\tau_v)_{v\in\mathcal{T}_R}$ is sampled from $\pi^{\gamma,\mathbf{0}}$, and $\mathbf{B} = \{(B_v(t))_{t\geq 0}\}_{v\in\mathcal{T}_R}$ is another collection of independent one-dimensional standard Brownian motions.

Note that $\mathbf{Y}^\gamma$ for all $\gamma \in [0, \beta]$ are generated by the same driving Brownian motions $\mathbf{W}^R_t$, thus if we define $\mathbf{H}^\gamma = (\mathbf{H}^\gamma_t)_{t\geq 0} = \{(H^\gamma_t(v))_{t\geq 0}\}_{v\in\mathcal{T}_R}$ to be

$$\mathbf{H}^\gamma_t := \partial_\gamma \mathbf{Y}^\gamma_t, \tag{2.8}$$

then from (2.7), we can deduce that

$$\frac{d\mathbf{H}^\gamma_t}{dt} = \partial_\gamma \left\{\mathbf{F}^\gamma(\mathbf{Y}^\gamma_t)\right\} = \nabla \mathbf{F}^\gamma(\mathbf{Y}^\gamma_t)\mathbf{H}^\gamma_t + \partial_\gamma \mathbf{F}^\gamma\left(\mathbf{Y}^\gamma_t\right), \tag{2.9}$$

and $\mathbf{H}^\gamma_0 = \mathbf{0}$. We have the following estimate on $\mathbf{H}^\gamma$.

Proposition 2.4. *Under the above setting, there exist absolute constants $c > 0$, and $C_d > 0$ depending only on d, and $\alpha = \alpha(d, \beta) \in (0, 1)$, such that if $d > c^{-1}$, $\tanh \beta \leq c(d-1)^{-\frac{1}{2}}$, and $0 \leq \gamma \leq \beta$, we have*

$$\mathbb{E}\left[H^\gamma_t(u)^2\right] \leq C_d e^{C_d t} \alpha^{R-\mathrm{dist}(\rho,u)-3}(d-1)^{\mathrm{dist}(\rho,u)}$$

for all $t > 0$, $u \in \mathcal{T}_R$.

It is straight-forward to deduce Proposition 2.2 from Proposition 2.4.

Proof of Proposition 2.2. From the construction we have

$$X_t^R(u) - X_t^{R-1}(u) = \int_0^\beta H_t^\gamma(u)d\gamma, \tag{2.10}$$

and thus, the Cauchy-Schwarz inequality gives

$$\begin{aligned}\mathbb{E}\left[\left(X_t^R(u) - X_t^{R-1}(u)\right)^2\right] &\le \beta^2 \sup_{\gamma\in[0,\beta]} \mathbb{E}\left[H_t^\gamma(u)^2\right] \\ &\le C_d e^{C_d t}\alpha^{R-\operatorname{dist}(\rho,u)-3}(d-1)^{\operatorname{dist}(\rho,u)},\end{aligned}$$

concluding the proof. □

Now we move on to the case of Proposition 2.3, where we use a similar argument. Let ρ' be a neighbor of ρ, and let $\mathcal{T}_R^\sharp := \mathcal{T}_R(\rho)\cup\mathcal{T}_R(\rho')$. Here, we use the inverse temperature $\boldsymbol{\beta}^{\gamma,\rho'} = \{\beta_{u,v}^{\gamma,\rho'}\}_{(u,v)\in E(\mathcal{T}_R^\sharp)}$ defined as $\beta_{u,v}^{\gamma,\rho'} = \beta$ for $(u,v)\in E_R$, and $\beta_{u,v}^{\gamma,\rho'} = \gamma$ for $(u,v)\in E(\mathcal{T}_R^\sharp)\setminus E_R$. Moreover, define the probability measure $\pi^{\gamma,\rho',\mathbf{x}} := \pi_{\mathcal{T}_R^\sharp}^{\boldsymbol{\beta}^{\gamma,\rho'},\mathbf{x}}$. The function $\mathbf{F}^{\gamma,\rho'} : \mathbb{R}^{\mathcal{T}_R^\sharp} \to \mathbb{R}^{\mathcal{T}_R^\sharp}$ is defined to be $F_v^{\gamma,\rho'}(\mathbf{x}) := \langle\sigma_v\rangle_{\pi^{\gamma,\rho',\mathbf{x}}}$. Then, as before, we consider the strong solution $\mathbf{Y}^{\gamma,\rho'} = (\mathbf{Y}_t^{\gamma,\rho'})_{t\ge0} = \{(Y_t^{\gamma,\rho'}(v))_{t\ge0}\}_{v\in\mathcal{T}_R^\sharp}$ of the following system of stochastic differential equations:

$$d\mathbf{Y}_t^{\gamma,\rho'} = \mathbf{F}^{\gamma,\rho'}(\mathbf{Y}_t^{\gamma,\rho'})dt + d\mathbf{W}_t^\sharp, \quad \mathbf{Y}_0^{\gamma,\rho'} = \mathbf{0},$$

where $\mathbf{W}^\sharp$ denotes the restriction of $\mathbf{W}$ to $\mathcal{T}_R^\sharp$. This setup gives $Y^{\gamma,\rho'} = X^R$ for $\gamma = 0$, and if we switch the roles of ρ and ρ' from the beginning then it will correspond to $X^{R,\rho'}$. Then, as before, its γ-derivative $\mathbf{H}_t^{\gamma,\rho'} := \partial_\gamma\mathbf{Y}_t^{\gamma,\rho'}$ satisfies

$$\frac{d\mathbf{H}_t^{\gamma,\rho'}}{dt} = \partial_\gamma\{\mathbf{F}^{\gamma,\rho'}(\mathbf{Y}_t^{\gamma,\rho'})\} = \nabla\mathbf{F}^{\gamma,\rho'}(\mathbf{Y}_t^{\gamma,\rho'})\mathbf{H}_t^{\gamma,\rho'} + \partial_\gamma\mathbf{F}^{\gamma,\rho'}(\mathbf{Y}^{\gamma,\rho'})_t,$$

and $\mathbf{H}_0^{\gamma,\rho'} = \mathbf{0}$. We can control $\mathbf{H}_t^{\gamma,\rho'}$ as follows.

Corollary 2.5. *Under the setting of* Proposition 2.4 *and the above notations, we have*

$$\mathbb{E}\left[H_t^{\gamma,\rho'}(u)^2\right] \le C_d e^{C_d t}\alpha^{R-\operatorname{dist}(\rho,u)-3}(d-1)^{\operatorname{dist}(\rho,u)},$$

for all $t > 0$ and $u \in \mathcal{T}_R^\sharp$.

We address the proof of Proposition 2.3 which comes as a direct consequence.

Proof of Proposition 2.3. To bound $\mathbb{E}\left[\left(X_t^R(u) - X_t^{R,\rho'}(u)\right)^2\right]$, we can just bound the expectations $\mathbb{E}\left[\left(X_t^R(u) - Y_t^{\beta,\rho'}(u)\right)^2\right]$ and $\mathbb{E}\left[\left(X_t^{R,\rho'}(u) - Y_t^{\beta,\rho'}(u)\right)^2\right]$ respectively.

For the first one, we can write the difference $X_t^R(u) - Y_t^{\beta,\rho'}(u) = Y_t^{0,\rho'}(u) - Y_t^{\beta,\rho'}(u)$ in an integral form as (2.10), and hence Corollary 2.5 implies

$$\mathbb{E}\left[\left(X_t^R(u) - Y_t^{\beta,\rho'}(u)\right)^2\right] \le C_d e^{C_d t}\alpha^{R-\operatorname{dist}(\rho,u)-3}(d-1)^{\operatorname{dist}(\rho,u)}.$$

By the same arguments and switching the roles of ρ and ρ', we can get the same bound for $\mathbb{E}\left[\left(X_t^{R,\rho'}(u) - Y_t^{\beta,\rho'}(u)\right)^2\right]$. We conclude the proof by combining the two bounds together, and noting that $|\mathrm{dist}(\rho, u) - \mathrm{dist}(\rho', u)| \le 1$. □

2.4. Reduction to a chain of covariances. Recall the definitions of $\mathbf{Y}^\gamma = \{Y_t^\gamma(u)\}$ given by (2.7) and $\mathbf{H}^\gamma = \{H_t^\gamma(u)\}$ given by (2.8). From now on, let $\gamma \in [0, \beta]$ be a fixed number and write $\mathbf{Y} = \mathbf{Y}^\gamma$, $\mathbf{H} = \mathbf{H}^\gamma$, i.e., drop the superscript γ from their expressions. We shall also just write $\pi^{\mathbf{x}}$ for $\pi^{\gamma,\mathbf{x}}$ for any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$. In this subsection, we give an explicit formula of $\mathbf{H}$ written by a chain of covariances, and state an estimate that controls its second moment, then prove Proposition 2.4. For the simplicity of exposition, we work with $\mathbf{H}$, and it will be clear from the discussion that the methods we describe below can be used to investigate $\mathbf{H}^{\gamma,\rho'}$ in the same way.

By straightforward computations, we can write (2.9) as

$$\frac{d\mathbf{H}_t}{dt} = \mathcal{M}_t \mathbf{H}_t + \mathcal{N}_t,$$

where $\mathcal{M}_t$ (resp. $\mathcal{N}_t$) is a $\mathcal{T}_R \times \mathcal{T}_R$ matrix (resp. $\mathcal{T}_R$-vector) given by

$$\mathcal{M}_t(u, v) = \langle \sigma_u \,;\, \sigma_v \rangle_{\pi^{\mathbf{Y}_t}}, \quad \mathcal{N}_t(v) = \sum_{(u,u') \in \partial E_R} \langle \sigma_u \sigma_{u'} \,;\, \sigma_v \rangle_{\pi^{\mathbf{Y}_t}}.$$

Since each entry of $\mathcal{M}_t$ and $\mathcal{N}_t$ is bounded by 1 and $|\partial E_R|$, respectively, we can write

$$\mathbf{H}_t = \sum_{k=1}^{\infty} \int_{0<t_1<\cdots<t_k<t} \mathcal{M}_{t_k} \cdots \mathcal{M}_{t_2} \mathcal{N}_{t_1} d\mathbf{t},$$

where $d\mathbf{t}$ denotes $dt_1 \ldots dt_k$.

We use the Cauchy-Schwarz inequality to obtain that

$$\mathbb{E}[H_t(u)^2] \le \left(\sum_{k=1}^{\infty} \frac{1}{2^k}\right) \cdot \sum_{k=1}^{\infty} 2^k \mathbb{E}\left[\left(\int_{0<t_1<\cdots<t_k<t} [\mathcal{M}_{t_k} \cdots \mathcal{M}_{t_2} \mathcal{N}_{t_1}](u) d\mathbf{t}\right)^2\right].$$

Moreover, applying the Cauchy-Schwarz inequality given by $(\int_A 1 dt)(\int_A f(t)^2 dt) \ge (\int_A f(t) dt)^2$ to the integral in the RHS, we have for any $u \in \mathcal{T}_R$ that

$$\mathbb{E}[H_t(u)^2] \le \sum_{k=1}^{\infty} 2^k \frac{t^k}{k!} \int_{0<t_1<\cdots<t_k<t} \mathbb{E}\left[\left(\sum_{v_1,\cdots,v_k=u} \mathcal{N}_{t_1}(v_1) \prod_{i=2}^{k} \mathcal{M}_{t_i}(v_i, v_{i-1})\right)^2\right] d\mathbf{t}. \tag{2.11}$$

To study this formula and establish Proposition 2.4, the following bound is crucial, and its proof is presented in the next section. For a sequence of vertices $v_0, \ldots, v_k$, we write for simplicity that

$$\mathrm{dist}(v_{0:k}) := \sum_{i=1}^{k} \mathrm{dist}(v_i, v_{i-1}).$$

Also recall that we denote $\theta = \tanh \beta$.

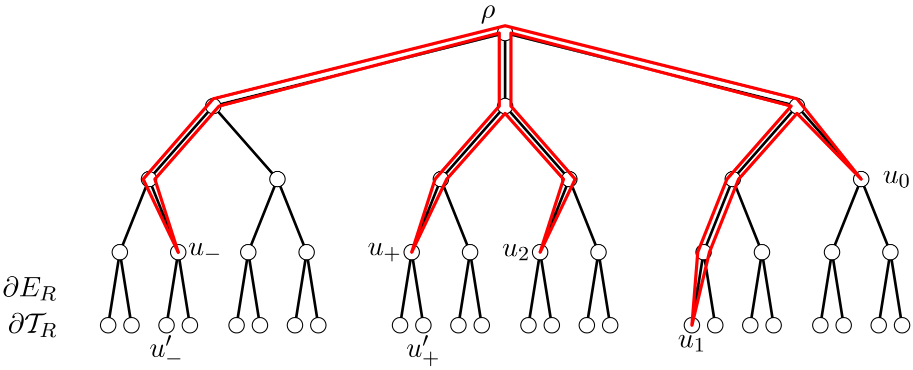

Fig. 1. An illustration of a path visiting $u_-, u_0, \ldots, u_k, u_+$ in a d-regular tree

Proposition 2.6. *There exists an absolute constant $C > 0$ such that the following holds when d is large enough. For any $k \geq 0$, $u_0, \cdots, u_k \in \mathcal{T}_R$, and $(u_-, u'_-), (u_+, u'_+) \in \partial E_R$, $u'_-, u'_+ \in \partial \mathcal{T}_R$, and $0 \leq t_-, t_+, t_1, \cdots, t_k < t$, we have*

$$\mathbb{E}\left[\langle \sigma_{u_-}\sigma_{u'_-} \,;\, \sigma_{u_0}\rangle_{\pi^{\mathbf{Y}_{t_-}}} \langle \sigma_{u_+}\sigma_{u'_+} \,;\, \sigma_{u_k}\rangle_{\pi^{\mathbf{Y}_{t_+}}} \prod_{i=1}^{k} \langle \sigma_{u_i} \,;\, \sigma_{u_{i-1}}\rangle_{\pi^{\mathbf{Y}_{t_i}}}\right] \leq C^k (C\theta)^{\mathrm{dist}(u_-,u_+)+\mathrm{dist}(u_-,u_0)+\mathrm{dist}(u_+,u_k)+\mathrm{dist}(u_{0:k})-2} e^{25t+d}. \tag{2.12}$$

For the exponent of $\mathrm{dist}(u_-, u_+) + \mathrm{dist}(u_-, u_0) + \mathrm{dist}(u_+, u_k) + \mathrm{dist}(u_{0:k})$, it can be understood as the length of the shortest path, which starts from u_- and visits $u_0, \ldots, u_k$ sequentially, then goes to u_+ and back to u_- (see Fig. 1). We also note that without the first term $\mathrm{dist}(u_-, u_+)$, such bound would be easy to prove (e.g. it directly follows from Lemma 3.1 below). However, the term $\mathrm{dist}(u_-, u_+)$ is crucial to get Proposition 2.4.

For C from Proposition 2.12, let $\eta = (C\theta)^{1/2}(d-1)^{-1/4}$. Then $C\theta < \eta < 1/\sqrt{d-1}$. Also let $S = (1-(d-1)C\theta\eta)^{-1}(1-C\theta/\eta)^{-1}$, and assume that $C\theta < 1/\sqrt{d-1}$. Then we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{v_1,\cdots,v_k=u} \mathcal{N}_{t_1}(v_1)\prod_{i=2}^{k}\mathcal{M}_{t_i}(v_i, v_{i-1})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{\substack{v_1,\cdots,v_k=u\\ w_1,\cdots,w_k=u}} \mathcal{N}_{t_1}(v_1)\mathcal{N}_{t_1}(w_1)\prod_{i=2}^{k}\mathcal{M}_{t_i}(v_i, v_{i-1})\mathcal{M}_{t_i}(w_i, w_{i-1})\right] \\
&\leq e^{25t+d} C^{2k} \sum_{\substack{(u_-,u'_-)\in\partial E_R,\\ (u_+,u'_+)\in\partial E_R}} \sum_{\substack{v_1,\cdots,v_k=u\\ w_1,\cdots,w_k=u}} (C\theta)^{\mathrm{dist}(u_-,u_+)+\mathrm{dist}(u_-,v_1)+\mathrm{dist}(u_+,w_1)+\mathrm{dist}(v_{1:k})+\mathrm{dist}(w_{1:k})-2} \\
&< e^{25t+d} C^{2k}\theta^{-2} \sum_{\substack{(u_-,u'_-)\in\partial E_R,\\ (u_+,u'_+)\in\partial E_R}} S^{2k+2}\eta^{\mathrm{dist}(u_-,u_+)+\mathrm{dist}(u_-,u)+\mathrm{dist}(u_+,u)}
\end{aligned}$$

$$\leq e^{25t+d}C^{2k}\theta^{-2}S^{2k+2}\eta^{2(R-1-\mathrm{dist}(\rho,u))}\sum_{\substack{(u_-,u'_-)\in\partial E_R,\\ (u_+,u'_+)\in\partial E_R}}\eta^{\mathrm{dist}(u_-,u_+)}$$

$$\leq e^{25t+d}C^{2k}\theta^{-2}S^{2k+2}\eta^{2(R-1-\mathrm{dist}(\rho,u))}d^2(d-1)^{R-1}\sum_{i=0}^{R-1}\eta^{2i}(d-1)^i$$

$$< e^{25t+d}C^{2k}\theta^{-2}S^{2k+2}\eta^{2(R-1-\mathrm{dist}(\rho,u))}d^2(d-1)^{R-1}(1-\eta^2(d-1))^{-1}. \tag{2.13}$$

Here the first inequality is by Proposition 2.6, the third equality is by $\mathrm{dist}(u_-,u)$, $\mathrm{dist}_{u_+,u} \geq R-1-\mathrm{dist}(\rho,u)$, and the fourth and fifth inequalities are by direct computations. The second inequality is by the following lemma.

Lemma 2.7. *For any $k \geq 2$ and $v_1, v_k \in \mathcal{T}$, we have*

$$\sum_{v_2,\cdots,v_{k-1}\in\mathcal{T}}(C\theta)^{\mathrm{dist}(v_{1:k})} < S^k\eta^{\mathrm{dist}(v_1,v_k)}.$$

Proof. First, by symmetry the LHS depends only on k and $N := \mathrm{dist}(v_1, v_k)$. Denote the LHS by $A_{k,N}$, and we prove $A_{k,N} < S^k\eta^N$ by induction in k. For $k = 2$, we have $A_{2,N} = (C\theta)^N < \eta^N$. Now suppose that $A_{k,N} < S^k\eta^N$ for some $k \geq 2$ and any N. For $v_1, \cdots, v_{k+1} \in \mathcal{T}$, let v' be the (only) vertex with the smallest $\mathrm{dist}(v_1, v')+\mathrm{dist}(v_2, v')+\mathrm{dist}(v_{k+1}, v')$. Let $m = \mathrm{dist}(v_1, v')$ and $m' = \mathrm{dist}(v', v_2)$, then $\mathrm{dist}(v_1, v_2) = m + m'$ and $\mathrm{dist}(v_2, v_{k+1}) = \mathrm{dist}(v_1, v_{k+1}) - m + m'$. Thus by the induction hypothesis we have

$$A_{k+1,N} \leq \sum_{m=0}^{N}\sum_{m'=0}^{\infty}(d-1)^{m'}(C\theta)^{m+m'}A_{k,N-m+m'}$$

$$< \sum_{m=0}^{N}\sum_{m'=0}^{\infty}(d-1)^{m'}(C\theta)^{m+m'}S^k\eta^{N-m+m'}.$$

Since $C\theta < \eta < 1/\sqrt{d-1}$, by first summing over m' then over m, we can bound this by

$$S^k\eta^N(1-(d-1)C\theta\eta)^{-1}(1-C\theta/\eta)^{-1} = S^{k+1}\eta^N$$

where the equality holds by the definition of S. □

Proof of Proposition 2.4. By (2.11) and (2.13), we conclude that when d is large enough,

$$\mathbb{E}[H_t^\gamma(u)^2] \leq e^{25t+d}\eta^4\theta^{-2}S^2d^2(d-1)^{\mathrm{dist}(\rho,u)+2}(\eta^2(d-1))^{R-\mathrm{dist}(\rho,u)-3}$$

$$(1-\eta^2(d-1))^{-1}\sum_{k=1}^{\infty}2^k\left(\frac{t^k}{k!}\right)^2C^{2k}S^{2k}$$

$$< e^{25t+d}\eta^4\theta^{-2}S^2d^2(d-1)^{\mathrm{dist}(\rho,u)+2}(\eta^2(d-1))^{R-\mathrm{dist}(\rho,u)-3}(1-\eta^2(d-1))^{-1}e^{4tCS}$$

Now we take c small enough to ensure that d is large and $c < C^{-2}$. Recall that $\theta = \tanh\beta \leq c(d-1)^{-\frac{1}{2}}$ and $\eta = (C\theta)^{1/2}(d-1)^{-1/4}$, so we have $C^2\theta < (d-1)^{-1/2}$ and $S < (1-C^{-1})^{-1}(1-C^{-1/2})^{-1}$, thus S has a universal upper bound; and we also have $(1-\eta^2(d-1))^{-1} < (1-C^{-1})^{-1}$, and $\eta^2\theta^{-1} = C(d-1)^{-1/2}$. By taking $\alpha = \eta^2(d-1)$, and

$$C_d \geq (4CS+25)\vee e^d\eta^4\theta^{-2}S^2d^2(d-1)^2(1-\eta^2(d-1))^{-1},$$

depending only on d, the conclusion of Proposition 2.4 follows. □

3. Inductive Coupling for a Chain of Covariances

This section is devoted to the proof of Proposition 2.6. We begin by introducing the notion of induced external field and setting up the necessary notations in Sect. 3.1. Then, in Sect. 3.2, we reformulate the chain of covariances in Proposition 2.6 into a more tractable form, consisting of products of partition functions. Furthermore, in Sect. 3.3 we discuss the method of inductive coupling to investigate these partition functions, and derive necessary estimates needed for the coupling argument in Sect. 3.4, the final subsection.

3.1. Preliminaries. In this section we work under the setting of Proposition 2.6. Specifically, we work on $\mathcal{T}_R$ for fixed R, and let $\boldsymbol{\beta} \in \mathbb{R}^{E_R}$ such that $\beta_{u,v} = \beta$ if $(u,v) \notin \partial E_R$, and $\beta_{u,v} = \gamma$ if $(u,v) \in \partial E_R$.

We first introduce the notion of *induced external field*. Take any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$ and an edge $(u,v) \in E_R$. Let the subgraph $\mathcal{T}_{u\setminus v} \subset \mathcal{T}_R$ be defined by removing the edge (u,v) from $\mathcal{T}_R$ and taking the connected component containing u, and consider the subgraph $\mathcal{T}_{u\to v}$ obtained by adding the vertex v back to $\mathcal{T}_{u\setminus v}$ via the edge (u,v). We define the *Belief Propagation message* $m^{\mathbf{x}}_{u\to v}$ from u to v as the probability measure on $\{\pm 1\}$ given by

$$m^{\mathbf{x}}_{u\to v}(\sigma) := \frac{1}{Z^{\mathbf{x}}_{u\to v}} \sum_{\substack{(\sigma_{v'})_{v'\in\mathcal{T}_{u\to v}}\in\{\pm1\}^{\mathcal{T}_{u\to v}},\\ \text{with } \sigma_v=\sigma}} \exp\left(\sum_{(u',v')\in E(\mathcal{T}_{u\to v})} \beta_{u',v'}\sigma_{u'}\sigma_{v'} + \sum_{v'\in\mathcal{T}_{u\setminus v}} x(v') \right),$$

where $Z^{\mathbf{x}}_{u\to v}$ is the normalizing constant that makes $m^{\mathbf{x}}_{u\to v}$ a probability measure, i.e. $m^{\mathbf{x}}_{u\to v}(+1)+m^{\mathbf{x}}_{u\to v}(-1) = 1$. Note that we regard the external field at v as 0, to measure the effect of $\mathcal{T}_{u\setminus v}$ on v via the edge (u,v). See [15, Chapter 14] for a detailed background on the notion of Belief Propagation.

Then, the *induced external field on v from u*, denoted by $\zeta^{\mathbf{x}}_{u\to v}$, is defined as

$$\zeta^{\mathbf{x}}_{u\to v} := \frac{1}{2}\log\left(\frac{m^{\mathbf{x}}_{u\to v}(+1)}{m^{\mathbf{x}}_{u\to v}(-1)}\right).$$

In particular, it satisfies $m^{\mathbf{x}}_{u\to v}(+1) = \frac{1}{2}\{1+\tanh(\zeta^{\mathbf{x}}_{u\to v})\}$. It is straightforward to compute that $\tanh(\zeta^{\mathbf{x}}_{u\to v}) = \tanh(\beta_{u,v})\tanh(x(u) + \sum_{w\sim u, w\neq v}\zeta^{\mathbf{x}}_{w\to u})$, so

$$|\zeta^{\mathbf{x}}_{u\to v}| \le \beta_{u,v} \le \beta. \tag{3.1}$$

We also have that

$$\langle \sigma_v \rangle_{\pi^{\mathbf{x}}_{\mathcal{T}_{u\to v}}} = \frac{m^{\mathbf{x}}_{u\to v}(+1)e^{x(v)} - m^{\mathbf{x}}_{u\to v}(-1)e^{-x(v)}}{m^{\mathbf{x}}_{u\to v}(+1)e^{x(v)} + m^{\mathbf{x}}_{u\to v}(-1)e^{-x(v)}} = \tanh(\zeta^{\mathbf{x}}_{u\to v} + x(v)).$$

We can understand $\zeta^{\mathbf{x}}_{u\to v}$ alternatively as follows. Let $\mathcal{T}_{v\setminus u}$ be defined as before (but take the connected component of v). If we let $\mathbf{x}' = (x'(v'))_{v'\in\mathcal{T}_{v\setminus u}} \in \mathbb{R}^{\mathcal{T}_{v\setminus u}}$ be the vector where $x'(v') = x(v') + \mathbb{1}[v'=v]\zeta^{\mathbf{x}}_{u\to v}$, then $\zeta^{\mathbf{x}}_{u\to v}$ is the number such that the measure $\pi^{\mathbf{x}'}_{\mathcal{T}_{v\setminus u}}$ on $\{\pm1\}^{\mathcal{T}_{v\setminus u}}$ gives the marginal distribution of $\pi^{\mathbf{x}}$ on $\mathcal{T}_{v\setminus u}$.

For any connected subgraph $\mathcal{G}$ of $\mathcal{T}_R$ and for any $v \in \mathcal{G}$, we let

$$\zeta_{\mathcal{G}}^{\mathbf{x}}(v) = \sum_{u \sim v, u \notin \mathcal{G}} \zeta_{u \to v}^{\mathbf{x}}. \tag{3.2}$$

Then, if we let $\mathbf{x}' = (x'(v))_{v \in \mathcal{G}} \in \mathbb{R}^{\mathcal{G}}$ be the vector where $x'(v) = x(v) + \zeta_{\mathcal{G}}^{\mathbf{x}}(v)$, then the measure $\pi_{\mathcal{G}}^{\mathbf{x}'}$ on $\mathcal{G}$ gives the marginal distribution of $\pi^{\mathbf{x}}$ on $\mathcal{G}$. Let $\widetilde{Z}_{\mathcal{G}}(\mathbf{x})$ be the partition function of $\pi^{\mathbf{x}}$ restricted to $\mathcal{G}$; i.e. we let

$$\widetilde{Z}_{\mathcal{G}}(\mathbf{x}) := \sum_{(\sigma_v) \in \{\pm 1\}^{\mathcal{G}}} \exp\left(\sum_{(u,v) \in E(\mathcal{G})} \beta_{u,v} \sigma_u \sigma_v + \sum_{v \in \mathcal{G}} (x(v) + \zeta_{\mathcal{G}}^{\mathbf{x}}(v)) \sigma_v \right). \tag{3.3}$$

We shall also need the following notation of the partition function with some fixed spins. Take any $\mathcal{H} \subset \mathcal{G}$ and $h \in \{\pm 1\}^{\mathcal{H}}$. We denote

$$\widetilde{Z}_{\mathcal{G}}^{h}(\mathbf{x}) := \sum_{\substack{(\sigma_v) \in \{\pm 1\}^{\mathcal{G}} \\ \sigma_v = h(v), \forall v \in \mathcal{H}}} \exp\left(\sum_{(u,v) \in E(\mathcal{G})} \beta_{u,v} \sigma_u \sigma_v + \sum_{v \in \mathcal{G}} (x(v) + \zeta_{\mathcal{G}}^{\mathbf{x}}(v)) \sigma_v \right). \tag{3.4}$$

Here, $\mathcal{H} \subset \mathcal{G}$ is a set of vertices in $\mathcal{G}$ whose spins are fixed by the assignment h.

For each $u, v \in \mathcal{T}_R$, denote $[u, v]$ as the subgraph given by the shortest path from u to v. For simplicity of notations, we also write $\widetilde{Z}_{u,v}(\mathbf{x}) := \widetilde{Z}_{[u,v]}(\mathbf{x})$, and we denote the normalized version as $\overline{Z}_{u,v}(\mathbf{x}) := \frac{\widetilde{Z}_{u,v}(\mathbf{x})}{\widetilde{Z}_{u,v}(\mathbf{0})} \geq 1$. We also let

$$A_{u,v} = \prod_{(u',v') \in E([u,v])} \tanh(\beta_{u',v'}). \tag{3.5}$$

Then by straightforward computation we have that $A_{u,v}$ equals the covariance without external field $\langle \sigma_u \, ; \, \sigma_v \rangle_{\pi}$. We can use these quantities to write the covariances with external field $\mathbf{x}$.

Lemma 3.1. *For any $u, v \in \mathcal{T}_R$, we have*

$$\langle \sigma_u \, ; \, \sigma_v \rangle_{\pi^{\mathbf{x}}} = \frac{A_{u,v}}{\overline{Z}_{u,v}(\mathbf{x})^2}. \tag{3.6}$$

For any $(u, u') \in \partial E_R$ with $u' \in \partial \mathcal{T}_R$, if $v \neq u'$ we have

$$\langle \sigma_u \sigma_{u'} \, ; \, \sigma_v \rangle_{\pi^{\mathbf{x}}} = \frac{\sinh(2x(u')) A_{u,v}}{2\overline{Z}_{u',v}(\mathbf{x})^2 \cosh^2(\gamma)}, \tag{3.7}$$

and if $v = u'$ we have

$$\langle \sigma_u \sigma_{u'} \, ; \, \sigma_v \rangle_{\pi^{\mathbf{x}}} = \frac{\sinh(2x(u) + 2\zeta_{[u,u']}^{\mathbf{x}}(u))}{2\overline{Z}_{u',u}(\mathbf{x})^2 \cosh^2(\gamma)}. \tag{3.8}$$

We leave the proof of this lemma to Appendix A.

3.2. Evaluating the chain of covariances. Take $u_0, \cdots, u_k \in \mathcal{T}_R$ and (u_-, u'_-), $(u_+, u'_+) \in \partial E_R$, as in Proposition 2.6. Also recall the definition of $\mathbf{Y}$ given by (2.7), and that in law we have

$$\mathbf{Y} = (\mathbf{Y}_t)_{t\geq 0} \overset{d}{=} (t\boldsymbol{\tau} + \mathbf{B}_t)_{t\geq 0}, \tag{3.9}$$

where $\boldsymbol{\tau} = (\tau_v)_{v\in\mathcal{T}_R}$ is sampled from $\pi = \pi^{\gamma,\mathbf{0}}$, the free Ising model on $\mathcal{T}_R$ with inverse temperature γ on ∂E_R, and inverse temperature β on other edges; and $\mathbf{B} = \{(B_v(t))_{t\geq 0}\}_{v\in\mathcal{T}_R}$ is another collection of independent one-dimensional standard Brownian motions. We couple $\mathbf{Y}$ with $\boldsymbol{\tau}, \mathbf{B}$, so that this equality holds almost surely.

Now we evaluate the LHS of (2.12) using Lemma 3.1. If $u_0 \neq u'_-$ and $u_k \neq u'_+$ it equals

$$\mathbb{E}\left[\frac{\sinh\left(2Y_{t_-}(u'_-)\right)\sinh\left(2Y_{t_+}(u'_+)\right)}{4\overline{Z}_{u'_-,u_0}(\mathbf{Y}_{t_-})^2\overline{Z}_{u'_+,u_k}(\mathbf{Y}_{t_+})^2\prod_{i=1}^k \overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2\cosh^4(\gamma)}\right] A_{u_-,u_0}A_{u_k,u_+}\prod_{i=1}^k A_{u_i,u_{i-1}}. \tag{3.10}$$

If $u_0 = u'_-$ and $u_k \neq u'_+$, it equals

$$\mathbb{E}\left[\frac{\sinh\left(2Y_{t_-}(u_-)+2\zeta^{\mathbf{Y}_{t_-}}_{[u_-,u'_-]}(u_-)\right)\sinh\left(2Y_{t_+}(u'_+)\right)}{4\overline{Z}_{u'_-,u_-}(\mathbf{Y}_{t_-})^2\overline{Z}_{u'_+,u_k}(\mathbf{Y}_{t_+})^2\prod_{i=1}^k \overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2\cosh^4(\gamma)}\right] A_{u_k,u_+}\prod_{i=1}^k A_{u_i,u_{i-1}}. \tag{3.11}$$

If $u_0 \neq u'_-$ and $u_k = u'_+$, it equals

$$\mathbb{E}\left[\frac{\sinh\left(2Y_{t_-}(u'_-)\right)\sinh\left(2Y_{t_+}(u_+)+2\zeta^{\mathbf{Y}_{t_+}}_{[u_+,u'_+]}(u_+)\right)}{4\overline{Z}_{u'_-,u_0}(\mathbf{Y}_{t_-})^2\overline{Z}_{u'_+,u_+}(\mathbf{Y}_{t_+})^2\prod_{i=1}^k \overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2\cosh^4(\gamma)}\right] A_{u_-,u_0}\prod_{i=1}^k A_{u_i,u_{i-1}}. \tag{3.12}$$

If $u_0 = u'_-$ and $u_k = u'_+$, it equals

$$\mathbb{E}\left[\frac{\sinh\left(2Y_{t_-}(u_-)+2\zeta^{\mathbf{Y}_{t_-}}_{[u_-,u'_-]}(u_-)\right)\sinh\left(2Y_{t_+}(u_+)+2\zeta^{\mathbf{Y}_{t_+}}_{[u_+,u'_+]}(u_+)\right)}{4\overline{Z}_{u'_-,u_-}(\mathbf{Y}_{t_-})^2\overline{Z}_{u'_+,u_+}(\mathbf{Y}_{t_+})^2\prod_{i=1}^k \overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2\cosh^4(\gamma)}\right]\prod_{i=1}^k A_{u_i,u_{i-1}}. \tag{3.13}$$

If $u_- = u_+$, we would get Proposition 2.6 immediately from these equations. Indeed, for each $u, v \in \mathcal{T}_R$ we have $A_{u,v} \leq (\tanh\beta)^{\mathrm{dist}(u,v)} = \theta^{\mathrm{dist}(u,v)}$, by (3.5) (recall that $\theta = \tanh\beta$). For the expectation factor in each case, note that the denominator is at least 4. Thus using (3.1) and (3.9), the expectation is bounded by $\mathbb{E}[e^{4t+2B_t(u'_-)+2B_t(u'_+)}]$, $\mathbb{E}[e^{4t+2B_t(u_-)+2d\beta+2B_t(u'_+)}]$, $\mathbb{E}[e^{4t+2B_t(u'_-)+2B_t(u_+)+2d\beta}]$, or $\mathbb{E}[e^{4t+2B_t(u_-)+2B_t(u_+)+4d\beta}]$, in each case respectively; and each can be further bounded by e^{8t+d}, since d is large enough thus β is small enough.

Below we assume that $u_- \neq u_+$, thus $u'_- \neq u'_+$. When $u_0 \neq u'_-$ we let $u_{-1} = u'_-$, and otherwise we let $u_{-1} = u_-$; similarly, when $u_k \neq u'_+$ we let $u_{k+1} = u'_+$, and otherwise we let $u_{k+1} = u_+$. We also denote $t_0 := t_-$, $t_{k+1} := t_+$. A motivation for such

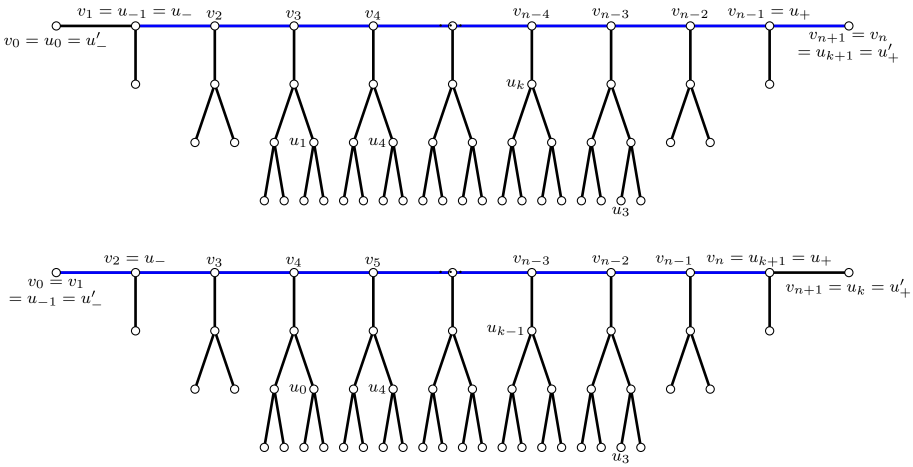

Fig. 2. Illustrations of the path $[u_{-1}, u_{k+1}] = [v_1, v_n]$, indicated by the blue edges. The top one is the case where $u_0 = u'_-$ and $u_{k+1} \neq u'_+$, while the bottom one is the case where $u_0 \neq u'_-$ and $u_{k+1} = u'_+$

definitions is that, for the denominator inside the expectation in each of (3.10), (3.11), (3.12), (3.13), it can now be written as $4\prod_{i=0}^{k+1} \overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2 \cosh^4(\gamma)$.

Consider $[u_{-1}, u_{k+1}]$, the shortest path from u_{-1} to u_{k+1}, and we enumerate the vertices on the path as $u_{-1} = v_1, v_2, \dots, v_n = u_{k+1}$ for some $n \in \mathbb{Z}_+$. Denote $v_0 = u'_-$ and $v_{n+1} = u'_+$. Then the path $[v_1, v_n]$ is contained in the path $[v_0, v_{n+1}]$; $v_0 = v_1$ if $u_0 \neq u'_-$, and $v_n = v_{n+1}$ if $u_{k+1} \neq u'_+$. See Fig. 2 for an illustration.

Our next goal is to expand the factors

$$\sinh\left(2Y_{t_-}(u_-) + 2\zeta^{\mathbf{Y}_{t_-}}_{[u_-,u'_-]}(u_-)\right), \quad \sinh\left(2Y_{t_+}(u_+) + 2\zeta^{\mathbf{Y}_{t_+}}_{[u_+,u'_+]}(u_+)\right), \tag{3.14}$$

which appeared in (3.11), (3.12), (3.13). For this we set up some notations for induced external fields along this path $[v_0, v_{n+1}]$. For any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$ and $1 \le \ell \le n$, let

$$\zeta^{\mathbf{x}}_\ell := x(v_\ell) + \sum_{v \sim v_\ell, v \neq v_{\ell-1}, v_{\ell+1}} \zeta^{\mathbf{x}}_{v \to v_\ell} = x(v_\ell) + \zeta^{\mathbf{x}}_{[v_0, v_{n+1}]}(v_\ell);$$
$$\zeta^{\mathbf{x}}_{\ell-1\to\ell} := \zeta^{\mathbf{x}}_{v_{\ell-1}\to v_\ell}, \quad \text{and} \quad \zeta^{\mathbf{x}}_{\ell+1\to\ell} := \zeta^{\mathbf{x}}_{v_{\ell+1}\to v_\ell}.$$

Here we assume that $\zeta^{\mathbf{x}}_{u'_-\to u'_-} = \zeta^{\mathbf{x}}_{u'_+\to u'_+} = 0$. In words, $\zeta^{\mathbf{x}}_\ell$ is the total external and induced external field on v_ℓ, except for those from $v_{\ell-1}$ and $v_{\ell+1}$, which are $\zeta^{\mathbf{x}}_{\ell-1\to\ell}$ and $\zeta^{\mathbf{x}}_{\ell+1\to\ell}$ respectively. From these definitions we have that

$$2Y_{t_-}(u_-) + 2\zeta^{\mathbf{Y}_{t_-}}_{[u_-,u'_-]}(u_-) = 2\zeta^{\mathbf{Y}_{t_-}}_1 + 2\zeta^{\mathbf{Y}_{t_-}}_{2\to 1}, \tag{3.15}$$

when $u_0 = u'_-$ (then $v_1 = u_{-1} = u_-$, see top-left of Fig. 2); and

$$2Y_{t_+}(u_+) + 2\zeta^{\mathbf{Y}_{t_+}}_{[u_+,u'_+]}(u_+) = 2\zeta^{\mathbf{Y}_{t_+}}_n + 2\zeta^{\mathbf{Y}_{t_+}}_{n-1\to n}, \tag{3.16}$$

when $u_k = u'_+$ (then $v_n = u_{k+1} = u_+$, see bottom-right of Fig. 2).

We now study (3.14). For each $1 \le \ell < n$, by direct computation we can write

$$\sinh(2\zeta^{\mathbf{x}}_{\ell+1\to\ell}) = \frac{2\tanh(\beta_{v_\ell,v_{\ell+1}})\cosh^2(\beta_{v_\ell,v_{\ell+1}})\sinh(2\zeta^{\mathbf{x}}_{\ell+1} + 2\zeta^{\mathbf{x}}_{\ell+2\to\ell+1})}{\cosh(2\beta_{v_\ell,v_{\ell+1}}) + \cosh(2\zeta^{\mathbf{x}}_{\ell+1} + 2\zeta^{\mathbf{x}}_{\ell+2\to\ell+1})},$$

which implies that

$$\begin{aligned}
&\sinh(2\zeta^{\mathbf{x}}_{\ell} + 2\zeta^{\mathbf{x}}_{\ell+1\to\ell}) \\
&\quad = \sinh(2\zeta^{\mathbf{x}}_{\ell})\cosh(2\zeta^{\mathbf{x}}_{\ell+1\to\ell}) + \cosh(2\zeta^{\mathbf{x}}_{\ell})\sinh(2\zeta^{\mathbf{x}}_{\ell+1\to\ell}) \\
&\quad = \sinh(2\zeta^{\mathbf{x}}_{\ell})\cosh(2\zeta^{\mathbf{x}}_{\ell+1\to\ell}) \\
&\qquad + \frac{2\tanh(\beta_{v_\ell,v_{\ell+1}})\cosh(2\zeta^{\mathbf{x}}_{\ell})\cosh^2(\beta_{v_\ell,v_{\ell+1}})\sinh(2\zeta^{\mathbf{x}}_{\ell+1} + 2\zeta^{\mathbf{x}}_{\ell+2\to\ell+1})}{\cosh(2\beta_{v_\ell,v_{\ell+1}}) + \cosh(2\zeta^{\mathbf{x}}_{\ell+1} + 2\zeta^{\mathbf{x}}_{\ell+2\to\ell+1})}.
\end{aligned} \tag{3.17}$$

Now for $1 \le \ell < n$, we denote

$$U_\ell(\mathbf{x}) := \frac{2\cosh(2\zeta^{\mathbf{x}}_{\ell})\cosh^2(\beta_{v_\ell,v_{\ell+1}})}{\cosh(2\beta_{v_\ell,v_{\ell+1}}) + \cosh(2\zeta^{\mathbf{x}}_{\ell+1} + 2\zeta^{\mathbf{x}}_{\ell+2\to\ell+1})}. \tag{3.18}$$

Thus we can write (3.17) as

$$\begin{aligned}
&\sinh(2\zeta^{\mathbf{x}}_{\ell} + 2\zeta^{\mathbf{x}}_{\ell+1\to\ell}) \\
&\quad = \sinh(2\zeta^{\mathbf{x}}_{\ell})\cosh(2\zeta^{\mathbf{x}}_{\ell+1\to\ell}) + U_\ell(\mathbf{x})\tanh(\beta_{v_\ell,v_{\ell+1}})\sinh(2\zeta^{\mathbf{x}}_{\ell+1} + 2\zeta^{\mathbf{x}}_{\ell+2\to\ell+1}).
\end{aligned} \tag{3.19}$$

When $u_0 = u'_-$, we have

$$\begin{aligned}
&\sinh\left(2Y_{t_-}(u_-) + 2\zeta^{\mathbf{Y}_{t_-}}_{[u_-,u'_-]}(u_-)\right) \\
&= \sinh\left(2\zeta_1^{\mathbf{Y}_{t_-}} + 2\zeta^{\mathbf{Y}_{t_-}}_{2\to1}\right) \\
&= \sum_{\ell=1}^{n-1} \sinh\left(2\zeta_\ell^{\mathbf{Y}_{t_-}}\right)\cosh\left(2\zeta^{\mathbf{Y}_{t_-}}_{\ell+1\to\ell}\right) A_{v_1,v_\ell} \prod_{1\le j<\ell} U_j(\mathbf{Y}_{t_-}) \\
&\quad + \sinh\left(2\zeta_n^{\mathbf{Y}_{t_-}} + 2\zeta^{\mathbf{Y}_{t_-}}_{n+1\to n}\right) A_{v_1,v_n} \prod_{1\le j<n} U_j(\mathbf{Y}_{t_-}).
\end{aligned} \tag{3.20}$$

where the first equality is by (3.16), and the second equality is by repeated applying (3.19). Similarly, when $u_k = u'_+$, by (3.16) and repeated expansion, we have

$$\begin{aligned}
&\sinh\left(2Y_{t_+}(u_+) + 2\zeta^{\mathbf{Y}_{t_+}}_{[u_+,u'_+]}(u_+)\right) \\
&= \sinh\left(2\zeta_n^{\mathbf{Y}_{t_+}} + 2\zeta^{\mathbf{Y}_{t_+}}_{n-1\to n}\right) \\
&= \sum_{\ell=2}^{n} \sinh\left(2\zeta_\ell^{\mathbf{Y}_{t_+}}\right)\cosh\left(2\zeta^{\mathbf{Y}_{t_+}}_{\ell-1\to\ell}\right) A_{v_\ell,v_n} \prod_{\ell<j\le n} V_j(\mathbf{Y}_{t_+}) \\
&\quad + \sinh\left(2\zeta_1^{\mathbf{Y}_{t_+}} + 2\zeta^{\mathbf{Y}_{t_+}}_{0\to1}\right) A_{v_1,v_n} \prod_{1<j\le n} V_j(\mathbf{Y}_{t_+}),
\end{aligned} \tag{3.21}$$

where

$$V_\ell(\mathbf{x}) := \frac{2\cosh(2\zeta_\ell^{\mathbf{x}})\cosh^2(\beta_{v_\ell,v_{\ell-1}})}{\cosh(2\beta_{v_\ell,v_{\ell-1}}) + \cosh(2\zeta_{\ell-1}^{\mathbf{x}} + 2\zeta_{\ell-2\to\ell-1}^{\mathbf{x}})} \tag{3.22}$$

for each $1 < \ell \le n$ and $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$. Now to prove Proposition 2.6 it now suffices to prove the following technical estimate. Recall that $\theta = \tanh\beta$.

Proposition 3.2. *For $1 \le l \le r \le n$, let $C_l : \mathbb{R}^{\mathcal{T}_R} \to \mathbb{R}$ satisfy either $C_l \equiv 1$ or $C_l(\mathbf{x}) = \cosh(2\zeta_{l+1\to l}^{\mathbf{x}})$ for any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$; and $C_r : \mathbb{R}^{\mathcal{T}_R} \to \mathbb{R}$ satisfy either $C_r \equiv 1$ or $C_r(\mathbf{x}) = \cosh(2\zeta_{r-1\to r}^{\mathbf{x}})$ for any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$. Then we have*

$$\mathbb{E}\left[\frac{\sinh\left(2\zeta_l^{\mathbf{Y}_{t_-}}\right)\sinh\left(2\zeta_r^{\mathbf{Y}_{t_+}}\right)C_l(\mathbf{Y}_{t_-})C_r(\mathbf{Y}_{t_+})}{\prod_{i=0}^{k+1}\overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2}\prod_{1\le j<l}U_j(\mathbf{Y}_{t_-})\prod_{r<j\le n}V_j(\mathbf{Y}_{t_+})\right]$$
$$\le e^{17t+d}\theta^{r-l}C^{n+k+\sum_{j=0}^{k+1}\operatorname{dist}(u_j,u_{j-1})}$$

where C is an absolute constant.

Assuming this, we now prove Proposition 2.6, by bounding each of (3.10), (3.11), (3.12), and (3.13), using the expansions (3.20) and (3.21) and Proposition 3.2.

Proof of Proposition 2.6. In the case where $u_0 \neq u'_-$ and $u_k \neq u'_+$, we have $\zeta_1^{\mathbf{Y}_{t_-}} = Y_{t_-}(u'_-)$ and $\zeta_n^{\mathbf{Y}_{t_+}} = Y_{t_+}(u'_+)$ by their definitions. We apply Proposition 3.2 with $l = 1$ and $r = n$, and bound (3.10) by

$$C^{n+k}(C\theta)^{\operatorname{dist}(u'_-,u'_+)+\operatorname{dist}(u_-,u_0)+\operatorname{dist}(u_k,u_+)+\sum_{i=1}^k\operatorname{dist}(u_i,u_{i-1})}e^{17t+d}.$$

For the case where $u_0 \neq u'_-$ and $u_k = u'_+$, using (3.21), we apply Proposition 3.2 for $l = 1$ and each $1 \le r \le n$ (note that $\zeta_{0\to1}^{\mathbf{Y}_{t_+}} = 0$ in this case), and bound (3.11) by

$$nC^{n+k}(C\theta)^{\operatorname{dist}(u'_-,u_+)+\operatorname{dist}(u_-,u_0)+\sum_{i=1}^k\operatorname{dist}(u_i,u_{i-1})}e^{17t+d}.$$

Similarly, for the case where $u_0 = u'_-$ and $u_k \neq u'_+$, using (3.20) and Proposition 3.2 we can bound (3.12) by

$$nC^{n+k}(C\theta)^{\operatorname{dist}(u_-,u'_+)+\operatorname{dist}(u_+,u_k)+\sum_{i=1}^k\operatorname{dist}(u_i,u_{i-1})}e^{17t+d}.$$

Finally, for the case where $u_0 = u'_-$ and $u_k = u'_+$, by (3.20) and (3.21), and using Proposition 3.2 for each $1 \le l < r \le n$, we can bound (3.13) by

$$\frac{n(n-1)}{2}C^{n+k}(C\theta)^{\operatorname{dist}(u_-,u_+)+\sum_{i=1}^k\operatorname{dist}(u_i,u_{i-1})}e^{17t+d} + \sum_{1\le r\le l\le n}\theta^{n-r+l-1+\sum_{i=1}^k\operatorname{dist}(u_i,u_{i-1})}$$
$$\times\,\mathbb{E}\left[e^{2|\zeta_l^{\mathbf{Y}_{t_-}}|+2|\zeta_{l+1\to l}^{\mathbf{Y}_{t_-}}|+2|\zeta_r^{\mathbf{Y}_{t_+}}|+2|\zeta_{r-1\to r}^{\mathbf{Y}_{t_+}}|}\prod_{1\le j<l}U_j(\mathbf{Y}_{t_-})\prod_{r<j\le n}V_j(\mathbf{Y}_{t_+})\right]. \tag{3.23}$$

We next bound the expectation in the second line. For any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$, we have $\frac{\cosh(2\zeta_j^{\mathbf{x}})}{\cosh(2\zeta_j^{\mathbf{x}}+2\zeta_{j+1\to j}^{\mathbf{x}})} < 2\cosh(2\zeta_{j+1\to j}^{\mathbf{x}}) \leq 2\cosh(2\beta)$ for each $1 \leq j < l$, by (3.1). So by (3.18) and (3.22) we have

$$\prod_{1\leq j<l} U_j(\mathbf{x}) < \prod_{1\leq j<l} \frac{2\cosh^2(\beta)\cosh(2\zeta_j^{\mathbf{x}})}{\cosh(2\zeta_{j+1}^{\mathbf{x}}+2\zeta_{j+2\to j+1}^{\mathbf{x}})} < \cosh(2\zeta_1^{\mathbf{x}})(4\cosh^2(\beta)\cosh(2\beta))^{l-1}, \tag{3.24}$$

and similarly

$$\prod_{r<j\leq n} V_j(\mathbf{x}) < \prod_{r<j\leq n} \frac{2\cosh^2(\beta)\cosh(2\zeta_j^{\mathbf{x}})}{\cosh(2\zeta_{j-1}^{\mathbf{x}}+2\zeta_{j-2\to j-1}^{\mathbf{x}})} < \cosh(2\zeta_n^{\mathbf{x}})(4\cosh^2(\beta)\cosh(2\beta))^{n-r}.$$

Thus the second line of (3.23) can be bounded by

$$\begin{aligned}&(4\cosh^2(\beta)\cosh(2\beta))^{2n}\mathbb{E}\left[e^{2|\zeta_l^{\mathbf{Y}_{t_-}}|+2|\zeta_{l+1\to l}^{\mathbf{Y}_{t_-}}|+2|\zeta_r^{\mathbf{Y}_{t_+}}|+2|\zeta_{r-1\to r}^{\mathbf{Y}_{t_+}}|+2|\zeta_1^{\mathbf{Y}_{t_-}}|+2|\zeta_n^{\mathbf{Y}_{t_+}}|}\right]\\ &\quad< C^n\mathbb{E}\left[e^{4\beta+8t+2|B_{t_-}(v_l)|+2|B_{t_-}(v_1)|+2|B_{t_+}(v_n)|+2|B_{t_+}(v_r)|+8(d-1)\beta}\right] < C^{n+1}e^{25t+d}.\end{aligned}$$

Now we conclude that (3.13) can be bounded by

$$n^2C^{n+k}(C\theta)^{\mathrm{dist}(u_-,u_+)+\sum_{i=1}^k \mathrm{dist}(u_i,u_{i-1})}e^{25t+d}.$$

Then our conclusion follows in each case. □

3.3. Inductive construction of coupling. In this and the next subsection we prove Proposition 3.2, by developing an inductive coupling scheme to study the partition functions in the main inequality.

We consider the space $\mathcal{J} = \{\pm 1\}^{\mathcal{T}_R} \times C([0,\infty),\mathbb{R})^{\mathcal{T}_R}$ with coordinates $(\boldsymbol{\tau},\mathbf{B})$, and regard $\mathbf{Y}$ as a function of $\boldsymbol{\tau},\mathbf{B}$ defined via (3.9) on $\mathcal{J}$. We let μ be the probability measure on the space $\mathcal{J}$ such that for $(\boldsymbol{\tau},\mathbf{B}) \sim \mu$, $\boldsymbol{\tau}$ is sampled from $\pi = \pi^{\gamma,\mathbf{0}}$ and $\mathbf{B}$ are independent Brownian motions. Let $1 \leq l \leq r \leq n$ be given, and set $\mu_\pm$ be the measures on $\mathcal{J}$ with $d\mu_\pm = \mathbb{1}[\pm\zeta_l^{\mathbf{Y}_{t_-}} \geq 0]d\mu$.

We let μ^* be another measure on $\mathcal{J}$, defined as

$$d\mu^* = \frac{|\sinh(2\zeta_l^{\mathbf{Y}_{t_-}})|C_l(\mathbf{Y}_{t_-})C_r(\mathbf{Y}_{t_+})}{\cosh(2\zeta_n^{\mathbf{Y}_{t_+}})\prod_{i=0}^{k+1}\overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2}\prod_{1\leq j<l}U_j(\mathbf{Y}_{t_-})\prod_{r<j\leq n}V_j(\mathbf{Y}_{t_+})d\mu.$$

We let $\mu_\pm^*$ be the measure on $\mathcal{J}$ with $d\mu_\pm^* = \mathbb{1}[\pm\zeta_l^{\mathbf{Y}_{t_-}} \geq 0]d\mu^*$. To prove Proposition 3.2, it now suffices to bound

$$\int \sinh(2\zeta_r^{\mathbf{Y}_{t_+}})\cosh(2\zeta_n^{\mathbf{Y}_{t_+}})d\mu_+^* - \int \sinh(2\zeta_r^{\mathbf{Y}_{t_+}})\cosh(2\zeta_n^{\mathbf{Y}_{t_+}})d\mu_-^*. \tag{3.25}$$

We leave out the factor of $\sinh(2\zeta_r^{\mathbf{Y}_{t_+}})\cosh(2\zeta_n^{\mathbf{Y}_{t_+}})$ for technical reasons. To bound (3.25), we construct a coupling Γ^* of μ_+^* and μ_-^*. In words, denote the coordinates of

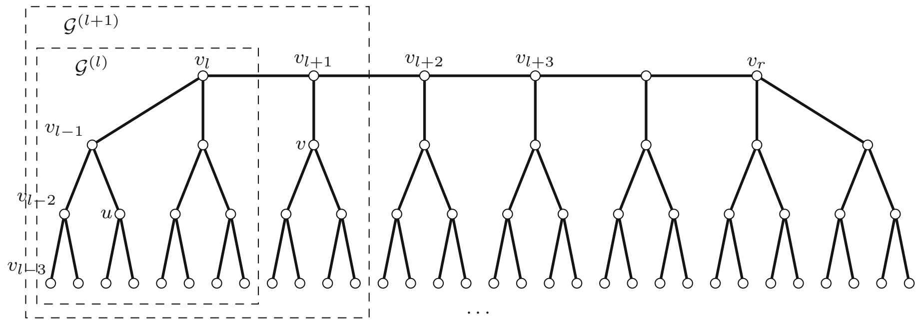

Fig. 3. An illustration of the path from v_l to v_r, and the subgraphs $\mathcal{G}^{(\ell)}$. The vertex u has $\rho(u) = l$ and the vertex v has $\rho(v) = l + 1$. The distance $\mathbf{d}^r_{l+3}(u, v) = 2$ since $\rho(u) = l$ and $\rho(v) = l + 1$

$\mathcal{J}^2$ as $(\boldsymbol{\tau}_-, \mathbf{B}_-)$, $(\boldsymbol{\tau}_+, \mathbf{B}_+)$, we will construct a measure Γ^* on it, such that for any measurable function f on $\mathcal{J}$ we have $\int f(\boldsymbol{\tau}_-, \mathbf{B}_-) d\Gamma^* = \int f d\mu^*_-$ and $\int f(\boldsymbol{\tau}_+, \mathbf{B}_+) d\Gamma^* = \int f d\mu^*_+$. Denote $\mathbf{Y}^-$, $\mathbf{Y}^+$ as the two copies of $\mathbf{Y}$, regarded as functions of $(\boldsymbol{\tau}_-, \mathbf{B}_-)$ and $(\boldsymbol{\tau}_+, \mathbf{B}_+)$. With such a coupling we would like to bound

$$\int \left\{ \sinh(2\zeta_r^{\mathbf{Y}^+_{t_+}}) \cosh(2\zeta_n^{\mathbf{Y}^+_{t_+}}) - \sinh(2\zeta_r^{\mathbf{Y}^-_{t_+}}) \cosh(2\zeta_n^{\mathbf{Y}^-_{t_+}}) \right\} d\Gamma^*. \tag{3.26}$$

We construct such coupling Γ^* inductively. For each $l \le \ell < r$, we denote $\mathcal{G}^{(\ell)}$ as the subgraph obtained by breaking the edge $(v_\ell, v_{\ell+1})$ in $\mathcal{T}_R$, and taking the connected component containing v_1. We also let $\mathcal{G}^{(r)} = \mathcal{T}_R$. See Fig. 3 for an illustration.

Our general strategy is (1) to define a version of $\mu^*_\pm$ that only depends on the information on $\mathcal{G}^{(\ell)}$, denoted as $\mu^{(\ell)}_\pm$; (2) then we shall construct a coupling $\Gamma^{(\ell)}$ of $\mu^{(\ell)}_+$ and $\mu^{(\ell)}_-$ inductively in ℓ, and $\Gamma^{(r)}$ would be the desired Γ^*.

The notation W_j. We introduce the notation W_j for $r < j \le n$, which is a slightly different version of V_j. For any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$ we denote

$$W_{r+1}(\mathbf{x}) := \frac{2\cosh^2(\beta_{v_{r+1}, v_r})}{\cosh(2\beta_{v_{r+1}, v_r}) + \cosh(2\zeta^{\mathbf{x}}_r + 2\zeta^{\mathbf{x}}_{r-1\to r})},$$

and for each $r + 1 < j \le n$ we denote

$$W_j(\mathbf{x}) := \frac{2\cosh(2\zeta^{\mathbf{x}}_{j-1})\cosh^2(\beta_{v_j, v_{j-1}})}{\cosh(2\beta_{v_j, v_{j-1}}) + \cosh(2\zeta^{\mathbf{x}}_{j-1} + 2\zeta^{\mathbf{x}}_{j-2\to j-1})}.$$

We note that this is slightly different from $V_j(\mathbf{x})$; in fact, we have $W_{r+1}(\mathbf{x}) = (\cosh(2\zeta^{\mathbf{x}}_{r+1}))^{-1} V_{r+1}(\mathbf{x})$ and $W_j(\mathbf{x}) = (\cosh(2\zeta^{\mathbf{x}}_j))^{-1}\cosh(2\zeta^{\mathbf{x}}_{j-1}) V_j(\mathbf{x})$ for $r + 1 < j \le n$. Such W_j is easier to use (than V_j) in the inductive arguments, since it can be bounded by a constant: $\forall r < j \le n$, we have

$$W_j(\mathbf{x}) < \frac{2\cosh^2(\beta)\cosh(2\zeta^{\mathbf{x}}_{j-1})}{\cosh(2\zeta^{\mathbf{x}}_{j-1} + 2\zeta^{\mathbf{x}}_{j-2\to j-1})} < 4\cosh^2(\beta)\cosh(2\zeta^{\mathbf{x}}_{j-2\to j-1})$$

$$\leq 4\cosh^2(\beta)\cosh(2\beta), \tag{3.27}$$

since $|\zeta^{\mathbf{x}}_{j-2\to j-1}| \leq \beta$ (by (3.1)). Now we can write

$$d\mu^* = \frac{|\sinh(2\zeta_l^{\mathbf{Y}_{t_-}})| C_l(\mathbf{Y}_{t_-}) C_r(\mathbf{Y}_{t_+})}{\prod_{i=0}^{k+1} \overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2} \prod_{1\leq j<l} U_j(\mathbf{Y}_{t_-}) \prod_{r<j\leq n} W_j(\mathbf{Y}_{t_+}) d\mu.$$

Definition of $\mu_\pm^{(\ell)}$. For each $l \leq \ell \leq r$ and any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$, let $\mathbf{x}^{(\ell)} \in \mathbb{R}^{\mathcal{T}_R}$ be the vector given by $\mathbf{x}^{(\ell)}(v') = \mathbf{x}(v')$ for $v' \in \mathcal{G}^{(\ell)}$, and $\mathbf{x}^{(\ell)}(v') = 0$ for $v' \notin \mathcal{G}^{(\ell)}$. Then we define $\overline{Z}^{(\ell)}_{u,v}$ as the function such that

- if at least one of $u, v \in \mathcal{G}^{(\ell)}$, then $\overline{Z}^{(\ell)}_{u,v}(\mathbf{x}) = \overline{Z}_{u,v}(\mathbf{x}^{(\ell)})$,
- if $u, v \notin \mathcal{G}^{(\ell)}$, then $\overline{Z}^{(\ell)}_{u,v}(\mathbf{x}) \equiv 1$.

Also, we define $C_l^{(\ell)}(\mathbf{x}) := C_l(\mathbf{x}^{(\ell)})$, and $U_j^{(\ell)}(\mathbf{x}) := U_j(\mathbf{x}^{(\ell)})$, for any $1 \leq j < l$; and $C_r^{(\ell)}(\mathbf{x}) := C_r(\mathbf{x}^{(\ell)})$, and $W_j^{(\ell)}(\mathbf{x}) := W_j(\mathbf{x}^{(\ell)})$, for any $r < j \leq n$. We write $\mu^{(\ell)}$ as the measure on $\mathcal{J}$, with

$$d\mu^{(\ell)} = \frac{|\sinh(2\zeta_l^{\mathbf{Y}_{t_-}})| C_l^{(\ell)}(\mathbf{Y}_{t_-}) C_r^{(\ell)}(\mathbf{Y}_{t_+})}{\prod_{i=0}^{k+1} \overline{Z}^{(\ell)}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2} \prod_{1\leq j<l} U_j^{(\ell)}(\mathbf{Y}_{t_-}) \prod_{r<j\leq n} W_j^{(\ell)}(\mathbf{Y}_{t_+}) d\mu,$$

and we let $\mu_\pm^{(\ell)}$ be the measure on $\mathcal{J}$ with $d\mu_\pm^{(\ell)} = \mathbb{1}[\pm\zeta_l^{\mathbf{Y}_{t_-}} \geq 0] d\mu^{(\ell)}$. From these definitions we have $\mu^{(r)} = \mu^*$ and $\mu_\pm^{(r)} = \mu_\pm^*$.

Construction of Γ **and** $\Gamma^{(l)}$. We start by constructing Γ, a probability measure on $\mathcal{J}^2$, and a coupling of $2\mu_\pm$ (recall that $d\mu_\pm = \mathbb{1}[\pm\zeta_l^{\mathbf{Y}_{t_-}} \geq 0] d\mu$; we multiplied the measures $\mu_\pm$ by the scalar 2 for convenience, since $2\mu_\pm$ are probability measures).

We define Γ via defining random variables $\boldsymbol{\tau}^\pm, \mathbf{B}^\pm$, as follows. We first take $(\tau_v^+, (B_t^+(v))_{t\geq 0})_{v\in\mathcal{G}^{(l)}}$ as sampled from $2\mu_+$; and we take $\tau_v^- = -\tau_v^+$, $B_t^-(v) = -B_t^+(v)$ for each $v \in \mathcal{G}^{(l)}$ and $t \geq 0$.

Next we regard $\mathcal{T}_R$ as a tree rooted at v_l and inductively define $\tau_v^\pm$ for each $v \notin \mathcal{G}^{(l)}$. For any $v \notin \mathcal{G}^{(l)}$, denote v' as its parent in this v_l-rooted tree. Given $\tau_{v'}^\pm$, we need to take $\tau_v^\pm$ to be $\tau_v^\pm = \tau_{v'}^\pm$ with probability $\frac{e^{\beta_{v,v'}}}{e^{\beta_{v,v'}}+e^{-\beta_{v,v'}}}$, according to the law of the Ising model. To couple τ_v^+ and τ_v^-, if $\tau_{v'}^- = \tau_{v'}^+$ we set $\tau_v^- = \tau_v^+$ with probability 1; and if $\tau_{v'}^- \neq \tau_{v'}^+$, we have $\tau_{v'}^- = \tau_v^- \neq \tau_v^+ = \tau_{v'}^+$ with probability $\tanh(\beta_{v,v'})$, and otherwise $\tau_v^\pm = -$ or $\tau_v^\pm = +$, each with probability $\frac{1}{2}(1-\tanh(\beta_{v,v'}))$. We let $(B_t^-(v))_{t\geq 0} = (B_t^+(v))_{t\geq 0}$ be the same Brownian motion (but independent for each $v \notin \mathcal{G}^{(l)}$).

From this we have that the marginal distributions of $(\boldsymbol{\tau}^-, \mathbf{B}^-)$ and $(\boldsymbol{\tau}^+, \mathbf{B}^+)$ are given by $2\mu_-$ and $2\mu_+$, respectively.

To get $\Gamma^{(l)}$, we reweight Γ. Note under Γ, almost surely we have $\sinh(2\zeta_l^{\mathbf{Y}^-_{t_-}}) = -\sinh(2\zeta_l^{\mathbf{Y}^+_{t_-}})$, $C_l^{(l)}(\mathbf{Y}^-_{t_-}) = C_l^{(l)}(\mathbf{Y}^+_{t_-})$, $C_r^{(l)}(\mathbf{Y}^-_{t_-}) = C_r^{(l)}(\mathbf{Y}^+_{t_-})$; and $U_j^{(l)}(\mathbf{Y}^-_{t_-}) = U_j^{(l)}(\mathbf{Y}^+_{t_-})$ for each $1 \leq j < l$, $W_j^{(l)}(\mathbf{Y}^-_{t_-}) = W_j^{(l)}(\mathbf{Y}^+_{t_-})$ for each $r < j \leq n$. So we can define $\Gamma^{(l)}$ as

$$d\Gamma^{(l)} = \frac{1}{2}|\sinh(2\zeta_l^{\mathbf{Y}^-_{t_-}})| C_l^{(l)}(\mathbf{Y}^-_{t_-}) C_r^{(l)}(\mathbf{Y}^-_{t_+}) \prod_{1\le j<l} U_j^{(l)}(\mathbf{Y}^-_{t_-}) \prod_{r<j\le n} W_j^{(l)}(\mathbf{Y}^-_{t_+}) d\Gamma$$
$$= \frac{1}{2}|\sinh(2\zeta_l^{\mathbf{Y}^+_{t_-}})| C_l^{(l)}(\mathbf{Y}^+_{t_-}) C_r^{(l)}(\mathbf{Y}^+_{t_+}) \prod_{1\le j<l} U_j^{(l)}(\mathbf{Y}^+_{t_-}) \prod_{r<j\le n} W_j^{(l)}(\mathbf{Y}^+_{t_+}) d\Gamma.$$

From $\Gamma^{(\ell)}$ to $\Gamma^{(\ell+1)}$. For $l \le \ell < r$, we assume that we have a measure $\Gamma^{(\ell)}$ on the space of $\mathcal{J}^2$, such that it is a coupling of $\mu_\pm^{(\ell)}$. In fact, we can make the following stronger assumption of $\Gamma^{(\ell)}$. Let $\mathcal{F}_\ell^\pm$ be the sigma algebra of $\mathcal{J}$, generated by $(\tau_v^\pm)_{v\in\mathcal{G}^{(\ell)}}$, $(B_t^\pm(v))_{t\ge 0, v\in\mathcal{G}^{(\ell)}}$; and $\mathcal{F}_\ell$ be the sigma algebra of $\mathcal{J}^2$, generated by $\{S_+ \times \mathcal{J} : S_+ \in \mathcal{F}_\ell^+\} \cup \{\mathcal{J} \times S_- : S_- \in \mathcal{F}_\ell^-\}$. Then $\Gamma^{(\ell)}$ satisfies the following conditions:

- For any $S_+ \in \mathcal{F}_\ell^+$ we have $\Gamma^{(\ell)}(S_+ \times \mathcal{J}) = \mu_+^{(\ell)}(S_+)$; and for any $S_- \in \mathcal{F}_\ell^-$ we have $\Gamma^{(\ell)}(\mathcal{J} \times S_-) = \mu_-^{(\ell)}(S_-)$.
- Letting $\overline{\Gamma}^{(\ell)} := \frac{\Gamma^{(\ell)}}{|\Gamma^{(\ell)}|}$, which is a probability measure, the conditional distribution $\overline{\Gamma}^{(\ell)}(\cdot \mid \mathcal{F}_\ell)$ is the same as $\Gamma(\cdot \mid \mathcal{F}_\ell)$.

Now we construct $\Gamma^{(\ell+1)}$. Denote

$$R_\ell^\pm := \prod_{i=0}^{k+1} \frac{\overline{Z}^{(\ell)}_{u_i,u_{i-1}}(\mathbf{Y}^\pm_{t_i})^2}{\overline{Z}^{(\ell+1)}_{u_i,u_{i-1}}(\mathbf{Y}^\pm_{t_i})^2} \frac{C_l^{(\ell+1)}(\mathbf{Y}^\pm_{t_-})}{C_l^{(\ell)}(\mathbf{Y}^\pm_{t_-})} \frac{C_r^{(\ell+1)}(\mathbf{Y}^\pm_{t_-})}{C_r^{(\ell)}(\mathbf{Y}^\pm_{t_-})} \prod_{1\le j<l} \frac{U_j^{(\ell+1)}(\mathbf{Y}^\pm_{t_-})}{U_j^{(\ell)}(\mathbf{Y}^\pm_{t_-})} \prod_{r<j\le n} \frac{W_j^{(\ell+1)}(\mathbf{Y}^\pm_{t_-})}{W_j^{(\ell)}(\mathbf{Y}^\pm_{t_-})}. \tag{3.28}$$

Then we have $d\mu_\pm^{(\ell+1)} = R_\ell^\pm d\mu_\pm^{(\ell)}$; i.e., $R_\ell^\pm$ is the 'reweight' from $\mu_\pm^{(\ell)}$ to $\mu_\pm^{(\ell+1)}$. We also have that $R_\ell^\pm$ is $\mathcal{F}_{\ell+1}^\pm$ measurable, and also $\mathcal{F}_{\ell+1}$ measurable.

Let $P_\ell = R_\ell^- \wedge R_\ell^+$, and define $\Gamma^{(\ell+1)}$ as the measure with $d\Gamma^{(\ell+1)} = P_\ell d\Gamma^{(\ell)} + d\Xi^{(\ell)}$, where $\Xi^{(\ell)}$ is a measure on $\mathcal{J}^2$, given by the following conditions. For any $S_+ \in \mathcal{F}_{\ell+1}^+$ and $S_- \in \mathcal{F}_{\ell+1}^-$, we have

$$\Xi^{(\ell)}(S_+ \times \mathcal{J}) = \int (R_\ell^+ - P_\ell)\mathbb{1}[S_+ \times \mathcal{J}] d\Gamma^{(\ell)},$$
$$\Xi^{(\ell)}(\mathcal{J} \times S_-) = \int (R_\ell^- - P_\ell)\mathbb{1}[\mathcal{J} \times S_-] d\Gamma^{(\ell)}.$$

We also require that the conditional distribution $\overline{\Xi}^{(\ell)}(\cdot \mid \mathcal{F}_{\ell+1})$ is the same as $\Gamma(\cdot \mid \mathcal{F}_{\ell+1})$, where $\overline{\Xi}^{(\ell)} := \frac{\Xi^{(\ell)}}{|\Xi^{(\ell)}|}$ is a probability measure on $\mathcal{J}^2$. Such $\Xi^{(\ell)}$ exists (although not unique) since $R_\ell^+ - P_\ell$, $R_\ell^- - P_\ell \ge 0$, and

$$\int (R_\ell^+ - P_\ell) d\Gamma^{(\ell)} = \int R_\ell^+ d\mu_+^{(\ell)} - \int P_\ell d\Gamma^{(\ell)} = \int R_\ell^- d\mu_-^{(\ell)} - \int P_\ell d\Gamma^{(\ell)}$$
$$= \int (R_\ell^- - P_\ell) d\Gamma^{(\ell)}.$$

We now check $\Gamma^{(\ell+1)}$ satisfies the desired conditions. From the definitions, for any $S_+ \in \mathcal{F}_{\ell+1}^+$ we have

$$\Gamma^{(\ell+1)}(S_+ \times \mathcal{J}) = \int P_\ell \mathbb{1}[S_+ \times \mathcal{J}] d\Gamma^{(\ell)} + \Xi^{(\ell)}(S_+ \times \mathcal{J}) = \int R_\ell^+ \mathbb{1}[S_+ \times \mathcal{J}] d\Gamma^{(\ell)}$$
$$= \int R_\ell^+ \mathbb{1}[S_+] d\mu_+^{(\ell)} = \int \mathbb{1}[S_+] d\mu_+^{(\ell+1)} = \mu_+^{(\ell+1)}(S_+).$$

Similarly, for any $S_- \in \mathcal{F}_{\ell+1}^-$ we have $\Gamma^{(\ell+1)}(\mathcal{J} \times S_-) = \mu_-^{(\ell+1)}(S_+)$.

Also, since P_ℓ is $\mathcal{F}_{\ell+1}$ measurable, and $\overline{\Xi}^{(\ell)}(\cdot \mid \mathcal{F}_{\ell+1})$ and $\overline{\Gamma}^{(\ell)}(\cdot \mid \mathcal{F}_{\ell+1})$ are the same as $\Gamma(\cdot \mid \mathcal{F}_{\ell+1})$, we have that $\overline{\Gamma}^{(\ell+1)}(\cdot \mid \mathcal{F}_{\ell+1})$ is the same as $\Gamma(\cdot \mid \mathcal{F}_{\ell+1})$ as well. By principle of induction, we have defined $\Gamma^{(\ell)}$ for any $l \le \ell \le r$.

Finally, we just let $\Gamma^* = \Gamma^{(r)}$.

3.4. Key estimates on coupling and inductive expansion. It now remains to bound (3.26) for the Γ^* we constructed. For this we need the following key estimate, to state which we set up some further notations.

Take $E_\ell = |R_\ell^+ - R_\ell^-|$ for $l \le \ell < r$, where $R_\ell^\pm$ are the reweights (3.28). For any vertex $u \in \mathcal{T}_R$, let $\rho(u) \in \{l, \dots, r\}$, such that $u \in \mathcal{G}^{(\rho(u))}$ but $u \notin \mathcal{G}^{(\rho(u)-1)}$ (assuming that $\mathcal{G}^{(l-1)} = \emptyset$). In other words, $\rho(u) = \arg\min_{l \le \ell \le r} \operatorname{dist}(v_\ell, u)$; i.e. $\rho(u)$ is the 'projection' of u onto $[v_l, v_r]$. For any $u, v \in \mathcal{T}_R$, and $l \le a \le b \le r$, we let $\mathbf{d}_a^b(u, v)$ be the distance between the intervals $[a, b]$ and $[\rho(u), \rho(v)]$ (if $\rho(u) \le \rho(v)$) or $[\rho(v), \rho(u)]$ (if $\rho(v) \le \rho(u)$). We also denote $\mathbf{d}_a(u, v) := \mathbf{d}_a^a(u, v)$. See Fig. 3 for an illustration of these notations. Recall that $P_\ell = R_\ell^- \wedge R_\ell^+$ for $l \le \ell < r$.

Proposition 3.3. *Let $l - 1 \le a < b < r$, and f be a non-negative function of $(B_t^\pm(v))_{t \ge 0, v \notin \mathcal{G}^{(a+1)}}$, and let Λ be any probability measure on $\mathcal{J}^2$, such that $\Lambda(\cdot \mid \mathcal{F}_{a+1}) = \Gamma(\cdot \mid \mathcal{F}_{a+1})$. Then we have*

$$\int f E_b \prod_{a < \ell < b} P_\ell d\Lambda < M_a^b \mathbb{E}_\Gamma[f], \tag{3.29}$$

$$\int f \mathbb{1}[\tau_{v_{b+1}}^- \ne \tau_{v_{b+1}}^+] \prod_{a < \ell \le b} P_\ell d\Lambda < M_a^b \mathbb{E}_\Gamma[f], \tag{3.30}$$

where

$$M_a^b := C(2\theta)^{b-a} \prod_{i=0}^{k+1} (1 + C(2\theta)^{\mathbf{d}_{a+1}^{b+1}(u_i, u_{i-1})})(1 + C(2\theta)^{a+2-l})^l (1 + C(2\theta)^{r-b-1})^{n-r+1}$$

and C is an absolute constant. In particular, this implies that we have

$$\left.\begin{array}{r} \displaystyle\int f E_b \prod_{a < \ell < b} P_\ell d\Xi^{(a)} \\ \displaystyle\int f \mathbb{1}[\tau_{v_{b+1}}^- \ne \tau_{v_{b+1}}^+] \prod_{a < \ell \le b} P_\ell d\Xi^{(a)} \end{array}\right\} < M_a^b \mathbb{E}_\Gamma[f] \Xi^{(a)}(\mathcal{J}^2)$$
$$\le M_a^b \mathbb{E}_\Gamma[f] \int E_a d\Gamma^{(a)} = M_a^b \int f E_a d\Gamma^{(a)}, \tag{3.31}$$

when $a \geq l$; and similarly

$$\left.\begin{array}{r}\displaystyle\int f E_b \prod_{a<\ell<b} P_\ell d\Gamma^{(a+1)} \\ \displaystyle\int f \mathbb{1}[\tau^-_{v_{b+1}} \neq \tau^+_{v_{b+1}}] \prod_{a<\ell\leq b} P_\ell d\Gamma^{(a+1)}\end{array}\right\} < M_a^b \mathbb{E}_\Gamma[f]\Gamma^{(a+1)}(\mathcal{J}^2) = M_a^b \int f d\Gamma^{(a+1)}. \tag{3.32}$$

Here we used that $\overline{\Xi}^{(a)}(\cdot \mid \mathcal{F}_{a+1})$ *(when* $a \geq l$*),* $\overline{\Xi}^{(a+1)}(\cdot \mid \mathcal{F}_{a+1})$ *and* $\overline{\Gamma}^{(a)}(\cdot \mid \mathcal{F}_{a+1})$ *are the same as* $\Gamma(\cdot \mid \mathcal{F}_{a+1})$*, and that* E_a *is* $\mathcal{F}_{a+1}$ *measurable.*

We leave the proof of this proposition to Appendix B. Now we use it to prove Proposition 3.2, by expansion on the inductive coupling.

Proof of Proposition 3.2. From the construction of $\Gamma^* = \Gamma^{(r)}$ we need to bound (3.26). Denote

$$\mathcal{L} := 2e^{4t_+ + 4(d-1)\beta + 2|B^+_{t_+}(v_r)| + 2|B^+_{t_+}(v_n)|}.$$

Under $\Gamma^{(r)}$, when $\tau^-_{v_r} = \tau^+_{v_r}$ we must have that $2\zeta_r^{\mathbf{Y}^-_{t_+}} = 2\zeta_r^{\mathbf{Y}^+_{t_+}}$ and $2\zeta_n^{\mathbf{Y}^-_{t_+}} = 2\zeta_n^{\mathbf{Y}^+_{t_+}}$. Thus we have the bound

$$\begin{aligned}&\int \sinh(2\zeta_r^{\mathbf{Y}^+_{t_+}})\cosh(2\zeta_n^{\mathbf{Y}^+_{t_+}}) - \sinh(2\zeta_r^{\mathbf{Y}^-_{t_+}})\cosh(2\zeta_n^{\mathbf{Y}^-_{t_+}}) d\Gamma^{(r)} \\ &\quad \leq \int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] d\Gamma^{(r)}.\end{aligned} \tag{3.33}$$

Now we inductively expand the RHS using the relation $d\Gamma^{(\ell+1)} = P_\ell d\Gamma^{(\ell)} + d\Xi^{(\ell)}$ for $l \leq \ell < r$, and Proposition 3.3. Recall that $E_\ell = |R^-_\ell - R^+_\ell|$ and $P_\ell = R^-_\ell \wedge R^+_\ell$, and $R^\pm_\ell$ are the reweights (3.28), for any $l \leq \ell < r$. We first assume that $r > l$, and in this case we claim that for each $1 \leq m \leq r - l$,

$$\begin{aligned}&\int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] d\Gamma^{(r)} \\ &\leq \int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] \prod_{r-m\leq\ell<r} P_\ell d\Gamma^{(r-m)} \\ &\quad + \int \mathcal{L} E_{r-1} \prod_{r-m\leq\ell<r-1} P_\ell d\Gamma^{(r-m)} \\ &\quad + \sum_{s\geq 2}\sum_{r-m\leq a_1<\cdots<a_s=r-1}\prod_{i=1}^{s-1} M_{a_i}^{a_{i+1}} \int 2\mathcal{L} E_{a_1} \prod_{r-m\leq\ell<a_1} P_\ell d\Gamma^{(r-m)}.\end{aligned} \tag{3.34}$$

We prove this by induction in m. For $m = 1$, we have

$$\begin{aligned}\int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] d\Gamma^{(r)} &= \int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] P_{r-1} d\Gamma^{(r-1)} + \int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] d\Xi^{(r-1)} \\ &\leq \int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] P_{r-1} d\Gamma^{(r-1)} + \int \mathcal{L} E_{r-1} d\Gamma^{(r-1)},\end{aligned}$$

where the inequality is due to the following reason. Note that $\mathcal{L}$ depends only on $B^+_{t_+}(v_r)$ and $B^+_{t_+}(v_n)$, and that $d\mu_+^{(r)} = R^+_{r-1} d\mu_+^{(r-1)}$, so we have $\int \mathcal{L} d\Gamma^{(r)} = \int \mathcal{L} R^+_{r-1} d\Gamma^{(r-1)}$, and $\int \mathcal{L} d\Xi^{(r-1)} = \int \mathcal{L} d\Gamma^{(r)} - \int \mathcal{L} P_{r-1} d\Gamma^{(r-1)} = \int \mathcal{L}(R^+_{r-1} - P_{r-1}) d\Gamma^{(r-1)}$. Thus we have

$$\begin{aligned}\int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] d\Xi^{(r-1)} \leq \int \mathcal{L} d\Xi^{(r-1)} &= \int \mathcal{L}(R^+_{r-1} - P_{r-1}) d\Gamma^{(r-1)} \\ &\leq \int \mathcal{L} E_{r-1} d\Gamma^{(r-1)}.\end{aligned}$$

Now we assume that (3.34) holds for some $1 \leq m < r - l$. We study each term in the RHS. By $d\Gamma^{(r-m)} = P_{r-m-1} d\Gamma^{(r-m-1)} + d\Xi^{(r-m-1)}$ and (3.31), we have

$$\begin{aligned}&\int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] \prod_{r-m\leq \ell<r} P_\ell d\Gamma^{(r-m)} \\ &= \int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] \prod_{r-m-1\leq \ell<r} P_\ell d\Gamma^{(r-m-1)} + \int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] \prod_{r-m\leq \ell<r} P_\ell d\Xi^{(r-m-1)} \\ &\leq \int \mathcal{L}\mathbb{1}[\tau^-_{v_r} \neq \tau^+_{v_r}] \prod_{r-m-1\leq \ell<r} P_\ell d\Gamma^{(r-m-1)} + M^{r-1}_{r-m-1} \int \mathcal{L} E_{r-m-1} d\Gamma^{(r-m-1)},\end{aligned}$$

and by (3.31),

$$\begin{aligned}&\int \mathcal{L} E_{r-1} \prod_{r-m\leq \ell<r-1} P_\ell d\Gamma^{(r-m)} \\ &= \int \mathcal{L} E_{r-1} \prod_{r-m-1\leq \ell<r-1} P_\ell d\Gamma^{(r-m-1)} + \int \mathcal{L} E_{r-1} \prod_{r-m\leq \ell<r-1} P_\ell d\Xi^{(r-m-1)} \\ &\leq \int \mathcal{L} E_{r-1} \prod_{r-m-1\leq \ell<r-1} P_\ell d\Gamma^{(r-m-1)} + M^{r-1}_{r-m-1} \int \mathcal{L} E_{r-m-1} d\Gamma^{(r-m-1)},\end{aligned}$$

and for each $s \geq 2$ and $r - m \leq a_1 < \cdots < a_s = r - 1$, by (3.31),

$$\begin{aligned}&\prod_{i=1}^{s-1} M^{a_{i+1}}_{a_i} \int 2\mathcal{L} E_{a_1} \prod_{r-m\leq \ell<a_1} P_\ell d\Gamma^{(r-m)} \\ &= \prod_{i=1}^{s-1} M^{a_{i+1}}_{a_i} \int 2\mathcal{L} E_{a_1} \prod_{r-m-1\leq \ell<a_1} P_\ell d\Gamma^{(r-m-1)} \\ &\quad + \prod_{i=1}^{s-1} M^{a_{i+1}}_{a_i} \int 2\mathcal{L} E_{a_1} \prod_{r-m\leq \ell<a_1} P_\ell d\Xi^{(r-m-1)} \\ &\leq \prod_{i=1}^{s-1} M^{a_{i+1}}_{a_i} \int 2\mathcal{L} E_{a_1} \prod_{r-m-1\leq \ell<a_1} P_\ell d\Gamma^{(r-m-1)} \\ &\quad + M^{a_1}_{r-m-1} \prod_{i=1}^{s-1} M^{a_{i+1}}_{a_i} \int 2\mathcal{L} E_{r-m-1} d\Gamma^{(r-m-1)}.\end{aligned}$$

Summing up these inequalities we get (3.34) for $m+1$. Thus (3.34) holds for all $1 \le m \le r-l$. Take $m = r-l$ in (3.34), we get

$$\int \mathcal{L}\mathbb{1}[\tau_{v_r}^- \neq \tau_{v_r}^+] d\Gamma^{(r)} \le \int \mathcal{L}\mathbb{1}[\tau_{v_r}^- \neq \tau_{v_r}^+] \prod_{l\le \ell < r} P_\ell d\Gamma^{(l)} + \int \mathcal{L} E_{r-1} \prod_{l\le\ell<r-1} P_\ell d\Gamma^{(l)}$$
$$+ \sum_{s\ge 2} \sum_{l\le a_1<\cdots<a_s=r-1} \prod_{i=1}^{s-1} M_{a_i}^{a_{i+1}} \int 2\mathcal{L} E_{a_1} \prod_{l\le\ell<a_1} P_\ell d\Gamma^{(l)}.$$

Apply (3.32) to the terms in the RHS, we get

$$\int \mathcal{L}\mathbb{1}[\tau_{v_r}^- \neq \tau_{v_r}^+] d\Gamma^{(r)} \le \sum_{s\ge 2} \sum_{l-1=a_1<\cdots<a_s=r-1} \prod_{i=1}^{s-1} M_{a_i}^{a_{i+1}} \int 2\mathcal{L} d\Gamma^{(l)}. \tag{3.35}$$

We consider the summation: there are 2^{r-l-1} terms; and for each $s \ge 2$ and $l-1 = a_1 < \cdots < a_s = r-1$, we have

$$\begin{aligned}\prod_{i=1}^{s-1} M_{a_i}^{a_{i+1}} &= C^{s-1}(2\theta)^{r-l} \prod_{i=1}^{s-1}\prod_{j=0}^{k+1}(1+C(2\theta)^{\mathbf{d}_{a_i+1}^{a_{i+1}+1}(u_j,u_{j-1})}) \\ &\quad (1+C(2\theta)^{a_i+2-l})^l(1+C(2\theta)^{r-a_{i+1}-1})^{n-r+1} \\ &\le (2C\theta)^{r-l}\prod_{j=0}^{k+1} C'^{1+\mathrm{dist}(u_j,u_{j-1})} C'^{l+n-r+1}\end{aligned}$$

for some absolute constant C'. By plugging this into (3.35) we get

$$\int \mathcal{L}\mathbb{1}[\tau_{v_r}^- \neq \tau_{v_r}^+] d\Gamma^{(r)} \le 2^{r-l}(2C\theta)^{r-l} \prod_{j=0}^{k+1} C'^{1+\mathrm{dist}(u_j,u_{j-1})} C'^{l+n-r+1} \int \mathcal{L} d\Gamma^{(l)}. \tag{3.36}$$

In the case where $l = r$, we have $\int \mathcal{L}\mathbb{1}[\tau_{v_r}^- \neq \tau_{v_r}^+] d\Gamma^{(r)} = \int \mathcal{L} d\Gamma^{(l)}$, so (3.36) also holds. Now it remains to bound $\int \mathcal{L} d\Gamma^{(l)}$. For any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$, we have $C_l^{(l)}(\mathbf{x}), C_r^{(l)}(\mathbf{x}) \le \cosh(2\beta)$, by (3.1). Then with (3.24) and (3.27) we have

$$\begin{aligned}\int \mathcal{L} d\Gamma^{(l)} &= \int \frac{1}{2}|\sinh(2\zeta_l^{\mathbf{Y}_{t_-}^+})| C_l^{(l)}(\mathbf{Y}_{t_-}^+) C_r^{(l)}(\mathbf{Y}_{t_+}^+) \prod_{1\le j<l} U_j^{(l)}(\mathbf{Y}_{t_-}^+) \prod_{r<j\le n} W_j^{(l)}(\mathbf{Y}_{t_+}^+) \mathcal{L} d\Gamma^{(l)} \\ &< \int |\sinh(2\zeta_l^{\mathbf{Y}_{t_-}^+})| \cosh(2\zeta_1^{\mathbf{Y}_{t_-}^+}) \cosh^2(2\beta)(4\cosh^2(\beta)\cosh(2\beta))^{n-r+l-1} \\ &\quad \times e^{4t_+ + 4(d-1)\beta + 2|B_{t_+}^+(v_r)| + 2|B_{t_+}^+(v_n)|} d\Gamma \\ &< \int e^{4t_- + 4t_+ + 8(d-1)\beta + 2|B_{t_-}^+(v_1)| + 2|B_{t_-}^+(v_l)| + 2|B_{t_+}^+(v_r)| + 2|B_{t_+}^+(v_n)|}(2\cosh(2\beta))^{2n} d\Gamma \\ &< C''^{2n} e^{17t+d}\end{aligned}$$

for some absolute constant C''. Using this and (3.33), (3.36), we conclude that

$$\mathbb{E}\left[\frac{\sinh(2\zeta_l^{\mathbf{Y}_{t_-}})\sinh(2\zeta_r^{\mathbf{Y}_{t_+}})C_l(\mathbf{Y}_{t_-})C_r(\mathbf{Y}_{t_+})}{\prod_{i=0}^{k+1}\overline{Z}_{u_i,u_{i-1}}(\mathbf{Y}_{t_i})^2}\prod_{1\le j<l}U_j(\mathbf{Y}_{t_-})\prod_{r<j\le n}V_j(\mathbf{Y}_{t_+})\right]$$
$$\le C''^{2n}e^{17t+d}2^{r-l}(2C\theta)^{r-l}\prod_{j=0}^{k+1}C'^{1+\mathrm{dist}(u_j,u_{j-1})}C'^{l+n-r+1}$$

which gives the desired bound. □

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Appendix A: Computations of the Covariances

We prove Lemma 3.1 in this appendix.

Proof of Lemma 3.1. We first compute $\langle\sigma_u\,;\,\sigma_v\rangle_{\pi^{\mathbf{x}}}$. For any $\sigma,\sigma'\in\{\pm1\}$, we denote $L_{\sigma,\sigma'}=\widetilde{Z}_{[u,v]}^{\{u\mapsto\sigma,v\mapsto\sigma'\}}$. Then $\langle\sigma_u\,;\,\sigma_v\rangle_{\pi^{\mathbf{x}}}$ equals

$$\begin{aligned}
&\langle\sigma_u\sigma_v\rangle_{\pi^{\mathbf{x}}}-\langle\sigma_u\rangle_{\pi^{\mathbf{x}}}\langle\sigma_v\rangle_{\pi^{\mathbf{x}}}\\
&=\frac{L_{1,1}+L_{-1,-1}-L_{1,-1}-L_{-1,1}}{\widetilde{Z}_{u,v}(\mathbf{x})}\\
&\quad-\frac{L_{1,1}-L_{-1,-1}+L_{1,-1}-L_{-1,1}}{\widetilde{Z}_{u,v}(\mathbf{x})}\,\frac{L_{1,1}-L_{-1,-1}-L_{1,-1}+L_{-1,1}}{\widetilde{Z}_{u,v}(\mathbf{x})}\\
&=\frac{4L_{1,1}L_{-1,-1}-4L_{1,-1}L_{-1,1}}{\widetilde{Z}_{u,v}(\mathbf{x})^2}.
\end{aligned}$$

We claim that $4L_{1,1}L_{-1,-1}-4L_{1,-1}L_{-1,1}$ is independent of $\mathbf{x}$. Then we can get the conclusion by recalling that $\langle\sigma_u\,;\,\sigma_v\rangle_\pi=A_{u,v}$. We can expand it as

$$\sum_{\substack{(\sigma_{v'})_{v'\in[u,v]},(\overline{\sigma}_{v'})_{v'\in[u,v]}\\ \sigma_u=\sigma_v=1,\overline{\sigma}_u=\overline{\sigma}_v=-1}}$$
$$\exp\left(\sum_{\substack{(u',v')\in E_R,\\ u',v'\in[u,v]}}\beta_{u',v'}(\sigma_{u'}\sigma_{v'}+\overline{\sigma}_{u'}\overline{\sigma}_{v'})+\sum_{v'\in[u,v]}(x(v')+\zeta_{\mathcal{G}}^{\mathbf{x}}(v'))(\sigma_{v'}+\overline{\sigma}_{v'})\right)$$
$$-\sum_{\substack{(\sigma_{v'})_{v'\in[u,v]},(\overline{\sigma}_{v'})_{v'\in[u,v]}\\ \sigma_u=\overline{\sigma}_v=1,\overline{\sigma}_u=\sigma_v=-1}}$$
$$\exp\left(\sum_{\substack{(u',v')\in E_R,\\ u',v'\in[u,v]}}\beta_{u',v'}(\sigma_{u'}\sigma_{v'}+\overline{\sigma}_{u'}\overline{\sigma}_{v'})+\sum_{v'\in[u,v]}(x(v')+\zeta_{\mathcal{G}}^{\mathbf{x}}(v'))(\sigma_{v'}+\overline{\sigma}_{v'})\right).$$

Take any $(\sigma_{v'})_{v'\in[u,v]}$, $(\overline{\sigma}_{v'})_{v'\in[u,v]}$ such that $\sigma_u=\sigma_v=1,\overline{\sigma}_u=\overline{\sigma}_v=-1$, let $w\in[u,v]$ be the vertex with the smallest $\mathrm{dist}(w,u)$, such that $\sigma_w=\overline{\sigma}_w$ (if such vertex exists). We then exchange $\sigma_{v'}$ and $\overline{\sigma}_{v'}$ for all $v' \in [w, v]$. Then we obtain some $(\sigma_{v'})_{v'\in[u,v]}$, $(\overline{\sigma}_{v'})_{v'\in[u,v]}$ such that $\sigma_u = \overline{\sigma}_v = 1$, $\overline{\sigma}_u = \sigma_v = -1$; and this construction is a bijection for those $(\sigma_{v'})_{v'\in[u,v]}$, $(\overline{\sigma}_{v'})_{v'\in[u,v]}$ where w exists. Thus in the above sums we only need to consider $(\sigma_{v'})_{v'\in[u,v]}$, $(\overline{\sigma}_{v'})_{v'\in[u,v]}$ where $\sigma_{v'} \neq \overline{\sigma}_{v'}$ for each $v' \in [u, v]$; and these terms are independent of $\mathbf{x}$.

For $\langle \sigma_u \sigma_{u'} \,;\, \sigma_v \rangle_{\pi^{\mathbf{x}}}$ with $v \neq u'$, we begin with an observation on the conditional expectation of $\sigma_{u'}$ given σ_u. The following identity can be verified from a straight-forward computation and we omit the detail:

$$\mathbb{E}_{\pi^{\mathbf{x}}}[\sigma_{u'}|\sigma_u] = \frac{\sinh(2x(u')) + \sinh(2\gamma)\sigma_u}{\cosh(2x(u')) + \cosh(2\gamma)}.$$

This implies that

$$\begin{aligned}\langle \sigma_u \sigma_{u'} \,;\, \sigma_v \rangle_{\pi^{\mathbf{x}}} &= \mathbb{E}_{\pi^{\mathbf{x}}}\left[\sigma_u \sigma_v \mathbb{E}_{\pi^{\mathbf{x}}}[\sigma_{u'}|\sigma_u]\right] - \mathbb{E}_{\pi^{\mathbf{x}}}\left[\sigma_u \mathbb{E}_{\pi^{\mathbf{x}}}[\sigma_{u'}|\sigma_u]\right] \mathbb{E}_{\pi^{\mathbf{x}}}[\sigma_v] \\ &= \frac{\sinh(2x(u'))}{\cosh(2x(u')) + \cosh(2\gamma)} \langle \sigma_u \,;\, \sigma_v \rangle_{\pi^{\mathbf{x}}}.\end{aligned}$$

Moreover, note that $Z_{u',v}(\mathbf{0}) = 2\cosh(\gamma) Z_{u,v}(\mathbf{0})$. Thus, to get (3.7), from (3.6) it suffices to show that

$$\widetilde{Z}_{u',v}(\mathbf{x})^2 = 2\widetilde{Z}_{u,v}(\mathbf{x})^2 (\cosh(2x(u')) + \cosh(2\gamma)). \tag{A.1}$$

To this end, we write $\zeta^{\mathbf{x}}_{u'\to u}$ as

$$\zeta^{\mathbf{x}}_{u'\to u} = \frac{1}{2}\log\left(\frac{\cosh(x(u') + \gamma)}{\cosh(x(u') - \gamma)}\right), \tag{A.2}$$

since u' is a leaf of $\mathcal{T}_R$. Furthermore, recall the definition of $L_{\sigma,\sigma'}$ above, and set

$$L_1 := L_{1,1} + L_{1,-1}, \quad L_{-1} := L_{-1,1} + L_{-1,-1}.$$

Then, we can decompose $\widetilde{Z}_{u',v}(\mathbf{x})$ into two cases of either $\sigma_u = 1$ or $\sigma_u = -1$, and express it as follows.

$$\begin{aligned}\widetilde{Z}_{u',v}(\mathbf{x}) &= L_1 e^{-\zeta^{\mathbf{x}}_{u'\to u}} \left(e^{x(u')+\gamma} + e^{-x(u')-\gamma}\right) + L_{-1} e^{\zeta^{\mathbf{x}}_{u'\to u}} \left(e^{x(u')-\gamma} + e^{-x(u')+\gamma}\right) \\ &= 2(L_1 + L_{-1}) \left(\cosh(x(u') + \gamma)\cosh(x(u') - \gamma)\right)^{1/2}.\end{aligned} \tag{A.3}$$

In the first identity, we reweighted L_1 (resp. L_{-1}) by $e^{-\zeta^{\mathbf{x}}_{u'\to u}}$ (resp. $e^{\zeta^{\mathbf{x}}_{u'\to u}}$) since L_1 and L_{-1} are the partition functions including the effect of the induced field $\zeta^{\mathbf{x}}_{u'\to u}$ already. The second equality uses (A.2). We then obtain the conclusion by noticing that (A.1) is equivalent to (A.3).

Finally, for $\langle \sigma_u \sigma_{u'} \,;\, \sigma_v \rangle_{\pi^{\mathbf{x}}}$ with $v \neq u'$, we remark that (3.8) can be obtained from an analogous calculation, where the only difference is that we need to include the induced external field $\zeta^{\mathbf{x}}_{[u,u']}(u)$ rather than $\zeta^{\mathbf{x}}_{u'\to u}$. $\square$

Appendix B: Computations for the Key Estimate

This appendix is devoted to proving Proposition 3.3. Recall the setup from Sect. 3.1: we work on $\mathcal{T}_R$ for fixed R, and let $\boldsymbol{\beta} \in \mathbb{R}^{E_R}$ be the inverse temperature, such that $\beta_{u,v} = \beta$ if $(u, v) \notin \partial E_R$, and $\beta_{u,v} = \gamma$ if $(u, v) \in \partial E_R$. Also recall the Belief Propagation message $m^{\mathbf{x}}_{u\to v}$ and induced external field $\zeta^{\mathbf{x}}_{u\to v}$ for $(u, v) \in E(\mathcal{T}_R)$, and $\theta = \tanh\beta$. Much of our computation is based on the following result on the induced external field, which will be repeatedly used.

Lemma B.1. *Let $\mathbf{x}_-, \mathbf{x}_+ \in \mathbb{R}^{\mathcal{T}_R}$, and $(u, v) \in E(\mathcal{T}_R)$. Then we have*

$$\tanh(|\zeta^{\mathbf{x}_-}_{u\to v} - \zeta^{\mathbf{x}_+}_{u\to v}|) \le 2\theta \tanh\left(|x_-(u) - x_+(u) + \sum_{w\sim u, w\neq v} \zeta^{\mathbf{x}_-}_{w\to u} - \zeta^{\mathbf{x}_+}_{w\to u}|\right).$$

Proof. Note that $\tanh(\zeta^{\mathbf{x}_\pm}_{u\to v}) = \tanh(\beta_{u,v}) \tanh(x_\pm(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_\pm}_{w\to u})$. Then we have

$$\begin{aligned}
&\tanh(|\zeta^{\mathbf{x}_-}_{u\to v} - \zeta^{\mathbf{x}_+}_{u\to v}|) \\
&= \frac{|\tanh(\zeta^{\mathbf{x}_-}_{u\to v}) - \tanh(\zeta^{\mathbf{x}_+}_{u\to v})|}{1 - \tanh(\zeta^{\mathbf{x}_-}_{u\to v}) \tanh(\zeta^{\mathbf{x}_+}_{u\to v})} \\
&= \frac{\tanh(\beta_{u,v})|\tanh(x_-(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_-}_{w\to u}) - \tanh(x_+(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_+}_{w\to u})|}{1 - \tanh(x_-(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_-}_{w\to u}) \tanh(x_+(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_+}_{w\to u}) \tanh^2(\beta_{u,v})} \\
&\le \frac{2\theta|\tanh(x_-(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_-}_{w\to u}) - \tanh(x_+(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_+}_{w\to u})|}{1 - \tanh(x_-(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_-}_{w\to u}) \tanh(x_+(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_+}_{w\to u})} \\
&= 2\theta \tanh\left(|x_-(u) - x_+(u) + \sum_{w\sim u,w\neq v} \zeta^{\mathbf{x}_-}_{w\to u} - \zeta^{\mathbf{x}_+}_{w\to u}|\right).
\end{aligned}$$

Here we used the basic fact that $\tanh(|a - b|) = \frac{|\tanh(a)-\tanh(b)|}{1-\tanh(a)\tanh(b)}$ for any $a, b \in \mathbb{R}$, and $\tanh(\beta_{u,v}) \le \theta$, and the basic fact that $\frac{1}{1-ab} \le \frac{2}{1-a}$ for any $a \in (-1, 1)$ and $b \in [0, 1)$. $\square$

B.1. Bounds for C_l, C_r, U_j, W_j terms. In this subsection, we provide main estimates for the quantities $C_l, C_l^{(a)}, C_r, C_r^{(a)}, U_j, U_j^{(a)}, W_j$ and $W_j^{(a)}$ introduced in Sects. 3.2 and 3.3.

Lemma B.2. *Take any $l \le a \le r$ and vector $\mathbf{x}_-, \mathbf{x}_+ \in \mathbb{R}^{\mathcal{T}_R}$. If $x_-(v) = x_+(v)$ for any $v \in \mathcal{G}^{(a)}$, we must have*

$$1 - C(2\theta)^{a+1-l} < \frac{C_l(\mathbf{x}_-)}{C_l(\mathbf{x}_+)}, \frac{U_j(\mathbf{x}_-)}{U_j(\mathbf{x}_+)} < 1 + C(2\theta)^{a+1-l},$$

for any $1 \le j < l$. If $x_-(v) = x_+(v)$ for any $v \notin \mathcal{G}^{(a-1)}$, we must have

$$1 - C(2\theta)^{r-a+1} < \frac{C_r(\mathbf{x}_-)}{C_r(\mathbf{x}_+)}, \frac{W_j(\mathbf{x}_-)}{W_j(\mathbf{x}_+)} < 1 + C(2\theta)^{r-a+1},$$

for any $r < j \le n$.

Proof. By symmetry, it suffices to prove the upper bounds. From the definition of C_l in Proposition 3.2, we have

$$\frac{C_l(\mathbf{x}_-)}{C_l(\mathbf{x}_+)} < e^{|2\zeta^{\mathbf{x}_-}_{l+1\to l} - 2\zeta^{\mathbf{x}_+}_{l+1\to l}|}. \tag{B.1}$$

If $x_-(v) = x_+(v)$ for any $v \in \mathcal{G}^{(a)}$, for each $1 \leq j < l$ we have $\zeta^{\mathbf{x}_-}_j = \zeta^{\mathbf{x}_+}_j$ and $\zeta^{\mathbf{x}_-}_{j+1} = \zeta^{\mathbf{x}_+}_{j+1}$; then from the definition of U_j in (3.18) we have

$$\frac{U_j(\mathbf{x}_-)}{U_j(\mathbf{x}_+)} < e^{|2\zeta^{\mathbf{x}_-}_{j+2\to j+1} - 2\zeta^{\mathbf{x}_+}_{j+2\to j+1}|}, \ \forall 1 \leq j < l. \tag{B.2}$$

By Lemma B.1, for $l \leq \ell < a$ we have $\tanh(|\zeta^{\mathbf{x}_-}_{\ell+1\to\ell} - \zeta^{\mathbf{x}_+}_{\ell+1\to\ell}|) \leq 2\theta \tanh(|\zeta^{\mathbf{x}_-}_{\ell+2\to\ell+1} - \zeta^{\mathbf{x}_+}_{\ell+2\to\ell+1}|)$, and $\tanh(|\zeta^{\mathbf{x}_-}_{a+1\to a} - \zeta^{\mathbf{x}_+}_{a+1\to a}|) \leq 2\theta$. By multiplying these together we get

$$\tanh(|\zeta^{\mathbf{x}_-}_{l+1\to l} - \zeta^{\mathbf{x}_+}_{l+1\to l}|) < (2\theta)^{a+1-l}.$$

This with (B.1) and (B.2) gives the desired upper bounds for $\frac{C_l(\mathbf{x}_-)}{C_l(\mathbf{x}_+)}, \frac{U_j(\mathbf{x}_-)}{U_j(\mathbf{x}_+)}$. Similarly, when $x_-(v) = x_+(v)$ for any $v \notin \mathcal{G}^{(a-1)}$, we have

$$\frac{C_r(\mathbf{x}_-)}{C_r(\mathbf{x}_+)} < e^{|2\zeta^{\mathbf{x}_-}_{r-1\to r} - 2\zeta^{\mathbf{x}_+}_{r-1\to r}|}, \quad \frac{W_j(\mathbf{x}_-)}{W_j(\mathbf{x}_+)} < e^{|2\zeta^{\mathbf{x}_-}_{j-2\to j-1} - 2\zeta^{\mathbf{x}_+}_{j-2\to j-1}|}, \ \forall r < j \leq n.$$

Using Lemma B.1 we can similarly get $\tanh(|\zeta^{\mathbf{x}_-}_{r-1\to r} - \zeta^{\mathbf{x}_+}_{r-1\to r}|) < (2\theta)^{r-a+1}$; thus the desired upper bounds for $\frac{C_r(\mathbf{x}_-)}{C_r(\mathbf{x}_+)}, \frac{W_j(\mathbf{x}_-)}{W_j(\mathbf{x}_+)}$ also hold. $\square$

Lemma B.3. *For any $l - 1 \leq a < b < r$, vector $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$, we have*

$$1 - C(2\theta)^{a+2-l} < \frac{C_l^{(b+1)}(\mathbf{x})}{C_l^{(a+1)}(\mathbf{x})}, \frac{U_j^{(b+1)}(\mathbf{x})}{U_j^{(a+1)}(\mathbf{x})} < 1 + C(2\theta)^{a+2-l}, \tag{B.3}$$

for any $1 \leq j < l$; and

$$1 - C(2\theta)^{r-b-1} < \frac{C_r^{(b+1)}(\mathbf{x})}{C_r^{(a+1)}(\mathbf{x})}, \frac{W_j^{(b+1)}(\mathbf{x})}{W_j^{(a+1)}(\mathbf{x})} < 1 + C(2\theta)^{r-b-1}, \tag{B.4}$$

for any $r < j \leq n$. Here C is an absolute constant.

Proof. For $\mathbf{x}^{(a+1)}$ and $\mathbf{x}^{(b+1)}$, we have that $\mathbf{x}^{(a+1)}(v) = \mathbf{x}^{(b+1)}(v)$ for $v \in \mathcal{G}^{(a+1)}$, or $v \notin \mathcal{G}^{(b+1)}$. Thus by Lemma B.2, we get (B.3), and (B.4) when $b < r - 1$.
Now we prove (B.4) under the case where $b = r - 1$. By taking $C > 1$ it suffices to prove the upper bound in (B.4), i.e. to upper bound $\frac{C_r^{(r)}(\mathbf{x})}{C_r^{(a+1)}(\mathbf{x})}$ and $\frac{W_j^{(r)}(\mathbf{x})}{W_j^{(a+1)}(\mathbf{x})}$ by a constant. Note that $C_r^{(r)}(\mathbf{x}) = C_r^{(r-1)}(\mathbf{x})$, so we have $\frac{C_r^{(r)}(\mathbf{x})}{C_r^{(a+1)}(\mathbf{x})} = \frac{C_r^{(r-1)}(\mathbf{x})}{C_r^{(a+1)}(\mathbf{x})} < 1 + C(2\theta) < 1 + C$. By (3.27), we have that $W_j^{(r)}(\mathbf{x})$ is upper bounded by a constant. Note that as $a + 1 < r$ and $j > r$, $\zeta^{\mathbf{x}^{(a+1)}}_{j-1} = 1$, so we have $W_j^{(a+1)}(\mathbf{x}) \geq \frac{2}{\cosh(2\beta)+\cosh(2\zeta^{\mathbf{x}^{(a+1)}}_{j-2\to j-1})}$, which is lower bounded by a constant. Thus $\frac{W_j^{(r)}(\mathbf{x})}{W_j^{(a+1)}(\mathbf{x})}$ is upper bounded by a constant. $\square$

B.2. Bounds for normalized partition functions. We move on to the study of the normalized partition functions $\overline{Z}_{u,v}^{(\ell)}(\mathbf{x})$ defined in Sect. 3.3. Recall that $\overline{Z}_{u,v}^{(\ell)}(\mathbf{x}) = \overline{Z}_{u,v}(\mathbf{x}^{(\ell)})$ if at least one of $u, v \in \mathcal{G}^{(\ell)}$, and $\overline{Z}_{u,v}^{(\ell)}(\mathbf{x}) \equiv 1$ otherwise. Also recall that $\rho(u)$ is the 'projection' of u onto $[v_l, v_r]$ for each $u \in \mathcal{T}_R$; and for any $u, v \in \mathcal{T}_R$, and $l \le a \le b \le r$, $\mathbf{d}_a^b(u, v)$ is the distance between the intervals $[a, b]$ and $[\rho(u), \rho(v)]$ (if $\rho(u) \le \rho(v)$) or $[\rho(v), \rho(u)]$ (if $\rho(v) \le \rho(u)$).

Lemma B.4. *For any $l-1 \le a < b < r$, vector $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$, and $u, v \in \mathcal{T}_R$, we have*

$$1 - C(2\theta)^{\mathbf{d}_{a+2}^{b+1}(u,v)} < \frac{\overline{Z}_{u,v}^{(a+1)}(\mathbf{x})^2}{\overline{Z}_{u,v}^{(b+1)}(\mathbf{x})^2} < 1 + C(2\theta)^{\mathbf{d}_{a+2}^{b+1}(u,v)}, \tag{B.5}$$

where C is an absolute constant.

To prove this lemma, we need the following notion that extends (3.4). Take any $\mathcal{H} \subset \mathcal{G}$, $\mathcal{G} \subset \mathcal{G}'$, and $h \in \{\pm 1\}^{\mathcal{H}}$, and define

$$\widetilde{Z}_{\mathcal{G}}^{\mathcal{G}',h}(\mathbf{x}) := \sum_{\substack{(\sigma_v)_{v\in\mathcal{G}} \\ \sigma_v = h(v), \forall v \in \mathcal{H}}} \exp\left(\sum_{(u,v)\in E(\mathcal{G})} \beta_{u,v}\sigma_u\sigma_v + \sum_{v\in\mathcal{G}} (x(v) + \zeta_{\mathcal{G}'}^{\mathbf{x}}(v))\sigma_v \right). \tag{B.6}$$

The only difference compared to (3.4) is that we use $\zeta_{\mathcal{G}'}^{\mathbf{x}}(v)$ in the RHS instead of $\zeta_{\mathcal{G}}^{\mathbf{x}}$. Recalling the definition (3.2), this means that we purposefully avoid considering the induced external fields coming from some branches outside of $\mathcal{G}$ by setting $\mathcal{G}' \supsetneq \mathcal{G}$ if necessary.

Proof of Lemma B.4. We first consider the case where $u, v \notin \mathcal{G}^{(a+1)}$. We have that $\overline{Z}_{u,v}^{(a+1)}(\mathbf{x}) = 1$ and $\overline{Z}_{u,v}^{(b+1)}(\mathbf{x}) \ge 1$, and the second inequality of (B.5) holds. If $u, v \notin \mathcal{G}^{(b+1)}$, then $\overline{Z}_{u,v}^{(b+1)}(\mathbf{x}) = 1$ and the first inequality of (B.5) holds; if at least one of u, v is in $\mathcal{G}^{(b+1)}$, we must have that $\mathbf{d}_{a+2}^{b+1}(u, v) = 0$, and the first inequality of (B.5) also holds by taking $C \ge 1$.
We next consider the case where at least one of u, v is in $\mathcal{G}^{(a+1)}$; and by symmetry we assume that $u \in \mathcal{G}^{(a+1)}$. If $v \notin \mathcal{G}^{(a+1)}$, we have that $\mathbf{d}_{a+2}^{b+1}(u, v) = 0$, and the first inequality of (B.5) holds (by taking $C \ge 1$). Also we must have $v_{a+1}, v_{a+2} \in [u, v]$. Let $\mathcal{H}_1$ be the subgraph generated by vertices in $[u, v] \cap \mathcal{G}^{(a+1)}$, and $\mathcal{H}_2$ be the subgraph generated by vertices in $[u, v] \setminus \mathcal{G}^{(a+1)}$. We have that

$$\begin{aligned} \frac{\overline{Z}_{u,v}^{(a+1)}(\mathbf{x})}{\overline{Z}_{u,v}^{(b+1)}(\mathbf{x})} &= \frac{\widetilde{Z}_{[u,v]}(\mathbf{x}^{(a+1)})}{\widetilde{Z}_{[u,v]}(\mathbf{x}^{(b+1)})} \\ &= \frac{\sum_{\sigma,\sigma'\in\{\pm1\}} \widetilde{Z}_{\mathcal{H}_1}^{[u,v];\{v_{a+1}\mapsto\sigma\}}(\mathbf{x}^{(a+1)}) \widetilde{Z}_{\mathcal{H}_2}^{[u,v];\{v_{a+2}\mapsto\sigma'\}}(\mathbf{x}^{(a+1)}) e^{\beta_{v_{a+1},v_{a+2}}\sigma\sigma'}}{\sum_{\sigma,\sigma'\in\{\pm1\}} \widetilde{Z}_{\mathcal{H}_1}^{[u,v];\{v_{a+1}\mapsto\sigma\}}(\mathbf{x}^{(b+1)}) \widetilde{Z}_{\mathcal{H}_2}^{[u,v];\{v_{a+2}\mapsto\sigma'\}}(\mathbf{x}^{(b+1)}) e^{\beta_{v_{a+1},v_{a+2}}\sigma\sigma'}}. \end{aligned} \tag{B.7}$$

Here the second equality is by expanding $\widetilde{Z}_{[u,v]}(\mathbf{x}^{(a+1)})$ and $\widetilde{Z}_{[u,v]}(\mathbf{x}^{(b+1)})$ in terms of the spin at v_{a+1} and v_{a+2}. We first consider the numerator. Since $x^{(a+1)}(v) = 0$ for any $v \notin \mathcal{G}^{(a+1)}$, we have

$$2\widetilde{Z}_{\mathcal{H}_2}^{[u,v];\{v_{a+2}\mapsto\sigma'\}}(\mathbf{x}^{(a+1)}) = \widetilde{Z}_{\mathcal{H}_2}^{[u,v]}(\mathbf{0})$$

for any $\sigma' \in \{\pm 1\}$. Thus the numerator (of the last line in (B.7)) equals

$$\widetilde{Z}_{\mathcal{H}_2}^{[u,v]}(\mathbf{0})\cosh(\beta_{v_{a+1},v_{a+2}}) \sum_{\sigma\in\{\pm 1\}} \widetilde{Z}_{\mathcal{H}_1}^{[u,v];\{v_{a+1}\mapsto\sigma\}}(\mathbf{x}^{(a+1)}). \tag{B.8}$$

For the denominator (of the last line in (B.7)), by Cauchy-Schwartz we have

$$4\widetilde{Z}_{\mathcal{H}_2}^{[u,v];\{v_{a+2}\mapsto 1\}}(\mathbf{x}^{(b+1)})\widetilde{Z}_{\mathcal{H}_2}^{[u,v];\{v_{a+2}\mapsto -1\}}(\mathbf{x}^{(b+1)}) \geq \widetilde{Z}_{\mathcal{H}_2}^{[u,v]}(\mathbf{0})^2.$$

Thus the denominator (of the last line in (B.7)) is at least

$$\widetilde{Z}_{\mathcal{H}_2}^{[u,v]}(\mathbf{0}) \sum_{\sigma\in\{\pm 1\}} \widetilde{Z}_{\mathcal{H}_1}^{[u,v];\{v_{a+1}\mapsto\sigma\}}(\mathbf{x}^{(b+1)}). \tag{B.9}$$

Since $x^{(a+1)}(v) = x^{(b+1)}(v)$ for $v \in \mathcal{G}^{(a+1)}$, we have $\widetilde{Z}_{\mathcal{H}_1}^{[u,v];\{v_{a+1}\mapsto\sigma\}}(\mathbf{x}^{(a+1)}) = \widetilde{Z}_{\mathcal{H}_1}^{[u,v];\{v_{a+1}\mapsto\sigma\}}(\mathbf{x}^{(b+1)})$ for each $\sigma \in \{\pm 1\}$. Then by taking the ratio of (B.8) over (B.9), and using (B.7), we conclude that

$$\frac{\overline{Z}_{u,v}^{(a+1)}(\mathbf{x})}{\overline{Z}_{u,v}^{(b+1)}(\mathbf{x})} \leq \cosh(\beta_{v_{a+1},v_{a+2}}) \leq \cosh(\beta),$$

and this is bounded by constant. So the second inequality of (B.5) holds.

Finally, we study the case where both $u, v \in \mathcal{G}^{(a+1)}$. Let $v_* = \arg\min_{w\in[u,v]} \operatorname{dist}(w, v_{a+2})$. Using the definitions we have

$$e^{-|\zeta_{[u,v]}^{\mathbf{x}^{(a+1)}}(v_*)-\zeta_{[u,v]}^{\mathbf{x}^{(b+1)}}(v_*)|} \leq \frac{\overline{Z}_{u,v}^{(a+1)}(\mathbf{x})}{\overline{Z}_{u,v}^{(b+1)}(\mathbf{x})} = \frac{\widetilde{Z}_{[u,v]}(\mathbf{x}^{(a+1)})}{\widetilde{Z}_{[u,v]}(\mathbf{x}^{(b+1)})} \leq e^{|\zeta_{[u,v]}^{\mathbf{x}^{(a+1)}}(v_*)-\zeta_{[u,v]}^{\mathbf{x}^{(b+1)}}(v_*)|}. \tag{B.10}$$

By using Lemma B.1 on each edge in the path $[v_*, v_{a+2}]$, and the fact that $x^{(a+1)}(w) = x^{(b+1)}(w)$ for each $w \in \mathcal{G}^{(a+1)}$, we have

$$\begin{aligned}
&\tanh(|\zeta_{[u,v]}^{\mathbf{x}^{(a+1)}}(v_*) - \zeta_{[u,v]}^{\mathbf{x}^{(b+1)}}(v_*)|)\\
&\leq (2\theta)^{\operatorname{dist}(v_*,v_{a+2})}\tanh\left(|\sum_{w\sim v_{a+2}, w\neq v_{a+1}} \zeta_{w\to v_{a+2}}^{\mathbf{x}^{(b+1)}} + x(v_{a+2}) - \zeta_{w\to v_{a+2}}^{\mathbf{x}^{(a+1)}}|\right)\\
&< (2\theta)^{\operatorname{dist}(v_*,v_{a+2})} \leq (2\theta)^{\mathbf{d}_{a+2}^{b+1}(u,v)}.
\end{aligned}$$

Thus with (B.10) we have $\left|\frac{\overline{Z}_{u,v}^{(a+1)}(\mathbf{x})^2}{\overline{Z}_{u,v}^{(b+1)}(\mathbf{x})^2} - 1\right| < \frac{2(2\theta)^{\mathbf{d}_{a+2}^{b+1}(u,v)}}{1-(2\theta)^{\mathbf{d}_{a+2}^{b+1}(u,v)}}$, and our conclusion follows.

□

For $\mathbf{x}_-, \mathbf{x}_+ \in \mathbb{R}^{\mathcal{T}_R}$, and $l \le a \le b \le r$, we let

$$\kappa_a^b(\mathbf{x}_-, \mathbf{x}_+) := |\{\ell : a \le \ell \le b, x_-(v) = x_+(v), \forall v \in \mathcal{T}_R \setminus \mathcal{G}^{(\ell-1)}\}|. \quad \text{(B.11)}$$

We also write $\kappa^b(\mathbf{x}_-, \mathbf{x}_+) := \kappa_l^b(\mathbf{x}_-, \mathbf{x}_+)$. We note that $\kappa_a^b(\mathbf{x}_-, \mathbf{x}_+)$ is the distance from $b-1$ to the set $\{\rho(v) : v \in \mathcal{T}_R, x_-(v) \ne x_+(v)\}$ (recall that $\rho(v) = \arg\min_{l\le\ell\le r} \operatorname{dist}(v_\ell, v)$).

Lemma B.5. *For any $l \le b < r$, vector $\mathbf{x}_-, \mathbf{x}_+ \in \mathbb{R}^{\mathcal{T}_R}$, and $u, v \in \mathcal{T}_R$, assuming $\kappa^{b+1}(\mathbf{x}_-, \mathbf{x}_+) \ge 1$ we have*

$$\frac{\overline{Z}_{u,v}^{(b+1)}(\mathbf{x}_-)^2 \overline{Z}_{u,v}^{(b)}(\mathbf{x}_+)^2}{\overline{Z}_{u,v}^{(b)}(\mathbf{x}_-)^2 \overline{Z}_{u,v}^{(b+1)}(\mathbf{x}_+)^2} < 1 + C(2\theta)^{\kappa^{b+1}(\mathbf{x}_-,\mathbf{x}_+)}, \quad \text{(B.12)}$$

where C is an absolute constant.

Proof. Denote $\chi := b + 1 - \kappa^{b+1}(\mathbf{x}_-, \mathbf{x}_+)$, then $x_-(w) = x_+(w)$ for any $w \in \mathcal{T}_R \setminus \mathcal{G}^{(\chi)}$. We first study the case where $u, v \notin \mathcal{G}^{(b)}$. If in addition $u, v \notin \mathcal{G}^{(b+1)}$, then the LHS of (B.12) equals 1 and the statement holds. If at least one of u, v is in $\mathcal{G}^{(b+1)}$, we have

$$\frac{\overline{Z}_{u,v}^{(b+1)}(\mathbf{x}_-)^2 \overline{Z}_{u,v}^{(b)}(\mathbf{x}_+)^2}{\overline{Z}_{u,v}^{(b)}(\mathbf{x}_-)^2 \overline{Z}_{u,v}^{(b+1)}(\mathbf{x}_+)^2} = \frac{\widetilde{Z}_{[u,v]}(\mathbf{x}_-^{(b+1)})^2}{\widetilde{Z}_{[u,v]}(\mathbf{x}_+^{(b+1)})^2} \le e^{|\zeta_{[u,v]}^{\mathbf{x}_-^{(b+1)}}(v_*) - \zeta_{[u,v]}^{\mathbf{x}_+^{(b+1)}}(v_*)|}$$

where $v_* = \arg\min_{w\in[u,v]} \operatorname{dist}(w, v_\chi)$. As $x_-^{(b+1)}(w) = x_+^{(b+1)}(w)$ for any $w \notin \mathcal{G}^{(\chi)}$, by using Lemma B.1 for each edge in the path $[v_*, v_\chi]$ we have

$$\tanh(|\zeta_{[u,v]}^{\mathbf{x}_-^{(b+1)}}(v_*) - \zeta_{[u,v]}^{\mathbf{x}_+^{(b+1)}}(v_*)|) \le (2\theta)^{\operatorname{dist}(v_*,v_\chi)} \le (2\theta)^{\kappa^{b+1}(\mathbf{x}_-,\mathbf{x}_+)},$$

and this implies (B.12) (in the case where $u, v \notin \mathcal{G}^{(b)}$).

We next study the case where at least one of u, v is in $\mathcal{G}^{(b)}$. Let $w_1, \cdots, w_s$ denote all the vertices in $[u, v]$. Consider the graph of $\mathcal{T}_R$ removing the edges in $[u, v]$, and let the graphs $\mathcal{H}_1, \cdots, \mathcal{H}_s$ be its connected components that contain $w_1, \cdots, w_s$, respectively. We also recall the notation (B.6). For $1 \le j \le s$ and any $\mathbf{x} \in \mathbb{R}^{\mathcal{T}_R}$, note that

$$\zeta_{[u,v]}^{\mathbf{x}}(w_j) + x(w_j) = \frac{1}{2}\log\left(\frac{\widetilde{Z}_{\mathcal{H}_j}^{\mathcal{T}_R,\{w_j\mapsto 1\}}(\mathbf{x})}{\widetilde{Z}_{\mathcal{H}_j}^{\mathcal{T}_R,\{w_j\mapsto -1\}}(\mathbf{x})}\right).$$

Thus, by plugging this into (3.3) with $\mathcal{G} = [u, v]$, we can write

$$\widetilde{Z}_{u,v}(\mathbf{x})^2 = \frac{\widetilde{Z}(\mathbf{x})^2}{\prod_{j=1}^s \widetilde{Z}_{\mathcal{H}_j}^{\mathcal{T}_R,\{w_j\mapsto 1\}}(\mathbf{x}) \widetilde{Z}_{\mathcal{H}_j}^{\mathcal{T}_R,\{w_j\mapsto -1\}}(\mathbf{x})}. \quad \text{(B.13)}$$

Plugging this formula into the LHS of (B.12), we then need to bound

$$\frac{\widetilde{Z}_{\mathcal{H}_j}^{\mathcal{T}_R,\{w_j\mapsto\sigma\}}(\mathbf{x}_-^{(b)}) \widetilde{Z}_{\mathcal{H}_j}^{\mathcal{T}_R,\{w_j\mapsto\sigma\}}(\mathbf{x}_+^{(b+1)})}{\widetilde{Z}_{\mathcal{H}_j}^{\mathcal{T}_R,\{w_j\mapsto\sigma\}}(\mathbf{x}_-^{(b+1)}) \widetilde{Z}_{\mathcal{H}_j}^{\mathcal{T}_R,\{w_j\mapsto\sigma\}}(\mathbf{x}_+^{(b)})}. \quad \text{(B.14)}$$

for each $j = 1, \cdots, s$ and $\sigma \in \{-1, 1\}$; and

$$\frac{\widetilde{Z}(\mathbf{x}_-^{(b+1)})\widetilde{Z}(\mathbf{x}_+^{(b)})}{\widetilde{Z}(\mathbf{x}_-^{(b)})\widetilde{Z}(\mathbf{x}_+^{(b+1)})} \tag{B.15}$$

We note that $x_-^{(b)}(w) = x_+^{(b)}(w)$ and $x_-^{(b+1)}(w) = x_+^{(b+1)}(w)$ for any $w \notin \mathcal{G}^{(\chi)}$; and $x_-^{(b)}(w) = x_-^{(b+1)}(w)$ and $x_+^{(b)}(w) = x_+^{(b)}(w)$ for any $w \in \mathcal{G}^{(b)}$ or $w \notin \mathcal{G}^{(b+1)}$. This means that for any $\mathcal{H}_j$ and $\sigma \in \{\pm 1\}$, (B.14) equals 1, unless $\mathcal{H}_j$ intersects both $\mathcal{G}^{(\chi)}$ and $\mathcal{G}^{(b+1)} \setminus \mathcal{G}^{(b)}$. However, this means that $v_b, v_{b+1} \in \mathcal{H}_j$, which happens to at most one of $\mathcal{H}_1, \cdots, \mathcal{H}_s$ (since they are disjoint).
We hold on estimating (B.14) for a particular j, and bound (B.15) first. Similar to (B.13), we have

$$\widetilde{Z}_{v_b,v_{b+1}}(\mathbf{x})^2 = \frac{\widetilde{Z}(\mathbf{x})^2}{\widetilde{Z}_{\mathcal{G}^{(b)}}^{\mathcal{T}_R,\{v_b\mapsto 1\}}(\mathbf{x})\widetilde{Z}_{\mathcal{G}^{(b)}}^{\mathcal{T}_R,\{v_b\mapsto -1\}}(\mathbf{x})\widetilde{Z}_{\mathcal{T}_R\setminus\mathcal{G}^{(b)}}^{\mathcal{T}_R,\{v_{b+1}\mapsto 1\}}(\mathbf{x})\widetilde{Z}_{\mathcal{T}_R\setminus\mathcal{G}^{(b)}}^{\mathcal{T}_R,\{v_{b+1}\mapsto -1\}}(\mathbf{x})}. \tag{B.16}$$

We note that for $\mathcal{G}^{(b)}$ and $\mathcal{T}_R \setminus \mathcal{G}^{(b)}$, neither of them intersects both $\mathcal{G}^{(\chi)}$ and $\mathcal{G}^{(b+1)} \setminus \mathcal{G}^{(b)}$. So for $f \in \{\widetilde{Z}_{\mathcal{G}^{(b)}}^{\mathcal{T}_R,\{v_b\mapsto 1\}}, \widetilde{Z}_{\mathcal{G}^{(b)}}^{\mathcal{T}_R,\{v_b\mapsto -1\}}, \widetilde{Z}_{\mathcal{T}_R\setminus\mathcal{G}^{(b)}}^{\mathcal{T}_R,\{v_{b+1}\mapsto 1\}}, \widetilde{Z}_{\mathcal{T}_R\setminus\mathcal{G}^{(b)}}^{\mathcal{T}_R,\{v_{b+1}\mapsto -1\}}\}$, we have

$$\frac{f(\mathbf{x}_-^{(b+1)})f(\mathbf{x}_+^{(b)})}{f(\mathbf{x}_-^{(b)})f(\mathbf{x}_+^{(b+1)})} = 1.$$

Thus by (B.16) we have

$$\frac{\widetilde{Z}(\mathbf{x}_-^{(b+1)})\widetilde{Z}(\mathbf{x}_+^{(b)})}{\widetilde{Z}(\mathbf{x}_-^{(b)})\widetilde{Z}(\mathbf{x}_+^{(b+1)})} = \frac{\widetilde{Z}_{v_b,v_{b+1}}(\mathbf{x}_-^{(b+1)})\widetilde{Z}_{v_b,v_{b+1}}(\mathbf{x}_+^{(b)})}{\widetilde{Z}_{v_b,v_{b+1}}(\mathbf{x}_-^{(b)})\widetilde{Z}_{v_b,v_{b+1}}(\mathbf{x}_+^{(b+1)})}$$

The RHS can be expanded as

$$\frac{\cosh\left\{\zeta_{v_b\to v_{b+1}}^{\mathbf{x}_-} + x_-(v_{b+1}) + \sum_{w\sim v_{b+1},w\neq v_b}\zeta_{w\to v_{b+1}}^{\mathbf{x}_-}\right\}\cosh(\zeta_{v_b\to v_{b+1}}^{\mathbf{x}_+})}{\cosh(\zeta_{v_b\to v_{b+1}}^{\mathbf{x}_-})\cosh\left\{\zeta_{v_b\to v_{b+1}}^{\mathbf{x}_+} + x_+(v_{b+1}) + \sum_{w\sim v_{b+1},w\neq v_b}\zeta_{w\to v_{b+1}}^{\mathbf{x}_+}\right\}}. \tag{B.17}$$

By our assumption of $\kappa^{b+1}(\mathbf{x}_-, \mathbf{x}_+) \geq 1$, we have $x_-(v_{b+1}) = x_+(v_{b+1})$ and $\sum_{w\sim v_{b+1},w\neq v_b}\zeta_{w\to v_{b+1}}^{\mathbf{x}_-} = \sum_{w\sim v_{b+1},w\neq v_b}\zeta_{w\to v_{b+1}}^{\mathbf{x}_+}$. Thus by the expression (B.17) we have

$$\frac{\widetilde{Z}(\mathbf{x}_-^{(b+1)})\widetilde{Z}(\mathbf{x}_+^{(b)})}{\widetilde{Z}(\mathbf{x}_-^{(b)})\widetilde{Z}(\mathbf{x}_+^{(b+1)})} \leq e^{2|\zeta_{v_b\to v_{b+1}}^{\mathbf{x}_-} - \zeta_{v_b\to v_{b+1}}^{\mathbf{x}_+}|}. \tag{B.18}$$

By using Lemma B.1 for each edge in $[v_{b+1}, v_\chi]$, we get

$$\tanh(|\zeta_{v_b\to v_{b+1}}^{\mathbf{x}_-} - \zeta_{v_b\to v_{b+1}}^{\mathbf{x}_+}|) \leq (2\theta)^{\kappa^{b+1}(\mathbf{x}_-,\mathbf{x}_+)}.$$

This with (B.18) implies that

$$\frac{\widetilde{Z}(\mathbf{x}_-^{(b+1)})\widetilde{Z}(\mathbf{x}_+^{(b)})}{\widetilde{Z}(\mathbf{x}_-^{(b)})\widetilde{Z}(\mathbf{x}_+^{(b+1)})} < 1 + C'(2\theta)^{\kappa^{b+1}(\mathbf{x}_-,\mathbf{x}_+)} \tag{B.19}$$

for some constant C'.
Now we go back to bound (B.14), for some j such that $\mathcal{H}_j$ intersects both $\mathcal{G}^{(\chi)}$ and $\mathcal{G}^{(b+1)} \setminus \mathcal{G}^{(b)}$. Such bound can be directly obtained from (B.19), by exchanging $\mathbf{x}_-$ and $\mathbf{x}_+$, and taking the following special case of $\mathbf{x}_\pm$: first set $x_-(v) = x_+(v) = 0$ for $v \notin \mathcal{H}_j$, then send $x_-(w_j) = x_+(w_j)$ to ∞ (if $\sigma = 1$) or $-\infty$ (if $\sigma = -1$). So we conclude that (B.14) for such particular j is also bounded by $1 + C'(2\theta)^{\kappa^{b+1}(\mathbf{x}_-,\mathbf{x}_+)}$. Then the LHS of (B.12) is bounded by $(1 + C'(2\theta)^{\kappa^{b+1}(\mathbf{x}_-,\mathbf{x}_+)})^4$, and our conclusion follows. □

B.3. Proof of the key estimate. We conclude this section by establishing Proposition 3.3.

Proof of Proposition 3.3. Recall the definition of $\kappa_{a'}^{b'}$ (B.11) and $\kappa^{b'}$ for $l \le a' \le b' \le r$. We first prove (3.29). We note that for each $l \le a' \le b' \le r$, and $t \ge 0$, we have $\kappa_{a'}^{b'}(\boldsymbol{\tau}^-, \boldsymbol{\tau}^+) = \kappa_{a'}^{b'}(\mathbf{Y}_t^-, \mathbf{Y}_t^+)$. Thus in this proof, we write $\kappa_{a'}^{b'} = \kappa_{a'}^{b'}(\boldsymbol{\tau}^-, \boldsymbol{\tau}^+)$ and $\kappa^{b'} = \kappa^{b'}(\boldsymbol{\tau}^-, \boldsymbol{\tau}^+)$.
First consider the case where $\kappa_{a+2}^{b+1} = 0$. We have $E_b \le R_b^+ \vee R_b^-$ and $\prod_{a<\ell<b} P_\ell \le \prod_{a<\ell<b} R_\ell^+ \vee \prod_{a<\ell<b} R_\ell^-$, so

$$\begin{aligned}
&E_b \prod_{a<\ell<b} P_\ell \le \prod_{a<\ell\le b} R_\ell^+ \vee \prod_{a<\ell\le b} R_\ell^- \\
&= \prod_{i=0}^{k+1} \frac{\overline{Z}_{u_i,u_{i-1}}^{(a+1)}(\mathbf{Y}_{t_i}^+)^2}{\overline{Z}_{u_i,u_{i-1}}^{(b+1)}(\mathbf{Y}_{t_i}^+)^2} \frac{C_l^{(b+1)}(\mathbf{Y}_{t_-}^+)}{C_l^{(a+1)}(\mathbf{Y}_{t_-}^+)} \frac{C_r^{(b+1)}(\mathbf{Y}_{t_+}^+)}{C_r^{(a+1)}(\mathbf{Y}_{t_+}^+)} \prod_{1\le j<l} \frac{U_j^{(b+1)}(\mathbf{Y}_{t_-}^+)}{U_j^{(a+1)}(\mathbf{Y}_{t_-}^+)} \prod_{r<j\le n} \frac{W_j^{(b+1)}(\mathbf{Y}_{t_+}^+)}{W_j^{(a+1)}(\mathbf{Y}_{t_+}^+)} \\
&\quad \vee \prod_{i=0}^{k+1} \frac{\overline{Z}_{u_i,u_{i-1}}^{(a+1)}(\mathbf{Y}_{t_i}^-)^2}{\overline{Z}_{u_i,u_{i-1}}^{(b+1)}(\mathbf{Y}_{t_i}^-)^2} \frac{C_l^{(b+1)}(\mathbf{Y}_{t_-}^-)}{C_l^{(a+1)}(\mathbf{Y}_{t_-}^-)} \frac{C_r^{(b+1)}(\mathbf{Y}_{t_+}^-)}{C_r^{(a+1)}(\mathbf{Y}_{t_+}^-)} \prod_{1\le j<l} \frac{U_j^{(b+1)}(\mathbf{Y}_{t_-}^-)}{U_j^{(a+1)}(\mathbf{Y}_{t_-}^-)} \prod_{r<j\le n} \frac{W_j^{(b+1)}(\mathbf{Y}_{t_+}^-)}{W_j^{(a+1)}(\mathbf{Y}_{t_+}^-)}.
\end{aligned} \tag{B.20}$$

By Lemma B.3 and B.4, we have

$$\begin{aligned}
&\prod_{a<\ell\le b} R_\ell^+, \prod_{a<\ell\le b} R_\ell^- < \\
&\prod_{i=0}^{k+1} (1+C(2\theta)^{\mathbf{d}_{a+2}^{b+1}(u_i,u_{i-1})})(1 + C(2\theta)^{a+2-l})^l (1 + C(2\theta)^{r-b-1})^{n-r+1}.
\end{aligned} \tag{B.21}$$

We next consider the case where $\kappa_{a+2}^{b+1} \ge 1$. Without loss of generality, we assume that $R_b^+ \ge R_b^-$. We have

$$E_b \prod_{a<\ell<b} P_\ell \le E_b \prod_{a<\ell<b} R_\ell^- = \left(\frac{R_b^+}{R_b^-} - 1\right) \prod_{a<\ell\le b} R_\ell^-. \tag{B.22}$$

For factor $\prod_{a<\ell\le b} R_\ell^-$, it is again bounded using (B.21). It remains to bound $\frac{R_b^+}{R_b^-}$. Recall the definition of $R_b^\pm$ from (3.28). We consider the factors one by one. For each $0 \le i \le k+1$, by Lemma B.5 we have

$$\frac{\overline{Z}_{u_i,u_{i-1}}^{(b)}(\mathbf{Y}_{t_i}^+)^2 \overline{Z}_{u_i,u_{i-1}}^{(b+1)}(\mathbf{Y}_{t_i}^-)^2}{\overline{Z}_{u_i,u_{i-1}}^{(b+1)}(\mathbf{Y}_{t_i}^+)^2 \overline{Z}_{u_i,u_{i-1}}^{(b)}(\mathbf{Y}_{t_i}^-)^2} < 1 + C(2\theta)^{\kappa^{b+1}} \le 1 + C(2\theta)^{\kappa_{a+2}^{b+1}}. \tag{B.23}$$

Recall that for any $l \le a' \le r$, we denote $\mathbf{d}_{a'} = \mathbf{d}_{a'}^{a'}$. If $\mathbf{d}_{b+1}(u_i, u_{i-1}) \ge 1$, by Lemma B.4, we also have

$$\frac{\overline{Z}^{(b)}_{u_i,u_{i-1}}(\mathbf{Y}^+_{t_i})^2\overline{Z}^{(b+1)}_{u_i,u_{i-1}}(\mathbf{Y}^-_{t_i})^2}{\overline{Z}^{(b+1)}_{u_i,u_{i-1}}(\mathbf{Y}^+_{t_i})^2\overline{Z}^{(b)}_{u_i,u_{i-1}}(\mathbf{Y}^-_{t_i})^2} < \frac{1+C(2\theta)^{\mathbf{d}_{b+1}(u_i,u_{i-1})}}{1-C(2\theta)^{\mathbf{d}_{b+1}(u_i,u_{i-1})}} < 1+C'(2\theta)^{\mathbf{d}_{b+1}(u_i,u_{i-1})}, \tag{B.24}$$

for some constant $C' > C$. Thus by combining (B.23) and (B.24), we have

$$\frac{\overline{Z}^{(b)}_{u_i,u_{i-1}}(\mathbf{Y}^+_{t_i})^2\overline{Z}^{(b+1)}_{u_i,u_{i-1}}(\mathbf{Y}^-_{t_i})^2}{\overline{Z}^{(b+1)}_{u_i,u_{i-1}}(\mathbf{Y}^+_{t_i})^2\overline{Z}^{(b)}_{u_i,u_{i-1}}(\mathbf{Y}^-_{t_i})^2} < 1+C'(2\theta)^{\mathbf{d}_{b+1}(u_i,u_{i-1})\vee\kappa^{b+1}_{a+2}}. \tag{B.25}$$

If $u_i, u_{i-1} \notin \mathcal{G}^{(b+1)}$, the LHS equals 1; otherwise, we have

$$(\mathbf{d}_{b+1}(u_i,u_{i-1})-\kappa^{b+1}_{a+2})\vee 0 \ge (\mathbf{d}_{b+1}(u_i,u_{i-1})-(b-a))\vee 0 = \mathbf{d}^{b+1}_{a+1}(u_i,u_{i-1}).$$

By plugging this into (B.25) we have

$$\frac{\overline{Z}^{(b)}_{u_i,u_{i-1}}(\mathbf{Y}^+_{t_i})^2\overline{Z}^{(b+1)}_{u_i,u_{i-1}}(\mathbf{Y}^-_{t_i})^2}{\overline{Z}^{(b+1)}_{u_i,u_{i-1}}(\mathbf{Y}^+_{t_i})^2\overline{Z}^{(b)}_{u_i,u_{i-1}}(\mathbf{Y}^-_{t_i})^2} < 1+C'(2\theta)^{\mathbf{d}^{b+1}_{a+1}(u_i,u_{i-1})+\kappa^{b+1}_{a+2}}. \tag{B.26}$$

By Lemma B.3, we have $\frac{C^{(b+1)}_l(\mathbf{Y}^+_{t\_})}{C^{(b)}_l(\mathbf{Y}^+_{t\_})} < 1+C(2\theta)^{b+1-l}$, $\frac{C^{(b)}_l(\mathbf{Y}^-_{t\_})}{C^{(b+1)}_l(\mathbf{Y}^-_{t\_})} < (1-C(2\theta)^{b+1-l})^{-1}$; and $\frac{U^{(b+1)}_j(\mathbf{Y}^+_{t\_})}{U^{(b)}_j(\mathbf{Y}^+_{t\_})} < 1+C(2\theta)^{b+1-l}$, $\frac{U^{(b)}_j(\mathbf{Y}^-_{t\_})}{U^{(b+1)}_j(\mathbf{Y}^-_{t\_})} < (1-C(2\theta)^{b+1-l})^{-1}$, for each $1 \le j < l$. Thus we have

$$\frac{C^{(b+1)}_l(\mathbf{Y}^+_{t\_})}{C^{(b)}_l(\mathbf{Y}^+_{t\_})}\frac{C^{(b)}_l(\mathbf{Y}^-_{t\_})}{C^{(b+1)}_l(\mathbf{Y}^-_{t\_})}, \quad \frac{U^{(b+1)}_j(\mathbf{Y}^+_{t\_})}{U^{(b)}_j(\mathbf{Y}^+_{t\_})}\frac{U^{(b)}_j(\mathbf{Y}^-_{t\_})}{U^{(b+1)}_j(\mathbf{Y}^-_{t\_})} < 1+C'(2\theta)^{a+1-l+\kappa^{b+1}_{a+2}}. \tag{B.27}$$

By Lemma B.2, each of $\frac{C^{(b+1)}_r(\mathbf{Y}^+_{t\_})}{C^{(b+1)}_r(\mathbf{Y}^-_{t\_})}$, $\frac{C^{(b)}_r(\mathbf{Y}^-_{t\_})}{C^{(b)}_r(\mathbf{Y}^+_{t\_})}$ and $\frac{W^{(b+1)}_j(\mathbf{Y}^+_{t\_})}{W^{(b+1)}_j(\mathbf{Y}^-_{t\_})}$, $\frac{W^{(b)}_j(\mathbf{Y}^-_{t\_})}{W^{(b)}_j(\mathbf{Y}^+_{t\_})}$, $r < j \le n$, is bounded by $1+C(2\theta)^{r-b-1+\kappa^{b+1}_{a+2}}$. So we have

$$\frac{C^{(b+1)}_r(\mathbf{Y}^+_{t\_})}{C^{(b)}_r(\mathbf{Y}^+_{t\_})}\frac{C^{(b)}_r(\mathbf{Y}^-_{t\_})}{C^{(b+1)}_r(\mathbf{Y}^-_{t\_})}, \quad \frac{W^{(b+1)}_j(\mathbf{Y}^+_{t\_})}{W^{(b)}_j(\mathbf{Y}^+_{t\_})}\frac{W^{(b)}_j(\mathbf{Y}^-_{t\_})}{W^{(b+1)}_j(\mathbf{Y}^-_{t\_})} < 1+C'(2\theta)^{r-b-1+\kappa^{b+1}_{a+2}}. \tag{B.28}$$

By putting together (B.26), (B.27), (B.28), we have

$$\begin{aligned}\frac{R^+_b}{R^-_b} &\le \prod_{i=0}^{k+1}(1+C'(2\theta)^{\mathbf{d}^{b+1}_{a+1}(u_i,u_{i-1})+\kappa^{b+1}_{a+2}})(1+C'(2\theta)^{a+1-l+\kappa^{b+1}_{a+2}})^l \\ &\quad (1+C'(2\theta)^{r-b-1+\kappa^{b+1}_{a+2}})^{n-r+1} \\ &\le 1+(2\theta)^{\kappa^{b+1}_{a+2}}\prod_{i=0}^{k+1}(1+C'(2\theta)^{\mathbf{d}^{b+1}_{a+1}(u_i,u_{i-1})})(1+C'(2\theta)^{a+1-l})^l \\ &\quad (1+C'(2\theta)^{r-b-1})^{n-r+1}.\end{aligned}$$

Thus with (B.22) and the bound of $\prod_{a<\ell\leq b} R_\ell^-$ by (B.21), we conclude that

$$E_b \prod_{a<\ell<b} P_\ell \leq (2\theta)^{\kappa_{a+2}^{b+1}} \prod_{i=0}^{k+1} (1 + C''(2\theta)^{\mathbf{d}_{a+1}^{b+1}(u_i,u_{i-1})})(1 + C''(2\theta)^{a+1-l})^l$$
$$(1 + C''(2\theta)^{r-b-1})^{n-r+1},$$

where C'' is another constant. Note that this also holds when $\kappa_{a+2}^{b+1} = 0$ (by (B.20)). Thus we always have

$$\int f E_b \prod_{a<\ell<b} P_\ell d\Lambda$$
$$\leq \prod_{i=0}^{k+1} (1 + C''(2\theta)^{\mathbf{d}_{a+1}^{b+1}(u_i,u_{i-1})})(1 + C''(2\theta)^{a+1-l})^l (1 + C''(2\theta)^{r-b-1})^{n-r+1}$$
$$\int (2\theta)^{\kappa_{a+2}^{b+1}} f d\Lambda. \tag{B.29}$$

From the construction of Γ we have $\mathbb{P}_\Gamma(b - a - \kappa_{a+2}^{b+1} \geq j) \leq \theta^j$ for any $j \in \mathbb{Z}_+$. Since $\Gamma(\cdot \mid \mathcal{F}_{a+1}) = \Lambda(\cdot \mid \mathcal{F}_{a+1})$ we would also have $\mathbb{P}_\Lambda(b - a - \kappa_{a+2}^{b+1} \geq j) \leq \theta^j$ for any $j \in \mathbb{Z}_+$. Also note that f is a function of $(B_t^\pm(v))_{t\geq 0, v\notin \mathcal{G}^{(a+1)}}$, which is independent of $(\tau_v^\pm)_{v\notin \mathcal{G}^{(a+1)}}$. Thus we have

$$\int f (2\theta)^{\kappa_{a+2}^{b+1}} d\Lambda < 2(2\theta)^{b-a} \mathbb{E}_\Gamma[f] \tag{B.30}$$

By plugging (B.30) into (B.29) we get (3.29).
For (3.30), using (B.21) we have

$$\int f \mathbb{1}[\tau_{v_{b+1}}^- \neq \tau_{v_{b+1}}^+] \prod_{a<\ell\leq b} P_\ell d\Lambda$$
$$\leq \int f \mathbb{1}[\tau_{v_{b+1}}^- \neq \tau_{v_{b+1}}^+] \prod_{a<\ell\leq b} R_\ell^+ d\Lambda$$
$$\leq \prod_{i=0}^{k+1} (1 + C(2\theta)^{\mathbf{d}_{a+2}^{b+1}(u_i,u_{i-1})})(1 + C(2\theta)^{a+2-l})^l (1 + C(2\theta)^{r-b-1})^{n-r+1}$$
$$\int f \mathbb{1}[\tau_{v_{b+1}}^- \neq \tau_{v_{b+1}}^+] d\Lambda.$$

Again, using that $\Gamma(\cdot \mid \mathcal{F}_{a+1}) = \Lambda(\cdot \mid \mathcal{F}_{a+1})$, and $\mathbb{P}_\Gamma(\tau_{v_{b+1}}^- \neq \tau_{v_{b+1}}^+ \mid \mathcal{F}_{a+1}) \leq \theta^{b-a}$, and the independence of $(\tau_v^\pm)_{v\notin \mathcal{G}^{(a+1)}}$ and $(B_t^\pm(v))_{t\geq 0, v\notin \mathcal{G}^{(a+1)}}$ under $\Gamma(\cdot \mid \mathcal{F}_{a+1})$, we can bound this by

$$\theta^{b-a} \prod_{i=0}^{k+1} (1 + C(2\theta)^{\mathbf{d}_{a+2}^{b+1}(u_i,u_{i-1})})(1 + C(2\theta)^{a+2-l})^l (1 + C(2\theta)^{r-b-1})^{n-r+1} \mathbb{E}_\Gamma[f],$$

and (3.30) follows. $\square$

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