Pairwise Comparison Evolutionary Dynamics with Strategy-Dependent Revision Rates: Stability and δ-Passivity

Semih Kara, *Student Member, IEEE*, and Nuno C. Martins, *Senior Member, IEEE*

***Abstract*— We report on new sufficient conditions for the stability of evolutionary dynamics in population games. A large number of agents interact noncooperatively in a population game by selecting strategies based on their payoffs. Each agent is allowed to revise its strategy repeatedly with an average frequency referred to as the revision rate. We are interested in the case where an agent's current strategy influences directly the revision rate. Existing stability results for this case assume that a memoryless potential game generates the strategies' payoffs. This article extends these results to allow for payoff mechanisms that can be either dynamic or memoryless games that do not have to be potential. To make our analysis concrete, we assume that the agents' revision preferences follow a so-called pairwise comparison protocol. These protocols are ubiquitous because they operate fully decentralized and with minimal information requirements (they need to access only the payoff values, not the mechanism). We use a well-motivated example to illustrate an application of our framework.**

***Index Terms*— Evolutionary dynamics, passivity, population games, stability of nonlinear systems.**

I. Introduction

IN THIS paper, we model and analyze the strategic behavior of large agent populations interacting noncooperatively in the context of a population game. We adopt a population games and evolutionary dynamics framework [1] in which each agent follows one strategy at a time. The agents are partitioned into populations, where each population can have a distinct set of available strategies. The net reward of a strategy is quantified by its payoff. A payoff mechanism determines the strategies' payoffs as a function of the strategies' prevalence in the populations as registered in the so-called *social state*. Namely, the social state is a vector whose entries are the proportions of the populations adopting the available strategies.

The agents are repeatedly allowed to revise their strategies at instants called revision opportunity times. For an agent carrying out a strategic revision, a so-called *revision protocol* specifies how to select the next strategy. Typically, the agents tend to switch to strategies with higher payoffs. More generally, the revision protocols, which can be stochastic, model the agents' preferences by specifying how they use the information available (such as the payoffs) to choose their strategies. These strategic choices by the agents in response to the payoffs affect the social state, causing it to vary over time. Consequently, the populations' strategic choices act as a dynamical system, denoted as *evolutionary dynamics*, whose input is the strategies' payoffs and output is the social state. The evolutionary dynamics and the payoff mechanism form a feedback system, as represented in Fig.1. This framework is widely used for modeling large-scale engineering and social systems (see [2], [3, §IV.B] and the references therein).

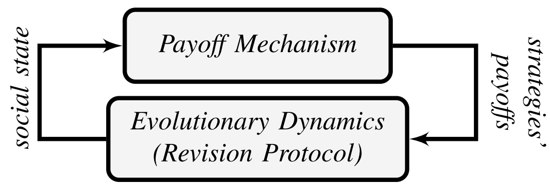


Fig. 1: Payoff mechanism and evolutionary dynamics.

Our main goal is to ascertain the infinite-horizon properties of the social state. To do so, we adopt the deterministic approach described in [3], [4], which generalizes that used in most previous work to study population games [1] and evolutionary games [5], [6]. Specifically, we investigate the global asymptotic stability **(GAS)** properties of a system called the *mean closed loop*, which is a mean field approximation of the system in Fig.1 valid under a large number of agents. As we explain in §II, the GAS equilibria set of the mean closed loop characterizes the long term behavior of the social state.

The rate with which the revision opportunity times occur, which we call the *revision rate*, is central to this article. The novelty of our work is that we will determine how allowing the agents' revision rates to depend on their current strategies affect the GAS properties mentioned above.

All of the concepts introduced so far will be defined rigorously in § II. Using the preliminary framework description we just concluded, we can now preview our main contributions and highlight their novelty relative to existing work.

A. Related Work And Main Contributions

Under some common revision protocols, an agent's revision rate cannot depend directly on the agent's current strategy.

Manuscript received 12 July 2021; revised 15 June 2022 and 27 September 2022; accepted 27 December 2022. Date of publication XXXXX X, XXXX; date of current version XXXXX X, XXXX. This work is supported in part by AFOSR Grant FA9550-19-1-0315 and NSF Grant 2135561. Recommended by Associate Editor W. X. Zheng. *(Corresponding author: Nuno C. Martins.)*

The authors are with the ECE Department and the ISR, University of Maryland, College Park, MD 20742 USA (email: skara@umd.edu; nmartins@umd.edu).

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3237485

A well-studied class of protocols that has this limitation is the so-called *impartial pairwise comparison* **(IPC)** [7, §7.1] protocols. The qualifier *impartial* was introduced in [7, §7.1] to indicate that the revision rate may depend on the current strategy only indirectly through its payoff. Specifically, under an IPC protocol, any two agents whose current strategies have the same payoffs also have the same revision rate. In order to introduce strategy-dependent revision rates to the IPC class, we will propose a straightforward modification of the IPC protocols, which we will call the *rate-modified pairwise comparison* **(RM-PC)** protocols.

Throughout this paper, we assume that the agents follow RM-PC protocols. Provided that the strategies' payoffs are generated by a so-called δ-antidissipative *payoff dynamics model* **(PDM)** [4], we will provide conditions on the revision rates that ensure the GAS of the mean closed loop's equilibria.

Our work fills a clear gap in the literature. Although similar approaches were followed in [4], [8], [9], they considered impartial protocol classes, such as the IPC. As a result, the analysis in these articles cannot be immediately employed when the revision rates are strategy-dependent. On the other hand, at the expense of restricting the payoff mechanism to be a memoryless potential game [10], it is possible to show the global attractivity of the mean closed loop's equilibria when the pairwise comparison protocol is not necessarily impartial [11], [12]. However, the class of memoryless potential games does not admit the presence of any dynamics in the payoff mechanism, and it does not include important types of memoryless payoff mechanisms such as contractive[1] games [8], [13] and their weighted extensions [9].

B. Paper Structure

After the motivation and framework description in §II, in §III we will rigorously describe our problem formulation and in §IV we will state our main results. We will illustrate numerically our results in §V and the article's main body will end with conclusions in §VI.

II. Motivation And Framework Description

We start with an example that motivates the need to consider strategy-dependent revision rates. Subsequently, we will leverage the example to describe our framework in detail.

A. Hassle vs. Price Game (HPG) Example

A motivating example of application of our framework, which we will be invoking throughout this article to illustrate our contributions, is that of a "hassle vs. price" game **(HPG)**. In this example, each agent operates a machine that uses a component that must be replaced when it fails. There are several manufacturers that make the component to varying degrees of reliability. Specifically, each component has an exponentially distributed lifetime and its failure rate depends on the manufacturer. The available strategies are the manufacturers, and the payoff of each strategy combines two non-positive terms: (i) a hassle (disruption) cost that increases with the failure rate and (ii) the price of the component, which is higher for more reliable manufacturers. The revision opportunity time occurs when the component fails and the agent must decide based on the available information, such as the current payoffs ascribed to the strategies, whether to keep the current strategy (buy again from the same manufacturer) or follow a different strategy (decide on another manufacturer to buy from). The agents are partitioned into populations, each uniquely associated with a machine type and/or the undertaking for which the machine is used.

In Example 1 (in §II-C.1) we will describe in detail a memoryless payoff mechanism for the HPG, and in Appendix A we will describe a PDM that generalizes Example 1.

With the HPG example in mind, we now proceed to describe our framework in more general and precise terms.

[1]Contractive games were originally called stable games in [13]. The possibility that calling games stable could cause confusion with notions of system-theoretic stability prompted the nomenclature change.

B. Population States And The Social State

In our framework, a large number of agents is partitioned into a finite number of populations $\{1, \ldots, \rho\}$. At any time $t \geq 0$, each agent of any population r follows a single strategy from a strategy set $\{1, \ldots, n^r\}$. The agents that belong to the same population are nondescript. Hence, the strategy profile of population r at time t can be described by the so-called *population state* $X^r(t)$, whose entries are the proportions of agents in population r selecting the available strategies. Specifically, if N^r is the number of agents in population r, then $N^r X_i^r(t)$ is the number of agents following strategy i at time t in population r. Consequently, $X^r(t)$ is a jump process (right continuous by convention) whose jumps occur at the revision opportunity times for population r. For any $t \geq 0$, $X^r(t)$ is in the following simplex:

$$\mathbb{X}^r := \{x^r \in \mathbb{R}^{n^r}_{\geq 0} \mid x_1^r + \ldots + x_{n^r}^r = 1\}.$$

The so-called *social state* at time t is the concatenation of the states of all populations $X(t) := (X^1(t), \ldots, X^\rho(t))$. Thus, X takes values in $\mathbb{X} := \mathbb{X}^1 \times \cdots \times \mathbb{X}^\rho$.

C. Payoffs

At any time t, we use $P_i^r(t)$ to denote the payoff of strategy i for population r. We denote the concatenation of the populations' payoff vectors as $P(t) := (P^1(t), \ldots, P^\rho(t))$.

We assume that a causal payoff mechanism determines $P := \{P(t) \mid t \geq 0\}$ in terms of $X := \{X(t) \mid t \geq 0\}$. The simplest mechanism is memoryless, in which case $P(t)$ is determined by a map (referred to as *game*) applied to $X(t)$. More generally, the payoff mechanism is a so-called *payoff dynamics model* **(PDM)** with input X and output P. Below, we describe these payoff mechanism types in detail.

1) Memoryless Payoff Mechanism: A memoryless payoff mechanism is specified by a globally Lipschitz continuous and continuously differentiable function $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{R}^n$, acting as $\mathcal{F} : X(t) \mapsto P(t)$, where $n := n^1 + \cdots + n^\rho$. Such an $\mathcal{F}$ is referred to as *the game*.

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3237485

Example 1: **(An HPG game)** The game $\mathcal{F}$, with its i-th component for population r specified below, is an example of a memoryless payoff mechanism for our HPG example:

$$\mathcal{F}_i^r(x) \underset{HPG}{:=} \underbrace{-\beta^r \lambda_i^r}_{\substack{\text{hassle} \\ \text{(replacement) cost}}} - \underbrace{\mathcal{C}_i\big(\mathcal{D}_i(x)\big)}_{\text{component price}}, \quad x \in \mathbb{X}. \quad (1)$$

The game's components have the following meaning:

- $\beta^1, \ldots, \beta^\rho$ are positive constants quantifying the costs of replacing a component for the respective population,
- $\{1, \ldots, \kappa\}$ is the set of available manufacturers (this is also the strategy set equally available to all[2] populations),
- $\lambda_1^r, \ldots, \lambda_\kappa^r$ are the failure rates of the components for the r-th population according to the manufacturer, which we assume are ordered as $\lambda_1^r > \ldots > \lambda_\kappa^r > 0$ (manufacturer κ makes the most reliable components),
- $\mathcal{D} : \mathbb{X} \to [0, \bar{d}]^\kappa$ gives the (effective) demand from each manufacturer as

$$\mathcal{D}_i(x) := \sum_{r=1}^{\rho} \alpha^r x_i^r, \quad 1 \le i \le \kappa, \ x \in \mathbb{X}. \quad (2)$$

Here, $\alpha^1, \ldots, \alpha^\rho$ are positive constants that quantify the relative weight of each population on the demand. These constants may reflect, for instance, the relative sizes of the populations. Finally, $\mathcal{C}_i : \mathbb{R}_{\ge 0} \to [c_i, \infty)$ is a continuously differentiable surjective function (of the demand) that quantifies the cost of a component made by the i-th manufacturer.

Example 2: **(A labour-market example)** We could model the effect of the contract value on employee turnover in a way that would lead to another example analogous to Example 1. In such an example, a population's agents would be the businesses wishing to hire and retain an employee for a specific job type. Each population would comprise businesses with comparable characteristics from the employees' viewpoint, such as location, structure, and size. The strategies available to a population's agents would be the different types of contracts they can offer. In this case, $\mathcal{C}_i$ in (1) would determine the cost of contract i as a function of the demand. Cheaper contracts offering worse benefits and/or lower salaries would lead to a higher turnover rate (quantified by λ_i^r) and associated increased cost for retraining and rehiring (quantified by $\beta_i^r \lambda_i^r$).

2) Payoff Dynamics Model (PDM): More generally, the payoff mechanism is modeled by a PDM [4], [8] with the following structure:

$$\begin{aligned} \dot{Q}(t) &= \mathcal{G}\big(Q(t), X(t)\big) \\ P(t) &= \mathcal{H}\big(Q(t), X(t)\big) \end{aligned}, \quad t \ge 0, \quad Q(0) \in \mathfrak{Q}_0, \quad (3)$$

where, for some $m \in \mathbb{N}$, the set $\mathfrak{Q}_0 \subseteq \mathbb{R}^m$ is compact, $\mathcal{G} : \mathbb{R}^m \times \mathbb{X} \to \mathbb{R}^m$ is globally Lipschitz continuous, $\mathcal{H} : \mathbb{R}^m \times \mathbb{X} \to \mathbb{R}^n$ is continuously differentiable and globally Lipschitz continuous, and there is a game $\mathcal{F}_{\mathcal{G},\mathcal{H}}$ that equals $\mathcal{H}$ in the stationary regime in the sense that

$$\mathcal{G}(q, x) = 0 \Rightarrow \mathcal{H}(q, x) = \mathcal{F}_{\mathcal{G},\mathcal{H}}(x), \quad (q, x) \in \mathbb{R}^m \times \mathbb{X}. \quad (4)$$

[2]This means that all populations have the same strategy set and the same number of strategies ($n^1 = \cdots = n^\rho = \kappa$).

As discussed in [3], [4], PDMs can account for dynamics inherent to certain payoff mechanisms, such as delays, pricing inertia, and agent-level learning. We present a PDM example in Appendix A, which is a modification of Example 1.

The analysis in this article presumes, as was the case in [3], [4], that the state Q remains in a bounded set $\mathfrak{Q}$. Notice that this is guaranteed whenever the PDM is input to state stable [14] because X and $Q(0)$ take values in respectively the bounded sets $\mathbb{X}$ and $\mathfrak{Q}_0$. Furthermore, in combination with Q and X being bounded, the fact that $\mathcal{H}$ is Lipschitz continuous ensures that P remains in a bounded set $\mathfrak{P} := \mathfrak{P}^1 \times \cdots \times \mathfrak{P}^\rho$.

Remark 1: Games are specific PDM instances, because any game $\mathcal{F}$ can be obtained as a PDM by choosing $\mathcal{G}(q, x) = 0$ and $\mathcal{H}(q, x) = \mathcal{F}(x)$ for all $x \in \mathbb{X}$, $q \in \mathbb{R}^m$. Moreover, if a PDM is given by a game $\mathcal{F}$, then $\mathcal{F}_{\mathcal{G},\mathcal{H}} = \mathcal{F}$ (see [4] for further information on how PDMs generalize games).

3) Nash Equilibria: The following Nash equilibria concept defined for a game $\mathcal{F}$ will be central to our approach:

$$\mathbb{NE}(\mathcal{F}) := \Big\{ x \in \mathbb{X} \ \Big| \ x^T \mathcal{F}(x) \ge y^T \mathcal{F}(x), \quad y \in \mathbb{X} \Big\}.$$

When the payoffs are determined by a PDM, we will be interested in $\mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$, which represents the Nash equilibria set of $\mathcal{F}_{\mathcal{G},\mathcal{H}}$. As shown in [1, Theorem 2.1.1], any game has a nonempty set of Nash equilibria.

D. Strategy-Dependent Revision Rates: Key Concepts

In §II-B and §II-C, we introduced the underlying strategic environment, and now we proceed to describe how the agents revise their strategies.

1) Strategy-Dependent Revision Rates: We assume that, for each i in $\{1, \ldots, n^r\}$, a positive constant λ_i^r characterizes the rate at which the agents in population r currently following the i-th strategy revise their strategies. Namely, the length of the time interval between any two consecutive revision opportunities of an agent in population r is distributed exponentially with rate λ_i^r, where i is the agent's strategy resulting from the revision at the beginning of the interval. Additionally, the event that an agent receives a revision opportunity in any time interval $(\underline{t}, \bar{t})$ is conditionally independent, given its strategy at $\underline{t}$, of the revision opportunity events of all other agents. We refer to $\lambda_1^r, \ldots, \lambda_{n^r}^r$ as the *revision rates* for population r and denote $\lambda^r := [\lambda_1^r \ \ldots \ \lambda_{n^r}^r]^T$.

Remark 2: In the HPG example, the revision rates are the failure rates of the components. Note that the conditional independence requirement for the revision events holds for the HPG. Indeed it is safe to assume that once an agent installs a new component, its time of failure depends only on its manufacturer and the agent's population, and not on the choices of the other agents or when the components they currently employ fail.

2) Revision Protocols: Following the standard approach in [1, §4.1.2], we assume that a probabilistic heuristic models how the agents in population r revise their strategies. This heuristic is characterized by a globally Lipschitz continuous map $\mathcal{T}^r : \mathbb{X}^r \times \mathbb{R}^{n^r} \to \mathbb{R}_{\ge 0}^{n^r \times n^r}$ referred to as *the revision protocol*. Specifically, for infinitesimally small $\delta > 0$, the probability that some agent of population r switches from

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3237485

strategy i to j during the time interval $(t^*, t^* + \delta)$ is approximately $\delta N^r X_i^r(t^*) \mathcal{T}_{ij}^r(X^r(t^*), P^r(t^*))$ (we refer to [4, §V.A] for details on this probability). Each agent in a given population revises its strategy according to the same protocol, yet the populations' protocols can be distinct.

Combining the above description with the revision rates characterized in §II-D.1, we decompose $\mathcal{T}^r$ as follows:

$$\mathcal{T}_{ij}^r(x^r, p^r) = \lambda_i^r \tau_{ij}^r(x^r, p^r), \quad (x^r, p^r) \in \mathbb{X}^r \times \mathfrak{P}^r, \tag{5}$$

where $\tau_{ij}^r : \mathbb{X}^r \times \mathbb{R}^{n^r} \to \mathbb{R}_{\geq 0}$ is a globally Lipschitz continuous map satisfying for all $(x^r, p^r) \in \mathbb{X}^r \times \mathfrak{P}^r$ the equality $\sum_{j=1}^{n^r} \tau_{ij}^r(x^r, p^r) = 1$. We interpret τ^r as a transition probability, meaning that if an agent of population r following the i-th strategy is given a revision opportunity at time t^* then $\lim_{t \uparrow t^*} \tau_{ij}^r(X^r(t), P^r(t))$ is the probability that it will switch to strategy j.

E. The Mean Closed Loop Model

We now proceed to describe a deterministic approximation that will simplify our analysis to ascertain whether and in what sense $X(t)$ approaches $\mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$ as time progresses.

1) Deterministic Approximation For Large N^r: When the revision times of the agents occur as described in §II-D.1, the payoffs are characterized by a PDM (which reduces to a game if it is memoryless), and the agents decide on their strategies according to the procedure in §II-D.2, a straightforward modification of the analysis in [4, §V][3] reveals that the pair (X, Q) complies with the assumptions of [15, Theorem 2.11]. Consequently, as discussed in [4, §V] and [3, §IV.A], we can leverage [15, Theorem 2.11] to conclude the following: as the number of agents in each population tends to infinity, if $(X(0), Q(0))$ converges almost surely to some $(x_0, q_0) \in \mathbb{X} \times \mathfrak{Q}_0$, then X and P converge in probability to x and p uniformly over any finite time interval, where x and p are obtained with initial state $(x(0), q(0)) = (x_0, q_0)$ from the solution of the system of differential equations

$$\dot{q}(t) = \mathcal{G}\big(q(t), x(t)\big) \tag{6a}$$
$$p(t) = \mathcal{H}\big(q(t), x(t)\big) \tag{6b}$$
$$\dot{x}^r(t) = \mathcal{V}^r\Big(x^r(t), p^r(t)\Big), \quad r \in \{1, \ldots, \rho\}, \tag{6c}$$

in which $t \geq 0$ and the components of $\mathcal{V}^r : \mathbb{X}^r \times \mathbb{R}^{n^r} \to \mathbb{R}^{n^r}$ are given for all $(x^r, p^r) \in \mathbb{X}^r \times \mathfrak{P}^r$, $i \in \{1, \ldots, n^r\}$ by

$$\mathcal{V}_i^r\Big(x^r, p^r\Big) := \underbrace{\sum_{j=1, j\neq i}^{n^r} \mathcal{T}_{ji}^r\Big(x^r, p^r\Big) x_j^r}_{\text{inflow switching to strategy } i} - \underbrace{\sum_{j=1, j\neq i}^{n^r} \mathcal{T}_{ij}^r\Big(x^r, p^r\Big) x_i^r}_{\text{outflow switching away from strategy } i}. \tag{7}$$

[3]The analysis in [4, §V] can be modified to address strategy-dependent revision rates by replacing the $z_i \mathcal{T}_{ij}/\varrho$ in (15a) and (15b) (of [4, §V]) with $z_i \lambda_i \tau_{ij}/\lambda^T z$, and reproducing the discussion that follows these equations.

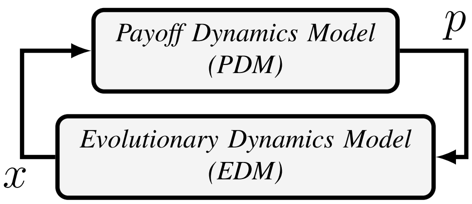


Fig. 2: Diagram representing the feedback interconnection between a PDM and an EDM. The resulting system is referred to as the mean closed loop model.

We refer to x as the mean social state. Note that, when the PDM is given by a game $\mathcal{F}$, the equations in (6) reduce to

$$\dot{x}^r(t) = \mathcal{V}^r\Big(x^r(t), \underbrace{\mathcal{F}^r\big(x(t)\big)}_{p^r(t)}\Big), \quad r \in \{1, \ldots, \rho\}. \tag{8}$$

2) Infinite Horizon Analysis: It follows from [1, Theorem 12.B.3] that, if the equilibria of (6) is GAS, then the stationary distributions of X concentrate near the equilibria of (6) as the number of agents in each population tends to infinity. This result and the assumption that N^r is large for all $r \in \{1, \ldots, \rho\}$ justify the practical relevance of our work.

3) The Mean Closed Loop: Observe that (6a) and (6b) are specified by the PDM, and the dynamics of x is characterized by (6c). The system with input p and output x given by (6c) is referred to as the *evolutionary dynamics model* **(EDM)** [4].

Henceforth, we will interpret (6) as the positive feedback interconnection between the PDM and the EDM. The resulting feedback system, depicted in Fig.2, is called the *mean closed loop* [4]. This interpretation will allow us to employ a passivity-based methodology, which we present in the following section.

III. Problem Formulation And Key Concepts

Having presented the framework, we proceed to formulate in precise terms the technical problems we seek to solve. Subsequently, we will introduce the concepts and techniques that will be central to our technical approach.

A. Problem Formulation

We start by defining the following worst-case ratios quantifying the relative discrepancies of a population's revision rates.

Definition 1: Given r in $\{1, \ldots, \rho\}$ and the revision rates $\lambda_1^r, \ldots, \lambda_{n^r}^r$ for population r, we define the worst-case revision rate ratio for the r-th population as follows:

$$\lambda_R^r := \max\left\{ \frac{\lambda_i^r}{\lambda_j^r} \;\middle|\; i, j \in \{1, \ldots, n^r\} \right\}. \tag{9}$$

Notice that $\lambda_R^r \geq 1$ holds by definition and $\lambda_R^r = 1$ if and only if the revision rates for the r-th population are identical.

In order to develop a methodology that can cope with the case in which $\lambda_R^r > 1$ for one or more populations (unequal revision rates), we seek to solve the following sub-problems:

Problem-i) Propose practicable modified protocols that are compatible, in the sense of the decomposition in (5), with any pre-selected revision rates. As we mentioned in §I-A, our answer to this problem is called the RM-PC protocol class.

Problem-ii) Considering that the revision protocol of each population belongs to the RM-PC class, show that if (x^*, q^*) is an equilibrium point of the mean closed loop, then x^* is in $\mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$. Moreover, determine conditions on $\{\lambda_R^r \mid 1 \leq r \leq \rho\}$ and other parameters that ensure the GAS of the mean closed loop's equilibria. In view of the discussion in §II-E.2, these conditions guarantee high probability convergence of X to a close vicinity of $\mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$ for large populations.

In what follows, we introduce the concepts and techniques that we use to address these problems.

B. Nash Stationarity

The *Nash stationarity* concept [4], [13] defined below will be central in our approach to address Problem-ii.

Definition 2: Given r in $\{1, \ldots, \rho\}$, a protocol for population r satisfies the Nash stationarity property if the following equivalence holds for the r-th component of the EDM (7) for all p^r in $\mathbb{R}^{n^r}$:

$$(x^r)^T p^r = \max_{y \in \mathbb{X}^r} y^T p^r \Leftrightarrow \mathcal{V}^r(x^r, p^r) = 0. \tag{10}$$

Thus, Nash stationarity implies that x^r at an equilibrium must be a best response to p^r.

Notice that, if Nash stationarity holds for all populations, then $x^* \in \mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$ if and only if (x^*, q^*) is an equilibrium point of (6). In this case, the mean social state is guaranteed to converge to $\mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$ when the equilibria set of the mean closed loop is GAS.

We refer to [16, §III.B] for a discussion on why Nash stationarity, in combination with GAS, assuages some of the well-known criticism of the Nash equilibrium concept and gives it a well-motivated role in our context.

C. The δ-Passivity Approach

To analyze the GAS of the equilibria of the mean closed loop, we employ the δ-passivity approach introduced in [8] and extended in [4], [9][4]. In doing so, we mainly invoke the results in [9], which are the state-of-the-art.

The δ-passivity approach facilitates the stability analysis of the mean closed loop by exploiting its feedback structure and allowing the EDM and the PDM to be investigated separately. In particular, the GAS of the mean closed loop's equilibria can be ensured through [9, Theorem 2] by verifying that the protocol of each population and the PDM respectively satisfy the so-called δ-passivity and δ-antidissipativity conditions. Now, we give further details on these conditions.

1) δ-Antidissipativity:

To employ the results in [9], we will need the PDM to be δ-antidissipative according to the definition below.

Definition 3: **(PDM δ-antidissipativity)** Given positive constants $w^1, \ldots, w^\rho$, a PDM is said to be δ-antidissipative with weights $w^1, \ldots, w^\rho$ if there are functions $\mathcal{Q} : \mathbb{R}^m \times \mathbb{X} \to \mathbb{R}_{\geq 0}$ and $\mathcal{R} : \mathbb{R}^m \times \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that the following holds for all $q \in \mathbb{R}^m$, $x \in \mathbb{X}$ and $v \in T\mathbb{X}$:

$$\frac{\partial \mathcal{Q}(q,x)}{\partial q}\mathcal{G}(q,x) + \frac{\partial \mathcal{Q}(q,x)}{\partial x} v \leq -\mathcal{R}(q,x) - \psi^T \Pi \psi \tag{11a}$$

$$\mathcal{Q}(q,x) = 0 \Leftrightarrow \mathcal{G}(q,x) = 0 \tag{11b}$$

$$\mathcal{R}(q,x) = 0 \Leftrightarrow \mathcal{G}(q,x) = 0, \tag{11c}$$

where $T\mathbb{X} = T\mathbb{X}^1 \times \cdots \times T\mathbb{X}^\rho$ is the tangent space of $\mathbb{X}$, meaning that $T\mathbb{X}^r := \{v^r \in \mathbb{R}^{n^r} \mid \sum_{i=1}^{n^r} v_i^r = 0\}$ for all $r \in \{1, \ldots, \rho\}$, ψ is the vector specified by

$$\psi := \begin{bmatrix} \frac{\partial \mathcal{H}(q,x)}{\partial q}\mathcal{G}(q,x) + \frac{\partial \mathcal{H}(q,x)}{\partial x} v \\ v \end{bmatrix},$$

and Π is the block matrix with blocks $\Pi_{11} = \Pi_{22} = 0$, $\Pi_{12} = \Pi_{21} = 1/2\mathbf{W}$ in which $\mathbf{W}$ is the block-diagonal matrix $\mathbf{W} := \mathbf{diag}\left(w^1 \mathbf{I}_{n^1 \times n^1}, \ldots, w^\rho \mathbf{I}_{n^\rho \times n^\rho}\right)$.

Remark 3: If a PDM is δ-antidissipative with weights $w^r = 1$ for all $r \in \{1, \ldots, \rho\}$, then it is δ-antipassive as defined and studied in [4] (see [9, Remark 8]). Numerous examples of δ-antipassive PDMs can be found in [4]. These examples demonstrate possible application areas for our results in §IV.

Remark 4: **(Weighted contractivity)** For the case when the PDM is a game $\mathcal{F}$, δ-antidissipativity with weights $w^1, \ldots, w^\rho$ reduces to the condition:

$$\sum_{r=1}^{\rho} w^r \Big(\mathcal{F}^r(x) - \mathcal{F}^r(y)\Big)^T (x^r - y^r) \leq 0, \quad x, y \in \mathbb{X}. \tag{12}$$

The inequality in (12) coincides with the so-called contractivity [13] when the weights are identical and can be viewed, more generally, as weighted contractivity [9] with respect to the block-diagonal matrix $\mathbf{W}$.

We give examples of a δ-antidissipative PDM and a weighted contractive game by requiring the HPG with the payoffs specified respectively in Example 4 (in Appendix A) and Example 1 to satisfy the following assumption.

Assumption 1: **(Properties of $\mathcal{C}$ for Example 1 and Example 4)** We assume that $\mathcal{C}$ in equations (1) and (22) have the following properties:

a) $\mathcal{C}_1, \ldots, \mathcal{C}_\kappa$ are increasing.
b) More reliable components are more expensive, i.e., if $i > j$ then $\mathcal{C}_i(d) > \mathcal{C}_j(d)$, for d in $[0, \bar{d}]$.

When $\mathcal{C}$ satisfies Assumption 1.a we can show, by following an approach analogous to that of [9, §IV.A], that Example 1 is weighted contractive with $(w^1, \ldots, w^\rho) = (\alpha^1, \ldots, \alpha^\rho)$. Moreover, by appropriately modifying the steps in the proof of [9, Proposition 3], we can also show that the PDM example described in Appendix A satisfies δ-antidissipativity with $(w^1, \ldots, w^\rho) = (\alpha^1, \ldots, \alpha^\rho)$.

In economic theory, Assumption 1.a is referred to as demand-pull inflation [17] that occurs when the supply of a product is limited[5], the manufacturer discounts the price when the demand is weak (and gradually eliminates the discount as demand rises), or when the manufacturer raises the price

[4]See [9, Remark 3] for a comparison between δ-passivity as defined above, δ-dissipativity and δ-passivity as proposed in [8].

[5]Factors restricting supply may include scarcity of raw materials, manufacturers strategically opting to limit production to keep prices up (as dynamic random-access memory manufacturers have been doing in the last 3 years), difficulty in ramping up production fast enough to meet demand, and sanctions to name a few.

with increasing demand as a way to increase profits when the product becomes popular. Higher cost (decrease in payoff) for a strategy with higher demand, as measured by the portion of the population following it, is common in many other applications, such as congestion games [18].

Additional examples of contractive and weighted contractive games can be found respectively in [1] and [9], while other instances of δ-antipassive and δ-antidissipative PDMs can be found respectively in [3,4,8] and [9].

2) δ-Passivity: In order to leverage the results in [9], we will also need the protocol of each population to be δ-passive according to the following definition.

Definition 4: **(Protocol δ-passivity)** Given r in $\{1,\dots,\rho\}$, the protocol for population r is δ-passive if there are functions $\mathcal{S}^r : \mathbb{X}^r \times \mathbb{R}^{n^r} \to \mathbb{R}_{\geq 0}$ and $\mathfrak{S}^r : \mathbb{X}^r \times \mathbb{R}^{n^r} \to \mathbb{R}_{\geq 0}$ such that the following holds for all $x^r \in \mathbb{X}^r$ and $p^r, u^r \in \mathbb{R}^{n^r}$:

$$\frac{\partial \mathcal{S}^r(x^r,p^r)}{\partial x^r}\mathcal{V}^r(x^r,p^r) + \frac{\partial \mathcal{S}^r(x^r,p^r)}{\partial p^r}u^r \leq -\mathfrak{S}^r(x^r,p^r) + \mathcal{V}^r(x^r,p^r)^T u^r \quad \text{(13a)}$$

$$\mathcal{S}^r(x^r,p^r) = 0 \Leftrightarrow \mathcal{V}^r(x^r,p^r) = 0 \quad \text{(13b)}$$

$$\mathfrak{S}^r(x^r,p^r) = 0 \Leftrightarrow \mathcal{V}^r(x^r,p^r) = 0. \quad \text{(13c)}$$

Following the convention in [3], [4], we will refer to $\mathcal{S}^r$ as a δ-storage function.

D. Reformulation Of Problem-ii

The δ-passivity and Nash stationarity concepts provide us with tools to determine the conditions for GAS, as called for in Problem-ii of §III-A. Specifically, as outlined in §III-C, it follows from [9, Theorem 2] that, if the protocol of each population is δ-passive and the PDM is δ-antidissipative, then the equilibria of the mean closed loop is GAS. Additionally, on account of the arguments in §III-B, if the protocol of each population is Nash stationary, then $x^* \in \mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$ for every equilibrium point (x^*, q^*) of the mean closed loop.

Consequently, together with the δ-antidissipativity of the PDM, Nash stationarity and constraints that guarantee δ-passivity of RM-PC protocols provide an answer to Problem-ii. Therefore, in the subsequent section, we will examine the Nash stationarity and δ-passivity properties of RM-PC protocols.

IV. RM-PC Protocols And Main Results

In this section, we address the problems formulated in §III-A and refined in §III-D. We begin by precisely defining the RM-PC protocol class. Then, we argue that RM-PC protocols are Nash stationary and derive conditions under which they are δ-passive. Finally, by means of the discussion in §III-D, we leverage these results to draw conclusions on the equilibrium stability of the mean closed loop.

A. The RM-PC Protocol Class

To cope with the case of unequal strategy-dependent revision rates, we extend the class of IPC protocols [7] as follows.

Definition 5: **(RM-PC protocol)** Given r in $\{1,\dots,\rho\}$, the protocol (5) of the r-th population is of the *rate-modified* pairwise comparison (RM-PC) class if τ^r can be written for all $(x^r, p^r) \in \mathbb{X}^r \times \mathfrak{P}^r$ and $i, j \in \{1,\dots,n^r\}$ with $i \neq j$ as:

$$\tau^r_{ij}(x^r,p^r) = \frac{1}{\bar{\tau}^r}\phi^r_j(p^r_j - p^r_i), \quad \text{(14a)}$$

$$\tau^r_{ii}(x^r,p^r) = 1 - \sum_{\ell=1,\ell\neq i}^{n^r} \frac{1}{\bar{\tau}^r}\phi^r_\ell(p^r_\ell - p^r_i), \quad \text{(14b)}$$

where $\bar{\tau}^r$ is a positive normalization constant for which τ^r_{ii} is non-negative, and $\phi^r_j : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a globally Lipschitz continuous and sign-preserving map, meaning that $\phi^r_j(\delta) > 0$ for $\delta > 0$ and $\phi^r_j(\delta) = 0$ for $\delta \leq 0$.

By substituting (14) into (7), we obtain the following RM-PC EDM for the r-th population for each i in $\{1,\dots,n^r\}$:

$$(\mathcal{V}^{\text{RM-PC}}_i)^r(x^r,p^r) := \sum_{j=1,j\neq i}^{n^r} \lambda^r_j \frac{1}{\bar{\tau}^r}\phi^r_i(p^r_i - p^r_j)x^r_j - \sum_{j=1,j\neq i}^{n^r} \lambda^r_i \frac{1}{\bar{\tau}^r}\phi^r_j(p^r_j - p^r_i)x^r_i. \quad \text{(15)}$$

Remark 5: **(IPC is an RM-PC subclass)** In the particular case where the revision rates for the r-th population are equal ($\lambda^r_1 = \cdots = \lambda^r_{n^r}$), an RM-PC protocol becomes of the IPC class considered in previous work characterizing δ-passivity [4], [8], [9].

Example 3: **(RM-Smith protocol)** As an example of an RM-PC protocol, we can define the *rate-modified* Smith protocol (RM-Smith) by substituting $\phi^r_j(\cdot) = [\cdot]_+ := \max\{0,\cdot\}$ in (14) and (5), leading to:

$$\mathcal{T}^r_{ij}(x^r,p^r) \underset{\text{RM-Smith}}{=} \lambda^r_i \frac{1}{\bar{\tau}^r}[p^r_j - p^r_i]_+, \ \ (x^r,p^r) \in \mathbb{X}^r \times \mathfrak{P}^r. \quad \text{(16)}$$

Consequently, agents that follow the RM-Smith protocol switch strategies with probabilities proportional to the positive parts of the payoff differences. When the revision rates are equal ($\lambda^r_1 = \cdots = \lambda^r_{n^r}$), the RM-Smith protocol reduces to the well-known Smith protocol originally proposed in [19] to analyze the dynamics of traffic assignment strategies.

Remark 6: **(RM-PC informational requirements)** It follows from (14) that, other than the knowledge of the payoffs of the available strategies for the population it is a part of, each agent following an RM-PC protocol does not need to coordinate with other agents and it does not require any additional information about the social state or the strategic choices of the other agents.

B. Nash Stationarity And δ-Passivity Of RM-PC Protocols

Having introduced the RM-PC protocol class, we now establish its Nash stationarity and identify its δ-passivity properties.

1) Pairwise Comparison Protocols And Nash Stationarity: The RM-PC class can be interpreted as a particular case of the so-called *pairwise comparison* protocol class defined in [12, §4.1]. It is relevant to recognize this because, although previous δ-passivity results that we seek to generalize [4], [8], [9] were restricted to IPC protocols only, there is existing work establishing other useful properties for the much broader pairwise comparison protocol class. Pertinently, [12, Theorem 1]

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3237485

states that a pairwise comparison protocol is Nash stationary, which leads directly to the following lemma.

Lemma 1: **(RM-PC protocols are Nash stationary)** Given r in $\{1, \dots, \rho\}$, if the r-th population's protocol is of the RM-PC class, then (10) holds for any positive revision rates $\lambda^r_1, \dots, \lambda^r_{n^r}$.

2) Conditions For δ-Passivity: We now proceed to investigate the δ-passivity properties of RM-PC protocols. Inspired by the Lyapunov and δ-storage functions introduced respectively in [7] and [8], we choose the δ-storage function described as follows. Given a population $r \in \{1, \dots, \rho\}$ with a protocol $\mathcal{T}^r$ of the RM-PC class, we set our δ-storage function to be $(\mathcal{S}^{\text{RM-PC}})^r : \mathbb{X}^r \times \mathbb{R}^{n^r} \to \mathbb{R}_{\geq 0}$ specified by

$$(\mathcal{S}^{\text{RM-PC}})^r(x^r, p^r) := \sum_{i=1}^{n^r} \frac{1}{\bar{\tau}^r} \lambda^r_i x^r_i \left(\sum_{k=1}^{n^r} \psi^r_k(p^r_k - p^r_i) \right), \tag{17a}$$

where

$$\psi^r_k(p^r_k - p^r_i) := \int_0^{p^r_k - p^r_i} \phi^r_k(s) ds. \tag{17b}$$

Denoting $\sum_{k=1}^{n^r} \psi^r_k(p^r_k - p^r_i)$ by $\gamma^r_i(p^r)$ we can write $(\mathcal{S}^{\text{RM-PC}})^r$ in a more compact form as

$$(\mathcal{S}^{\text{RM-PC}})^r(x^r, p^r) = \sum_{i=1}^{n^r} \frac{1}{\bar{\tau}^r} \lambda^r_i x^r_i \gamma^r_i(p^r). \tag{17c}$$

Our analysis of $(\mathcal{S}^{\text{RM-PC}})^r$ yields the following theorem, which presents a condition that ensures δ-passivity of RM-PC protocols.

Theorem 1: Given r in $\{1, \dots, \rho\}$, consider that the r-th population follows an RM-PC protocol specified by a given ϕ^r and a worst-case revision rate ratio λ^r_R (see (9)). The RM-PC protocol for population r is δ-passive if **(i)** $n^r = 2$ or **(ii)** $n^r \geq 3$ and the following inequality holds:

$$\lambda^r_R < \bar{\lambda}_{\phi^r}(n^r), \tag{18}$$

where $\bar{\lambda}_{\phi^r}$ is determined from ϕ^r as

$$\bar{\lambda}_{\phi^r}(n^r) := \min_{1 \leq k \leq n^r} \inf_{p^r \in \mathbb{R}^{n^r}} \left\{ \frac{\gamma^r_k(p^r) \sum_{i=1}^{n^r} \phi^r_i(p^r_i - p^r_k)}{\sum_{i=1}^{n^r} \phi^r_i(p^r_i - p^r_k) \gamma^r_i(p^r)} \right\}. \tag{19a}$$

Although (to avoid cluttered notation) we do not explicitly indicate in (19a), the infimum is computed subject to the following constraint on p^r:

$$\sum_{i=1}^{n^r} \phi^r_i(p^r_i - p^r_k) \gamma^r_i(p^r) \neq 0. \tag{19b}$$

In Appendix B.1 we will prove Theorem 1 by showing that $(\mathcal{S}^{\text{RM-PC}})^r$ satisfies (13).

Remark 7: **(When to compute (19))** According to Theorem 1, an RM-PC protocol is always δ-passive for a population with two strategies $n^r = 2$, irrespective of the revision rates. Hence, only when $n^r \geq 3$ will one need to compute (19) to test whether (18) holds. Notably, in many cases (19) can be computed numerically [20].

We now illustrate the practical significance of (18) in the context of the HPG. Suppose that there are 5 manufacturers and the agents of population r follow the RM-Smith protocol. In §IV-C, we will compute $\bar{\lambda}_{\phi^r}(5)$ to be 4.65 for the RM-Smith protocol. Consequently, in this setting, (18) requires the worst-case failure rates of the component when used by population r (calculated via (9)) to be less than 4.65.

Although in many cases (19) can be computed numerically, this computation can be cumbersome. To facilitate bypassing the computation of (19), we will present in Proposition 1 a simple lower bound for $\bar{\lambda}_{\phi^r}(n^r)$ that is valid for RM-PC protocols satisfying the following assumption.

Assumption 2: For a non-decreasing map $\bar{\phi}^r : \mathbb{R} \to \mathbb{R}_{\geq 0}$, the following holds:

$$\phi^r_i(p) = \bar{\phi}^r(p), \quad p \in \mathbb{R}, \ i \in \{1, \dots, n^r\}. \tag{20}$$

Proposition 1: Consider that a population r in $\{1, \dots, \rho\}$ (with $n^r \geq 3$) follows an RM-PC protocol. If the protocol satisfies Assumption 2 then the following holds for $n^r \geq 3$:

$$\bar{\lambda}_{\phi^r}(n^r) \geq \frac{n^r - 1}{n^r - 2}. \tag{21}$$

A proof of Proposition 1 is presented in Appendix B.2. We note that this proof also provides an alternative way to compute $\bar{\lambda}_{\phi^r}(n^r)$ for the case in which Assumption 2 holds (see (29)).

Remark 8: We can conclude from (21) that, for the protocols satisfying the conditions of Proposition 1, $\bar{\lambda}_{\phi^r}(n^r)$ is strictly greater than 1, which, according to Theorem 1, affords some δ-passivity robustness with respect to λ^r_R regardless of the number of strategies. This fact is in contrast to previous results establishing δ-passivity only for protocols in which λ^r_R is exactly 1 (see Remark 5).

We emphasize that Assumption 2 is critical in obtaining Proposition 1 and the following counterexample illustrates the reason for this.

Counterexample 1: Consider that $n^r = 3$ and population r adopts an RM-PC protocol specified by $\phi^r_1(\cdot) = [\cdot]^2_+$, $\phi^r_2(\cdot) = \phi^r_3(\cdot) = [\cdot]_+$. This protocol violates Assumption 2 and, as we proceed to show, it will infringe (21) with $\bar{\lambda}_{\phi^r}(3) = 1$. To do so, we use the following inequality that we obtain by setting $p^r_1 = 0$, $p^r_2 = -\epsilon$, $p^r_3 = -\epsilon + \epsilon^{7/4}$, with $\epsilon > 0$, when computing the infimum in (19a):

$$\bar{\lambda}_{\phi^r}(3) \leq \lim_{\epsilon \to 0^+} \frac{(2\epsilon^3 + 3\epsilon^{7/2})(\epsilon^2 + \epsilon^{7/4})}{2(\epsilon - \epsilon^{7/4})^3 \epsilon^{7/4}} = 1.$$

C. Numerical Evaluation Of $\bar{\lambda}_{\phi^r}$ For RM-Smith

Let us denote $\bar{\lambda}_{\phi^r}$ for the RM-Smith protocol as $\bar{\lambda}_{\text{RM-Smith}}$. In Fig. 3, we plot the values of $\bar{\lambda}_{\text{RM-Smith}}(n^r)$ for $3 \leq n^r \leq 10$, which we determined by computing (19) numerically [20]. Fig. 3 also displays the lower bound in (21) for $\bar{\lambda}_{\text{RM-Smith}}$. Note that, since the RM-Smith protocol satisfies Assumption 2, the lower bound in (21) holds for $\bar{\lambda}_{\text{RM-Smith}}$, for any $n^r \geq 3$.

The plots in Fig. 3 illustrate that the lower-bound in (21) may be conservative – a consequence of it being valid for a large subclass of RM-PC protocols. Notably, from the values of $\bar{\lambda}_{\text{RM-Smith}}$ plotted in Fig. 3 we observe that the RM-Smith protocol satisfies δ-passivity even if the revision rates vary by

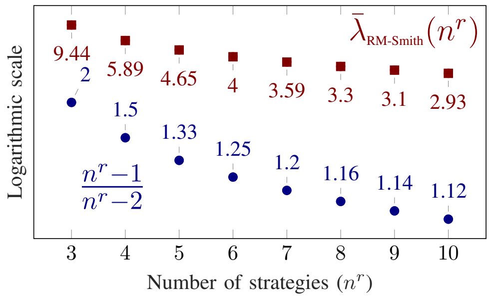


Fig. 3: Comparing $\bar{\lambda}_{\text{RM-Smith}}$ with the lower-bound in (21).

a multiplicative factor of 9.44 when the number of strategies is 3. For the case where there are 9 strategies, the RM-Smith protocol remains δ-passive even when the revision rates vary by a multiplicative factor of 3.1.

D. Establishing GAS Of $\mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$

As outlined in §III, δ-passivity and Nash stationarity, together with [9], suffice to draw conclusions about the equilibrium stability of the mean closed loop. Consequently, our results on the Nash stationarity and δ-passivity of RM-PC protocols lead to the theorem given below, which we state without proof because it follows directly from [9, Theorem 2] in conjunction with Lemma 1 and Theorem 1.

Theorem 2: Consider that a PDM is given and that each population follows an RM-PC protocol. If the protocols satisfy the conditions of Theorem 1 and the PDM is δ-antidissipative (see Definition 3), then the equilibria set of the mean closed loop is GAS. In addition, if (x^*, q^*) is an equilibrium point of the mean closed loop, then $x^* \in \mathbb{NE}(\mathcal{F}_{\mathcal{G},\mathcal{H}})$.

Recall from Remark 4 that, for the case when the PDM is given by a game, the δ-antidissipativity requirement reduces to weighted contractivity. This, in combination with Theorem 2, leads to the corollary presented below.

Corollary 1: Consider that a game $\mathcal{F}$ is given and that each population follows an RM-PC protocol. If the protocols satisfy the conditions of Theorem 1 and the game is weighted contractive (see Remark 4), then the equilibria set of the mean closed loop is $\mathbb{NE}(\mathcal{F})$ and is GAS.

We note that Corollary 1 generalizes the earliest stability results for IPC protocols, established in [7, Theorem 7.1], in two ways. In comparison to [7, Theorem 7.1], which presumes that the game is contractive and the revision rates are identical within each population, Corollary 1 allows for weighted contractive games and it contends with unequal revision rates so long as they satisfy the conditions of the corollary. The stability theorems in [9] allow for weighted contractive games, but the article lacks the results needed to consider the case in which the revision rates within each population are different.

V. Numerical Examples

As industrial-grade data-driven processing centers and vehicle-to-vehicle networks are becoming more prevalent, life cycles of dynamic random-access memories **(DRAMs)** used in these applications emerge as important benchmarks. To provide examples of how our results can come into play, we look into the HPG, introduced in §II-A, in the context of the DRAM market.

A. A DRAM Market HPG

We proceed by introducing an HPG in the context of the DRAM market. Two classes of systems in which DRAMs are commonly used are industrial and automotive systems, which we refer to as respectively class 1 and 2. Hence, there are two populations, where we use population r to represent the collection of agents that utilize DRAMs in class r. We assume that there are 3 manufacturers producing DRAMs with failure rates in these utilization classes given by $\lambda_1^1 = 5$, $\lambda_1^2 = 10$, $\lambda_2^1 = 4$, $\lambda_2^2 = 9$ and $\lambda_3^1 = 3$, $\lambda_3^2 = 5$, where λ_i^r is the failure rate of DRAMs produced by manufacturer i when utilized in class r. Moreover, we let the replacement costs for industrial and automotive DRAMs to be respectively $\beta^1 = 2$ and $\beta^2 = 1$.

We assume that the payoffs are determined by a game $\mathcal{F}$ with the structure presented in Example 1. We specify the component price from manufacturer $i \in \{1, 2, 3\}$, which is the $\mathcal{C}_i$ in (1), as the sum of a fixed production cost, $\mathcal{C}_{0i}$, and a term reflecting the pull-back inflation, $\mathcal{C}_{pi}$. To represent the pull-back inflation on the cost, we use a quadratic term $\mathcal{C}_{pi}(\mathcal{D}_i(x)) = (\mathcal{D}_i(x))^2 = (\alpha^1 x_i^1 + \alpha^2 x_i^2)^2$, where α^r is in proportion to the share of class-r in the DRAM market. Finally, we set $\alpha^1 = 1$ and $\alpha^2 = 2$, and the fixed DRAM production costs to be $\mathcal{C}_{01} = 1$, $\mathcal{C}_{02} = 1.2$ and $\mathcal{C}_{03} = 1.5$, which completes the construction of $\mathcal{F}$ (as in (1)) for our DRAM market HPG.[6] Notice that this $\mathcal{F}$ satisfies Assumption 1, hence is a weighted contractive game.

B. Dynamics Under The RM-Smith Protocol

Now we illustrate how Corollary 1 can be utilized. Consider the mean closed loop (8) formed by the $\mathcal{F}$ constructed in §V-A and the EDM in which all populations follow the RM-Smith protocol (16) with the revision rates specified in §V-A. Assume that initially the buyers are distributed on the manufacturers according to $x^1(0) = x^2(0) = (2/3, 1/6, 1/6)$.

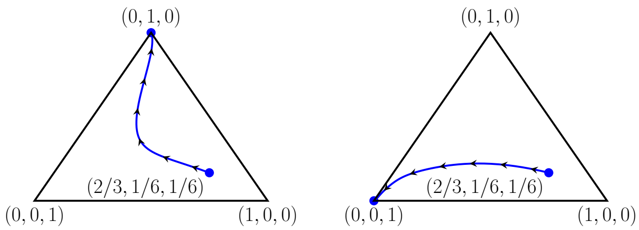
Since the failure rates satisfy the conditions of Theorem 1 and $\mathcal{F}$ is weighted contractive, we can invoke Corollary 1 to conclude that x converges to $\mathbb{NE}(\mathcal{F})$, which in this case is the singleton $\{x^*\}$ with $(x^1)^* = (0, 1, 0)$, $(x^2)^* = (0, 0, 1)$ [20]. For this example, the trajectory and the time domain plot of x are portrayed respectively in Fig. 4 and Fig. 5. We note that the way in which the trajectories in Fig. 4 are portrayed is a common way to represent the trajectory of the mean social state when the populations have 3 strategies (see for instance [1, §2.3.4]).

C. Smoothed HPG For The DRAM Market And Dynamics Under The RM-Smith Protocol

We also carry out an analysis that is analogous to that in §V-B, but with the PDM characterized by the smoothed HPG

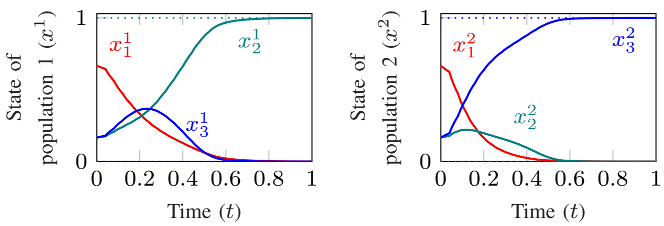
[6] We would like to clarify that the functions and parameters selected in this section are for illustration purposes, and they are not estimated from data.

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3237485



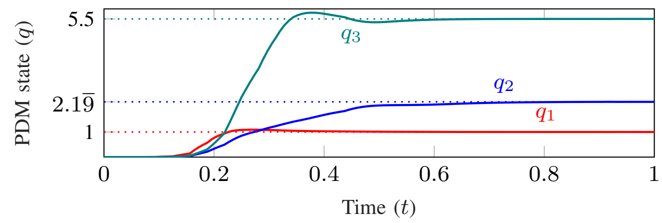
(a) Industrial-grade users (b) Automotive-grade users

Fig. 4: Trajectory of distribution of DRAM buyers on manufacturers under the HPG and RM-Smith protocol.

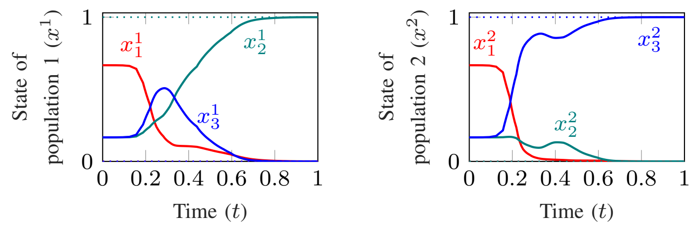


(a) Industrial-grade users (b) Automotive-grade users

Fig. 5: Time domain plots of the distribution of DRAM buyers on manufacturers under the HPG and RM-Smith protocol.



(a) State of the smoothed HPG



(b) Industrial-grade users (c) Automotive-grade users

Fig. 6: Time domain plots of the PDM's state and distribution of DRAM buyers on manufacturers under the smoothed HPG and RM-Smith protocol.

given in Appendix A. We select $a = 5$ in (22) while keeping all the other parameters from §V-A and §V-B unchanged. Note that the $\mathcal{C}$ constructed in §V-A satisfies Assumption 1, therefore this PDM is δ-antidissipative.

Since the failure rates satisfy the conditions of Theorem 1 and the PDM specified above is δ-antidissipative, we conclude from Theorem 2 and Remark 9 (in Appendix A) that, like in §V-B, x converges to $\mathbb{NE}(\mathcal{F})$. The time evolution of the PDM's state q and the mean social state x are plotted in Fig. 6, indicating that x^1 and x^2 indeed converge respectively to $(0, 1, 0)$ and $(0, 0, 1)$.

VI. Conclusions And Future Directions

In this article we generalized the approach in [4], [8] and [9] to a class of pairwise comparison protocols we called RM-PC, for which the agents' revision rates may depend on their current strategies. We stated and proved two results establishing global asymptotic stability of the equilibria of the mean closed loop for the cases when the payoff mechanism is a memoryless game or a payoff dynamics model (PDM). These results rely on Theorem 1 establishing conditions for δ-passivity of RM-PC protocols. Proposition 1 establishes for an RM-PC protocol subclass a rather simple (but more conservative) sufficient condition for δ-passivity.

Our results also raise pertinent questions for future research. **Future Direction 1:** The excess payoff target (EPT) [21] protocols is another class of revision protocols that can't accommodate strategy-dependent revision rates, but induces desirable stability properties under uniform revision rates. Hence, a meaningful next step would be to introduce strategy-dependent revision rates to the EPT class and study the δ-passivity of the ensuing protocols by appropriately generalizing the approach in [9].

Future Direction 2: Although when every population has two strategies Theorem 1 guarantees δ-passivity of an RM-PC protocol for any revision rates (undoubtedly a strong result), if a population has three or more strategies the theorem only provides a sufficient condition. Considering that we were unable to construct an example of an RM-PC protocol that is not δ-passive when this condition fails, we believe that it would be important to continue to investigate whether such an example exists or whether the condition could be weakened.

Appendix

A. Smoothed HPG: A PDM Example

The following is an example of a PDM that can be viewed as a dynamic version of Example 1. Our construction parallels that in [9, §VI.A].

Example 4: Given a positive time constant a and parameters as defined in Example 1, the following is the "smoothed" HPG PDM:

$$a\dot{q}(t) = -q(t) + \begin{bmatrix} \mathcal{C}_1\Big(\mathcal{D}_1\big(x(t)\big)\Big) \\ \vdots \\ \mathcal{C}_\kappa\Big(\mathcal{D}_\kappa\big(x(t)\big)\Big) \end{bmatrix}, \quad t \geq 0, \; q(0) \in \mathfrak{Q}_0 \tag{22a}$$

$$p_i^r(t) = -\beta^r \lambda_i^r - q_i(t), \quad 1 \leq i \leq \kappa, \; 1 \leq r \leq \rho. \tag{22b}$$

Here $\mathfrak{Q} = \mathfrak{Q}_0 := [0, \bar{d}]^\kappa$. We can also specify a set $\mathfrak{P}$ that includes all possible p as follows:

$$\mathfrak{P} := \Big\{ p \in \mathbb{R}^{\kappa\rho} \;\Big|\; p_i^r = -\beta^r \lambda_i^r - q_i, \text{ for some } q \text{ in } \mathfrak{Q} \Big\}.$$

Note that, in (22), p is the effective payoff perceived by the agents and q represents a smoothed version of the costs of the component (produced by different manufacturers).

In [8], the authors argue that dynamically modifying a game via smoothing reduces the impacts of short-term fluctuations and isolates the longer-term trends. We can motivate this effect in the context of our HPG in two ways. Firstly, even though the costs of the component may change persistently (e.g. due to the demand-pull inflation outlined in Assumption 1), retailers often do not change the sale prices as frequently, and rather, the sale prices follow the long-term trends of the costs. Secondly, it might take agents some time to learn and react to the sale prices, causing the effects of the changes in costs on the payoffs to be smoothed.

Remark 9: It follows immediately from (22), (1) and (4) that, for the smoothed HPG, $\mathcal{F}$ is identical to $\mathcal{F}_{\mathcal{G},\mathcal{H}}$.

B. Proofs Of Theorem 1 And Proposition 1

Before presenting the proofs of Theorem 1 and Proposition 1, we define a partial order $\succ$ (respectively $\succeq$) on the elements of $\mathbb{R}^{n^r}$ as follows: given any $x, y \in \mathbb{R}^{n^r}$ we write $x \succ y$ (respectively $x \succeq y$) if and only if $x_i > y_i$ (respectively $x_i \geq y_i$) for all $i \in \{1, \ldots, n^r\}$. Moreover, given $\lambda^r \in \mathbb{R}^{n^r}$ and $u^r, l^r \in \mathbb{R}$ with $u^r > l^r$, we use a slight abuse of notation and let $u^r \succ \lambda^r \succ l^r$ (respectively $u^r \succeq \lambda^r \succeq l^r$) denote $u^r > \lambda_i^r > l^r$ (respectively $u^r \geq \lambda_i^r \geq l^r$) for all $i \in \{1, \ldots, n^r\}$. We drop the superscript r in the proofs for notational convenience.

1) Proof Of Theorem 1:

We want to show that the candidate δ-storage function $\mathcal{S}^{\text{RM-PC}}$, given by (17), satisfies δ-passivity for RM-PC protocols that meet either $n = 2$ or condition (18).

With the $\mathcal{S}^{\text{RM-PC}}$ and $\mathcal{V}^{\text{RM-PC}}$ given respectively in (17) and (15), we have

$$\frac{\partial \mathcal{S}^{\text{RM-PC}}}{\partial x}(x,p)\mathcal{V}^{\text{RM-PC}}(x,p) + \frac{\partial \mathcal{S}^{\text{RM-PC}}}{\partial p}(x,p)u = \sum_{i=1}^{n} \frac{1}{\bar{\tau}} \lambda_i \mathcal{V}_i^{\text{RM-PC}}(x,p)\gamma_i(p) + u^T \mathcal{V}^{\text{RM-PC}}(x,p).$$

Hence, in order to show that RM-PC protocols satisfying $n = 2$ or (18) are δ-passive, we can define $\mathfrak{S}^{\text{RM-PC}}$ as

$$\mathfrak{S}^{\text{RM-PC}}(x,p) = -\frac{1}{\bar{\tau}} \sum_{i=1}^{n} \lambda_i \mathcal{V}_i^{\text{RM-PC}}(x,p)\gamma_i(p) \tag{23}$$

and prove, assuming $n = 2$ or (18), that $\mathcal{S}^{\text{RM-PC}}$ and $\mathfrak{S}^{\text{RM-PC}}$ are non-negative and satisfy (13b), (13c).

To begin with, non-negativity of $\bar{\tau}$, ϕ, x and λ imply that $\mathcal{S}^{\text{RM-PC}}$ is non-negative. Moreover, plugging $\mathcal{S}^{\text{RM-PC}}$ to [4, Lemma 4] it follows that $\mathcal{S}^{\text{RM-PC}}$ satisfies (13b). Thus we are left with the analysis of non-negativity of $\mathfrak{S}^{\text{RM-PC}}$ and conditions under which $\mathfrak{S}^{\text{RM-PC}}$ satisfy (13c). We partition the remainder of the proof into 2 steps. Step **(i)** discusses the non-negativity of $\mathfrak{S}^{\text{RM-PC}}$ and step **(ii)** establishes the validity of (13c).

Step i: In this step we discuss the non-negativity of $\mathfrak{S}^{\text{RM-PC}}$. With our choice of δ-storage function, the results that we get for $n = 2$ and $n \geq 3$ differ and we split our analysis for these two cases.

$n = 2$: Under $n = 2$, we will show that $\mathfrak{S}^{\text{RM-PC}}$ is non-negative for all $\lambda \succ 0$. For this instance, we have

$$\begin{aligned} -\mathfrak{S}^{\text{RM-PC}}(x,p) = \frac{1}{\bar{\tau}}(&\lambda_1(x_2\lambda_2\phi_1(p_1 - p_2) \\ &- x_1\lambda_1\phi_2(p_2 - p_1))\psi_2(p_2 - p_1) \\ &+ \lambda_2(x_1\lambda_1\phi_2(p_2 - p_1) \\ &- x_2\lambda_2\phi_1(p_1 - p_2))\psi_1(p_1 - p_2)). \end{aligned} \tag{24}$$

Let us analyze the cases $p_1 = p_2$, $p_1 > p_2$ and $p_1 < p_2$ separately. When $p_1 = p_2$, (24) becomes 0 by the sign preservation of ϕ. If we assume $p_1 > p_2$, then (24) becomes $-\lambda_2^2 x_2 \phi_1(p_1 - p_2)\psi_1(p_1 - p_2)/\bar{\tau}$ which is non-positive for all $x \in \mathbb{X}$ and $\lambda \succ 0$. If $p_1 < p_2$, then (24) becomes $-\lambda_1^2 x_1 \phi_2(p_2 - p_1)\psi_2(p_2 - p_1)/\bar{\tau}$ which is again non-positive for all $x \in \mathbb{X}$ and $\lambda \succ 0$.

$n \geq 3$: Results that we have for the $n = 2$ and $n \geq 3$ differ in the sense that, when $n \geq 3$ we show non-negativity of $\mathfrak{S}^{\text{RM-PC}}$ only for RM-PC protocols satisfying (18). Hence, in what follows we assume that (18) holds. Let us denote $u := \max_{i \in \{1,\ldots,n\}} \lambda_i$ and $l := \min_{i \in \{1,\ldots,n\}} \lambda_i$, so (18) can be written as $u/l < \bar{\lambda}_\phi$. Notice that

$$\begin{aligned} &-\mathfrak{S}^{\text{RM-PC}}(x,p) \\ &= \frac{1}{\bar{\tau}} \sum_{i=1}^{n} \gamma_i(p)\lambda_i \left(\sum_{j=1}^{n} x_j \lambda_j \phi_i(p_i - p_j) \right) \\ &\quad - \frac{1}{\bar{\tau}} \sum_{i=1}^{n} \gamma_i(p)\lambda_i \left(\sum_{j=1}^{n} x_i \lambda_i \phi_j(p_j - p_i) \right) \\ &= \frac{1}{\bar{\tau}} \sum_{i=1}^{n} \sum_{j=1}^{n} x_j \phi_i(p_i - p_j)\lambda_j(\lambda_i\gamma_i(p) - \lambda_j\gamma_j(p)) \\ &= \frac{1}{\bar{\tau}} \begin{bmatrix} \sum_{i=1}^{n} \phi_i(p_i - p_1)\lambda_1(\lambda_i\gamma_i(p) - \lambda_1\gamma_1(p)) \\ \vdots \\ \sum_{i=1}^{n} \phi_i(p_i - p_n)\lambda_n(\lambda_i\gamma_i(p) - \lambda_n\gamma_n(p)) \end{bmatrix}^T x. \end{aligned}$$

From $x \succeq 0$, it follows that $-\mathfrak{S}^{\text{RM-PC}}(x,p) \leq 0$ for all $u \succeq \lambda \succeq l$, $x \in \mathbb{X}$ and $p \in \mathbb{R}^n$ if and only if the inequality below holds:

$$\sup_{k \in \{1,\ldots,n\}, p \in \mathbb{R}^n, u \succeq \lambda \succeq l} \left\{ \sum_{i \in \{1,\ldots,n\} \setminus \{k\}} \phi_i(p_i - p_k) \lambda_k(\lambda_i\gamma_i(p) - \lambda_k\gamma_k(p)) \right\} \leq 0. \tag{25}$$

We can take supremum with respect to one set of the variables, and then take the supremum of the resulting expression with respect to the ones left [22]. We first choose to take supremum with respect to λ. Fixing any $k \in \{1, \ldots, n\}$ and $p \in \mathbb{R}^n$, since γ_i and ϕ_i are non-negative for all $i \in \{1, \ldots, n\}$, the expression on the left-hand side of (25) is maximized with respect to λ when λ_i/λ_k is maximized for all $i \in \{1, \ldots, n\} \setminus \{k\}$. Due to the box constraint $u/l \geq \lambda_i/\lambda_k \geq l/u$, we have that for any $i \in \{1, \ldots, n\}$, supremum of λ_i/λ_k is reached when $\lambda_i/\lambda_k = u/l$. Thus (25) holds if and only if the following holds for every k in $\{1, \ldots, n\}$:

$$\sum_{i=1}^{n} \phi_i(p_i - p_k) \left(\frac{u}{l}\gamma_i(p) - \gamma_k(p) \right) \leq 0, \quad p \in \mathbb{R}^n. \tag{26}$$

Notice that if $\sum_{i=1}^n \phi_i(p_i - p_k)\gamma_i(p) = 0$, then (26) is satisfied, meaning that (26) holds if and only if

$$\frac{u}{l} \leq \inf_{\substack{k\in\{1,\ldots,n\}, p\in\mathbb{R}^n \text{ s.t.} \\ \sum_{i=1}^n \phi_i(p_i-p_k)\gamma_i(p)\neq 0}} \left\{ \frac{\gamma_k(p)\sum_{i=1}^n \phi_i(p_i - p_k)}{\sum_{i=1}^n \phi_i(p_i - p_k)\gamma_i(p)} \right\}. \tag{27}$$

Since (18) holds by assumption, (27) is satisfied with strict inequality. Thus, $\mathfrak{S}^{\text{RM-PC}}(x,p) \geq 0$ for all $x \in \mathbb{X}$ and $p \in \mathbb{R}^n$.

Step ii: In the second step we discuss under what conditions $\mathfrak{S}^{\text{RM-PC}}$ satisfies (13c). Similar to that of the conclusions on non-negativity of $\mathfrak{S}^{\text{RM-PC}}$, with our choice of δ-storage function, results that we obtain for the $n = 2$ and $n \geq 3$ cases differ.

$n = 2$: Assuming $n = 2$, we show that $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$ if and only if $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$ for all $\lambda \succ 0$. We present a proof by analyzing the cases $p_1 = p_2$, $p_1 > p_2$, and $p_1 < p_2$ separately. Recall that when $n = 2$, $\mathfrak{S}^{\text{RM-PC}}$ is given by (24). If $p_1 = p_2$, then (24) is 0, but in this case $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$. Now assume $p_1 > p_2$. Then, $\mathfrak{S}^{\text{RM-PC}}(x,p)$ becomes $-\lambda_2^2 x_2 \phi_1(p_1 - p_2)\psi_1(p_1-p_2)/\bar{\tau}$, but since $\phi_1(p_1-p_2) > 0$ and $\psi_1(p_1-p_2) > 0$ we see that $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$ implies $x_2 = 0$. Moreover, from $p_1 > p_2$, we have $\phi_2(p_2 - p_1) = 0$. These combined yield $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$. For the case $p_2 > p_1$, $\mathfrak{S}^{\text{RM-PC}}(x,p)$ becomes $-\lambda_1^2 x_1 \phi_2(p_2 - p_1)\psi_2(p_2 - p_1)/\bar{\tau}$. But since $\phi_2(p_2 - p_1) > 0$ and $\psi_2(p_2 - p_1) > 0$ we see that $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$ implies $x_1 = 0$. From $p_2 > p_1$, we also have $\phi_1(p_1 - p_2) = 0$. These combined again yield $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$. Hence, we arrive at $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$ implies $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$. For the other direction, assume $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$. Then, since $\mathfrak{S}^{\text{RM-PC}}(x,p) = -\sum_{i=1}^n \lambda_i \mathcal{V}_i^{\text{RM-PC}}(x,p)\gamma_i(p)/\bar{\tau}$, it follows that $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$. As a result $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$ if and only if $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$.

$n \geq 3$: Now assume $n \geq 3$. We will show that for all RM-PC protocols satisfying (18) we have $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$ if and only if $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$. Recall that

$$\mathfrak{S}^{\text{RM-PC}}(x,p) = -\sum_{i=1}^n \sum_{j=1}^n \frac{x_j}{\bar{\tau}} \phi_i(p_i - p_j)\lambda_j(\lambda_i\gamma_i(p) - \lambda_j\gamma_j(p)).$$

Given any $j \in \{1,\ldots,n\}$, there are three possibilities: out of $p_1,\ldots,p_n$ it must be that, p_j is the largest, p_j is the second largest, or there exist $l, m \in \{1,\ldots,n\} \setminus \{j\}$ such that $p_m > p_l > p_j$. We analyze these three cases separately. If j is such that p_j is the largest, then for all $i \in \{1,\ldots,n\}$, $\phi_i(p_i - p_j)\lambda_j(\lambda_i\gamma_i(p) - \lambda_j\gamma_j(p)) = 0$, and any x_j gives $x_j\phi_i(p_i - p_j)\lambda_j(\lambda_i\gamma_i(p) - \lambda_j\gamma_j(p))/\bar{\tau} = 0$. In the second case, p_j is the second largest. Let us denote $\mathcal{I} = \{i \in \{1,\ldots,n\} \mid p_i > p_j\}$, so $\mathcal{I}$ is the set of strategies having greater payoff than that of j. For any $l \in \mathcal{I}$ we have that $\gamma_l(p) = \sum_{k=1}^n \psi_k(p_k - p_l) = 0$ and $\gamma_j(p) = \sum_{k=1}^n \psi_k(p_k - p_j) \geq \psi_l(p_l - p_j) > 0$. However, for any $k \in \{1,\ldots,n\}\setminus\mathcal{I}$ we have $\phi_k(p_k - p_j) = 0$, implying $\phi_k(p_k - p_j)\lambda_k(\lambda_k\gamma_k(p) - \lambda_j\gamma_j(p)) = 0$. Therefore,

$$\begin{aligned}
&\sum_{i=1}^n \phi_i(p_i - p_j)\lambda_j(\lambda_i\gamma_i(p) - \lambda_j\gamma_j(p))\frac{1}{\bar{\tau}} \\
&= \sum_{k\in\{1,\ldots,n\}\setminus\mathcal{I}} \phi_k(p_k - p_j)\lambda_k(\lambda_k\gamma_k(p) - \lambda_j\gamma_j(p))\frac{1}{\bar{\tau}} \\
&\quad + \sum_{l\in\mathcal{I}} \phi_l(p_l - p_j)\lambda_l(\lambda_l\gamma_l(p) - \lambda_j\gamma_j(p))\frac{1}{\bar{\tau}} < 0.
\end{aligned}$$

Finally, if j is such that there exist $l, m \in \{1,\ldots,n\} \setminus \{j\}$ with $p_m > p_l > p_j$, then $\gamma_l(p) = \sum_{k=1}^n \psi_k(p_k - p_l) > 0$, thus $\sum_{i=1}^n \phi_i(p_i - p_j)\gamma_i(p) \geq \phi_l(p_l - p_j)\gamma_l(p) > 0$. Consequently, (18) can be utilized to arrive at the following: for all $p \in \mathbb{R}^n$ such that there exists $l, m \in \{1,\ldots,n\} \setminus \{j\}$ with $p_m > p_l > p_j$, it holds that $(u/l) < (\gamma_j(p)\sum_{i=1}^n \phi_i(p_i - p_j))/(\sum_{i=1}^n \phi_i(p_i-p_j)\gamma_i(p))$. This implies that for all $p \in \mathbb{R}^n$ with $l, m \in \{1,\ldots,n\} \setminus \{j\}$ satisfying $p_m > p_l > p_j$ we have

$$\begin{aligned}
&\sum_{i=1}^n \phi_i(p_i - p_j)\lambda_j(\lambda_i\gamma_i(p) - \lambda_j\gamma_j(p))\frac{1}{\bar{\tau}} \\
&\leq \sum_{i=1}^n \phi_i(p_i - p_j) l \left(\frac{u}{l}\gamma_j(p) - \gamma_i(p)\right)\frac{1}{\bar{\tau}} < 0.
\end{aligned}$$

From the analysis of these three cases on p_j, it becomes evident that $\mathfrak{S}^{\text{RM-PC}}(x,p) = -\sum_{j=1}^n x_j \sum_{i=1}^n \phi_i(p_i - p_j)\lambda_j(\lambda_i\gamma_i(p) - \lambda_j\gamma_j(p))/\bar{\tau} = 0$ if and only if $x_j > 0$ only when $j \in \arg\max_{k\in\{1,\ldots,n\}} p_k$. Hence, $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$ if and only if $x \in \arg\max_{y\in\mathbb{X}} y^T p$. Finally, by the Nash stationarity of RM-PC protocols, we arrive at $\mathfrak{S}^{\text{RM-PC}}(x,p) = 0$ if and only if $\mathcal{V}^{\text{RM-PC}}(x,p) = 0$. ■

2) Proof Of Proposition 1: Assume that $n \geq 3$ and notice that, under Assumption 2, we can substitute ϕ_i with $\bar{\phi}$ (also denote $\bar{\psi}(v) = \int_0^v \bar{\phi}(s)ds$ for $v \in \mathbb{R}$ and $\bar{\gamma}_i(p) = \sum_{k=1}^n \bar{\psi}(p_k - p_i)$ for $i \in \{1,\ldots,n\}$, $p \in \mathbb{R}^n$) to rewrite (19) as

$$\bar{\lambda}_\phi(n) = \min_{1\leq k\leq n} \inf_{p\in\Theta_k} \mathcal{O}(k,p), \tag{28}$$

where

$$\mathcal{O}(k,p) := \frac{\bar{\gamma}_k(p)\sum_{i=1}^n \bar{\phi}(p_i - p_k)}{\sum_{i=1}^n \bar{\phi}(p_i - p_k)\bar{\gamma}_i(p)},$$

$$\Theta_k := \left\{ p \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \bar{\phi}(p_i - p_k)\bar{\gamma}_i(p) \neq 0 \right\}.$$

In what follows, we show that the right-hand side of (21) is a lower bound to (28). Our approach consists of three steps. **(i)** First, we show that without loss of generality we can fix k in (28) to be n, effectively discarding the minimization over k. **(ii)** Then, we prove that the value of $\inf_{p\in\Theta_n} \mathcal{O}(n,p)$ is unchanged when we introduce the additional constraint $p_1 \geq \cdots \geq p_n$. **(iii)** Finally, by exploiting the fact that $\bar{\phi}$ is non-decreasing, we derive a lower bound to the value of $\inf_{p\in\Theta_n} \mathcal{O}(n,p)$ under the additional constraint $p_1 \geq \cdots \geq p_n$.

Step i: We begin by showing that (28) is equal to $\inf_{p\in\Theta_n} \mathcal{O}(n,p)$. Given $k, l \in \{1,\ldots,n\}$ and $p \in \Theta_k$, let us construct $\tilde{p}$ by swapping the values of the k-th and l-th indices of p. Then, $\tilde{p} \in \Theta_l$ and $\mathcal{O}(k,p) = \mathcal{O}(l,\tilde{p})$. Therefore, infimum of $\mathcal{O}(k,p)$ over $p \in \Theta_k$ is independent of k, implying that without loss of generality we can fix the k in (28) to be n and discard the minimization over k. Hence, we can conclude from (28) that $\bar{\lambda}_\phi(n) = \inf_{p\in\Theta_n} \mathcal{O}(n,p)$.

Step ii: Now we prove that the value of $\inf_{p\in\Theta_n} \mathcal{O}(n,p)$ does not change when the additional constraint $p_1 \geq p_2 \geq \cdots \geq p_n$ is imposed on the problem. First, observe that for any given p in Θ_n, we have that $\mathcal{O}(n,p) = \mathcal{O}(n,\tilde{p})$ for any $\tilde{p}$ constructed from p by arbitrarily permuting the first $n-1$ entries and leaving the n-th entry unchanged. Therefore, imposing the additional constraint $p_1 \geq p_2 \geq \cdots \geq p_{n-1}$ does not change $\inf_{p \in \Theta_n} \mathcal{O}(n,p)$. Second, we will show that the infimum is unchanged even if we impose the more stringent constraint $p_1 \geq p_2 \geq \cdots \geq p_{n-1} \geq p_n$. Specifically, we will show that given any p in Θ_n with $p_1 \geq \cdots \geq p_{n-1}$, there exists a $\tilde{p}$ in Θ_n satisfying $\tilde{p}_1 \geq \cdots \geq \tilde{p}_{n-1} \geq \tilde{p}_n$ for which $\mathcal{O}(n,p) = \mathcal{O}(n,\tilde{p})$. To do so, we take an arbitrary p in Θ_n satisfying $p_1 \geq \cdots \geq p_{n-1}$. From $p \in \Theta_n$, it follows that $p_1 \geq p_n$, since otherwise we would have $\sum_{i=1}^n \bar{\phi}(p_i - p_n)\bar{\gamma}_i(p) = 0$. Thus, for any $p \in \Theta_n$ we either have $p_{n-1} \geq p_n$, or there is m in $\{2, \ldots, n-1\}$ such that $p_1 \geq p_2 \geq \cdots \geq p_{m-1} \geq p_n > p_m \geq \cdots \geq p_{n-1}$. If p is such that $p_{n-1} \geq p_n$, then taking $\tilde{p} = p$ gives the desired result. On the other hand, if there is m in $\{2, \ldots, n-1\}$ such that $p_1 \geq p_2 \geq \cdots \geq p_{m-1} \geq p_n > p_m \geq \cdots \geq p_{n-1}$, then we construct $\tilde{p}$ by setting $\tilde{p}_i = p_n$ for all i in $\{m, \ldots, n\}$ and $\tilde{p}_j = p_j$ for all j in $\{1, \ldots, m-1\}$. We can now verify by direct substitution that for the constructed $\tilde{p}$ it holds that $\mathcal{O}(n,p) = \mathcal{O}(n,\tilde{p})$ and $\tilde{p}$ is in Θ_n, which concludes our proof of the second step.

Thus, defining the vectors $\tilde{\phi}_n(p)$ and $\bar{\gamma}(p)$ as

$$\tilde{\phi}_n(p) := \begin{bmatrix} (\tilde{\phi}_n)_1(p) \\ \vdots \\ (\tilde{\phi}_n)_n(p) \end{bmatrix} = \begin{bmatrix} \frac{\bar{\phi}(p_1 - p_n)}{\sum_{i=1}^{n-1} \bar{\phi}(p_i - p_n)} \\ \vdots \\ \frac{\bar{\phi}(p_n - p_n)}{\sum_{i=1}^{n-1} \bar{\phi}(p_i - p_n)} \end{bmatrix}; \bar{\gamma}(p) := \begin{bmatrix} \bar{\gamma}_1(p) \\ \vdots \\ \bar{\gamma}_n(p) \end{bmatrix},$$

we have shown up to this point that

$$\bar{\lambda}_\phi(n) = \inf_{\substack{p \in \mathbb{R}^n \text{ s.t. } p_1 \geq \cdots \geq p_n, \\ \tilde{\phi}_n^T(p)\bar{\gamma}(p) \neq 0}} \frac{1}{\tilde{\phi}_n^T(p)\bar{\gamma}(p)/\bar{\gamma}_n(p)}. \tag{29}$$

Note that for $p \in \mathbb{R}^n$ satisfying $\tilde{\phi}_n^T(p)\bar{\gamma}(p) \neq 0$ and $p_1 \geq \cdots \geq p_n$, there is $m \in \{1, \ldots, n-1\}$ such that $p_m > p_n$, which in turn implies $\bar{\gamma}_n(p) = \sum_{k=1}^n \bar{\psi}(p_k - p_n) \geq \bar{\psi}(p_m - p_n) > 0$.

Step iii: As for the final step, we will derive a lower bound for (29). From the proof of [1, Theorem 7.2.9] it is known that for any $i, j \in \{1, \ldots, n\}$, $p_i \geq p_j$ implies $\bar{\gamma}_i(p) \leq \bar{\gamma}_j(p)$. Hence, under the constraint $p_1 \geq \cdots \geq p_n$, we have $0 = \bar{\gamma}_1(p) \leq \cdots \leq \bar{\gamma}_n(p)$. Thus, for all $p \in \mathbb{R}^n$ such that $p_1 \geq \cdots \geq p_n$ and $\sum_{i=1}^n \bar{\phi}(p_i - p_n)\bar{\gamma}_i(p) \neq 0$ we have

$$\tilde{\phi}_n^T(p)\bar{\gamma}(p)/\bar{\gamma}_n(p) \leq \sum_{i=2}^{n-1} (\tilde{\phi}_n)_i(p) = 1 - (\tilde{\phi}_n)_1(p).$$

The function $\bar{\phi}$ being non-decreasing implies under the constraints $p_1 \geq \cdots \geq p_n$ and $\sum_{i=1}^n \bar{\phi}(p_i - p_n)\bar{\gamma}_i(p) \neq 0$ that $(\tilde{\phi}_n)_1(p) \geq 1/(n-1)$. As a result $\tilde{\phi}_n^T(p)\bar{\gamma}(p)/\bar{\gamma}_n(p) \leq 1 - 1/(n-1) = (n-2)/(n-1)$, meaning that $(n-1)/(n-2)$ is a lower bound to (29). ■

References

[1] W. H. Sandholm, *Population games and evolutionary dynamics*. MIT Press, 2010.

[2] N. Quijano, C. Ocampo-Martinez, J. Barreiro-Gomez, G. Obando, A. Pantoja, and E. Mojica-Nava, "The role of population games and evolutionary dynamics in distributed control systems," *IEEE Control Systems Magazine*, vol. 37, no. 1, pp. 70–97, 2017.

[3] S. Park, N. C. Martins, and J. S. Shamma, "From population games to payoff dynamics models: A passivity-based approach," in *Proceedings of the IEEE Conference on Decision and Control*, 2019, pp. 6584–6601.

[4] ——, "Payoff dynamics model and evolutionary dynamics model: Feedback and convergence to equilibria," *ArXiv:1903.02018*, July 2020.

[5] J. W. Weibull, *Evolutionary game theory*. MIT Press, 1995.

[6] J. Hofbauer and K. Sigmund, "Evolutionary game dynamics," *Bulletin of the American mathematical society*, vol. 40, no. 4, pp. 479–519, July 2003.

[7] J. Hofbauer and W. H. Sandholm, "Stable games and their dynamics," *Journal of Economic Theory*, vol. 144, no. 4, pp. 1665–1693.e4, July 2009.

[8] M. J. Fox and J. S. Shamma, "Population games, stable games, and passivity," *Games*, vol. 4, pp. 561–583, 2013.

[9] M. Arcak and N. C. Martins, "Dissipativity tools for convergence to Nash equilibria in population games," *IEEE Transactions on Control of Network Systems*, vol. 8, no. 1, pp. 39–50, 2021.

[10] D. Monderer and L. S. Shapley, "Potential games," *Games and economic behavior*, vol. 14, pp. 124–143, 1996.

[11] W. H. Sandholm, "Potential games with continuous player sets," *Journal of economic theory*, vol. 97, pp. 81–108, 2001.

[12] ——, "Pairwise comparison dynamics and evolutionary foundations for Nash equilibrium," *Games*, vol. 1, no. 1, pp. 3–17, 2010.

[13] J. Hofbauer and W. H. Sandholm, "Stable games," in *Proceedings of the IEEE conference on decision and control*, 2007, pp. 3416–3421.

[14] E. D. Sontag, *Mathematical Control Theory: Finite-Dimensional Systems*. Springer-Verlag, 1998.

[15] T. G. Kurtz, "Solutions of ordinary differential equations as limits of pure jump Markov processes," *Journal of applied probability*, vol. 7, no. 1, pp. 49–58, April 1970.

[16] S. Kara and N. C. Martins, "Pairwise comparison evolutionary dynamics with strategy-dependent revision rates: Stability and delta-passivity (expanded version)," *arXiv:2107.02835*, July 2021.

[17] G. L. Perry, *The New Palgrave Dictionary of Economics*. London: Palgrave Macmillan UK, 2017, ch. Demand-Pull Inflation, pp. 1–4.

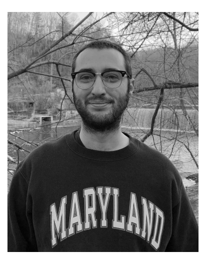
[18] M. Beckmann, C. McGuire, and C. B. Winsten, *Studies in the Economics of Transportation*. New Haven, Conn.: Yale University Press, 1956.

[19] M. J. Smith, "The stability of a dynamic model of traffic assignment: an application of a method of Lyapunov," *Transportation science*, vol. 18, no. 3, pp. 245–252, August 1984.

[20] S. Kara. *Strategy Dependent Revision Rates Pairwise Comparison Protocols*. (Version v1.0). Zenodo. Accessed: May 11, 2022. [Online]. Available: https://doi.org/10.5281/zenodo.5068102

[21] W. H. Sandholm, "Excess payoff dynamics and other well-behaved evolutionary dynamics," *Journal of economic theory*, vol. 124, no. 2, pp. 149–170, 2005.

[22] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.



Semih Kara received his B.S. degree in Electrical and Electronics Engineering from Bilkent University, Turkey in 2017. He is currently pursuing a Ph.D. degree in Electrical and Computer Engineering from the University of Maryland, College Park.

Nuno Miguel Lara Cintra Martins graduated with a M.S. degree in Electrical Engineering from IST, Portugal, in 1997, and a Ph.D. degree in Electrical Engineering and Computer Science with a minor in Mathematics from MIT in 2004. He is a Professor in the ECE Dept. of the U. of Maryland at College Park, where he also holds a joint appointment with the ISR.