SEMISIMPLICITY OF THE LYAPUNOV SPECTRUM FOR IRREDUCIBLE COCYCLES

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ABSTRACT. Let G be a semisimple Lie group acting on a space X, let μ be a symmetric compactly supported measure on G, and let A be a strongly irreducible linear cocycle over the action of G. We then have a random walk on X, and let T be the associated shift map. We show that, under certain assumptions, the cocycle A over the action of T is conjugate to a block conformal cocycle.

This statement is used in the recent paper by Eskin-Mirzakhani on the classification of invariant measures for the $SL(2,\mathbb{R})$ action on moduli space. The ingredients of the proof are essentially contained in the papers of Guivarch and Raugi and also Goldsheid and Margulis.

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1. INTRODUCTION

1.1. **Statement of the main results.** Let G be a semisimple Lie group. Denote by μ a symmetric compactly supported probability measure on G.

Let X be a space where G acts and denote by ν a μ-stationary measure (that is, $\mu * \nu = \nu$ where $\mu * \nu := \int_G g_* \nu d\mu(g)$). We assume that ν is μ-ergodic.

Consider L a real finite-dimensional vector space and $A : G \times X \to SL(L)$ a (linear) cocycle[1]. Since it is sufficient for our purposes, we will assume that $A(g,x)$

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[1] I.e., the cocycle relation $A(g_2g_1, x) = A(g_2, g_1(x)) \cdot A(g_1, x)$ holds for all $x \in X$ and $g_1, g_2 \in G$.

is bounded for g in the support of μ. Denote by $\mathbf{H}$ the *algebraic hull* of $A(.,.)$ in Zimmer's sense, that is, the smallest linear $\mathbb{R}$-algebraic subgroup[2] $\mathbf{H}$ such that there exists a measurable map $C : X \to SL(L)$ with $C(g(x))A(g,x)C(x)^{-1} \in \mathbf{H}$ for μ-almost all $g \in G$ and ν-almost all $x \in X$. In what follows, we will assume that $\mathbf{H}$ is a $\mathbb{R}$-simple Lie group with finite center, and a basis of L is (measurably) chosen at each $x \in X$ so that the cocycle $A(.,.)$ takes its values in $\mathbf{H}$.

Definition 1.1. We say that the cocycle $A(.,.)$ has an *invariant system of subspaces* if there are measurable families $W_i(x)$, $i = 1, \dots, k$, of subspaces of L such that $A(g,x)(W_1(x) \cup \dots \cup W_k(x)) \subset W_1(g(x)) \cup \dots \cup W_k(g(x))$ for μ-almost every $g \in G$ and ν-almost every $x \in X$.

Definition 1.2 (Strong irreducibility)**.** We say that $A(.,.)$ is *strongly irreducible* if there are no non-trivial and proper invariant systems of subspaces.

We will be interested in the behavior of a strongly irreducible cocycle $A(.,.)$ on the Lyapunov subspaces obtained after multiplying the matrices $A(g,x)$ while following a random walk on G. For this reason, let us introduce the following objects.

Let $\Omega = G^{\mathbb{N}}$. Denote by $T : \Omega \times X \to \Omega \times X$ the natural forward shift map on $\Omega \times X$:

$$T(u,x) = (\sigma(u), u_1(x))$$

where $\sigma(u) = (u_2, \dots)$ for $u = (u_1, u_2, \dots) \in \Omega$. Denoting by $\beta = \mu^{\mathbb{N}}$ the probability measure on Ω naturally induced by μ, it follows from the fact that ν is μ-stationary that the probability measure $\beta \times \nu$ is T-invariant.

As we already mentioned above, from now on, we will assume that the stationary measure ν is μ-ergodic[3], that is, $\beta \times \nu$ is T-ergodic.

In this language, we can study the products of matrices of the cocycle $A(.,.)$ along random walks with the aid of the cocycle dynamics $F_A : \Omega \times X \times \mathbf{H} \to \Omega \times X \times \mathbf{H}$ naturally associated to $A(.,.)$:

$$F_A(u,x,h) = (T(u,x), A(u_1,x)h)$$

Actually, for our purposes, the "fiber dynamics" of F_A will be more important than the base dynamics T. For this reason, given $u \in \Omega$ and $x \in X$, let us denote by $A^n(u,x)$ the matrix given by the formula:

$$\begin{aligned} F_A^n(u,x,\mathrm{Id}) &= (T^n(u,x), A(u_n, u_{n-1} \dots u_1(x)) \dots A(u_1,x)) \\ &= (T^n(u,x), A(u_n \dots u_1, x)) = (T^n(u,x), A^n(u,x)). \end{aligned}$$

In this context, the multiplicative ergodic theorem of V. Oseledets [Os] says that, if $\int \log^+ \|A(g,x)\| d\mu(g) d\nu(x) < \infty$, then there is a collection of numbers $\lambda_1 > \dots > \lambda_k$

[2]Recall that the algebraic hull is unique up to conjugation (cf. Zimmer's book [Zi]).

[3]By definition, ν is μ-ergodic if it is not a non-trivial convex combination of two distinct μ-stationary measures. The fact that ν is μ-ergodic is equivalent to the T-ergodicity of $\beta \times \nu$ is classical: see e.g. Benoist–Quint's book [BQb].

with multiplicities $m_1, \ldots, m_k$ called *Lyapunov exponents* and, at $\beta \times \nu$-almost every point $(u, x) \in \Omega \times X$, we have a *Lyapunov flag*

$$\{0\} = V_{k+1}^+ \subset V_k^+(u, x) \subset \cdots \subset V_1^+(u, x) = L \tag{1.1}$$

such that $V_i^+(u, x)$ has dimension $m_i + \cdots + m_k$ and $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(u, x)\vec{p}\| = \lambda_i$ whenever $\vec{p} \in V_i^+(u, x) \setminus V_{i+1}^+(u, x)$.

In this paper, we will study the consequences of the strong irreducibility of a cocycle for its Lyapunov spectrum (i.e., collection of Lyapunov exponents and flags). In particular, we will focus on the following property:

Definition 1.3. We say that F_A or simply $A(.,.)$ has *semisimple Lyapunov spectrum* if its algebraic hull $\mathbf{H}$ is *block-conformal* in the sense that, for each $i = 1, \ldots, k$, $V_i^+(u, x)/V_{i+1}^+(u, x)$ possesses an invariant splitting,

$$V_i^+(u, x)/V_{i+1}^+(u, x) = \bigoplus_{j=1}^{n_i} E_{ij}(u, x),$$

and on each $E_{ij}(u, x)$ there exists a (non-degenerate) quadratic form $\langle ., . \rangle_{ij,u,x}$ such that, for all $\vec{p}, \vec{q} \in E_{ij}(u, x)$ and for all $n \in \mathbb{N}$,

$$\langle A^n(u, x)\vec{p}, A^n(u, x)\vec{q} \rangle_{ij,T^n(u,x)} = e^{\lambda_{ij}(u,x,n)} \langle \vec{p}, \vec{q} \rangle_{ij,(u,x)}$$

for some cocycle[4] $\lambda_{ij} : \Omega \times X \times \mathbb{N} \to \mathbb{R}$.

Standing assumptions. From now on, besides the hypotheses

- (A1) G is a semisimple Lie group group acting on a space X;
- (A2) μ is a symmetric compactly supported probability measure on G and ν is an ergodic μ-stationary probability measure on X;
- (A3) $A : G \times X \to SL(L)$ is a linear cocycle (where L is a real finite-dimensional vector space) such that $A(g, x)$ is bounded for g in the support of μ;
- (A4) the algebraic hull $\mathbf{H}$ of $A(.,.)$ is a $\mathbb{R}$-simple Lie group with finite center.
- (A5) A verifies Oseledets' integrability condition $\int \log \|A(g, x)^{\pm 1}\| d\mu(g) d\nu(x) < \infty$,

we will actually require the (stronger assumption of) invariance of ν under $\mathrm{supp}(\mu)$:

- (A6) for all $g \in \mathrm{supp}(\mu)$, one has $g_*\nu = \nu$.

In fact, most of the arguments in this paper need just[5] the μ-stationarity of ν: as it turns out, the invariance of ν under $\mathrm{supp}(\mu)$ is used *only* at Subsection 4.3.

In this context, the main result of this paper is the following theorem:

Theorem 1.4. *If $A(.,.)$ is strongly irreducible, then it has semisimple Lyapunov spectrum.*

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[4]That is, $\lambda_{ij}(u, x, m + 1) = \lambda_{ij}(T^{m-1}(u, x), 1) + \lambda_{ij}(u, x, m)$.

[5]We emphasize this point by writing most of this article in the setting of stationary measures.

Furthermore, the top Lyapunov exponent corresponds to a single conformal block, that is, for $\beta\times\nu$-a.e. (u,x) there are a (non-degenerate) quadratic form $\langle .,.\rangle_{u,x}$ and a cocycle $\lambda:\Omega\times X\times\mathbb{N}\to\mathbb{R}$ such that

$$\langle A^n(u,x)\vec{p}, A^n(u,x)\vec{q}\rangle_{T^n(u,x)} = e^{\lambda(u,x,n)}\langle\vec{p},\vec{q}\rangle_{u,x} \tag{1.2}$$

for all $\vec{p},\vec{q}\in V_1^+(u,x)/V_2^+(u,x)$.

In fact, the ingredients of the proof of this result are essentially contained in the articles of Goldsheid-Margulis [GM] and Guivarc'h-Raugi [GR1], [GR2]. In particular, the fact that such a result holds is no surprise to the experts.

Nevertheless, we decided to write down a proof of this theorem here mainly for two reasons: firstly, this precise statement is hard to locate in these references, and, secondly, this result is relevant in the recent paper [EMi] where a Ratner-type theorem is shown for the action of $SL(2,\mathbb{R})$ on moduli spaces of Abelian differentials.

1.2. **The backwards cocycle.** As it turns out, for the application in Eskin-Mirzakhani paper [EMi], one needs the analog of Theorem 1.4 for the backward shift.

More precisely, let $\Omega^-=G^{\mathbb{Z}-\mathbb{N}}$ and $\hat{\Omega}=\Omega^-\times\Omega$. Denote by $T^-:\Omega^-\times X\to\Omega^-\times X$ the natural backward shift map on $\Omega^-\times X$:

$$T^-(v,y)=(\sigma^-(v),v_0^{-1}(y))$$

where $\sigma^-(v)=(\dots,v_{-1})$ for $v=(\dots,v_0)\in\Sigma^-$. Similarly, denote by $\hat{T}:\hat{\Omega}\times X\to\hat{\Omega}\times X$ the natural forward shift map on $\hat{\Omega}\times X$:

$$\hat{T}(v,u,x)=(\hat{\sigma}(v,u),u_1(x))$$

where $\hat{\sigma}(v,u)=(c_{i-1})_{i\in\mathbb{Z}}$ for $(v,u)=(c_i)_{i\in\mathbb{Z}}$.

Recall that Ω is equipped with the probability measure $\beta=\mu^{\mathbb{N}}$, so that $\beta\times\nu$ is a T-invariant probability measure on $\Omega\times X$. Note that Borel measures on Ω and $\hat{\Omega}$ are uniquely determined by their values on cylinders. In particular, the natural projection $\pi_+:\hat{\Omega}\times X\to\Omega\times X$ induces a bijection $(\pi_+)_*$ between the spaces of $\hat{T}$-invariant and T-invariant Borel probability measures, and, *a fortiori*, there exists an unique probability measure $\widehat{\beta\times\nu}$ on $\hat{\Omega}\times X$ projecting to $\beta\times\nu$ under $(\pi_+)_*$. In this context, the natural T^--invariant probability measure β^X constructed in Lemma 3.1 of Benoist and Quint [BQ] is $\beta^X=(\pi_-)_*\circ(\pi_+)_*^{-1}(\beta\times\nu):=(\widehat{\beta\times\nu})^-$, where $\pi_-:\Omega\times X\to\Omega^-\times X$ is the natural projection.

Similarly to the previous subsection, we can study the products of matrices of the cocycle $A(.,.)$ along backward random walks with the aid of the dynamical system $F_A^-:\Omega^-\times X\times\mathbf{H}\to\Omega^-\times X\times\mathbf{H}$ given by

$$F_A^-(v,y,h)=(T^-(v,y),A(v_0,v_0^{-1}(y))^{-1}h)$$

naturally associated to A, or, equivalently, the "fiber" dynamics $A^{-n}(v,y)$ given by the formula:

$$\begin{aligned}(F_A^-)^n(v,y,\mathrm{Id}) &= ((T^-)^n(v,y), A(v_{-(n-1)}, v_{-(n-1)}^{-1}\dots v_0^{-1}(y))^{-1}\dots A(v_0, v_0^{-1}(y))^{-1}) \\ &=: ((T^-)^n(v,y), A^{-n}(v,y))\end{aligned}$$

By Oseledets multiplicative ergodic theorem, if $\int \log^+ \|A(g,x)^{\pm 1}\| d\mu(g)d\nu(x) < \infty$, then we have a Lyapunov flag

$$\{0\} = V_0^- \subset V_1^-(v,y) \subset \cdots \subset V_k^-(v,y) = L \tag{1.3}$$

such that $V_j^-(v,y)$ has dimension $m_1 + \cdots + m_j$ and $\lim\limits_{n\to\infty} \frac{1}{n}\log\|A^{-n}(v,y)\vec{q}\| = -\lambda_j$ for $\vec{q} \in V_j^-(v,y) \setminus V_{j-1}^-(v,y)$, where λ_i are the Lyapunov exponents of F_A and m_i are their multiplicities from the paragraph surrounding (1.1).

In this setting, we will show the following:

Theorem 1.5. *Suppose that $A(.,.)$ is strongly irreducible. Then, F_A^- has semisimple Lyapunov spectrum.*

Furthermore, the largest Lyapunov exponent corresponds to a single conformal block, i.e., for $\beta^X = (\widehat{\beta\times\nu})^-$-a.e. (v,y) there are a (non-degenerate) quadratic form $\langle .,.\rangle_{v,y}$ and a cocycle $\lambda : \Omega^- \times X \times \mathbb{N} \to \mathbb{R}$ such that

$$\langle A^{-n}(v,y)\vec{p}, A^{-n}(v,y)\vec{q}\rangle_{(T^-)^n(v,y)} = e^{\lambda(v,y,n)}\langle \vec{p},\vec{q}\rangle_{v,y} \tag{1.4}$$

for all $\vec{p},\vec{q} \in V_1^-(v,y)$.

1.3. **The invertible cocycle.** Both Theorem 1.4 and Theorem 1.5 are derived as a consequence of a theorem about the two-sided walk. By Oseledets theorem applied to $\hat{T}$, the flags (1.1) and (1.3) exist for $\widehat{\beta\times\nu}$-a.e. $(v,u,x) \in \hat{\Omega}\times X$ (and, moreover, $V_i^+(v,u,x) = V_i^+(u,x)$ and $V_j^-(v,u,x) = V_j^-(v,x)$).

Then for $\widehat{\beta\times\nu}$-a.e. $(v,u,x) \in \hat{\Omega}\times X$, let us define

$$\mathcal{V}_\ell(v,u,x) = V_\ell^+(u,x) \cap V_\ell^-(v,x)$$

for every $1 \le \ell \le k$. By [GM, Lemma 1.5], $\mathcal{V}_\ell(v,u,x)$ has dimension m_ℓ and

$$V_i^+(u,x) = \bigoplus_{\ell=i}^{k} \mathcal{V}_\ell(v,u,x) \quad \text{and} \quad V_j^-(u,x) = \bigoplus_{\ell=1}^{j} \mathcal{V}_\ell(v,u,x)$$

In particular, for $\widehat{\beta\times\nu}$-a.e. (v,u,x),

$$V_j^+(u,x)/V_{j+1}^+(u,x) \simeq \mathcal{V}_j(v,u,x) \simeq V_j^-(v,x)/V_{j-1}^-(v,x)$$

Using this information, we show below that Theorems 1.4 and 1.5 follow from the corresponding result for the two-sided walk:

Theorem 1.6. *If $A(.,.)$ is strongly irreducible, then it has semisimple Lyapunov spectrum, in the sense that the restriction of $A^n(v,u,x)$ to each $\mathcal{V}_i(v,u,x)$ is block-conformal.*

Furthermore, the top Lyapunov exponent corresponds to a single conformal block, that is, for $\widehat{\beta\times\nu}$-a.e. (v,u,x) there are a (non-degenerate) quadratic form $\langle .,.\rangle_{v,u,x}$ and a cocycle $\lambda:\hat{\Omega}\times X\times\mathbb{N}\to\mathbb{R}$ such that

$$\langle A^n(v,u,x)\vec{p}, A^n(v,u,x)\vec{q}\rangle_{T^n(v,u,x)} = e^{\lambda(v,u,x,n)}\langle\vec{p},\vec{q}\rangle_{v,u,x} \tag{1.5}$$

for all $\vec{p},\vec{q}\in\mathcal{V}_1(v,u,x)$.

Remark 1.7. It is shown in [EMi, Appendix C] that if the algebraic hull $\mathbf{H}$ is the whole group $SL(L)$, then all Lyapunov exponents are associated to single conformal blocks, i.e., for $\widehat{\beta\times\nu}$-a.e. $(v,u,x)\in\Omega\times X$ and for each $1\le i\le k$, there are a (non-degenerate) quadratic form $\langle .,.\rangle_{i,v,u,x}$ and a cocycle $\lambda_i:\hat{\Omega}\times X\times\mathbb{N}\to\mathbb{R}$ such that

$$\langle A^n(v,u,x)\vec{p}, A^n(v,u,x)\vec{q}\rangle_{i,\hat{T}^n(v,u,x)} = e^{\lambda_i(v,u,x,n)}\langle\vec{p},\vec{q}\rangle_{i,v,u,x}$$

for all $\vec{p},\vec{q}\in\mathcal{V}_i(v,u,x)$. Furthermore, analogous statements hold for the forward and backward walks.

Proof of Theorems 1.4 and 1.5 assuming Theorem 1.6. Denote by $\langle .,.\rangle_{ij,v,u,x}$ the inner-products coming from the block-conformality property ensured by Theorem 1.6. We will show that for $(\beta\times\nu)$-almost every (u,x), resp. β^X-almost every (v,x), the conformal class of $\langle .,.\rangle_{ij,v,u,x}$ does not depend on v, resp. u (and this will suffice to obtain Theorems 1.4 and 1.5).

Given $\varepsilon>0$, we can select a compact subset $K\subset\hat{\Omega}\times X$ with $\widehat{\beta\times\nu}(K)>1-\varepsilon$ such that the functions $(v,u,x)\mapsto\langle .,.\rangle_{ij,v,u,x}$ are uniformly continuous on K. By ergodicity, if we take $0<\varepsilon<1/2$ and we consider the corresponding compact subset K just described, it follows that there exists $Y\subset\hat{\Omega}\times X$ with $\widehat{\beta\times\nu}(Y)=1$ such that, for any $(v,u,x)\in Y$, the elements of the orbit $(\hat{T}^n(v,u,x))_{n\in\mathbb{Z}}$ belong to K for a set of integers n with asymptotic[6] density $>1/2$.

Next, we define

$$[\vec{p},\vec{q}]_{ij,v,u,x} := \frac{\langle\vec{p},\vec{q}\rangle_{ij,v,u,x}}{\langle\vec{p},\vec{p}\rangle^{1/2}_{ij,v,u,x}\langle\vec{q},\vec{q}\rangle^{1/2}_{ij,v,u,x}}$$

Let $(v,u,x),(v',u,x)\in Y$, resp. $(v,u,x),(v,u',x)\in Y$. By Theorem 1.6,

$$\begin{aligned}&[\vec{p}_n,\vec{q}_n]_{ij,\hat{T}^n(v,u,x)} = [\vec{p},\vec{q}]_{ij,v,u,x}, \quad [\vec{p}_n,\vec{q}_n]_{ij,\hat{T}^n(v',u,x)} = [\vec{p},\vec{q}]_{ij,v',u,x} \ \ \forall n\ge 0,\\ &[\vec{p}_n,\vec{q}_n]_{ij,\hat{T}^n(v,u',x)} = [\vec{p},\vec{q}]_{ij,v,u',x} \ \ \forall n\le 0\end{aligned} \tag{1.6}$$

[6]By definition, $\mathcal{R}\subset\mathbb{Z}$ has asymptotic density $>\delta$ when $\liminf\limits_{m\to+\infty}\frac{1}{m}\#\{n\in\mathcal{R}:0\le n<m\}>\delta$ and $\liminf\limits_{m\to+\infty}\frac{1}{m}\#\{n\in\mathcal{R}:-m<n\le 0\}>\delta$.

where $\vec{p}_n := A^n(u,x)\vec{p}$ and $\vec{q}_n := A^n(u,x)\vec{q}$ for all $n \geq 0$, resp. $\vec{p}_n := A^{-n}(v,x)\vec{p}$ and $\vec{q}_n := A^{-n}(v,x)\vec{q}$ for all $n \leq 0$.

By construction of Y, we can select a subsequence $n_k \to +\infty$, resp. $n_k \to -\infty$ such that $\hat{T}^{n_k}(v,u,x), \hat{T}^{n_k}(v',u,x) \in K$, resp. $\hat{T}^{n_k}(v,u,x), \hat{T}^{n_k}(v',u,x) \in K$ $\forall k \in \mathbb{N}$.

Since the points $\hat{T}^n(v,u,x)$ and $\hat{T}^n(v',u,x)$, resp. $\hat{T}^n(v,u,x)$ and $\hat{T}^n(v,u',x)$, approach each other as $n \to +\infty$, resp. $n \to -\infty$, it follows from the definition of K (and our choice of $(n_k)_{k\in\mathbb{N}}$) that

$$[\vec{p}_{n_k}, \vec{q}_{n_k}]_{ij,\hat{T}^{n_k}(v,u,x)} - [\vec{p}_{n_k}, \vec{q}_{n_k}]_{ij,\hat{T}^{n_k}(v',u,x)} \to 0,$$

resp.

$$[\vec{p}_{n_k}, \vec{q}_{n_k}]_{ij,\hat{T}^{n_k}(v,u,x)} - [\vec{p}_{n_k}, \vec{q}_{n_k}]_{ij,\hat{T}^{n_k}(v,u,x)} \to 0$$

as $k \to \infty$.

By plugging this into (1.6), we see that

$$[\vec{p},\vec{q}]_{ij,v',u,x} = [\vec{p},\vec{q}]_{ij,v,u,x}, \quad \text{resp. } [\vec{p},\vec{q}]_{ij,v,u',x} = [\vec{p},\vec{q}]_{ij,v,u,x},$$

whenever $(v,u,x),(v',u,x) \in Y$, resp. $(v,u,x),(v,u',x) \in Y$.

In other terms,

$$\begin{aligned} \langle\vec{p},\vec{q}\rangle_{ij,v',u,x} &= c(v',v,u,x)\langle\vec{p},\vec{q}\rangle_{ij,v,u,x}, \quad \text{resp.} \\ \langle\vec{p},\vec{q}\rangle_{ij,v,u',x} &= c(v,u',u,x)\langle\vec{p},\vec{q}\rangle_{ij,v,u,x}, \end{aligned} \tag{1.7}$$

whenever $(v,u,x),(v',u,x) \in Y$, resp. $(v,u,x),(v,u',x) \in Y$.

On the other hand, for $\beta\times\nu$, resp. β^X almost every (u,x), resp. (v,x), we can Borel measurably select $v = v(u,x) \in \Omega^-$, resp. $u = u(v,x) \in \Omega$ such that $(v,u,x) \in Y$ (because of von Neumann selection theorem, see Theorem A.9 at page 196 of Zimmer's book [Zi]). By setting $\langle.,.\rangle_{u,x} := \langle.,.\rangle_{v(u,x),u,x}$, resp. $\langle.,.\rangle_{v,x} := \langle.,.\rangle_{v,u(v,x),x}$, we obtain that the conclusions of Theorems 1.4 and 1.5 are valid for these choices of inner-products thanks to Theorem 1.6 and the conformality relations (1.7). □

The remainder of this paper is devoted to the proof of Theorem 1.6.

2. ℝ-simple Lie groups

Let **H** be a ℝ-simple Lie group. We will always assume that **H** is a linear algebraic group with finite center. Let θ denote a Cartan involution of **H**, and let **K** denote the set of fixed points of θ. Then, **K** is a maximal compact subgroup of **H**.

Let **A** denote a maximal ℝ-split torus of **H** such that $\theta(\mathbf{A}) = \mathbf{A}$, and let Σ denote the associated root system. Let Σ^+ denote the set of positive roots, and let Δ denote the set of simple roots. Let **B** denote the Borel subgroup of **H** corresponding to Σ^+. Let W denote the Weyl group of $(\mathbf{H},\mathbf{A})$.

Let $\mathbf{A}_+$ be the positive Weyl chamber, i.e.,

$$\mathbf{A}_+ := \{a \in \mathbf{A} \;:\; \alpha(\log a) \geq 0 \text{ for all } \alpha \in \Sigma^+\}.$$

We have the decomposition

$$\mathbf{H} = \mathbf{K}\mathbf{A}_+\mathbf{K}. \tag{2.1}$$

If $g \in \mathbf{H}$ is written as $g = k_1 a k_2$ where $k_1, k_2 \in \mathbf{K}$ and $a \in \mathbf{A}_+$, we write for $\alpha \in \Sigma^+$

$$\alpha(g) = \alpha(\log a). \tag{2.2}$$

We also have the Bruhat decomposition

$$\mathbf{H} = \bigsqcup_{w \in W} \mathbf{B} w \mathbf{B}.$$

Let $w_0 \in W$ be the longest root. Then, $\mathbf{B}w_0\mathbf{B}$ is open and dense in $\mathbf{H}$. Let

(2.3) $J \subset \mathbf{H}/\mathbf{B}$ denote the complement of $\mathbf{B}w_0\mathbf{B}/\mathbf{B}$ in $\mathbf{H}/\mathbf{B}$.

Given a subset $I \subset \Delta$, let $\mathbf{P}_I$ denote the parabolic subgroup of $\mathbf{H}$ associated[7] to I. We have the Langlands decomposition

$$\mathbf{P}_I = \mathbf{M}_I \mathbf{A}_I \mathbf{N}_I,$$

where

$$\mathbf{A}_I = \{a \in \mathbf{A} \;\;:\;\; \alpha(\log a) = 0 \text{ for all } \alpha \in I\}.$$

The group $\mathbf{M}_I$ is semisimple, and commutes with $\mathbf{A}_I$. The group $\mathbf{N}_I$ is unipotent, and $\mathbf{N}_I \lhd \mathbf{P}_I$.

For later use, we denote $\bar{\mathbf{N}}_I = w_0 \mathbf{N}_I w_0^{-1}$ and let J_I be the complement of $(\mathbf{B}w_0\mathbf{P}_I)/\mathbf{P}_I$ in $\mathbf{H}/\mathbf{P}_I$.

We will use the rest of this section to deduce some *general* properties of the actions of elements of $\mathbf{H}$ on $\mathbf{H}/\mathbf{P}_I$. In particular, even though these properties help in the proof of Theorem 1.6, we decided to present them in their own section because they have nothing to do with the cocycle A but only with the group $\mathbf{H}$.

2.1. A lemma of Furstenberg.

Definition 2.1 (**(ϵ,δ)-regular**)**.** Suppose $\epsilon > 0$ and $\delta > 0$ are fixed. A measure η on $\mathbf{H}/\mathbf{B}$ is (ϵ,δ)-*regular* if for any $g \in \mathbf{H}$,

$$\eta(\mathrm{Nbhd}_\epsilon(gJ)) < \delta,$$

where J is as in (2.3). A measure η_I on $\mathbf{H}/\mathbf{P}_I$ is (ϵ,δ)-*regular* if for any $g \in \mathbf{H}$,

$$\eta_I(\mathrm{Nbhd}_\epsilon(gJ_I)) < \delta,$$

where J_I is the complement of $(\mathbf{B}w_0\mathbf{P}_I)/\mathbf{P}_I$ in $\mathbf{H}/\mathbf{P}_I$.

[7] I.e., $\mathbf{P}_I$ contains $\mathbf{A}$ and its root system $(\mathbf{P}_I, A)$ is $\Sigma^+ \cup \Sigma_I$ where $\Sigma_I \subset \Sigma$ consists of roots whose expansions relative to Δ have vanishing coefficients at elements of I.

Lemma 2.2 (Furstenberg[8]). *Suppose $I \subset \Delta$, $g_n \in \mathbf{H}$ is a sequence, and η_n is a sequence of uniformly (ϵ, δ)-regular measures on $\mathbf{H}/\mathbf{P}_I$. Suppose $\delta \ll 1$. Write*

$$g_n = k_n a_n k_n',$$

where $k_n \in \mathbf{K}$, $k_n' \in \mathbf{K}$ and $a_n \in \mathbf{A}_+$.

(a) *Suppose $I \subset \Delta$ is such that for all $\alpha \in \Delta \setminus I$,*

$$\alpha(a_n) \to \infty. \tag{2.4}$$

Then, for any subsequential limit λ of $g_n\eta_n$, we have

$$k_n\mathbf{P}_I \to k_\infty \mathbf{P}_I \quad \text{and} \quad \lambda(\{k_\infty \mathbf{P}_I\}) \geq 1 - \delta \tag{2.5}$$

for some element $k_\infty \in \mathbf{K}$.

(b) *Suppose $g_n\eta_n \to \lambda$ where λ is some measure on $\mathbf{H}/\mathbf{P}_I$. Suppose also that there exists an element k_∞ such that $\lambda(\{k_\infty\mathbf{P}_I\}) > 5\delta$. Then, as $n \to \infty$, (2.4) holds for all $\alpha \in \Delta \setminus I$. As a consequence, by part (a), (2.5) holds and $\lambda(\{k_\infty\mathbf{P}_I\}) \geq 1 - \delta$.*

Proof of (a). Without loss of generality, k_n' is the identity (or else we replace η_n by $k_n'\eta_n$).

Let $\bar{\mathbf{N}}_I = w_0\mathbf{N}_I w_0^{-1}$. By our assumption (2.4), for $\bar{n} \in \bar{\mathbf{N}}_I$,

$$a_n\bar{n}\mathbf{P}_I = (a_n\bar{n}a_n^{-1})\mathbf{P}_I \to \mathbf{P}_I \quad \text{ in } \mathbf{H}/\mathbf{P}_I.$$

For any $z \in \mathbf{H}/\mathbf{P}_I$ such that $z \notin J_I$, we may write $z = \bar{n}\mathbf{P}_I$ for some $\bar{n} \in \bar{\mathbf{N}}_I$. Therefore, $d(g_n z, k_n\mathbf{P}_I) \to 0$, where $d(\cdot, \cdot)$ denotes some distance on $\mathbf{H}/\mathbf{P}_I$. It then follows from the (ϵ, δ)-regularity of η_n that (2.5) holds, and any limit of $g_n\eta_n$ must give weight at least $1 - \delta$ to $k_\infty \mathbf{P}_I$ (where k_∞ is a subsequential limit of k_n).

Proof of (b). This is similar to [GM, Lemma 3.9]. There is a subsequence of the g_n (which we again denote by $g_n = k_n a_n k_n'$) such that for all $\gamma \in \Delta$, either $\gamma(a_n) \to \infty$ or $\gamma(a_n)$ is bounded. After passing again to a subsequence, we may assume that $k_n \to k_\infty$. Also, without loss of generality, we may assume that k_n' is the identity (or else we replace η_n by $k_n'\eta_n$).

Suppose there exists $\alpha \in \Delta \setminus I$ such that (2.4) fails. Let $I' \subset \Delta$ denote the set of $\gamma \in \Delta$ such that, for $\gamma \in \Delta \setminus I'$, $\gamma(a_n) \to \infty$. Since we are assuming that $\alpha \in \Delta \setminus I$ and $\alpha \notin \Delta \setminus I'$, we have $\Delta \setminus I \not\subset \Delta \setminus I'$, and thus $I' \not\subset I$.

Let $\bar{\mathbf{N}}_\alpha \subset \bar{\mathbf{N}}$ denote the subgroup obtained by exponentiating the root subspace $-\alpha$. We may write $\bar{\mathbf{N}}_I = \bar{\mathbf{N}}_\alpha\bar{\mathbf{N}}'$ for some subgroup $\bar{\mathbf{N}}'$ of $\bar{\mathbf{N}}$. Note that the action by left multiplication by g_n on $\mathbf{H}/\mathbf{P}_I$ does not shrink the direction $\bar{\mathbf{N}}_\alpha$.

Write[9] $k_\infty\mathbf{P}_I = \bar{n}_\alpha\bar{n}'\mathbf{P}_I$, where $\bar{n}_\alpha \in \bar{\mathbf{N}}_\alpha$, $\bar{n}' \in \bar{\mathbf{N}}'$. Then, for $z \in \mathbf{H}/\mathbf{P}_I$, $g_n z$ does not converge to $k_\infty \mathbf{P}_I$ unless $z \in \bar{n}_\alpha\bar{\mathbf{N}}'\mathbf{P}_I$ or $z \in J_I$. In particular, since

[8]Compare with [Fu, Theorems 8.3 and 8.4].

[9]The (ϵ, δ)-regularity of η_n and our assumption $\lambda(\{k_\infty\mathbf{P}_I\}) > 5\delta$ imply that $k_\infty\mathbf{P}_I \notin J_I$. Hence, it is possible to write $k_\infty\mathbf{P}_I$ as claimed.

$\bar{n}_\alpha \bar{\mathbf{N}}' \mathbf{P}_I \subset \bar{n}_\alpha J_I$ (because $w_0 \bar{\mathbf{N}}' w_0^{-1} \in \mathbf{B} w_\alpha w_0 \mathbf{B}$), we obtain that if $g_n z$ converges to $k_\infty \mathbf{P}_I$ then $z \in J_I \cup \bar{n}_\alpha J_I$.

On the other hand, since η_n is (ϵ, δ)-regular,

$$\eta_n(\mathrm{Nbhd}_\epsilon(J_I \cup \bar{n}_\alpha J_I)) < 2\delta.$$

Therefore $\lambda(k_\infty \mathbf{P}_I) < 3\delta$ which is a contradiction. Thus $\alpha(g_n) \to \infty$ for all $\alpha \in \Delta \setminus I$. Now, by part (a), (2.5) holds, and $\lambda(k_\infty \mathbf{P}_I) \geq 1 - \delta$. □

2.2. **The functions $\xi_\alpha(\cdot,\cdot)$ and $\hat{\sigma}_\alpha(\cdot,\cdot)$.** Let ω_α be the fundamental weight corresponding to α, i.e. for $\gamma \in \Delta$,

$$\langle \omega_\alpha, \gamma \rangle = \begin{cases} 1 & \text{if } \alpha = \gamma \\ 0 & \text{if } \gamma \in \Delta \setminus \{\alpha\}. \end{cases}$$

Then,

$$\alpha = \sum_{\gamma \in \Delta} \langle \alpha, \gamma \rangle \omega_\gamma. \tag{2.6}$$

We write

$$\omega_\alpha(g) = \omega_\alpha(\log a), \qquad \text{where } g = k_1 a k_2,\ k_1, k_2 \in \mathbf{K},\ a \in \mathbf{A}_+. \tag{2.7}$$

Note that for all $\alpha \in \Delta$ and all $g \in \mathbf{H}$,

$$\alpha(g) = \sum_{\gamma \in \Delta} \langle \alpha, \gamma \rangle \omega_\gamma(g). \tag{2.8}$$

Lemma 2.3. *For all $g_1 \in \mathbf{H}$, $g_2 \in \mathbf{H}$, and for all $\alpha \in \Delta$,*

$$\omega_\alpha(g_1 g_2) \leq \omega_\alpha(g_1) + \omega_\alpha(g_2). \tag{2.9}$$

and

$$\omega_\alpha(g_1 g_2) \geq \omega_\alpha(g_1) - \omega_\alpha(g_2^{-1}). \tag{2.10}$$

Proof. There exists a representation $\rho_\alpha : \mathbf{H} \to GL(V)$ such that its highest weight is ω_α (see [Kn, Chapter V]). Let $\|\cdot\|$ be any $\mathbf{K}$-invariant norm on V. Then, since ω_α is the highest weight,

$$\|\rho_\alpha(g)\| \equiv \sup_{v \in V \setminus \{0\}} \frac{\|\rho_\alpha(g) v\|}{\|v\|} = e^{\omega_\alpha(g)}.$$

Since $\|\rho_\alpha(g_1 g_2)\| \leq \|\rho_\alpha(g_1)\| \|\rho_\alpha(g_2)\|$, (2.9) follows.

Now write $g_1 = h_1 h_2$, $g_2 = h_2^{-1}$, so that $g_1 g_2 = h_1$. Substituting into (2.9), we get

$$\omega_\alpha(h_1) \leq \omega_\alpha(h_1 h_2) + \omega_\alpha(h_2^{-1})$$

which immediately implies (2.10). □

Let $\mathbf{P}_\alpha$ be the parabolic subgroup corresponding to the subset $\Delta \setminus \{\alpha\} \subset \Delta$. We can write

$$\mathbf{P}_\alpha = \mathbf{M}_\alpha \mathbf{A}_\alpha \mathbf{N}_\alpha,$$

where

$$\mathbf{A}_\alpha = \{a \in \mathbf{A} \ : \ \gamma(\log a) = 0 \text{ for all } \gamma \in \Delta \setminus \{\alpha\}\}.$$

Note that $\mathbf{A}_\alpha$ is one dimensional, and that $\mathbf{M}_\alpha$ commutes with $\mathbf{A}_\alpha$. We have the Iwasawa decomposition

$$\mathbf{H} = \mathbf{K}\mathbf{P}_\alpha = \mathbf{K}\mathbf{M}_\alpha \mathbf{A}_\alpha \mathbf{N}_\alpha.$$

Let $\mathbf{P}^0_\alpha = \mathbf{M}_\alpha \mathbf{N}_\alpha$. If we decompose $g \in \mathbf{H}$ as $g = k_\alpha m_\alpha a_\alpha n_\alpha$ with $k_\alpha \in \mathbf{K}$, $m_\alpha \in \mathbf{M}_\alpha$, $a_\alpha \in \mathbf{A}_\alpha$ and $n_\alpha \in \mathbf{N}_\alpha$, then the decomposition is unique up to the transformation $k_\alpha \to k_\alpha m_1$, $m_\alpha \to m_1^{-1} m_\alpha$ for $m_1 \in \mathbf{K} \cap \mathbf{M}_\alpha$. We can thus define the function $\xi_\alpha : \mathbf{H}/\mathbf{P}^0_\alpha \to \mathbb{R}$ by

$$\xi_\alpha(g) = \omega_\alpha(\log a), \qquad \text{where } g = kman,\ k \in \mathbf{K},\ m \in \mathbf{M}_\alpha,\ a \in \mathbf{A}_\alpha \text{ and } n \in \mathbf{N}_\alpha.$$

By definition, we have for $a \in \mathbf{A}_\alpha$,

$$\xi_\alpha(ga) = \xi_\alpha(g) + \xi_\alpha(a). \tag{2.11}$$

We now define for $g \in \mathbf{H}$, $z \in \mathbf{H}/\mathbf{P}^0_\alpha$,

$$\xi_\alpha(g, z) = \xi_\alpha(gz) - \xi_\alpha(z).$$

Then, in view of (2.11), for $a \in \mathbf{A}_\alpha$, $\xi_\alpha(g, za) = \xi_\alpha(g, z)$. Thus, we may consider $\xi_\alpha(\cdot, \cdot)$ to be a function from $\mathbf{H} \times (\mathbf{H}/\mathbf{P}_\alpha)$ to $\mathbb{R}$.

Lemma 2.4. *We have for all $\alpha \in \Delta$:*

(a) *For all $g_1, g_2 \in \mathbf{H}$,*

$$\xi_\alpha(g_1 g_2, z) = \xi_\alpha(g_1, g_2 z) + \xi_\alpha(g_2, z).$$

(b) *For all $g \in \mathbf{H}$ and all $z \in \mathbf{H}/\mathbf{P}_\alpha$,*

$$\xi_\alpha(g, z) \le \omega_\alpha(g),$$

where $\omega_\alpha(g)$ is as defined in (2.7).

(c) *For all $\epsilon > 0$ there exists $C = C(\epsilon) > 0$ such that for all $k_2 \in \mathbf{K}$, for all $g \in \mathbf{K}\mathbf{A}_+ k_2$ and all $z \in \mathbf{H}/\mathbf{P}_\alpha$ with $d(k_2 z, J_\alpha) > \epsilon$,*

$$\xi_\alpha(g, z) \ge \omega_\alpha(g) - C.$$

Proof. Part (a) is clear from the definition of $\xi_\alpha(\cdot, \cdot)$. To show part (b), note that there exists a representation $\rho_\alpha : \mathbf{H} \to GL(V)$ with highest weight ω_α. Let $\|\cdot\|$ be any $\mathbf{K}$-invariant norm on V. Let v_α be the highest weight vector. Then $\mathbf{P}^0_\alpha$ is the stabilizer of v_α, and for all $g \in \mathbf{H}$,

$$\xi_\alpha(g) = \log \frac{\|\rho_\alpha(g) v_\alpha\|}{\|v_\alpha\|}.$$

As in the proof of Lemma 2.3,

$$\sup_{v\in V\setminus\{0\}} \log \frac{\|\rho_\alpha(g)v\|}{\|v\|} = \omega_\alpha(g).$$

Then, part (b) of Lemma 2.4 follows.

To show part (c), write $g = k_1ak_2$, $k_1, k_2 \in \mathbf{K}$, $a \in \mathbf{A}_+$. Note that if $d(k_2z, J_\alpha) = d(k_2z, (\bar{\mathbf{N}}_\alpha\mathbf{P}_\alpha)^c) > \epsilon$, then we can write

$$k_2z = \bar{n}_\alpha\mathbf{P}_\alpha,$$

with $d(\bar{n}_\alpha, e) \leq C_1(\epsilon)$. Then, $|\omega_\alpha(\bar{n}_\alpha)| < C(\epsilon)$. We have

$$\begin{aligned}
\xi_\alpha(g,z) &= \xi_\alpha(k_1ak_2, z) & \\
&= \xi_\alpha(k_1a, k_2z) & &\text{by (a) and since } \xi_\alpha(k_2,z)=0\\
&= \xi_\alpha(k_1a, \bar{n}_\alpha\mathbf{P}_\alpha) & \\
&= \xi_\alpha(a, \bar{n}_\alpha\mathbf{P}_\alpha) & &\text{by (a) and since } \xi_\alpha(k_1,\cdot)=0\\
&= \xi_\alpha(a\bar{n}_\alpha, \mathbf{P}_\alpha) - \xi_\alpha(\bar{n}_\alpha, \mathbf{P}_\alpha) & &\text{by (a)}\\
&\geq \xi_\alpha(a\bar{n}_\alpha, \mathbf{P}_\alpha) - C(\epsilon) & &\text{by (b) and since } |\omega_\alpha(\bar{n}_\alpha)| < C(\epsilon)\\
&= \xi_\alpha(a\bar{n}_\alpha a^{-1}, \mathbf{P}_\alpha) + \xi_\alpha(a, \mathbf{P}_\alpha) - C(\epsilon) & &\text{by (a)}\\
&= \xi_\alpha(a\bar{n}_\alpha a^{-1}, \mathbf{P}_\alpha) + \omega_\alpha(a) - C(\epsilon) & &\text{since } \xi_\alpha(a,\mathbf{P}_\alpha) = \omega_\alpha(a)\\
&\geq \omega_\alpha(a) - 2C(\epsilon) & &\text{by (b) and since } |\omega_\alpha(a\bar{n}_\alpha a^{-1})| \leq C(\epsilon).
\end{aligned}$$

□

For $\alpha \in \Delta$, $g \in \mathbf{H}$, let $\mathbf{B} = \mathbf{MAN}$ be Langlands' decomposition of $\mathbf{B}$ and

$$\hat{\sigma}_\alpha(g) = \alpha(a), \qquad \text{where } g = kman,\ k \in \mathbf{K},\ m \in \mathbf{M},\ a \in \mathbf{A} \text{ and } n \in \mathbf{N}.$$

Note that $\hat{\sigma}_\alpha$ descends to a well-defined function on $\mathbf{H}/(\mathbf{MN})$.

By definition, we have for $a \in \mathbf{A}$,

$$\hat{\sigma}_\alpha(ga) = \hat{\sigma}_\alpha(g) + \hat{\sigma}_\alpha(a). \tag{2.12}$$

We now define for $g \in \mathbf{H}$, $z \in \mathbf{H}/(\mathbf{MN})$,

$$\hat{\sigma}_\alpha(g,z) = \hat{\sigma}_\alpha(gz) - \hat{\sigma}_\alpha(z).$$

Then, in view of (2.12), for $a \in \mathbf{A}$, $\hat{\sigma}_\alpha(g, za) = \hat{\sigma}_\alpha(g,z)$. Thus, we may consider $\hat{\sigma}_\alpha(\cdot,\cdot)$ to be a function $\mathbf{H} \times \mathbf{H}/\mathbf{B} \to \mathbb{R}$.

Lemma 2.5. *We have for all $\alpha \in \Delta$:*

(a) *For all $g_1, g_2 \in \mathbf{H}$ and $z \in \mathbf{H}/\mathbf{B}$,*

$$\hat{\sigma}_\alpha(g_1g_2, z) = \hat{\sigma}_\alpha(g_1, g_2z) + \hat{\sigma}_\alpha(g_2, z).$$

(b) *For all* $\epsilon > 0$ *there exists* $C = C(\epsilon) > 0$ *such that for all* $k_2 \in \mathbf{K}$*, for all* $g \in \mathbf{KA}_+ k_2$ *and all* $z \in \mathbf{H}/\mathbf{B}$ *with* $d(k_2 z, J_\alpha) > \epsilon$*,*

$$\hat{\sigma}_\alpha(g, z) \geq \alpha(g) - C,$$

where $\alpha(g)$ *is as defined in* (2.2)*.*

Proof. The natural map $\mathbf{H}/\mathbf{P}_\alpha \to \mathbf{H}/\mathbf{B}$ allows us to consider the functions $\xi_\alpha(\cdot, \cdot)$ to be functions $\mathbf{H} \times \mathbf{H}/\mathbf{B} \to \mathbb{R}$. Then, in view of (2.6), we have

$$\hat{\sigma}_\alpha(g, z) = \sum_{\gamma \in \Delta} \langle \alpha, \gamma \rangle \xi_\gamma(g, z).$$

Then (a) immediately follows from (a) of Lemma 2.4. Also,

$$\begin{aligned}
\hat{\sigma}_\alpha(g, z) &= \langle \alpha, \alpha \rangle \xi_\alpha(g, z) + \sum_{\gamma \neq \alpha} \langle \alpha, \gamma \rangle \xi_\gamma(g, z) \\
&\geq \langle \alpha, \alpha \rangle \xi_\alpha(g, z) + \sum_{\gamma \neq \alpha} \langle \alpha, \gamma \rangle \omega_\gamma(g) && \text{by Lemma 2.4(b) and since } \langle \alpha, \gamma \rangle \leq 0 \\
&\geq \langle \alpha, \alpha \rangle \omega_\alpha(g) - C(\epsilon) + \sum_{\gamma \neq \alpha} \langle \alpha, \gamma \rangle \omega_\gamma(g) && \text{by Lemma 2.4(c)} \\
&= \alpha(g) - C(\epsilon).
\end{aligned}$$

This completes the proof of (b). □

3. Cocycles with values in $\mathbb{R}$-simple Lie groups

Let $A : G \times X \to SL(L)$ be a linear cocycle satisfying the properties (A1) to (A6) described in §1 above. In particular, we will assume that $A(.,.)$ takes values in its algebraic hull $\mathbf{H}$. Furthermore, we will suppose that $\mathbf{H}$ is a $\mathbb{R}$-simple Lie group with finite center.

For $\alpha \in \Delta$, let

$$\lambda_\alpha \equiv \limsup_{n \to +\infty} \frac{1}{n} \alpha(A^n(u, x)) \tag{3.1}$$

By (2.8) and Lemma 2.3, the map

$$(u, x, n) \to \lambda_\alpha(A^n(u, x))$$

is a linear combination of subadditive cocycles.

Therefore, by the subadditive ergodic theorem, the limsup is actually a limit. Also, by the ergodicity of T, λ_α is constant a.e. on $\Omega \times X$.

From now on, let us fix $I \subset \Delta$ minimal such that for $(\beta \times \nu)$-a.e. $(u, x) \in \Omega \times X$, we have

$$I = \{\alpha \in \Delta \ : \ \lambda_\alpha = 0\}. \tag{3.2}$$

Thus, for all $\alpha \in \Delta \setminus I$, $\lambda_\alpha > 0$.

We will deduce Theorem 1.6 from the following:

Theorem 3.1. *Let $I \subset \Delta$ be as in (3.2). Then, for almost all $((v,u),x) \in \hat{\Omega} \times X$ there exists $C(v,u,x) \in \mathbf{H}$ such that*

$$C(\hat{T}^n(v,u,x))^{-1} A^n(v,u,x) C(v,u,x) = k_n(v,u,x) a_n(u,v,x), \tag{3.3}$$

where $k_n(v,u,x) \in \mathbf{K} \cap \mathbf{M}_I$ and $a_n(v,u,x) \in \mathbf{A}_I$, and for all $\alpha \in \Delta \setminus I$,

$$\lim_{|n|\to\infty} \frac{1}{n} \alpha(\log a_n(v,u,x)) = \lambda_\alpha > 0 \tag{3.4}$$

where λ_α is as in (3.1).

Let $w_0 \in W$ be the longest root. Let $I' \subset \Delta$ be defined by:

$$I' = \{-w_0 \alpha w_0^{-1} \ : \ \alpha \in I\}. \tag{3.5}$$

Theorem 3.1 will be deduced from the following results.

Proposition 3.2. *Let $I' \subset \Delta$ be as in (3.5). Then,*

(a) *For almost all $(u,x) \in \Omega \times X$ there exists $C^+(u,x) \in \mathbf{H}$ such that for all n and almost all (u,x),*

$$C^+(T^n(u,x))^{-1} A^n(u,x) C^+(u,x) \in \mathbf{P}_{I'}.$$

(b) *For almost all $(v,x) \in \Omega^- \times X$ there exists $C^-(v,x) \in \mathbf{H}$ such that for all n and almost all (v,x),*

$$C^-(T^{-n}(v,x))^{-1} A^{-n}(v,x) C^-(v,x) \in \mathbf{P}_I.$$

(c) *For almost all $(v,u,x) \in \hat{\Omega} \times X$,*

$$C^+(u,x)^{-1} C^-(v,x) \in \mathbf{P}_{I'}\, w_0\, \mathbf{P}_I. \tag{3.6}$$

We note that Proposition 3.2 is similar in spirit to the geometrical versions of Osceledets multiplicative ergodic theorem in the literature (see the survey [Fi]). The standard proofs (see e.g. [GM]) are based on the subadditive ergodic theorem. We give a proof in §4 below based on the martingale convergence theorem. Parts of this proof will be used again in the proof of Theorem 3.1 in §5.

4. Proof of Proposition 3.2

4.1. **A Zero One Law.** Let ν be an ergodic stationary measure on X. Let $\hat{X} = X \times \mathbf{H}/\mathbf{B}$. We then have an action of G on $\hat{X}$, by

$$g \cdot (x,z) = (gx, A(g,x)z).$$

Let $\hat{\nu}$ be an ergodic μ-stationary measure on $\hat{X}$ which projects to ν under the natural map $\hat{X} \to X$. Note there is always at least one such: one chooses $\hat{\nu}$ to be an extreme point among the μ-stationary measures which project to ν. If $\hat{\nu} = \hat{\nu}_1 + \hat{\nu}_2$ where the $\hat{\nu}_i$ are μ-stationary measures then $\nu = \pi_*(\hat{\nu}) = \pi_*(\hat{\nu}_1) + \pi_*(\hat{\nu}_2)$. Since ν is μ-ergodic, this implies that $\pi_*(\hat{\nu}_1) = \nu$ or $\pi_*(\hat{\nu}_2) = \nu$, hence the $\hat{\nu}_1$ or $\hat{\nu}_2$ also project to ν. Since

$\hat{\nu}$ is an extreme point among such measures, we must have $\hat{\nu}_1 = \nu$ or $\hat{\nu}_2 = \hat{\nu}$. This $\hat{\nu}$ is μ-ergodic.

We may write

$$d\hat{\nu}(x,z) = d\nu(x)\, d\eta_x(z),$$

where η_x is a measure on $\mathbf{H}/\mathbf{B}$.

Lemma 4.1 (cf. [GM, Lemma 4.2], cf. [GR1, Théorème 2.6], cf. [EMi, Lemma C.10])**.** *For almost all $x \in X$ and any $g \in \mathbf{H}$,*

$$\eta_x(gJ) = 0,$$

where J is defined in (2.3).

Proof. Let d be the smallest number such that there exists a subset $E \subset X$ with $\nu(E) > 0$ and for all $x \in E$ an irreducible algebraic subvariety $J_x \subset \mathbf{H}/\mathbf{B}$ of dimension d with $\eta_x(J_x) > 0$. For $x \in E$, let $\mathcal{S}(x)$ denote the set of irreducible algebraic subvarieties of $\mathbf{H}/\mathbf{B}$ of dimension d such that for $Q \in \mathcal{S}(x)$, $\eta_x(Q) > 0$.

Note that for a.e. $x \in X$, for any $Q_1 \in \mathcal{S}(x)$, $Q_2 \in \mathcal{S}(x)$ with $Q_2 \neq Q_1$,

$$\eta_x(Q_1 \cap Q_2) = 0.$$

(since $Q_1 \cap Q_2$ is an algebraic subvariety of dimension lower than d). Thus

$$\sum_{Q \in \mathcal{S}(x)} \eta_x(Q) \leq 1.$$

Therefore $\mathcal{S}(x)$ is at most countable. Moreover, by setting

$$f(x) = \max_{Q \in \mathcal{S}(x)} \eta_x(Q) \tag{4.1}$$

and $\mathcal{S}_{\max}(x) := \{Q \in \mathcal{S}(x) : \eta_x(Q) = f(x)\}$, we see that $\mathcal{S}_{\max}(x)$ is finite.

Consider the measurable subset $\mathcal{S}_{\max} = \{(x,z) : x \in E, z \in \mathcal{S}_{\max}(x)\} \subset \bigcup_{x \in E} (\{x\} \times \mathcal{S}(x))$. By definition, for each $x \in X$, the fiber $\{z \in \mathbf{H}/\mathbf{B} : (x,z) \in \mathcal{S}_{\max}\} := \mathcal{S}_{\max}(x)$ of $\mathcal{S}_{\max}$ at x is a finite set, and, in particular, $\mathcal{S}_{\max}(x)$ is a countable union of compact sets. By a result of Kallman (see, e.g., the statement of Theorem A.5 in Appendix A of Zimmer's book [Zi]), we can find a Borel measurable section for the restriction to $\mathcal{S}_{\max}$ of the natural projection $\pi : \hat{X} \to X$. In other words, one has a Borel measurable map $X \ni x \mapsto Q_x^{(1)} \in \mathcal{S}_{\max}(x)$ whose graph $E_1 := \{(x, Q_x^{(1)}) \in \hat{X} : x \in X\}$ is a measurable subset of $\mathcal{S}_{\max}$. If $E_1 = \mathcal{S}_{\max}$, we get that $\mathcal{S}_{\max}$ is the graph of a section of π. Otherwise, we apply once more Kallman's theorem to $\mathcal{S}_{\max} - E_1$ in order to obtain a measurable subset E_2 of $\mathcal{S}_{\max} - E_1$ given by the graph of a Borel measurable map $\pi(E_2) := \{y \in X : \#\mathcal{S}_{\max}(y) \geq 2\} \ni x \mapsto Q^{(2)}(x) \in \mathcal{S}_{\max}(x)$ that we extend (in a measurable way) to X by setting $Q^{(2)}(x) := Q^{(1)}(x)$ whenever $\#\mathcal{S}_{\max}(x) = 1$. Since the fibers $\mathcal{S}_{max}(x)$ of $\mathcal{S}_{\max}$ are finite sets, by iterating this procedure at most countably many times, we obtain a non-empty subset $Z \subset \mathbb{N}$ and, for each $m \in Z$, a Borel measurable map

$$X \ni x \mapsto Q^{(m)}(x) \in \mathcal{S}_{\max}(x)$$

such that $\mathcal{S}_{\max}(x) = \{Q^{(1)}(x), \dots, Q^{(\#\mathcal{S}_{\max}(x))}(x)\}$ for almost every $x \in X$.

Fix $m \in Z$. Since $\hat{\nu}$ is μ-stationary, we have $\mu * \hat{\nu}(\mathrm{graph}(Q^{(m)})) = \hat{\nu}(\mathrm{graph}(Q^{(m)}))$, that is,

$$\begin{aligned} \int_X \eta_x(Q^{(m)}(x))d\nu(x) &= \hat{\nu}(\mathrm{graph}(Q^{(m)})) = \mu * \hat{\nu}(\mathrm{graph}(Q^{(m)})) \\ &= \int_G \int_{\hat{X}} \chi_{\mathrm{graph}(Q^{(m)})}(gx, A(g,x)z)d\eta_x(z)d\nu(x)d\mu(g) \\ &= \int_G \int_X \eta_x(A(g,x)^{-1}Q^{(m)}(gx))d\nu(x)d\mu(g) \\ &= \int_X \left(\int_G \eta_x(A(g,x)^{-1}Q^{(m)}(gx))d\mu(g)\right)d\nu(x). \end{aligned} \tag{4.2}$$

On the other hand, since $\eta_x(Q^{(m)}(x)) = f(x) = \max_{Q\in\mathcal{S}(x)} \eta_x(Q)$, we see that

$$\int_G \eta_x(A(g,x)^{-1}Q^{(m)}(gx))d\mu(g) \leq f(x) = \eta_x(Q^{(m)}(x)) \tag{4.3}$$

By combining (4.2) and (4.3), we deduce that

$$f(x) = \eta_x(Q^{(m)}(x)) = \eta_x(A(g,x)^{-1}Q^{(m)}(gx)),$$

i.e., $A(g,x)^{-1}Q^{(m)}(gx) \in \mathcal{S}_{\max}(x)$ for μ-almost every g and ν-almost every x. In other terms, for all $m \in Z$, μ-almost every g and ν-almost every x, one has $Q^{(m)}(gx) \in A(g,x)\mathcal{S}_{\max}(x)$. By putting together this inclusion with the facts that $\mathcal{S}_{\max}(y) = \{Q^{(1)}(y), \dots, Q^{(\#\mathcal{S}_{\max}(y))}(y)\}$ for ν-almost every y and ν is μ-stationary, one has that $\mathcal{S}_{\max}(gx) \subset A(g,x)\mathcal{S}_{\max}(x)$ for μ-almost every g and ν-almost x.

Now, let $n_0 \in Z$ the smallest integer in Z such that $\{x \in X : \#\mathcal{S}_{\max}(x) \leq n_0\}$ has positive ν-measure. Because $\mathcal{S}_{\max}(gx) \subset A(g,x)\mathcal{S}_{\max}(x)$ (for $\mu \times \nu$-almost every (g,x)), the set $\{x \in X : \mathcal{S}_{\max}(x) \leq n_0\}$ is essentially invariant. Thus, from the μ-ergodicity of ν and our choice of n_0, we conclude that $\{x \in X : \#\mathcal{S}_{\max}(x) = n_0\}$ has full ν-measure. Hence, from the μ-stationarity of ν, we obtain that $\#\mathcal{S}_{\max}(gx) = \#\mathcal{S}_{\max}(x) = n_0$ for $(\mu\times\nu)$-almost every (g,x). In particular, the inclusion $\mathcal{S}_{\max}(gx) \subset A(g,x)\mathcal{S}_{\max}(x)$ is actually an equality

$$\mathcal{S}_{\max}(gx) = A(g,x)\mathcal{S}_{\max}(x) \tag{4.4}$$

for $(\mu\times\nu)$-almost every (g,x), that is, the cocycle A permutes the finite sets $\mathcal{S}_{\max}(x)$.

Denote by $\mathcal{M}$ the space of algebraic subvarieties of $\mathbf{H}/\mathbf{B}$ of dimension d, so that, by definition, $\mathcal{S}_{\max}(x)$ are finite subsets of $\mathcal{M}$. Note that the algebraic group $\mathbf{H}$ acts algebraically on $\mathcal{M}$. By the Borel–Serre theorem (see, e.g., Theorem 3.1.3 of Zimmer's book [Zi]), the action of $\mathbf{H}$ on $\mathcal{M}$ has locally closed orbits, and, hence,

$\mathcal{M}/\mathbf{H}$ is Hausdorff. Consider the function $f : \Omega \times X \to \mathcal{M}/\mathbf{H}$, $f(u,x) := \mathbf{H}\mathcal{S}_{\max}(x)$. By (4.4), the function $f(u,x)$ is T-invariant. By the T-ergodicity of $\beta \times \nu$, it follows that f is almost everywhere constant, so that there is $\mathcal{S}_0 \in \mathcal{M}$ such that, for ν-a.e. x, one has $\mathcal{S}_{\max}(x) = h(x)\mathcal{S}_0$ with $h(x) \in \mathbf{H}$.

In particular, it follows that the conjugated cocycle $A'(g,x) = h(gx)^{-1}A(g,x)h(x)$ stabilizes $\mathcal{S}_0$, i.e., $A'(g,x)$ takes its values in the stabilizer $\mathrm{Stab}_{\mathbf{H}}(\mathcal{S}_0)$ of $\mathcal{S}_0$ in $\mathbf{H}$. Hence, the algebraic hull of $A(.,.)$ must be a subgroup of $\mathrm{Stab}_{\mathbf{H}}(\mathcal{S}_0)$. This is a contradiction with our assumption that $\mathbf{H}$ is the algebraic hull of $A(.,.)$ because $\mathrm{Stab}_{\mathbf{H}}(\mathcal{S}_0)$ is a proper subgroup of $\mathbf{H}$ (since $\mathbf{H}$ acts transitively on $\mathbf{H}/\mathbf{B}$). □

4.2. **Another lemma of Furstenberg.** Let $\hat{X} = X \times \mathbf{H}/\mathbf{B}$. The group G acts on the space $\hat{X}$ is by

$$g \cdot (x,z) = (gx, A(g,x)z). \tag{4.5}$$

We choose some ergodic μ-stationary measure $\hat{\nu}$ on $\hat{X}$, which projects to ν, and write

$$d\hat{\nu}(x,z) = d\nu(x)\, d\eta_x(z).$$

Note that Lemma 4.1 applies to the measures η_x on $\mathbf{H}/\mathbf{B}$.

Lemma 4.2 (Furstenberg[10])**.** *For $\alpha \in \Delta$, let $\bar{\sigma}_\alpha : G \times \hat{X} \times (\mathbf{H}/\mathbf{B}) \to \mathbb{R}$ be given by*

$$\bar{\sigma}_\alpha(g,x,z) = \hat{\sigma}_\alpha(A(g,x)z)$$

with $\hat{\sigma}_\alpha(.,.)$ as in §2.2. Then, we have

$$\lambda_\alpha = \int_G \int_{\hat{X}} \bar{\sigma}_\alpha(g,x,z)\, d\hat{\nu}(x,z)\, d\mu(g).$$

where λ_α is as in (3.1).

Proof. This is similar to the proof of [GM, Lemma 5.2]. Note that

$$\xi_\alpha(g,z) = \sum_{\gamma \in \Delta} \langle \omega_\alpha, \omega_\gamma \rangle \hat{\sigma}_\gamma(g,z)$$

where $\xi_\alpha(.,.)$ and $\hat{\sigma}_\alpha(.,.)$ as in §2.2. Therefore, it is enough to show that for all $\alpha \in \Delta$,

$$\int_G \int_{\hat{X}} \xi_\alpha(A(g,x)z)\, d\hat{\nu}(x,z)\, d\mu(g) = \sum_{\gamma \in \Delta} \langle \omega_\alpha, \omega_\gamma \rangle \lambda_\gamma. \tag{4.6}$$

By (3.1), the boundedness assumption on $A(\cdot,\cdot)$ and the dominated convergence theorem, we have

$$\lambda_\alpha = \lim_{n\to\infty} \frac{1}{n} \int_{\Omega \times X} \alpha(A^n(g,x))\, d\beta(g) d\nu(x).$$

[10] Compare with [Fu, Theorem 8.5].

Thus,

$$\sum_{\gamma\in\Delta}\langle\omega_\alpha,\omega_\gamma\rangle\lambda_\gamma=\lim_{n\to\infty}\frac{1}{n}\int_{\Omega\times X}\omega_\alpha(A^n(g,x))\,d\beta(g)d\nu(x).$$

Write $A^n(g,x)=\bar{k}_n(g,x)\bar{a}_n(g,x)\bar{k}'_n(g,x)$, where $\bar{k}_n(g,x),\bar{k}'_n(g,x)\in\mathbf{K}$, $\bar{a}_n(g,x)\in\mathbf{A}_+$, and fix $\epsilon>0$. Then, by Lemma 2.4, for all $z\in\mathbf{H}/\mathbf{P}_\alpha$ with $d(\bar{k}'_n(g,x)z,(\bar{\mathbf{N}}_\alpha\mathbf{P}_\alpha)^c)>\epsilon$, we have

$$\omega_\alpha(A^n(g,x))\geq\xi_\alpha(A^n(g,x),z)\geq\omega_\alpha(A^n(g,x))-C(\epsilon).$$

Hence, by Lemma 4.1,

$$\sum_{\gamma\in\Delta}\langle\omega_\alpha,\omega_\gamma\rangle\lambda_\gamma=\lim_{n\to\infty}\frac{1}{n}\int_{\Omega\times\hat{X}}\xi_\alpha(A^n(g,x),z)\,d\beta(g)\,d\hat{\nu}(x,z).$$

By Lemma 2.4 (a) and by iterating the cocycle relation for $A(.,.)$,

$$\xi_\alpha(A^n(g,x),z)=\sum_{k=1}^{n}\xi_\alpha(A(g_k,g_{k-1}\dots g_1x),A(g_{k-1}\dots g_1,x)z),$$

Since $\hat{\nu}$ is stationary, each of the terms in the sum has the same integral over $\Omega\times\hat{X}$ (with respect to $\beta\times\hat{\nu}$). Therefore

$$\frac{1}{n}\int_{\Omega\times\hat{X}}\xi_\alpha(A^n(g,x),z)\,d\beta(g)\,d\hat{\nu}(x,z)=\int_G\int_{\hat{X}}\xi_\alpha(A(g,x),z)\,d\hat{\nu}(x,z)\,d\mu(g),$$

which completes the proof of (4.6). □

4.3. **Proof of Proposition 3.2(a).** For $u\in\Omega$, let the measures ν_u, $\hat{\nu}_u$ be essentially[11] as defined in [BQ, Lemma 3.2], i.e.

$$\nu_u=\lim_{n\to\infty}(u_n\dots u_1)_*^{-1}\nu \tag{4.7}$$

$$\hat{\nu}_u=\lim_{n\to\infty}(u_n\dots u_1)_*^{-1}\hat{\nu}. \tag{4.8}$$

The limits exist for β-a.e. $u\in\Omega$ by the martingale convergence theorem. We disintegrate

$$d\hat{\nu}(x,z)=d\nu(x)\,d\eta_x(z),\qquad d\hat{\nu}_u(x,z)=d\nu_u(x)\,d\eta_{u,x}(z).$$

We want to exploit (4.5), (4.7) and (4.8) to deduce that the sequence of conditional measures $A((u_n\dots u_1)^{-1},u_n\dots u_1x)\eta_{u_n\dots u_1x}$ converges to the conditional measures $\eta_{u,x}$. For this sake, we shall use the invariance of ν under $\mathrm{supp}(\mu)$. More

[11] It is shown in [BQ, Lemma 3.2] that the convergence of $(u_1\dots u_n)_*\nu$ and $(u_1\dots u_n)_*\hat{\nu}$ for β-almost every $u=(u_1,u_2,\dots)\in\Omega$. In our setting, this implies the convergence of $(u_n\dots u_1)_*^{-1}\nu$ and $(u_n\dots u_1)_*^{-1}\hat{\nu}$ for β-almost $u\in\Omega$ because $(u_n\dots u_1)^{-1}=u_1^{-1}\dots u_n^{-1}$, $\beta=\mu^{\mathbb{N}}$, and μ is symmetric.

concretely, let $C \subset X$ and $D \subset \mathbf{H}/\mathbf{B}$ be measurable subsets, and denote by χ_A the characteristic function of A. Since $\hat{\nu}$ is μ-stationary, we have from (4.5) that

$$\int_C \eta_x(D)d\nu(x) = \hat{\nu}(C \times D) = (\mu * \hat{\nu})(C \times D) = \int \chi_C(gy)A(g,y)_*\eta_y(D)d\nu(y)d\mu(g)$$

By using the invariance of ν to rewrite the right-hand side of this equality, we get

$$\begin{aligned}\int_C \eta_x(D)d\nu(x) &= \int \chi_C(x)A(g,g^{-1}x)_*\eta_{g^{-1}x}(D)d\nu(x)d\mu(g)\\ &= \int_C \left(\int_G A(g,g^{-1}x)_*\eta_{g^{-1}x}(D)d\mu(g)\right)d\nu(x)\end{aligned}$$

Because C and D are arbitrary, we deduce that

$$\eta_x = \int_G A(g,g^{-1}x)_*\eta_{g^{-1}x}d\mu(g)$$

From this identity, the cocycle relation and the symmetry of μ, we conclude that

$$A((g_{n-1}\dots g_1)^{-1}, g_{n-1}\dots g_1 x)_*\eta_{g_{n-1}\dots g_1 x} = \int_G A((g_n\dots g_1)^{-1}, g_n\dots g_1 x)_*\eta_{g_n\dots g_1 x}d\mu(g_n),$$

so that $A((g_n\dots g_1)^{-1}, g_n\dots g_1 x)_*\eta_{g_n\dots g_1 x}$ is a martingale. Thus, the martingale convergence theorem and the uniqueness of Rokhlin disintegration imply that

$$\lim_{n\to\infty} A((u_n\dots u_1)^{-1}, u_n\dots u_1 x)\eta_{u_n\dots u_1 x} = \eta_{u,x}.$$

for (u,x) in a set of $\beta \times \nu$ full measure.

Note that, by the cocycle relation $A(g^{-1},gx) = A(g,x)^{-1}$, one has

$$A((u_n\dots u_1)^{-1}, u_n\dots u_1 x) = A(u_n\dots u_1, x)^{-1}.$$

Hence, on a set of $\beta \times \nu$-full measure,

$$\lim_{n\to\infty} A(u_n\dots u_1,x)^{-1}\eta_{u_n\dots u_1 x} = \eta_{u,x}. \tag{4.9}$$

In view of Lemma 4.1 (see also the proof of [EMi, Lemma 14.4]), given $\delta > 0$, there exists a compact $\mathcal{K}_\delta \subset X$ with $\nu(\mathcal{K}_\delta) > 1-\delta$ and $\epsilon = \epsilon(\delta) > 0$ with $\epsilon(\delta) \to 0$ as $\delta \to 0$ such that the family of measures $\{\eta_x\}_{x\in\mathcal{K}_\delta}$ is uniformly $(\epsilon, \delta/5)$-regular (in the sense of Definition 2.1). Let

$$\mathcal{N}_\delta(u,x) = \{n \in \mathbb{N} \ : \ u_n\dots u_1 x \in \mathcal{K}_\delta\}. \tag{4.10}$$

Write

$$A(u_n\dots u_1,x)^{-1} = k_n(u,x)a_n(u,x)k_n'(u,x) \tag{4.11}$$

where $k_n(u,x) \in \mathbf{K}$, $k_n'(u,x) \in \mathbf{K}$ and $a_n(u,x) \in \mathbf{A}_+$. We also write

$$A(u_n\dots u_1,x) = \bar{k}_n(u,x)\bar{a}_n(u,x)\bar{k}_n'(u,x). \tag{4.12}$$

where $\bar{k}_n$ and $\bar{k}'_n$ are elements of $\mathbf{K}$, and $\bar{a}_n \in \mathbf{A}_+$. Then, $\bar{a}_n(u,x) = w_0 a_n(u,x)^{-1} w_0^{-1}$ and thus,

$$\begin{aligned} &\alpha'(a_n(u,x)) = \alpha(\bar{a}_n(u,x)), \\ &\qquad \bar{k}_n(u,x) = k'_n(u,x)^{-1} w_0^{-1}, \quad \bar{k}'_n(u,x) = w_0 k_n(u,x)^{-1}, \end{aligned} \tag{4.13}$$

where w_0 is longest element of the Weyl group, and $\alpha' = -w_0 \alpha w_0^{-1}$.

Let $\pi_{I'} : \mathbf{H}/\mathbf{B} \to \mathbf{H}/\mathbf{P}_{I'}$ be the natural map. Let $\eta_x^{I'} = (\pi_{I'})_* \eta_x$ and $\eta_{u,x}^{I'} = (\pi_{I'})_* \eta_{u,x}$. Then, $\eta_x^{I'}$ and $\eta_{u,x}^{I'}$ are measures on $\mathbf{H}/\mathbf{P}_{I'}$.

Suppose $\alpha \in \Delta \setminus I$. Then, $\lambda_\alpha > 0$ and, *a fortiori*,

$$\lim_{n\to\infty} \alpha(\bar{a}_n(u,x)) \to \infty.$$

Thus,

$$\lim_{n\to\infty} \alpha'(a_n(u,x)) \to \infty$$

for each $\alpha' \in \Delta \setminus I'$.

Applying Lemma 2.2(a) to $g_n = A(u_n \dots u_1, x)^{-1}$ for $n \in \mathcal{N}_\delta(u,x)$ and the (ϵ,δ)-regular measures $\eta_n = \eta^{I'}_{u_n \dots u_1 x}$ we get that there exists $\bar{k} = \bar{k}(I',u,x) \in \mathbf{K}$ such that, for $n \in \mathcal{N}_\delta(u,x)$, one has $k_n(u,x)\mathbf{P}_{I'} \to \bar{k}\mathbf{P}_{I'}$ and

$$\eta_{u,x}^{I'}(\{\bar{k}\,\mathbf{P}_{I'}\}) \geq 1-\delta.$$

Since $\delta > 0$ is arbitrary, we get that for almost all (u,x), $\eta_{u,x}^{I'}$ is supported on one point of $\mathbf{H}/\mathbf{P}_{I'}$. Now choose $C^+(u,x) \in \mathbf{H}/\mathbf{B}$ so that $\pi_{I'}(C^+(u,x)) = \bar{k}(I',u,x)\mathbf{P}_{I'}$. The desired property about $C^+(u,x)$ follows from the stationarity of $\hat{\nu}$. $\square$

4.4. **Proof of Proposition 3.2 (b),(c).** The proof of Proposition 3.2 (b) is virtually identical to the proof of Proposition 3.2(a), and so we omit the details. Part (c) of Proposition 3.2 is also a classical fact, cf. [GM, Lemma 1.5]. We give an outline of a geometric argument as follows.

Let $\mathbf{H}/\mathbf{K}$ be the symmetric space associated to $\mathbf{H}$. We say that two geodesic rays (parametrized by arc length) are equivalent if they stay a bounded distance apart.

By the geometric version of the multiplicative ergodic theorem [KM], [Ka], for almost all $(u,x) \in \Omega \times X$ there exists a geodesic ray $\gamma^+ : [0,\infty) \to \mathbf{H}/\mathbf{K}$ with $\gamma^+(0) = \mathbf{K}$ such that

$$\lim_{n\to\infty} \frac{1}{n} d(A^n(u,x)^{-1}\mathbf{K}, \gamma^+(n)) = 0. \tag{4.14}$$

Similarly, by applying the same argument to the backwards walk, we get that for almost all $(v,x) \in \Omega^- \times X$ there exists a geodesic ray $\gamma^- : [0,\infty) \to \mathbf{H}/\mathbf{K}$ such that

$$\lim_{n\to\infty} \frac{1}{n} d(A^{-n}(v,x)^{-1}\mathbf{K}, \gamma^-(n)) = 0. \tag{4.15}$$

Let $F = F(v,u,x)$ be a flat in $\mathbf{H}/\mathbf{K}$ which contains rays $\hat{\gamma}^+$ and $\hat{\gamma}^-$ equivalent to γ^+ and γ^- respectively. Then, we have

$$\lim_{n\to\infty} \frac{1}{n} d(A^n(v,u,x)^{-1}\mathbf{K}, \hat{\gamma}^+(n)) = 0. \tag{4.16}$$

and

$$\lim_{n\to\infty} \frac{1}{n} d(A^{-n}(v,u,x)^{-1}\mathbf{K}, \hat{\gamma}^-(n)) = 0. \tag{4.17}$$

Therefore, for every $\delta > 0$ there exists a set $K_\delta \subset \hat{\Omega} \times X$ with $\widehat{\beta \times \nu}(K_\delta) > 1-\delta$ and $N > 0$ such that for $(v,u,x) \in K_\delta$ and $n > N$,

$$d(A^n(v,u,x)^{-1}\mathbf{K}, \hat{\gamma}^+(n)) \leq \delta n, \text{ and } d(A^{-n}(v,u,x)^{-1}\mathbf{K}, \hat{\gamma}^-(n)) < \delta n. \tag{4.18}$$

Let $X_n = A^n(v,u,x)^{-1}\mathbf{K}$, and let $\hat{X}_n$ be the closest point to X_n on $\hat{\gamma}_n^+$. Then, by (4.16), for $(v,u,x) \in K_\delta$ and $n > N$,

$$d(X_n, \hat{X}_n) \leq \delta n. \tag{4.19}$$

Let $\hat{\gamma}_n^-(t)$ be unique geodesic ray equivalent to $\hat{\gamma}^-$ such that $\hat{\gamma}_n^-(0) = \hat{X}_n$. Then, as long as $T^n(v,u,x) \in K_\delta$, and $m > N$, by (4.18) and (4.19), we have

$$d(A^{-m}(v,u,x)^{-1}X_n, \hat{\gamma}_n^-(m)) \leq \delta n + \delta m.$$

Since $A^n(v,u,x)$ and $A^{-n}(\hat{T}^n(v,u,x))$ are inverses, we have

$$d(\hat{\gamma}_n^-(n), e) \leq 2\delta n. \tag{4.20}$$

Note that $\hat{X}_n$, $\hat{\gamma}^+$, $\hat{\gamma}_n^-$ all lie in F. However in that case, (4.20) (for sufficiently small δ and large enough n) implies that

(4.21) $\hat{\gamma}^+$ and $\hat{\gamma}^-$ belong to the closures of opposite Weyl chambers in F.

We now interpret (4.21) in terms of $C^+(u,x)$ and $C^-(v,x)$. We can write

$$\gamma^+(t) = k(u,x)\hat{\Lambda}^t\mathbf{K},$$

where $k(u,x) \in \mathbf{K}$ and $\hat{\Lambda}^t \in \mathbf{A}_+$. Then, by comparing (4.14) with (4.11), we get

$$k(u,x)\mathbf{P}_{I'} = C^+(u,x)\mathbf{P}_{I'},$$

where $C^+(u,x)$ is as in Proposition 3.2 (a), and I' is as in (3.5). Similarly, if we may write

$$\gamma^-(t) = \bar{k}(v,x)\Lambda^t\mathbf{K},$$

where $\bar{k}(u,x) \in \mathbf{K}$ and $\Lambda^t \in \mathbf{A}_+$. Then, by comparing (4.15) with (4.12), we get

$$\bar{k}(u,x)\mathbf{P}_I = C^-(u,x)\mathbf{P}_{I'},$$

where $C^-(u,x)$ is as in Proposition 3.2 (b), and I is as in (3.2). Then, (4.21) implies (3.6). $\square$

5. Proof of Theorem 3.1

5.1. **Conformal blocks and Schmidt's criterion.** We will use the following criterion of K. Schmidt [Sch] for the detection of conformal blocks.

Definition 5.1 (cf. Definition 4.6 in [Sch])**.** We say that a cocycle $A : G \times X \to \mathbf{H}$ is *Schmidt-bounded* if, for every $\varepsilon > 0$, there exists a compact set $\mathcal{K}(\epsilon) \subset \mathbf{H}$ such that

$$\widehat{\beta \times \nu}\left(\left\{((v,u),x) \in \hat{\Omega} \times X : A^n(v,u,x) \notin \mathcal{K}(\epsilon)\right\}\right) < \varepsilon$$

for all $n \in \mathbb{N}$.

The importance of this notion in the search of conformal blocks becomes apparent in view of the next result, which follows from [Sch, Theorem 4.7].

Theorem 5.2 (Schmidt)**.** *$A(.,.)$ is Schmidt-bounded if and only if there exists a measurable map $C : X \to \mathbf{H}$ such that the cocycle $C(g(x))A(g,x)C(x)^{-1}$ takes its values in a compact subgroup of $\mathbf{H}$.*

5.2. **Proof of Theorem 3.1.** We use the notation from §4.3.

Lemma 5.3. *For any $\alpha \in I$, let $\alpha' = -w_0 \alpha w_0^{-1}$ (so that $\alpha' \in I'$). Then, $\beta \times \nu$-almost all $(u,x) \in \Omega \times X$, the measure $\eta_{u,x}^{\alpha'}$ has no atoms; i.e. for any element $\bar{k}_{u,x} \in \mathbf{K}$, we have $\eta_{u,x}^{\alpha'}(\{\bar{k}_{u,x}\mathbf{P}_{\alpha'}\}) = 0$.*

Proof. Suppose there exists $\delta > 0$ so that, for some $\alpha' \in I'$ and for a set (u,x) of positive measure, there exists $\bar{k}_{u,x} \in \mathbf{K}$ with $\eta_{u,x}^{\alpha'}(\{\bar{k}_{u,x}\mathbf{P}_{\alpha'}\}) > \delta$. Then this happens for a subset of full measure by ergodicity. Note that (4.9) holds.

Then, by Lemma 2.2 (b), for $\beta \times \nu$ almost all (u,x), $\eta_{u,x}^{\alpha'}(\{\bar{k}_{u,x}\mathbf{P}_{\alpha'}\}) \geq 1 - \delta$ (and thus $\bar{k}_{u,x}\mathbf{P}_{\alpha'}$ is unique) and, as $n \to \infty$ along $\mathcal{N}_\delta(u,x)$ (where $\mathcal{N}_\delta(u,x)$ was defined (4.10)), we have:

$$\alpha'(a_n(u,x)) \to \infty,$$

and

$$k_n(u,x)\mathbf{P}_{\alpha'} \to \bar{k}_{u,x}\mathbf{P}_{\alpha'}, \tag{5.1}$$

Then, by (4.13),

$$\alpha(\bar{a}_n(u,x)) \to \infty, \tag{5.2}$$

and

$$\bar{k}_n'(u,x)^{-1} w_0 \,\mathbf{P}_{\alpha'} \to \bar{k}_{u,x}\mathbf{P}_{\alpha'}.$$

Therefore for any $\epsilon_1 > 0$ there exists a subset $H_{\epsilon_1} \subset \Omega \times X$ of $\beta \times \nu$-measure at least $1 - \epsilon_1$ such that the convergence in (5.2) and (5.1) is uniform over $(u,x) \in H_{\epsilon_1}$. Hence there exists $M > 0$ such that for any $(u,x) \in H_{\epsilon_1}$, and any $n \in \mathcal{N}_\delta(u,x)$ with $n > M$,

$$\bar{k}_n'(u,x)^{-1} w_0 \mathbf{P}_{\alpha'} \in \mathrm{Nbhd}_{\epsilon_1}(\bar{k}_{u,x}\mathbf{P}_{\alpha'}). \tag{5.3}$$

By Lemma 4.1 (see also the proof of [EMi, Lemma 14.4]) there exists a subset $H''_{\epsilon_1} \subset X$ with $\nu(H''_{\epsilon_1}) > 1 - c_2(\epsilon_1)$ with $c_2(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$ such that for all $x \in H''_{\epsilon_1}$, and any $g \in \mathbf{H}$,

$$\eta_x(\mathrm{Nbhd}_{2\epsilon_1}(gJ)) < c_3(\epsilon_1),$$

where $c_3(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$. Let

$$H'_{\epsilon_1} = \{(u, x, z) \;:\; (u, x) \in H_{\epsilon_1}, \quad x \in H''_{\epsilon_1} \quad \text{and} \quad d(z, \bar{k}_{u,x}J) > 2\epsilon_1\}. \tag{5.4}$$

Then, $(\beta \times \hat{\nu})(H'_{\epsilon_1}) > 1 - \epsilon_1 - c_2(\epsilon_1) - c_3(\epsilon_1)$, hence $(\beta \times \hat{\nu})(H'_{\epsilon_1}) \to 1$ as $\epsilon_1 \to 0$.

We now claim that for $(u, x, z) \in H'_{\epsilon_1}$ and $n \in \mathcal{N}_\delta(u, x)$, we have

$$d(\bar{k}'_n(u, x)z, (\bar{\mathbf{N}}_\alpha \mathbf{P}_\alpha)^c) > \epsilon_1. \tag{5.5}$$

Suppose not, then there exist $(u, x, z) \in H'_{\epsilon_1}$ and $n \in \mathcal{N}_\delta(u, x)$ such that

$$d(\bar{k}'_n(u, x)z, (\bar{\mathbf{N}}_\alpha \mathbf{P}_\alpha)^c) \leq \epsilon_1.$$

Let $W_\alpha \subset W$ denote the subgroup of the Weyl group which fixes $\mathbf{M}_\alpha$. Then,

$$d(\bar{k}'_n(u, x)z, w_0 \bigsqcup_{w \notin W_\alpha w_0^{-1} W_\alpha} \mathbf{B}w\mathbf{B}) \leq \epsilon_1.$$

Hence,

$$d(z, \bar{k}'_n(u, x)^{-1} w_0 \bigsqcup_{w \notin W_\alpha w_0^{-1} W_\alpha} \mathbf{B}w\mathbf{B}) \leq \epsilon_1. \tag{5.6}$$

Note that

$$\mathbf{P}_{\alpha'} \bigsqcup_{w \notin W_\alpha w_0^{-1} W_\alpha} \mathbf{B}w\mathbf{B} = \bigsqcup_{w \notin W_\alpha w_0^{-1} W_\alpha} \mathbf{B}w\mathbf{B}.$$

By (5.3) and (5.6), this implies that

$$d(z, \bar{k}_{u,x} \bigsqcup_{w \notin W_\alpha w_0^{-1} W_\alpha} \mathbf{B}w\mathbf{B}) \leq 2\epsilon_1,$$

contradicting (5.4). This completes the proof of (5.5).

Therefore, in view of Lemma 2.5, there exists $C = C(\epsilon_1)$, such that for any $(u, x, z) \in H'_{\epsilon_1}$, any $n \in \mathcal{N}_\delta(u, x)$ with $n > M$,

$$\hat{\sigma}_\alpha(A(u_n \dots u_1, x), z) \geq \alpha(A(u_n \dots u_1, x)). \tag{5.7}$$

By (5.2) and (4.12), this implies that for $(u, x, z) \in H'_{\epsilon_1}$,

$$\hat{\sigma}_\alpha(A(u_n \dots u_1, x), z) \to \infty \quad \text{as } n \to \infty \text{ along } \mathcal{N}_\delta(u, x). \tag{5.8}$$

Since $(\beta \times \hat{\nu})(H'_{\epsilon_1}) \to 1$ as $\epsilon_1 \to 0$, (5.8) holds for $\beta \times \hat{\nu}$ almost all $(u, x, z) \in \Omega \times \hat{X}$.

Let $\sigma_\alpha : \Omega \times \hat{X} \to \mathbb{R}$ be defined by $\sigma_\alpha(u, x, z) = \bar{\sigma}_\alpha(u_1, x, z)$, where $\bar{\sigma}_\alpha$ is as in Lemma 4.2. Then, the left hand side of (5.8) is exactly

$$\sum_{j=0}^{n-1} \sigma_\alpha(\hat{T}^j(u, x, z)).$$

Also, we have $n \in \mathcal{N}_\delta(u, x)$ if and only if $T^n(u, x) \in \Omega \times \mathcal{K}_\delta$. Then, by [EMi, Lemma C.6],

$$\int_{\Omega \times \hat{X}} \sigma_\alpha(q)\, d(\beta \times \hat{\nu})(q) > 0.$$

By Lemma 4.2 (Furstenberg's formula), the above expression is λ_α. Thus $\lambda_\alpha > 0$, contradicting our assumption that $\alpha \in I$. This completes the proof of the lemma. □

Proof of Theorem 3.1. Let $C^+(u, x) \in \mathbf{H}$ and $C^-(v, x) \in \mathbf{H}$ be as in Proposition 3.2. By Proposition 3.2(c), for a.e. (v, u, x),

$$C^+(u, x)^{-1} C^-(v, x) = p_{I'}(v, u, x) w_0 p_I(v, u, x) \qquad \text{where } p_I(v, u, x) \in \mathbf{P}_I,\ p_{I'}(v, u, x) \in \mathbf{P}_{I'}.$$

Let

$$C_1(v, u, x) = C^+(u, x) p_{I'}(v, u, x) = C^-(v, x) p_I(v, u, x)^{-1} w_0^{-1}.$$

Then, by Proposition 3.2(a) and (b),

$$C_1(\hat{T}^n(v, u, x))^{-1} A^n(v, u, x) C_1(v, u, x) \in \mathbf{P}_{I'} \cap w_0 \mathbf{P}_I w_0^{-1} = \mathbf{M}_{I'} \mathbf{A}_{I'}.$$

Let

$$A^n_{I'}(v, u, x) := C_1(\hat{T}^n(v, u, x))^{-1} A^n(v, u, x) C_1(v, u, x).$$

Suppose $\delta > 0$. Then there exist compact sets $\mathcal{K}_2(\delta) \subset \hat{\Omega} \times X$ with $\widehat{\beta \times \nu}(\mathcal{K}_2(\delta)) > 1 - \delta$ and $\mathcal{K}_3(\delta) \subset \mathbf{H}$ such that for $((v, u), x) \in \mathcal{K}_2(\delta)$, $C_1(v, u, x) \in \mathcal{K}_3(\delta)$.

Therefore, by (2.8) and Lemma 2.3, there exists $c_1(\delta) \in \mathbb{R}_+$ such that for all $((v, u), x) \in \mathcal{K}_2(\delta)$ and all $n \in \mathbb{N}$ with $T^n(v, u, x) \in \mathcal{K}_2(\delta)$, we have, for all $\alpha \in \Delta$,

$$|\alpha(A^n_{I'}(v, u, x)) - \alpha(A^n(v, u, x))| \le c_1(\delta). \tag{5.9}$$

Let now $\epsilon = \epsilon(\delta)$ and $\mathcal{K}(\delta) \subset X$ be as in the proof of Proposition 3.2(a), so that for $x \in \mathcal{K}(\delta)$, the measure η_x is (ϵ, δ)-regular.

By Lemma 5.3, for all $\alpha' \in I'$, the measures $\eta^{\alpha'}_{u,x}$ are non-atomic. Therefore, we can find $\epsilon' = \epsilon'(\delta)$ and $\mathcal{K}'(\delta) \subset \Omega \times X$ such that for $(u, x) \in \mathcal{K}'(\delta)$, and all $\alpha' \in I'$, for any $z \in \mathbf{H}/\mathbf{P}_{\alpha'}$,

$$\eta^{\alpha'}_{u,x}(\mathrm{Nbhd}_{\epsilon'}(z)) < \epsilon.$$

Let

$$\mathcal{K}_1(\delta) = \{(u, x) \in \Omega \times X \ :\ x \in \mathcal{K}(\delta), (u, x) \in \mathcal{K}'(\delta)\}$$

Then, by (4.9), (4.11) and Lemma 2.2(a), there exists $c_2 = c_2(\delta) \in \mathbb{R}^+$ such that for all $(u, x) \in \mathcal{K}_1(\delta)$, all n with $T^n(u, x) \in \mathcal{K}_1(\delta)$, and any $\alpha' \in I'$,

$$\alpha(A(u_n \dots u_1, x)^{-1}) < c_2(\delta), \text{where } \alpha = -w_0 \alpha' w_0^{-1}.$$

Thus, by (4.12) and (4.13), for all $(u,x) \in \mathcal{K}_1(\delta)$ and all $n \in \mathbb{N}$ such that $T^n(u,x) \in \mathcal{K}_1(\delta)$ and all $\alpha' \in I'$,

$$\alpha'(A(u_n \dots u_1, x)) < c_2(\delta). \tag{5.10}$$

Let $\tilde{A}^n_{I'}(v,u,x)$ denote the part of $A^n_{I'}(v,u,x)$ which lies in $\mathbf{M}_{I'}$. Then,

$$\alpha'(\tilde{A}^n_{I'}(v,u,x)) = \alpha'(A^n_{I'}(v,u,x)) \qquad \text{for } \alpha' \in I'. \tag{5.11}$$

Let $\mathcal{K}'_1(\delta) = \mathcal{K}_2(\delta) \cap \{(v,u,x) \;:\; (u,x) \in \mathcal{K}_1(\delta)\}$. It follows from (5.9), (5.10) and (5.11), that for all $\alpha' \in I'$, all $(v,u,x) \in \mathcal{K}'_1(\delta)$ and all $n \in \mathbb{N}$ with $T^n(v,u,x) \in \mathcal{K}'_1(\delta)$,

$$\alpha'(\tilde{A}^n_{I'}(v,u,x)) \le c_3(\delta).$$

Note that $\widehat{\beta \times \nu}(\mathcal{K}'_1(\delta)) > 1 - 4\delta$. Since $\delta > 0$ is arbitrary, it follows that $\tilde{A}_{I'}$ is Schmidt-bounded (see Definition 5.1). Therefore, by Theorem 5.2, there exists $\tilde{C} : \hat{\Omega} \times X \to \mathbf{M}$ such that $\tilde{C}(\hat{T}^n(v,u,x))^{-1}\tilde{A}^n_{I'}(v,u,x)\tilde{C}(v,u,x) \in \mathbf{K} \cap \mathbf{M}_{I'}$. Let

$$C(v,u,x) = C_1(v,u,x)\tilde{C}(v,u,x)w_0.$$

Then,

$$C(T^n(v,u,x))^{-1}A^n(v,u,x)C(v,u,x) \in w_0^{-1}(\mathbf{M}_{I'} \cap \mathbf{K})\mathbf{A}_{I'}w_0 = (\mathbf{M}_I \cap \mathbf{K})\mathbf{A}_I.$$

Thus, (3.3) holds.

Finally, it is easy to see that (3.4) follows from (5.9) and the definition of the λ_α (cf. the argument in the proof of [GM, Lemma 1.5]). □

6. Proof of Theorem 1.6

Let L be a vector space, and suppose $\mathbf{H}$ is a subgroup of $SL(L)$. We assume that the action of $\mathbf{H}$ on L is irreducible, in the sense that no non-trivial proper subspace of L is fixed by $\mathbf{H}$.

Let $\mathbf{K}$, I, $\mathbf{A}_I$ and $\mathbf{M}_I$ be as in Theorem 3.1. By Theorem 3.1, we may assume that the cocycle $A(\cdot,\cdot)$ takes values in $(\mathbf{K} \cap \mathbf{M}_I)\mathbf{A}_I$. We choose an inner product on L which is preserved by $\mathbf{K}$.

Then, the block conformality of Theorem 1.6 follows from the corresponding statement in Theorem 3.1.

We note that, by Theorem 3.1, there exists a_* in the interior of $\mathbf{A}_I$ such that the Lyapunov exponents of $A(\cdot,\cdot)$ are given by expressions of the form $\omega(\log a_*)$, where ω is a weight of the action of $\mathbf{H}$ on L.

Let $\mathcal{V}'_\omega \subset L$ be the subspace corresponding to the weight ω; then for $a \in \mathbf{A}$, and $v \in \mathcal{V}'_\omega$,

$$a \cdot v = \omega(\log a)v.$$

Let ω_0 be the highest weight. (It exists and has multiplicity 1 because the action of $\mathbf{H}$ on L is irreducible). Then, the top Lyapunov exponent λ_1 of $A(\cdot,\cdot)$ is $\omega_0(\log a_*)$.

Then, since the action of $\mathbf{H}$ on L is irreducible, the Lyapunov subspace $\mathcal{V}_1$ of $A(\cdot,\cdot)$ corresponding to the Lyapunov exponent λ_1 is given by

$$\mathcal{V}_1 = \bigoplus_{\omega \in S_0} \mathcal{V}'_\omega,$$

where S_0 consists of weights of the form

$$\omega_0 - \sum_{\alpha \in I} c_\alpha \alpha.$$

Recall that for $\alpha \in I$, $\alpha(\mathbf{A}_I) = 0$. Therefore, for $a \in \mathbf{A}_I$ and $v \in \mathcal{V}_1$,

$$a \cdot v = \omega_0(\log a) v.$$

Then, for $k \in (\mathbf{K} \cap \mathbf{M}_I)$, $a \in \mathbf{A}_I$,

$$(ka) \cdot v = \omega_0(\log a) k \cdot v. \tag{6.1}$$

Since $A(\cdot,\cdot)$ takes values in $(\mathbf{K} \cap \mathbf{M}_I)\mathbf{A}_I$, (1.5) follows from (6.1). □

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