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An ergodic system is dominant exactly when it has positive entropy

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Abstract. An ergodic dynamical system **X** is called dominant if it is isomorphic to a generic extension of itself. It was shown by Glasner *et al* [On some generic classes of ergodic measure preserving transformations. *Trans. Moscow Math. Soc.* **82**(1) (2021), 15–36] that Bernoulli systems with finite entropy are dominant. In this work, we show first that every ergodic system with positive entropy is dominant, and then that if **X** has zero entropy, then it is not dominant.

Key words: dominant systems, generic properties, Bernoulli systems, relative Bernoulli, very weak Bernoulli
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Contents

1. *Introduction*

We say that an ergodic system $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ is *dominant* if a generic extension $\hat{T}$ of T is isomorphic to T. We obtain the surprising result that every ergodic positive entropy system of an amenable group has the property that its generic extension is isomorphic to it. For $\mathbb{Z}$ systems, we show that, conversely, when an ergodic system has zero entropy, then it is not dominant. Our first result for $\mathbb{Z}$ actions follows from an extension of a result from [8] according to which a generic extension of a Bernoulli system is Bernoulli with the same entropy (and hence is isomorphic to it by Ornstein's fundamental result) to the relative situation—together with Austin's weak Pinsker theorem [3]. The extension to all countable amenable groups relies on the results in [5, 18, 22]. For the result that zero entropy is not dominant for $\mathbb{Z}$ actions, we use an idea from the slow entropy developed in [12].

To make the definition of dominance more precise, as in [8, 9], we present a convenient way of parameterizing the space of extensions of T as follows. Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an ergodic system. We will assume throughout this work (excepting the last section, where we will comment about the infinite entropy case) that it is infinite and has finite entropy, which, for convenience, we assume is equal to 1. Let $\mathcal{R} \subset \mathcal{X}$ be a finite generating partition. Let $\mathcal{S}$ be the collection of Rokhlin cocycles with values in the Polish group of measure-preserving automorphisms of the unit interval $\mathrm{MPT}(I, \mathcal{C}, \lambda)$, where λ is the normalized Lebesgue measure and $\mathcal{C}$ is the Borel σ-algebra on $I = [0, 1]$. Thus, an element $S \in \mathcal{S}$ is a measurable map $x \mapsto S_x \in \mathrm{MPT}(I, \lambda)$, and we associate to it the *skew product transformation*

$$\hat{S}(x, u) = (Tx, S_x u) \quad (x \in X, u \in I),$$

on the measure space $(X \times I, \mathcal{X} \times \mathcal{C}, \mu \times \lambda)$.

We recall that, by Rokhlin's theorem, every ergodic extension $\mathbf{Y} \to \mathbf{X}$ either has this form or it is n to 1 almost everywhere (a.e) for some $n \in \mathbb{N}$ (see e.g. [7, Theorem 3.18]). Thus, the collection $\mathcal{S}$ parameterizes the ergodic extensions of $\mathbf{X}$ with infinite fibers. This defines a Polish topology on $\mathcal{S}$ which is inherited from the Polish group $\mathrm{MPT}(X \times I, \mu \times \lambda)$ of all the measure-preserving transformations.

In [8], we have shown that for a fixed ergodic finite entropy T with property $\mathbf{A}$, a generic extension $\hat{T}$ of T also has the property $\mathbf{A}$, where $\mathbf{A}$ stands for each of the following properties: (i) having the same entropy as T; (ii) Bernoulli; (iii) K; and (iv) loosely Bernoulli.

Now with this notation at hand, the definition above becomes the following.

Definition 1.1. An ergodic system $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ is *dominant* if there is a dense G_δ subset $\mathcal{S}_0 \subset \mathcal{S}$ such that for each $S \in \mathcal{S}_0$, we have $\hat{S} \cong T$.

From [8, Theorems 4.1 and 5.1], if $\mathbf{B}$ is a Bernoulli system with finite entropy, then its generic extension is again Bernoulli having the same entropy. By Ornstein's theorem [17], such an extension is isomorphic to $\mathbf{B}$. This proves the following proposition.

PROPOSITION 1.2. *Every Bernoulli system with finite entropy is dominant.*

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We recall (see [16]) that an ergodic system $\mathbf{X}$ is *coalescent* if every endomorphism E of $\mathbf{X}$ is an automorphism. Note that when an extension $\hat{S}$, as above with $\hat{S} \cong T$, exists, then the system $\mathbf{X}$ is not coalescent. In fact, if $\pi : \hat{S} \to T$ is the (infinite to one) extension, and $\theta : T \to \hat{S}$ is an isomorphism, then $E = \pi \circ \theta$ is an endomorphism of $\mathbf{X}$ which is not an automorphism. Thus, we have the following proposition.

PROPOSITION 1.3. *A dominant system is not coalescent.*

Hahn and Parry [10] showed that totally ergodic automorphisms with quasi-discrete spectrum are coalescent. In [16], Dan Newton says:

'A question put to me by Parry in conversation is the following: if T has positive entropy does it follow that T is not coalescent?'

Using theorems of Ornstein [17] and Austin [3], we can now prove the following theorem.

THEOREM 1.4. *An ergodic system with positive entropy is not coalescent.*

Proof. We first observe that a Bernoulli system is never coalescent (if $\mathbf{B}$ is Bernoulli and $\mathbf{B}' \to \mathbf{B}$ is an isometric extension which is again Bernoulli (see [20] for examples) then, by Ornstein's theorem, $\mathbf{B}' \cong \mathbf{B}$). Now let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an ergodic system with positive entropy. By Austin's weak Pinsker theorem [3], we can write $\mathbf{X}$ as a product system $\mathbf{B} \times \mathbf{Z}$ with $\mathbf{B}$ a Bernoulli system of finite entropy. Finally, as noted in [16, Proposition 1], if $T = T_1 \times T_2$, where T_1 is not coalescent, then T is not coalescent. In fact, given an endomorphism E of T_1 which is not an automorphism, the map $E \times \mathrm{Id}$, where Id denotes the identity automorphism on the second coordinate, is an endomorphism of T which is not an automorphism. Applying this observation to $\mathbf{X} = \mathbf{B} \times \mathbf{Z}$, we obtain our claim. □

These results suggest the following question: is every ergodic system of zero entropy not dominant? At least generically, we immediately see that the answer is affirmative. As was shown in [16], the set of coalescent automorphisms in $\mathrm{MPT}(I, \lambda)$ is comeager. Thus by Proposition 1.3, we conclude that the set of non-dominant automorphisms is comeager in $\mathrm{MPT}(I, \lambda)$, and hence also in the dense G_δ subset of $\mathrm{MPT}(I, \lambda)$ comprising the zero entropy automorphisms. However, as we will show in §4 using a slow entropy argument, the answer is affirmative for every ergodic system with zero entropy.

THEOREM 1.5. *Every ergodic system* $\mathbf{X}$ *with zero entropy is not dominant.*

We thank the referee for his helpful comments.

2. *Background on relative Bernoullicity*

Definition 2.1. Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an ergodic system and $\mathcal{X}_0 \subset \mathcal{X}$ a T-invariant σ-subalgebra. Let $\mathbf{X}_0 = (X_0, \mathcal{X}_0, \mu_0, T_0)$ be the corresponding factor system and let $\pi : \mathbf{X} \to \mathbf{X}_0$ denote the factor map. We say that $\mathbf{X}$ is *relatively Bernoulli* over $\mathbf{X}_0$ if there is a T-invariant σ-algebra $\mathcal{X}_1 \subset \mathcal{X}$ independent of $\mathcal{X}_0$ such that $\mathcal{X} = \mathcal{X}_0 \vee \mathcal{X}_1$, and there is a $\mathcal{X}_1$-generating finite partition $\mathcal{K} \subset \mathcal{X}_1$ such that the partitions $\{T^i \mathcal{K}\}_{i \in \mathbb{Z}}$ are

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independent; in other words, the corresponding system $\mathbf{X}_1 = (X_1, \mathcal{X}_1, \mu_1, T_1)$ is Bernoulli and $\mathbf{X} \cong \mathbf{X}_0 \times \mathbf{X}_1$.

If $\mathcal{R}_0$ is a finite generating partition for $\mathcal{X}_0$ and $\mathcal{R}$ is a finite generating partition for $\mathcal{X}$, then J.-P. Thouvenot showed that there is a condition called relatively weak Bernoulli, which is equivalent to the extension being relatively Bernoulli, see [25] and also [14]. This condition is as follows.

Definition 2.2. The partition $(\mathcal{R}, T)$ is *relatively Bernoulli* over $(\mathcal{R}_0, T)$ if for every $\epsilon > 0$, there is N such that for a collection $\mathcal{G}$ of atoms A of the partition $\bigvee_{i=-\infty}^{-1} T^{-i}\mathcal{R}$, and a collection $\mathcal{G}_0$ of atoms B of the partition $\bigvee_{i=-\infty}^{-\infty} T^{-i}\mathcal{R}_0$, we have

$$\mu\bigg(\bigcup\{A \cap B : A \in \mathcal{G}, B \in \mathcal{G}_0\}\bigg) > 1 - \epsilon, \tag{1a}$$

$$\bar{d}_N\bigg(\mathrm{dist}\bigg(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{R} \restriction A \cap B\bigg), \mathrm{dist}\bigg(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{R} \restriction B\bigg)\bigg) < \epsilon, \tag{1b}$$

for all such A and B.

Since $\bigvee_{i=-k}^{-1} T^{-i}\mathcal{R} \nearrow \bigvee_{i=-\infty}^{-1} T^{-i}\mathcal{R}$ and $\bigvee_{i=-k}^{k} T^{-i}\mathcal{R}_0 \nearrow \bigvee_{i=-\infty}^{\infty} T^{-i}\mathcal{R}_0$, this can be formulated in finite terms as: for every $\epsilon > 0$, there exist N and k_0 such that for all $k > k_0$, there is a collection $\mathcal{G}$ of atoms A of $\bigvee_{i=-k}^{-1} T^{-i}\mathcal{R}$ and a collection $\mathcal{G}_0$ of atoms B of $\bigvee_{i=-k}^{k} T^{-i}\mathcal{R}_0$ such that

$$\mu\bigg(\bigcup\{A \cap B : A \in \mathcal{G}, B \in \mathcal{G}_0\}\bigg) > 1 - \epsilon, \tag{2a}$$

$$\bar{d}_N\bigg(\mathrm{dist}\bigg(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{R} \restriction A \cap B\bigg), \mathrm{dist}\bigg(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{R} \restriction B\bigg)\bigg) < \epsilon, \tag{2b}$$

for all such A and B.

One last change—instead of (2b), we can also require that for $A, A' \in \mathcal{G},\ B \in \mathcal{G}_0$,

$$\bar{d}_N\bigg(\mathrm{dist}\bigg(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{R} \restriction A \cap B\bigg), \mathrm{dist}\bigg(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{R} \restriction A' \cap B\bigg)\bigg) < \epsilon. \tag{3}$$

That (2b) implies (3) with 2ϵ is immediate.

For the converse implication, observe first that the distribution $\mathrm{dist}(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{R} \restriction B)$ is the average of $\mathrm{dist}(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{R} \restriction A \cap B)$ over all $A \in \bigvee_{i=-k}^{k} T^{-i}\mathcal{R}$, and that the $\bar{d}$ metric is a convex function of distributions. Therefore, fixing one $A' \in \mathcal{G}$ and averaging over all $A \in \mathcal{G}$, we get (2b).

3. *Positive entropy systems are dominant*

The next theorem is a relative version of Theorem 5.1 in [8] and serves as the main tool in the proof of Theorem 3.2 below.

THEOREM 3.1. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an ergodic system which is relative Bernoulli over $\mathbf{X}_0$ with finite relative entropy, so that $\mathbf{X} = \mathbf{X}_0 \times \mathbf{X}_1$. Then, the generic extension $\hat{S}$ of T is relatively Bernoulli over $\mathbf{X}_0$.*

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Proof. For convenience, we assume that the relative entropy is 1.

As in [8], let $\mathcal{R} \subset \mathcal{X}$ be a finite relatively generating partition for $\mathbf{X}$ over $\mathbf{X}_0$ with entropy 1 (so that $\mathcal{R}$ is a Bernoulli partition independent of $\mathbf{X}_0$), and let $\mathcal{R}_0 \subset \mathcal{X}_0$ be a finite generator for $\mathbf{X}_0$. Let $\mathcal{S}$ be the collection of Rokhlin cocycles with values in $\mathrm{MPT}(I, \lambda)$, where λ is the normalized Lebesgue measure on the unit interval $I = [0, 1]$. Thus, an element $S \in \mathcal{S}$ is a measurable map $x \mapsto S_x \in \mathrm{MPT}(I, \lambda)$, and we associate to it the *skew product transformation*

$$\hat{S}(x, u) = (Tx, S_x u) \quad (x \in X, u \in I).$$

Let $Y = X \times I$ and set $\mathbf{Y} = (Y, \mathcal{Y}, \mu \times \lambda)$, with $\mathcal{Y} = \mathcal{X} \otimes \mathcal{C}$.

Part I: By Theorem 4.1 of [8], there is a dense G_δ subset $\mathcal{S}_0 \subset \mathcal{S}$ with $h(\hat{S}) = 1$ for every $S \in \mathcal{S}_0$. We will first show that the collection of the elements $S \in \mathcal{S}_0$ for which the corresponding $\hat{S}$ is relatively Bernoulli over $\mathbf{X}_0$ forms a G_δ set.

As the inverse limit of relatively Bernoulli systems is relatively Bernoulli, see [24, Proposition 7], to show that a transformation T on $(X, \mathcal{X}, \mu)$ is relatively Bernoulli over $\mathbf{X}_0$, it suffices to show that for a refining sequence of partitions

$$\mathcal{P}_1 \prec \cdots \prec \mathcal{P}_n \prec \mathcal{P}_{n+1} \prec \cdots$$

such that the corresponding algebras $\hat{\mathcal{P}}_n$ satisfy $\bigvee_{n\in\mathbb{N}} \hat{\mathcal{P}}_n = \mathcal{X}$, for each n, the process $(T, \mathcal{P}_n)$ is relatively very weak Bernoulli relative to $(T, \mathcal{R}_0)$.

For each $n \in \mathbb{N}$, let $\mathcal{Q}_n$ denote the dyadic partition of $[0, 1]$ into intervals of size $1/2^n$, and let

$$\mathcal{P}_n = \mathcal{R} \times \mathcal{Q}_n.$$

For any $S \in \mathcal{S}_0$, the relative entropy of $\mathbf{Y} = \mathbf{X} \times [0, 1]$ over $\mathbf{X}_0$ is also 1. Thus, for all n, we have

$$H\bigg(\mathcal{P}_n \mid \bigg(\bigvee_{i=-\infty}^{-1} \hat{S}^{-i}\mathcal{P}_n\bigg) \vee \bigg(\bigvee_{i=-\infty}^{\infty} \hat{S}^{-i}\mathcal{R}_0\bigg)\bigg) = 1,$$

and for all $N \geq 1$,

$$H\bigg(\bigvee_{i=0}^{N-1} \hat{S}^{-i}\mathcal{P}_n \mid \bigg(\bigvee_{i=-\infty}^{-1} \hat{S}^{-i}\mathcal{P}_n\bigg) \vee \bigg(\bigvee_{i=-\infty}^{\infty} \hat{S}^{-i}\mathcal{R}_0\bigg)\bigg) = N.$$

Therefore, we can find a suitably small $\delta > 0$ such that for k_0 large enough,

$$H\bigg(\bigvee_{i=0}^{N-1} \hat{S}^{-i}\mathcal{P}_n \mid \bigg(\bigvee_{i=-k_0}^{-1} \hat{S}^{-i}\mathcal{P}_n\bigg) \vee \bigg(\bigvee_{i=-k_0}^{k_0} \hat{S}^{-i}\mathcal{R}_0\bigg)\bigg) < N + \delta.$$

Now, conditioned on the partition

$$\bigg(\bigvee_{i=-k_0}^{-1} \hat{S}^{-i}\mathcal{P}_n\bigg) \vee \bigg(\bigvee_{i=-k_0}^{k_0} \hat{S}^{-i}\mathcal{R}_0\bigg),$$

the partition $\bigvee_{i=0}^{N-1} \hat{S}^{-i}\mathcal{P}_n$ will be η-independent of

$$\bigg(\bigvee_{i=-k}^{-k_0-1} \hat{S}^{-i}\mathcal{P}_n\bigg) \vee \bigg(\bigvee_{i=-k}^{-k_0+1} \hat{S}^{-i}\mathcal{R}_0\bigg) \vee \bigg(\bigvee_{i=k_0+1}^{k} \hat{S}^{-i}\mathcal{R}_0\bigg)$$

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for all $k \geq k_0$ for η small enough (see Definition 5.1 in [8] and the following discussion), so that the inequality (3) in §2 (with $\mathcal{P}_n$ replacing $\mathcal{R}$) for $k = k_0$ will imply (3) with 2ϵ, for all $k > k_0$.

Define the set $U(n, N_1, N_2, \epsilon, \delta)$ to consist of those $S \in \mathcal{S}_0$ that satisfy:

(1) $H(\bigvee_{i=0}^{N_1-1} \hat{S}^{-i}\mathcal{P}_n \mid (\bigvee_{i=-N_2}^{-1} \hat{S}^{-i}\mathcal{P}_n) \vee (\bigvee_{i=-N_2}^{N_2} \hat{S}^{-i}\mathcal{R}_0)) < N_1 + \delta$;

(2) $\bar{d}_{N_1}(\bigvee_{i=0}^{N_1-1} \hat{S}^{-i}\mathcal{P}_n \restriction A \cap B, \bigvee_{i=0}^{N_1-1} \hat{S}^{-i}\mathcal{P}_n \restriction A' \cap B) < \epsilon$, for a set of atoms $A, A' \in \mathcal{G}$, $B \in \mathcal{G}_0$, where $\mathcal{G} \subset \bigvee_{-N_2}^{-1} \hat{S}^{-i}\mathcal{P}_n$, $\mathcal{G}_0 \subset \bigvee_{-N_2}^{N_2} \hat{S}^{-i}\mathcal{R}_0$ and $(\mu \times \lambda)(\bigcup\{A \cap B : A \in \mathcal{G},\ B \in \mathcal{G}_0\}) > 1 - \epsilon$.

Now the sets $U(n, N_1, N_2, \epsilon, \delta)$ are open (easy to check) and the G_δ set

$$\mathcal{S}_1 = \bigcap_{n,k,l} \bigcup_{N_1,N_2} U(n, N_1, N_2, 1/k, 1/l)$$

comprises exactly the elements $S \in \mathcal{S}_0$ for which the corresponding $\hat{S}$ is relatively Bernoulli over $\mathbf{X}_0$. Thus, if $S \in \mathcal{S}_0$ is such that $\hat{S}$ is relatively Bernoulli, then for every n, ϵ, δ, there are N_1, N_2 such that $S \in U(n, N_1, N_2, \epsilon, \delta)$, and conversely, for every relatively Bernoulli $\hat{S}$, the corresponding S is in $\mathcal{S}_1$.

Part II: The collection $\mathcal{S}_1$ is non-empty. To see this, we first note that the Bernoulli system $\mathbf{X}_1$ admits a proper extension $\hat{\mathbf{X}}_1 \to \mathbf{X}_1$ which is also Bernoulli and has the same entropy. This follows e.g. by a deep result of Rudolph [20, 21], who showed that every weakly mixing group extension of $\mathbf{X}_1$ is again a Bernoulli system. An explicit example of such an extension of the 2-shift is given by Adler and Shields [2]. Since $\hat{\mathbf{X}}_1$ is weakly mixing, the product system $\hat{\mathbf{X}} = \mathbf{X}_0 \times \hat{\mathbf{X}}_1$ is ergodic and $\hat{\mathbf{X}} \to \mathbf{X}_0$ is an element of $\mathcal{S}_1$.

Now apply the relative Halmos theorem [9, Proposition 2.3] to deduce that the G_δ subset $\mathcal{S}_1$ is dense in $\mathcal{S}$, as claimed. □

We can now deduce the positive entropy part of our main result.

THEOREM 3.2. *Every ergodic system $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ of positive finite entropy is dominant.*

Proof. By Austin's weak Pinsker theorem [3], we can present $\mathbf{X}$ as a product system $\mathbf{X} = \mathbf{B} \times \mathbf{Z}$, where $\mathbf{B}$ is a Bernoulli system with finite entropy. Thus, $\mathbf{X}$ is relatively Bernoulli over $\mathbf{Z}$, and by Theorem 3.1, it follows that a generic extension $\hat{S}$ of $\mathbf{X}$ is relatively Bernoulli over $\mathbf{Z}$. Therefore, for such $\hat{S}$, the system $\mathbf{Y} = (X \times I, \mathcal{X} \times \mathcal{C}, \mu \times \lambda, \hat{S})$ is again of the form $\mathbf{Y} = \mathbf{B}' \times \mathbf{Z}$ with $\mathbf{B}'$ a Bernoulli system with the same entropy as that of $\mathbf{B}$. By Ornstein's theorem [17], $\mathbf{B} \cong \mathbf{B}'$, whence also $\mathbf{X} \cong \mathbf{Y}$, and our proof is complete. □

Remark 3.3. With notation as in the proofs of Theorems 3.1 and 3.2, observe that for every $S \in \mathcal{S}$, the system $(Y, \mu \times \lambda, \hat{S})$ admits $\mathbf{Z} = (Z, \mathcal{Z}, \mu, T)$ (with $\mathcal{Z}$ considered as a subalgebra of $\mathcal{X}$) as a factor:

$$(Y, \mu \times \lambda, \hat{S}) \to \mathbf{X} \to \mathbf{Z}.$$

In the Polish group $G =$MPT$(Y, \mu \times \lambda)$, consider the closed subgroup $G_{\mathbf{Z}} = \{g \in G : gA = A \text{ for all } A \in \mathcal{Z}\}$. We now observe that the residual set $\mathcal{S}_1 \subset \mathcal{S}_0$, of those $S \in \mathcal{S}_0$ for

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which $\hat{S}$ is Bernoulli over $\mathbf{Z}$ with the same relative entropy over $\mathbf{X}$, is a single orbit for the action of $G_{\mathbf{Z}}$ under conjugation.

In the last section (§5), we will show that the positive entropy theorem holds for any countable amenable group.

In [8, Theorem 6.4], it was shown that the generic extension of a K-automorphism is a mixing extension. We will next prove an analogous theorem for a general ergodic system with positive entropy. We first prove the following relatively Bernoulli analogue of Theorem 6.2 in [8].

THEOREM 3.4. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be a relatively Bernoulli system over $\mathbf{X}_0$, and S a Rokhlin cocycle with values in* $\mathrm{MPT}(I, \lambda)$*, where $I = [0, 1]$ and λ is Lebesgue measure on I. We denote by $\hat{S}$ the transformation*

$$\hat{S}(x, u) = (Tx, S_x u)$$

on $Y = X \times I$, and let

$$\check{S}(x, u, v) = (Tx, S_x u, S_x v), \quad (x, u, v) \in W = X \times I \times I$$

be the relative independent product of $\mathbf{Y}$ with itself over $\mathbf{X}$. Then for a generic $S \in \mathcal{S}$, the transformation $\check{S}$ is relatively Bernoulli over $\mathbf{X}_0$.

Proof. For the G_δ part, we follow, almost verbatim, the proof of Theorem 3.1, where we now let $\mathcal{Q}_n$ denote the product dyadic partition of $I \times I$ into squares of size $1/2^n \times 1/2^n$ and, with notation as in the proof of Theorem 3.1, we let $\mathcal{P}_n = \mathcal{R} \times \mathcal{Q}_n$.

Thus, it only remains to show that the G_δ set $\mathcal{S}_1$, comprising those $S \in \mathcal{S}_0$ for which $\check{S}$ is relatively Bernoulli on $W = X \times I \times I$ relative to $\mathbf{X}_0$, is non-empty. Now, examples of skew products over a Bernoulli system with such properties are provided by Hoffman in [11]. The base Bernoulli transformation that Hoffman constructs for his example can be arranged to have arbitrarily small entropy by an appropriate choice of the parameters used in the construction in §4 (the skew product example is in §5 and the proof of Bernoullicity is in §5). Using such construction on $\mathbf{X}$ (where the cocycle is measurable with respect to the Bernoulli direct component of $\mathbf{X}$), we obtain our required extension of $\mathbf{X}$. This completes our proof. □

We also recall the following criterion [8, Lemma 6.5].

LEMMA 3.5. *Let $\mathbf{X}$ be ergodic and $\mathbf{Y}$ be a factor of $\mathbf{X}$. Then, the following are equivalent.*

(1) *$\mathbf{X}$ is a relatively mixing extension of $\mathbf{Y}$.*

(2) *In the relatively independent product $X \underset{Y}{\times} X$, the Koopman operator restricted to $L^2(Y)^\perp$ is mixing.*

THEOREM 3.6. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an ergodic system with positive entropy, then the generic extension of $\mathbf{X}$ is relatively mixing over $\mathbf{X}$.*

Proof. By the weak Pinsker theorem [3], we can present $\mathbf{X}$ as a product system $\mathbf{X} = \mathbf{Z} \times \mathbf{B}$, where $\mathbf{B}$ is a Bernoulli system with finite entropy. Thus, $\mathbf{X}$ is relatively

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Bernoulli over $\mathbf{Z}$, and by Theorem 3.4, it follows that a generic extension $\check{S}$ of $\mathbf{X}$ to $X \times I \times I$ is still relatively Bernoulli over $\mathbf{Z}$. Thus, the extended system $\mathbf{W}$ on $W = X \times I \times I$ with $\check{S}$ action has the form $\mathbf{W} = \mathbf{Z} \times \mathbf{B}'$ with $\mathbf{B}'$ again a Bernoulli system.

Now, for the system $\mathbf{Y}$, defined on $Y = X \times I$ by

$$\hat{S}(x, u) = (Tx, S_x u),$$

we have that the corresponding relative product system $\mathbf{Y} \underset{\mathbf{X}}{\times} \mathbf{Y}$ is isomorphic to $\mathbf{W}$, which is a Bernoulli extension of $\mathbf{Z}$ and therefore, by Lemma 3.5, a relatively mixing extension of $\mathbf{Z}$. *A fortiori*, $\mathbf{Y} \underset{\mathbf{X}}{\times} \mathbf{Y}$ is a relatively mixing extension of $\mathbf{X}$ and our proof is complete. □

4. *Zero entropy systems are not dominant*

Definition 4.1.

- For $\omega, \omega' \in \{0, 1\}^n$, the *Hamming* (or $\bar{d}$*-distance*) is defined by

$$\bar{d}(\omega, \omega') = \frac{1}{n}\#\{0 \leq i < n : \omega_i \neq \omega'_i\}.$$

- For two measurable partitions $Q = \{A_i\}_{i=1}^n$, $\hat{Q} = \{B_i\}_{i=1}^n$ of a measured space (X, μ), the distance $d(Q, \hat{Q})$ is defined by

$$d(Q, \hat{Q}) = \frac{1}{2}\sum_{i=1}^{n} \mu(A_i \triangle B_i).$$

THEOREM 4.2. *Every ergodic system* $\mathbf{X}$ *with zero entropy is not dominant.*

Remark 4.3. Recently, Adams [1] has proved a somewhat analogous result in the setting of MPT, the group of all measure-preserving transformations of the unit interval with Lebesgue measure. It is well known that, generically, a T in MPT has zero entropy. What Adams shows is that for any preassigned growth rate for slow entropy, the generic transformation has a complexity which exceeds that rate. In our proof of Theorem 4.2, we do not introduce a formal definition of slow entropy but its definition lies behind our Lemma 4.4.

Proof. We first choose a strictly ergodic model $\mathbf{X} = (X, \mathcal{X}, \mu_0, T)$ for our system which is a subshift of $\{0, 1\}^{\mathbb{Z}}$. By the variational principle, this model will have zero topological entropy. (To see that such a model exists, see for example [6], where this fact can be deduced from property (b) on pp. 281 and Theorem 29.2 on pp. 301.) Denote by a_n the number of n-blocks in X so that a_n is sub-exponential.

For $x_0 \in X$ and $\mathcal{Q} = \{Q_0, Q_1\}$ a partition of X, let

$$B_n(x_0, \epsilon) = \{x \in X : \bar{d}_n(Q_n(x), Q_n(x_0)) < \epsilon\},$$

where for a point $x \in X$ and $n \geq 1$, we write

$$Q_n(x) = \omega_0\omega_1\omega_2 \ldots \omega_{n-1} \quad \text{when } x \in \bigcap_{i=0}^{n-1} T^{-i}(Q_{\omega_i}).$$

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LEMMA 4.4. *For $\epsilon < 1/100$ and $\delta < 1/100$, there is an N such that for all $n \geq N$, if m is the minimal number such that there are points $x_1, x_2, \ldots, x_m$ with*

$$\mu_0\bigg(\bigcup_{i=1}^{m} B_n(x_i, \epsilon)\bigg) > 1 - \delta,$$

then $m \leq a_{2n}$.

Proof. Denote by $\mathcal{P} = \{P_1, P_2\}$ the partition of X according to the 0th coordinate. Given $\epsilon > 0$, there is some k_0 and a partition $\hat{\mathcal{Q}}$ measurable with respect to $\bigvee_{i=-k_0}^{k_0} T^i\mathcal{P}$ such that

$$d(\mathcal{Q}, \hat{\mathcal{Q}}) < \frac{\epsilon}{2}.$$

By ergodicity, there exists an N such that for $n \geq N$, there is a set $A \subset X$ with $\mu_0(A) > 1 - \delta$ with

$$\bar{d}_n(Q_n(x), \hat{Q}_n(x)) < \epsilon \quad \text{for all } x \in A.$$

Let $\{\alpha_i\}_{i=1}^{\ell}$ be those atoms of $\bigvee_{i=-k_0}^{n+k_0} T^i\mathcal{P}$ such that $\alpha_i \cap A \neq \emptyset$, so that $\ell \leq a_{n+2k_0+1}$. Choose $x_i \in \alpha_i \cap A,\ 1 \leq i \leq \ell$. We claim that

$$A \subset \bigcup_{i=1}^{\ell} B_n(x_i, \epsilon).$$

For $x \in \bigcup_{i=1}^{\ell} \alpha_i$, we denote by $i(x)$ that index such that $x \in \alpha_{i(x)}$. Now, since x and $x_{i(x)}$ are in A, we have

$$\bar{d}_n(Q_n(x), \hat{Q}_n(x)) < \epsilon \quad \text{and} \quad \bar{d}_n(Q_n(x), \hat{Q}_n(x)) < \epsilon.$$

Since $x \in \alpha_{i(x)}$, $\hat{Q}_n(x) = Q_n(x)$. Therefore,

$$\bar{d}_n(Q_n(x), Q_n(x_{i(x)})) < 2\epsilon,$$

whence $x \in B_n(x_{i(x)}, \epsilon)$. This proves our claim and we conclude that $m \leq \ell \leq a_{n+2k_0+1}$. Thus, for sufficiently large n, we indeed get $m \leq a_{2n}$. □

We will show that a generic extension of T to $(Y, \mu) = (X \times [0, 1], \mu_0 \times \lambda)$, with λ Lebesgue measure on $[0, 1]$, is not isomorphic to $\mathbf{X}$. To do this, we will show that for a generic extension $\hat{S}$, the partition $\mathcal{Q}$ of Y, defined by splitting $X \times [0, 1]$ into $\{Q_0, Q_1\} = \{X \times [0, \frac{1}{2}], X \times [\frac{1}{2}, 1]\}$, will not satisfy the conclusion of this lemma.

Notation.

- $\mathcal{S}$ is the Polish space comprising the measurable Rohklin cocycles $x \mapsto S_x \in \mathrm{MPT}([0, 1], \lambda)$.
- For $S \in \mathcal{S}$, let $\hat{S}(x, u) = (Tx, S_x u)$.
- $Q_n^{\hat{S}}(y) = \omega_0\omega_1\omega_2 \ldots \omega_{n-1}$, where $y \in \bigcap_{i=0}^{n-1} \hat{S}^{-i}(Q_{\omega_i})$.
-
$$C(\hat{S}, n, \epsilon, \delta) = \min\bigg\{k : \text{there exists } y_1, y_2, \ldots, y_k \in Y, \text{ such that } \mu\bigg(\bigcup_{i=1}^{k} B_n^{\hat{S}}(y_i, \epsilon)\bigg) > 1 - \delta\bigg\}.$$

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Define now

$$\mathcal{U}(N, \epsilon, \delta) = \{S \in \mathcal{S} : \text{there exists } n \geq N \text{ such that } C(\hat{S}, n, \epsilon, \delta) > 2a_{2n}\}.$$

This is an open subset of $\mathcal{S}$ (see e.g. [8] for similar claims). We will show that, for sufficiently small ϵ and δ, it is dense in $\mathcal{S}$.

First, consider the case $S_0 = \text{id}$. Let $\eta > 0$ be given and choose M so that $1/M < \eta$. Now build a Rohklin tower for T, with base B_0 and heights $mM > N$ and $mM + 1$ for a suitable m, filling all of X (for this version of the Rokhlin lemma, see [26, p. 32]). Let $B = B_0 \times [0, 1]$ be the base of the corresponding tower in $(Y, \mu, \hat{S})$. We modify $S_0 = \text{id}$ only on the levels $T^{jM-1}B_0$ for $1 \leq j \leq m$, so that the new S will be within η of S_0. The Q-M names of the points in $T^{jM-1}B$ are constant for all $0 \leq j < m$. We modify S_0 on the levels $T^{jM-1}B$ so that we see all possible 0 -1 names for the M-blocks as we move up the tower with equal measure. A similar procedure is described as *independent cutting and stacking* and is explained in detail in §I.10.d in Shields' book [23].

Lemma 4.5. *Any $B_{mM}(y, \epsilon)$ ball has measure at most $2^{m(-1/2+H(2\epsilon,1-2\epsilon))}$.*

Proof. The Q_{mM}-names of points $y \in B$ are constant on blocks of length M, and all sequences of zeros and ones have equal probability by construction. So by a well-known estimation (using Stirling's formula), in $\{0, 1\}^m$ with uniform measure, the measure of an ϵ-ball in normalized Hamming metric is $\leq 2^{m(-1/2+H(2\epsilon,1-2\epsilon))}$.

For points in the lower half of the tower over B, we have a similar estimate with m replaced by some $\ell > \frac{1}{2}m$ and ϵ replaced by $(m/\ell)\epsilon < 2\epsilon$. For points in the upper half of the tower, for some $\ell < \frac{1}{2}m$, we have that $\hat{S}^\ell y \in B$ and then we get an estimate with $m - \ell > \frac{1}{2}m$. This proves the lemma. □

From this lemma, it follows that to achieve even $\frac{1}{2}$ as $\mu(\bigcup_{i=1}^L B_{mM}(y_i, \epsilon))$, we must have $L \cdot 2^{m(-1/2+H(2\epsilon,1-2\epsilon))} > \frac{1}{2}$, and hence

$$L \geq \tfrac{1}{2} \cdot 2^{m(1/2-H(2\epsilon,1-2\epsilon))}.$$

Since a_n is sub-exponential, this lower bound certainly exceeds a_{2mM} if m is sufficiently large. This shows that this modified S is an element of $\mathcal{U}(N, \epsilon, \delta)$.

A similar construction can be carried out for any $S \in \mathcal{S}$. The main point that needs to be checked is that for small ϵ, no $B_M^{\hat{S}}(y, \epsilon)$-ball can have measure greater than $\frac{1}{2} + \epsilon$.

Lemma 4.6. *For any $\hat{S}$ and all y_0,*

$$\mu(B_M^{\hat{S}}(y_0, \epsilon)) \leq \tfrac{1}{2} + \epsilon.$$

Proof. Let $Q_M^{\hat{S}}(y_0) = \omega_0\omega_1 \ldots \omega_{M-1}$. Then,

$$\bar{d}_M(Q_M^{\hat{S}}(y), Q_M^{\hat{S}}(y_0)) = \frac{1}{M} \sum_{i=0}^{M-1} \mathbf{1}_{Q_{\omega_i}}(\hat{S}^i y_0)(1 - \mathbf{1}_{Q_{\omega_i}}(\hat{S}^i y)),$$

and

$$\int_Y \bar{d}_M(Q_M^{\hat{S}}(y), Q_M^{\hat{S}}(y_0))\, d\mu = \tfrac{1}{2}.$$

Since $\bar{d}_M \leq 1$, the measure of the set where $\bar{d}_M(Q_M^{\hat{S}}(y), Q_M^{\hat{S}}(y_0)) \leq \epsilon$ cannot exceed $\frac{1}{2} + \epsilon$. □

This lemma, which is formulated for the measure μ on the entire space Y, in fact holds as well for any level $L_j = \hat{S}^{jM} B$ in the tower, when we replace μ by the measure μ restricted to L_j. This is so because the partition $\{Q_0, Q_1\}$ intersects each level of the tower in relative measure $\frac{1}{2}$ and $\hat{S}$ is measure preserving.

We now mimic the proof outlined for $S_0 = \mathrm{id}$ and, given $S \in \mathcal{S}$, using an independent cutting and stacking, we change $\hat{S}$ as follows. For the level $L_j = \hat{S}^{jM} B$, consider the partition

$$\mathcal{R}_j = \bigvee_{i=0}^{M-1} \hat{S}^{-i}(\mathcal{Q} \cap \hat{S}^{jM+i} B).$$

We change the transformation $\hat{S}$ at the transition from level $jM - 1$ to level jM, so that these partitions $\mathcal{R}_j$ will become independent.

We want to estimate the size of an mM-ϵ ball around a point $y_0 \in B$. If $y \in B$ belongs to this ball, there is a set $A \subset \{0, 1, 2, \ldots, nM - 1\}$ with $|A| \leq \epsilon\, mM$ where the mM-names of y and y_0 differ. We need now a simple lemma.

LEMMA 4.7. *Let $A \subset \{0, 1, \ldots, mM - 1\}$ such that $|A| \leq \epsilon\, mM$. Denote $I_j = \{jM, jM + 1, \ldots, jM + M - 1\}$, $0 \leq j < m - 1$. Let $J \subset \{0, 1, \ldots, m - 1\}$ be the set of ℓ such that*

$$|I_\ell \cap A| < \sqrt{\epsilon} M.$$

Then, $|J| > (1 - \sqrt{\epsilon})m$.

Proof. Let $K = \{0, 1, \ldots, mM - 1\} \setminus J$. Then,

$$\epsilon\, mM \geq \Big|\bigcup_{k \in K} I_k \cap A\Big| \geq M\sqrt{\epsilon}|K|.$$

Thus, $|K| \leq \sqrt{\epsilon} m$, whence $|J| > (1 - \sqrt{\epsilon})m$. □

Next, using Lemma 4.6 for each level of the form $T^{jM} B_0$, we will estimate the size of an mM-ϵ ball. So fix a point $y_0 \in B$. If $y \in B_{mM}(y_0, \epsilon)$, then by Lemma 4.7, there is a set of indices $J_y \subset \{1, 2, \ldots, m\}$ such that:

(1) $|J_y| \geq (1 - \sqrt{\epsilon})m$;
(2) for each $j \in J_y$, $\hat{S}^{jM} y \in B_M(\hat{S}^{jM} y_0, \sqrt{\epsilon})$.

The number of possible sets that satisfy item (1) is bounded by $2^{mH(\sqrt{\epsilon}, 1-\sqrt{\epsilon})}$. By Lemma 4.6 and by the independence, for such a fixed J_y, the measure of the set of points that satisfy item (2) is at most

$$\big(\tfrac{1}{2} + 2\sqrt{\epsilon}\big)^{m(1-\sqrt{\epsilon})}.$$

Write $(\frac{1}{2}+2\sqrt{\epsilon})^{1-\sqrt{\epsilon}}=2^{-c}$, where $c\geq c_0>0$ for all sufficiently small ϵ. Then,

$$2^{-cm}\cdot 2^{mH(2\epsilon,1-2\epsilon)}=2^{m(-c+H(2\epsilon,1-2\epsilon))}\leq 2^{-(m/2)c_0},$$

for $H(2\epsilon,1-2\epsilon)\leq\frac{1}{2}c_0$. We now see that the measure of the ball $B_{mM}(y_0,\epsilon)$ is bounded by $2^{-(m/2)c_0}$.

This was done for $y_0\in B$ and as in the proof of Lemma 4.5, we obtain the suitable estimations for any y in the tower over B. We conclude the argument as in the case $S=\mathrm{id}$ and again it follows that the resultant modified S is an element of $\mathcal{U}(N,\epsilon,\delta)$.

Finally, for fixed sufficiently small ϵ and δ, setting

$$\mathcal{E}=\bigcap_{N=1}^{\infty}\mathcal{U}(N,\epsilon,\delta),$$

we obtain the required dense G_δ subset of $\mathcal{S}$, where for each $S\in\mathcal{E}$, the corresponding $\hat{S}$ is not isomorphic to T. In fact, if $\hat{S}$ would be isomorphic to T, then the isomorphism would take the partition $\mathcal{Q}$ of Y to a partition $\tilde{\mathcal{Q}}$ of X. Applying Lemma 4.4 to $\tilde{\mathcal{Q}}$, we see that there is some N such that for all $n\geq N$, the conclusion of the lemma holds. However, since $S\in\mathcal{E}$, this is a contradiction. □

5. *The positive entropy theorem for amenable groups*

We fix an arbitrary infinite countable amenable group G. We let $\mathbb{A}(G,\mu)$ denote the Polish space of measure-preserving actions $\{T_g\}_{g\in G}$ of G on the Lebesgue space $(X,\mathcal{X},\mu)$. (For a description of the topology on $\mathbb{A}(G,\mu)$, we refer e.g. to [13].)

As in the proof of Theorem 3.1, let $\mathcal{S}$ be the collection of Rokhlin cocycles from $\mathbf{X}$ with values in $\mathrm{MPT}(I,\lambda)$, that is, $\mathcal{S}$ is a family $\{S^g\}_{g\in G}$, where each element S^g is a collection of measurable maps $x\mapsto S_x^g\in\mathrm{MPT}(I,\lambda)$, such that for $g,h\in G$ and $x\in X$, we have

$$S^{gh}(x)=S^g(T_hx)S^h(x),\quad \mu\text{ a.e.}$$

We associate to $S\in\mathcal{S}$ the *skew product transformation*

$$\hat{S}^g(x,u)=(T_gx,S_x^gu)\quad (x\in X,u\in I).$$

Let $Y=X\times I$ and set $\mathbf{Y}=(Y,\mathcal{Y},\mu\times\lambda)$, with $\mathcal{Y}=\mathcal{X}\otimes\mathcal{C}$.

A free G-action $\mathbf{X}$ defines an equivalence relation $R\subset X\times X$, where $(x,x')\in R$ if and only if there exists $g\in G$, $x'=gx$, and a cocycle $S\in\mathcal{S}$ defines uniquely a cocycle α on R:

$$\alpha(x,x')=S_x^g.$$

(A cocycle α on R is a function from R to $\mathrm{MPT}(I,\lambda)$, which satisfies the *cocycle equation*:

$$\alpha(x,z)=\alpha(y,z)\alpha(x,y).)$$

This map is one-to-one and onto from the set of cocycles on $\mathbf{X}$ to the set of cocycles on R. For more details on this correspondence, see [13, §20, C].

Now let

$$\mathbf{X}=(X,\mathcal{X},\mu,\{T_g\}_{g\in G})\to\mathbf{X}_0=(X_0,\mathcal{X}_0,\mu_0,\{(T_0)_g\}_{g\in G})$$

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be a G-Bernoulli extension, where this notion is defined as in Definition 2.1, but instead of $\{T^i\mathcal{K}\}_{i\in\mathbb{Z}}$ being independent, we now have that $\{T_g\mathcal{K}\}_{g\in G}$ are independent.

Definition 5.1. If G and H are two countable groups acting as measure-preserving transformations $\{T_g\}_{g\in G}, \{S_h\}_{h\in H}$ on the measure space (Z, ν), we say that the actions are *orbit equivalent* if for ν-a.e. $z \in Z,\ Gz = Hz$.

In [**4**, **18**], it is shown that any ergodic measure-preserving action of an amenable group is orbit equivalent to an action of $\mathbb{Z}$.

We will now state an extension of Theorem 3.1 to free actions of G, and, moreover, we will also be able to get rid of the finite entropy assumption on $\mathbf{X}$.

For the proof of the theorem, we will need two facts about extensions. The first is that the relative entropy of an extension depends only on the cocycle defining it and is the same for all amenable group actions which generate the same orbit equivalence relation of the base. This is established in [**22**]. The second fact is that the property of being a relatively Bernoulli extension also depends only on the cocycle and not on the specific action of an amenable group which generates the orbit equivalence relation in the base. This second fact is stated explicitly in [**5**] (§4), but actually follows easily from the first. For the convenience of the reader, we give a proof of this.

LEMMA 5.2. *Let G_1, G_2 be two amenable groups which, acting on $(X_0, \mathcal{X}_0, \mu_0)$ by $\{T_g^{(1)}\}_{g\in G_1}, \{T_g^{(2)}\}_{g\in G_2}$, have the same orbits. If $(X, \mathcal{X}, \mu, \{T_g^{(1)}\}_{g\in G_1})$ is a relatively Bernoulli extension of $(X_0, \mathcal{X}_0, \mu_0, \{T_g^{(1)}\}_{g\in G_1})$ with finite relative entropy, via a cocycle S, then the S-extension of $(X_0, \mathcal{X}_0, \mu_0, \{T_g^{(2)}\}_{g\in G_2})$ is also relatively Bernoulli.*

Proof. By the assumption, there is a finite partition $\mathcal{P}$ of X such that $\{T_g^{(1)}\mathcal{P}\}_{g\in G_1}$ are independent, $\bigvee_{g\in G_1} T_g^{(1)}\mathcal{P}$ is independent of $\mathcal{X}_0$, and together with $\mathcal{X}_0$ spans $\mathcal{X}$. These properties are equivalent to having the relative entropy of $\{T_g^{(1)}\mathcal{P}\}_{g\in G_1}$ being equal to $H(\mathcal{P})$, and having $\{T_g^{(1)}\mathcal{P}\}_{g\in G_1}$ separating points relative to X_0. By the first fact above, these properties persist for $\{T_g^{(2)}\mathcal{P}\}_{g\in G_2}$ and thus, using the same cocycle, the G_2-extension is also relatively Bernoulli. □

THEOREM 5.3. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, \{T_g\}_{g\in G})$ be an ergodic G-system which is relative Bernoulli over a free system $\mathbf{X}_0$ with finite relative entropy, so that $\mathbf{X} = \mathbf{X}_0 \times \mathbf{X}_1$. Then, the generic extension $\hat{S}$ of $\{T_g\}_{g\in G}$ is relatively Bernoulli over $\mathbf{X}_0$.*

Proof. By [**18**, **19**], there is a measure-preserving transformation $T_0 : X_0 \to X_0$ such that orbits of T_0 coincide with G-orbits on X_0, and such that T_0 has zero entropy. The G-factor map $\mathbf{X} = \mathbf{X}_0 \times \mathbf{X}_1 \to \mathbf{X}_0$ is given by a constant cocycle whose constant value is the Bernoulli action on the Bernoulli factor $\mathbf{X}_1$. We use this cocycle, now viewed as a cocycle on the equivalence relation defined by T_0, to define an extension $T : X \to X$. By [**22**], the relative entropy of such a generic T over T_0 is the same as that of the G-action $\mathbf{X}$ over $\mathbf{X}_0$. By Lemma 5.2, the extension of $\mathbb{Z}$-systems $\pi : T \to T_0$ is again relatively Bernoulli. Applying Theorem 3.1 to π, we conclude that a dense G_δ subset $\mathcal{S}_1(\mathbb{Z})$ of extensions of T is such that each $\hat{S} \in \mathcal{S}_1(\mathbb{Z})$ is relatively Bernoulli over T_0. Finally, applying Lemma 5.2

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again, we conclude that the corresponding set of extensions $\mathcal{S}_1(G)$ is a dense G_δ subset of $\mathcal{S}(G)$ and that for each $S \in \mathcal{S}_1(G)$, the corresponding G-system is relatively Bernoulli over $\mathbf{X}_0$. □

As in the case of $\mathbb{Z}$-actions, with the same proof, we now obtain the following theorem.

THEOREM 5.4. *Every ergodic free G-system* $\mathbf{X}$ *of positive entropy is dominant.*

It is natural to ask whether Theorem 4.2 can also be extended to all infinite countable amenable groups. This extension is less straightforward, but it has now been accomplished by Lott [15].

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