

Spacecraft Attitude and Gyro-Bias Estimation with a Single Magnetometer on $SO(3) \times \mathbb{R}^3$

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This paper presents a Bayesian estimation scheme for the attitude and the gyro bias of a spacecraft orbiting Earth. In particular, it is assumed that a single magnetometer and a gyroscope are available to measure the direction of the local magnetic field and the angular velocity. The objective is to estimate the three-dimensional attitude and the three-dimensional gyro bias with the series of two-dimensional measurements of the Earth magnetic field coupled with the attitude kinematics. To achieve this, uncertainties in the attitude are represented with the matrix Fisher-Gaussian distribution on the product manifold of the special orthogonal group and the Euclidean space, and utilizing this, a Bayesian estimator is constructed. The proposed approach represents the uncertainties in the attitude and the correlation between attitude and gyro bias in a global fashion without relying on any attitude parameterization or linearization. The performance is illustrated with numerical simulations.

I. Nomenclature

ECI	=	Earth-centered inertial frame
ECEF	=	Earth-centered, Earth-fixed frame
$SO(3)$	=	special orthogonal group, $\{R \in \mathbb{R}^{3 \times 3} \mid R^T R = I_{3 \times 3}, \det[R] = 1\}$
S^2	=	unit-sphere, $S^2 = \{q \in \mathbb{R}^3 \mid \ q\ = 1\}$
$R \in SO(3)$	=	attitude of the spacecraft,
	=	linear transformation from the body-fixed frame to the Earth-centered inertial frame
$\Omega \in \mathbb{R}^3$	=	angular velocity of the spacecraft, resolved in the body-fixed frame
$x \in \mathbb{R}^3$	=	gyro bias resolved in the body-fixed frame
$F \in \mathbb{R}^{3 \times 3}$	=	parameter of the matrix Fisher distribution
$c(F) \in \mathbb{R}$	=	normalizing constant of the matrix Fisher distribution with the parameter F
$b \in S^2$	=	the direction of the Earth magnetic field in the ECEF frame

II. Introduction

Extended Kalman filters (EKF) and their variations have been applied to various attitude estimation problems in aerospace engineering and robotics. In particular, the multiplicative extended Kalman filter (MEKF) [1-3] has been the most successful, where the mean attitude is described by a quaternion and the uncertainties around the mean attitude is defined by a three-dimensional Gaussian distribution of minimal attitude representations, such as Rodrigues parameters. MEKF relies on the assumption that the attitude uncertainty is highly concentrated about the mean attitude, which legitimizes the linearization of EKF. Furthermore, multiple values of attitude parameterizations may represent the same attitude, and therefore, the Gaussian distribution of attitude parameters should be *wrapped*, which is not considered in the common implementation of MEKF. As such, large attitude uncertainties and large estimation errors may not be effectively addressed by MEKF.

To overcome these issues, there have been several efforts to develop attitude estimation schemes with probability density models directly formulated on the special orthogonal group $SO(3)$, without relying on local parameterizations. In [4], an attitude estimation scheme is proposed by representing a probability density function on the special orthogonal group with noncommutative harmonic analysis [5], which is the Fourier analysis generalized into compact manifolds. In

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particular, [6] formulates a probability density function on $\text{SO}(3) \times \mathbb{R}^n$ inspired by noncommutative harmonic analysis, and the evolution of the density is studied in the ambient Euclidean space to develop an attitude estimator on $\text{SO}(3)$.

On the other hand, directional statistics present various probability density models on compact manifolds [7]. In particular, stochastic properties of the matrix Fisher distribution on the special orthogonal group are presented, and based on those, attitude estimation schemes are proposed in [8]. However, this work does not consider the bias in the angular velocity measured by a gyro, which is inevitable in long-term space missions. Recently, a new probability density, referred to as the *matrix Fisher-Gaussian distribution (MFG)*, is introduced on the product manifold of $\text{SO}(3) \times \mathbb{R}^n$ [9], where the angular-linear correlation between the attitude and Euclidean space of an arbitrary dimension is formulated along the tangent space of $\text{SO}(3)$ to capture the correlation without over-parameterization. The MFG inherits desirable properties from the matrix Fisher distribution and the Gaussian distribution, as the marginal distribution of MFG for the attitude part is a matrix Fisher distribution, and the distribution of the linear part conditioned by the attitude is Gaussian. By utilizing these properties, a Bayesian estimator is developed to estimate the attitude and gyro bias concurrently. This approach resolves the aforementioned issues of MEKF, as the attitude uncertainties and the correlations are globally formulated on $\text{SO}(3)$ without relying on any local parameterization. In fact, when the attitude is highly concentrated, a matrix Fisher-Gaussian distribution is approximated by a joint Gaussian distribution of parameterizations of the attitude and linear random variable as in MEKF. Therefore, MFG encloses the uncertainty formulation of MEKF as a special case. Numerical simulations illustrate that attitude estimators with MFG exhibit similar performances compared with MEKF or unscented Kalman filters for small attitude estimation errors and small uncertainties. However, when the estimation error or the degree of uncertainties is increased, there are non-trivial improvements in accuracy of the estimated attitude and gyro bias, at the cost of additional computation time.

This paper utilizes the matrix Fisher-Gaussian distribution to estimate the attitude and gyro bias with single direction measurements. In particular, we consider a satellite orbiting around the Earth with a single magnetometer and a gyroscope. A Bayesian attitude estimator is presented, where the correction step accounts the measurement of the Earth magnetic fields, which provides two-dimensional information of the three-dimensional attitude. The presented approach should be distinguished from the existing development of deterministic attitude observers based on single vector measurements, such as [10, 11]. In contrast to the attitude observers that provide the attitude estimate only, the presented stochastic attitude estimator follows the Bayesian framework. Consequently, it provides the estimated probability density function, which specifies the complete stochastic properties, including the mean and moments. Having the measure of confidence in the estimated attitude is critical when only single direction measurements are available, as the estimated attitude distribution may be degenerated, i.e., the rotation about the local magnetic field is mostly uncertain. Therefore, the underlying assumption for MEKF requiring that attitude distribution is concentrated about the mean may not be satisfied well. Numerical examples show that the proposed approach estimates the attitude accurately with a series of magnetometer measurements, even for the challenging case of the equatorial orbit where the magnetic field does not vary extensively. And, the estimation error is substantially reduced compared with MEKF.

III. Matrix Fisher–Gaussian Distribution

The configuration space for the attitude of a rigid body is the three-dimensional special orthogonal group,

$$\text{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} \mid R^T R = I_{3 \times 3}, \det[R] = 1\},$$

where each rotation matrix corresponds to the linear transformation of the coordinates of a vector from the body-fixed frame to the inertial frame. The lie algebra $\mathfrak{so}(3)$ is the set of 3×3 skew-symmetric matrices, i.e., $\mathfrak{so}(3) = \{S \in \mathbb{R}^{3 \times 3} \mid S = -S^T\}$. The *hat* map: $\wedge : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is defined such that $\hat{x} = -(\hat{x})^T$, and $\hat{x}y = x \times y$ for any $x, y \in \mathbb{R}^3$. The inverse of the hat map is denoted by the *vee* map: $\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$. The two-sphere is the set of unit-vectors in $\mathbb{R}^3$, i.e., $S^2 = \{q \in \mathbb{R}^3 \mid \|q\| = 1\}$, and the i -th standard basis of $\mathbb{R}^3$ is denoted by $e_i \in S^2$ for $i \in \{1, 2, 3\}$. The set of circular shifts of $(1, 2, 3)$ is defined as $\mathcal{I} = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

In this section, we first present the formulation of the matrix Fisher–Gaussian distribution on $\text{SO}(3) \times \mathbb{R}^n$, along with selected stochastic properties. The detailed developments and proofs are available in [9, 12].

A. Formulation

Definition 1 *The random elements $(R, x) \in \text{SO}(3) \times \mathbb{R}^n$ follow the matrix Fisher–Gaussian distribution with parameters $\mu \in \mathbb{R}^n$, $\Sigma = \Sigma^T \in \mathbb{R}^{n \times n}$, $U, V \in \text{SO}(3)$, $S = \text{diag}(s_1, s_2, s_3) \in \mathbb{R}^{3 \times 3}$ with $s_1 \geq s_2 \geq |s_3| \geq 0$ and $P \in \mathbb{R}^{n \times 3}$, if it has the*

following density function:

$$p(R, x; \mu, \Sigma, V, S, U, P) = \frac{1}{c(S)\sqrt{(2\pi)^n \det(\Sigma_c)}} \exp \left\{ -\frac{1}{2}(x - \mu_c)^T \Sigma_c^{-1} (x - \mu_c) \right\} \text{etr} \{ F R^T \}, \quad (1)$$

where $\text{etr}(\cdot) = \exp(\text{tr}[\cdot])$. And $\mu_c \in \mathbb{R}^n$ is given by

$$\mu_c = \mu + P v_R, \quad (2)$$

where the expression for v_R has two forms, which defines the two variants of MFG, namely MFGB and MFGI:

$$(MFGI) \quad v_R = (QS - SQ^T)^\vee, \quad (3a)$$

$$(MFGB) \quad v_R = (SQ - Q^T S)^\vee, \quad (3b)$$

with $Q = U^T R V$. In addition, $0 < \Sigma_c \in \mathbb{R}^{n \times n}$ is defined as

$$\Sigma_c = \Sigma - P(\text{tr}[S] I_{3 \times 3} - S)P^T, \quad (4)$$

Also, $F = USV^T \in \mathbb{R}^{3 \times 3}$, and $c(S) \in \mathbb{R}$ is the normalizing constant of the corresponding matrix Fisher distribution [8]. This distribution is denoted by $\mathcal{MG}(\mu, \Sigma, P, U, S, V)$.

The probability density function of MFG given by (1) is interpreted as follows. As a density on $\text{SO}(3) \times \mathbb{R}^n$, there are three main components: the first part of (1) is for normalization; the second part is for the linear random variable x and has the form of Gaussian distribution $\mathcal{N}(\mu_c, \Sigma_c)$; the last part is for the attitude R and has the form of matrix Fisher distribution $\mathcal{M}(F)$. The correlation between x and R is caused by the dependency of μ_c on R . In fact, the marginal distribution of R is a matrix Fisher distribution with parameter F , and the distribution of x conditioned by R is Gaussian with $x|R \sim \mathcal{N}(\mu_c(R), \Sigma_c)$. The interpretation of the matrix Fisher distribution $\mathcal{M}(F)$ is available in [8]. To summarize it briefly, the mean attitude is given by UV^T , and the dispersion about the mean attitude is described by U, V and S , where the matrices U, V specify the principal axes of rotations, and the diagonal matrix S specifies the degree of dispersion in the way that the distribution is more concentrated as S increases. The conditional distribution of $x|R$ is easily interpreted by the Gaussian distribution. As such, MFG can be considered as a combination of the matrix Fisher distribution and the Gaussian distribution.

The most interesting part is the attitude-linear correlation specified by the $3n$ elements of P . Roughly speaking, the correlation between two random variables represents the linear relation for how much one variable would be expected to deviate from its mean when the other variable is shifted from its own mean. Here, the correlation between x and R is caused by the fact that the mean of x conditioned by R is given as $\mu_c(R)$ which is dependent on R . When conditioned by R , it is shifted by $P v_R$ in (2), where v_R in (3) indicates how R deviates from the mean attitude UV^T . In other words, the deviation of R from its mean UV^T is measured by v_R , which causes the conditional mean of x to be shifted by $P v_R$. For example, when $R = UV^T$, we have $v_R = 0$ and $\mu_c = \mu$. While the attitude is described by the nine elements of R , the deviation of R from its mean is specified by the three elements of v_R . Therefore, the correlation is defined by the $3n$ elements of P , avoiding over-parameterizations.

Two variants of MFG can be used to model the angular-linear correlation, given by the two expressions for v_R in (3a) and (3b), and are named MFGI [12] and MFGB [9] respectively. Although MFGI and MFGB share most of the stochastic properties, they differ by how the correlation is interpreted. For MFGI, x is correlated with rotations of R interpreted in the inertial frame; whereas for MFGB, x is correlated with rotations of R interpreted in the body-fixed frame of R . When applied to attitude estimation with gyro bias, since the bias is represented in the body-fixed frame, it is more appropriate to use MFGB to model its correlation with the attitude.

With regards to the construction, MFG can also be interpreted as the Gaussian distribution in the ambient Euclidean space $\mathbb{R}^9 \times \mathbb{R}^n$, after projecting it onto $\text{SO}(3) \times \mathbb{R}^n$ and restricting the correlation to the tangent space of $\text{SO}(3)$. Finally, it has been shown that a matrix Fisher distribution can be approximated by a three-dimensional Gaussian distribution when it is highly concentrated, i.e., when $s_3 \gg 0$ [13]. This is the same with MFG, which can be approximated by a $(3+n)$ -dimensional Gaussian when its attitude part is highly concentrated.

B. Maximum Likelihood Estimation

We first present selected moments of MFG, which are used in the approximate MLE later.

Theorem 1 Suppose $(R, x) \sim \mathcal{MG}(\mu, \Sigma, P, U, S, V)$. Then,

$$\mathbb{E}[R] = UDV^T, \quad (5)$$

where $D = \text{diag}(d_1, d_2, d_3)$ and

$$d_i = \frac{1}{c(S)} \frac{\partial c(S)}{\partial s_i}. \quad (6)$$

Also,

$$\mathbb{E}[x] = \mu, \quad (7)$$

$$\mathbb{E}[v_R] = 0, \quad (8)$$

$$\mathbb{E}[xx^T] = \Sigma_c + \mu\mu^T + P\mathbb{E}[v_R v_R^T]P^T, \quad (9)$$

$$\mathbb{E}[xv_R^T] = P\mathbb{E}[v_R v_R^T], \quad (10)$$

where $\mathbb{E}[v_R v_R^T] \in \mathbb{R}^{3 \times 3}$ is a diagonal matrix with the i -th diagonal element given by

$$(\mathbb{E}[v_R v_R^T])_{ii} = s_j d_j + s_k d_k. \quad (11)$$

for $(i, j, k) \in \mathcal{I}$.

Proof: The proofs for (5) to (10) are available in [9]. Note that in [9], we also have

$$(\mathbb{E}[v_R v_R^T])_{ii} = (s_j^2 + s_k^2)\mathbb{E}[Q_{jk}^2] - 2s_j s_k \mathbb{E}[Q_{jk} Q_{kj}]. \quad (12)$$

In addition, from [14], we have when $s_j \neq s_k$

$$\mathbb{E}[Q_{jk}^2] = \frac{1}{c(S)} \left(-\frac{\partial c(S)}{\partial s_j} \frac{s_j}{s_k^2 - s_j^2} + \frac{\partial c(S)}{\partial s_k} \frac{s_k}{s_k^2 - s_j^2} \right), \quad (13)$$

$$\mathbb{E}[Q_{jk} Q_{kj}] = \frac{1}{c(S)} \left(-\frac{\partial c(S)}{\partial s_j} \frac{s_k}{s_k^2 - s_j^2} + \frac{\partial c(S)}{\partial s_k} \frac{s_j}{s_k^2 - s_j^2} \right). \quad (14)$$

Then (11) can be derived from the above three equations, and noting that $\mathbb{E}[v_R v_R^T]$ is continuous in S . $\square$

Next, we consider the maximum likelihood estimation (MLE) problem to construct an MFG from its samples. Given a set of samples $(R_i, x_i)_{i=1}^{N_s}$, the log-likelihood function of the parameters, after omitting some constants, is given by

$$\begin{aligned} l = & -\log(c(S)) + \text{tr}(F\bar{\mathbb{E}}[R]^T) - \frac{1}{2} \log(\det(\Sigma_c)) \\ & - \frac{1}{2} \bar{\mathbb{E}}[(x - \mu - P v_R)^T \Sigma_c^{-1} (x - \mu - P v_R)], \end{aligned} \quad (15)$$

where $\bar{\mathbb{E}}[\cdot]$ represents the sample mean of a random variable. For example, $\bar{\mathbb{E}}[R] = \frac{1}{N_s} \sum_{i=1}^{N_s} R_i$. Instead of jointly maximizing the likelihood, we exploit the fact that the marginal distribution for R is a matrix Fisher distribution, and the conditional distribution for $x|R$ is Gaussian. More specifically, the log-likelihood for the marginal distribution corresponds to the first two terms on the right hand side of (15), and the marginal MLE for parameters U, S, V is solved by the MLE of the matrix Fisher distribution.

Theorem 2 The marginal maximum likelihood estimates for U, V are given by the proper singular value decomposition $\bar{\mathbb{E}}[R] = UDV^T$, and the marginal MLE for S is given by solving (6) for S using D .

After obtaining U, S, V , they are used in the conditional log-likelihood for $x|R$ corresponding to the last two terms on the right hand side of (15). Denote the sample covariance $\text{cov}(a, b) = \bar{\mathbb{E}}[ab^T] - \bar{\mathbb{E}}[a] \bar{\mathbb{E}}[b]^T$ for $a \in \mathbb{R}^m, b \in \mathbb{R}^n$, then the resulting conditional MLE is addressed as follows.

Theorem 3 Let $U, V \in \text{SO}(3)$ and $S \in \mathbb{R}^{3 \times 3}$ be the solution of the marginal MLE for R . Define $Q_i = U^T R_i V$, and $\nu_{R_i} = (Q_i S - S Q_i^T)^\vee$ for MFGI, or $\nu_{R_i} = (S Q_i - Q_i^T S)^\vee$ for MFGB, $i = 1, \dots, N_s$. Then the solution of the conditional MLE for P , μ , and Σ is given by

$$P = \overline{\text{cov}}(x, \nu_R) \overline{\text{cov}}(\nu_R, \nu_R)^{-1}, \quad (16)$$

$$\mu = \bar{E}[x] - P \bar{E}[\nu_R], \quad (17)$$

$$\Sigma = \overline{\text{cov}}(x, x) - P \overline{\text{cov}}(x, \nu_R)^T + P(\text{tr}[S] I_{3 \times 3} - S) P^T. \quad (18)$$

IV. Bayesian Attitude Estimation with Single Direction Measurements

Having formulated the matrix Fisher–Gaussian distribution, we present a Bayesian attitude estimator utilizing it. The proposed attitude estimator is a so-called *assumed density filter*, where the uncertainties are distributed according to the matrix Fisher–Gaussian distributions.

Consider the following kinematics model [6, 8]

$$R^T dR = (\hat{x} + \hat{\Omega}) dt + (H_u dW_u)^\wedge, \quad (19)$$

$$dx = H_v dW_v, \quad (20)$$

where $R \in \text{SO}(3)$ is the attitude of spacecraft and $x \in \mathbb{R}^3$ is the bias of the onboard gyroscope. The vector $\Omega \in \mathbb{R}^3$ is the angular velocity measured by the gyroscope that is resolved in the body-fixed frame. Next, W_u and $W_v \in \mathbb{R}^3$ are two independent three-dimensional Wiener processes, and $H_u, H_v \in \mathbb{R}^{3 \times 3}$ are two matrices describing the strengths of noises. The angular velocity measurement has two sources of noises: the bias term x and the Gaussian white noise contributed by $H_u dW_u$. The bias is slowly varying while being driven by another white noise $H_v dW_v$.

Let the time be discretized by a sequence $\{t_0, t_1, \dots\}$. For convenience, it is assumed that the time step $h \in \mathbb{R}^1$ is fixed, i.e., $h = t_{k+1} - t_k$ for any k . According to [15] Eqn. 14], the kinematics model can be discretized into

$$R_{k+1} = R_k \exp \{ h(\hat{\Omega}_k + \hat{x}_k) + (H_u \Delta W_u)^\wedge \}, \quad (21)$$

$$x_{k+1} = x_k + H_v \Delta W_v, \quad (22)$$

where $\Delta W_u, \Delta W_v \in \mathbb{R}^3$ are the stochastic increments of the Wiener processes over a time step, which are Gaussian with

$$H_u \Delta W_u \sim \mathcal{N}(0, h G_u), \quad H_v \Delta W_v \sim \mathcal{N}(0, h G_v), \quad (23)$$

where $G_u = H_u H_u^T$ and $G_v = H_v H_v^T \in \mathbb{R}^{3 \times 3}$.

The initial attitude and bias $(R(t_0), x(t_0))$ at t_0 are assumed to follow MFG with $n = 3$ and $(\mu_0, \Sigma_0, P_0, U_0, S_0, V_0)$ of appropriate dimensions. The presented Bayesian estimator is composed of two parts: uncertainty propagation and correction. The detailed developments and proofs are available in [9, 12].

A. Analytical Uncertainty Propagation

Suppose $(R_k, x_k) \sim \mathcal{MG}(\mu_k, \Sigma_k, P_k, U_k, S_k, V_k)$. In this subsection, we present an approach to construct a new MFG corresponding to the propagated density of (R_{k+1}, x_{k+1}) by calculating its moments analytically. It follows the approach of MLE discussed in Section III.B, where the marginal distribution of R_{k+1} is constructed first.

Theorem 4 The expectation of the propagated attitude R_{k+1} is given by

$$E[R_{k+1}] = \left\{ E[R_k] \left(I_{3 \times 3} + \frac{h}{2} (G_u - \text{tr}[G_u] I_{3 \times 3}) \right) + h U_k E \left[Q_k V_k^T \widehat{P_k \nu_{R_k}} \right] \right\} e^{h(\hat{\Omega}_k + \hat{\mu}_k)} + O(h^2), \quad (24)$$

where $Q_k = U_k^T R_k V_k$, $\nu_{R_k} = (Q_k S_k - S_k Q_k^T)^\vee$ for MFGI, or $\nu_{R_k} = (S_k Q_k - Q_k^T S_k)^\vee$ for MFGB.

With the given $E[R_{k+1}]$, the marginal MLE for the attitude part of MFG can be solved as discussed in Theorem 2, which yields the estimates of U_{k+1} , S_{k+1} and V_{k+1} . Define $Q_{k+1} = U_{k+1}^T R_{k+1} V_{k+1}$, and $\nu_{R_{k+1}} = (Q_{k+1} S_{k+1} - S_{k+1} Q_{k+1}^T)^\vee$ for MFGI, or $\nu_{R_{k+1}} = (S_{k+1} Q_{k+1} - Q_{k+1}^T S_{k+1})^\vee$ for MFGB as the intermediate parameters for the MFG at time t_{k+1} . Then the conditional MLE for the rest of parameters is solved as in Theorem 3 with the moments given as follows.

Theorem 5 Let $\tilde{U}, \tilde{V} \in \text{SO}(3)$ and $\tilde{S}, \tilde{\tilde{S}} \in \mathbb{R}^{3 \times 3}$ be

$$\begin{aligned}\tilde{U} &= U_{k+1}^T U_k, & \tilde{V} &= V_{k+1}^T e^{-h(\hat{\Omega}_k + \hat{\mu}_k)} V_k, & \tilde{S} &= \tilde{U}^T S_{k+1} \tilde{V}, \\ \tilde{\tilde{V}} &= V_{k+1}^T e^{-h(\hat{\Omega}_k + \hat{\mu})} G_u^T V_k, & \tilde{\tilde{S}} &= \tilde{U}^T S_{k+1} \tilde{\tilde{V}}^T.\end{aligned}$$

Also, let $\tilde{v}_R, \tilde{\tilde{v}}_R \in \mathbb{R}^3$, and $\Gamma_Q \in \mathbb{R}^{3 \times 3}$ be

$$(MFGI) \quad \tilde{v}_R = (Q_k \tilde{S}^T - \tilde{S} Q_k^T)^\vee, \quad (25a)$$

$$(MFGB) \quad \tilde{v}_R = (\tilde{S}^T Q_k - Q_k^T \tilde{S})^\vee, \quad (25b)$$

$$(MFGI) \quad \tilde{\tilde{v}}_R = (Q_k \tilde{\tilde{S}}^T - \tilde{\tilde{S}} Q_k^T)^\vee, \quad (26a)$$

$$(MFGB) \quad \tilde{\tilde{v}}_R = (S_{k+1} \tilde{U} Q_k \tilde{\tilde{V}}^T - \tilde{\tilde{V}} Q_k^T \tilde{U}^T S_{k+1})^\vee. \quad (26b)$$

$$(MFGI) \quad \Gamma_Q = (\text{tr}[Q_k \tilde{S}^T] I_{3 \times 3} - Q_k \tilde{S}^T) Q_k \quad (27a)$$

$$(MFGB) \quad \Gamma_Q = \text{tr}[Q_k^T \tilde{S}] I_{3 \times 3} - Q_k^T \tilde{S} \quad (27b)$$

Then, the moments of x_{k+1} and $v_{R_{k+1}}$ required for the conditional MLE are given by

$$\mathbb{E}[x_{k+1}] = \mu_k, \quad (28)$$

$$\mathbb{E}[v_{R_{k+1}}] = 0, \quad (29)$$

$$\mathbb{E}[x_{k+1} x_{k+1}^T] = \mathbb{E}[x_k x_k^T] + h G_v, \quad (30)$$

and

$$(MFGI) \quad \mathbb{E}[x_{k+1} v_{R_{k+1}}^T] = \left[P_k \left(\mathbb{E}[v_{R_k} \tilde{v}_R^T] + \frac{h}{2} \mathbb{E}[v_{R_k} \tilde{\tilde{v}}_R^T] - \frac{h \text{tr}[G_u]}{2} \mathbb{E}[v_{R_k} \tilde{v}_R^T] + h \mathbb{E}[v_{R_k} v_{R_k}^T P_k^T V_k \Gamma_Q^T] \right) + \mu_k \left(\mathbb{E}[\tilde{v}_R^T] + \frac{h}{2} \mathbb{E}[\tilde{\tilde{v}}_R^T] - \frac{h \text{tr}[G_u]}{2} \mathbb{E}[\tilde{v}_R^T] + h \mathbb{E}[v_{R_k}^T P_k^T V_k \Gamma_Q^T] \right) + h \Sigma_{c_k} V_k \mathbb{E}[\Gamma_Q^T] \right] \tilde{U}^T + O(h^2), \quad (31a)$$

$$(MFGB) \quad \mathbb{E}[x_{k+1} v_{R_{k+1}}^T] = \left[P_k \left(\mathbb{E}[v_{R_k} \tilde{v}_R^T] - \frac{h \text{tr}[G_u]}{2} \mathbb{E}[v_{R_k} \tilde{v}_R^T] + h \mathbb{E}[v_{R_k} v_{R_k}^T P_k^T V_k \Gamma_Q^T] \right) + \mu_k \left(\mathbb{E}[\tilde{v}_R^T] - \frac{h \text{tr}[G_u]}{2} \mathbb{E}[\tilde{v}_R^T] + h \mathbb{E}[v_{R_k}^T P_k^T V_k \Gamma_Q^T] \right) + h \Sigma_{c_k} V_k \mathbb{E}[\Gamma_Q^T] \right] \tilde{V}^T + \frac{h}{2} \left(\mu_k \mathbb{E}[\tilde{\tilde{v}}_R^T] + P_k \mathbb{E}[v_{R_k} \tilde{\tilde{v}}_R^T] \right) + O(h^2), \quad (31b)$$

$$(MFGI) \quad \mathbb{E}[v_{R_{k+1}} v_{R_{k+1}}^T] = \tilde{U} \left[\mathbb{E}[\tilde{v}_R \tilde{v}_R^T] + h \mathbb{E}[\Gamma_Q V_k^T P_k v_{R_k} \tilde{v}_R^T] + h \mathbb{E}[\tilde{v}_R v_{R_k}^T P_k^T V_k \Gamma_Q^T] + h \mathbb{E}[\Gamma_Q V_k^T G_u V_k \Gamma_Q^T] - h \text{tr}[G_u] \mathbb{E}[\tilde{v}_R \tilde{v}_R^T] + \frac{h}{2} \mathbb{E}[\tilde{v}_R \tilde{\tilde{v}}_R^T] + \frac{h}{2} \mathbb{E}[\tilde{\tilde{v}}_R \tilde{v}_R^T] \right] \tilde{U}^T + O(h^2), \quad (32a)$$

$$(MFGB) \quad \mathbb{E}[v_{R_{k+1}} v_{R_{k+1}}^T] = \tilde{V} \left(\mathbb{E}[\tilde{v}_R \tilde{v}_R^T] + h \mathbb{E}[\Gamma_Q V_k^T P_k v_{R_k} \tilde{v}_R^T] + h \mathbb{E}[\tilde{v}_R v_{R_k}^T P_k^T V_k \Gamma_Q^T] + h \mathbb{E}[\Gamma_Q V_k^T G_u V_k \Gamma_Q^T] - h \text{tr}[G_u] \mathbb{E}[\tilde{v}_R \tilde{v}_R^T] \right) \tilde{V}^T + \frac{h}{2} \left(\tilde{V} \mathbb{E}[\tilde{v}_R \tilde{\tilde{v}}_R^T] + \mathbb{E}[\tilde{\tilde{v}}_R \tilde{v}_R^T] \tilde{V}^T \right) + O(h^2). \quad (32b)$$

With these moments, the estimates for $(\mu_{k+1}, \Sigma_{k+1}, P_{k+1})$ can be constructed through the conditional MLE given in Theorem 3. In summary, Theorem 4 and Theorem 5 provide an analytical approach to propagate $(R_k, x_k) \sim \mathcal{MG}(\mu_k, \Sigma_k, P_k, U_k, S_k, V_k)$ into $(R_{k+1}, x_{k+1}) \sim \mathcal{MG}(\mu_{k+1}, \Sigma_{k+1}, P_{k+1}, U_{k+1}, S_{k+1}, V_{k+1})$, up to accuracy $O(h^2)$ in moments. Besides the presented approach, the propagation step can also be completed with unscented transform [9, 12].

B. Measurement Update

Finally, we present how to update the propagated MFG when measurements are available. As the measurement update is assumed to be completed instantaneously, the subscript k denoting the time step is omitted throughout this subsection. The variables relevant to the posterior distribution conditioned by measurements are denoted by the superscript $+$. Suppose there are N_v fixed reference vectors $b_j \in \mathbb{S}^2$ in the inertial reference frame, which are measured by direction sensors in the body-fixed frame as $z_j \in \mathbb{S}^2$ for $j = 1, \dots, N_v$. Furthermore, given the true attitude R_t , the noisy measurement z_j is assumed to follow the von Mises Fisher distribution [7] with mean direction $R_t^T B_j b_j \in \mathbb{S}^2$ and concentration parameter $\kappa_j > 0$. The parameter $B_j \in \text{SO}(3)$ specifies the constant bias of the direction sensor, and κ_j specifies the concentration of its random noise.

Suppose the prior distribution of (R, x) before measurement update follows MFG with parameters $(\mu, \Sigma, P, U, S, V)$. By Bayes' rule and Theorem 3.2 in [8], the posterior density conditioned on all of the available measurements $\mathcal{Z} = \{z_1, \dots, z_{N_v}\}$ is

$$p(R, x | \mathcal{Z}) \propto \text{etr} \left(\left(F + \sum_{j=1}^{N_v} \kappa_j B_j b_j z_j^T \right) R^T \right) \exp \left(-\frac{1}{2} (x - \mu_c)^T \Sigma_c^{-1} (x - \mu_c) \right), \quad (33)$$

where F , μ_c and Σ_c are defined as in Definition 1 with respect to $(\mu, \Sigma, P, U, S, V)$. The above posterior density of $(R, x) | \mathcal{Z}$ is no longer MFG, as the tangent space at the mean attitude of the updated matrix Fisher part is altered. Similar to the previous subsection, we match a new MFG with parameters $(\mu^+, \Sigma^+, P^+, U^+, S^+, V^+)$ to this density through MLE after calculating the required moments.

Theorem 6 Define $F^+ \in \mathbb{R}^{3 \times 3}$ as

$$F^+ = F + \sum_{j=1}^{N_v} \kappa_j B_j b_j z_j^T, \quad (34)$$

and let its proper singular value decomposition be $F^+ = U^+ S^+ (V^+)^T$. Also, let

$$(MFGI) \quad v_R^+ = (Q^+ S^+ - S^+ (Q^+)^T)^\vee \quad (35a)$$

$$(MFGI) \quad v_R^+ = (S^+ Q^+ - (Q^+)^T S^+)^\vee \quad (35b)$$

for $Q^+ = (U^+)^T R V^+ \in \text{SO}(3)$. Then the moments of the posterior density (33), namely $E[R | \mathcal{Z}]$, $E[v_R^+ | \mathcal{Z}]$ and $E[v_R^+ (v_R^+)^T | \mathcal{Z}]$ are identical to their counterparts in Theorem 1 after replacing U, S, V with U^+, S^+, V^+ , and

$$E[x | \mathcal{Z}] = \mu + P E[v_R | \mathcal{Z}], \quad (36)$$

$$E[xx^T | \mathcal{Z}] = \mu \mu^T + \mu E[v_R | \mathcal{Z}]^T P^T + P E[v_R | \mathcal{Z}] \mu^T + P E[v_R v_R^T | \mathcal{Z}] P^T + \Sigma_c, \quad (37)$$

$$E[x(v_R^+)^T | \mathcal{Z}] = P E[v_R (v_R^+)^T | \mathcal{Z}], \quad (38)$$

where

$$(MFGI) \quad E[v_R | \mathcal{Z}] = \tilde{U} (E[Q^+ | \mathcal{Z}] \tilde{S}^T - \tilde{S} E[Q^+ | \mathcal{Z}]^T)^\vee, \quad (39a)$$

$$(MFGB) \quad E[v_R | \mathcal{Z}] = \tilde{V} (\tilde{S}^T E[Q^+ | \mathcal{Z}] - E[Q^+ | \mathcal{Z}]^T \tilde{S})^\vee, \quad (39b)$$

$$(MFGI) \quad E[v_R v_R^T | \mathcal{Z}] = \tilde{U} E[\tilde{v}_R^+ (\tilde{v}_R^+)^T | \mathcal{Z}] \tilde{U}^T, \quad (39a)$$

$$(MFGB) \quad E[v_R v_R^T | \mathcal{Z}] = \tilde{V} E[\tilde{v}_R^+ (\tilde{v}_R^+)^T | \mathcal{Z}] \tilde{V}^T, \quad (39b)$$

$$(MFGI) \quad E[v_R (v_R^+)^T | \mathcal{Z}] = \tilde{U} E[\tilde{v}_R^+ (\tilde{v}_R^+)^T | \mathcal{Z}], \quad (39a)$$

$$(MFGB) \quad E[v_R (v_R^+)^T | \mathcal{Z}] = \tilde{V} E[\tilde{v}_R^+ (\tilde{v}_R^+)^T | \mathcal{Z}], \quad (39b)$$

with $\tilde{U} = U^T U^+$, $\tilde{V} = V^T V^+ \in \text{SO}(3)$, $\tilde{S} = \tilde{U}^T S \tilde{V} \in \mathbb{R}^{3 \times 3}$, and $\tilde{v}_R^+ \in \mathbb{R}^3$ is

$$(MFGI) \quad \tilde{v}_R^+ = (Q^+ \tilde{S}^T - \tilde{S} (Q^+)^T)^\vee, \quad (42a)$$

$$(MFGB) \quad \tilde{v}_R^+ = (\tilde{S}^T Q^+ - (Q^+)^T \tilde{S})^\vee. \quad (42b)$$

Since the attitude part of (33) is already a matrix Fisher density, $U^+ S^+ (V^+)^T = F^+$ is the solution to the marginal MLE for the matrix Fisher part. The conditional MLE is solved by Theorem 3 with the moments calculated above, which yields μ^+ , Σ^+ and P^+ . These provide the measurement update to represent the posterior distribution conditioned by the measurement as MFG.

The proposed uncertainty propagation and measurement update steps constitute a Bayesian estimator for the attitude and gyro bias. The current belief represented by MFG can be propagated until an additional measurement is available, based on which the propagated belief is updated. The estimates for the attitude and gyro bias are given by UV^T and μ , respectively. The pseudocode for the proposed Bayesian estimator is presented in Table 1. A set of MATLAB codes for the proposed MFG estimators are available at [16].

Table 1 Bayesian estimation for attitude and gyroscope bias

1:	procedure ESTIMATION($\mathcal{MG}(t_0), \Omega(t), \mathcal{Z}(t)$)
2:	Let $k = 0$.
3:	repeat
4:	Calculate $E[R_{k+1}]$ using (24).
5:	Obtain $U_{k+1}, S_{k+1}, V_{k+1}$ according to Theorem 2 using $E[R_{k+1}]$.
6:	Calculate the moments in Theorem 5
7:	Obtain $\mu_{k+1}, \Sigma_{k+1}, P_{k+1}$ according to Theorem 3 using the moments calculated in Step 6.
8:	Set $\mathcal{MG}(t_{k+1}) = \mathcal{MG}(\mu_{k+1}, \Sigma_{k+1}, P_{k+1}, U_{k+1}, S_{k+1}, V_{k+1})$.
9:	$k = k + 1$.
10:	until $\mathcal{Z}(t_{k+1})$ is available
11:	Compute F^+ from (34), and calculate its proper SVD as $U^+ S^+ (V^+)^T = F^+$.
12:	Calculate the moments of the posterior density in Theorem 6
13:	Obtain μ^+, Σ^+, P^+ according to Theorem 3
14:	Set $\mathcal{MG}(t_{k+1}) = \mathcal{MG}(\mu^+, \Sigma^+, P^+, U^+, S^+, V^+)$
15:	Obtain the estimates as $R(t_{k+1}) = U^+ (V^+)^T, x(t_{k+1}) = \mu^+$.
16:	go to step 3.
17:	end procedure

V. Spacecraft Attitude Estimation with Magnetometer

Orbital Properties The proposed attitude estimation scheme is applied to a spacecraft on a near-circular orbit around the Earth. The orbital elements are given as follows, and three cases are considered for varying inclinations.

$$a = 6916 \text{ km}, \quad e = 0.02, \quad \Omega = 50^\circ, \quad \omega = 0^\circ, \quad i = 0^\circ, 40^\circ \text{ or } 90^\circ. \quad (43)$$

According to the solution of the two-body problem, the orbital position with respect to the ECI frame, namely $r_{ECI} \in \mathbb{R}^3$, is computed at each time step.

Satellite Model A 3U CubeSat with the mass of $m = 3 \text{ kg}$ is considered. The inertia matrix is given by

$$J = \text{diag}[0.0325, 0.0325, 0.005] \text{ kgm}^2.$$

The rotation matrix $R \in \text{SO}(3)$ corresponds to the linear transformation of the coordinates of a vector from the body-fixed frame to the ECI frame, and the angular velocity $\Omega \in \mathbb{R}^3$ is measured by a gyroscope in the body-fixed frame.

The true attitude and the true angular velocity, namely (R_{ref}, Ω_{ref}) are generated according to

$$\begin{aligned} J\dot{\Omega}_{ref} + \Omega_{ref} \times J\Omega_{ref} &= 0, \\ \dot{R}_{ref} &= R_{ref} \hat{\Omega}_{ref}, \end{aligned}$$

where the initial condition is chosen as

$$R_{ref}(0) = I_{3 \times 3}, \quad \Omega_{ref}(0) = [0.05, 0.05, 0.02] \text{ rad/s}.$$

These are numerically integrated with the geometric numerical integrator preserving the structures of $\text{SO}(3)$, referred to as Lie group variational integrator [17, 18], using the time step of 0.05 sec.

It is assumed that the angular velocity measurement is available at 20 Hz, i.e., $h = 0.05 \text{ sec}$. The noise parameters for the gyroscope are: (i) angle random walk: $H_u = 0.1 I_{3 \times 3}$, and (ii) bias instability: $H_v = 0.0005 I_{3 \times 3}$, as indicated in (19) and (20). The initial bias of the gyroscope is set as $[0.01, 0.01, 0.01] \text{ rad s}^{-1}$.

Magnetometer Model Next, we generate the magnetometer measurements as follow. Assuming that the simulation begins at the midnight of January 1, 2021, the orbital position in the ECI frame is transformed into the Earth-centered, Earth-fixed (ECEF) frame. This is because the Earth magnetic field is fixed to the ECEF frame. For the given location with respect to the ECEF frame, the Matlab function `wrldmagm` is called to compute the magnetic field. The Matlab

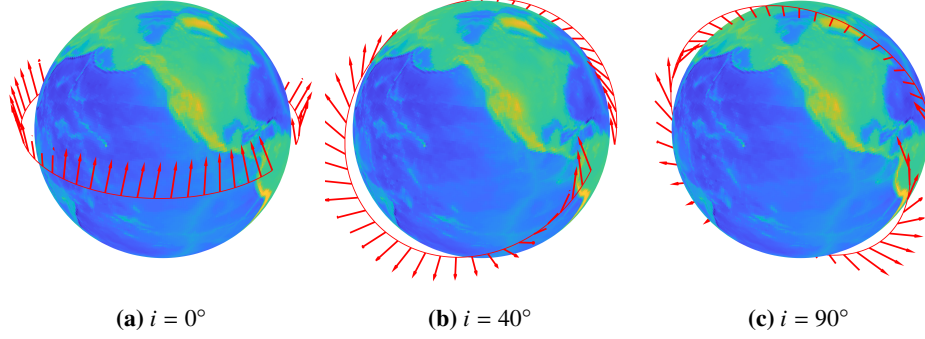


Fig. 1 Magnetic field on the orbit with different inclinations

function implements the NOAA World Magnetic Model presented in [19]. Then, the magnetic field is normalized to a unit-vector, which is transformed back to the ECI frame to obtain the true direction of the magnetic field at $t = t_k$, namely b in (33). Figure 1 illustrates the NOAA magnetic field model, where the magnetic fields along the selected orbits with respect to the ECEF frame are shown.

As discussed in the prior section, the measurement from the magnetometer is assumed to be distributed according to the von Mises–Fisher distribution centered at the true magnetic field resolved in the body-fixed frame, namely $R^T b \in \mathbb{S}^2$. In other words, the probability density of the measurement $z \in \mathbb{S}^2$ is given by

$$p(z|R) = \frac{\kappa}{4\pi \sinh \kappa} \exp(\kappa z^T R^T b), \quad (44)$$

where the concentration parameter κ is chosen as $\kappa = 200$. The average attitude error of the noisy magnetometer measurement is 5.08° , and it is assumed to be available also at 20 Hz.

Estimation Results Five filters are simulated to estimate the attitude and gyroscope bias of the spacecraft from gyroscope and magnetometer measurements, namely the conventional MEKF, two MFGB filters (using analytical and unscented uncertainty propagation, denoted by MFGBA and MFGBU respectively), and two MFGI filters (denoted by MFGIA and MFGIU respectively). The filters are initialized by

$$\begin{aligned} U_0 &= \text{diag}([1, -1, -1]) \quad S_0 = 10I_{3 \times 3} \quad V_0 = I_{3 \times 3} \\ \mu_0 &= 0_{3 \times 1} \quad \Sigma_0 = 0.01^2 I_{3 \times 3} \quad P_0 = 0_{3 \times 3}. \end{aligned}$$

The attitude estimation is given by $(R_{\text{est}})_k = U_k V_k^T$ at time t_k , and the bias estimation is given by $(x_{\text{est}})_k = \mu_k$. The initial attitude estimation is the true attitude rotated by 180° about its first body-fixed axis, and the initial bias estimation is off by 0.01 rad s^{-1} in each axis.

Since there is only one direction measurement available from the magnetometer, the rotation of spacecraft about this reference vector is unobservable if the magnetic field is fixed. Fortunately, as the spacecraft circles around the earth, the direction of the magnetic field changes at different locations (see Fig. 1), and this allows the full attitude to be estimated [11]. However, since the magnetic field changes very slowly, estimating full attitude still remains very challenging. Therefore, we define two attitude errors: the full attitude error (FAE) at time t_k is the angle that rotates $(R_{\text{ref}})_k$ to $(R_{\text{est}})_k$, and the partial attitude error (PAE) is the angle between $(R_{\text{ref}})_k^T b_k$ and $(R_{\text{est}})_k^T b_k$. Note that the partial attitude error does not take into account the rotation about the reference vector a_k .

The estimation errors for the five filters are presented in Table 2 and Fig. 2 to Fig. 4. In addition, the estimated uncertainties in attitude and bias are presented in Fig. 2 to Fig. 4. Looking at the bias estimation, the two MFGB based filters are much more accurate than MEKF, though their estimated uncertainties are similar. Because the bias is represented in the body-fixed frame, MFGB is more appropriate than MFGI to model the attitude-linear correlation. Therefore, MFGB based filters behave better than MFGI based filters, and this can also be seen from that the MFGB filters have lower bias uncertainty. In particular, the MFGIU filter has the worst performance in bias estimation.

The partial attitude error is directly related to the magnetometer measurement, and is corrected in every time step. Hence, it is relatively easy to estimate the partial attitude which is not involved with the rotation about the reference vector. And as a result, all filters tested are able to estimate partial attitude with equally low estimation error.

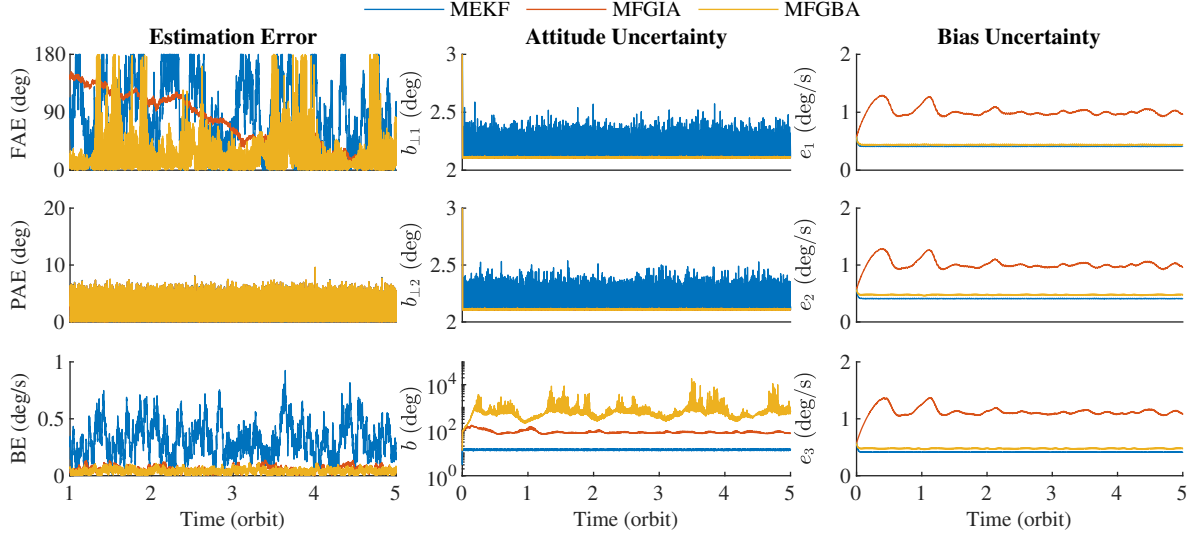


Fig. 2 Estimation error, attitude uncertainty, and bias uncertainty for various attitude filters when $i = 0^\circ$. The attitude uncertainty is expressed in the coordinate frame specified by the direction of magnetic field (b), and its two perpendicular vectors ($b_{\perp 1}$ and $b_{\perp 2}$). The attitude uncertainty for MEKF is given by the square root of the diagonals of its covariance; and for MFG filters is given by the square root of the diagonals of $(\text{tr}[S] I_{3 \times 3} - S)^{-1}$, after transformed into the correct frame. The bias uncertainty is expressed in the body-fixed frame, and is given by the square root of the diagonals of the covariance. The results for MFG unscented filters are omitted for readability.

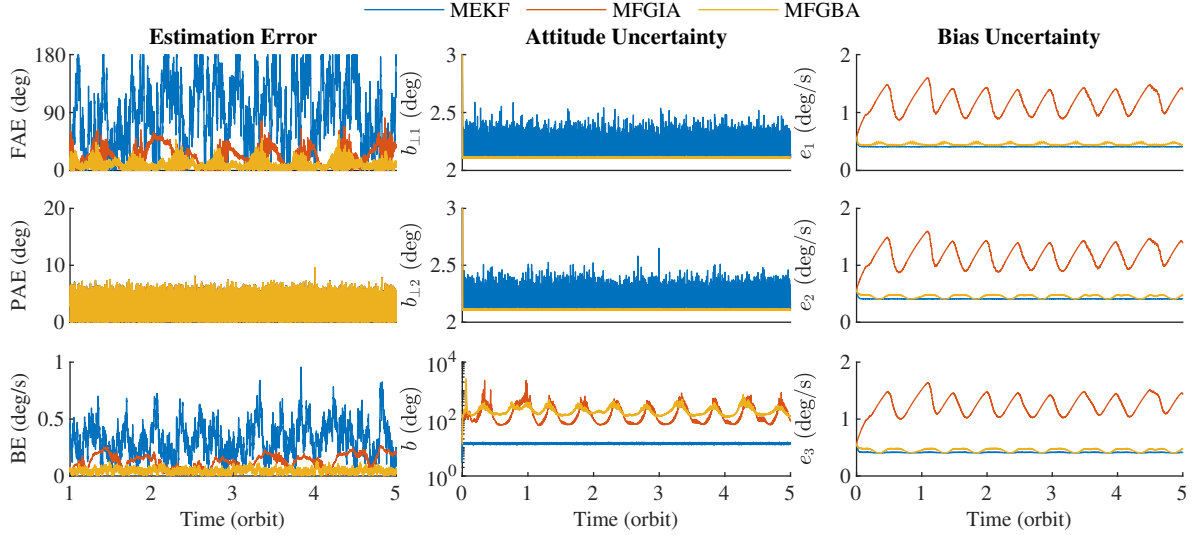


Fig. 3 Estimation error, attitude uncertainty, and bias uncertainty for various attitude filters when $i = 40^\circ$.

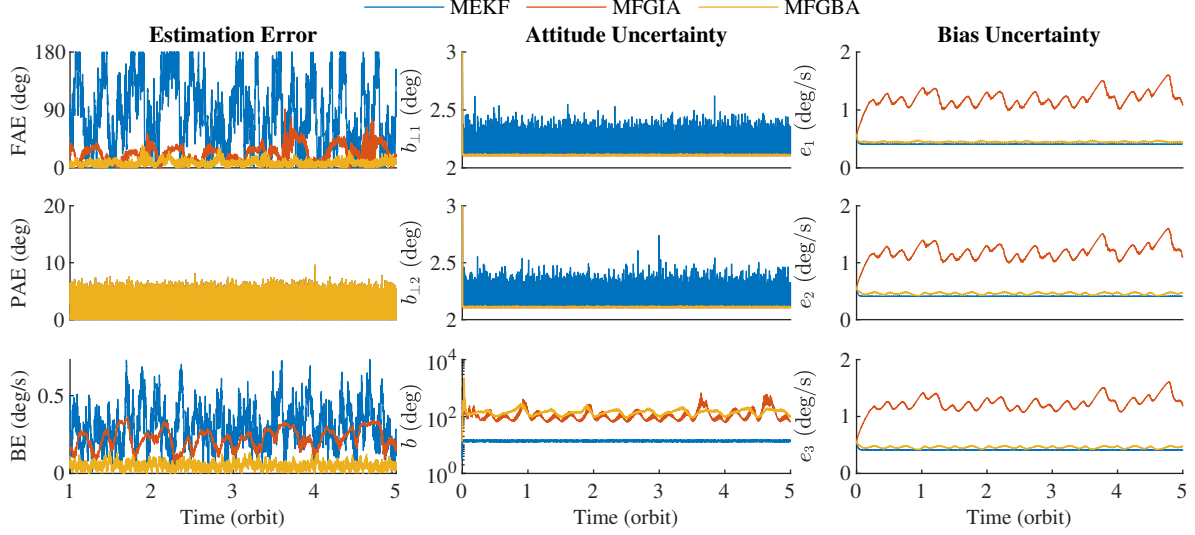


Fig. 4 Estimation error, attitude uncertainty, and bias uncertainty for various attitude filters when $i = 90^\circ$.

Table 2 Estimation errors for various attitude filters

		MEKF	MFGBA	MFGBU	MFGIA	MFGIU
$i = 0^\circ$	Attitude full error (deg)	82.0	29.4	63.4	90.1	87.0
	Attitude partial error (deg)	2.019	2.017	2.018	2.015	2.020
	Bias error (deg/s)	0.313	0.044	0.045	0.119	1.366
$i = 40^\circ$	Attitude full error (deg)	85.0	8.7	25.7	22.0	82.9
	Attitude partial error (deg)	2.018	2.016	2.019	2.015	2.020
	Bias error (deg/s)	0.321	0.048	0.051	0.176	1.324
$i = 90^\circ$	Attitude full error (deg)	82.8	7.2	20.5	22.8	76.8
	Attitude partial error (deg)	2.018	2.016	2.019	2.014	2.020
	Bias error (deg/s)	0.298	0.053	0.060	0.252	1.178

Furthermore, the attitude uncertainties in the two axes perpendicular to the reference magnetic field are consistently low for all filters, though there appears to be more fluctuations for MEKF. Also, the partial attitude error does not depend on the inclination angle i of the orbit.

The full attitude is extremely challenging to estimate due to that the direction of magnetic field b changes very slowly. Among all tested filters, only the MFGB based filters (and in particular MFGBA) are able to give relatively low full attitude error. Compared with the result in [20] where the full attitude can be accurately estimated with a gyroscope and a single magnetometer, the difference here is the inclusion of a non-zero time varying gyroscope bias. Because there is only one slow-varying reference vector, the integration accuracy of angular velocity from gyroscope is of crucial importance for full attitude estimation. However, the inclusion of a time varying gyroscope bias deteriorates the integration accuracy very badly. And therefore, the estimation accuracy of bias has a strong impact on the estimation of full attitude, and this explains why the MFGB based filters have lower full attitude error compared with other filters. It is surprising that MEKF has the lowest attitude uncertainty in the reference magnetic field direction, nonetheless it is the least accurate in full attitude estimation. This may reflect that the Gaussian distribution assumed by MEKF cannot properly model the large attitude dispersion due to the wrapping problem. In addition, the analytical propagation for MFG filters behave consistently better than unscented propagation in full attitude estimation. Also, because the direction of b changes more quickly as the inclination angle i is increased, the full attitude error also becomes lower.

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