# The maximum number of 10- and 12-cycles in a planar graph

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**Abstract**

For a fixed planar graph $H$, let $\mathbf{N}_{\mathcal{P}}(n, H)$ denote the maximum number of copies of $H$ in an $n$-vertex planar graph. In the case when $H$ is a cycle, the asymptotic value of $\mathbf{N}_{\mathcal{P}}(n, C_m)$ is currently known for $m \in \{3, 4, 5, 6, 8\}$. In this note, we extend this list by establishing $\mathbf{N}_{\mathcal{P}}(n, C_{10}) \sim (n/5)^5$ and $\mathbf{N}_{\mathcal{P}}(n, C_{12}) \sim (n/6)^6$. We prove this by answering the following question for $m \in \{5, 6\}$, which is interesting in its own right: which probability mass $\mu$ on the edges of some clique maximizes the probability that $m$ independent samples from $\mu$ form an $m$-cycle?

## 1 Introduction

For graphs $G$ and $H$, let $\mathbf{N}(G, H)$ denote the number of (unlabeled, not necessarily induced) copies of $H$ in $G$. Furthermore, for a planar graph $H$, define

$$\mathbf{N}_{\mathcal{P}}(n, H) \overset{\text{def}}{=} \max\big\{\mathbf{N}(G, H) : G \text{ is an } n\text{-vertex planar graph}\big\}.$$

The study of $\mathbf{N}_{\mathcal{P}}(n, H)$ was initiated by Hakimi and Schmeichel [7], who determined $\mathbf{N}_{\mathcal{P}}(n, C_3)$ and $\mathbf{N}_{\mathcal{P}}(n, C_4)$ exactly. Alon and Caro [1] continued this study by determining $\mathbf{N}_{\mathcal{P}}(n, K_{2,k})$ exactly for all $k$; in particular, they determined $\mathbf{N}_{\mathcal{P}}(n, P_3)$. Győri et al. [4, 5] later gave the exact values for $\mathbf{N}_{\mathcal{P}}(n, P_4)$ and $\mathbf{N}_{\mathcal{P}}(n, C_5)$. Afterward, Ghosh et al. [3] asymptotically determined $\mathbf{N}_{\mathcal{P}}(n, P_5)$, and the current authors [2] computed $\mathbf{N}_{\mathcal{P}}(n, P_7)$ asymptotically. Generalizations of some of these results to other surfaces have been established by Huynh, Joret and Wood [8].

In [2], a general technique was introduced which allows one to bound $\mathbf{N}_{\mathcal{P}}(n, H)$ whenever $H$ exhibits a particular subdivision structure. Using this technique, the authors established $\mathbf{N}_{\mathcal{P}}(n, C_6) \sim (n/3)^3$ and $\mathbf{N}_{\mathcal{P}}(n, C_8) \sim (n/4)^4$. They furthermore conjectured that

$$\mathbf{N}_{\mathcal{P}}(n, C_{2m}) = \left(\frac{n}{m}\right)^m + o(n^m) \qquad \text{for all } m \geq 3,$$

and exhibited graphs meeting this lower bound.[1] Currently, the best known general upper bound is

$$\mathbf{N}_{\mathcal{P}}(n, C_{2m}) \leq \frac{n^m}{m!} + o(n^m) \qquad \text{for } m \geq 5.$$

In this note, we make progress toward this conjecture by establishing the next two open cases:

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[1]It would not be surprising if this conjecture was suggested earlier, though it does not appear to be explicitly stated anywhere else in the literature. The closest reference we could find is [6], in which the lower-bound construction is mentioned.

**Theorem 1.** *For $m \in \{5, 6\}$,*

$$\mathbf{N}_{\mathcal{P}}(n, C_{2m}) = \left(\frac{n}{m}\right)^m + o(n^m).$$

The paper is organized as follows. In Section 2, we recall the technique developed in [2] and extract the key ingredients necessary for the proof of Theorem 1. The proof of Theorem 1 is then the topic of Section 3.

**Remark.** While this paper was under review, Theorem 1 was extended to cover all values $m \geq 3$ by Lv–Győri–He–Salia–Tompkins–Zhu [9], thus settling the full conjecture.

## 2 Preliminaries

In [2] it was shown that one can upper bound $\mathbf{N}_{\mathcal{P}}(n, C_{2m})$ for $m \geq 3$ by answering the following question, which is interesting in its own right:

**Question 2.** Which probability mass $\mu$ on the edges of some clique maximizes the probability that $m$ independent samples from $\mu$ form a copy of $C_m$?

To formalize this question, we lay out the following definition. For a graph $G$ and an integer $m \geq 3$, let $\mathbf{C}_m(G)$ denote the set of (unlabeled) copies of $C_m$ in $G$, so $\mathbf{N}(G, C_m) = |\mathbf{C}_m(G)|$. Additionally, for a finite set $X$, let $K_X$ denote the clique on vertex-set $X$.

**Definition 3.** Let $X$ be a finite set and let $\mu$ be a probability mass on $\binom{X}{2}$.

1. For a subgraph $G \subseteq K_X$, define
$$\mu(G) \stackrel{\text{def}}{=} \prod_{e \in E(G)} \mu(e).$$

2. For an integer $m \geq 3$, define
$$\beta(\mu; m) \stackrel{\text{def}}{=} \sum_{C \in \mathbf{C}_m(K_X)} \mu(C), \qquad \text{and}$$
$$\beta(m) \stackrel{\text{def}}{=} \sup\left\{\beta(\mu; m) : \mu \text{ a probability mass on } \binom{X}{2} \text{ for some finite set } X\right\}.$$

The quantity $\beta(m)$ yields an upper bound on $\mathbf{N}_{\mathcal{P}}(n, C_{2m})$:

**Theorem 4** ([2, Lemma 2.5])**.** *For $m \geq 3$,*

$$\mathbf{N}_{\mathcal{P}}(n, C_{2m}) \leq \beta(m) \cdot n^m + O(n^{m-1/5}).$$

The argument here establishes that, as $n \to \infty$, the extremal graphs for $\mathbf{N}_{\mathcal{P}}(n, C_{2m})$ may be approximated by taking some fixed graph $G$ and "blowing up" the edges into independent sets of various sizes. Here, "blowing up" an edge $xy$ means replacing the edge with an independent set of vertices, each of which is connected to both $x$ and $y$. The probability mass $\mu$ is a compact way to represent the relative sizes of each of these independent sets, and $\beta(\mu; m)$ is a normalized count of the number of $C_{2m}$'s in the resulting blow-up. Interestingly, planarity plays only a minor role in this argument, and so the theorem applies to a much larger class of graphs. For example, the result applies to the class of graphs which can be embedded onto any surface of a fixed genus.

In [2], the authors conjectured that $\beta(m) = m^{-m}$, which is the value achieved by the uniform distribution on $E(C_m)$, and they established the cases of $m = 3$ and $m = 4$. In this paper we prove that $\beta(m) = m^{-m}$ for $m \in \{5, 6\}$, which will then imply Theorem 1 thanks to Theorem 4; constructions establishing a matching lower-bound are discussed in [2, Section 1].

In order to accomplish this goal, we will need to understand the structure of the probability masses at play. The following definition lays out two important aspects of such a probability mass.

**Definition 5.** Fix a finite set $X$ and let $\mu$ be a probability mass on $\binom{X}{2}$.

1. For $x \in X$, define
$$\bar{\mu}(x) \stackrel{\text{def}}{=} \sum_{y \in X \setminus \{x\}} \mu(xy),$$
which is the probability that an edge sampled from $\mu$ is incident to $x$. It can also be thought of as the weighted degree of the vertex $x$. Note that $\sum_{x \in X} \bar{\mu}(x) = 2$ thanks to the handshaking lemma.

2. The *support graph* of $\mu$ is the graph $G_\mu$, which has $E(G_\mu) = \operatorname{supp} \mu$ and $V(G_\mu) = \operatorname{supp} \bar{\mu}$. Since $G_\mu$ records the edges of positive mass under $\mu$, observe that
$$\beta(\mu; m) = \sum_{C \in \mathbf{C}_m(G_\mu)} \mu(C).$$

In [2, Corollary 4.7], it was shown that $\beta(m)$ is achieved for each $m \geq 3$, so we introduce the following notation:

**Definition 6.** For an integer $m \geq 3$, denote by $\operatorname{Opt}(m)$ the set of all probability masses $\mu$ satisfying $\beta(\mu; m) = \beta(m)$.

We will require structural results about such optimal masses which were established in [2].

**Lemma 7** ([2, Lemma 4.5])**.** *Fix $m \geq 3$ and $\mu \in \operatorname{Opt}(m)$. Then,*
$$m \cdot \beta(\mu; m) \cdot \mu(e) = \sum_{\substack{C \in \mathbf{C}_m(G_\mu): \\ E(C) \ni e}} \mu(C), \qquad \text{for every } e \in \operatorname{supp} \mu, \quad \text{and}$$
$$m \cdot \beta(\mu; m) \cdot \bar{\mu}(x) = \sum_{\substack{C \in \mathbf{C}_m(G_\mu): \\ V(C) \ni x}} 2 \cdot \mu(C), \qquad \text{for every } x \in \operatorname{supp} \bar{\mu}.$$

**Lemma 8** ([2, Lemma 4.6])**.** *Fix $m \geq 3$ and $\mu \in \operatorname{Opt}(m)$. If $z \in (0, 1)$ satisfies*
$$1 - \frac{m}{2} z > (1 - z)^m,$$
*then $\bar{\mu}(x) > z$ for all $x \in \operatorname{supp} \bar{\mu}$.*

Explicitly, we will employ the following two straightforward corollaries.

**Corollary 9.** *Fix $m \geq 3$ and $\mu \in \operatorname{Opt}(m)$. Then,*

- *For any $e \in \operatorname{supp} \mu$, we have $\mu(e) \leq 1/m$ with equality if and only if $e \in E(C)$ for every $C \in \mathbf{C}_m(G_\mu)$, and*

- *For any $x \in \operatorname{supp}\bar{\mu}$, we have $\bar{\mu}(x) \leq 2/m$ with equality if and only if $x \in V(C)$ for every $C \in \mathbf{C}_m(G_\mu)$.*

*Proof.* Using Lemma 7, we observe that

$$m \cdot \beta(\mu; m) \cdot \mu(e) = \sum_{\substack{C \in \mathbf{C}_m(G_\mu): \\ E(C) \ni e}} \mu(C) \leq \sum_{C \in \mathbf{C}_m(G_\mu)} \mu(C) = \beta(\mu; m),$$

and hence the first claim follows. The proof of the second claim is analogous. □

**Corollary 10.** *Fix $m \in \{5, 6\}$. If $\mu \in \operatorname{Opt}(m)$, then $|\operatorname{supp}\bar{\mu}| = m$ and $\bar{\mu}(x) = 2/m$ for every $x \in \operatorname{supp}\bar{\mu}$.*

*Proof.* Set $z = 2/(m+1)$. For $m \in \{5, 6\}$, it can be checked that $1 - \frac{m}{2}z > (1-z)^m$. Thus, Lemma 8 implies that $\bar{\mu}(x) > z = 2/(m+1)$ for every $x \in \operatorname{supp}\bar{\mu}$. From here, we see that

$$2 = \sum_{x \in \operatorname{supp}\bar{\mu}} \bar{\mu}(x) > \frac{2}{m+1} \cdot |\operatorname{supp}\bar{\mu}| \qquad \Longrightarrow \qquad |\operatorname{supp}\bar{\mu}| < m + 1.$$

As such, we know that $|\operatorname{supp}\bar{\mu}| = m$, since certainly $|\operatorname{supp}\bar{\mu}| \geq m$. Therefore, every copy of $C_m$ in $G_\mu$ must use every vertex of $\operatorname{supp}\bar{\mu}$; hence $\bar{\mu}(x) = 2/m$ holds for every $x \in \operatorname{supp}\bar{\mu}$ thanks to Corollary 9. □

Corollary 10 is the key observation which enables our arguments in the next section. If one could extend Corollary 10 to all $m \geq 7$, then approaching the full conjecture that $\beta(m) = m^{-m}$ would likely be significantly more tractable.

**Remark 11.** For $\mu \in \operatorname{Opt}(7)$, Lemma 8 implies that $\bar{\mu} > 0.246$ and so $|\operatorname{supp}\bar{\mu}| \leq 8$.

For $\mu \in \operatorname{Opt}(m)$ and general $m$, using the bound $1 - z \leq e^{-z}$ and Lemma 8 yields $\bar{\mu} > 1.593/m$ and so $|\operatorname{supp}\bar{\mu}| < 1.256 \cdot m$.

## 3 Proof of Theorem 1

To prove Theorem 1, it suffices to establish that $\beta(m) = m^{-m}$ for $m \in \{5, 6\}$, thanks to Theorem 4.

We begin by establishing an inequality on the edge-masses in an optimal probability mass.

**Lemma 12.** *Fix $m \in \{5, 6\}$ and $\mu \in \operatorname{Opt}(m)$. For each $e \in \operatorname{supp}\mu$,*

$$\left(\frac{2}{2 - m \cdot \mu(e)}\right)^4 \left(\frac{m-4}{m - 4 + m \cdot \mu(e)}\right)^{m-4} \big(1 - m \cdot \mu(e)\big) \leq 1.$$

*Proof.* Fix any $e \in \operatorname{supp}\mu$ and define

$$a \stackrel{\text{def}}{=} \frac{2}{2 - m \cdot \mu(e)}, \qquad \text{and} \qquad b \stackrel{\text{def}}{=} \frac{m-4}{m - 4 + m \cdot \mu(e)}.$$

Observe that $a, b \geq 0$ since $m \geq 5$ and $\mu(e) \in (0, 1/m]$ (Corollary 9). We define a new probability mass $\nu$ by

$$\nu(s) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } s = e, \\ a \cdot \mu(s), & \text{if } |s \cap e| = 1, \\ b \cdot \mu(s), & \text{otherwise.} \end{cases}$$

Suppose that $e = xy$. Corollary 10 tells us that $\bar{\mu}(x) = \bar{\mu}(y) = 2/m$, so

$$\begin{aligned}\sum_{s\in\operatorname{supp}\nu}\nu(s) &= a\cdot\sum_{|s\cap e|=1}\mu(s) + b\cdot\sum_{s\cap e=\varnothing}\mu(s)\\ &= a\cdot\big(\bar{\mu}(x)+\bar{\mu}(y)-2\mu(e)\big) + b\cdot\big(1-\bar{\mu}(x)-\bar{\mu}(y)+\mu(e)\big)\\ &= a\cdot\left(\frac{4}{m}-2\mu(e)\right) + b\cdot\left(1-\frac{4}{m}+\mu(e)\right)\\ &= \frac{4}{m}+\frac{m-4}{m} = 1.\end{aligned}$$

Therefore, $\nu$ is indeed a probability mass.

Since $|\operatorname{supp}\bar{\mu}| = |\operatorname{supp}\bar{\nu}| = m$ (Corollary 10) and $e \notin \operatorname{supp}\nu$, we know that any $C \in \mathbf{C}_m(G_\nu)$ has exactly 4 edges incident to the pair $\{x, y\}$. Furthermore, $\beta(\mu; m) \geq \beta(\nu; m)$ since $\mu \in \operatorname{Opt}(m)$. Thus, by appealing to Lemma 7, we compute

$$\begin{aligned}\beta(\mu;m) \geq \beta(\nu;m) &= a^4\cdot b^{m-4}\cdot\sum_{C\in\mathbf{C}_m(G_\nu)}\mu(C)\\ &= a^4\cdot b^{m-4}\cdot\Bigg(\beta(\mu;m) - \sum_{\substack{C\in\mathbf{C}_m(G_\mu):\\ E(C)\ni e}}\mu(C)\Bigg)\\ &= a^4\cdot b^{m-4}\cdot\big(\beta(\mu;m) - m\cdot\beta(\mu;m)\cdot\mu(e)\big).\end{aligned}$$

Dividing both sides of this inequality by $\beta(\mu; m)$ proves the lemma. □

Lemma 12 allows us to place lower bounds on $\mu(e)$ for $e \in \operatorname{supp}\mu$.

**Corollary 13.** *Fix $m \in \{5, 6\}$ and $\mu \in \operatorname{Opt}(m)$. If $z \in (0, 1)$ satisfies*

$$\left(\frac{2}{2-z}\right)^4\left(\frac{m-4}{m-4+z}\right)^{m-4}(1-z) > 1,$$

*then $\mu(e) > z/m$ holds for all $e \in \operatorname{supp}\mu$.*

*Proof.* For $z \in [0, 1)$ define the function

$$f(z) \stackrel{\text{def}}{=} \left(\frac{2}{2-z}\right)^4\left(\frac{m-4}{m-4+z}\right)^{m-4}(1-z).$$

Observe that $f(z) > 0$ for all $z \in [0, 1)$, so we may compute

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z) = \frac{4}{2-z} - \frac{m-4}{m-4+z} - \frac{1}{1-z} = \frac{(1-m)z^2+2z}{(2-z)(1-z)(m-4+z)}$$

Again, $f(z) > 0$ for all $z \in [0, 1)$, so, since $m \geq 5$,

$$\operatorname{sgn} f'(z) = \operatorname{sgn}\big((1-m)z^2+2z\big), \qquad \text{for all } z \in [0,1).$$

In particular, we see that $f'(z) \geq 0$ for all $0 \leq z \leq \frac{2}{m-1}$ and $f'(z) \leq 0$ for all $\frac{2}{m-1} \leq z < 1$. Since $f(0) = 1$ and $f(1) = 0$, this implies that the curves $y = f(z)$ and $y = 1$ intersect at 0 and at some unique $z^* \in (0, 1)$. Furthermore, $f(z) > 1$ for all $z \in (0, z^*)$ and $f(z) < 1$ for all $z \in (z^*, 1)$.

Now that we have a better understanding of the function $f$, the claim follows quickly. Suppose that $z \in (0, 1)$ satisfies $f(z) > 1$. If we were to have $0 < \mu(e) \leq z/m$, then $0 < m \cdot \mu(e) \leq z$. From the above, this would then imply that $f(m \cdot \mu(e)) > 1$ as well, contradicting Lemma 12. □

**Remark 14.** Lemma 12 (and hence Corollary 13) follows solely from the observations laid out in Corollary 10. Therefore, if Corollary 10 can be extended to $m \geq 7$, then so can Lemma 12 and Corollary 13.

With these observations in hand, we can determine $\beta(5)$ and $\beta(6)$. Firstly, for a graph $G$, let $\mu_G$ denote the uniform distribution on $E(G)$.

**Theorem 15.** $\beta(5) = 5^{-5}$.

*Proof.* Fix any $\mu \in \mathrm{Opt}(5)$. Thanks to Corollary 10, we know that $|\operatorname{supp} \bar{\mu}| = 5$ and that $\bar{\mu}(x) = 2/5$ for all $x \in \operatorname{supp} \bar{\mu}$. Furthermore, by applying Corollary 13 with $z = 2/3$, we see that $\mu(e) > 2/15$ for all $e \in \operatorname{supp} \mu$.

Now, certainly $\delta(G_\mu) \geq 2$ because $G_\mu$ has a spanning copy of $C_5$. Furthermore, for any $x \in \operatorname{supp} \bar{\mu}$, we have

$$\frac{2}{5} = \bar{\mu}(x) > \frac{2}{15} \deg(x) \qquad \Longrightarrow \qquad \deg(x) < 3.$$

We conclude that $G_\mu$ is 2-regular, and so we must have $G_\mu = C_5$. Applying the arithmetic–geometric mean (AM–GM) inequality then yields

$$\beta(\mu; 5) = \mu(G_\mu) \leq \left( \frac{1}{5} \sum_{e \in \operatorname{supp} \mu} \mu(e) \right)^5 = \frac{1}{5^5}.$$

with equality if and only if $\mu = \mu_{C_5}$. □

The proof that $\beta(C_6) = 6^{-6}$ is similar, albeit slightly more involved.

**Theorem 16.** $\beta(C_6) = 6^{-6}$.

*Proof.* We begin by observing that

$$\beta(6) \geq \beta(\mu_{C_6}; 6) = \frac{1}{6^6}.$$

Now, fix any $\mu \in \mathrm{Opt}(6)$. Appealing to Corollary 10, we know that $|\operatorname{supp} \bar{\mu}| = 6$ and that $\bar{\mu}(x) = 1/3$ for all $x \in \operatorname{supp} \bar{\mu}$. Furthermore, by applying Corollary 13 with $z = 6/11$, we see that

$$\mu(e) > \frac{1}{11}, \qquad \text{for all } e \in \operatorname{supp} \mu. \tag{1}$$

Now, certainly $\delta(G_\mu) \geq 2$ because $G_\mu$ has a spanning copy of $C_6$. Furthermore, eq. (1) tells us that each $x \in \operatorname{supp} \bar{\mu}$ satisfies

$$\frac{1}{3} = \bar{\mu}(x) > \frac{1}{11} \deg(x) \qquad \Longrightarrow \qquad \deg(x) < \frac{11}{3} < 4.$$

Therefore, each vertex of $G_\mu$ must have either degree 2 or degree 3. In fact, we claim that $G_\mu$ is either 2- or 3-regular. Indeed, if this were not true, then there would exist two adjacent vertices $x, y$ with $\deg(x) = 2$ and $\deg(y) = 3$ since $G_\mu$ is certainly connected. Now, since $\deg(x) = 2$ and $G_\mu$ has 6 vertices, every copy of $C_6$ in $G_\mu$ must use the edge $xy$ and so $\mu(xy) = 1/6$ thanks to Corollary 9. But then, one of the other two edges incident to $y$ must have mass at most

$$\frac{\bar{\mu}(y) - 1/6}{2} = \frac{1/3 - 1/6}{2} = \frac{1}{12} < \frac{1}{11},$$

contradicting eq. (1).

Next, we claim that $G_\mu$ must actually be 2-regular. To prove this, suppose for the sake of contradiction that $G_\mu$ is 3-regular.

To begin, we know that $G_\mu$ has 9 edges, so we may apply eq. (1) to see that for each $C \in \mathbf{C}_6(G_\mu)$,

$$\sum_{e\in E(C)} \mu(e) = 1 - \sum_{e\in E(G_\mu)\setminus E(C)} \mu(e) < 1 - \frac{3}{11} = \frac{8}{11}. \tag{2}$$

Figure 1: The only two 3-regular graphs on 6 vertices: $K_{3,3}$ (left) and $K_3 \,\square\, K_2$ (right).

Now, it is a routine exercise to show that the only 3-regular graphs on 6 vertices are $K_{3,3}$ and $K_3 \square K_2$ (here, "$\square$" denotes the Cartesian product of graphs; see Figure 1 for a drawing of $K_3 \square K_2$). In either case, we have $\mathbf{N}(G_\mu, C_6) \le 6$; thus by applying the AM–GM inequality and eq. (2), we bound

$$\begin{aligned}\frac{1}{6^6} \le \beta(6) = \beta(\mu; 6) = \sum_{C\in\mathbf{C}_6(G_\mu)} \mu(C) &\le \sum_{C\in\mathbf{C}_6(G_\mu)} \left(\frac{1}{6}\sum_{e\in E(C)} \mu(e)\right)^6 \\ &< 6\cdot\left(\frac{8}{11}\right)^6 \cdot \frac{1}{6^6} \le \frac{0.89}{6^6} < \frac{1}{6^6};\end{aligned}$$

a contradiction.

Therefore, we know that $G_\mu$ is 2-regular, and so $G_\mu = C_6$. Applying the AM–GM inequality one final time then yields

$$\beta(\mu; 6) = \mu(G_\mu) \le \left(\frac{1}{6}\sum_{e\in \operatorname{supp}\mu} \mu(e)\right)^6 = \frac{1}{6^6},$$

with equality if and only if $\mu = \mu_{C_6}$. This concludes the proof. □

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