

Balanced Hodge Laplacians optimize consensus dynamics over simplicial complexes

Cite as: Chaos **32**, 023128 (2022); <https://doi.org/10.1063/5.0080370>

Submitted: 30 November 2021 • Accepted: 31 January 2022 • Published Online: 23 February 2022

 Cameron Ziegler,  Per Sebastian Skardal, Haimonti Dutta, et al.

COLLECTIONS

Paper published as part of the special topic on [Dynamics on Networks with Higher-Order Interactions](#)



View Online



Export Citation



CrossMark

ARTICLES YOU MAY BE INTERESTED IN

[Consensus on simplicial complexes: Results on stability and synchronization](#)

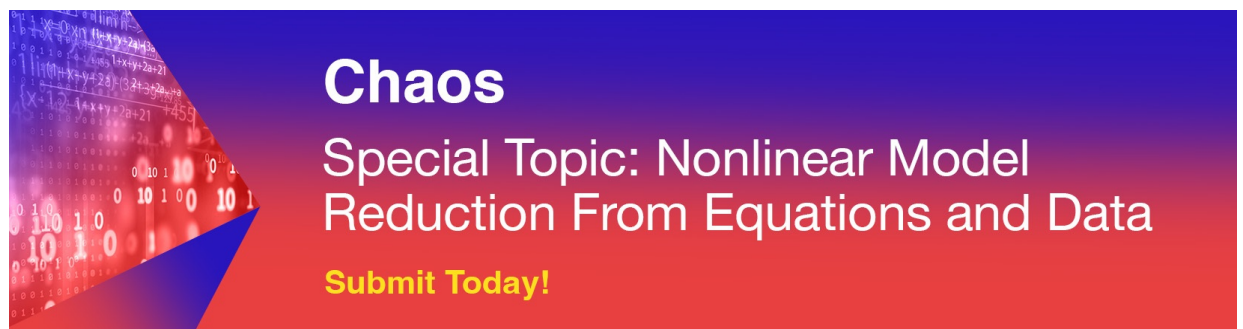
Chaos: An Interdisciplinary Journal of Nonlinear Science **31**, 023137 (2021); <https://doi.org/10.1063/5.0037433>

[Intralayer and interlayer synchronization in multiplex network with higher-order interactions](#)

Chaos: An Interdisciplinary Journal of Nonlinear Science **32**, 033125 (2022); <https://doi.org/10.1063/5.0074641>

[Synchronization in Hindmarsh–Rose neurons subject to higher-order interactions](#)

Chaos: An Interdisciplinary Journal of Nonlinear Science **32**, 013125 (2022); <https://doi.org/10.1063/5.0079834>



Chaos
Special Topic: Nonlinear Model
Reduction From Equations and Data
Submit Today!

Balanced Hodge Laplacians optimize consensus dynamics over simplicial complexes

Cite as: Chaos 32, 023128 (2022); doi: 10.1063/5.0080370

Submitted: 30 November 2021 · Accepted: 31 January 2022 ·

Published Online: 23 February 2022



View Online



Export Citation



CrossMark

Cameron Ziegler,^{1,a)} Per Sebastian Skardal,² Haimonti Dutta,³ and Dane Taylor¹

AFFILIATIONS

¹Department of Mathematics, University at Buffalo, State University of New York, Buffalo, New York 14260, USA

²Department of Mathematics, Trinity College, Hartford, Connecticut 16106, USA

³Management Science and Systems Department, University at Buffalo, State University of New York, Buffalo, New York 14260, USA

Note: This article is part of the Focus Issue, Dynamics on Networks with Higher-Order Interactions.

^{a)}**Author to whom correspondence should be addressed:** danet@buffalo.edu

ABSTRACT

Despite the vast literature on network dynamics, we still lack basic insights into dynamics on higher-order structures (e.g., edges, triangles, and more generally, k -dimensional “simplices”) and how they are influenced through higher-order interactions. A prime example lies in neuroscience where groups of neurons (not individual ones) may provide building blocks for neurocomputation. Here, we study consensus dynamics on edges in simplicial complexes using a type of Laplacian matrix called a Hodge Laplacian, which we generalize to allow higher- and lower-order interactions to have different strengths. Using techniques from algebraic topology, we study how collective dynamics converge to a low-dimensional subspace that corresponds to the homology space of the simplicial complex. We use the Hodge decomposition to show that higher- and lower-order interactions can be optimally balanced to maximally accelerate convergence and that this optimum coincides with a balancing of dynamics on the curl and gradient subspaces. We additionally explore the effects of network topology, finding that consensus over edges is accelerated when two-simplices are well dispersed, as opposed to clustered together.

Published under an exclusive license by AIP Publishing. <https://doi.org/10.1063/5.0080370>

Consensus is an important dynamical process for modeling group decision making in social networks and designing decentralized machine learning algorithms; however, existing theory mostly focuses on models that are limited to pairwise interactions encoded by graphs. Here, we develop a model for consensus dynamics over simplicial complexes. We assign dynamical states to edges (or higher-dimensional simplices), as opposed to nodes, and allow them to interact through both higher- and lower-order interactions. For example, two edges can be connected at a node, or they can be two sides of a triangle, which is called a two-simplex. We encode these two types of interaction in a Hodge Laplacian matrix that we generalize by introducing a balancing parameter that tunes the relative influence of higher- and lower-order interactions. Our new theory reveals that homology plays a vital role for consensus over simplicial complexes and that the strengths of higher- and lower-order interactions can be optimally balanced to maximally accelerate convergence.

I. INTRODUCTION

Interest in dynamical processes occurring on and through higher-order structures in simplicial complexes has grown rapidly in recent years, with examples including random walks,^{1–3} social contagions,^{4,5} neuronal activity,^{6–8} cellular networks,⁹ synchronization,^{10–15} and consensus.^{16,17} Such dynamics are useful for applications ranging from biological processes¹⁸ to the design of machine learning^{19,20} and signal processing.^{21,22} In this pursuit, a variety of mathematical tools have been developed ranging from mean field theory²³ and probability theory²⁴ to persistent homology,^{25–27} algebraic topology,^{28,29} and Hodge theory.³⁰ Thus, new problems in modeling and understanding the dynamical systems on simplicial complexes are revitalizing interest in analytical methods from algebraic topology and Hodge theory.

Consensus over networks is a popular model for collective decision making in cognitive,³¹ social,³² and biological systems,³³ and more recently, it has been used to investigate interconnected human-AI decision systems.³⁴ Notably, consensus dynamics

provide a foundation for decentralized optimization algorithms, which strategically implement consensus using synchronous or asynchronous gossiping between nodes.^{35,36} Such algorithms are often employed to take advantage of distributed computing infrastructure to more efficiently train machine learning models, such as support vector machines^{37,38} and deep neural networks.^{39–42} For such systems, each node trains a local model on local data, and at the same time, communication between nodes enables them to reach a consensus on what the model parameter should be. Notably, much of the literature on decentralized learning aims to understand and optimize the converge rate to consensus, which subsequently minimizes the time and number of computations that are required to train a predictive model.

Herein, we extend the study of consensus to simplicial complexes by assigning states to higher-dimensional simplices and allowing interactions through both their lower and higher-dimensional faces. Our work complements other extensions of collective dynamics over simplicial complexes that use Hodge Laplacians^{10–13} in which states are assigned to higher- and/or lower-dimensional simplices. Contrasting this prior work, we propose a model that utilizes a generalization of the Hodge Laplacian matrix in which the relative influence of higher and lower-order interactions can be tuned via a *balancing parameter* δ . This leads to our definition of a *generalized Hodge Laplacian*, and our work is motivated by other recent generalizations of Hodge Laplacians that have been introduced to study random walks¹ and synchronization¹³ over simplicial complexes or to implement signal processing.²²

With the help of an associated subspace decomposition called the *Hodge decomposition*, we provide a convergence analysis of our model and use tools from algebraic topology to investigate the role of *simplicial complex homology*. We find that the collective dynamics converge to a low-dimensional subspace lying within the homology subspace, which includes and generalizes the notion of connected components by way of Betti numbers (i.e., connected components characterize 0-dimensional homology). Furthermore, we show that higher- and lower-order interactions can be optimally balanced to maximally accelerate convergence and that this optimum coincides with a balancing of dynamics on the curl and gradient subspaces of a Hodge decomposition, respectively. We additionally explore the effects of topology, finding that consensus over edges is accelerated when two-simplices are well dispersed, i.e., as opposed to clustered together.

Our findings provide new insights into how dynamical systems with higher-order interactions can be engineered and optimized, which not only improves our understanding of self-organizing biological and physical systems, but may also support the development of improved consensus-based algorithms for machine learning. Specifically, recent work^{43,44} has illustrated advantages for neural networks that use simplicial complexes and Hodge Laplacians. Our formulation, analysis, and optimization of consensus over simplicial complexes may help future research implement decentralized versions of such emerging frameworks by allowing them to also take advantage of decentralized optimization techniques that rely on consensus.

This paper is organized as follows. We present background information in Sec. II, our model and the role of homology in Sec. III, optimization theory in IV, and a discussion in Sec. V.

II. BACKGROUND INFORMATION

We provide here background information on the homology of simplicial complexes (Sec. II A) as well as Hodge Laplacians and decompositions (Sec. II B).

A. Simplicial complexes, boundary matrices, and homology

We begin with some definitions. *Simplicial complexes* generalize undirected graphs (which encode dyadic interactions) by allowing polyadic interactions among sets of nodes/vertices. They consist of *simplices* of any number of dimensions. For a set \mathcal{V} of nodes, a k -dimensional simplex (or simply, k -simplex) $\mathcal{S} \subset \mathcal{V}$ is a subset with $k + 1$ elements. We will refer to zero-simplices as *nodes* or *vertices*, one-simplices as *edges*, and two-simplices as “filled-in” *triangles*. For a k -simplex \mathcal{S} , any $(k - 1)$ -simplex that is a subset of \mathcal{S} with k elements is called its *face*. For any such face, \mathcal{S} is called its *coface*. For example, an edge (i, j) has two faces, nodes i and j , and (i, j) is a coface to both i and j . Two k -simplices are called *lower adjacent* if they share a face, and they are called *upper adjacent* if they share a coface. For example, two edges are lower adjacent if they share a common node, and the edges are upper adjacent if they are two sides of the “boundary” of a two-simplex. (We will formally define boundaries below.)

A *simplicial complex* is a set of simplices in which any face of a simplex must also be contained in the simplicial complex, and the intersection between faces is either another simplex or an empty set. The dimension of a simplicial complex is the maximum dimension of its simplices. For example, a one-dimensional simplicial complex contains only nodes and edges and is equivalent to an undirected graph. For simplicity, herein, we will focus on two-dimensional simplicial complexes; however, we emphasize that our results naturally extend to higher dimensions.

Many of our calculations require simplicial complexes with *oriented simplices*. An orientation of a k -simplex is an equivalence class of the ordering of its nodes, where two classes are equivalent if they are the same up to an even number of permutations. The most convenient way to orient a simplicial complex is to order its vertices x_1, x_2, \dots, x_n . From this ordering, for all $k > 1$, choose the orientation of each k -simplex $\{x_{i_0}, x_{i_1}, \dots, x_{i_k}\}$ to be $[x_{i_0}, x_{i_1}, \dots, x_{i_k}]$ such that $i_0 < i_1 < \dots < i_k$. Note that this ordering is done only for bookkeeping purposes and does not represent a direction to the simplices.

Importantly, connectivity of faces and cofaces can be encoded using boundary maps and their associated boundary matrices. Their precise definition stems from studying functions that are defined over k -simplices, and we call such a function a *k -chain*, i.e., a linear combination of k -simplices. The set of all k -chains for a simplicial complex is the finite-dimensional vector space C_k . We can then define *boundary maps* $\partial_k : C_k \rightarrow C_{k-1}$ by their action on a simplicial complex,

$$\partial_k([x_0, x_1, \dots, x_k]) = \sum_{i=0}^k (-1)^i [x_0, \dots, x_{i-1}, x_{i+1}, x_k]. \quad (1)$$

This takes a simplex to an alternating sum of its faces. Representing the map ∂_k gives us the *boundary matrix* B_k , with columns corresponding to the k -simplices and rows corresponding to the

$(k - 1)$ -simplices. An entry $[B_k]_{ij}$ of B_k is ± 1 if the i th $(k - 1)$ -simplex is a face of the j th k -simplex (where the sign indicates orientation), and it is otherwise 0. Boundary matrices fully encode a simplicial complex in that it can be recreated from its boundary matrices.

In Fig. 1, we illustrate an example simplicial complex and its boundary matrices. Note that the edges are undirected, and arrows only indicate the simplices' chosen orientations. Also note that it is a 2D simplicial complex since there are no simplices of dimension 3 or greater (i.e., all higher-dimensional boundary matrices are empty). Observe that each column of B_1 corresponds to an edge (i, j) , and all entries in a column are zero except for the i th and j th entries, which contain -1 and 1 , respectively. The rows and columns of B_2 correspond to edges and two-simplices, respectively, and values ± 1 indicate incidences between faces and cofaces. The signs indicate whether or not the edges' orientations match that for a triangular cycle/boundary around each two-simplex. For example, the orientation of edge $(1, 3)$ matches that for two-simplex $(1, 3, 4)$. In contrast, the orientation of $(1, 4)$ does not since the cycle would be completed by traversing from node 4 to node 1 and not 1 to 4.

Of note, this simplicial complex has four separate cycles: two cycles are the boundary of a two-simplex, $\{(1, 3), (3, 4), (4, 1)\}$ and $\{(3, 4), (4, 6), (6, 3)\}$, and two cycles are not such a boundary: $\{(1, 2), (2, 3), (3, 1)\}$ and $\{(2, 3), (3, 6), (6, 5), (5, 2)\}$.

The study of cycles (and higher-dimensional "holes" including voids) generalizes the study of connected components in graphs and is the focus of homology. With these boundary functions and a simplicial complex X , we can define the n th (simplicial) homology group $H_n(X)$ as $\text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$. The dimension of the H_n is called the n th Betti number. These Betti numbers provide us with important topological information about simplicial complexes. The 0th Betti number is equal to the number of connected components in the simplicial complex, while the 1st gives us the number of cycles that are not "filled-in" by triangles. Higher-dimensional Betti numbers indicate the number of higher-dimensional "holes" in a simplicial complex.²⁸

B. Hodge Laplacians and Hodge decomposition

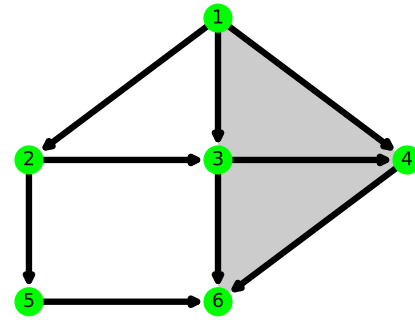
The k -th combinatorial Hodge Laplacian

$$L_k = B_k^T B_k + B_{k+1} B_{k+1}^T \tag{2}$$

is a symmetric, positive semi-definite, and square matrix of size n_k , where n_k is the number of k -simplices in the simplicial complex. From this definition, we have that $B_k^T B_k$ is calculated from the lower-adjacencies of k -simplices, while $B_{k+1} B_{k+1}^T$ is from their upper-adjacencies. Finally, notice that L_0 recovers the widely known combinatorial Laplacian of a graph: $L_0 = B_1 B_1^T = D - A$, where A is the adjacency matrix and D is a diagonal matrix that encodes node degrees, $D_{ii} = \sum_j A_{ij}$.

If for some k , we consider maps that give each k -simplex some real value (i.e., real-valued k -chains), the space of all such maps is \mathbb{R}^{n_k} , where n_k is the number of k -simplices in the simplicial complex. Using the combinatorial Hodge Laplacian L_k , we can decompose that space \mathbb{R}^{n_k} into three mutually orthogonal subspaces,⁴⁵

$$\mathbb{R}^{n_k} = \text{im}(B_k^T) \oplus \text{im}(B_{k+1}) \oplus \text{ker}(L_k). \tag{3}$$



	[1, 2]	[1, 3]	[1, 4]	[2, 3]	[2, 5]	[3, 4]	[3, 6]	[4, 6]	[5, 6]
1	-1	-1	-1	0	0	0	0	0	0
2	1	0	0	-1	-1	0	0	0	0
3	0	1	0	1	0	-1	-1	0	0
4	0	0	1	0	0	1	0	-1	0
5	0	0	0	0	1	0	0	0	-1
6	0	0	0	0	0	0	1	1	1

	[1, 3, 4]	[3, 4, 6]
[1, 2]	0	0
[1, 3]	1	0
[1, 4]	-1	0
[2, 3]	0	0
[2, 5]	0	0
[3, 4]	1	1
[3, 6]	0	-1
[4, 6]	0	1
[5, 6]	0	0

FIG. 1. Example simplicial complex and its boundary matrices. A simplicial complex with six nodes, nine edges, and two triangles. The nodes are indexed from 1 to 6. The edges of the simplicial complex have arrows not because they are directed edges, but to show their chosen orientation, where the arrow points to the node with a larger index. Its boundary matrices are given by B_1 and B_2 .

This is called the *Hodge decomposition*, and these three subspaces are known as the *gradient*, *curl*, and *harmonic* subspaces, respectively. From this, we can immediately see that the gradient subspace only depends on the $(k - 1)$ -dimensional simplices, the curl subspace depends on the $(k + 1)$ -dimensional simplices, and the harmonic subspace depends on both. Importantly, the harmonic space and homology group are isomorphic vector spaces,⁴⁵ and the Betti number is the dimension of both spaces.

In Fig. 2(a), we set $k = 1$ and consider a random real-valued function \mathbf{x} that is defined on the edges of the simplex that was shown in Fig. 1. In panels (b)–(d), we depict, respectively, the projection of \mathbf{x} onto the gradient, curl, and harmonic subspaces. We denote these $\mathbf{x}^{(g)}$, $\mathbf{x}^{(c)}$, and $\mathbf{x}^{(h)}$, respectively. Examining Fig. 2, we highlight several important properties of these subspaces. First, the gradient components sum to zero along any cycles. Second, the curl components are zero for edges that are not a two-simplex boundary, and the

entries sum to zero around each node. Third, the harmonic component sums to zero around each node, and it also sums to zero along each two-simplex (but not the other cycles).¹

III. CONSENSUS OVER SIMPLICIAL COMPLEXES WITH GENERALIZED HODGE LAPLACIANS

A. Model for consensus

Herein, we present a new formulation for consensus dynamics over simplicial complexes using Hodge Laplacians. However, rather than utilize combinatorial Hodge Laplacians, we instead define and utilize generalized Hodge Laplacians (GHLs),

$$L_k^{(\delta)} = (1 + \delta)B_k^T B_k + (1 - \delta)B_{k+1} B_{k+1}^T. \quad (4)$$

Specifically, we introduce a *balancing parameter* $\delta \in [-1, 1]$ to account for the possibility that for some applications, it is reasonable to assume that the higher- and lower-dimensional interactions should have different strengths of influence. Notably, interactions through lower-adjacencies (e.g., via shared nodes when $k = 1$) are more strongly weighted when $\delta > 0$, whereas interactions through higher-adjacencies (e.g., via shared two-simplices when $k = 1$) are more strongly weighted when $\delta < 0$. This formulation was chosen so that $L_k^{(\delta)}$ exactly recovers the combinatorial Hodge Laplacian under the choice of $\delta = 0$ (i.e., $L_k^{(0)} = L_k$). It is also worth noting that Eq. (4) is a particular case of other generalizations for Hodge Laplacians, wherein one can assign a unique weight for each individual simplex¹³ (as opposed to a common weight $1 \pm \delta$ for all simplices of a given dimension, as we have done) or by considering linear combinations of the matrices $B_k^T B_k$ and $B_{k+1} B_{k+1}^T$ as well as their matrix powers²² (which is useful for signal processing/filtering).

Importantly, we point out that the relevance of the Hodge decomposition is exactly the same for GHLs as for the combinatorial Hodge Laplacian. That is because our introduction of δ in Eq. (4) is just a scalar re-weighting of the two summands in the right-hand side of Eq. (2), and these matrices' image spaces are orthogonal and unchanged by δ . We point out that this "preservation" of the Hodge decomposition contrasts another recent generalization called *normalized Hodge Laplacians*.¹ That is, normalization uses the degrees (i.e., number of neighbors) of k -simplices of different dimensions, and this can significantly alter (i.e., rotate) the Hodge subspaces.

Given a GHL, we consider a model for consensus over a simplicial complex using the following system of linear ordinary differential equations. We define the linear dynamics

$$\frac{d}{dt}\mathbf{x}(t) = -L_k^{(\delta)}\mathbf{x}(t) \quad (5)$$

as generalized Hodge Laplacian- k (GHL- k) consensus. Note that $L_0^{(0)} = 2L = 2(D - A)$ again recovers the combinatorial graph Laplacian, and therefore, GHL-0 consensus with $\delta = -1$ is $\frac{d}{dt}\mathbf{x}(t) = -2L\mathbf{x}(t)$, proportional to the standard definition for consensus dynamics over a graph, $\frac{d}{dt}\mathbf{x}(t) = -L\mathbf{x}(t)$.⁴⁶ Under the choice $\delta = 0$, Eq. (5) recovers a model for heat diffusion over simplicial complexes.⁴⁷ We also note that a generalization of Eq. (5) has been recently defined;¹³ however, that work does not explore the optimal balancing of higher- and lower-order interactions.

For the remainder of this paper, we will identify fundamental insights about dynamics evolving under Eq. (5) and the role that is played by the topology and homology of the simplicial complex as well as the associated Hodge subspace. In principle, one can define and study consensus over k -simplices of any dimension. However, we will reduce our scope to the case when $k = 1$ so that each edge (i_p, j_p) is assigned a state $x_p \in \mathbb{R}$, and their states collectively evolve over time.

B. Decoupling of dynamics within Hodge subspaces

We first show that the Hodge subspaces are invariant subspaces for the dynamics evolving under Eq. (5). Consider a solution $\mathbf{x}(t) = \mathbf{x}^{(g)}(t) + \mathbf{x}^{(c)}(t) + \mathbf{x}^{(h)}(t)$ to Eq. (5), where $\mathbf{x}^{(g)}(t)$, $\mathbf{x}^{(c)}(t)$, and $\mathbf{x}^{(h)}(t)$, respectively, give the gradient, curl, and harmonic components (i.e., projections onto the respective subspaces). We substitute this form into Eq. (5) to obtain

$$\begin{aligned} \frac{d}{dt} [\mathbf{x}^{(g)}(t) + \mathbf{x}^{(c)}(t) + \mathbf{x}^{(h)}(t)] \\ = -[L_k^{(\delta)}\mathbf{x}^{(g)}(t)] - [L_k^{(\delta)}\mathbf{x}^{(c)}(t)] - [L_k^{(\delta)}\mathbf{x}^{(h)}(t)] \\ = -[(1 + \delta)B_1^T B_1 \mathbf{x}^{(g)}(t)] - [(1 - \delta)B_2 B_2^T \mathbf{x}^{(c)}(t)] - [0]. \end{aligned} \quad (6)$$

These simplifications use

- $B_2 B_2^T \mathbf{x}^{(g)}(t) = 0$ since $\mathbf{x}^{(g)}(t) \in \ker(B_2 B_2^T)$,
- $B_1^T B_1 \mathbf{x}^{(c)}(t) = 0$ since $\mathbf{x}^{(c)}(t) \in \ker(B_1^T B_1)$, and
- $L_1^{(\delta)} \mathbf{x}^{(h)}(t) = 0$ since $\mathbf{x}^{(h)}(t) \in \ker(L_1^{(\delta)})$.

Due to the orthogonality of these subspaces, we obtain the following decoupled set of linear ODEs:

$$\begin{aligned} \frac{d}{dt}\mathbf{x}^{(g)}(t) &= -(1 + \delta)B_1^T B_1 \mathbf{x}^{(g)}(t), \\ \frac{d}{dt}\mathbf{x}^{(c)}(t) &= -(1 - \delta)B_2 B_2^T \mathbf{x}^{(c)}(t), \\ \frac{d}{dt}\mathbf{x}^{(h)}(t) &= 0, \end{aligned} \quad (7)$$

which have the solutions

$$\begin{aligned} \mathbf{x}^{(g)}(t) &= e^{-t(1+\delta)B_1^T B_1} \mathbf{x}^{(g)}(0), \\ \mathbf{x}^{(c)}(t) &= e^{-t(1-\delta)B_2 B_2^T} \mathbf{x}^{(c)}(0), \\ \mathbf{x}^{(h)}(t) &= \mathbf{x}^{(h)}(0). \end{aligned} \quad (8)$$

In other words, the dynamics evolve separately within each Hodge subspace. Because the matrices $B_1^T B_1$ and $B_2 B_2^T$ are both positive semi-definite, the gradient and curl components converge to zero with increasing time; i.e., $\lim_{t \rightarrow \infty} \mathbf{x}^{(g)}(t) = 0$ and $\lim_{t \rightarrow \infty} \mathbf{x}^{(c)}(t) = 0$. On the other hand, the harmonic component remains fixed in time, and therefore, the full state converges to $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^{(h)}(0)$, which we will call the *consensus vector*.

Of particular interest is the *asymptotic convergence rate* $\mu(\delta)$ for Eq. (5), which is the convergence rate for the slower subspace dynamics. That is, each subspace in the Hodge decomposition also converges, but not all at the same rate. Let $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ be the smallest nonzero eigenvalues of $B_1^T B_1$ and $B_2 B_2^T$, respectively. The

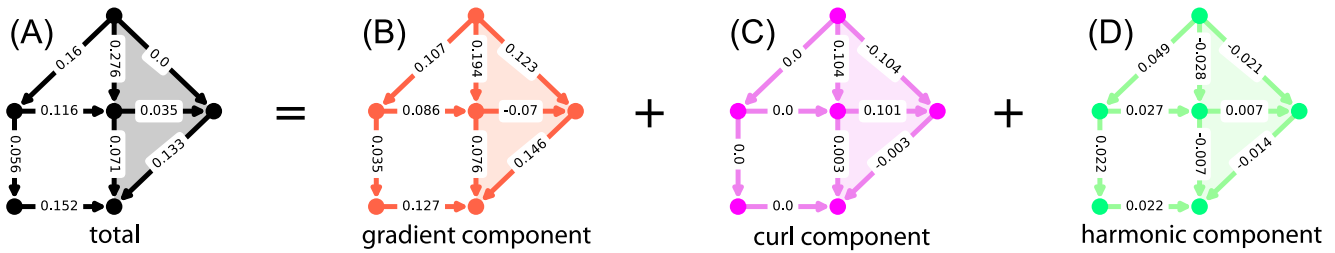


FIG. 2. Hodge decomposition of a real-valued function defined over one-simplices. (a) A one-chain $\mathbf{x} \in \mathbb{R}^k$ that is defined over $n^k = 9$ edges (i.e., one-simplices). (b) Projection $\mathbf{x}^{(g)}$ onto the gradient subspace. (c) Projection $\mathbf{x}^{(c)}$ onto the curl subspace. (d) Projection $\mathbf{x}^{(h)}$ onto the harmonic subspace.

gradient subspace is dependent on the lower-dimensional interactions and has convergence rate $(1 + \delta)\lambda_2^{(1)}$. The convergence curl subspace is similarly determined by the higher-dimensional interactions, and therefore, its convergence rate is $(1 - \delta)\lambda_2^{(2)}$. Notably, the balancing parameter δ tunes the relative convergence rates for the invariant dynamics in the gradient and curl subspace. It also follows that $\mu(\delta) = \min\{(1 + \delta)\lambda_2^{(1)}, (1 - \delta)\lambda_2^{(2)}\}$.

In Fig. 3, we illustrate timeseries $\mathbf{x}(t)$ for simulations of Eq. (5) as well as the projections onto the three Hodge subspaces. The top and bottom rows indicate the same initial condition $\mathbf{x}(0)$ but two different choices for the balancing parameter: (top) $\delta = 0.6$ and (bottom) $\delta = -0.4$. Each curve indicates a trajectory $x_p(t)$ for some edge (i_p, j_p) in the simplicial complex that was shown in Figs. 1 and 2. Observe in both rows that the gradient and curl components converge to zero, whereas the harmonic components remain fixed in time. In fact, for both values of δ , the system states converge to the same consensus vector $\mathbf{x}^{(h)}(0)$ —that is because the initial conditions

are the same. That is, δ has no effect on the final state. In contrast, δ has a significant impact on the convergence rates for the invariant dynamics within the gradient and curl subspaces. Observe for significantly positive δ (i.e., when the lower-dimensional interactions are prioritized) that the gradient component converges much more quickly than the curl component. The reverse is true when δ is sufficiently negative.

C. Uncovering the role of homology

In Subsection III B, we saw that the consensus dynamics remains unchanged in the harmonic subspace. That is to say, the consensus vector is the harmonic component of the initial condition. It is, therefore, important to take a look at properties of the harmonic subspace and their dependence on the topology and homology of the simplicial complex. Importantly, the harmonic subspace and homology group are isomorphic vector spaces, and therefore, the space of possible solutions to Eq. (5) is determined by the homology

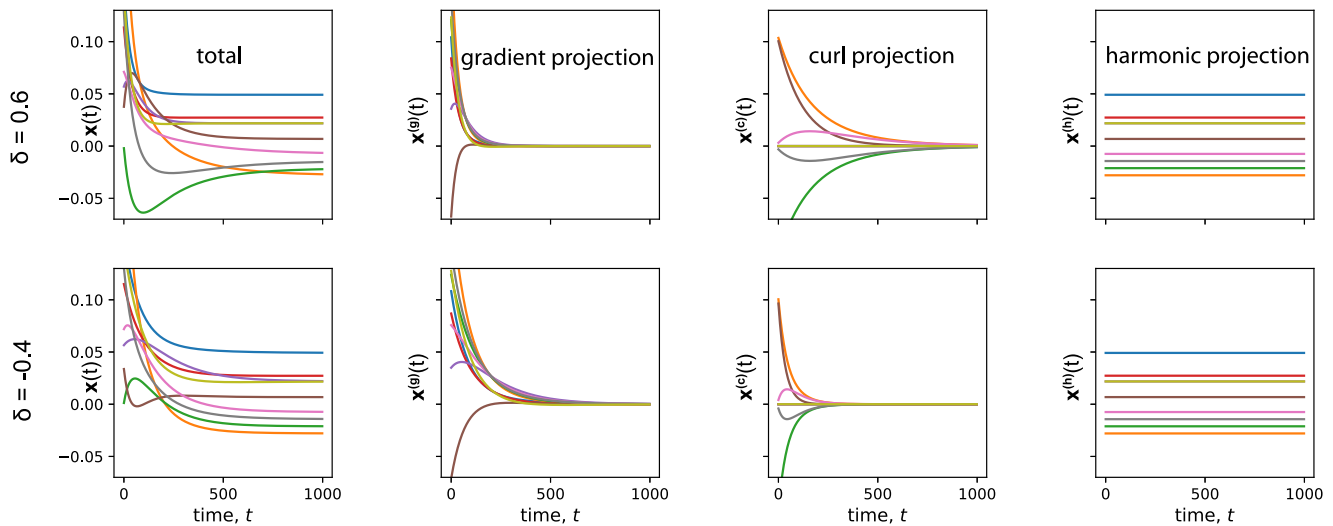


FIG. 3. Balancing parameter δ influences the convergence rate of GHL-1 consensus dynamics. Two simulations of GHL-1 consensus were run on the simplicial complex from Fig. 1. The top row shows the Hodge decomposition of the dynamics when $\delta = 0.6$, while the bottom row has $\delta = -0.4$. Both simulations began with the same initial condition. Notice that the curl subspace converges faster when δ is negative, while the gradient subspace converges faster when δ is positive. As expected, the harmonic subspace does not change under these dynamics, and therefore, the total function converges to the harmonic projection.

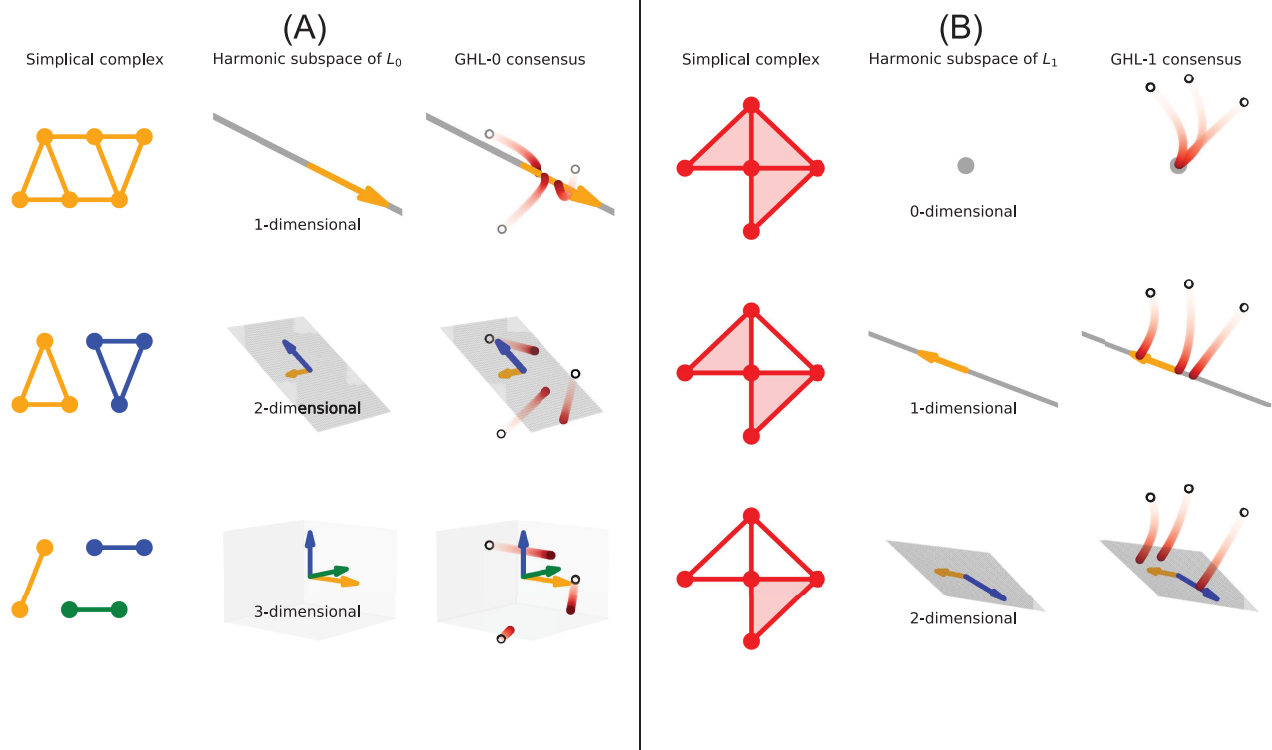


FIG. 4. Consensus dynamics converge to the harmonic subspace that is isomorphic to the homology space. The zero and first dimensional homology groups for a few example simplicial complexes. The homology groups are illustrated as vector spaces. We plot the convergence of GHL-0 (a) and GHL-1 (b) consensus with three different random initial conditions starting from the white circles. Notice that they all converge to the harmonic subspace. Within each panel (a) and (b), the initial conditions are the same. While the harmonic subspaces are plotted in full, the converging flows are projected down into three dimensions to simplify this visualization.

of the simplicial complex. In particular, the dimension, i.e., the Betti number, gives the dimension of the solution space.

In Fig. 4, we show several example simplicial complexes, their associated harmonic subspaces of L_k (which are identical to those of $L_k^{(b)}$), and the convergence of GHL- k consensus to the harmonic space with either (a) $k = 0$ or (b) $k = 1$.

Starting with Fig. 4(a), in the first column, we depict one-dimensional simplicial complexes such that Betti-0 (i.e., the dimension of both the harmonic and homology spaces for L_0) increases as one considers lower rows (see the second column). Note that Betti-0 is simply the number of connected components. In this case, the harmonic subspace of L_0 is spanned by vectors that have a constant value for entries associated with nodes in a given component and all other entries are zeros. If there are β_0 such components, then there are β_0 such vectors that span $\text{Ker}(L_0)$. The third column illustrates that GHL-0 consensus—which is equivalent to the traditional definition of consensus over an undirected graph—converges to a consensus value in this harmonic subspace. For practical purposes, one typically assumes that the graph is strongly connected so that the harmonic space is a one-dimensional line, and the consensus value gives the mean value for the initial condition $\{x_p(0)\}$.

Turning our attention to Fig. 4(b), in the first column, we depict two-dimensional simplicial complexes such that Betti-1 (i.e., the dimension of both the harmonic and homology spaces for L_1) increases as one considers lower rows (see the second column). Observe that Betti-1 corresponds to the number of cycles that are not the boundary of a two-simplex, i.e., “unfilled triangles” in these three examples. The third column illustrates that GHL-1 consensus converges to a consensus value in the harmonic subspace $\text{Ker}(L_1)$. For some applications, we predict that it is beneficial for $\mathbf{x}(t) \rightarrow 0$. This can be achieved by either choosing the initial condition to be orthogonal to the harmonic subspace, $\mathbf{x}^{(h)}(0) = 0$, or ensuring that the simplicial complex has a zero-dimensional harmonic subspace, i.e., no homology groups (i.e., unfilled cycles) in dimension k . For other applications—in particular, multiscale dynamical systems, it may also be beneficial for the dynamics to instead converge to a low-dimensional subspace, within which other dynamics could occur (e.g., oscillations and synchronization). Thus, although we call our proposed model GHL- k “consensus,” we expect the findings that we present herein to also be relevant to much more diverse types of dynamics (see, e.g., Refs. 1 and 12).

IV. OPTIMIZING CONSENSUS CONVERGENCE RATE

A. Convergence rate is maximized by optimal balancing of subspace dynamics

Recall that the convergence rate of Eq. (5) is given by $\mu(\delta) = \min\{(1 + \delta)\lambda_2^{(1)}, (1 - \delta)\lambda_2^{(2)}\}$ and that convergence occurs faster when μ is larger. Therefore, for a given simplicial complex, we seek to maximize the convergence rate by studying an optimization problem $\delta^* = \operatorname{argmax}_\delta \mu(\delta)$, for which we find the solution

$$\delta^* = \frac{\lambda_2^2 - \lambda_2^1}{\lambda_2^2 + \lambda_2^1} \tag{9}$$

In this case, the optimum coincides with the intersection of two lines and is found by equating $(1 + \delta)\lambda_2^{(1)} = (1 - \delta)\lambda_2^{(2)}$. In other words, the fastest total convergence requires that we choose the balancing parameter δ so that the decoupled dynamics with the gradient and curl subspaces have the same convergence rate. Such an optimum may be beneficial to engineering applications and the optimization of naturally occurring systems.

In Fig. 5, we provide numerical validation of the optimum for the simplicial complex from Fig. 1. The solid orange and dashed pink lines [Fig. 5(a)] indicate our theoretical predictions of convergence rate within the gradient and curl subspaces, respectively. These are in excellent agreement with empirically observed convergence rates for these two spaces, which are indicated by orange triangles and pink squares, respectively. Black dots indicate empirically observed convergence rates for the full system, and note that they align the minimum of the two subspace convergence rates.

Vertical dotted lines in Fig. 5(a) highlight three values of δ that are chosen to be below, at, and above the optimum $\delta^* \approx 0.08$. In Fig. 5(b), we plot empirically observed values of the error $\|\mathbf{x}(t) - \mathbf{x}(\infty)\|$, in a log scale, for the total system as well as the gradient and curl components. Considering the left and right subpanels of Fig. 5(b), observe that the gradient component converges more slowly than the curl component when $\delta < \delta^*$, and the opposite occurs when $\delta > \delta^*$. In either case, the slower dynamics slows the full system’s convergence. As shown in the center panel of Fig. 5(b), the system converges fastest at the optimum δ^* , where the curl and gradient components have the same convergence rate. Under this optimal choice for the balancing parameter, we say that $L_1^{(\delta^*)}$ is *balanced*.

B. Dispersed higher-order interactions accelerate consensus

The optimal balancing parameter δ^* is unique to each simplicial complex, and it is natural to wonder how δ^* is influenced by the topology and homology of a simplicial complex. To this end, we created three example simplicial complexes that have the same underlying 1-skeleton (the same nodes and edges) but different two-simplices. Specifically, each simplicial complex has the same number of triangles, but they have different placements, as shown in Fig. 6(a). Because they share the same nodes and edges, all three simplicial complexes have the same B_1 boundary matrix and the associated gradient subspace. In contrast, their B_2 boundary matrices differ as well as their curl subspaces. In Fig. 6(b), we plot the convergence rates of the gradient and curl components for

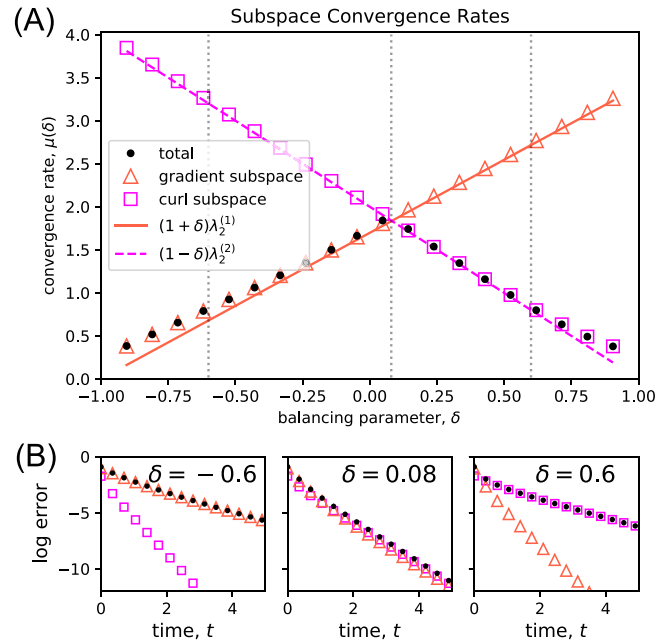


FIG. 5. Balancing invariant subspace dynamics maximally accelerates convergence. Working with the simplicial complex from Fig. 1, plot (a) shows the convergence rate of the total value as well as the gradient and curl subspaces in GHL-1 consensus simulations for different values of δ . The orange and purple lines show the expected values for these convergence rates calculated using the eigenvalues of the component matrices of the generalized Hodge Laplacian as explained in Sec. III B. For this simplicial complex, the balancing parameter that optimizes the convergence rate is $\delta^* \approx 0.08$. In (b), we plot the log normed error, $\log(\|\mathbf{x}(t) - \mathbf{x}^{(h)}\|_2)$, again for the total as well as the gradient and curl subspaces. Remember that the consensus here converges to the harmonic projection of the initial condition.

the three simplicial complexes. First, observe that their gradient subspaces’ convergence rates are all the same (see the black line). In contrast, their curl subspaces have different convergence rates (see dashed lines). The three dots highlight the optimum δ^* for each simplicial complex. Observe that these dots move up and to the right as one considers simplicial complex 1, 2, and 3. That is, convergence occurs fastest for simplicial complex 3 and slowest for simplicial complex 1. It follows that convergence is faster for the curl subspace (and the total system) when the two-simplices are spaced apart, i.e., rather than clustered together.

Another interesting property is that while the optimal convergence rate $\mu(\delta^*)$ increases with each subsequent simplicial complexes (i.e., 1–3), the associated optimum δ^* also increases. Recall our definition for the generalized Hodge Laplacian $L_k^{(\delta)}$ given by Eq. (4) that larger δ implies stronger coupling via lower-dimensional interactions (i.e., adjacency via shared nodes). In other words, because the total coupling weight is conserved [i.e., the weights are $(1 + \delta)$ and $(1 - \delta)$] and because the curl subspace (which arises due to higher-dimensional interactions) converges faster for simplicial complex 3, this allows the coupling weight to be increased for the lower-dimensional interactions. We find this to be somewhat

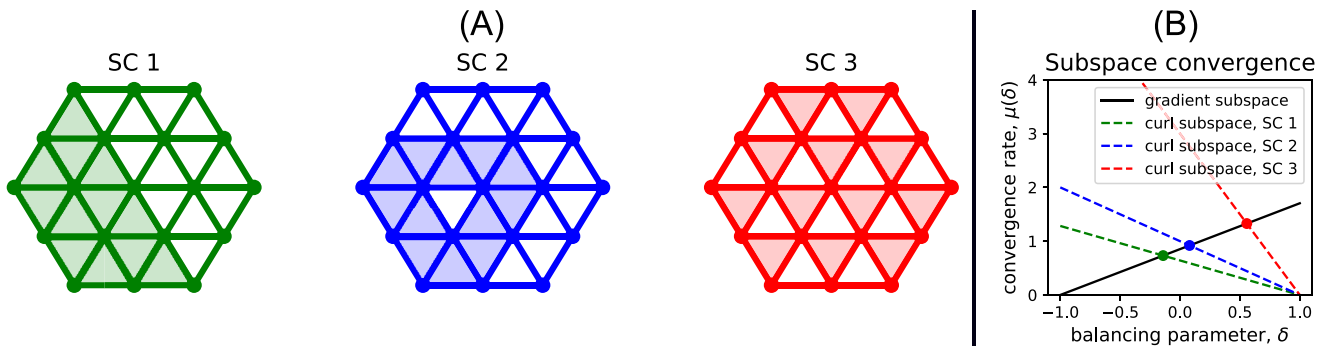


FIG. 6. Two-simplex arrangement affects the convergence rate of the curl component. (a) Simplicial complexes with different two-simplices. Since they have identical nodes and edges, they have the same gradient subspaces but different curl subspaces. (b) Convergence rate $\mu(\delta)$ vs δ for GHL-1 consensus on these three simplicial complexes. The three dots highlight how the optimum δ^* differs across the three simplicial complexes. Notably, convergence is fastest for SC 3 (i.e., where the two-simplices are most dispersed), and the optimum occurs at a larger value of δ .

counterintuitive—that is, as convergence through higher-dimensional interactions becomes faster, the optimal system relies more strongly on the lower-dimensional interactions.

In Fig. 7, we further study the optimal convergence rate $\mu(\delta^*)$ and optimal coupling δ^* for several additional simplicial complexes that have the same nodes and edges but different arrangements of two-simplices. We also now allow them to contain different numbers of two-simplices, but we do not identify an obvious relation between that number and $\mu(\delta^*)$ or δ^* . That said, similar to Fig. 6, we observe that faster convergence is associated with larger δ^* . However, it is important to stress that in general, this need not be the case. This phenomenon occurs here because we restricted our attention to simplicial complexes that have the same nodes and edges and thus the same gradient subspaces.

V. DISCUSSION

We have presented here a model for consensus dynamics over simplicial complexes that uses a generalization of the Hodge Laplacian that we call *generalized Hodge Laplacians*. It is likely that in many real-world applications of simplicial complex dynamics, higher- and lower-dimensional interactions between simplices are not equally weighted. In these applications, the generalized Hodge Laplacian could prove a useful tool in modeling this phenomenon. For example, balancing the Hodge subspaces has been recently shown to benefit signal processing over simplicial complexes,²² and the connections between dynamics, data filtering, and spectral optimization remain underexplored for simplicial complexes.

Our work identified a decoupling of consensus dynamics given by Eq. (5) onto the Hodge subspaces, and we showed that the collective dynamics converge to the harmonic space (which is isomorphic to the homology group of the simplicial complex). These findings open several new directions of research in working with dynamics, consensus, and homology of simplicial complexes. The literature on consensus and other dynamical processes over networks (e.g., diffusion) has been restricted to studying convergence toward a 1D subspace. Understanding how systems converge toward

higher-dimensional subspaces represents an important new direction that may be foundational to more complicated dynamical systems. In particular, biological neurosystems are also likely to organize along low-dimensional subspaces and/or manifolds,⁴⁸ and it has been noted that deep neural networks are non-convergent and may even approach limit cycles⁴⁹ that are within subspaces of dimension greater than 1.

On the practical side, we also developed theory to optimally balance higher- and lower-order interactions to maximally accelerate convergence. Interestingly, the optimum coincides with a balancing of dynamics on the gradient and curl subspaces of the Hodge decomposition. We additionally explored the effects of topological structure, finding that consensus over edges is accelerated

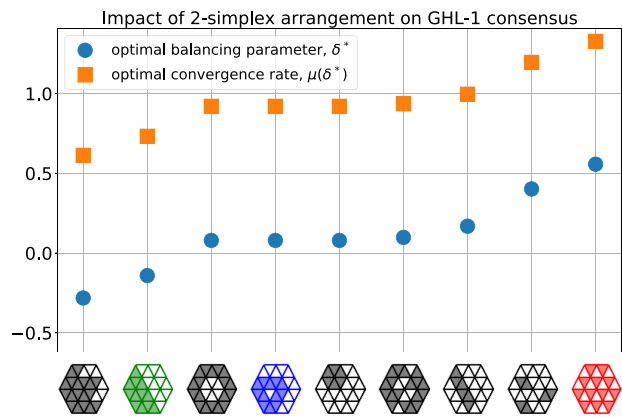


FIG. 7. Two-simplex placement changes the optimal convergence rate. Visualization of the optimal convergence rate $\mu(\delta^*)$ and the optimal balancing parameter δ^* for different simplicial complexes that have the same nodes and edges but different locations and numbers of two-simplices. Interestingly, we do not identify a clear relation between the number of two-simplices and $\mu(\delta^*)$ or δ^* . However, we again find that convergence tends to be faster when the two-simplices are more dispersed.

when two-simplices are well dispersed. There is much to learn about the specific quantitative reasons for this behavior. Moreover, we expect that the study of empirical simplicial complexes (e.g., models of biological, social, and physical systems) and random simplicial complexes (e.g., with tunable levels of community structure, geometry, and clustering) will reveal additional structural/topological/homological mechanisms that also contribute to the optimization of consensus dynamics.

Other possible future directions can be found by revisiting our modeling choices. While GHl consensus is defined for higher dimensions, we restricted our simulations to the $k = 1$ case for simplicity. It is worth investigating not only GHl- k consensus for $k > 1$, but also properties of the gradient, curl, and harmonic subspaces for these higher dimensions. At the same time, we introduced a balancing based on the combinatorial Hodge Laplacian, and it would also be beneficial to balance normalized Hodge Laplacians¹ (which were recently introduced to model random walks on simplicial complexes). Our generalized Hodge Laplacian may be useful for other dynamics as well, and we are currently investigating the relation between our work and explosive synchronization on simplicial complexes.¹² Another interesting direction would be to consider adaptive dynamics in which the balancing parameter is allowed to change with time, i.e., $\delta(t)$. Alternatively, we could introduce a learning rate $\beta(t)$ and replace Eq. (5) by $\frac{d}{dt}\mathbf{x}(t) = -\beta(t)L_k^\delta\mathbf{x}(t)$. We could allow the learning rate to change over time, which is commonly done in decentralized machine learning.

Finally, we conclude by highlighting how our proposed model and theory may be useful to help support the development of decentralized algorithms for machine learning and artificial intelligence (AI). In this application, data are dispersed across a network of computer nodes (e.g., separate CPUs that are either organized into a high-performance computer cluster or are spatially dispersed through a network of remote sensors). Such algorithms already rely on various forms of consensus in which the states of nodes (i.e., one-simplices) coordinate to learn a single model for data. In this context, most research focuses on designing algorithms that converge as fast and efficient as possible (see the review⁵⁰). Our findings here suggest that it may be feasible to generalize such algorithms to also coordinate the states of two-simplices and higher-dimensional simplices. This potential new direction for decentralized learning is especially relevant, given this community's growing interest^{19,20,43,44} to utilize simplicial complexes and Hodge Laplacians to design deep neural networks.

ACKNOWLEDGMENTS

C.Z., H.D., and D.T. were supported in part by the National Science Foundation (No. EDT-1551069). D.T. also acknowledges the NSF grant (No. DMS-2052720) and the Simons Foundation (Grant No. 578333). P.S.S. acknowledges the NSF grant (No. MCB-2126177).

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study. However, the Python code that implements the GHl-1 consensus on simplicial complexes and reproduces our experiments can be found in Ref. 51.

REFERENCES

- ¹M. T. Schaub, A. R. Benson, P. Horn, G. Lippner, and A. Jadbabaie, "Random walks on simplicial complexes and the normalized Hodge 1-Laplacian," *SIAM Rev.* **62**, 353–391 (2020).
- ²S. Mukherjee and J. Steenbergen, "Random walks on simplicial complexes and harmonics," *Random Struct. Algorithms* **49**, 379–405 (2016).
- ³R. Rosenthal, "Simplicial branching random walks and their applications," [arXiv:1412.5406](https://arxiv.org/abs/1412.5406) (2014).
- ⁴I. Iacopini, G. Petri, A. Barrat, and V. Latora, "Simplicial models of social contagion," *Nat. Commun.* **10**, 2485 (2019).
- ⁵J. T. Matamalas, S. Gómez, and A. Arenas, "Abrupt phase transition of epidemic spreading in simplicial complexes," *Phys. Rev. Res.* **2**, 012049 (2020).
- ⁶G. Petri, P. Expert, F. Turkheimer, R. Carhart-Harris, D. Nutt, P. J. Hellyer, and F. Vaccarino, "Homological scaffolds of brain functional networks," *J. R. Soc. Interface* **11**, 20140873 (2014).
- ⁷M. W. Reimann, M. Nolte, M. Scolamiero, K. Turner, R. Perin, G. Chindemi, P. Dłotko, R. Levi, K. Hess, and H. Markram, "Cliques of neurons bound into cavities provide a missing link between structure and function," *Front. Comput. Neurosci.* **11**, 555 (2017).
- ⁸B. U. Kilic and D. Taylor, "Simplicial cascades are orchestrated by the multidimensional geometry of neuronal complexes," [arXiv:2201.02071](https://arxiv.org/abs/2201.02071) (2022).
- ⁹A. Vergne, L. Decreusefond, and P. Martins, "Simplicial homology for future cellular networks," *IEEE Trans. Mobile Comput.* **14**, 1712–1725 (2015).
- ¹⁰L. Gambuzza, F. Di Patti, L. Gallo, S. Lepri, M. Romance, R. Criado, M. Frasca, V. Latora, and S. Boccaletti, "The master stability function for synchronization in simplicial complexes," [arXiv:2004.03913](https://arxiv.org/abs/2004.03913) (2020).
- ¹¹G. Bianconi, "The topological Dirac equation of networks and simplicial complexes," [arXiv:2106.02929](https://arxiv.org/abs/2106.02929) (2021).
- ¹²A. P. Millán, J. J. Torres, and G. Bianconi, "Explosive higher-order Kuramoto dynamics on simplicial complexes," *Phys. Rev. Lett.* **124**, 218301 (2020).
- ¹³L. DeVillie, "Consensus on simplicial complexes: Results on stability and synchronization," *Chaos* **31**, 023137 (2021).
- ¹⁴P. S. Skardal and A. Arenas, "Abrupt desynchronization and extensive multistability in globally coupled oscillator complexes," *Phys. Rev. Lett.* **122**, 248301 (2019).
- ¹⁵A. Arnaudon, R. L. Peach, G. Petri, and P. Expert, "Connecting Hodge and Sakaguchi-Kuramoto: A mathematical framework for coupled oscillators on simplicial complexes," [arXiv:2111.11073](https://arxiv.org/abs/2111.11073) (2021).
- ¹⁶S. Maletić and M. Rajković, "Consensus formation on a simplicial complex of opinions," *Physica A* **397**, 111–120 (2014).
- ¹⁷L. Neuhäuser, A. Mellor, and R. Lambiotte, "Multibody interactions and nonlinear consensus dynamics on networked systems," *Phys. Rev. E* **101**, 032310 (2020).
- ¹⁸M. Arai, V. Brandt, and Y. Dabaghian, "The effects of theta precession on spatial learning and simplicial complex dynamics in a topological model of the hippocampal spatial map," *PLoS Comput. Biol.* **10**, e1003651 (2014).
- ¹⁹T. M. Roddenberry and S. Segarra, "HodgeNet: Graph neural networks for edge data," in *2019 53rd Asilomar Conference on Signals, Systems, and Computers* (IEEE, 2019), pp. 220–224.
- ²⁰S. Ebli, M. Defferrard, and G. Spreemann, "Simplicial neural networks," [arXiv:2010.03633](https://arxiv.org/abs/2010.03633) (2020).
- ²¹M. T. Schaub, J.-B. Seby, F. Frantzen, T. M. Roddenberry, Y. Zhu, and S. Segarra, "Signal processing on simplicial complexes," [arXiv:2106.07471](https://arxiv.org/abs/2106.07471) (2021).
- ²²M. Yang, E. Isufi, M. T. Schaub, and G. Leus, "Finite impulse response filters for simplicial complexes," [arXiv:2103.12587](https://arxiv.org/abs/2103.12587) (2021).
- ²³J. P. Gleeson, S. Melnik, J. A. Ward, M. A. Porter, and P. J. Mucha, "Accuracy of mean-field theory for dynamics on real-world networks," *Phys. Rev. E* **85**, 026106 (2012).

- ²⁴T. A. Snijders, G. G. Van de Bunt, and C. E. Steglich, "Introduction to stochastic actor-based models for network dynamics," *Soc. Netw.* **32**, 44–60 (2010).
- ²⁵D. Horak, S. Maletić, and M. Rajković, "Persistent homology of complex networks," *J. Stat. Mech.: Theory Exp.* **2009**, P03034.
- ²⁶J.-B. Bardin, G. Spreemann, and K. Hess, "Topological exploration of artificial neuronal network dynamics," *Netw. Neurosci.* **3**, 725–743 (2019).
- ²⁷M. Q. Le and D. Taylor, "Persistent homology of convection cycles in network flows," [arXiv:2109.08746](https://arxiv.org/abs/2109.08746) (2021).
- ²⁸A. Hatcher, *Algebraic Topology* (Cambridge University Press, 2001).
- ²⁹L. Marsh, E. Dufresne, H. M. Byrne, and H. A. Harrington, "Algebra, geometry and topology of ERK kinetics," [arXiv:2112.00688](https://arxiv.org/abs/2112.00688) (2021).
- ³⁰A. Muhammad and M. Egerstedt, "Control using higher order Laplacians in network topologies," in Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (2006), pp. 1024–1038.
- ³¹V. B. Hinsz, "Cognitive and consensus processes in group recognition memory performance," *J. Pers. Social Psychol.* **59**, 705 (1990).
- ³²C. M. Fiol, "Consensus, diversity, and learning in organizations," *Organ. Sci.* **5**, 403–420 (1994).
- ³³L. Conradt and T. J. Roper, "Consensus decision making in animals," *Trends Ecol. Evol.* **20**, 449–456 (2005).
- ³⁴Z. Song and D. Taylor, "Asymmetric coupling optimizes interconnected consensus systems," [arXiv:2106.13127](https://arxiv.org/abs/2106.13127) (2021).
- ³⁵J. Tsitsiklis, D. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Trans. Autom. Control* **31**, 803–812 (1986).
- ³⁶S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Gossip algorithms: Design, analysis and applications," in *Proceedings IEEE 24th Annual Joint Conference of the IEEE Computer and Communications Societies* (IEEE, 2005), Vol. 3, pp. 1653–1664.
- ³⁷A. S. Bijral, A. D. Sarwate, and N. Srebro, "Data-dependent convergence for consensus stochastic optimization," *IEEE Trans. Autom. Control* **62**, 4483–4498 (2017).
- ³⁸B. Huynh, H. Dutta, and D. Taylor, "Impact of community structure on consensus machine learning," [arXiv:2011.01334](https://arxiv.org/abs/2011.01334) (2021).
- ³⁹M. Assran, N. Loizou, N. Ballas, and M. Rabbat, "Stochastic gradient push for distributed deep learning," in *International Conference on Machine Learning* (PMLR, 2019), pp. 344–353.
- ⁴⁰K. Niwa, N. Harada, G. Zhang, and W. B. Kleijn, "Edge-consensus learning: Deep learning on P2P networks with nonhomogeneous data," in *Proceedings of the 26th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining* (ACM, 2020), pp. 668–678.
- ⁴¹T. Vogels, S. P. Karimireddy, and M. Jaggi, "PowerGossip: Practical low-rank communication compression in decentralized deep learning," [arXiv:2008.01425](https://arxiv.org/abs/2008.01425) (2020).
- ⁴²L. Kong, T. Lin, A. Koloskova, M. Jaggi, and S. U. Stich, "Consensus control for decentralized deep learning," [arXiv:2102.04828](https://arxiv.org/abs/2102.04828) (2021).
- ⁴³T. M. Roddenberry, N. Glaze, and S. Segarra, "Principled simplicial neural networks for trajectory prediction," in *International Conference on Machine Learning* (PMLR, 2021), pp. 9020–9029.
- ⁴⁴E. Bunch, Q. You, G. Fung, and V. Singh, "Simplicial 2-complex convolutional neural nets," [arXiv:2012.06010](https://arxiv.org/abs/2012.06010) (2020).
- ⁴⁵L.-H. Lim, "Hodge Laplacians on graphs," *SIAM Rev.* **62**, 685–715 (2020).
- ⁴⁶R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control* **49**, 1520–1533 (2004).
- ⁴⁷D. V. Anand and M. K. Chung, "Hodge-Laplacian of brain networks and its application to modeling cycles," [arXiv:2110.14599](https://arxiv.org/abs/2110.14599) (2021).
- ⁴⁸A. R. McIntosh and V. K. Jirsa, "The hidden repertoire of brain dynamics and dysfunction," *Netw. Neurosci.* **3**, 994–1008 (2019).
- ⁴⁹S. Bock and M. Weiß, "Non-convergence and limit cycles in the Adam optimizer," in *International Conference on Artificial Neural Networks* (Springer, 2019), pp. 232–243.
- ⁵⁰M. Nokleby, H. Raja, and W. U. Bajwa, "Scaling-up distributed processing of data streams for machine learning," *Proc. IEEE* **108**, 1984–2012 (2020).
- ⁵¹C. Ziegler, "Codebase for balanced Hodge Laplacians optimize consensus over simplicial complexes," <https://github.com/cameronziegler/Balanced-Hodge-Laplacian> (2021).