COUNTING CLOSED GEODESICS IN STRATA

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Abstract. We compute the asymptotic growth rate of the number $N(C;R)$ of closed geodesics of length R in a connected component C of a stratum of quadratic differentials. We prove that, for any 0 1, the number of closed geodesics of length at most R such that spends at least –fraction of its time outside of a compact subset of C is exponentially smaller than $N(C;R)$. The theorem follows from a lattice counting statement. For points $x;y$ in the moduli space $M(S)$ of Riemann surfaces, and for 0 1 we find an upper-bound for the number of geodesic paths of length R in C which connect a point near x to a point near y and spend at least a –fraction of the time outside of a compact subset of C.

1. Introduction

Let $S = S_{g;p}$ be a surface of genus g with p punctures and let $M(S)$ be the moduli space of Riemann surfaces homeomorphic to S. The co-tangent bundle of $M(S)$ is naturally identified with $QM(S)$ the space of finite area quadratic differentials on S. Let $Q^1M(S)$ be subspace of quadratic differentials of area 1. There is a natural $SL(2;R)$ action on the $Q^1M(S)$. The orbits of the diagonal flow, $g_t = \begin{matrix} e^t & 0 \\ 0 & e \ t \end{matrix}$, projects to geodesics in $M(S)$ equipped with the Teichmüller metric. For $R > 0$, let $N(R)$ be the number of closed Teichmüller geodesics of length less than or equal to R on $Q^1M(S)$. It was shown in [EM2] that, as $R \ ! \ 1$, the number $N(R)$ is asymptotic to $e^{hR}{=}hR$; where $h = 6g \quad 6 + 2p$.

The moduli space of quadratic differentials is naturally stratified: to each quadratic differential $(x;q)$ 2 $QM(S)$ we can associate $(q) = ({}_i;:::;{}_k;\&)$ where ${}_1;:::;{}_k$ are the orders of the zeros and poles of q, and & 2 f 1; 1g is equal to 1 if q is the square of an abelian differential and 1 otherwise. For a given tuple , we say a quadratic differential $(x;q)$ 2 $QM(S)$ is of type if (q) = . The space $QM()$ of all quadratic differentials in $QM(S)$ of type is called the stratum of quadratic differentials of type . The stratum $QM()$ is an analytic orbifold of real dimension $4g + 2k + \& \quad 3$.

Let $Q^1M()$ be the space of quadratic differentials in $QM()$ of area 1. It is not necessarily connected (see [KZ] and [La] for the classification of the connected components), however, each connected component is $SL(2;R)$ invariant. Let C be a connected component of $Q^1M()$. In this paper, we study the asymptotic growth rate of the number $N(C;R)$ of closed Teichmüller geodesics of length less than or equal to R in C. Our main tool is estimating the number $N(C_K;R)$ of closed geodesics that stay completely outside of a large compact set K C.

Theorem 1.1. Given > 0 there exists a compact subset K C and $R_0 > 0$ such that for all $R > R_0$,

$$N(C_K;R) \quad e^{(h \quad 1+)R}:$$

This result implies that:

Theorem 1.2. As $R \ ! \ 1$, we have

$$N(C;R) \quad \frac{e^{hR}}{hR};$$

where $h = \frac{1}{2}[1 + \dim_R(C)]$ and the notation A B means that the ratio $A{=}B$ tends to 1 as R tends to infinity.

Remark 1.3. In the case of abelian differentials, h is equal to the dimension of the relative homology of S with respect to the set of singular points of $(x;q)$ 2 C, otherwise h is one less.

Recurrence of geodesics. We prove a stronger version of Theorem 1.1. Every quadratic differential defines a singular Euclidean metric on the surface S and for every compact set K C, there is a lower bound for the q–length of a saddle connection where q 2 K. Here, we restrict attention to closed geodesics where more than one simple closed curve or saddle connection is assumed to be short; in this case the growth rate is of even lower order.

Let $T(S)$ be the Teichmüller space, the universal cover of $M(S)$. Let $QT(S)$ and $Q^1T(S)$ be defined similarly. To distinguish between points in the Moduli space and Teichmüller space, we use x 2 $M(S)$ and X 2 $T(S)$. Also, we use the notation $(x; q)$ for points in $Q^1M(S)$ and $(X; q)$ for points in $Q^1T(S)$. We denote a geodesic in $Q^1M(S)$ by g and a geodesic in $Q^1T(S)$ by G. The space $Q^1T(S)$ is also naturally stratified. We denote the space of quadratic differentials in $Q^1T(S)$ of type by $Q^1T()$. To simplify the notation, let

$$Q() := Q^1T():$$

Recall that $\mathrm{Ext}_X()$ denotes the extremal length of a a simple closed curve on the Riemann surface X 2 $T(S)$. (see Equation (1) for definition). We introduce a notion of extremal length for saddle connections on quadratic differentials. Essentially, the extremal length of a saddle connection ! in a quadratic differential $(X; q)$ 2 $Q^1T(S)$ with distinct end points p_1 and p_2 is the extremal length of the associated curve in the ramified double cover of X with simple ramification points at only p_1 and p_2 (see §3.5 for more details).

Definition 1.4. For $>$ 0 and for any quadratic differential $(X; q)$ 2 $Q()$, let $_q()$ be the set of saddle connections ! so that either $\mathrm{Ext}_q(!)$ or ! appears in a geodesic representative of a simple closed curve with $\mathrm{Ext}_X()$. Let $Q_{j;}()$ be the set of quadratic differentials $(X; q)$ of type so that $_q()$ contains at least j disjoint homologically independent saddle connections. When is fixed, we denote this set simply by $Q_{j;}$. Let $C_{j;}$ be the set of points in C whose lift to $Q^1T(S)$ lies in $Q_{j;}$. For 0 1, define $N(C_{j;}; R)$ to be the number of closed geodesics of length at most R in C that spend at least –fraction of their length in $C_{j;}$.

In this paper, we show:

Theorem 1.5. Given $>$ 0, there exist $>$ 0 small enough and $R >$ 0 large enough so that, for all j 1 and 0 1,

$$N(C_{j;}; R) \quad e^{(h \quad j+)R}:$$

Remark 1.6. The condition on $_q()$ is necessary. Just assuming there are j saddle connections of q–length less than does not reduce the exponent by j. In fact, for any , there is a closed geodesic g in $Q^1M(S)$ where the number of saddle connections with q–length less than is as large as desired at every quadratic differential $(X; q)$ along g. This is because the Euclidean size of a subsurface could be as small as desired (see §3.4) and short saddle connection can intersect.

Lattice counting in Teichmüller space. Let (S) denote the mapping class group of S and let $B_R(X)$ denote the ball of radius R in the Teichmüller space with respect to the Teichmüller metric centered at the point X 2 $T(S)$. It is known ([ABEM]) that, for and Y 2 $T(S)$,

$$(S) \; Y \setminus B_R(X) \quad ^2 e^{(6g \quad 6)R};$$

as R ! 1: Here is a constant depending only on the topology of S (See [Du]).

The main theorem in this paper is a partial generalization of this result for the strata of quadratic differentials. Here we are interested in the case where the Teichmüller geodesic joining X to $g\,Y$, for g 2 (S), is assumed to belong to the stratum $Q()$ or stay close to it.

More precisely, for a fix $r_0 >$ 0 (see §6.2), let $N(Q_{j;}; X; Y; R)$ be the number of points Z 2 $T(S)$ such that (See Fig. 1):

- Z 2 $B_R(X)$ and $Z = g\,Y$, for some g 2 (S).
- there is a Teichmüller geodesic segment G $Q()$ joining X_1 2 $B_{r_0}(X)$ to Y_1 2 $B_{r_0}(Z)$ G spends at least –fraction of the time in $Q_{j;}$.

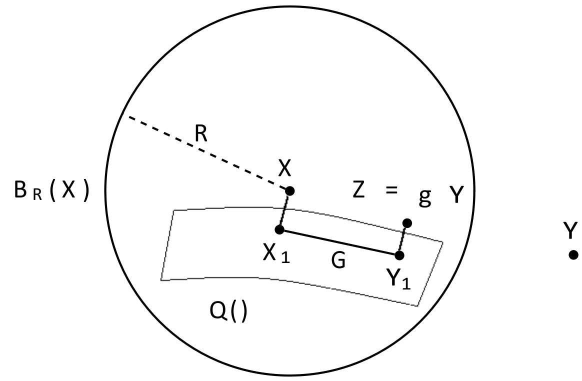

Figure 1. There is a geodesic G in Q() connecting a point near X to a point near Z 2 $B_R(X)$ that is in the orbit of Y .

Also, for a fix $_0$ (see §1.3 below), we define S_X to be the set of $_0$–short curves in X and

$$G(X) = 1 + \prod_{2 S_X} \frac{1}{\sqrt{\mathrm{Ext}_X()}} :$$

Theorem 1.7. Given > 0, there exist > 0 small enough and R > 0 large enough such that, for every 0 1, j 1 and X; Y 2 T(S), we have

$$N(Q_{j;}; X; Y; R) \quad G(X)G(Y)\, e^{(h \quad j+)R};$$

Compare with Theorem 7:2 in [EM2].

1.1. Notes on the proof.

1. Each stratum $Q^1M()$ has an affine integral structure, and carries a unique probability measure , called the Masur-Veech measure, invariant by the Teichmüller flow which is equivalent to the Lebesgue measure. Moreover, the restriction of the Teichmüller flow to any connected component C of $Q^1M()$ is mixing with respect to the Lebesgue measure class [Ma], [Ve]. In fact, the Teich-müller flow on C is exponentially mixing with respect to [AR], [AGY]. However, we will only use the mixing property (as stated in Theorem 2.4) in this paper.

2. The main difficulty for proving Theorem 1.2 is the fact that the Teichmüller flow is not uniformly hyperbolic. As in [EM2], we show that the Teichmüller geodesic flow (or more precisely an associated random walk) is biased toward the part of C that does not contain short saddle connections (see Lemma 6.4). Similar method has been used in [EM2] where it is enough to use Minsky's product region theorem (see §2.5) to prove the necessary estimates. In this paper, since the projection map from C to M(S) is not easy to understand, we need different and more delicate methods to obtain similar results.

3. We define a notion of a (q;)–regular triangulation for a quadratic differential (X; q) (Defini-tion 3.11). Such a triangulation captures the geometry of singular Euclidean metric associated to q in a way that is compatible with the hyperbolic metric associated to X . We will show that a set of disjoint saddle connections in $_q()$ can be included in a (q;)–regular triangulation (Lemma 3.13).

4. In order to prove Theorem 5.1 (§5) we compute, given the triangulation T_a, the number of possible triangulations T_b which have certain bounds on their intersection number with T_a. It turns out that the number of possible triangulations T_b is related to the number of edges in T_a that are homologically independent. This is the main reason that the growth rate of $N(Q_{j;}; X; Y; R)$ is related to $\dim_{\mathbb{R}} C$. In §3 we establish the basic properties of a (q;)–regular triangulation and in §4 we establish the necessary bounds on the intersection number between T_a and T_b needed in §5.

5. Theorems 1.1 and 1.5 are essentially corollaries of Theorem 1.7. In §6, we use Theorem 5.1 to prove Theorem 1.7. Here we describe the steps involved in the proof of Theorem 1.7. First, we fix a net N in M(S) and its lift N in $\tilde{}$T(S). For any constant , we note that $N(Q_{j;}; X; Y; R)$ is bounded above by the number of trajectories $f_0; :::; _n g$ in N from X to $\tilde{Y}$ so that the distance between $_i$ and $_{i+1}$ is at most and, for proportion of steps, the segment $[_i; _{i+1}]$ can be approximated by a path in $Q_{j;}$.

Given $_i$, we bound the number of possible choices for $_{i+1}$ so that the segment $[_i; _{i+1}]$ can be approximated by a path in $Q_{j;}$. The bound depends on the geometry of $_i$ (captured by the function G( ).

On the other hand, if G: [a; b] ! $Q_{j;}$ is a geodesic segment with initial and terminal quadratic differentials $(X_a; q_a)$ and $(X_b; q_b)$ with jb aj , one can find a $(q_a;)$–regular triangulation T_a and $(q_b;)$–regular triangulation T_b so that T_a and T_b have j nomologically independent edges in common (See Lemma 6.1 for the precise statement). Then Theorem 5.1 shows that the number of choices for $_{i+1}$ is also reduced by a factor $e^{\ j}$.

6. To obtain Theorem 1.2, we use the basic properties of the Hodge norm [ABEM] to prove a closing lemma for the Teichmüller geodesic flow in §8. We remark that the Hodge norm behaves badly near smaller strata, i.e. near points with degenerating zeros of the quadratic differential, where quadratic differentials have small geodesic segments.

On the other hand, the set of quadratic differentials with no small geodesic segment is compact and in any compact subset of C, the geodesic flow is uniformly hyperbolic (See [Ve], [Fo] and §7). Also, in view of Theorem 1.5, for any 0 1, the number of closed geodesics of length at most R such that spends at least a –fraction of the time outside of a compact subset of C is exponentially smaller than N(C; R). Therefore, "most" closed geodesics spend at least (1)– fraction of the time away from the degenerating locus. This allows us to prove Theorem 1.2 following the ideas from Margulis' thesis [Mar].

1.2. Further remarks and references.

1. According to the Nielsen-Thurston classification, every irreducible mapping class g 2 (S) of infinite order has a representative which is a pseudo-Anosov homeomorphism. Let K_g denote the dilatation factor of g [Th1]. By a theorem of Bers, every closed geodesic in M(S) is the unique loop of minimal length in its homotopy class. Also a pseudo-Anosov g 2 (S) gives rise to a closed geodesic G_g of length $\log(K_g)$ in $Q^1M(S)$: Hence $\log(K_g)$ is the translation length of g as an isometry of T(S) [Be]. In other words,

$$L(S) = \log(K_g) \ j \ g\ 2 \quad (S) \quad \text{pseudo-Anosov}$$

is the length spectrum of M(S) equipped with the Teichmüller metric. By [AY] and [Iv], L(S) is a discrete subset of R. Hence the number of conjugacy classes of pseudo-Anosov elements of the group (S) with dilatation factor $K_g \ e^R$ is finite. We remark that for any pseudo-Anosov g 2 (S) the number K_g is an algebraic number and $\log(K_g)$ is equal to the minimal topological entropy of any element in the same homotopy class [FLP]. (See [Pe] and [BC] for simple explicit constructions of pseudo-Anosov mapping classes.) In terms of this notation, N(C; R) is the number of conjugacy classes of pseudo-Anosov elements g in the mapping class group (S) with expansion factor of at most e^R such that G_g C:

2. The first results on this problem are due to Veech [Ve]. He proved that there exists a constant c such that

$$h \ \liminf_{R!1} \frac{\log N(R)}{R} \ \limsup_{R!1} \frac{\log N(R)}{R} \ c$$

and conjectured that c = h.

Foliations fixed by pseudo-Anosov maps can be characterized by being representable by eventually periodic "convergent" words [PP1]. Moreover, there is an inequality relating the length of the repeating part of the word corresponding to a pseudo-Anosov foliation and the dilatation factor of a pseudo-Anosov map preserving that foliation [PP2]. However, the estimates obtained using these inequalities are weaker.

3. The basic idea behind the proof of the main theorem in this paper is proving recurrence results for Teichmüller geodesics. Variations on this theme have been used in [EMM], [EM1], [Ath], and [EM2]. One reason the proof is different from [EM2] is that in general the projection map : $Q^1M() ! \; M(S)$ is far from being a fibration: in many cases $\dim(Q^1M()) < \dim(M(S))$ and dim($^1(X) \setminus Q^1M())$ depends on the geometry of X. In this paper, we need to analyze the geometry of quadratic differentials more carefully. The results obtained in §3 allow us to deal with this issue.

4. Our results are complimentary to the following result:

Theorem 1.8 (Hammenstadt). There exists a compact $K \; C$ such that for R sufficiently large,

$$N(C_K; R) \; e^{(h \; 1)R}:$$

Also, by results in [H2] the normalized geodesic flow invariant measure supported on the set of closed geodesics of length R in C become equidistributetd with respect to the Lebesgue measure as $R ! \; 1$.

1.3. Choosing constants. We choose our constants as follows: We call a curve short if its extremal length is less than $_0$. This is a constant that depends on the topology of S only (a uniform constant) and is chosen so that Theorem 2.2 and the estimate in Equation (5) hold. We call any other constant that depends in the topology of S or the choice of $_0$ a uniform constant. Most of these constants are hidden in notations and (see the notation section below). For the arguments in §6 to work, we need to choose large enough depending on the value of (see proofs of Theorem 1.5 and Lemma 6.4 in §6). Then is chosen small enough depending on the value of . We need $_1 = {}_1()$ so that Lemma 3.13 holds and $_2 = {}_2()$ so that Lemma 6.1 holds. The dependence on the choice of and is always highlighted and a constant that we call uniform does not depend on or .

1.4. Notation. In this paper, the expression $A \; B$ means that $A < c\,B$ and $A \overset{+}{\;} B$ means $A \; B + c$ for some uniform constant c which only depends on the topology of S (a uniform constant). We write $A \; B$ if we have both $A \; B$ and $B \; A$. Similarly, $A \overset{+}{\;} B$ if both $A \; C$ and $B \overset{+}{\;} A$ hold. The notation $A = O(B)$ means that $A \; B$.

Acknowledgements. We would like to thanks the referee for many useful comments that have improve the exposition of the paper at several places.

2. Teichmüller Space and Quadratic Differentials

In this section, we recall some definitions and known results about the geometry of $M(S)$ equipped with the Teichmüller metric. For more details, see [Hu], [FM] and [St].

2.1. Teichmüller space. Let S be a connected oriented surface of genus g with p marked points. A point in the Teichmüller space $T(S)$ is a Riemann surface X of genus g with p marked points equipped with a diffeomorphism $f : S ! \; X$ sending marked points to marked points. The map f provides a marking on X by S. Two marked surfaces $f_1 : S ! \; X$ and $f_2 : S ! \; Y$ define the same point in $T(S)$ if and only if $f_1 \; f_2^{\;1} : Y ! \; X$ is isotopic (relative to the marked points) to a holomorphic map. By the uniformization theorem, each point X in $T(S)$ has a complete metric of constant curvature $\;1$ with punctures at the marked points. The space $T(S)$ is a complex manifold of dimension $3g \; 3 + p$, diffeomorphic to a cell. Let (S) denote the mapping class group of S, the group of isotopy classes of orientation preserving self-homeomorphisms of S fixing the marked points point-wise. The mapping class group (S) acts on $T(S)$ by changing the marking. The quotient space

$$M(S) = T(S)= \; (S)$$

is the moduli space of Riemann surfaces homeomorphic to S.

2.2. Teichmüller distance and Teichmüller's theorem. The Teichmüller metric on $T(S)$ is defined by

$$d_T\big((f_1 : S \to X_1), (f_2 : S \to X_2)\big) = \frac{1}{2} \inf_f \log(K_f),$$

where $f : X_1 \to X_2$ ranges over all quasiconformal maps isotopic to $f_1 \circ f_2^{-1}$ and $K_f \geq 1$ is the dilatation of f. For convenience, we will often omit the marking and write $X \in T(S)$. To distinguish between a marked point and an un-marked point, we use small case letters for points in Moduli space and write $x \in M(S)$.

We recall the following important theorem due to Teichmüller. Given any $X_1, X_2 \in T(S)$, there exists a unique quasi-conformal map f, called the Teichmüller map and quadratic differentials $(X_i, q_i) \in Q^1(X_i)$ such that the map f takes zeroes and poles of q_1 to zeroes and poles of q_2 of the same order and $d_T(X_1, X_2) = \frac{1}{2}\log(K_f)$.

2.3. The space of quadratic differentials. Let $Q(X)$ denote the vector space of quadratic differentials on X with at most simple poles at the marked points of X. The cotangent space of $T(S)$ at a point X can be identified with $Q(X)$ and the space

$$QT(S) = \Big\{ (X, q) \;\Big|\; X \in T(S), \; q \in Q(X) \Big\}$$

can be identified with the cotangent space of $T(S)$.

In local coordinates z, q is the tensor given by $q(z)dz^2$, where $q(z)$ is a meromorphic function with poles of degree at most one at the punctures of X. In this setting, the Teichmüller metric corresponds to the norm

$$\| q \|_T = \int_X |q(z)| \, |dz|^2$$

on $QT(S)$. Let $Q^1T(S)$ denote the space of (marked) unit area quadratic differentials, or equivalently the unit cotangent bundle over $T(S)$. Define

$$QM(S) = QT(S)/ (S) \quad \text{and} \quad Q^1M(S) = Q^1T(S)/ (S).$$

To simplify the notation, in this paper, we let p denote both projection maps

$$p : T(S) \to M(S), \qquad \text{and} \qquad p : Q^1T(S) \to Q^1M(S).$$

Similarly, will denote both projection maps:

$$: Q^1M(S) \to M(S), \qquad \text{and} \qquad : Q^1T(S) \to T(S).$$

2.4. Extremal and hyperbolic lengths of simple closed curves. By a curve we always mean the free homotopy class of a non-trivial, non-peripheral, simple closed curve on the surface S where the homotopy is relative to the marked points. We denote the set of curves on S by S to emphasize that they are simple curves.

Given a curve on the surface S and $X \in T(S)$, let $\ell_X()$ denote the hyperbolic length of the unique geodesic in the homotopy class of on X. The extremal length of a curve on X is defined by

$$\text{(1)} \qquad \operatorname{Ext}_X() := \sup \frac{\ell()^2}{\operatorname{Area}(X,)},$$

where the supremum is taken over all metrics conformally equivalent to X, and $\ell()$ denotes the infimum of –lengths of representatives of .

Here X can be any Riemann surface, even an open annulus. Recall that the modulus of an annulus A is defined to

$$\operatorname{Mod}(A) := \frac{1}{\operatorname{Ext}_A()},$$

where is the core curve of A.

Given curves and on S, the intersection number $i(,)$ is the minimum number of points in which representatives of and must intersect. In general, by [GM]

$$\text{(2)} \qquad i(,) \leq \sqrt{\operatorname{Ext}_X()} \sqrt{\operatorname{Ext}_X()}.$$

The following result [Ker] relates the ratios of extremal lengths to the Teichmüller distance:

Theorem 2.1 (Kerckhoff). Given $X, Y \in \mathcal{T}(S)$, the Teichmüller distance between X and Y is given by

$$d_{\mathcal{T}}(X, Y) = \sup_{\alpha \in \mathcal{S}} \log \left(\frac{\sqrt{\operatorname{Ext}_X(\alpha)}}{\sqrt{\operatorname{Ext}_Y(\alpha)}} \right).$$

The relationship between the extremal length and the hyperbolic length is complicated; in general, by the definition of extremal length,

$$\frac{\ell_X(\alpha)^2}{2\pi(2g-2+p)} \le \operatorname{Ext}_X(\alpha).$$

Also, for any $X \in \mathcal{T}(S)$, the extremal length can be extended continuously to the space of measured laminations [Ker] such that

$$\operatorname{Ext}_X(r\lambda) = r^2 \operatorname{Ext}_X(\lambda).$$

As a result, since the space of projectivized measured laminations is compact, for every X there exists a constant c_X so that

$$\frac{1}{c_X}\ell_X(\alpha) \le \sqrt{\operatorname{Ext}_X(\alpha)} \le c_X \ell_X(\alpha).$$

However, by [Mas]

$$\frac{1}{\pi} \le \frac{\operatorname{Ext}_X(\alpha)}{\ell_X(\alpha)} \le \frac{1}{2} e^{\ell_X(\alpha)/2}. \tag{3}$$

Hence, as $\ell_X(\alpha) \to 0$,

$$\frac{\ell_X(\alpha)}{\operatorname{Ext}_X(\alpha)} \asymp 1.$$

2.5. Minsky's product theorem. Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_j\}$ be a collection of disjoint simple closed curves on S and, for a fixed ϵ_0,

$$\mathcal{T}_{\epsilon_0}(\mathcal{A}) = \left\{ X \in \mathcal{T}(S) \mid \operatorname{Ext}_X(\alpha_i) \le \epsilon_0, \quad 1 \le i \le j \right\}.$$

Then, using the Fenchel-Nielsen coordinates on $\mathcal{T}(S)$, we can define

$$\Pi_{\mathcal{A}} : \mathcal{T}_{\epsilon_0}(\mathcal{A}) \to (\mathbb{H}^2)^j$$

by

$$\Pi_{\mathcal{A}}(X) = \left(\tau_1(X), \frac{1}{\ell_X(\alpha_1)}; \dots; \tau_j(X), \frac{1}{\ell_X(\alpha_j)} \right).$$

Here, $\tau_i(\cdot)$ is the Fenchel-Nielson twist coordinate around α_i and represents the x–coordinate in upper-half plane $\mathbb{H}$ and the y–coordinate in $\mathbb{H}$ is the reciprocal of the hyperbolic length. Following Minsky, we get a map

$$\Pi : \mathcal{T}_{\epsilon_0}(\mathcal{A}) \to (\mathbb{H}^2)^j \times \mathcal{T}(S \setminus \mathcal{A}),$$

where $\mathcal{T}(S \setminus \mathcal{A})$ is the quotient Teichmüller space obtained by collapsing all the α_i. The product region theorem [Mi] states that for sufficiently small ϵ_0 the Teichmüller metric on $\mathcal{T}_{\epsilon_0}(\mathcal{A})$ is within an additive constant of the supremum metric on $(\mathbb{H}^2)^j \times \mathcal{T}(S \setminus \mathcal{A})$. More precisely, let $d_{\mathcal{A}}(\cdot,\cdot)$ denote the supremum metric on $(\mathbb{H}^2)^j \times \mathcal{T}(S \setminus \mathcal{A})$. Then:

Theorem 2.2 (Minsky). There is $\epsilon_0 > 0$ is small enough and $B > 0$ depending only on S such that for all $X, Y \in \mathcal{T}_{\epsilon_0}(\mathcal{A})$,

$$\left| d_{\mathcal{T}}(X, Y) - d_{\mathcal{A}}\big(\Pi(X), \Pi(Y)\big) \right| < B.$$

As mentioned in the introduction, we fix ϵ_0 so that the above theorem and the estimate in Equation (5) hold.

2.6. Short curves on a surface. For $_0$ as above, we say a curve is short on X if $\mathrm{Ext}_X()$ $_0$. From discussions in [Mi], we know that, if two curves are short in X they can not intersect. Let S_X be the set of short curves on X. Define $G : T(S) \; ! \; R_+$ by

$$G(X) = 1 + \prod_{2 S_X} \frac{1}{\sqrt{\mathrm{Ext}_X()}} : \qquad (4)$$

If $d_T(X; Y) = O(1)$ then $G(X)$ $G(Y)$: The function G is (S) invariant and induces a proper function on $M(S)$. We also recall the following lemma which, for example, follows from [EM2].

Lemma 2.3. For any $X \; 2 \; T(S)$ let

$$I_X = \quad g \; 2 \quad (S) \; d_T(g \; X; X) = O(1) \quad :$$

be the set of mapping classes that move X by a bounded amount. Then

$$I_X \quad G(X)^2 :$$

2.7. Stratum of quadratic differentials. Although the value of $q \; 2 \; Q(X)$ at a point in X depends on the local coordinates, the zero set of q is well defined. As a result, there is a natural stratification of the space $QT(S)$ by the multiplicities of zeros of q. For $= (_1; : : : ; _k; \&)$ define $QT()$ $QT(S)$ to be the subset consisting of pairs $(X; q)$ of quadratic differentials on X with zeros and poles of multiplicities $(_1; : : : ; _k)$. The poles are always assumed to be simple and are located at the marked points, however, not all marked points have to be poles. The sign $\& 2 \; f \; 1; 1g$ is equal to 1 if q is the square of an abelian differential (an abelian differential). Otherwise, $\& = \; 1$. Then

$$QT(S) = \bigsqcup QT():$$

It is known that each $QT()$ is an orbifold. See [Ma] and [MS2] for more details.

2.8. Flat lengths of simple closed curves and saddle connections. Let $(X; q)$ be a quadratic differential. If we represent q locally as $q(z)dz^2$ then $jqj = jq(z)j^{\frac{1}{2}} jdzj$ defines a singular Euclidean metric on X with cone points at zeros and poles. The total angle at a singular point of degree is $(2 +)$. (for more details, see [St]). This is not a complete metric space since poles are a finite distance away. However, one can still talk about the geodesic representative of a curve that may pass through the poles even though the poles. Namely, for a arc in $(X; q)$, consider the lift of this arc to the universal cover, take the geodesic representative in the completion of the universal cover and then project it back to $(X; q)$. Following the discussion in [R1, Page 185], we can ignore this issue and treat these special geodesics as we would any other geodesic.

The homotopy class of an arc (relative to its endpoints) has a unique q–geodesic representative. Any curve either has a unique q–geodesic representative or there is flat cylinder of parallel representatives. In this case, we say is a cylinder curve and we denote the cylinder of geodesics representatives of by F. We denote the Euclidean length of the q–representative of by $`_q()$. A saddle connection on $(X; q)$ is a $q$–geodesic segment which connects a pair of singular points without passing through one in its interior. We denote the Euclidean length of a saddle connection $!$ on $q$ by $`_q(!)$.

2.9. Period coordinates on the strata. In general, any saddle connection $!$ joining two zeros of a quadratic differential $q = dz^2$ determines a complex number $\mathrm{hol}_q(!)$ (after choosing a branch of $\sqrt{}$ and an orientation of $!$) by

$$\mathrm{hol}_q(!) = @ \int_! < \sqrt{} A + @ \int_! \mathrm{Im} \sqrt{} A \, i:$$

We recall that for any $= (_i; : : : ; _k; \&)$ the period coordinates gives $QT()$ the structure of an affine manifold. Consider the first relative homology group $H_1(S; ; R)$ of the pair $(S;)$ with $jj = k$: Let

$$h = (2g + k \quad 1) = \dim \; H_1(S; ; R)$$

if &= 1; and

$$h = (2g + k \quad 2) = \dim \ H_1(S; ; R) \quad 1$$

if &= 1: We recall that given $(X; q)$ 2 Q^1T () there is a triangulation T of the underlying surface by saddle connections (see for example [Vo, Proposition 3.1] and [Th2, Proposition 3.1]). One can choose h directed edges $!_1; : : : ; !_h$ of T, and an open neighborhood U_q QT () of q such that the map

$$_{T;q} : QT () \ ! \ C^h \qquad \text{defined by} \qquad _{T;q}(q) = \ \mathrm{hol}_q(!_i)\ {}^{h}_{i=1}$$

is a local homeomorphism. For any other geodesic triangulation T^0, the map $_{T^0;q}$ $^{1}_{T;q}$ is linear.

In case of abelian differentials (& = 1) it is enough to choose a basis for $H_1(S; ; R)$ from the edges of T. Note that for non-orientable differentials (& = 1) there will be a linear relation between the holonomies of the vectors corresponding to a basis for the relative homology (see §4.3). In this case, it is enough to choose $\dim(H_1(S; ; R))$ 1 independent vectors of T. For a more detailed discussion of the holonomy coordinates see [MS1].

2.10. Teichmüller geodesic flow. We recall that when $3g + p > 4$ the Teichmüller metric is not even Riemannian. However, geodesics in this metric are well understood. A quadratic differential $(X; q)$ 2 $Q^1T(S)$ with zeros at $p_1; : : : p_k$ is determined by an atlas of charts mapping open subsets of S $fp_1; : : : ; p_kg$ to R^2 such that the change of coordinates are of the form $v \ ! \ v + c$: Therefore the group $SL(2; R)$ acts naturally on $Q^1T(S)$ by acting on the corresponding atlas; given A 2 $SL(2; R)$, A q 2 $Q^1T(S)$ is determined by the new atlas fA_ig: The action of the diagonal subgroup $g_t = \begin{matrix} e^t & 0 \\ 0 & e^{\ t} \end{matrix}$ is the Teichmüller geodesic flow for the Teichmüller metric. In other words, in holonomy coordinates the Teichmüller flow is simply defined by

$$< \ \mathrm{hol}_{g_t(q)}(!_i) \ = e^t < \ \mathrm{hol}_q(!_i) \ ;$$

and

$$= \ \mathrm{hol}_{g_t q}(!_i) \ = e^{\ t} = \ \mathrm{hol}_q(!_i) \ :$$

This action descends to $Q^1M(S)$ via the projection map $p\colon Q^1T(S) \ ! \ Q^1M(S)$. We denote both actions (on $Q^1T(S)$ and $Q^1M(S)$) by g_t. The subspaces Q^1T () and Q^1M () are invariant under the Teichmüller geodesic flow. Moreover, we have ([Ve], [Ma]):

Theorem 2.4 (Veech-Masur). Each connected component C of a stratum Q^1M () carries a unique probability measure in the Lebesgue measure class such that:

- the action of $SL(2; R)$ is volume preserving and ergodic;
- Teichmüller geodesic flow is mixing.

3. Geometry of a quadratic differential

In this section, we recall some of the basic geometric properties of a quadratic differential $(X; q)$. We describe how the extremal length of a curve, which can be calculated from the conformal structure of X, relates to the singular Euclidean metric associated to $(X; q)$. We also define the notion of a $(q;)$–regular triangulation, where > 0 is a large constant. This is a partial triangulation of $(X; q)$ using the saddle connections that captures the geometry of the singular Euclidean metric associated to q. The main statement of the section is Lemma 3.13 which shows the existence of such triangulations. In the rest of the section, we establish some basic properties of $(q;)$–regular triangulations which are used in section 5.

3.1. Intersection number. In the hyperbolic metric of X, the geodesic representatives of any two curves and intersect minimally. Hence, the geometric intersection number between homotopy classes of curves is equal to the intersection number between their geodesic representatives.

In the singular Euclidean metric jqj, this is not true. First, as mentioned in 2.8, the geodesic representative might pass through the poles even though the poles are removed from the surface. Also, the q–geodesic representatives of curves and that have geometric intersection number zero may intersect. However, these intersections are tangential. That is, and may share an edge, but they do not cross. By this, we mean that any lifts ~ and to the universal cover $\tilde{q}$ of q

have end points in the boundary that do no interlock. To simplify the exposition, when we say and intersect, we always mean that they have an essential intersection not tangential.

We also talk about the intersection number between two saddle connections. Here, we say two saddle connections are disjoint if they have disjoint interiors or if they are equal. The intersection number between two saddle connections is the number of interior intersection points. The intersection number between a saddle connection and itself is zero. In both cases, (saddle connections and curves) the intersection number is denoted by i(;).

If A is an embedded annulus, we distinguish between a curve intersecting A and crossing it. To intersect A, needs only to enter the interior of A. The curve crosses A if enters one side of A and exits the other. To be more precise, in the annular cover $\tilde{X}_A$ of X associated to A, there is a lift of connecting the two boundary components of $\tilde{X}_A$.

3.2. Extremal lengths and flat lengths of simple closed curves. One can give an estimate for the extremal length of a simple closed curve in X by examining the singular Euclidean metric $|q|$. As mentioned before, may not have a unique geodesic representative; different geodesic representatives of are parallel and foliate a flat cylinder that we refer to as F. Denote the two boundary curves of F by $_E$ and $_G$. When F is degenerate, $_E = {}_G$.

We say an annulus is regular if its boundary curves are equidistant. Let E be the largest embedded regular annulus with boundary curve $_E$ and let G be the largest embedded regular annulus with boundary curve $_G$. Note that E and G may intersect F and each other. In a degenerate case, the interior of some or all of these annuli could be empty, for example, the interior of F is empty when has a unique geodesic representative.

We call $_E$, the shared boundary of E and F, the inner boundary of E (and similarly $_G$ is the inner boundary of G). The annuli E and G are called expanding because the equidistance curves parallel to the inner boundary get longer as they span E and G. Let $l = \ell_q()$ and let e; f and g be the q–distances between the boundaries of E, F and G respectively. According to [R4], when $\mathrm{Ext}_X() \; _0$, (see §1.3 for the discussion of the choice of $_0$) we have the following estimates

$$\frac{1}{\mathrm{Ext}_X()} \quad \mathrm{Mod}(E) + \mathrm{Mod}(F) + \mathrm{Mod}(G) \text{ where} \tag{5}$$

$$\mathrm{Mod}(E) \quad \mathrm{Log} \frac{e}{l} \qquad \mathrm{Mod}(F) = \frac{f}{l}; \qquad \text{and} \qquad \mathrm{Mod}(G) \quad \mathrm{Log} \frac{g}{l}: \tag{6}$$

Here Log( ) is a modified logarithm function:

$$\mathrm{Log}(t) = \max \log(t); 1 :$$

We intend Log to apply only to large numbers. Of course, the value of either e, f or g could be zero and the second line will be 1 . We use the modified logarithm to avoid this issue.

Note that, a simple closed curve that has a short flat length may not have a small extremal length. We need to measure what is the largest neighborhood of that still has a simple topology. Later, we use this idea to define a notion of extremal length for a saddle connection.

3.3. Short simple closed curves. As in §2.6, we say a curve is short in q if $\mathrm{Ext}_X() \; _0$. Denote the set of short curves in q by S_q. We say is a cylinder curve if the interior of F is not empty. In what follows, the cases when $2 S_q$ is a cylinder curve and F has a large enough modulus will need special treatments. When the modulus of F is extremely small, behaves essentially like a non-cylinder curve. We make this precise:

Definition 3.1. Let be a positive real number and let $M = e^{\;2}$. We say a curve $2 S_q$ is a large-cylinder curve if Mod(F) M. Denote the set of large-cylinder curves by S_q and define

$$S_q = S_q \, n \, S_q :$$

For $2 S_q$, the size s of F is defined to be the distance between the boundaries of F.

Remark 3.2. The constant , which is determined in §6, is the distance between steps of a ran-dom walk trajectory. We use M instead of just writing e ^{2} to highlight the fact that M is a bound for modulus. There is an implicit assumption that is large enough (say, $_0$ for some uniform constant $_0$). That is, unless otherwise stated, all statements hold with uniform constants independent of as long as $_0$.

Along Teichmüller geodesics, the length of a curve 2 S_q changes slowly while the modulus of F remains small. More precisely, let

$$(X_t; q_t) = g_t(X; q);$$

where g_t is the Teichmüller geodesic flow. Assuming 2 S_q and 0 t , we have $\mathrm{Mod}_{q_t}(F)$ 1. As a consequence of Equations (5) and (6), $\mathrm{Mod}_{q_t}(G)$ and $\mathrm{Mod}_{q_t}(E)$ change at most linearly and we have

$$(7) \qquad \frac{1}{\mathrm{Ext}_X()} \quad t \quad \frac{1}{\mathrm{Ext}_{X_t}()} \quad \frac{1}{\mathrm{Ext}_X()} + t:$$

3.4. The thick-thin decomposition of quadratic differentials. We call the components of S n S_q the thick subsurfaces of q. The homotopy class of each such subsurface Q of S has a representative with q–geodesic boundaries. There is, in fact, a unique such representative that is disjoint from the interior of cylinders associated to the boundary curves of Q. This can also be described as the smallest representative of Q with q–geodesic boundaries. We denote this subsurface by Q as well. Define the size s_Q of Q to be the q–diameter of this representative. The following theorem states that the geometry of the subsurface Q is essentially the same as that of the thick hyperbolic subsurface of X in the homotopy class of Q but scaled down to a size s_Q:

Theorem 3.3 ([R3]). For every essential closed curve in Q,

$$`_X() \quad \sqrt{\mathrm{Ext}_X()} \quad \frac{`_q()}{s_Q}:$$

In particular, the q–length of shortest essential curve in Q is on the order of s_Q.

Example 3.4. A quadratic differential can be described as a singular flat structure of a surface plus a choice of a vertical direction. For example, the surface obtained from the polygon in Fig. 2 with the given edge identifications is a once punctured genus 2 surface. Assume that the edges 2; 3; 5 and 6 have a comparable lengths, the edge 1 is significantly shorter and the edge 4 is significantly longer than the others. Choose an arbitrary vertical direction and let (X; q) be the associated quadratic differential.

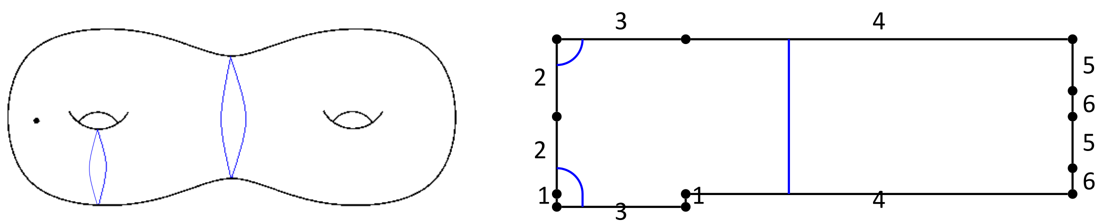

Figure 2. Quadratic differential (X; q) and short curves of X.

Then the hyperbolic metric on X has two short simple closed curves; S_q = f; g. The curve is a cylinder curve and has a small extremal length because the flat annulus F (Fig. 3) has a large modulus. In fact, is a large-cylinder curve (S_q = fg). The curve is a non-cylinder curve and it has a small extremal length because the expanding annuli E and G (Fig. 3) have large moduli (S_q = fg). Note that the q–geodesic representative of is the saddle connection 1 (the end points of arc 1 are identified).

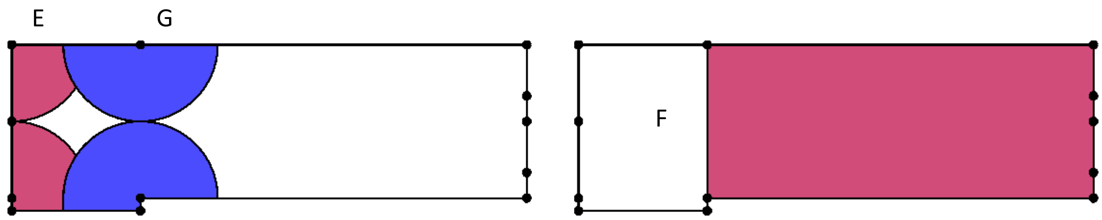

Figure 3. The maximal expanding annuli E and G and the maximal flat annulus F.

There are two thick subsurfaces. There is a once punctured torus with a boundary curve whose q–representative is degenerate and is represented in q with a graph of area zero (the union of arcs 5 and 6). The other is a pair of pants whose boundaries consist of two copies of and one copy of . The maximal expanding annuli E and G do not necessarily stay inside of the q–representative of this pair of pants and they may overlap.

The size of a thick subsurface Q is related to the radii of annuli E, F and G for every boundary curve . We make a few observations that will be useful later.

Lemma 3.5. Let Q be a thick subsurface of $(X; q)$, be a boundary component of Q and E be the expanding annulus in the direction of Q. Using the notation of Equation (5) we have

(1) l 2s_Q.
(2) e s_Q.
(3) max(e; f; g) $\ell_q()$. (4)
(e + l) s_Q.
(5) If Mod(E) 1 then e s_Q.

Proof. Since is part of Q, its length is less than twice the diameter of Q which is the first assertion. To see part two, note that if e is larger than s_Q, then Q is contained in E which is an annulus. This is a contradiction. Part (3) follows from Equation (6) and the fact that is $_0$–short. Parts (1) and (2) imply (e + l) s_Q. Hence, to prove part (4), we need to show (e + l) s_Q.

Since E is maximal, its outer boundary self-intersects. Let be the curve constructed as a concatenation a sub arc of and two arcs connecting to the boundary points of E associated to the self intersection of E. Note that the inner boundary of E is a geodesic and its outer boundary has positive curvature, therefore, the interior of E is convex, and the curve must be essential.

Then l + e $\ell_q()$. If is contained in Q and is essential in Q, then $\mathrm{Ext}_X()$ 1 (Q is a thick subsurface). From Theorem 3.3 we get,

$$\frac{\ell_q()}{s}\,1 \quad \text{and hence} \quad (e + l)\ s_Q: {}^{Q}$$

If is not contained in Q, we show that there exists a closed curve ^{0} in Q whose length is not much longer than .

Assume that exists Q by intersecting a boundary curve ^{0} and returns via a boundary curve ^{00} (^{0} and ^{00} maybe the same curve). By part (3), max(e^0; f^0; g^0) is larger than l^0, max(e^{00}; f^{00}; g^{00}) is larger than l^{00} and $\ell_q()$ is larger than both. There is a sub-arc ! of connecting ^{0} to ^{00}, in particular, $\ell_q(!)$ $\ell_q()$. If ^{0} = ^{00}, let ^{0} be the curve obtained as a concatenation of two copies of ! and a copy of ^{0} and ^{00} each. This curve is essential in Q unless Q is a pair of pants, in which case, we take ^{0} to be the curve that wraps around ^{0} twice. If ^{0} = ^{00}, then let ^{0} be the curve obtained as a concatenation of ! and a sub-arc of ^{0}. Again, this curve is essential in Q unless Q is a pair of pants, in which case, we take ^{0} to be the curve that wraps around ^{0} twice. The curve ^{0} resides in Q and $\ell_q(^0)$ $\ell_q()$. We have

$$(e + l)\ \ell_q()\ \ell_q(^0)\ s_Q:$$

To see part (5), we note that, if

$$\mathrm{Log}\,\frac{e}{l}\quad \mathrm{Mod}(E)\quad 1 \quad \text{then} \quad e\quad (e+l):\ \text{Now,}$$

part (5) follows from part (4).

As a corollary we get the following analogue of the collar lemma:

Corollary 3.6. Let 2 S_q be the boundary of a thick subsurface Q and let be any curve crossing . Then

$$`_q() \quad s_Q:$$

Proof. We have $`_q()$ max(e; f; g) and by part (3) of Lemma 3.5, max(e; f; g) l. Hence, $`_q()$ (e + l). The corollary now follows from part (4) of Lemma 3.5.

3.5. Extremal lengths and flat lengths of saddle connections. As mentioned above, we can also define a notion of extremal length for saddle connections. Let ! be a saddle connection connecting two distinct critical points in (X; q). Let $E_!$ be the annulus obtained by taking the largest regular neighborhood of ! that is still a topological disk and then cutting a slit open along ! . Let l = $`_q(!)$ and e be the radius of $E_!$ (the q–distance between ! and the boundary of $E_!$). Then, we define (the second inequality follows from Equation (6))

$$\mathrm{Ext}_q(!) := \frac{1}{\mathrm{Log}(e=l)} \quad \frac{1}{\mathrm{Mod}(E_!)}:$$

Another interpretation of this notion of extremal length, that would provide roughly the same result, is to compute the extremal length in a ramified double cover of (X; q). Denote the end points of ! by p_1 to p_2. There exists a unique ramified double cover : $X_!$! X with simple ramification points at only p_1 and p_2. Note that $_!$ = ^{1}! is a simple closed curve on $X_!$.

Lemma 3.7. If $\mathrm{Ext}_q(!)$ $_0$, then

$$\mathrm{Ext}_{X_!}(_!) \quad \mathrm{Ext}_q(!):$$

Proof. Let $q_!$ be the lift of q to $X_!$. Note that $_!$ has a unique geodesic representative in $q_!$ ($\mathrm{Mod}(F_!)$ = 0) and $E_!$ and $G_!$ are conformally equivalent to $E_!$. Hence, by Equation (5)

$$\frac{1}{\mathrm{Ext}_{X_!}(_!)}\ \mathrm{Mod}(E_!) + \mathrm{Mod}(G_!) = 2\,\mathrm{Mod}(E_!) \quad \frac{!}{\mathrm{Ext}_q()}\frac{1}{}:$$

Since l and e change at most exponentially fast along a Teichmüller geodesic, similar to Equation (7), for $q_t = g_t(q)$ we have

$$\frac{1}{\mathrm{Ext}_q(!)} \quad t \quad \frac{1}{\mathrm{Ext}_{q_t}(!)} \quad \frac{1}{\mathrm{Ext}_q(!)} + t: \tag{8}$$

Definition 3.8. For any 0 < $_0$, let
$_q()$ be the set of saddle connections ! of q so that, either

- $\mathrm{Ext}_q(!)$, or
- ! lies on a geodesic representative for with $\mathrm{Ext}_X()$.

Later in the text, we will add further restrictions on the value of depending on (see Lemma 3.13 and Lemma 6.1). We note however that, in all the proofs, making smaller or making larger does not effect the constants involved in any of our estimates.

In general, knowing $`_q(!)$ is small does not imply that ! has a small extremal length. However, we have the following lemma which is enough to show that Theorem 1.1 follows from Theorem 1.5.

Lemma 3.9. Assume that (X; q) has a saddle connection ! with $`_q(!)$ 1. Then, either

$$\frac{1}{\mathrm{Ext}_q(!)}\ \mathrm{Log}\,\frac{1}{`_q(!)} \quad \text{or} \quad \frac{1}{\mathrm{Ext}_X()}\mathrm{Log}\,\frac{1}{`_q(!)}, \text{ for}$$

some simple closed curve . In particular,
$_q()$ is non-empty.

Proof. Let $l = `_q(!)$ and e be the radius of $E_!$. Since the boundary of $E_!$ self intersects ($E_!$ is maximal), there is a simple closed curve , obtained by a concatenation of a sub arc of ! and two arcs connecting ! to the boundary of $E_!$, with $`_q()$ (e + l).

Assume first that S_q is empty. Then, $`_q()$ 1. Since, e 1, we have

$$\frac{e}{l} \quad \frac{(e+l)}{l} \quad \frac{1}{`_q(!)} \qquad \text{and} \qquad \frac{1}{\mathrm{Ext}_q(!)} \quad \mathrm{Log}\ \frac{e}{l}\ \mathrm{Log}\ \frac{1}{`_q(!)}:$$

That is, the first inequality holds. Otherwise, we show that, there is a curve $_1$ 2 S_q with $`_q(_1)$ (e + l). This is because, either 2 S_q and we can take $_1 =$ or intersects a thick subsurface Q in which case we let $_1$ be any boundary component of Q. Using Corollary 3.6 and part one of Lemma 3.5, we get:

$$(e+l)\ `_q()\ s_Q\ `_q(_1):$$

Since the total area of q is 1, there is always a thick subsurface of size comparable to 1. Let $Q_1;:::;Q_k$ be a sequence of distinct subsurfaces of sizes $s_1;:::;s_k$ respectively, where $_1$ is a boundary component of Q_1, $Q_{i\ 1}$ and Q_i share a boundary curve $_i$ and s_k 1. Let $l_i = `_q(_i)$ and let $s_0 = l_1$.

Consider G_i, the expanding annulus with inner boundary $_i$ in the direction of Q_i with radius g_i. For i 1, part (4) of Lemma 3.5 implies, $(g_i + l_i)$ s_i and by part (1) $s_{i\ 1}$ l_i. Hence, from Equation (5), we know that

$$\frac{1}{\mathrm{Ext}_q(_i)}\ \max \mathrm{Log}\ \frac{g_i}{l_i};1 \quad \mathrm{Log}\ \frac{g_i+l_i}{l_i} \quad \mathrm{Log}\ \frac{s_i}{s_{i\ 1}}:$$

That is, the common boundary curve of two surfaces of very different size has a very small extremal length. Also, (recall that $s_0 = l_1$ (e + l)):

$$\prod_{i=1}^{k} \frac{s_i}{s_{i\ 1}} \quad \frac{e+l}{l} \quad \frac{1}{s_0} \quad \frac{e+l}{l} \quad \frac{1}{l}: \tag{9}$$

Here, the maximum value of k depends only on the topology of S. Therefore, taking the logarithm of both sides of Equation (9), we conclude that either

$$\text{there is some i where,} \quad \frac{1}{\mathrm{Ext}_q(_i)}\ \mathrm{Log}\ \frac{1}{l} \quad \text{or} \quad \mathrm{Log}\ \frac{e+l}{l}\ \ \mathrm{Log}\ \frac{1}{l}:$$

In the first case, the lemma holds for $= _i$. In second case,

$$\frac{1}{\mathrm{Ext}_q(!)} \quad \mathrm{Log}\ \frac{e}{l}\ \mathrm{Log}\ \frac{e}{l} + \mathrm{Log}\ \frac{1}{`_q(!)}:$$

Remark 3.10. Note that in both Lemma 3.7 and Lemma 3.9 the implied constants only depend on the topology of S:

3.6. A (q;)–regular triangulation. We would like to mark a quadratic differential q by a triangulation where the edges have a bounded length. However, the notion of having a bounded length should depend on which thick subsurface we are in. That is, we would like the q–length of an edge to not be longer than the size of the thick subsurfaces it intersects. The complication comes from the fact that a saddle connection may intersect several thick subsurfaces of various sizes.

Also, as mentioned before, large-cylinder curves will require a special treatment. Hence, we triangulate only the complement of large-cylinders. Recall that two saddle connections are said to be disjoint if they have disjoint interiors but they may share one or two end points.

Definition 3.11. Let (X; q) be a quadratic differential. Given a cylinder curve , let be an arc connecting the boundaries of F that is perpendicular to . By a (q;)–regular triangulation T of q we mean a collection of disjoint saddle connections satisfying the following conditions:

(1) For 2 S , denote the interior of a cylinder F by F . Then, T is disjoint from F and it triangulates their complement

$$q \setminus F : \bigcup 2 S_q$$

That is, the complement of T is a union of triangles and large-cylinders F , 2 S . In particular, T contains the boundaries of F.

(2) If an edge ! of T intersects a thick subsurface Q of q then $\ell_q(!)$ s_Q.

(3) If is a cylinder curve in S_q then intersects T a uniformly bounded number of times.

We shall see that condition 3 means that the triangulation T does not twist around short simple closed curves.

Remark 3.12. It is important to choose the implied constants in conditions 2 and 3 in Definition 3.11 large enough so that every quadratic differential q has a (q;)–regular triangulation. In fact, we choose the constants so that the key Lemma 3.13 below holds.

Lemma 3.13. For every there is $_1()$ so that for < $_1()$ the following holds. Let be a subset of $_q()$ consisting of pairwise disjoint saddle connections. Then can be extended to a (q;)–regular triangulation T .

Proof. We would like to triangulate each thick piece Q separately and let T be the union of these triangulations. However, saddle connections in may intersect a boundary curve of Q. To remedy this, we perturb slightly to a curve that is a union of saddle connections, lies in a small neighborhood of and is disjoint from (see Claim 1). These curves divide the surface into subsurfaces with nearly geodesic boundaries. We denote the surface associated to Q with Q. We then extend to a triangulation in each Q so that the edge lengths are not much longer than the diameter of Q which is comparable to s_Q (see Claim 3) and let T be the union of these triangulations. However, one needs to be careful that Q does not intersect any subsurface of size much smaller that s_Q, otherwise the resulting triangulation would not be (q;)–regular.

Claim 1: For every 2 S_q, there is a representative of that is a union of saddle connections, lies in a ($\ell_q()=2$)–neighborhood of and is disjoint from . For ; 2 S_q, and do not intersect. Furthermore, if is a boundary of Q then intersects only surfaces that are larger than Q, namely, if intersects a thick subsurface Q^0 we have:

$$s_{Q^0} \quad s_Q :$$

Proof of Claim 1: Let 2 S_q be a common boundary of thick subsurfaces Q and R. Recall that M = 2 ^{2} . If Mod(F) M , we can choose $_1$ small enough to ensure that is disjoint from . This is because, if ! is part of a short curve ^{0}, then ! is disjoint from because short curves and ^{0} do not intersect. Otherwise, ! has to satisfy the first assumption in Definition 3.8. But, F does not contain any singular points and any arc ! 2 intersecting has to cross F. Therefore, $\ell_q(!)$ f (f is the distance between the boundaries of F) and, for the radius $e_!$ of $E_!$, we have $e_!$ $\ell_q()$ (otherwise would be contained in $E_!$). But Mod(F) = $\frac{f}{\ell_q()}$ M and thus (the second inequality follows from Equation (6))

$$\frac{1}{_1} \quad \frac{1}{\mathrm{Ext}_q(!)} \quad \mathrm{Log}\frac{\ell_q()}{!} \quad \mathrm{Log}\frac{e_!}{f} \quad \mathrm{Log}\frac{\ell_q()}{M} = 2 : \quad \frac{1}{}$$

which is not possible if $_1$ is chosen to be small enough. To summarize, if Mod(F) M, then is already disjoint from , we can take = .

If Mod(F) M, then either E or G has a large modulus. The annulus with the larger modulus is in the direction of the thick surface with the larger size (Lemma 3.5). Assume E, the annulus in the direction of Q, has a large modulus. Let e be the distance between the boundaries of E. By part (5) of Lemma 3.5 and the previous assumption we have

$$e \quad s_Q \quad s_R :$$

Denote the ($\ell_q()/2$)–neighborhood of in E with E . The annulus E may not be contained entirely in Q and may intersect some thick subsurfaces with very small size. But E does not intersect any small subsurfaces. To see this, assume Q^0 intersects E . Since Q^0 is disjoint from , it has to enter E intersecting the outer boundary of E . But e is much larger than $\ell_q()$, and hence:

$$s_{Q^0} \quad \ell_q(@Q^0) > e \quad \ell_q()/2 \quad s_Q:$$

Thus, the last condition of the claim is satisfied as long as stays in E .

Note that no arc in can cross E (intersect both boundaries). This is because, if ! is an arc in a curve 2 S_q, then it does not intersect since and have intersection number zero. Otherwise, $\mathrm{Ext}_q(!)$ is small, which implies that its length is much less than the injectivity radius of any point along !. But the injectivity radius of any point in E is less that $2\ell_q()$. Hence, (by choosing small enough) $\ell_q(!)$ is less than the distance between the boundaries of E with is equal to $\ell_q()/2$.

Consider the union of and the set of arcs in that intersect . The convex hull H of this set in E is an annulus (perhaps degenerate). We observe that the interior of H does not contain any singular points. Otherwise, there would be a geodesic quadrilateral, where two edges are subsegments of arcs in and one edge is a subsegment of , that contains a singular point in its interior. But this violates the Gauss-Bonnet theorem. Let be the boundary component of H that is not . Then is in the homotopy class of and lies inside E . Also, because the interior H does not contain any singular points, is disjoint from every saddle connection in . Furthermore, by the triangle inequality, any saddle connection ! that appears in has a q–length less than or equal to $2\ell_q()$.

It remains to show that for ; 2 S_q, and are disjoint. Assume $\ell_q() \quad \ell_q()$. Then, is disjoint from E , otherwise, would be contained in E which is an annulus an does not contain any curve non-homotopic to . This means is disjoint from which is contained in E . Also, since H contains no singular points, if a saddle connection ! 2 intersects then it also intersects . But then ! is in and hence it is disjoint from . Therefore, is disjoint from the convex hull H and thus also from . This finishes the proof of claim 1.

Next, let be the set of edges that appear in curves for every 2 S . We have shown that saddle connections in are disjoint from those in . After removing the interiors of large cylinders from the quadratic differential (X; q) and cutting along curves , 2 S_q, we obtains a collection of subsurfaces with nearly geodesic boundaries. Denote the representative of a thick subsurface Q that is disjoint from curves by Q.

For each 2 S_q , if F is disjoint from every saddle connection in [, we choose a saddle connection ! that crosses F, is disjoint from (does not twists around). In particular, ! is disjoint from every saddle connection [and has a length that is comparable with $\ell_q()$. Let ^{n} denote the set of such saddle connections !.

Claim 2: Saddle connections in

$$T_0 = \ [\ [\ ^n$$

satisfying conditions (2-3) of Definition 3.11.

Proof of Claim 2: All the conditions follow immediately from the construction, but the argument is long since we have to look at all the cases. We have already shown that these edges satisfy condition (1) and arcs in satisfy condition (2). To see that an arc ! 2 satisfies condition (2) note that if it did not, ! would intersect a thick subsurface Q with $\ell_q(!) \quad s_Q$. The radius of $E_!$ is much larger than length of ! ($\log \frac{e_!}{} \quad \frac{1}{1}$), which implies $E_!$ contains Q. This is a contradiction.

We show that arcs in T_0 satisfy condition (3). Namely, if ! 2

intersects a cylinder F, we need to show that ! intersects a bounded number of times. In fact, if they intersect more than once, then ‘$_q$(!) ‘$_q$(). But then E$_!$ would contain the curve which is a contradiction (E$_!$ is a topological disk). Also, the curve is a convex hull of the union of the curve which is disjoint

from F and a bounded number of arcs in —
, each of which intersect at most once. Hence intersects at most a bounded number of times and thus arcs in satisfy condition (3).

Since, for every 2 S_q , there is a saddle connection in T_0 crossing F, any triangulation containing T_0 is guaranteed to satisfy the condition (3).

In the next claim, we describe how to add the remaining edges to T_0 while still satisfying conditions (1) and (2).

Claim 3: A partial triangulation of $\overline{Q}$ where the length of edges are less than a fixed multiple of s_Q can be extended to a triangulation using saddle connections of length less than a larger fixed multiple of s_Q.

Proof of claim 3: We prove the claim by induction. Start by cutting $\overline{Q}$ along the given edges. Each cutting increases the diameter by at most twice the length of edge being cut. Hence, in the end, we have several components each with diameter comparable to s_Q. If all components are triangles, we are done. Otherwise, some component contains a saddle connection that is not part of its boundaries or the given triangulation, the shortest such saddle connection has a length less than the diameter of the component it is in, which is comparable to s_Q (again, see [Vo, Proposition 3.1]). The claim follows from the fact that this process ends after a uniformly bounded number of times. The diameter grows at most multiplicatively each time but still it is uniformly bounded multiple of s_Q in the end. We choose the constant in the second condition of a (q;)–regular triangulation large enough so that the outcome of this algorithm is in fact a (q;)–regular triangulation.

The triangulation T is now defined to be the union of all the saddle connections in T_0 and those coming from claim 3. The newly added edges in $\overline{Q}$ have a q–length less than a fixed multiple of s_Q and, for any thick subsurface R that $\overline{Q}$ intersects, we have s_Q s_R. Hence, the condition (2) in Definition 3.11 is satisfied. Therefore, the resulting triangulation T is (q;)–regular.

3.7. Twisting and extremal lengths. In this section we define several notions of twisting and discuss how they relate to each other. This is essentially the definition introduced by Minsky extended to a slightly more general setting. We denote the relative twisting of two objects or structures around a curve by twist (;). This is often only coarsely defined, that is, the value of twist(;) is determined up to a uniformly bounded additive error.

In the simplest case, let A be an annulus with core curve and let and be homotopy classes of arcs connecting the boundaries of A (here, homotopy is relative to the end points of an arc). The relative twisting of and around , twist(;), is defined to be the geometric intersection number between and .

Now consider a more general case where is a curve on the surface S and and are two transverse curves to . Let $\tilde{S}$ be the annular cover of S associated to and denote the core curve of $\tilde{S}$ again by . Let and ~ be the lifts of and to $\tilde{S}$ (respectively) that connect the boundaries of $\tilde{S}$. Note that freely homotopic curves lift to arcs that are homotopic relative their endpoints. The arc is not uniquely defined, however any pair of lifts are disjoint. We now define

$$\text{twist}(;) = \text{twist}(;\sim); \quad \sim$$

using the previous case. This is well defined up to an additive error of 2 (see [Mi]).

We can generalize this further and define twisting between any two structures on S as long as the structures in question provide a (nearly) canonical choice of a homotopy class of an arc connecting the boundaries of $\tilde{S}$. Then we say the given structure defines a notion of zero twisting around . The relative twisting between two structures is the relative twisting between the associated arcs in $\tilde{S}$. Here are a few examples:

Let X be a Riemann surface. Then ~can be taken to be the geodesic in $\tilde{X}$ that is perpendicular to in the Poincare metric of $\tilde{X}$. Alternatively, we can pick a shortest curve transverse to and let be the lift of that connects the boundaries of $\tilde{X}$. In any case, the choice of is not unique, but any two such transverse arcs have bounded geometric intersection number (see [Mi]) and the associated relative twisting twist$(;X)$ is well defined up to an additive error.

Let q be a quadratic differential. As before, can be taken to be the geodesic in $\tilde{q}$ that is perpendicular to in the Euclidean metric coming from q or a lift of a q–shortest curve transverse to (see [CSR]). We denote the associated relative twisting with twist$(;q)$.

Let T be a $(q;)$–regular triangulation of $(X;q)$ and $2 S$. Then we can choose a curve transverse to that is carried by T and has a bounded combinatorial length in T and let the lift of to the annular cover of define zero twisting. Since curves with bounded combinatorial length intersect a bounded number of times, the associated relative twisting twist$(;T)$, is again well defined up to an additive error.

The expression "fix a notion of zero twisting around " for a curve in S means "choose a homotopy class of arcs connecting the boundaries of $\tilde{S}$."

3.8. Intersection and twisting estimates. In this section we establish some statements relating Extremal length, twisting and intersection number. We start with a theorem of Minsky giving an estimate for the extremal length of a curve. For a $X\ 2\ T(S)$, let S_X be a set of $_0$–short simple closed curves in X. There is a uniform constant B depending on $_0$ and the topology of S so that, for every X, any curve not in S_X intersects a curve with $\mathrm{Ext}_X()\ B$. That is, the curves with extremal length at most B fill every complementary component of S_X. Let B_X be the set of curves with extremal length at most B.

Theorem 3.14. (Minsky, [Mi, Theorem 5.1]) Given $X\ 2\ T(S)$ and a simple closed curve $2\ S_X$

$$\mathrm{Ext}_X() \quad \max_{2 S_X} i(;)^2 \quad \frac{1}{\mathrm{Ext}_X()} + \mathrm{twist}^2(;X)\,\mathrm{Ext}_X() + \max_{2 B_X} i(;)^2: \tag{10}$$

The multiplicative constant depends only on the topology of S.

It follows from the definition of twisting and elementary hyperbolic geometry that if twist$(;X)$ is large (that is, if twists around a lot), then $\mathrm{Ext}_X()\ \mathrm{Ext}_X()$.

Corollary 3.15. For every curve and any $X\ 2\ T(S)$, there is a curve so that,

$$\sqrt{\mathrm{Ext}_X()\,\mathrm{Ext}_X()} \quad i(;) \qquad \text{and} \qquad \mathrm{twist}(X;) = O(1):$$

Note that the reverse of first inequality always holds (Equation (2)).

Proof. If $2\ S_X$, then we choose to be a curve that intersects once or twice, is disjoint from other curves in S_X, where twist$(;X)$ is bounded and where $i(;) = O(1)$ for $2\ B_X$. Applying Equation (10) to we have $\mathrm{Ext}_X()\ \frac{1}{\mathrm{Ext}_X()}$ which implies that the corollary holds for and . If is not short in X, Theorem 3.14 applies to . Since the number of elements in S_X and B_X is uniformly bounded, $\mathrm{Ext}_X()$ is comparable to one the following terms:

$$\frac{i(;)^2}{\mathrm{Ext}_X()} \qquad i(;)^2\,\mathrm{twist}(;X)^2\,\mathrm{Ext}_X() \quad \text{or} \qquad i(;)^2:$$

In the fist two cases $2\ S_X$ and in the third case $2\ B_X$. We argue in 3 cases.

If $\mathrm{Ext}_X()\ \frac{i(;)^2}{\mathrm{Ext}_X()}$ for $2\ S_X$, then the corollary holds for $=$ (the second conclusion follows from the fact that the twisting number of a short curve around a long curve is uniformly bounded).

In the second case, we take to be a curve transverse to with (see above) $\mathrm{Ext}_X()\ \frac{1}{\mathrm{Ext}_X()}$ and twist$(;X) = O(1)$. In particular

$$\mathrm{twist}(;X)\ \ \mathrm{twist}(;): \tag{11}$$

The curve also intersects and hence $\mathrm{Ext}_X()\ \frac{1}{\mathrm{Ext}_X()}\ \mathrm{Ext}_X()$. Thus, twist around at most a uniformly bounded number of times. Also, every strand of intersecting intersects at least twist(;) times (up to an additive error). In this case twist(;) is large and the additive error can be replaced by a multiplicative error to obtain

(12) $\quad i(;)\,\mathrm{twist}(;)\ \ i(;):$

Therefore,

(Assumption on) $\quad \mathrm{Ext}_X()\ \ i(;)^2\,\mathrm{twist}(;X)^2\,\mathrm{Ext}_X()$

(Equation (11)) $\quad \dfrac{i(;)}{}\ \dfrac{\mathrm{twist}(;)^2}{}\ \dfrac{\mathrm{Ext}_X()}{}$

(Equation (12)) $\quad \dfrac{i(;)^2}{\mathrm{Ext}_X()};$

which implies the corollary.

The last case is when $\ 2\ B_X$ and $\mathrm{Ext}_X()\ \ i(;)^2$. In this case, we take $=$. Since has bounded length in X,

$$\mathrm{twist}(;X) = O(1) \qquad \text{and} \qquad \mathrm{Ext}_X()\ \ 1:$$

Again, the corollary follows.

We also recall the following lemma ([R2, Theorem 4.3]):

Lemma 3.16 (Rafi). For a quadratic differential $(X;q)$ and a Riemann surface $Y\ 2\ T(S)$ with $d_T(X;Y) = O(1)$, we have

$$\mathrm{twist}(Y;q)\ \ \frac{1}{\mathrm{Ext}_X()}:$$

3.9. Geometry of quadratic differentials and (q;)–regular triangulations. As we mentioned at the beginning of the section, a (q;)–regular triangulation is supposed to capture the geometry of q. We make this explicit in the following two lemmas. In Lemma 3.17, we relate the length of a saddle connection to its intersection number with a (q;)–regular triangulation. Lemma 3.18 shows that the notion of zero twisting coming from q or T is the same. These are used to prove Lemma 3.19 but more essentially they are needed in §4.

Lemma 3.17. Let T be a (q;)–regular triangulation and $!_T$ be an edge of T. Let s be the minimum of s_Q where Q is a thick subsurface of q that intersects $!_T$. Let $!$ be any other saddle connection in q so that, for every curve $\ 2\ S_q$, $\mathrm{twist}(!;q) = O(1)$. Then

$$i(!_T;!)\ \ \frac{`_q(!)}{()s} + 1:$$

Proof. Condition (2) in the definition of a (q;)–regular triangulation implies that $`_q(!_T)\ \ s$. It is sufficient to prove the lemma for a subsegment of $!_T$ with a q–length less than s=7, because $!_T$ can be covered but uniformly bounded number of such segments. Hence, without loss of generality, we assume $`_q(!_T)\ \ s=7$.

Consider the s=7–neighborhood N of $!_T$. Then $! \setminus N$ has at most $O\ \frac{`_q(!)}{s}$ components. Hence, it is sufficient to show, for every component $!$ of $! \setminus N$, that

$$i(!_T;!) = O(1):$$

First, we claim that any non-trivial curve in N is homotopic to some curve in S_q. This is because, any nontrivial loop in N has a q–length of at most 3s=7. By the definition of s, it can not be an essential curve in any subsurface Q that $!_T$ intersects. Assume it intersects curves $_1;_2\ 2$ S_q that are boundary curves of Q_1 ($_1$ may equal $_2$). Then, $`_q(_1)$ and $`_q(_1)$ are much smaller than $`_q()$ which is at most 3s=7. But, the sum of $`_q(_1)$, $`_q(_2)$ and twice the distance between $_1$ and $_2$ (the sum is less than s) is an upper-bound for the size of Q which is assumed to be larger than s. The contradiction proves the claim.

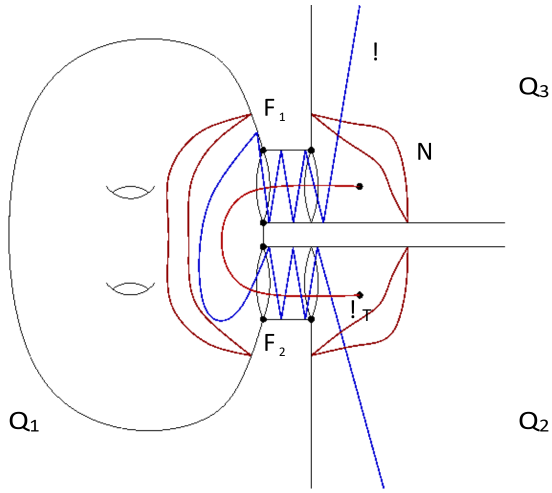

Figure 4. The arc $!_T$ intersects curves $_1; _2$ 2 S_q and thick subsurfaces $Q_1; Q_2$ and Q_3. Each component !of ! \ N intersects $!_T$ only a bounded num-ber of times outside of cylinders F_1 and F_2. The number of intersection points inside of F_i is bounded because of the assumption on the twisting.

We have shown that a closed curve in N cannon intersect curves in S_q. However, the saddle connection $!_T$ may still intersects some curve 2 S_q (in fact more than one, see Fig. 4). As before, let be an arc in F that connects the boundaries of F and is perpendicular to them.

First we observe that the number of intersection points between $!_T$ and ! inside of F is uniformly bounded. This is because both $!_T$ and !intersect a uniformly bounded number of times. (This follows from the definition $(q;)$–regular triangulation and the twisting assumption on !.) If two arcs inside of a cylinder have a large intersection number, at least one of them has to twist around F a large number of times.

It remains to show that the number of intersection points outside of all cylinders F is bounded. To see this we observe that, for any thick subsurface Q, it is not possible to have a subsegment of $!_T$ and a subsegment of ! that are contained in Q and have the same endpoint. Otherwise, the concatenation would create a two segment curve that is non-trivial in N. Hence, it has to be homotopic to some curve 2 S_q. Which means, and create a cylinder with total negative curvature which contradicts the Gauss-Bonnet theorem. (See [CSR, Lemma 5.6] for a more detailed discussion.)

Since the number of thick components Q is uniformly bounded and $!_T$ and ! can intersect at most once in each Q we conclude that the total intersection number outside of cylinders F is uniformly bounded as well. This finishes the proof.

Lemma 3.18. For a quadratic differential $(X; q)$, 2 S_q and a $(q;)$–regular triangulation T we have

$$\mathrm{twist}(T; q) = O(1):$$

Proof. Let Q_1 and Q_2 be the thick subsurfaces of $(X; q)$ glued along the cylinder F (which by assumption, has a modulus at most M), and let be an essential curve in Q_1 [F [Q_2 that is transverse to and has the shortest combinatorial T–length. A representative for the curve can be constructed using edges of T that intersect either Q_1 or Q_2. Consider such a representative traversing the minimum possible number of edges. Let be a curve transverse to with the shortest q–length. From the definition of relative twisting,

$$\mathrm{twist}(T; q) \ i(;): \text{Hence, it}$$

is sufficient to show that $i(;)$ is uniformly bounded.

The curve intersects once if $Q_1 = Q_2$ and twice otherwise. Its restriction to Q_i has a length bounded by $O(s_{Q_i})$ and its restriction to F has a length bounded by $`_q()$ (Mod(F) is bounded and there is no twisting around) which is less than both s_{Q_1} and s_{Q_2}. An argument similar to that of Lemma 3.17 implies that intersects any edge of T at most a bounded number of times.

On the other hand, each edge of T appears at most twice along the representative of , otherwise a surgery argument would reduce the length of . Also, the total number of edges of T is bounded by the topology of S. Hence, i(;) is uniformly bounded.

3.10. The number of (q;)–regular triangulation. We now count the number of (q;)–regular triangulations near a point in Teichmüller space. We can think of a (q;)–regular triangulations on (X; q) as topological objects on S, after being pulled back by the marking map $f_X : S\ !\ X$, up to homotopy. That is, we say a (q;)–regular triangulation T on (X; q) is equivalent to a q^0–regular triangulation T^0 on $(X^0; q^0)$ if the pre images $f_X^{\ 1}(T)$ and $f_{X^0}^{\ 1}(T^0)$ are homotopic on S. The homotopy does not have to fix the vertices of T. For a multi-curve S_0, we say T is equivalent to T^0 up twisting around S_0 if, T is equivalent to (T^0) where is a multi-twist with support on curves in S_0.

Lemma 3.19. Let U be a ball of radius one in T(S) centered at X_0. Then the number of equivalence classes, up to twisting around S_{X_0}, of (q;)–regular triangulations T on a quadratic differential (X; q) where X 2 U is uniformly bounded.

Proof. We start with a topological counting statement. Let $S_0 = S_0^c\ [\ S_0^n$ be a system of curves on S. For every subsurface Q in S n S_0, let $_Q$ be a marking for the subsurface Q in the sense of [MM]. That is, $_Q$ is a pants decomposition $f_1; : : : ; _k g$ for Q together with a transverse curve for 0 i k. Each is contained in Q, intersects $_i$ once or twice and is disjoint from $_j$, j = i. Also, for 2 S^n, let be a curve transverse to that is disjoint from all other curves in S_0 and $i(; _Q) = O(1)$. Define

$$M = \bigcup_0 {}_Q\ [\ S_0\ [\ fg_{2S^n : Q}\ _0$$

Claim: Given a set M as above, there is a uniformly bounded number of possibilities for the homotopy class of a triangulation T, triangulating S n S_0^c, where the curves in M and T have representatives with the following properties:

(1) curves in M have no self intersections and intersect each other minimally.
(2) for any 2 S_0^c, i(T;) = 0.
(3) for any 2 $_Q$, i(T;) = O(1).
(4) for 2 S_0^n, twist(T;) = O(1), and i(T;) = O(1).

To see the claim, note that the curves in M divide S into a uniformly bounded number of complementary regions, each one is either a polygon or an annulus parallel to a curve 2 S_0^c. Choose a representative of the homotopy class of T that intersects curves in M minimally. There are a uniformly bounded number of possibilities for the location of vertices of T. Once the vertices of T are fixed, there are a uniformly bounded number of possibilities for any given arc, with end points on these vertices, that can appear as an edge of T. This is because there are a uniformly bounded number of possibilities for the intersection pattern of the given arc with the complementary regions. Also, each region is either a polygon where there is a unique arc (up to homotopy) connecting any two edges (or a vertex to an edge) or an annulus neighborhood of a curve 2 S_0^c where there are two possibilities (edges of T are simple and disjoint from curves in S_0^c).

It remains to show, that for every (q;)–regular triangulation T_q on (X; q) where X 2 U, there is a set of simple closed curves M_q so that T_q and M_q satisfy the above properties and then to bound the number of possibilities for the set M_q.

Let (X; q) be a quadratic differential so that X 2 U. We construct M_q as follows: The curves $S_q = S_q\ [\ S_q$ have a uniformly bounded length in X_0 hence there are a uniformly bounded number of possibilities for these sets. For each thick subsurface Q of q, choose a q–short marking $_Q$ in Q. Curves in $_Q$ have a uniformly bounded length on X and hence a uniformly bounded length in X_0. Hence there are only a uniformly bounded number of choices for these as well. Now for each , let ^{q} be the shortest q transverse curve to . Lemma 3.16 implies that twist$(X_0; {}^q)$ $\frac{1}{\mathrm{Ext}_{X_0}()}$. ~~Hence~~ the number of possible choices for ^{q} is of the order of $\frac{1}{\mathrm{Ext}_{X_0}()}$. ~~Define~~

$$M_q = \bigcup_{Q} [\; S_q \; [\; f^q g^{2S} : Q_q$$

Bu construction, the total number of possible sets M_q chosen as above is of the order of $G(X_0)$. However, up to twisting around S_{X_0} there are only finitely many choices. For a $(q;)$–regular triangulation T_q in $(X; q)$, we need to check that the conditions (1)-(4) hold for T_q and M_q. Perturb the q–geodesic representative of curves in M_q so that they have no self-intersections, intersect each other minimally and the intersection number with T does not increase. Condition (2) follows from the construction of $(q;)$–regular triangulations. Condition (3) follows from Lemma 3.17. The first part of condition (4) is a consequence of Lemma 3.18 and the second part again follows from Lemma 3.17.

4. Intersection bounds between regular triangulations

As before, let $Q()$ be the stratum of quadratic differentials of type . In this section, we establish some intersection bounds for $(q;)$–regular triangulations associated to a pair of quadratic differentials that appear at the end points of a geodesic segment in $Q()$.

Recall, from Remark 3.2, that there is an implicit assumption that the constant is large. That is, there is a uniform constant $_0$ so that all statements in this section hold as long as $_0$. In particular, the implied constant in our estimates do not get worst as gets larger.

4.1. Notation. First we need to establish some notations.

1. For a fixed constant r_0, define $B(Q(); X;)$ to be the set of points $Z \; 2 \; T(S)$ so that there is a Teichmüller geodesic

$$G_Z : [a; b] \; ! \quad Q^1T(); \qquad G_Z(t) = (X_t; q_t);$$

such that

$$d_T(X_a; X) \;\; r_0; \qquad d_T(X_b; Z) \;\; r_0; \qquad b \quad a :$$

and

$$(X_t; q_t) \; 2 \; Q():$$

One could think of $B(Q(); X;)$ as a ball of radius centered at X, except that one is allowed only to move in the direction of $Q()$. Since r_0 is fixed, we refer to any constant that depends on r_0 as a uniform constant. The value of r_0 will be determined in §6.2 depending on the choice of the net N.

2. We use the notation of Equation (5) for q_a and denote the flat and expanding annuli associated to a curve by E^a, F^a and G^a and distances between their boundaries by e^a, f^a and g^a. Let $_a$ be an arc of length f^a connecting the boundaries of F. Also, let $l^a = \ell_q()$ and let $d^a_a = \max(e^a; f^a; g^a)$ be the maximum distance between the boundaries of these annuli. As a consequence of Equations (5) and (6) we have

$$\frac{1}{\mathrm{Ext}_X()} \quad \frac{d^a}{l^a} \qquad \text{and} \qquad \frac{1}{\mathrm{Ext}_X()} \quad \frac{f^a}{l^a}: \tag{13}$$

3. Let T_a be a $(q_a;)$–regular triangulation and T_b be a $(q_b;)$–regular triangulation. The geodesic flow induces a one-to-one correspondence between saddle connections of q_a and q_b. Hence, we can consider T_b as a union of saddle connections in q_a. Then T_a and T_b have identical vertex sets and their edges are either identical or intersect transversally. The slope of a saddle connection in q_a (or in q_b) is a well defined number in the interval $[0; 1]$.

Definition 4.1. Let $!_a$ be a saddle connection in q_a and let $!_b$ be a saddle connection in q_b. We say $!_b$ intersects $!_a$ positively, if when considering them both in q_a (or q_b), the slope of $!_b$ is larger than the slope of $!_a$. We say $!_b$ intersects $!_a$ essentially positively if either $!_b$ intersects $!_a$ positively or $i(!_a; !_b) = O(1)$. We use similar terminology for intersection between a saddle connection and a cylinder curve and two cylinder curves.

4.2. Intersection and twisting bounds between T_a and T_b. For the rest of this subsection, we assume that q_a and q_b, T_a and T_b are as described in the beginning of the section.

Lemma 4.2. Let $2\ S_{q_a}$, $!_b\ 2\ T_b$ and ${}_b\ 2\ S_Z$, then

$$\mathrm{twist}(q_a; !_b) = O(1) \qquad \text{and} \qquad \mathrm{twist}(q_a; {}_b) = O(1):$$

Similarly, let $2\ S_{q_b}$, $!_a\ 2\ T_a$ and ${}_a\ 2\ S_X$, then

$$\mathrm{twist}(q_b; !_a) = O(1) \qquad \text{and} \qquad \mathrm{twist}(q_b; {}_a) = O(1):$$

Proof. Let be the arc connecting the boundaries of F^a and is perpendicular to them. Then, by definition of S_q ,

$$\frac{`_{q_a}()}{`\ ()} = \mathrm{Mod}(F^a)\ e^{\ 2}: q_a$$

Therefore

$$`_{q_b} \quad \frac{`_{q_b}()}{()} \quad 1:$$

That is, twists around in q_b a bounded number of times. But the same is true for $!_b$. This gives a bound on $i(!_b;)$ and thus on $\mathrm{twist}(q_a; !_b)$. Also, the curve ${}_b$ is short in Z and hence in q_b. A short curve can not twist around any other curve. Hence $i({}_b;)$ is uniformly bounded. Which means $\mathrm{twist}(q_a; {}_b)$ is uniformly bounded. The proofs of the other two assertions are similar.

Remark 4.3. The main consequence of this lemma is that the twisting condition of Lemma 3.17 is satisfied and can be applied freely.

Lemma 4.4. Let $!_a$ and $!_b$ be edges of T_a and T_b respectively. Then $!_b$ intersects $!_a$ essentially positively and

$$i(!_a; !_b)\ \ e:$$

Proof. Let Q_a be the thick subsurface of q_a with the smallest size that intersects $!_a$ and let s_a be the size of the subsurface Q_a. Recall that, by the definition of a $(q_a;)$–regular triangulation, we have

$$`_{q_a}(!_a)\ \ s_a:$$

We denote the horizontal and the vertical lengths of $!_a$ by x_a and y_a. Let Q_b, s_b, x_b and y_b be similarly defined. The length of $!_a$ in Q_b is

$$p\ \overline{(x_a e\ )^2 + (y_a e\ \ )^2}\ \ x_a e + y_a e \ \ :$$

If $i(!_a; !_b) = O(1)$ we are done. Otherwise, Considering $!_a$ and $!_b$ in Q_b, in view of Remark 4.3, Lemma 3.17 implies that

$$i(!_a; !_b)\ \ \frac{`_{q_b}(!_a)}{s} + O(1):\ b$$

However, since $i(!_a; !_b)$ is large, $\frac{`_{q_b}(!_a)}{s_b}$ is large and we can incorporate the additive error into the multiplicative error. That is,

$$i(!_a; !_b)\ \ \frac{`_{q_b}(!_a)}{s}\ \ \frac{x_a e + y_a e}{s}\ {:}\ b\ b \tag{14}$$

Similarly, considering $!_a$ and $!_b$ in Q_a we get

$$i(!_a; !_b)\ \ \frac{`_{q_a}(!_b)}{s}\ \ \frac{x_b e\ \ + y_b e}{s}\ a\ \ a \tag{15}$$

Observing that $x_a; y_a \quad s_a$ and $x_b; y_b \quad s_b$, we can multiply the two inequalities and take a square root to get $i(!_a; !_b) \quad e$.

Now assume that $!_b$ does not intersect $!_a$ positively. This means that the slope of $!_a$ in q_b is larger than the slope of $!_b$. That is

$$\frac{y_a e}{x_a e} \quad \frac{y_b}{x_b} \quad =) \quad x_a y_b e^2 \quad x_b y_a :$$

From the product of inequalities in Equation (14) and Equation (15), we have

$$i(!_a; !_b)^2 \quad \frac{x_a x_b + y_a y_b + x_a y_b e^2 + x_b y_a e^{2}}{s \; s}_{a \; b}$$

$$\frac{x_a y_b e^2}{s_b s} \quad \frac{x_b y_a}{s_a s_b} = O(1): \quad a$$

For a simple closed curve and a triangulation T, we say T intersects essentially positively if any saddle connection in T intersects any saddle connection in the geodesic representative of essentially positively.

Lemma 4.5. If $2 S_X$ and $2 S_Z$ then intersects T_b essentially positively and

$$\mathrm{twist}(X; Z)\, i(; T_b) \quad p \frac{e}{\mathrm{Ext}_X()} :$$

Similarly, if $2 S_Z$ and $2 S_X$ then intersects T_a essentially positively and

$$\mathrm{twist}(X; Z)\, i(; T_a) \quad p \frac{e}{\mathrm{Ext}_Z()} :$$

Proof. Let be a simple closed curve in $S_X \; n \, S_Z$. Applying Lemma 3.16 to the pair X and q_a and to the pair Z and q_b, we get

$$\mathrm{twist}(X; Z) \quad \mathrm{twist}(q_a; q_b) + \frac{1}{\mathrm{Ext}_X()} : \tag{16}$$

(The term $\frac{1}{\mathrm{Ext}_Z()}$ is omitted from the right hand side because it is bounded and can be absorbed in the multiplicative error.) Hence, to prove the lemma, it is sufficient to show that the expression $p \frac{e}{\mathrm{Ext}_X()}$ is an upper-bounds for both

$$\frac{i(; T_b)}{\mathrm{Ext}_X()} \quad \text{and} \quad \mathrm{twist}(q_a; q_b)\, i(; T_b):$$

Let $!_b$ be an edge of T_b and ~~let Q_b~~ be the thick subsurface of q_b with the smallest size intersecting $!_b$. Let s_b be the size of Q_b (thus $`_{q_b}(!_b) \quad s_b$, by the definition of a $(q_b;)$–regular triangulation). Applying Lemma 3.17 to $!_b$ and in q_b, we get

$$i(; !_b) \quad O(1) \quad \frac{`_{q_b}()}{s} e \frac{l^a}{s} : b \quad b \tag{17}$$

Also, each subsegment of $!_b$ with end points in has a length larger than d^a. Hence,

$$i(; !_b) \quad 1 \quad \frac{`_{q_a}(!_b)}{d^a} e \frac{s_b}{d^a} :$$

Multiplying these two equations, taking the square root we and summing over all arcs in T_b we get

$$i(; T_b) \quad O(1) \; e \quad r \frac{l^a}{d^a} :$$

In view of Equation (13), we obtain

$$i(; T_b) \quad O(1) \; e \quad p \; \mathrm{Ext}_X():$$

Dividing both sides by $\mathrm{Ext}_X()$ we obtain

$$\frac{i(; T_b)}{\mathrm{Ext}_X()} \quad \frac{e + O(1)}{p\,\mathrm{Ext}_X()} \quad \frac{e}{p\,\mathrm{Ext}_X()} : \tag{18}$$

This is the first estimate we required.

We now find an upper-bound for $\mathrm{twist}(q_a; q_b)\, i(; T_b)$ by finding separate upper bounds for $\mathrm{twist}(q_a; q_b)$ and $i(; T_b)$. The argument involved in this new upper bound for $i(; T_b)$ is somewhat similar to above, but the two bounds do not imply each other. We need to consider the image of F^a in q_b under the Teichmüller geodesic flow. Denote this cylinder by F^b, the distance between its boundaries by f^b and let ^{b} be an arc of length f^b connecting the boundaries of F^b. Let $l^b = \ell_{q_b}()$. Note that the area of F^a and F^b are equal, that is

$$l^a f^a = l^b f^b:$$

Consider again the arc $!_b$ in T_b of q_b–length of order s_b. Then the q_b–length of every component of $!_b \setminus F^b$ is larger than f^b. Therefore

$$i(; !_b) \quad \frac{s_b}{f_b} = \frac{s_b\, l^b}{l^a f^a}:$$

As before, applying Lemma 3.17 to $!_b$ and in q_b we have

$$i(; !_b) \quad \frac{l^b}{s_b} + O(1) \quad : \frac{l^b}{s_b}$$

The reason we can ignore the additive errors here is that since is not short in Z, it has to either be an essential curve in Q_b or intersect some boundary curve of Q_b. In either case, $l^b \; s_b$, in the first case by definition of the size and in the second case by Corollary 3.6. Hence, the additive error can be absorbed into the fraction $\frac{l^b}{s_b}$. Multiplying the last two inequalities, taking the square root and summing over all arcs in T_b, we obtain

$$i(; T_b) \quad \frac{l^b}{\sqrt{l^a f^a}}: \tag{19}$$

We now argue that a component of $!_b \setminus F^b$ can intersect at most a uniformly bounded number of times: since is not short in Z, $\ell_{q_b}() \; s_b$ and $\ell_{q_b}(!_b) \; s_b$, which means the intersection number between $!_b$ and is at most $\frac{\ell_{q_b}(!_b)}{\ell_{q_b}()} = O(1)$. Therefore, the relative twisting of q_a and q_b around is comparable to the intersection number between ^{a} and ^{b} which is at most the q_a–length of ^{b} divided by the q_a–length of . That is

$$\mathrm{twist}(q_a; q_b) \quad O(1) \quad i(^a; ^b) \quad \frac{e f^b}{l^a}:$$

Taking a product and using the second part of Equation (13) we get:

$$\begin{aligned} \mathrm{twist}(q_a; q_b)\, i(; T_b) \quad O(i(; T_b)) \quad & e\frac{l^b f^b}{l^a} \quad \frac{1}{\sqrt{l^a f^a}} \\ & e\sqrt{\frac{f^a}{l^a}} \quad \frac{e}{\sqrt{\mathrm{Ext}\,()}}: \end{aligned} \tag{20}$$

By Equation (18), we have $i(; T_b)$ is much smaller than $\frac{e}{\sqrt{\mathrm{Ext}_X()}}$. Hence,

$$\mathrm{twist}(q_a; q_b)\, i(; T_b) \quad \frac{e}{\sqrt{\mathrm{Ext}_X()}}: \tag{21}$$

The estimate in the Lemma follows from Equations (16), (18) and (21).

It remains to show that $!_b$ and intersect essentially positively. Let $!_a$ be a saddle connection of that intersects $!_b$ many times. Then, by Lemma 3.17, $\ell_{q_b}(!_a) \; s_b$. However, $\ell_{q_b}(!_b) \; s_b$ and hence $\ell_{q_b}(!_a) \; l_{q_b}(!_b)$. If the slope of $!$ was smaller than $!_b$ (say in q_b) then we would also have $\ell_{q}(!_a) \; \ell_{q}(!_b)$. Hence, $!_a$ intersects $!_b$ at most twice (its length is less than d^a). This proves that $!_b$ intersects $!^a$ essentially positively. But this is true for every saddle connection of . Thus $!_b$ intersects essentially positively. The case when $2 S_Z$ can be treated similarly.

Lemma 4.6. Let $2 S_X$ and $_0\, 2 S_Z$. Then

(1) If $_0 =$, then

$$i(;_0)\,\mathrm{twist}_0(X;Z)\,\mathrm{twist}(X;Z) \quad \frac{e}{\sqrt{\mathrm{Ext}_X()\,\mathrm{Ext}_Z(_0)}};$$

(2) If $=_0$, then

$$\sqrt{\mathrm{Ext}_X()\,\mathrm{Ext}_Z(_0)} \quad \frac{e}{\mathrm{twist}(X;Z)}:$$

Proof. It is enough to prove that

$$\mathrm{twist}(X;Z) \quad \sqrt{\mathrm{Ext}_X()\,\mathrm{Ext}_Z()} \quad e: \tag{22}$$

If $=_0$; this is equivalent to Equation (2). Also, if $=_0$ and $2\ S_Z$, then the inequality (1) trivially holds (the left hand side is 0). Otherwise, from Theorem 3.14 (estimating $\mathrm{Ext}_Z()$) we have

$$i(;_0) \quad \sqrt{\mathrm{Ext}_Z(_0)}\,\mathrm{twist}_0(X;Z) \quad \sqrt{\mathrm{Ext}_Z()}: \tag{23}$$

Multiplying the above equation to Equation (22) we obtain part (1) of the lemma.

By Corollary 3.15, (replace X with Z, with and with $_0$) there always exits a simple closed curve $_0$ so that $\mathrm{twist}(_0;Z) = O(1)$ and

$$\sqrt{\mathrm{Ext}_Z(_0)\,\mathrm{Ext}_Z()} \quad i(;_0): \tag{24}$$

On the other hand, from Theorem 2.1 we have

$$\sqrt{\mathrm{Ext}_X(_0)} \quad e \quad \sqrt{\mathrm{Ext}_Z(_0)}; \tag{25}$$

and from Theorem 3.14 (this time estimating the length of $_0$ is X) we have

$$i(;_0)\,\mathrm{twist}(X;_0) \quad \sqrt{\mathrm{Ext}_X()} \quad \sqrt{\mathrm{Ext}_X(_0)}: \tag{26}$$

Since $\mathrm{twist}(_0;Z) = O(1)$, we can replace $\mathrm{twist}(X;_0)$ with $\mathrm{twist}(X;Z)$ in the above inequality. Now, Equation (22) is obtained by successive substitution using Equations (23), (24), (25) and (26).

4.3. Relations between intersections numbers. So far, we have provided upper-bounds for the intersection numbers between the edges of T_a and the edges of T_b. But these intersection numbers are not independent. The fact that the edges in T_a intersect edges in T_b essentially positively allows us to find relations between these intersection numbers. In this section we will describe these relations. There are two kinds of relations.

Lemma 4.7. For every triangle in T_a with edges $!_1$, $!_2$ and $!_3$, there are sings $\&_1;\&_2;\&_3$ 2 f 1;+1g so that, for every edge $!_b$ in T_b (respectively, for any $_b$ 2 S_{q_b}), we have the relation:

$$\sum_{i=1;2;3} \&_i\, i(!_i;!_b) = O(1) \qquad \left(\text{respectively,} \sum_{i=1;2;3} \&_i\, i(!_i;_b) = O(1)\right): \tag{27}$$

The additive error depends on the constant involved in the definition of essential positively.

Proof. There is a leaf of the vertical foliation that passes through a vertex of the given triangle before entering it. Assume this leaf intersects the interior of $!_3$ and makes an acute angle with $!_1$ inside of the triangle. We claim that, since $!_b$ intersects $!_1$ essentially positively, the number of sub-arcs of $!_b$ going from $!_1$ to $!_2$ is uniformly bounded. This is because either the slope of $!_b$ is larger than the slope of $!_1$ and every time $!_b$ intersects $!_1$ it has to intersect $!_3$ next, or it intersect $!_1$ a bounded number of times. Hence, we have

$$i(!_1;!_b) + i(!_2;!_b) = i(!_3;!_b) + O(1):$$

Note that the signs $\&_1 = 1$, $\&_2 = 1$ and $\&_3 = \ 1$ depend only on the triangle and are independent of $!_b$. The proof for $_b$ is similar.

For each $\gamma \in S_{q_a}$, consider a saddle connection connecting the boundaries of F. Let

$$U_a = T_a \cup \bigcup_{\gamma \in S} $$

We can choose the arcs so that $\mathrm{twist}(q_a, U_a) = O(1)$. After orienting the arcs in U_a, we can think of them as elements of $H_1(S, \Sigma)$ where Σ is the set critical points of q_a. In fact, arcs in U_a generate $H_1(S, \Sigma)$.

Lemma 4.8. *Assume that the vertical foliation of q_a is not orientable. Then, there is a set B of edges of U_a and for $\omega \in B$ there is a sign $\varepsilon_\omega \in \{-1, +1\}$ so that, for every ω_b in T_b (respectively, for any $\gamma_b \in S_{q_b}$), we have the relation:*

$$\sum_{\omega \in B} \varepsilon_\omega \, i(\omega, \omega_b) = O(1) \qquad \text{respectively,} \quad \left(\sum_{\omega \in B} \varepsilon_\omega \, i(\omega, \gamma_b) = O(1) \right). \tag{28}$$

Furthermore, this relation is independent of all the relations in Lemma 4.7.

Proof. Choose a minimum number of edges of U_a so that the complement is simply connected. Denote the set of all these edges by $\overline{B}$ and orient them in some arbitrary way. Minimality implies that the compliment P is connected. We can visualize P as a polygon in $\mathbb{C}$ with the vertical foliation parallel to the imaginary axis. Each edge of $\overline{B}$ has two representatives in the boundary of P. The two vectors are equal up to a multiplication by -1. Let B be the subset of $\overline{B}$ where the two representatives are negatives of each other (Fig. 5). Note that B is non-empty since the vertical foliation in q_a is not orientable.

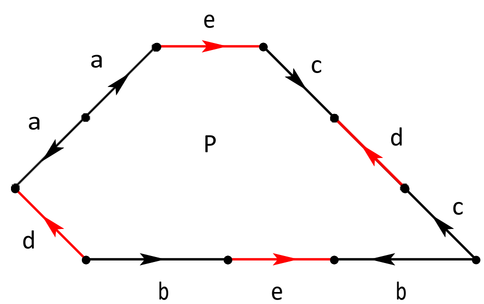

Figure 5. Polynomial P. The set $B = \{a, b, c\}$.

Now consider a double cover of q_a constructed as follows. Take a second copy P' of P. Glue the edges that were not in B as before and glue the edges in B to the corresponding edge in P'. Let $\tilde{B}$ be the set of lifts of edges in B to this cover. We now orient edges in $\tilde{B}$ so that, for every $\tilde{\omega} \in \tilde{B}$, P is on the same side of $\tilde{\omega}$ (say, the left side). Denote this double cover by $\tilde{q}_a = P \cup P'$.

Let $\tilde{S}$ be the underlying surface for $\tilde{q}_a$ and $\tilde{\Sigma}$ be the pre-image of Σ. Considering oriented saddle connections as elements of $H_1(\tilde{S}, \tilde{\Sigma})$ we let $\hat{i}(\cdot, \cdot)$ denote the algebraic intersection number. Note that $\tilde{q}_a$ is the unique double cover of q_a where $\tilde{q}_a$ is a square of an abelian differential. Hence, for every two oriented saddle connections $\tilde{\omega}$ and $\tilde{\omega}'$ in $\tilde{q}_a$, all the intersection points have the same signature. That is,

$$i(\tilde{\omega}, \tilde{\omega}') = |\hat{i}(\tilde{\omega}, \tilde{\omega}')|.$$

Consider $\omega \in B$ and its lift $\tilde{\omega}$. Note that $\tilde{\omega}$ has an orientation and hence is identified with vector in $\mathbb{C}$. We define ε_ω to be $+1$ if $\tilde{\omega}$ has a positive x–coordinate and -1 otherwise. Let $\omega_b \in T_b$ and let $\tilde{\omega}_b$ be a lift of ω_b. We choose an orientation for $\tilde{\omega}_b$ so it has a positive y–coordinate. We will show that

$$\sum_{\omega \in B} \varepsilon_\omega \, i(\omega, \omega_b) = O(1).$$

Consider an intersection point of $\tilde{\omega}_b$ and $\tilde{\omega}$ where $\varepsilon_\omega = 1$. If the absolute value of the slope of $\tilde{\omega}_b$ is larger than that of $\tilde{\omega}$ then $\tilde{\omega}_b$ is to the left of $\tilde{\omega}$ and hence $\tilde{\omega}$ intersects $\tilde{\omega}_b$ with a positive

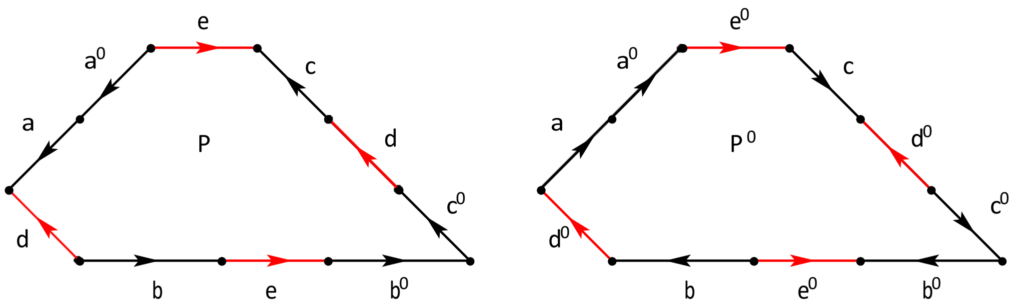

Figure 6. Polynomials P and P'. The set $\hat{\mathcal{B}} = \{a, a', b, b', c, c'\}$.

signature. Otherwise, $\hat\omega$ and $\hat\omega_b$ intersect a uniformly bounded number of times ($\hat\omega_b$ and $\hat\omega$ intersect essentially positively). The opposite of this is true if $\epsilon_\omega = -1$; either $\hat\omega$ intersects $\hat\omega_b$ with a negative signature or a uniformly bounded number of times. If $\hat\omega, \hat\omega' \in \hat{\mathcal{B}}$ are lifts of the same arc $\omega \in \mathcal{B}$ then, choosing orientations for $\hat\omega$ and $\hat\omega'$ as above, we have

$$i(\omega, \omega_b) = i(\hat\omega, \hat\omega_b) + i(\hat\omega', \hat\omega_b) \stackrel{+}{\asymp} \epsilon_\omega \left(\hat{i}(\hat\omega, \hat\omega_b) + \hat{i}(\hat\omega', \hat\omega_b) \right). \tag{29}$$

To reiterate, this is because the number of intersection points that do not have the same sign as ϵ_ω is uniformly bounded.

But arcs in $\hat{\mathcal{B}}$ separate $\tilde{q}_a$. Thus,

$$\sum_{\hat\omega \in \hat{\mathcal{B}}} \hat{i}(\hat\omega, \hat\omega_b) \le 1.$$

This is because every time $\hat\omega_b$ exits P it intersects the boundary with the opposite signature than when it enters it. The sum is not necessarily zero because $\hat\omega_b$ may start inside P and end in P'. Therefore, summing Equation (29) over $\omega \in \mathcal{B}$, we get

$$\sum_{\omega \in \mathcal{B}} \epsilon_\omega \hat{i}(\omega, \omega_b) = O(1).$$

The proof for the case of a simple closed curve $\gamma_b \in \mathcal{S}_{q_b}$ is similar.

Finally, we note that the relations of the type (27) are also relations in the relative homology with $\mathbb{Z}_2$–coefficients. But the edges in $\mathcal{B}$ are independent in $\mathbb{Z}_2$–relative homology. Hence, this new relation is independent from the previous ones.

5. Main counting statement

This section contains the main combinatorial counting arguments with the goal of proving Theorem 5.1. Recall the definition of $\mathcal{B}(\mathcal{Q}(\alpha), X, \epsilon)$ from §4.1. Define

$$\mathcal{B}_j(\mathcal{Q}(\alpha), X, \epsilon) \subset \mathcal{B}(\mathcal{Q}(\alpha), X, \epsilon)$$

to be the set of points $Z \in \mathcal{T}(S)$ so that, for the associated quadratic differentials q_a and q_b, there is a (q_a, ϵ)–regular triangulation T_a and a (q_b, ϵ)–regular triangulation T_b that have j common homologically independent saddle connections. Now let,

$$\mathcal{B}(\mathcal{Q}(\alpha), X, Y, \epsilon) = \mathcal{B}(\mathcal{Q}(\alpha), X, \epsilon) \cap \Gamma(S) \cdot Y,$$

and

$$\mathcal{B}_j(\mathcal{Q}(\alpha), X, Y, \epsilon) = \mathcal{B}_j(\mathcal{Q}(\alpha), X, \epsilon) \cap \Gamma(S) \cdot Y.$$

That is, $\mathcal{B}_j(\mathcal{Q}(\alpha), X, Y, \epsilon)$ is the intersection of the orbit of Y with $\mathcal{B}_j(\mathcal{Q}(\alpha), X, \epsilon)$. Also, recall from §2.6 that (when $\mathcal{S}_X$ is empty, $G(X) = 2$):

$$G(X) = 1 + \prod_{\gamma \in \mathcal{S}_X} \sqrt{\frac{1}{\operatorname{Ext}_\gamma(X)}} \prod_{\gamma \in \mathcal{S}_X} \sqrt{\frac{1}{\operatorname{Ext}_\gamma(X)}}.$$

Notice that if $g\,Y$ 2 $B_j(Q(); X; Y;)$ then $g^{1}\,X$ 2 $B_j(Q(); Y; X;)$. Thus, the number of points in $B_j(Q(); X; Y;)$ is the same as the number of points in $B_j(Q(); Y; X;)$. We prove the following upper-bound for the size of $B_j(Q(); Y; X;)$:

Theorem 5.1. Consider the stratum $Q()$. Given $X; Y$ 2 $T(S)$

$$B_j(Q(); X; Y;) \quad ^{jS_X j + jS_Y j} e^{(h \quad j)}\, G(X) G(Y);$$

where $h = \frac{\dim Q()}{2}$.

Remark 5.2. First we make a few remarks

(1) If, in the definition of $B_j(Q(); X; Y;)$, we replace the assumption on the number of common homologically independent saddle connections with an assumption on the number of common homologically independent simple closed curves, the same statement would still holds. However, the theorem is strictly stronger. For example, assume $S_X \setminus S_Y$ contains only one homologically trivial simple closed curves . We can still conclude that j 1 because the geodesic representative of in any quadratic differential q contains a (homologically) non-trivial arc. That is, the number points Y, where the geodesic connecting X to Y follows $Q()$ and contains a short curve throughout, is smaller than expected even when is a homologically trivial curve.

(2) The statement appears to be correct even without the term $^{jS_X j + jS_Y j}$. However, the proof would become significantly more complicated.

5.1. Sketch of the proof Theorem 5.1. Here is a an outline of our strategy :

(1) We define a notion of a marking for the surface S and what it means for a marking to have a bounded length in a Riemann surface X. A marking contains a partial triangulation of S, a set of short simple closed curves with their lengths and some twisting information. Fixing a Riemann surface X, every quadratic differential q where the underlying conformal structure is near X defines a marking that has a bounded length in X. A marking takes the lengths of the short simple closed curves and the twisting information around short cylinder curves from X and the triangulation and twisting around the non-cylinder short simple closed curves from q. Up to some twisting information, there are a uniformly bounded number of markings that have bounded length in a given Riemann surface X.

(2) Fixing a marking $_0$, a relation is a formal linear combination of edges of $_0$ with integer coefficients. Given $_0$ and $_1$ and a set of relations R we will define a set $M_R(_0; _1;)$ consisting of all markings such that is a homeomorphic image of $_1$, its weighted intersection number with $_0$ is less than e and so that the intersection patterns between and $_0$ satisfy the relations in R. The weights depend on the length and the twisting information of each short simple closed curve. This is similar to assuming that there is a geodesic segment in a the stratum $Q()$ starting near X and ending near Y. Lemma 5.8 provides and upper-bound for the number of elements in $M_R(_0; _1;)$.

(3) We then let R be the set of relation of the type described in Lemma 4.7 and Lemma 4.8. Each Z 2 $B_j(Q(); X; Y;)$ can then be mapped to a marking in 2 $M_R(_0; _1;)$ for some marking $_1$ that has bounded length in Y and some marking $_0$ that has both a bounded length and a bounded twisting in X. This map is finite-to-one except for some twisting information. An estimate for the number of possible markings $_0$ and $_1$ provides the desired upper-bound for the size of $B_j(Q(); X; Y; R)$.

As is apparent from the outline, the main complication is to keep careful track of all the different twisting informations. Otherwise, the argument is relatively elementary.

5.2. Markings on S. Fix a set of points on S. A partial triangulation T of S with the vertex set is an embedding of a graph to S where vertices are mapped onto and the complementary components are either triangles or annuli. Even though the vertex set is fixed, we think of T as representing a free homotopy class of triangulations. We say a curve is carried by T if the free homotopy class of can be represented by tracing the edges of T. We define a combinatorial length of a simple closed curve in S to be the minimum number of arcs of T that can appear in a representative of and we denote it by ‘$_T$ ().

Recall that a set of curves fill a subsurface Q os S if every essential curve in Q intersects one of these curves. We say a partial triangulation T fills a subsurface Q of S if, again, every essential curve in Q intersects T (their free homotopy classes do not have disjoint representatives). The two notions are related:

Lemma 5.3. There is a constant B such that, if T fills a subsurface Q of S, then the set of simple closed curves carried by T with ‘$_T$ () B also fill the subsurface Q.

Definition 5.4. A marking = S; fE()g; T for S is:

a free homotopy class of oriented curve system S (pairwise disjoint curves) together with a notion of zero twisting for each curve 2 S, (that is, the expression twist(;) makes sense),

a length E( ) associated to each simple closed curve 2 S, and

a homotopy class of a partial triangulation T with the vertex set such that the core curve of any annulus in the complement of T is in S.

for each 2 S intersecting T, twist(; T) = O(1).

We denote the set of simple closed curves that are disjoint from T by S^c and the remaining curves in S by S^n (the set S^c is a place holder for large cylinder curves and the set S^n is a place holder for non-cylinder curves or small cylinder curves).

We say a marking = S; fE()g; T has a bounded length in X if:

(1) S = S_X.
(2) For 2 S, E( ) = Ext$_X$( ).
(3) For 2 S^c, twist(; X) = O(1).
(4) For each simple closed curve 2 S_X that is disjoint from S_X, ‘$_X$ ( ) ‘$_T$ ().

We say has bounded length in X with –bounded twist if we further have (5)

For 2 S^n, twist(; X) = O().

Example 5.5. We continue Example 3.4 of a surface (X; q) described by a gluing of a polygon in R^2. As it was discussed, there are two thick subsurfaces in the complement of curves and (Fig. 2). A (q;)–regular triangulation of (X; q) is depicted in Fig. 7. Here S = fg and S_q = fg.

Figure 7. A (q;)–regular triangulation.

Here a marking that has bounded length in X can be obtained as follows: The set S is the set f; g of short curves in X (depicted as blue curves in Fig. 8), the triangulation T is the (q;)–regular triangulation (depicted as the red triangulation) and E ( ) and E ( ) are the extremal length of and in X respectively. The condition (4) for to have a bounded length in X is a consequence of T being a (q;)–regular triangulation.

Lemma 5.6. Let M (X;) be the set of markings that have a bounded length in X with –bounded twist. Then

$$jM(X;)j \quad ^{jS_X j}:$$

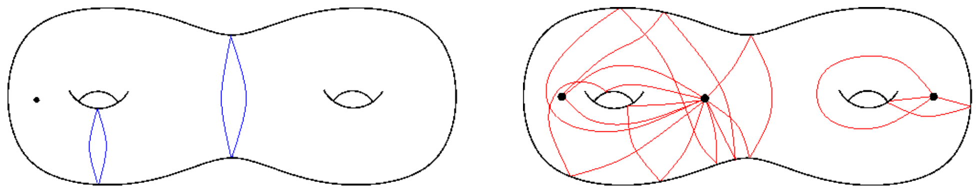

Figure 8. The curves and the triangulation in the marking .

Proof. The set S and the lengths fE()g and the twisting around curves in S^c are determined by definition. By Lemma 3.19, there is a uniformly bounded number of possibilities for T up to twisting around curves in S^n. But each of these twisting parameters is bounded by multiple of (condition (5) in the definition Definition 5.4). This finishes the proof.

Definition 5.7. Consider the markings

$$= fS; fE()g; T g \quad \text{and} \quad {}_0 = fS_0; fE_0({}_0)g; T_0 g:$$

Recall that T and T_0 have the same vertex set . For every 2 S^c, let be an arc with end point in and disjoint from T that crosses so that T [has bounded twisting around . Denote

$$U = T \, [\bigcup_{2 S^c} :$$

Let R[U] be the vector space of formal sums with real coefficient of edges in U. Let R be a finite subset of R[U] with integer coefficients. We define the set

$$M_R(; {}_0;)$$

to be the set of markings $\overline{}$ = fS; fE()g; T g such that:

(I) is a homeomorphic image of $_0$, and for every 2 S^c that is the image of $_0$ 2 S_0, we have $E_0({}_0) = E()$.

(II) For every element $\sum a_! \, !$ 2 R and every arc 2 T (respectively, 2 S ^{c}), we have

$$\sum_{! 2 U} a_! \, i(!;) = O(1); \qquad \text{respectively,} \quad \sum_{!} a_! \, i(!;) = O(1) \; :$$

(III) Given 2 S^c; 2 S^c, ! 2 T and ! 2 T; we have the following bounds on the intersection numbers:

(30) $$i(; !) \quad e$$

(31) $$\text{twist}(;) \sqrt{E()} \, i(; T) \quad e$$

(32) $$\text{twist}(;) \sqrt{E()} \, i(; T) \quad e$$

(33) $$i(;) \, \text{twist}(;) \, \text{twist}(;) \, E() E() \quad e$$

and finally if = 2 $S^c \setminus S^c$ we have:

(34) $$\text{twist}(;) \quad \sqrt{E() E()} \quad e:$$

Note that the partial triangulations in and are defined up to homotopy. By above intersection bounds we mean that the homotopy class of two partial triangulations have representations with vertex set so that the above bounds hold simultaneously.

Let hRi be the subspace of R[U] generated by elements in R. We give the following upper bound for such markings:

Lemma 5.8. Let $h_R = \dim(R[U]=hRi)$. Then

$$jM_R(;{}_0;)j \quad e^h \prod_{2S^c} \frac{1}{\sqrt{E()}} \prod_{{}_0 2 S_0} \frac{1}{\sqrt{E_0({}_0)}} :$$

Proof. For $\overline{}2\ M_R(;{}_0;)$ consider the weighted graphs $W = T$

$$\overline{}+\sum_{2S^c} \overline{} m(;)\ ;$$

where the weights on the edges of $\overline{T}$ are 1 and the weight $m(;) \overline{2}\ N$ are defined to be

$$m(;) = \quad \frac{\mathrm{twist}(;)}{E()} \quad \sqrt{} :$$

Define W to be the set of weighted graphs induced by elements of $M_R(;{}_0;)$:

$$W = \left\{ \overline{W} \quad \overline{2}\ M_R(;{}_0;) \right\} :$$

The weighted graph $\overline{W}$ essentially determines except that, for $2\ S^c$, the value of $m(;)$ determines twist$(;)$ only up to $\frac{1}{\sqrt{E()}} = \frac{1}{\sqrt{E_0({}_0)}}$ possibilities (we have used the floor function in defining $m(;)$). Hence,

$$(35) \qquad jM_R(;{}_0;)j \quad \prod_{{}_0 2 S_0} \frac{1}{\sqrt{E_0({}_0)}} \ jWj: We$$

proceed in two steps:

Step 1. Consider the set $E \quad U$ that forms a basis for the space $R[U]=hRi$. First, we claim that the map

$$I : W\ !\ N^{h_R}; \qquad \overline{W}\ ! \quad i(\overline{W};!)_{!2E}$$

is finite to one, where $i(\overline{W};!)$ is defined to be

$$i(\overline{W};!) = \sum i(\overline{T};!) + \sum_{2S^c} m(;)\ \overline{i}(;!):_{!2T}$$

Note that in general a weighted graph $\overline{W}$ is determined by the intersection numbers of its edges with all the edges of U. The map I records the intersection number with arcs in E. To prove the claim, we need to show that, there are only finitely many possibilities for the intersection number of $\overline{W}$ with the other edges of U.

We can consider an $2\ U$ as the element $1 \quad 2\ R[U]$. Then can be written as a linear combination elements in the generating set E (which generates $R[U]=hRi$)) and R (the relations). That is, there are constants $c_!$ and d_R so that

$$= \sum_{!2E} c_!\,! + \sum_{R2R} d_R R:$$

But the intersection number is linear hence, for every $\overline{!}\ 2\ \overline{W}$, we have

$$i(;\overline{!}) = \sum_{!2E} c_!\, i(!;\overline{!}) + O\left(\sum_{R2R} d_R\right) :$$

But the constants d_R depend only on the set R and otherwise are uniformly bounded. Hence, there are only finite number of possibilities for $i(;\overline{!})$. This proves the claim.

Step 2. We bound the size of $I(W) \; N^h$ by obtaining upper bounds on intersection numbers of $\overline{W}$ with arcs $! \; 2 \; E$.

First, if $! \; 2 \; T$, Equation (30) implies that

$$i(\overline{T}; !) \quad e: \tag{36}$$

Also, for $2 \; S^c \quad S^c$ the Equation (31) implies

$$m(;) \; i(; !) \; twist(;) \quad E() \; i(; !) \; e:$$

Hence,

$$i(\overline{W}; !) \quad e: \tag{37}$$

For arc $2 \; U$ where $2 \; S^c$, and arc $! \; 2 \; W$ by Equation (32) we have

$$i(; !) \quad i(; !) \; twist(;) \quad \frac{p}{E()} : \text{And for } \frac{e}{2 \; S^c} \tag{38}$$

S^c, by Equation (33) we have

$$m(;) \; i(;) \quad twist(;) \quad E() \; i(;) \quad \frac{q}{p\, E()} : \text{Finally, if } = \frac{e}{2 \; S^c} \setminus \tag{39}$$

S^c, by Equation (34)

$$m(;) \; i(;) \quad \frac{p}{E()} : \quad e \tag{40}$$

Now from Equations (37), (38), (39) and (40), we get:

$$jWj \; jI(W)j \quad \prod_{\substack{2 S^c \\ 2 E}} \frac{e}{p\, E()} \prod_{! 2 T \setminus E} e$$

$$e^{jEj} \prod_{2 S^c} \frac{1}{p\, E()} :$$

Now, applying Equation (35), we get

$$jM_R(; _0;)j \quad e^{h_R} \prod_{2 S^c} p\, E() \prod_{_0 2 S^c} \frac{1}{p\, E_0(_0)} ; \quad \frac{1}{_0}$$

which is as we claimed.

Proof of Theorem 5.1. Let $Z \; 2 \; B_j(Q(); X; Y;)$ and let $(X_a; q_a)$ and $(X_b; q_b)$ be the initial and the terminal quadratic differentials for the Teichmüller geodesic in $Q()$ starting near X and finishing near $Z \; 2 \quad Y$, as before. There may be many choices for these quadratic differentials. We need to be a bit careful.

Claim: We can choose $(X_a; q_a)$ and $(X_b; q_b)$ so that for any $2 \; S_{q_a}$,

$$twist(X; q_a) = O(): \tag{41}$$

Proof of claim. Assume $(X_{\hat a}; q_{\hat a})$ and $(X_{\hat b}; q_{\hat b})$ are some choice of initial and terminal points with associated regular triangulations $T_{\hat a}$ and $T_{\hat b}$ that have j common saddle connection. But, assume that they do not satisfy Equation (41). We define $(X_a; q_a)$ to be the image of $(X_{\hat a}; q_{\hat a})$ under an appropriate number of Dehn twists around curves in S_q to ensure (41) and let $(X_b; q_b)$ be the image of $(X_{\hat b}; q_{\hat b})$ under the same homeomorphism. We will show that X_a and X_b are still near X and Z.

For $2 \; S_{q_a}$, if $\frac{1}{Ext_X()}$, by Lemma 3.16.

$$twist(X_{\hat a}; q_{\hat a}) \quad \frac{1}{Ext_X()} :$$

Hence, using the triangle inequality and Theorem 2.2

(42) $\quad \mathrm{twist}(X; q_a) \quad \mathrm{twist}(X; X_a) + \mathrm{twist}(X_a; q_a) = O \quad \frac{1}{\mathrm{Ext}_X()} \quad = O():$

Therefore, (41) is holds and no modification is required.

Now, assume $\frac{1}{\mathrm{Ext}_X()}$. Since is a non-cylinder curve, $\mathrm{Ext}_X{}^{-}()$ changes at most linearly with time (Equation (7)). Hence, for large enough, we have

(43) $$\frac{1}{\mathrm{Ext}_X()} \quad \frac{1}{\mathrm{Ext}_Z()}:$$

Again by Lemma 3.16, the number of Dehn twists n around that needs to be applied to q_a to ensure Equation (41) is at most $O(1{=}\mathrm{Ext}_X())$. That is,

$$X_a = \prod_{2 S_{q_a}} D^{n} X_{q_0};$$

where D is a Dehn twist around and n $\mathrm{Ext}_X()$. By, Theorem 2.2

$$d_T(X_a; X_a) \quad \sum_{2 S_{q_a}} n\,\mathrm{Ext}_X() \quad 1:$$

and

$$d_T(Y_b; X_b) \quad \sum_{2 S_{q_a}} n\,\mathrm{Ext}_Z() \quad 1:$$

Hence, $(X_a; q_a)$ and $(X_b; q_b)$ are as desired. Also, the images T_a and T_b of T_a and T_b are still regular triangulations and have j arcs in common.

For the rest of the proof, we assume Equation (41) holds. To the pair $(X_a; q_a)$ we associate the marking = fS; fE()g; T g as follows:

- Let S be the set of short curve in X and set $E() = \mathrm{Ext}_X()$.
- Let T be the $(q_a;)$–regular triangulation T_a which has j edges in common with the triangulation T_b.
- If 2 S_{q_a} then set the twisting around in so that

$$\mathrm{twist}(; X) = O(1):$$

- If 2 S_{q_a} then set the twisting around in so that

$$\mathrm{twist}(; T) = O(1):$$

The result is a marking that has bounded length in X and (by Equation (41)) has –bounded twist in X. Also, note that $S^c = S_{q_a}$ and $S^n = S_{q_a}$.

We can similarly associate a marking to the pair $(X_b; q_b)$. Here we can only conclude that is bounded in Z (not with bounded twist); this is because the inequality (41) does not necessarily hold for Z and q_b. Instead, similar to Equation (42), we have

(44) $$\mathrm{twist}(Z; q_b) \quad \frac{1}{\mathrm{Ext}_Z()}:$$

Assume $Z = g(Y)$, for g 2 (S). Let $_0 = g^{-1}()$. Then $_0$ in bounded in Y. Also, let R be the elements in R[U] coming from Lemma 4.7, (and Lemma 4.8 in case quadratic differentials in Q() are not orientable) and the j edges in T that are present in the $(q_b;)$–regular triangulation T_b. Taking this T_b is the partial triangulation in , we have 2 $\overline{M}_R(; _0;)$. The number of possible choices for is $O(^{jS_X j})$ (Lemma 5.6) and there are finitely many choices of for the homeomorphism type of $_0$. Lemma 5.8 provides an upper-bound for the size of the set $M_R(; _0;)$. Also, using the fact that is bounded in Z and Equation (44), similar to Lemma 5.6, we can conclude that

the association Z ! is at most $O\left(\prod_{2 S_{q_b}} \frac{1}{\mathrm{Ext}_Z()}\right)$ –to-one.

To summarize, we have defined a map from $B_j(Q(); X; Y;)$ to the union of sets of markings $M_R(; _0;)$, where is bounded X with –bounded twist and $_0$ is bounded in Y. The map is not one-to-one but we have a bound on the multiplicity.

The size of $B_j(Q(); X; Y;)$ is comparable to the product of the following: the number of choices for , the number of choices for the homeomorphism class of $_0$, the maximum multiplicity of the association Z ! and the size of $M_R(; _0;)$. That is,

$$
\begin{aligned}
jB_j(Q(); X; Y;)j \quad jM(X;)j \quad O(1) \prod_{2 S_{q_b}} \frac{1}{\mathrm{Ext}_Z()} jM_R(; _0;)j \\
{}^{jS_X j} \prod_{q_b} \frac{1}{\mathrm{Ext}_Z()} \frac{1}{2S^c} e^{h_R} \prod E() \frac{1}{\sqrt{{}_0 2 S_0}} \prod \frac{1}{\sqrt{E_0(_0)}} \frac{1}{\sqrt{c}}\ 2S \\
{}^{jS_X j + jS_Y j} e^{h_R} \prod \frac{1}{\sqrt{E()}} \prod_{_0 2 S_0} \frac{1}{\sqrt{E_0(_0)}}\ 2S :
\end{aligned}
$$

The last line follows from the previous line because, for every term in the product $\prod_{2S_{q_b}} \frac{1}{\mathrm{Ext}_Z()}$ we either have $\frac{1}{\mathrm{Ext}_Z()} = O()$ or, as in Equation (43),

$$
\frac{1}{\mathrm{Ext}_Z()} \quad \frac{1}{\sqrt{\mathrm{Ext}_Z()}} \frac{1}{\sqrt{\mathrm{Ext}_X()}}; \quad \text{and} \quad 2\ S_{q_a} \setminus S_{q_b} = S^n \setminus S^n_0:
$$

That is, each term can either be counted in the power of in the beginning of last line or it can be divided into a term in each of the last two products. The proof is finished after checking that h_R = (h j). This is true because all the relations in Lemma 4.7 are also relations in $H_1(S;)$. The fact that the j arcs we have fixed in T_a are homologically independent implies that these arcs and the other relations in homology are independent in R[U]. In fact, Lemma 4.8 is used only when &= 1. But this is accounted for in the definition of h (see §2.9). Hence, the dimension of R[U]=hRi is exactly j less than h = $\frac{\dim C+1}{2}$.

6. Geodesics in the thin part of moduli space

In this section we prove Theorem 1.1 and Theorem 1.5. The main idea, which is due to Margulis, is to prove an inequality, which shows that the flow (or more precisely an associated random walk) is biased toward a compact part of the space. Consider the stratum Q(). We discretize the projection

$$(Q()) \quad T(S);$$

by fixing an appropriate net $\tilde{N}$ in T(S). Then, we consider the random walk $f_i g_{i0}$ on the points in N and apply Theorem 5.1 to show that the projection of this random walk in M(S) is biased towards the compact subset of M(S). Moreover, we show that quadratic differentials $fq(_i; _{i+1})g_{i0}$ (see §2.2) tend not to have short saddle connections. See Lemma 6.4 for the precise formulation.

These estimates imply Theorem 1.1; this is because, roughly speaking, every closed geodesic in C can be approximated by a path along the net points.

6.1. Short saddle connections and simple closed curves. For a quadratic differential (X; q) 2 Q^1T (S), recall the set of short saddle connections $_q()$ (Definition 3.8). Define s(q;) to denote the maximum number of homologically independent disjoint saddle connections in $_q()$. Given the tuple , define

$$Q_{j;}() = \left\{ (X; q)\ 2\ Q() \quad s(q;)\ j \quad Q^1T(S): \right\}$$

For the rest of this section, with fix and denote $Q_{j;}()$ simply by $Q_{j;}$. Also, recall the definition of B(Q(); X;) from §4.1 and $B_j(Q(); X;)$ from §5. We would like to refine the definition of $B_j(Q(); X;)$. Roughly speaking, we are interested in a ball of radius centered at X that is allowed to move in the direction $Q_{j;}$ only. Namely, define $B(Q_{j;}; X;)$ to be the set Z 2 T(S) so that

$Z \; 2 \; B(Q(); X;)$

for the associated quadratic differential q_a, we have $s(q_a;) \; j$.

One can similarly define $B(Q_{j;}; X; Y;)$ as in §5. Recall the choice of $_1()$ from Lemma 3.13.

Lemma 6.1. For any > 0, there is $_2() < _1()$ such that for $< _2()$, any integer $j \; 0$, and any $X; Y$ $2 \; T(S)$, we have

$$B(Q_{j;}; X;) \; B_j(Q(); X;); \text{ and} \tag{45}$$

$$B(Q_{j;}; X; Y;) \; B_j(Q(); X; Y;): \tag{46}$$

Proof. It is enough to let $_2() = e^{\;\;2}{}_1$. Assume, $Z \; 2 \; B(Q_{j;}; X;)$, q_a and q_b are the associated quadratic differentials and $(b \quad a) <$. Let $!_1; :::; !_j$ be disjoint homologically independent saddle connections counted in $s(q_a;)$. Then, for each i, by Equation (8),

$$\frac{1}{\mathrm{Ext}_{q_a}(!_i)} \quad \frac{1}{\mathrm{Ext}_{q_b}(!_i)} \Big) ;$$

and by Theorem 2.1 the extremal length of any short curve containing $!_i$ changes by at most a factor of at most e^2. That is, $!_i$ 2 $_{q_b}(_1)$. The arcs $!_i$ are still disjoint and homologically independent in q_b. Hence, the set $f!_i g$ can be extended to both a $(q_a;)$–regular triangulation T_a and a $(q_b;)$–regular triangulations T_b (Lemma 3.13). Thus, by the definition $Z \; 2 \; B_j(Q(); X;)$. The proof of Equation (46) is similar.

6.2. Choosing a net.

By a $(c; 2c)$–separated net $N \quad M(S)$ we mean a set of points in $M(S)$ so that:

- the Teichmüller distance between any two net points in N is at least c, and
- any point in $M(S)$ is within distance $2c$ of a point in N.

Let

$$N(X;) = p(B(X;)) \setminus N:$$

Then, it is easy to check (see Lemma 3 in [EM2]):

Lemma 6.2. There exists a constant $c_0 > 0$ such that for any $c > c_0$, and $(c; 2c)$ net $\tilde{N}$ as above, we have

$$jN(X;)j \quad ^{3g \quad 3+p}: \tag{47}$$

Let $\tilde{N} = p^{\;\;1}(N)$. We assume the $r_0 > 2c$, where r_0 is the constant used to define $B(Q(); X;)$ (see §4.1). We denote the intersection of a ball in Teichmüller space, $B()$, with $\tilde{N}$ by $\tilde{N}()$. That is, for $X; Y \; 2 \; T(S)$,

$$\tilde{N}(Q(); X;) = B(Q(); X;) \setminus \tilde{N};$$

$$\tilde{N}(Q_{j;}; X;) = B(Q_{j;}; X;) \setminus \tilde{N};$$

$$\tilde{N}(Q(); X; Y;) = B(Q(); X; Y;) \setminus \tilde{N};$$

and

$$\tilde{N}(Q_{j;}; X; Y;) = B(Q_{j;}; X; Y;) \setminus \tilde{N}:$$

6.3. The main inequality.

For a real-valued function $f : M(S) \; ! \; \mathbb{R}$, consider the average function

$$A_{j;} f : T(S) \; ! \; \mathbb{R};$$

defined by

$$A_{j;} f(X) = e^{\;\;h} \sum_{Z \; 2 \; \tilde{N}(Q_{j;}; X;)} f(Z):$$

Here, as before

$$h = \frac{\dim C + 1}{2}:$$

Our main tool is the following ($_2()$ is as in Lemma 6.1):

Proposition 6.3. Given > 0, and $< {}_2()$ we have

(48) $$A_{j;} G(X) \quad {}^{m} e^{\quad j} G(X):$$

where G is as in Equation (4) and m depends only on the topology of S.

Proof. Enumerate the elements of $N(X;)$ as $y_1; :::; y_k$ and let $Y_i\ 2\ T(S)$ be a pre-image of y_i, $i = 1; :::; k$. By Lemma 2.3, every net point in $Z\ 2\ B(Q_{j;}; X; Y_i;) \setminus \tilde{N}$ is near at most $G(Y_i)$ points in $B(Q_{j;}; X; Y_i;)$. That is,

(49) $$\tilde{N}(Q_{j;}; X; Y_i;) G(Y_i)^2 \quad B(Q_{j;}; X; Y_i;)$$

Hence, we have

$$A_{j;} G(X) = e^{\quad h} \sum_{Z\,2\,\tilde{N}(Q_{j;};X;)} G(Z)$$

$$= e^{\quad h} \sum_{i=1}^{k} \sum_{Z\,2\,\tilde{N}(Q_{j;};X;Y_i;)} G(Y_i)$$

(Equation (49)) $$e^{\quad h} \sum_{i=1}^{k} \frac{B(Q_{j;}; X; Y_i;)}{G(Y_i)}$$

(Theorem 5.1 and (46)) $$e^{\quad h} \sum_{i=1}^{k} {}_{jS_X j + jS_Y j} e^{(h \quad j)} G(X)$$

(Equation (47)) $$e^{\quad j\ m} G(X):$$

Here, $m = (9g \quad 9 + 3p)\ jS_X j + jS_Y j + (3g \quad 3 + p)$.

Trajectories of the random walk. Suppose R and let n be the integer part of $R=$. By a trajectory of the random walk we mean a map

$$: f0; ng\ !\quad \tilde{N}\,T(S)$$

such that, for all $0 < k \quad n$, we have $d_T({}_k; {}_{k\ 1})$, where ${}_k = (k)$. Let $P(X; R)$ denote the set of all trajectories for which $d_T({}_0; X) = O(1)$. For $j\ 2\ N$, let $P_{;}(Q_{j;}; X; R)$ denote the set of all trajectories $2\ P(X; R)$ so that,

for $1 \quad k \quad n$

$${}_k\ 2\ \tilde{N}(Q(); {}_{k\ 1};):$$

$$k\, j\, 1 \quad k \quad n;\ {}_k\ 2\ B(Q_{j;}; {}_{k\ 1}; N;) \qquad \tilde{n}:$$

Given $X; Y\ 2\ T$, let $P_{;}(Q_{j;}; X; Y; R)$ denote the set of all trajectories $2\ P_{;}(Q_{j;}; X; R)$ such that

$$d_T \quad p(Y); p({}_n) \quad = O(1):$$

We say that a trajectory is almost closed in the quotient if

$$d_T \quad p({}_0); p({}_n) = O(1):$$

Finally, let $\hat{P}_{;}(Q_{j;}; X; R) = P_{;}(Q_{j;}; X; X; R)$ denote the subset of these trajectories starting from X which are almost closed in the quotient. Let $_2()$ be as in Lemma 6.1 and Proposition 6.3.

Lemma 6.4. For any $_0 >$ 0 there is $_0 >$ 0 so that for $> {}_0$, 0 1 and $< {}_2({}_0)$ we have

$$P_{;}(Q_{j;};X;Y;R) \quad e^{(h \quad j+_0)R} \quad G(Y): \frac{G(X)}{}$$ (50)

In particular,

$$\hat{P}_{;}(Q_{j;};X;R) \quad e^{(h \quad j+_0)R}:$$ (51)

Proof. Define

$$q_k() = i\,1 \quad i \quad k;\ {}_i\ 2\ N(Q_{j;};{}_{i\ 1};) :$$

This keeps track of the number of steps in the trajectory (amount the first k steps) that can be approximated by a segment in $Q_{j;}$. For $0 < r = k < R$, let $P_{;}(Q_{j;};X;Y;R;r)$ be the set of trajectories obtained from a trajectory $2\ P_{;}(Q_{j;};X;Y;R)$ but truncated after $k = r=$ steps. Define

$$V(R;r) = \sum_{2P_{;}(Q_{j;};X;Y;R;r)} G({}_k)e^{j\,q_k()}:$$

Also, let $R = n$, $q() = q_n()$ and

$$V(R) = \sum_{2P_{;}(Q_{j;};X;Y;R)} G({}_n)e^{j\,q()}:$$

Note that $G(Y)\ \ G({}_n)$ and $q()\ \ R$. Therefore,

$$jP_{;}(Q_{j;};X;Y;R)j \quad \frac{V(R)}{G(Y)e^{\ R\ j}}:$$ (52)

If ${}_{k+1}\ 2\ N(Q_{j;};{}_k;)$ then $q_{k+1}() = q_k() + 1$ and $q_{k+1}() = q_k()$ otherwise. Hence,

$$V(R;r+) = \sum_{2P_{;}(Q_{j;};X;Y;R;r+)} G({}_{k+1})e^{j\,q_{k+1}()}$$

$$\sum_{2P_{;}(Q_{j;};X;Y;R;r)} \left(\sum_{{}_{k+1}\,2\,N(Q_{j;};{}_k;)} G({}_{k+1})e^{j(q_k()+1)} + \sum_{{}_{k+1}\,2\,N(Q_{j;};{}_k;)} G({}_{k+1})e^{j\,q_k()} \right):$$

The two summands inside of the parenthesis are similar to the average defined above. Using Equation (48), the first term is less than (up to a multiplicative error)

$$e^{j(q_k()+1)}e^{h}(A_{;}G)_j({}_k) \quad e^{j(q_k()+1)}e^{h\,m}e^{\ j}G({}_k):$$ and the

second term is less than (again, up to a multiplicative error)

$$e^{j\,q_k()}e^{h}(A_{;0}G)({}_k) \quad e^{j\,q_k()}e^{h\,m}G({}_k):$$

Note that the right hand sides of the above two equations are the same. Hence,

$$V(R;r+) \quad {}^m e^{h} \sum_{2P(C;X;R;r)} e^{j\,q_k()}G({}_k)$$

$$= {}^m e^{h}\,V(R;r):$$ (53)

Now iterating (53) $n = R=$ times we get

$$V(R) \quad (C)^{mn}\,G(X)\,e^{hn} = G(X)\,e^{(h+\frac{m(\log()+\log(C))}{})R};$$ (54)

where $C > 0$; and m 2 N are uniform constants. We choose large enough so that

$$\frac{m \log() + \log(C)}{} < {}_{0}:$$

The lemma follows from Equation (52) and Equation (54).

Let $N(Q_{j;}; X; Y; R)$ be the number points Z 2 $B(Q(); X; Y; R)$ (see §5 for definition) so that associated geodesic $(X_t; q_t)$ spends proportion of time in $Q_{j;}$. Similarly, for x 2 $M(S)$, let $N(C_{j;}; x; R)$ be the number of conjugacy classes mapping classes associated to closed geodesics g in C of length at most R which pass within a uniformly bounded distance of the point x and so that for at least fraction of the points $(x_t; q_t)$ 2 g, $s(q_t;)$ j (see §6.1). As we shall see in the proof of the lemma below, for $x = p(X)$, $N(Q_{j;}; X; X; R)$ may be much larger than $N(C_{j;}; x; R)$.

Lemma 6.5. For any ${}_1 > 0$, there is ${}_1$ so that, for $> {}_1$ X 2 $T(S)$ and any sufficiently large R (depending only on ${}_1;$) we have

$$N(C_{j;}; p(X); (1 \quad {}_1)R) \quad P_{;}(Q_{j;}; X; R); \tag{55}$$

and

$$N(Q_{j;}; X; Y; (1 \quad {}_1)R) \quad P_{;}(Q_{j;}; X; Y; R)\, G(Y)^2: \tag{56}$$

Proof. Recall the definition of

$$I_X = \quad g\, 2 \quad (S)\, d_T(X; g\, X) = O(1) \quad :$$

from Lemma 2.3. Consider a closed geodesic g in C which intersects a uniformly bounded neighborhood of $x = p(X)$. Let [g] denote the corresponding conjugacy class in (S). Then there are approximately jI_Xj lifts of [g] to T_g which start within a bounded distance of X. Each lift G is a geodesic segment of length equal to the length of g.

We can mark points distance apart on G, and replace these points by the nearest net points in N. (This replacement is the cause of the ${}_1R$ error). This gives a map from lifts of geodesics to trajectories. If the original geodesic g has length at most $(1 \quad {}_1)R$ and has $s(q_t;)$ j for fraction of its points, then the resulting trajectory belongs to $P_{;}(Q_{j;}; X; R)$.

If two geodesic segments map to the same trajectory, then the segments fellow travel within O(1) of each other. In particular if g_1 and g_2 are the pseudo-Anosov elements corresponding to the two geodesics, then $d_T(g_2^{\ 1}g_1X; X) = O(1)$, thus $g_2^{\ 1}g_1$ 2 I_X.

We now consider all possible geodesics contributing to $N(C_{j;}; x; (1 \quad {}_1)R)$; for each of these we consider all the possible lifts which pass near X, and then for each lift consider the associated random walk trajectory. We get:

$$jI_Xj\, N(C_{j;}; x; (1 \quad {}_1)R) \quad jI_Xj\, P^{;}(Q_{j;}; X; R):$$

The factor of jI_Xj on the left hand side is due to the fact that we are considering all possible lifts which pass near X, and the factor of jI_Xj on the right is the maximum possible number of times a given random walk trajectory can occur as a result of this process. Thus, the factors of jI_Xj cancel, and the lemma follows. Note that by Lemma 2.3 (See also, Theorem 5.1) $jI(Y)j \quad G(Y)^2$. An argument similar to the proof of the first part implies Equation (56).

We need he following lemma which is due to Veech [Ve].

Lemma 6.6. Suppose g is a closed geodesic of length at most R on $M(S)$. Then for any x 2 g, any X so that $p(X) = x$ and every simple closed curve

$$\mathrm{Ext}_X() \quad e^{(6g \quad 4+2p)R}:$$

Proof of Theorem 1.5. Let > 0. Choose ${}_0; {}_1 = 3$. Now choose $\max f_0; {}_1 g$ and let R be large enough so that Equations (51) and (55) hold. We get,

$$N(C_{j;}; x; R) \quad e^{(h \quad j+2=3)R}: \tag{57}$$

Finally

$$N(C_{j;};R) \quad \sum_{x \in N} N(Q_{j;}; x; R);$$

where N is the net chosen above. In view of Lemma 6.6 and Lemma 6.2, the number of relevant points in the net is at most polynomial in R. However, for R large enough, this polynomial is less than $e^{R=3}$. Thus Theorem 1.5 follows.

Proof of Theorem 1.1. Let g be a closed geodesic in C n K. By taking K large enough we can assure that every quadratic differential along g has an arbitrary short saddle connection. We choose K so that Lemma 3.9 implies that any such quadratic differential (x; q), $_q()$ is non-empty for $_2()$. Hence the number of disjoint homologically independent saddle connections in $_q()$ is at least one. That is, g is counted in $N(C_{j;}; R)$ for j = 1 and = 1. The theorem now follows from Theorem 1.5.

Proof of Theorem 1.7. We can use the argument applied in the proof of Theorem 1.5. Let 0 < $_0$; $_1$ =3. Choose a net satisfying Lemma 6.2. Then choose maxf$_0$; $_1$g and let R be large enough so that Equations (50) and (56) hold. As in the proof of Theorem 1.5, Equation (1.7) follows from Lemma 6.2 and Lemma 6.6.

7. The Hodge Norm and the Hodge Distance.

In this section, we use the Hodge norm [Fo] to show that in any compact subset of C the geodesic flow is uniformly hyperbolic: see [ABEM] and Remark 7.5 below. There are many approaches to proving hyperbolic like behavior for the Teichmüller geodesic flow in different settings, see for example [AGY, AG, Fo, H2, Ve].

Let $H^1T(S)$ be the bundle of area one abelian differentials over T(S). We also denote by g_t the geodesic flow on $H^1T(S)$ (where we square an abelian differential to get a quadratic differential).

7.1. Hodge norm. Fix a point (X;) in $H^1T(S)$, where X 2 T(S) and is an abelian differential on X. Let : $H^1T(S)$! T(S) and p: $H^1T(S)$! $H^1M(S)$ be natural maps as in §2.3. Let $kk_{H;t}$ denote the Hodge norm on the surface X_t = (g_t). Also, for each abelian differential , let <(); =() 2 $H^1(X; R)$ be forms obtained by the real part and the imaginary part of the holonomy.

The following fundamental result is due to Forni [Fo, §2]:

Theorem 7.1. For any 2 $H^1(X; R)$ and any t 0, $kk_{H;t}$

$$e^t kk_{H;0}:$$

If, in addition, ^ <() = ^ =() = 0 and, for some compact subset K of $H^1M(S)$, the segment [; g_t] starts and ends in p $^1(K)$ and spends at least half the time in p $^1(K)$, then we have

$$kk_{H;t} \quad e^{(1\ )t} kk_{H;0}; \text{ where}$$

> 0 depends only on K.

Theorem 7.1 gives a partial hyperbolicity property of the geodesic flow on spaces of abelian differentials. In our application, we need a similar property for compact subsets of the spaces $Q^1M()$ of quadratic differentials.

7.2. Quadratic and abelian differentials. Here, we briefly treat the case when q 2 QM(S) is not the global square of an abelian differential. A standard construction, given X 2 T(S) and q a quadratic differential on X, is to pass to the possibly ramified double cover on which the foliation defined by q is orientable. More precisely, we consider the canonical (ramified) double cover : X ! X such that (q) = ^{2}. (See the proof of Lemma 4.8 for the explicit construction.) The set of critical values of coincides with the set of zeros of q with odd degree.

This yields a surface $\tilde{X}$ with an abelian differential . However, even if p(X) belongs to a compact subset of M(S), there may be a curve that has a very small extremal length in $\tilde{X}$. This may occur since the flat structure defined by q may have an arbitrarily short saddle connection connecting distinct zeroes. Such a saddle connection lifts to a very short loop in the double cover.

Let $\ell_{min}(q)$ denote the length of the shortest saddle connection in the flat metric defined by q. We have,

$$\ell_{min}() \quad \ell_{min}(q):$$

That is, if q does not have any short saddle connection, then also does not have any short saddle connections.

7.3. The Hodge norm on relative cohomology. Let $(X; q)$ 2 Q^1T () and let be the set of singularities of q. Let $\tilde{X}$ be as before and $\tilde{}$ be the pre-image of . On $\tilde{X}$, $\tilde{q}$ has a canonical square root which we denote by . To simplify the notation, if q is a square of an abelian differential, let $\tilde{X} = X$; $\tilde{} =$.

Let $j : H^1(\tilde{X}; \tilde{}; R) \; ! \; H^1(\tilde{X}; R)$ denote the natural map. We define a norm kk on the relative cohomology group $H^1(\tilde{X}; \tilde{}; R)$ as follows:

$$kk = kj()k_H + \sum_{(p;p^0)2} \int_{p;p^0} (\quad h); \tag{58}$$

where kk_H denotes the Hodge norm on $H^1(\tilde{X}; R)$, h is the harmonic representative of the cohomology class $j()$ and $_{p;p^0}$ is any path connecting the zeroes p and p^0. Since $j()$ and h represent the same class in $H^1(\tilde{X}; R)$, the Equation (58) does not depend on the choice of $_{p;p^0}$.

Let q_t, X_t and $_t$ be defined as usual and let $k \; k_t$ denote the norm (58) on the surface $\tilde{X}_t =$ (g_t). We have the following analogue of Theorem 7.1:

Theorem 7.2. Let K be a compact subset $Q^1M()$. Then there is $t_0 > 0$ so that for $t > t_0$ the following holds. Suppose $p(q_0); p(q_t)$ 2 K and that the geodesic segment $[q_0; q_t]$ spends at least half the time in $p^{\;\;1}(K)$. Suppose 2 $H^1(\tilde{X}; \tilde{}; R)$ with

$$j() \wedge <() = j() \wedge =() = 0:$$

Then we have

$$kk_t \quad e^{(1 \quad)t} kk_0;$$

where > 0 depends only on K.

This theorem is essentially in [AF] (Lemma 4:4). We reproduce the proof here for the convenience of the reader.

Proof of Theorem 7.2. Since K is compact, quadratic differentials in K have no short saddle connections. Hence, for u 2 $[0; t]$, $p(q_u)$ 2 K implies that $\tilde{X}_u$ is thick (has no curves with short extremal lengths). Therefore, there exist a constant c_K such that for any u with $p(q_u)$ 2 K, any harmonic 2 $H^1(\tilde{X}_u; R)$ and any arc on $\tilde{X}_u$ with end points in $\tilde{}$,

$$\int \quad c_K \, k \; k_{H;u} \; 1 + \ell_u(); \tag{59}$$

where $\ell_u()$ is the length of in flat metric associated to $_u$.

Under the assumptions of Theorem 7.2, there exists s 2 $[0{:}1t; 0{:}9t]$ such that $p(q_s)$ 2 K. Fix $p; p^0$ 2 $\tilde{}$. Since $\tilde{X}_0$ is thick, there exists a path $_0$ connecting p and p^0 with $\ell_0(_0) = O(1)$. Similarly, since $\tilde{X}_s$ and $\tilde{X}_t$ are thick there are paths $_s$ and $_t$ connecting p and p^0 such that $\ell_s(_s) = O(1)$, $\ell_t(_t) = O(1)$. Then,

$$\ell_0(_s) = O(e^s) \qquad \text{and} \qquad \ell_s(_t) = O(e^{t \quad s}):$$

Suppose 2 $H^1(X;\tilde{\ };\tilde{R})$ with j() ^ <() = j() ^ =() = 0. Let = j(). For 0 u t, let $_u$ denote the harmonic representative of the cohomology class on $\tilde{X}_u$. We have

$$kk_t \quad kk_0 \;\; k \;\; k_{H;t} \quad k \;\; k_{H;0} + \sum_{p;p^0 2} \int_{p;p^0} {}_t \quad {}_0$$

$$e^{(1\ \ )t}k \;\; k_{H;0} + \sum_{p;p^0 2} \int_{p;p^0} {}_t \quad {}_0; \tag{60}$$

where we have used Theorem 7.1. Since the integral in Equation (60) is independent of the choice of $_{p;p^0}$, we use $_{p;p^0} = {}_s$. Then, by Equation (59),

$$\int {}_0 \;\; c_K\, k \;\; k_{H;0}(1 + `_0({}_s)) \;\; c_K\, k \;\; k_{H;0}\, e^s : {}_s \tag{61}$$

Also,

$$\int_s {}_t = \int {}_t + \int {}_t =$$
$$\int^{s\ \ t}_{s} + \int^{t}_{t}$$
$$\int^{s\ \ t}_{s} {}_s + \int^{\ t}_{t} {}_s + \int {}_t$$
$$c_K \;\; k \;\; k_{H;s} + k \;\; k_{H;s} e^{t\ \ s} + k \;\; k_{H;t} \;\; :$$

where to pass from the first line to the second we used the fact that $_s$ and $_t$ represent the same cohomology class in $H^1(\tilde{X}; R)$, and in the last line we used Equation (59) to estimate each term. Then, using Equation (61), we have

$$\int_s {}_t \quad {}_0 \;\; c_K \;\; k \;\; k_{H;s} + k \;\; k_{H;s}e^{t\ \ s} + k \;\; k_{H;t} + k \;\; k_{H;0}e^s$$
$$c_K \;\; e^{(1\ \ )s} + e^{(1\ \ )s+t\ \ s} + e^{(1\ \ )t} + e^s \;\; k \;\; k_{H;0} \;\; c_K$$
$$e^{(1\ \ 0:1)t}k \;\; k_{H;0};$$

where in the second line we used Theorem 7.1 and in the last line we use the fact that s 2 [0:1t; 0:9t]. Substituting into Equation (60) we get

$$kk_t \quad kk_0 \;\; c_K\, e^{(1\ \ 0:1)t}k \;\; k_{H;0} \;\; c_K\, e^{(1\ \ 0:1)t}kk_0 :$$

Assuming t is large enough, we can assume that the multiplicative error is less than $e^{_0 t}$ for some $_0$ 0:1. The theorem then holds for (0:1 $_0$).

7.4. The Hodge Distance. Let g_t be the Teichmüller flow on $Q^1M()$. To each quadratic differential q, we associate its imaginary and real measured foliations (q), and $^+(q)$.

The flow g_t admits the following foliations:

(1) F^{ss}, whose leaves are sets of the form (X; q) j $^+$(q) = const ;

(2) F^{uu}, whose leaves are sets of the form (X; q) j (q) = const .

In other words, for $(X_0; q_0)$ 2 Q(), a leaf of F^{ss} is given by

$$^{ss}(X_0; q_0) = f(X; q) \ 2 \ Q() \ j \ ^+(q) = {}^+(q_0)g; \text{ and}$$

a leaf of F^{uu} is given by

$$^{uu}(X_0; q_0) = f(X; q) \ 2 \ Q() \ j \ (q) = (q_0)g:$$

Note that the foliations F^{ss}, F^{uu} are invariant under both g_t and (S); in particular, they descend to the moduli space $Q^1M()$.

We also consider the foliation $\mathcal{F}^u$ whose leaves are defined by

$$W^u(q) = \bigcup_{t\in\mathbb{R}} g_t W^{uu}(q)$$

and $\mathcal{F}^s$ whose leaves are defined by

$$W^s(q) = \bigcup_{t\in\mathbb{R}} g_t W^{ss}(q).$$

If $\mathcal{C}$ is a subset of moduli space of abelian differentials, we can locally identify a leaf of $\mathcal{F}^{ss}$ (or $\mathcal{F}^{uu}$) with a subspace W^- (or W^+) of $H^1(X, \Sigma; \mathbb{R})$. In fact, for $\lambda \in W^-$ (or $\lambda \in W^+$), we have

$$j(\lambda) \wedge \Im(\omega) = 0 \quad \text{and} \quad j(\lambda) \wedge \Re(\omega) = 0. \tag{62}$$

See §1 and §2 of [Fo] for more details.

If γ is a map from $[0, r]$ into some leaf of $\mathcal{F}^{ss}$, then we define the Hodge length $\ell(\gamma)$ of γ as $\int_0^r \|\gamma'(t)\| \, dt$, where $\|\cdot\|$ is the Hodge norm. Finally:

- If two abelian differentials ω and ω' belong to the same leaf of $\mathcal{F}^{ss}$, then we define $d_H(\omega, \omega')$ to be the infimum of $\ell(\gamma)$ where γ varies over paths connecting ω and ω' and staying in the leaf of $\mathcal{F}^{ss}$ $Q(\alpha)$. We make the same definition if ω and ω' are on the same leaf of $\mathcal{F}^{uu}$.
- By taking a ramified double cover (see §7.2), we can define $d_H(q, q')$ for any q, q' on the same leaf of $\mathcal{F}^{ss}$ in $Q(\alpha)$.

Lemma 7.3. Let K be a compact subset of $\mathcal{C}$. Suppose $(X, q), (X', q') \in p^{-1}(K)$ are in the same leaf of $\mathcal{F}^{ss}$. Let γ be a Hodge length minimizing path connecting q to q'. Suppose $t > t_0$ is such that for all $q'' \in \gamma$,

$$\left|\{ s \in [0, t] \mid g_s q'' \in p^{-1}(K) \}\right| \geq t/2. \tag{63}$$

Then

$$d_H(g_t q, g_t q') \leq e^{-ct} d_H(q, q'),$$

where c depend only on K.

Proof. This follows from Theorem 7.2 and Equation (62).

We now show that the above condition holds whenever the projections of $g_t q$ and $g_t q'$ to $\mathcal{C}$ are also close. See also Lemma 5.4 of [EM2].

Lemma 7.4. Let K be a compact subset of $\mathcal{C}$. Then there is a larger compact subset $K' \subset \mathcal{C}$ and a covering of K with a finite family of open sets $\mathcal{U}$ so that the following holds. Let $U_1, U_2 \subset Q(\alpha)$ be connected open sets so that $p(U_i) \in \mathcal{U}$, $i = 1, 2$. Let $(X, q), (X', q') \in U_1$ and $t > 0$ be such that $g_t(q), g_t(q') \in U_2$. Further, assume that

$$\left|\{ s \in [0, t] \mid p(g_s q) \in K \}\right| \geq t/2. \tag{64}$$

Then,

$$\left|\{ s \in [0, t] \mid p(g_s q') \in K' \}\right| \geq t/2. \tag{65}$$

Proof. Let $\epsilon > 0$. We can find an open cover $\mathcal{U}$ of K so that the following holds. Let U be connected open sets so that $p(U) \in \mathcal{U}$, and let $(X_1, q_1), (X_2, q_2) \in U$. Then for any saddle connection ω, we have

$$\frac{1}{\epsilon} \ell_{q_1}(\omega) \leq \ell_{q_2}(\omega) \leq \epsilon\, \ell_{q_1}(\omega). \tag{66}$$

Let $U_1, U_2 \subset Q(\alpha)$ be connected open sets so that $p(U_i) \in \mathcal{U}$, $i = 1, 2$. Let $(X, q), (X', q') \in U_1$ and $t > 0$ be such that $g_t(q), g_t(q') \in U_2$. We first claim that (66) is true for quadratic differentials $q_s = g_s(q)$ and $q'_s = g_s(q')$ as well for a larger constant $\epsilon' = \epsilon^2$. Assume, for contradiction that

$$\ell_{q_s}(\omega) > \epsilon' \ell_{q'_s}(\omega).$$

for some $s \in [0, t]$. Assume ω is mostly vertical in q_s. That is,

$$\Im(\mathrm{hol}_{q_s}(\omega)) > \frac{1}{2} \ell_{q_s}(\omega).$$

Then

$$\begin{aligned}\ell_q(\omega) &= (\mathrm{hol}_q(\omega)) \\ &= e^s(\mathrm{hol}_{q_s}(\omega)) \\ &> \frac{1}{2}e^s\ell_{q_s}(\omega) \\ &> \frac{1}{2}e^{s_0}\ell_{q'_s}(\omega) \quad {}^{0}\frac{1}{2}\ell_{q'}(\omega):\end{aligned}$$

Which contradicts Equation (66). In case ω is mostly horizontal, we move forward in time and argue the same way. This proves the claim.

Now let be such that the length of any saddle connection in $q \in K$ is larger than , and let K^0 be the compact subset of C consisting of quadratic differentials where the length of every saddle connection is larger than ${}^0 = \; = {}^0$. Then (65) follows from the above length comparison.

Remark 7.5. We have essentially shown that under the assumption Equation (64) we have exponential contraction along the foliation F^{ss} (and similarly exponential expansion along the foliation F^{uu}).

8. Outline of the proof of Theorem 1.2

In this section, we prove Theorem 1.2. We only outline the arguments here since they are well known a more detailed version is already present in [H2]. We essentially follow the work of Margulis [Mar]. First, we need a closing lemma.

Lemma 8.1 (Closing Lemma). Let K be a compact subset of C consisting of non-orbifold points. Given a quadratic differential $(x; q) \in K$ and > 0, there exist constants $L_0 > 0$, and open neighborhoods $U \; U^0 \; C$ of $(x; q)$ with the following property. For $L > L_0$, suppose that $g : [0; L] \to C$ is a Teichmüller geodesic segment such that

(a) $g(0); g(L) \in U$ and
(b) g spends more than half of its length in K.

Let g_1 be the closed path in C which is the union of g and a segment connecting $g(L)$ to $g(0)$ in U. Then there exists a unique closed geodesic $g^0 \; C$ with the following properties:

(I) g^0 and g_1 have lifts in $T(S)$ which stay –close with respect to the Teichmüller metric.
(II) The length of g^0 is within of L,
(III) g^0 passes through U^0.

Remark 8.2. We remark that in Lemma 8.1 if we remove the assumption that K consists of non-orbifold points then there are at most a uniformly bounded number of closed geodesics satisfying conditions (I–III). A version of the closing lemma can be found in [H2].

Outline of the proof of Lemma 8.1. Consider the stable and unstable foliations for the geodesic flow. Our goal is to show that if U is small enough, the first return map on these foliations will define a contraction with respect to the Hodge distance function. As a result, we find a fixed point for the first return map in U^0; this is the same as a closed geodesic going through U^0.

In view of Lemma 7.3 and Lemma 7.4 there is in fact a neighborhood of $(x; q)$ such that the time L geodesic flow restricted to the neighborhood expands along the leaves of F^{uu} and contracts along the leaves of F^{ss}.

Then, the contraction mapping principle (applied first to the map on F^{ss} and then to the inverse of the map on F^{uu}) allows us to find a fixed point for the geodesic flow near $(x; q)$ (in a slightly bigger neighboorhood). In other words, there are neighborhoods $U \; U^0$ of $(x; q)$ such that:

if $g : [0; L] \to C$ satisfies properties (a) and (b) then in view of the hyperbolicity statement (Lemma 7.3)

the time L geodesic flow restricted to U expands along the leaves of F^{uu} and contracts along the leaves of F^{ss}, in the metric d_H,

for any $q_1; q_2 \in U$; if $q_1 \in F^{ss}(q_2)$ or $q_1 \in F^{uu}(q_2)$ then $d_H(q_1; q_2)$.

We can apply the contraction mapping principle to F^{ss} to find $(x_0; q_0)$ 2 U^0 such that $g_L(q_0)$ 2 $F^{uu}q_0$. Now we can consider the first return map of the map g_t on $F^{uu}(q_0)$.

Proof of Theorem 1.2. Note that by the bound proved in Theorem 1.5, we only need to consider the set of closed geodesics going through a fixed compact subset of C. We have

by Theorem 2.4, the geodesic flow on C is mixing, and

on a fixed compact subset of $Q^1M(S;)$ the geodesic flow is uniformly hyperbolic.

every nearly closed orbit approximates a close orbit (Lemma 8.1).

Hence, all the ingredients are in place to drive Theorem 1.2 following the work of Margulis [Mar]. (See also x20:6 in [KH].)

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