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A family of convex sets in the plane satisfying the $(4,3)$-property can be pierced by 9 points

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Abstract We prove that every finite family of convex sets in the plane satisfying the $(4,3)$-property can be pierced by 9 points. This improves the bound of 13 proved by Gyárfás, Kleitman, and Tóth in 2001.

1 Introduction

For positive integers $p \geq q$, a family of sets $\mathscr{C}$ is said to satisfy the (p,q)-*property* if for every p sets, some q have a point in common. We say that $\mathscr{C}$ *can be pierced by m points* if there exists a set of size at most m intersecting every element in $\mathscr{C}$. The piercing number $\tau(\mathscr{C})$ of $\mathscr{C}$ is the minimum m so that $\mathscr{C}$ can be pierced by m points.

In 1957 Hadwiger and Debrunner [2] conjectured that for every given positive integers $p \geq q > d$, there exists a (smallest) constant $HD_d(p,q)$ such that every finite family $\mathscr{C}$ of convex sets in $\mathbb{R}^d$ satisfying the (p,q)-property has $\tau(\mathscr{C}) \leq c$. This conjecture was proved by Alon and Kleitman in 1992 [1].

In general, the bounds on $HD_d(p,q)$ given by Alon and Kleitman's proof are far from optimal. The first case where $HD_d(p,q)$ is not known is when $d = 2$, $p = 4$, and $q = 3$. In this case, the bound in $HD_d(p,q)$ given by the Alon-Kleitman proof is 343, while there is no known example of a family of convex sets in the plane that satisfy the $(4,3)$-property and cannot be pierced by 3 points. We note that improvements on general upper bounds on $HD_d(p,q)$ were made in [4].

In 2001, Gyárfás, Kleitman, and Tóth [5] proved that $HD_2(4,3) \leq 13$, and since then this bound has seen no improvement. In this paper we prove that $HD_2(4,3) \leq 9$:

Theorem 1 *If $\mathscr{C}$ is a finite family of convex sets in $\mathbb{R}^2$ such that for any 4 sets, 3 have a point in common, then $\tau(\mathscr{C}) \leq 9$.*

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The main tools in the proof are the following two theorems, and a geometrical analysis.

Let $\Delta^{n-1} \subset \mathbb{R}^n$ denote the $n-1$-dimensional simplex with vertex set $e_1, \dots, e_n$. The following version of the KKM was proven in [**?**].

Theorem 2 *Let $A_1, \dots, A_n$ be open sets such that for every face $I \subseteq \{1, \dots, n\}$, $\cup_I A_i \supseteq$ conv$\{e_i \mid i \in I\}$. Then $\cap_{i=1}^n A_i \neq \emptyset$.*

We note that Theorem 2 stated for closed sets $A_1, \dots, A_n$ is the original KKM Theorem, which was proved in [6].

A matching in a family of sets $\mathscr{F}$ is a subset of pairwise disjoint sets in $\mathscr{F}$. The matching number $\nu(\mathscr{F})$ is the maximum size of a matching in $\mathscr{F}$.

Let L_1, L_2 be two homeomorphic copies of the real line. A 2-*interval* is a union $I_1 \cup I_2$, where I_i is an interval on L_i.

Theorem 3 (Tardos [7]) *If $\mathscr{F}$ is a family of 2-intervals then $\tau(\mathscr{F}) \leq 2\nu(\mathscr{F})$.*

2 Using the KKM theorem

Given a finite family $\mathscr{C}$ of convex sets we may assume that the sets are compact, by considering a set S containing a point in each intersection of sets in $\mathscr{C}$, and replacing every set $C \in \mathscr{C}$ by $C' = \text{conv}\{s \in S \mid s \in C\}$.

Let $\mathscr{C}$ be a finite family of compact convex sets satisfying the $(4,3)$ property. We may clearly assume $|\mathscr{C}| \geq 4$. We scale the plane so that all the sets in $\mathscr{C}$ are contained in the open unit disk, which we denote by D. Let f be a parameterization of the unit circle defined by

$$f(t) = (cos(2\pi t), -sin(2\pi t))$$

for $t \in [0,1]$. For two points a, b in the plane, let $\overline{ab}$ be the line through a and b and let $[a,b]$ be the line segment with a and b as endpoints.

Let $\Delta = \Delta^3 = \text{conv}\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^4$ be the standard 3-dimensional simplex, and let $x = (x_1, x_2, x_3, x_4) \in \Delta$. For $1 \leq i \leq 4$, define R_x^i to be the interior of the region bounded by the arc along the circle from $f(\sum_{j=1}^{i-1} x_j)$ to $f(\sum_{j=1}^{i} x_j)$ (an empty sum is understood to be 0) and by the line segments $[(1,0), f(x_1+x_2)]$ and $[f(x_1), f(x_1+x_2+x_3)]$ (see Figure 1). Notice that if $x_i = 0$, then $R_x^i = \emptyset$.

For every $1 \leq i \leq 4$ define a subset A_i of Δ as follows: $x \in \Delta^3$ is in A_i if and only if there exist three sets $C_1, C_2, C_3 \in \mathscr{C}$ such that $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and $C_j \cap C_k \subset R_x^i$ for all $1 \leq j < k \leq 3$ (see Figure 2). Observe that A_i is open.

For every $x \in \Delta$ and $C \in \mathscr{C}$ let I_C be the (possibly empty) 2-interval

$$(C \cap [(1,0), f(x_1+x_2)]) \cup (C \cap [f(x_1), f(x_1+x_2+x_3)]).$$

Lemma 1 *Suppose there exists $x \in \Delta \setminus \left(\cup_{i=1}^4 A_i\right)$. Then there exist two points a, b such that if $a, b \notin C$ then $I_C \neq \emptyset$.*

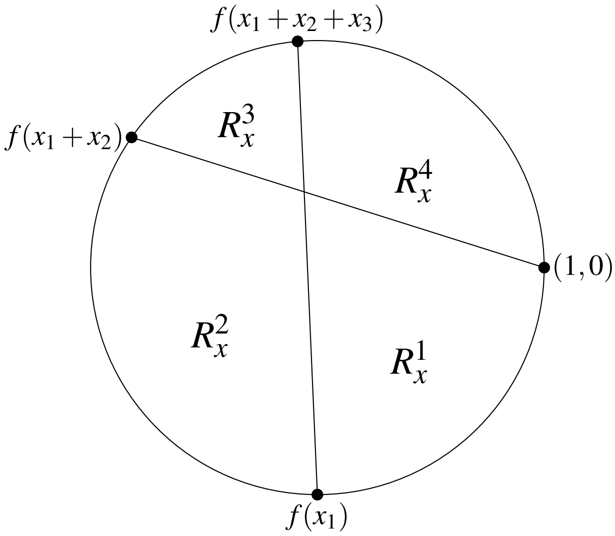

Fig. 1 A point $x \in \Delta^3$ corresponds to four regions R_x^i.

Proof Assume that $x \in \Delta \setminus \left(\cup_{i=1}^4 A_i\right)$. Note that since $\mathscr{C}$ does not contain three pairwise non-intersecting sets, at most two of the regions R_x^i can contain a set in $\mathscr{C}$.

We claim for every $i \leq 4$, the region R_x^i contains at most two sets in $\mathscr{C}$. Indeed, assume to the contrary that R_x^i contains three sets $C_1, C_2, C_3 \in \mathscr{C}$. Then $C_1 \cap C_2 \cap C_3 = \emptyset$ since $x \notin A_i$. Applying the $(4,3)$ property to C_1, C_2, C_3 and some additional set $F \in \mathscr{C}$, we obtain that $C_j \cap C_k \cap F \neq \emptyset$ for some $1 \leq j < k \leq 3$, and all pairwise intersections of C_j, C_k, F are contained in R_x^i, contradicting $x \notin A_i$.

If there is only one region R_x^i containing sets in $\mathscr{C}$, then since there are at most two such sets, there are two points that pierce them.

If there are two regions R_x^i and R_x^j containing sets in $\mathscr{C}$, then if there are two sets contained in R_x^i (or R_x^j), they must intersect. Otherwise these two sets together with a set in R_x^j (or R_x^i, respectively) will be three pairwise non-intersecting sets, a contradiction since $\mathscr{C}$ has the $(4,3)$-property.

Therefore, there is a point piercing the sets contained in R_x^i and a point piercing the sets in R_x^j and we are done. $\square$

Theorem 4 *If there exists* $x \in \Delta \setminus \left(\cup_{i=1}^4 A_i\right)$*, then* $\tau(\mathscr{C}) \leq 8$.

Proof Let $\mathscr{D} = \{C \in \mathscr{C} \mid I_C \neq \emptyset\}$. We will show that $\tau(\mathscr{D}) \leq 6$. Together with Lemma 1 this will imply the theorem.

Let $\mathscr{I} = \{I_C : C \in \mathscr{D}\}$. Let $C_1, C_2, C_3, C_4 \in \mathscr{D}$ be four sets. Some three, say C_1, C_2, C_3, intersect by the $(4,3)$-property. Since $x \notin \cup_{i=1}^4 A_i$, the intersection of two of these three sets, say $C_1 \cap C_2$, must intersect either $[(1,0), f(x_1+x_2)]$ or $[f(x_1), f(x_1+x_2+x_3)]$. In other words, $I_{C_1} \cap I_{C_2} \neq \emptyset$. This shows that $\mathscr{I}$ has no four pairwise disjoint elements, implying $\nu(\mathscr{I}) \leq 3$. Thus, by Theorem 3, $\tau(\mathscr{D}) \leq \tau(\mathscr{I}) \leq 6$. $\square$

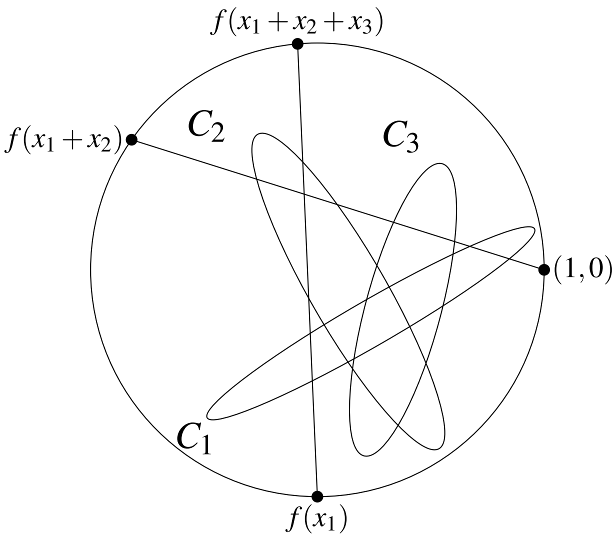

Fig. 2 Three sets $C_1, C_2, C_3 \in C$ with $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and $C_j \cap C_k \subset R_x^1$ for all $1 \leq j < k \leq 3$, implying $x = (x_1, x_2, x_3, x_4) \in A_1$.

By Theorem 4 we may assume that $\Delta \subset \cup_{i=1}^4 A_i$. We claim that in this case the sets $A_1, \ldots, A_4$ satisfy the conditions of Theorem 2. Indeed, let $I \subset [4]$, and let $y \in \operatorname{conv}\{e_i \mid i \in I\}$. Then for all $j \in [4] \setminus I$, we have $R_y^j = \emptyset$, implying $y \notin A_j$. Since $y \in \cup_{i=1}^4 A_i$, we have that $y \in \bigcup_{i \in I} A_i$. Thus, by Theorem 2 we have:

Theorem 5 *If $\Delta \subset \cup_{i=1}^4 A_i$, then there exists $x \in \bigcap_{i=1}^4 A_i$.*

For the rest of the paper we fix $x \in \bigcap_{i=1}^4 A_i$. Let $R_x^i = R^i$, and let $f_0 = (1,0)$, $f_1 = f(x_1)$, $f_2 = f(x_1+x_2)$, and $f_3 = f(x_1+x_2+x_3)$. Let c be the intersection point of $[(1,0), f_2]$ and $[f_1, f_3]$, and let $\mathscr{C}^* = \{C \in \mathscr{C} \mid c \notin C\}$. Note that $\bigcap_{i=1}^4 A_i$ is an open set, so we may shift x slightly to ensure that c does not lie on the boundary of any set in $\mathscr{C}$ and neither of the segments $[(1,0), f_2]$ or $[f_1, f_3]$ coincide with the boundary of any set in $\mathscr{C}$.

We use $\overline{R^i}$ to denote the topological closure of R^i.

Proposition 1 *If $C \in \mathscr{C}*$, then there exists some i for which $C \cap R^i = \emptyset$.*

Proof Assume C has a point p_i in each R^i. Then since C is convex, it contains the points $q_1 = [p_1, p_2] \cap [f_1, f_3]$ and $q_2 = [p_3, p_4] \cap [f_1, f_3]$. Since q_1 and q_2 lie in two different hyperplanes defined by the line $\overline{f_0 f_2}$, C must contain c, a contradiction. $\square$

Let $\mathscr{C}_i$ denote the family of sets in $\mathscr{C}^*$ that are disjoint from R^i. By Proposition 1, we have $\mathscr{C}^* = \cup_{i=1}^4 \mathscr{C}_i$. In the remainder of the paper we prove the following:

Theorem 6 *For every $i \leq 4$, $\tau(\mathscr{C}_i) \leq 2$.*

This will imply that $\mathscr{C}$ can be pierced by 9 points: two points for each $\mathscr{C}_i$ and the point c.

3 Piercing $\mathscr{C}_i$ by two points

In this Section we prove Theorem 6. Without loss of generality we prove the theorem for $\mathscr{C}_1$.

3.1 Preliminary definitions and observations

Let $C_1, C_2, C_3 \in \mathscr{C}$ be the three sets witnessing the fact that $x \in A_1$; so $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and $C_j \cap C_k \subset R^1$ for all $1 \leq j < k \leq 3$.

If there are two sets $F_1, F_2 \in \mathscr{C}_1$ that do not intersect, then F_1, F_2, C_1, C_2 do not satisfy the $(4,3)$-property. Thus every two sets in $\mathscr{C}_1$ intersect. Also, if for some $1 \leq i \leq 3$ we have $C_i \subset R^1$, then again by the $(4,3)$-property every three sets in $\mathscr{C}_1$ have a common point. This implies by Helly's theorem [3] that $\tau(\mathscr{C}_1) = 1$. So we may assume that no C_i is contained in R^1.

Let L_1 be the line $\overline{f_1 f_3}$ and let L_2 be the line $\overline{f_0 f_2}$ (see Figure 3).

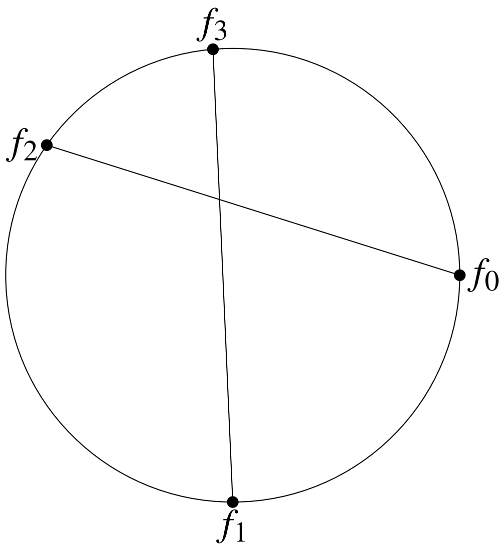

Fig. 3 The lines L_1 and L_2.

By our assumption C_i is not contained in R^1 for $1 \leq i \leq 3$, and thus $C_i \cap (\mathbb{R}^2 \setminus R^1)$ has at least one non-empty connected component. The next proposition shows that $C_i \cap (\mathbb{R}^2 \setminus R^1)$ has at most two connected components.

Proposition 2 *For every $1 \leq i \leq 3$, the set $C_i \cap (\mathbb{R}^2 \setminus R^1)$ has at most two connected components. Moreover, if $C_i \cap (\mathbb{R}^2 \setminus R^1)$ has two components, then the components are $C_i \cap \overline{R^2}$ and $C_i \cap \overline{R^4}$ and hence are convex.*

Proof If C_i contains c, then $C_i \cap (\mathbb{R}^2 \setminus R^1)$ has one component because the line segment from any point in $\mathbb{R}^2 \setminus R^1$ to c is contained in $\mathbb{R}^2 \setminus R^1$. So assume C_i does not contain c. Then it must have a point in either $\overline{R^4}$ or $\overline{R^2}$, without loss of generality, in $\overline{R^4}$.

Suppose C_i contains a point in $\overline{R^3}$. Since C_i does not contain c but contains points in the three regions $\overline{R^1}, \overline{R^4}, \overline{R^3}$, then by Proposition 1 it cannot contain a point in $\overline{R^2}$. Thus $C_i \cap (\mathbb{R}^2 \setminus R^1) = C_i \cap (\overline{R^4} \cup \overline{R^3})$. This means that $C_i \cap (\mathbb{R}^2 \setminus R^1)$ is an intersection of two convex sets, hence it is convex and has only one component.

Thus, if $C_i \cap (\mathbb{R}^2 \setminus R^1)$ has more than one component, then C_i does not have a point in $\overline{R^3}$. In this case the components of $C_i \cap (\mathbb{R}^2 \setminus R^1)$ are $C_i \cap \overline{R^2}$ and $C_i \cap \overline{R^4}$ both of which are convex. □

Let $Z = [f_1, c] \cup [c, f_0]$. We think of Z as a segment starting at f_1 and ending at f_0. Thus a point $a \in Z$ comes before a point $b \in Z$ if the distance from a to f_1 on Z is not larger than the distance from b to f_1 on Z.

Let $I_i^1 = C_i \cap [f_1, c]$, $I_i^2 = C_i \cap [c, f_0]$, and $I_i = C_i \cap Z$.

For any interval (i.e., connected set) I on Z, let $l(I)$ be the endpoint of I that comes first on Z, and let $r(I)$ be the other endpoint. Given a convex set C and a point p on the boundary of C, a *supporting line for* C at p is a line L passing through p that contains C in one of the closed halfspaces defined by L. For $1 \leq i \leq 3$, let $C_i' = C_i \cap (\mathbb{R}^2 \setminus R^1)$.

Definition 1 Let A be a connected component of C_i', and let $I = A \cap Z$ (so I is an interval on Z). We define supporting lines $S_i^l(I)$ and $S_i^r(I)$ for C_i at the points $l(I)$ and $r(I)$, respectively, as follows:

- If $A \subset L_j$ for some $j \in \{1, 2\}$, then $S_i^l(I) = S_i^r(I) = L_j$.
- If $r(I) = c$, A lies in $\overline{R^1 \cup R^2}$, and $A \not\subset L_1$, then $S_i^r(I) = L_2$.
- If $l(I) = c$, A lies in $\overline{R^1 \cup R^4}$, and $A \not\subset L_2$, then $S_i^l(I) = L_1$.
- Otherwise, set $S_i^l(I)$ and $S_i^r(I)$ to be any supporting line for C_i at the point $l(I)$ and $r(I)$, respectively.

Definition 2 Assume C_i' has two components $A_1 = C_i' \cap \overline{R^2}$ and $A_2 = C_i' \cap \overline{R^4}$. We define S_i' to be a piece-wise linear curve as follows:

- If $A_1 \subset L_1$ and $A_2 \subset L_2$, then
$$S_i' = [f_1, r(I_i^1)] \cup [r(I_i^1), l(I_i^2)] \cup [l(I_i^2), f_0].$$
- If $A_1 \subset L_1$ and $A_2 \not\subset L_2$, then
$$S_i' = [f_1, r(I_i^1)] \cup [r(I_i^1), l(I_i^2)] \cup (S_i^l(I_i^2) \cap \overline{R^4}).$$
- If $A_1 \not\subset L_1$ and $A_2 \subset L_2$, then
$$S_i' = (S_i^r(I_i^1) \cap \overline{R^2}) \cup [r(I_i^1), l(I_i^2)] \cup [l(I_i^2), f_0].$$
- If $A_1 \not\subset L_1$ and $A_2 \not\subset L_2$, then
$$S_i' = (S_i^r(I_i^1) \cap \overline{R^2}) \cup [r(I_i^1), l(I_i^2)] \cup (S_i^l(I_i^2) \cap \overline{R^4}).$$

Note that in all the cases S_i' lies in the closed halfspace defined by the line $\overline{r(I_i^1) l(I_i^2)}$ containing f_0 and f_1.

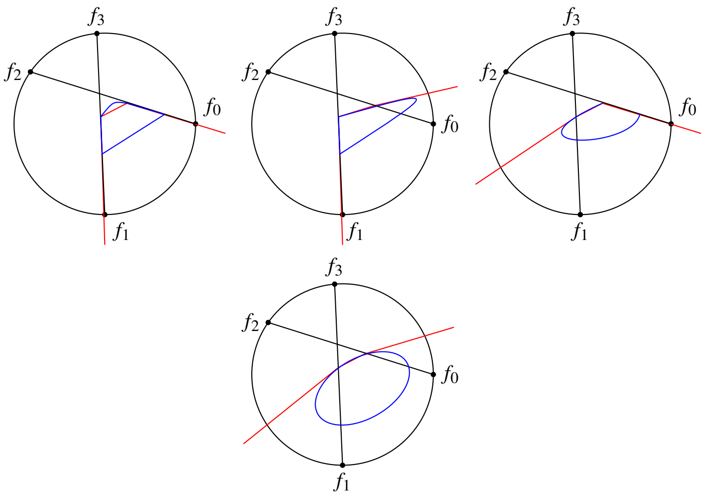

Fig. 4 Definition 2. The blue set is C_i, and the red curve is S_i^l.

3.2 Five lemmas

For two intervals I and J of Z, we say that I *comes before* J *on* Z if the point $r(I)$ comes before $l(J)$ on Z.

Lemma 2 *Let $1 \leq i \neq j \leq 3$. Let A be a component of C_i' and $I = A \cap Z$. Let B be a component of C_j' and $J = B \cap Z$. Suppose that $F \subset \mathscr{U}$ is a convex set such that $F \cap R^1 = \emptyset$, $F \cap A \neq \emptyset$ and $F \cap B \neq \emptyset$. If I comes before J on Z, then $F \cap S_i^r(I) \neq \emptyset$ and $F \cap S_j^l(J) \neq \emptyset$.*

Proof Observe that if A is a line segment or a single point, then $A \subset S_i^r(I)$ and we are done. This is true since if $A \subset L_t$ for some $t \in \{1,2\}$ then $S_i^r(I)$ is L_t, and if A is a line segment not contained in L_1 or L_2, then C_i must be a line segment so $A \subset S_i^r(I)$. So we may assume that A is not a line segment, and in particular, I consists of more than one point.

We will show that $F \cap S_i^r(I) \neq \emptyset$. The other statement follows similarly.

Case 1. $r(I) \in [c, f_0]$. We claim that B does not have a point $\overline{R^1 \cup R^2}$. Indeed, by definition B does not have a point in R^1. If B has a point p in $\overline{R^2}$, then for a point $q \in C_j \cap R^1$, we have that the line segment $[p,q]$ crosses Z in $y \in [f_1, c]$. Therefore, y comes before $l(J)$ on Z, a contradiction.

Now, if $S_i^r(I) = L_2$, then since $A \subset \overline{R^1 \cup R^2}$ and $B \subset \overline{R^3 \cup R^4}$ then F must intersect L_2 as needed. So we can assume $S_i^r(I) \neq L_2$. Let H be the closed halfspace defined by $S_i^r(I)$ containing C_i. If $r(I) \in (c, f_0]$, then since I consists of more than one point, H contains a point of $[c, f_0]$ that comes before $r(I)$ on Z, so every point on Z coming after $r(I)$ must lie in the complement of H. If $r(I) = c$, then since $S_i(r(I)) \neq L_2$, we have that A has a point y in $\overline{R^3 \cup R^4} \setminus L_2$ (this follows from the definition of $S_i^r(I)$ when $r(I) = c$).

Note that $y \in \overline{R^3}$. This is true because if A has a point in $\overline{R^4}$ other than c, then A must contain a point in $(c, f_0]$. By the convexity of C_i and the fact that I contains more than one point, we can conclude that A has a point in $L_2 \cap \overline{R^2 \cup R^3}$ other than c. Again, this implies that every point on Z coming after c lies in the complement of H. Now, if B has a point p in H, then for a point $q \in C_i \cap C_j \subset H \cap R^1$, we have that the segment $[p,q]$ crosses Z in H. This is a contradiction since every point of J lies in the complement of H.

Case 2. $r(I) \in [f_1, c)$. If $S_i^r(I) = L_1$, then $A \subset S_i^r(I)$ and we are done.

Otherwise, take H to be the closed halfspace defined by $S_i^r(I)$ containing C_i. If $S_i^r(I)$ does not intersect $(c, f_0]$, then every point after $r(I)$ on Z lies in the complement of H and we can apply the previous argument to conclude that B cannot contain a point in H. If $S_i^r(I)$ intersects $(c, f_0]$, then the set $(H \setminus R^1) \cap \mathscr{U}$ has two connected components. Let H' be the component containing A. Every point that comes after $r(I)$ on Z lies in the complement of H', so a similar argument to the one used above shows that B cannot contain a point in H'. Thus F must intersect $S_i^r(I)$. $\square$

Similar arguments can be applied to prove the following lemma.

Lemma 3 *Let $1 \le i \ne j \le 3$. Assume that C_i' has two components $A_1 = C_i' \cap \overline{R^2}$ and $A_2 = C_i' \cap \overline{R^4}$. Let B be a component of C_j' and $J = B \cap Z$. Suppose $F \subset \mathscr{U}$ is a convex set such that $F \cap R_x^1 = \emptyset$ and $F \cap B \ne \emptyset$. If $F \cap A_1 \ne \emptyset$ and J comes after I_i^1 on Z, or if $F \cap A_2 \ne \emptyset$ and J comes before I_i^2 on Z, then $F \cap S_i' \ne \emptyset$.*

We say that a set F *lies below a line L in $\overline{R^2}$* if F does not intersect L, $F \subset \overline{R^2}$, and F lies on the side of L containing f_1. Note that if F lies below L_1 in $\overline{R^2}$ then F must be empty. Similarly, we say that F *lies below L in $\overline{R^4}$* if F does not intersect L, $F \subset \overline{R^4}$, and lies on the side of L that contains f_0. If $L = L_2$, then again $F = \emptyset$.

Lemma 4 *Assume that for $1 \le i \ne j \le 3$, C_i' and C_j' both have two components: $A_1 = C_i' \cap \overline{R^2}$, $A_2 = C_i' \cap \overline{R^4}$, $B_1 = C_j' \cap \overline{R^2}$, and $B_2 = C_j' \cap \overline{R^4}$. Assume that I_i^1 comes before I_j^1 on Z and I_i^2 comes before I_j^2 on Z. Let $F_1, F_2 \in \mathscr{C}_1$, and write $F = F_1 \cap F_2$. Then one of the following holds:*

- *If F intersects C_i' and C_j' or F intersects B_2 then $F \cap S_i^r(I_i^1) \ne \emptyset$.*
- *If F intersects C_i' and C_j' or F intersects A_1, then $F \cap S_j^l(I_j^2) \ne \emptyset$.*

Proof First note that A_1 lies below $S_j^l(I_j^2)$ in $\overline{R^2}$ and B_2 lies below $S_i^r(I_i^1)$ in $\overline{R^4}$ For instance, if A_1 contains a point p lying on or above $S_j^l(I_j^2)$, then the segment $[p, l(I_i^2)]$ crosses $[f_1, c]$ on or above $S_j^l(I_j^2)$, contradicting the fact that I_i^1 comes before I_j^1 on Z. A similar argument applies to the corresponding statement for B_2.

Assume for contradiction that there exists two sets $F_1, F_2 \in \mathscr{C}_1$ such that $F_1 \cap F_2$ lies below $S_j^l(I_j^2)$ in $\overline{R^2}$ and $A_1 \cap (F_1 \cap F_2) \ne \emptyset$, and two sets $F_3, F_4 \in \mathscr{C}_1$ such that $F_3 \cap F_4$ lies below $S_i^r(I_i^1)$ in $\overline{R^4}$ and $B_2 \cap (F_1 \cap F_2) \ne \emptyset$. By the $(4,3)$-property, three sets out of F_1, F_2, F_3, F_4 have a common point. If F_1, F_2, F_3 intersect, then F_3 intersects B_2 and has a point below $S_j^l(I_j^2)$ in $\overline{R^2}$, which implies that F_3 has a point in R^1, a contradiction. Similarly, F_1, F_3, F_4 cannot intersect. Therefore, there is no pair of sets in $\mathscr{C}_1$ whose intersection intersects B_2 and lies below $S_i^r(I_i^1)$ in $\overline{R^4}$, or there is no pair of sets in $\mathscr{C}_1$ whose intersection intersects A_1 and lies below $S_j^l(I_j^2)$ in $\overline{R^2}$.

Assume that there is no pair of sets in $\mathscr{C}_1$ whose intersection lies below $S_i^r(I_i^1)$ in $\overline{R^4}$ and intersects B_2, and take $L = S_i^r(I_i^1)$. Let F be the intersection of any pair of sets in $\mathscr{C}_1$. If F intersects B_2, then by the above F does not lie below L in $\overline{R^4}$. This implies that F intersects L since B_2 lies below L in $\overline{R^4}$. If F intersects B_1 and A_1, then F intersects L by Lemma 2. If F intersects B_1 and A_2, then F intersects L since B_1 lies in the halfspace defined by L that does not contain C_i.

If there is no pair of sets in $\mathscr{C}_1$ whose intersection lies below $S_j^l(I_j^2)$ in $\overline{R^2}$ and intersects A_1, then a similar argument shows that the corresponding statements follow for $S_j^l(I_j^2)$. □

Lemma 5 *If $F \in \mathscr{C}_1$ and C_i' has two components, then $F \cap S_i'$ is an interval.*

Proof Clearly, $F \cap (S_i' \cap \overline{R^2})$ and $F \cap (S_i' \cap \overline{R^4})$ are intervals, and $F \cap S_i' = (S_i' \cap \overline{R^2}) \cup (S_i' \cap \overline{R^4})$, so it suffices to show that F cannot intersect both $S_i' \cap \overline{R^4}$ and $S_i' \cap \overline{R^2}$. Suppose it does. Let T be the line passing through $r(I_i^1)$ and $l(I_i^2)$. By the definition of S_i', both $S_i' \cap \overline{R^2}$ and $S_i' \cap \overline{R^4}$ lie on the closed halfspace defined by T containing f_0 and f_1. Since F is convex, this implies that F has a point in R^1, a contradiction. □

Lemma 6 *Let $F_1, F_2 \in \mathscr{C}_1$, then $F_1 \cap F_2$ intersects at least two of C_1, C_2, C_3.*

Proof Suppose $F_1 \cap F_2$ does not intersect C_1. Since $(C_1 \cap C_2) \subset R^1$ and $F \cap R^1 = \emptyset$, by the $(4,3)$-property for the sets C_1, C_2, F_1, F_2, we have that C_2 must intersect $F_1 \cap F_2$. Similarly, C_3 intersects $F_1 \cap F_2$. □

3.3 Proof of Theorem 6

We wish to show that $\tau(\mathscr{C}_1) \leq 2$. We split into four cases. In each case and subcase, we find two homeomorphic copies of the real line T_1 and T_2, and show that the family of 2-intervals $\mathscr{I} = \{(F \cap T_1) \cup (F \cap T_2) : F \in \mathscr{C}_1\}$ satisfies $\nu(\mathscr{I}) = 1$. By Theorem 3, this implies $\tau(\mathscr{I}) \leq 2$. The curves T_1, T_2 will be of the form $S_i^r(I)$, $S_i^l(I)$, or S_i', and Lemma 5 ensures that $\mathscr{I}$ is indeed a family of 2-intervals. Recall that $I_i^1 = C_i \cap [f_1, c]$, $I_i^2 = C_i \cap [c, f_0]$, and $I_i = C_i \cap Z$.

Case 1. C_i' has one component for each i (see Figure 5).

Notice in this case each I_i is an interval. Assume without loss of generality that I_1 comes before I_2 and I_2 comes before I_3 on Z.

Set $T_1 = S_1^r(I_1)$ and $T_2 = S_2^r(I_2)$, and let $F_1, F_2 \in \mathscr{C}_1$. By Lemma 6, $F_1 \cap F_2$ intersects two of the C_i. Then, by Lemma 2, $F_1 \cap F_2$ intersects T_1 or T_2. It follows then our collection of 2-intervals, $\mathscr{I}$, has matching number 1.

Case 2. One of the C_i''s has two components (see Figure 6).

We can assume without loss of generality that C_3' has two components, and that I_1 comes before I_2 on Z.

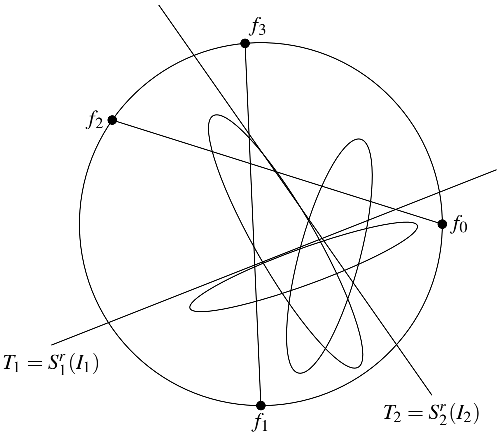

Fig. 5 Case 1

Subcase 2.1. If the order of the intervals on Z is I_1, I_2, I_3^1, I_3^2, then set $T_1 = S_1^r(I_1)$ and $T_2 = S_2^r(I_2)$.

Subcase 2.2. If the order of the intervals is I_1, I_3^1, I_2, I_3^2, then set $T_1 = S'_3$ and $T_2 = S_1^r(I_1)$.

Subcase 2.3. If the order of the intervals is I_1, I_3^1, I_3^2, I_2, then set $T_1 = S_1^r(I_1)$ and $T_2 = S_2^l(I_2)$.

Subcase 2.4. If the order of the intervals is I_3^1, I_1, I_2, I_3^2, then set $T_1 = S'_3$ and $T_2 = S_1^r(I_1)$.

The remaining subcases of Case 2 are symmetrical. For instance, the case where the order of the intervals is I_3^1, I_3^2, I_1, I_2 follows similarly to the case where the order of the intervals is I_1, I_2, I_3^1, I_3^2.

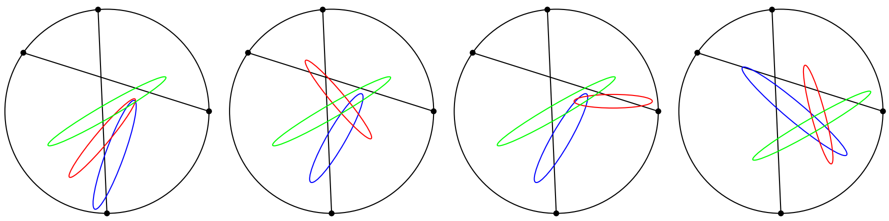

Fig. 6 The configuration of each C_i in the subcases of Case 2. Blue represents C_1, red represents C_2, and green represents C_3.

Case 3. Two of the C_i''s have two components (see Figure 7).

Without loss of generality, assume C_2' and C_3' have two components.

Subcase 3.1. Assume the order of the intervals is $I_1, I_2^1, I_3^1, I_3^2, I_2^2$, then set $T_1 = S_1^r(I_1)$ and $T_2 = S_2^l$.

Subcase 3.2. If the order of the intervals is $I_1, I_2^1, I_3^1, I_2^2, I_3^2$, then set $T_1 = S_1^r(I_1)$ and T_2 to be the line obtained by applying Lemma 4 to C_2 and C_3.

Subcase 3.3. If the order of the intervals is $I_2^1, I_1, I_3^1, I_3^2, I_2^2$, then set $T_1 = S_1^r(I_1)$ and $T_2 = S_2^l$.

Subcase 3.4. If the order of the intervals is $I_2^1, I_1, I_3^1, I_2^2, I_3^2$, then set $T_1 = S^r(I_1)$ and T_2 to be the line obtained by applying Lemma 4 to C_2 and C_3. Let F be the intersection of a pair of sets in $\mathscr{C}_1$. If F intersects C_2 and C_3, then F intersects T_2 by Lemma 4. If F intersects C_1 and C_3 or C_1 and $C_2^l \cap \overline{R^4}$, then F intersects T_1. If $T_2 = S_2^r(I_2^1)$ and F intersects C_1 and $C_2^l \cap \overline{R^2}$, then F intersects T_2 by Lemma 2. If $T_2 = S_3^l(I_3^2)$ and F intersects C_1 and $C_2^l \cap \overline{R^2}$, then F intersects T_2 by Lemma 4.

Subcase 3.5. If the order of the intervals is $I_2^1, I_3^1, I_1, I_3^2, I_2^2$, then set $T_1 = S_2^l$ and $T_2 = S_3^l$.

Subcase 3.6. If the order of the intervals is $I_2^1, I_3^1, I_1, I_2^2, I_3^2$, then set $T_1 = S_2^l$ and $T_2 = S_3^l$.

The remaining subcases are symmetrical.

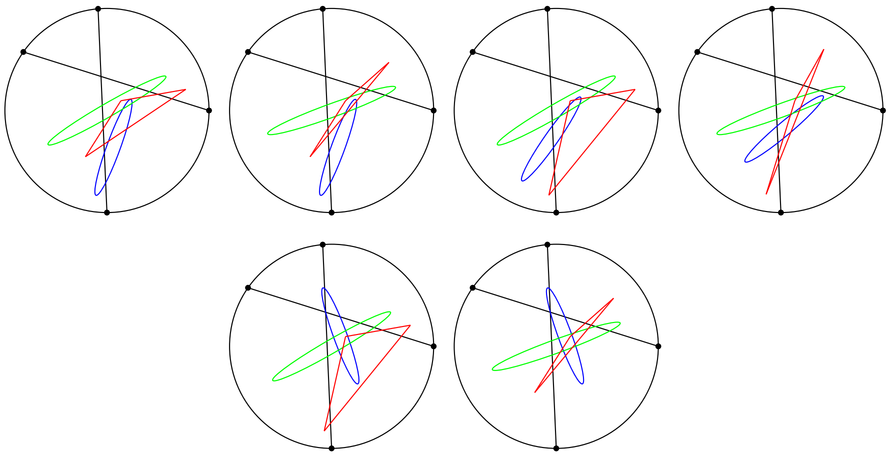

Fig. 7 The configuration of each C_i in the subcases of Case 3.

Case 4: Each C_i^l has two components (see Figure 8).

Subcase 4.1. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_3^2, I_2^2, I_1^2$, then set $T_1 = S_1^l$ and $T_2 = S_2^l$.

Subcase 4.2. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_3^2, I_1^2, I_2^2$, then set $T_1 = S_1^l$ and $T_2 = S_2^l$.

Subcase 4.3. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_1^2, I_2^2, I_3^2$, then set $T_1 = S_2'$ and T_2 to be the line obtained by applying Lemma 4 to C_1 and C_3. Let F be the intersection of a pair of sets in $\mathscr{C}_1$. If F intersects C_1 and C_3, then H intersects T_2 by Lemma 4. If F intersects $C_2' \cap \overline{R^4}$ and C_1 or $C_2' \cap \overline{R^4}$ and $C_3' \cap \overline{R^2}$, then F intersects T_1 by Lemma 3. If F intersects $C_2' \cap \overline{R^2}$ and C_3 or $C_2' \cap \overline{R^2}$ and $C_1' \cap \overline{R^4}$, then H intersects T_1 by Lemma 3.

If $T_2 = S_1^r(I_1^1)$ and F intersects $C_3' \cap \overline{R^4}$, then F intersects T_2 by Lemma 4. If F intersects $C_2' \cap \overline{R^2}$ and $C_1' \cap \overline{R^2}$, then F intersects T_2 by Lemma 2.

If $T_2 = S_3^l(I_3^1)$ and F intersects $C_1' \cap \overline{R^2}$, then F intersects T_2 by Lemma 4. If F intersects $C_2' \cap \overline{R^4}$ and $C_3' \cap \overline{R^4}$, then F intersects T_2 by Lemma 2.

Therefore, the resulting family of 2-intervals coming from these two lines has matching number 1.

Subcase 4.4. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_2^2, I_3^2, I_1^2$, then set $T_1 = S_1'$ and T_2 to be the line obtained by applying Lemma 4 to C_2 and C_3.

Subcase 4.5. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_2^2, I_1^2, I_3^2$, then set $T_1 = S_1'$ and T_2 to be the line obtained by applying Lemma 4 to C_2 and C_3. A similar argument as in subcase 3 shows that the resulting family of 2-intervals coming from these two lines has matching number 1.

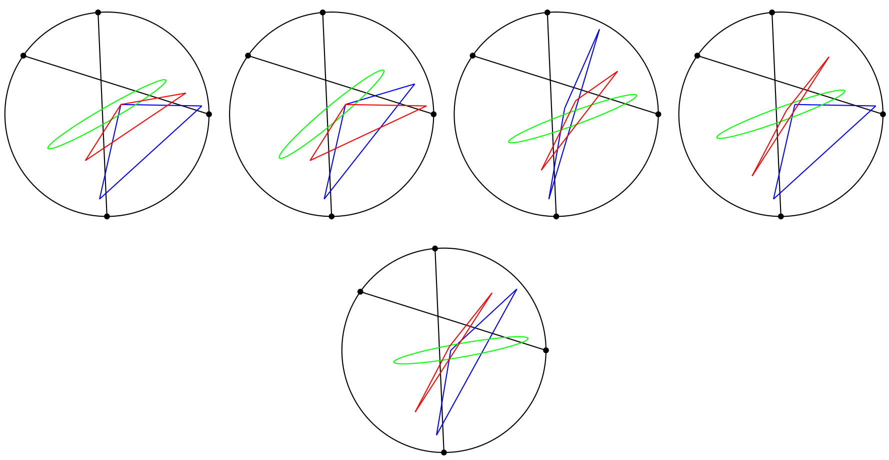

Fig. 8 The configuration of each C_i in the subcases of Case 4

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