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Population Games With Erlang Clocks: Convergence to Nash Equilibria For Pairwise Comparison Dynamics

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Abstract— **The prevailing methodology for analyzing population games and evolutionary dynamics in the large population limit assumes that a Poisson process (or clock) inherent to each agent determines when the agent can revise its strategy. Hence, such an approach presupposes exponentially distributed inter-revision intervals, and is inadequate for cases where each strategy entails a sequence of sub-tasks (sub-strategies) that must be completed before a new revision time occurs. This article proposes a methodology for such cases under the premise that a sub-strategy's duration is exponentially distributed, leading to Erlang distributed inter-revision intervals. We assume that a so-called pairwise comparison protocol captures the agents' revision preferences to render our analysis concrete. The presence of sub-strategies brings on additional dynamics that is incompatible with existing models and results. Our main contributions are twofold, both derived for a deterministic approximation valid for large populations. We prove convergence of the population's state to the Nash equilibrium set when a potential game generates payoffs for the strategies. We use system-theoretic passivity to determine conditions under which this convergence is guaranteed for contractive games.**

I. Introduction

Population games and evolutionary dynamics have been used as a tractable framework to model the strategic interactions in populations with large numbers of agents [1]–[3]. In this framework, each agent follows one strategy at a time, chosen from a finite set available to the population. Each available strategy has a payoff at any time specified by a map we refer to as "game". The agents repeatedly revise their strategies at the so-called revision times governed by a process (or clock) inherent to each agent. At a revision time, the agent may alter its strategy in response to the population's payoffs and the strategy profile. When revising their strategies, the agents act according to a probabilistic heuristic specified by a so-called revision protocol, which reflects the population's decision behavior and often has a simple structure. The agents are nondescript; consequently, a vector, called population state, whose entries are the population proportions following the available strategies, suffices to represent the population's strategic profile.

A. Erlang Revision Times

Existing work [1] and recent generalizations [4] assume that the inter-revision intervals are exponentially distributed.

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Applications where each strategy entails a sequence of tasks (sub-strategies) an agent must complete before a new revision opportunity occurs are not compatible with the existing theory because the sum of the tasks' service times is generally not exponentially distributed. In this article, our main goal is to propose a methodology to model and analyze the equilibrium stability of population games and evolutionary dynamics for such non-preemptive multi-task applications. Inspired by the queueing literature, we assume that the sub-strategies' service times are independent and exponentially distributed, resulting in Erlang distributed inter-revision intervals [5, Chapter 4.2].

Our work is motivated by applications, including the modeling of traffic congestion in roadways as a so-called congestion game [6]. Namely, in a congestion game the agents are drivers and the strategies are the possible routes connecting an origin to a destination. When each revision time coincides with the completion of a trip, the inter-revision interval becomes the trip duration. Hence, it is relevant to observe that several studies [7]–[9] find that, in many cases, trip duration is nearly Gamma distributed. Thus, noting that the Erlang and Gamma distributions coincide for certain parameter values, we conclude that allowing for Erlang distributed inter-revision intervals is a worthwhile generalization in the context of congestion games.

B. Deterministic Approximation And Stability Analysis

Similar to an extensive body of literature [1], [2], [10]–[12], we use a deterministic (mean field) approximation [13], [1, Appendix 12.B] to ascertain whether the population state converges to a neighborhood of the Nash equilibria of the game with high probability. In our analysis, the deterministic approximation will be the state of a system designated as Erlang Evolutionary Dynamics Model.

Existing convergence results for exponentially distributed inter-revision intervals are conclusive when additional structure is imposed on the game and the revision protocol. The cases in which the protocol is in the so-called pairwise comparison class, which is fully decentralized [4], [14], and the game is potential or strictly contractive[1] are well-known to have such a structure [15], [16]. Likewise, our focus in this article will be on pairwise comparison protocols and games that are potential or strictly contractive.

[1]Contractive games are also known as stable games. The recent article [12] defines weighted-contractive games, which subsume contractive ones. For simplicity, we limit our analysis to the non-weighted case.

II. Overview of the Framework

We start by presenting an overview of the population games and evolutionary dynamics paradigm.

A. Agents, Strategies and the Population State

Consider a population[2] comprising N agents, where N is large. At any time, each agent follows a single strategy from the same set of strategies $\{1, \dots, n\}$. As will be clarified throughout §II, the agents are "nondescript"; therefore, the proportions of agents playing the available strategies suffices to characterize population's strategy profile. We denote the proportion of agents playing strategy $i \in \{1, \dots, n\}$ at time $t \geq 0$ by $\bar{X}_i(t)$, i.e., the number of agents playing strategy i at time t is $N\bar{X}_i(t)$. Moreover, we define $\bar{X} := \begin{bmatrix}\bar{X}_1 & \dots & \bar{X}_n\end{bmatrix}^T$, and refer to $\bar{X}$ as the population state.

B. Payoffs and The Game Generating Them

At any time $t \geq 0$, each strategy $i \in \{1, \dots, n\}$ is endowed with a payoff $P_i(t)$. We assume that the mechanism assigning these payoffs is a continuously differentiable function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$, called the "game", that operates on the population state $\bar{X}$. The payoff of strategy i is denoted $\mathcal{F}_i$ and we write $\mathcal{F} := \begin{bmatrix}\mathcal{F}_1 & \dots & \mathcal{F}_n\end{bmatrix}^T$. Consequently, the payoff vector at time t is given by $P(t) = \mathcal{F}(\bar{X}(t))$. Since $\bar{X}$ takes values in $\Delta := \{\xi \in \mathbb{R}^n_{\geq 0} \mid \sum_{i=1}^n \xi_n = 1\}$ and $\mathcal{F}$ is continuous, P takes values in a bounded set $\mathfrak{P}$ given by

$$\mathfrak{P} := \{\mathcal{F}(\xi) \in \mathbb{R}^n \mid \xi \in \Delta\}.$$

The notion of Nash equilibria of a game is defined as:

$$\mathbb{NE}(\mathcal{F}) := \left\{ \xi \in \Delta \;\middle|\; \xi_i > 0 \implies i \in \underset{j \in \{1,\dots,n\}}{\arg\max}\, \mathcal{F}_j(\xi) \right\}.$$

According to [1, Theorem 2.1.1], the set $\mathbb{NE}(\mathcal{F})$ is nonempty, and in our context may be interpreted as in [17].

C. The Revision Paradigm

Agents repeatedly revise their strategies conforming to a procedure characterized by two components. The first component is the process that specifies the agents' revision times. The second component is the so-called revision protocol, which describes how a revising agent decides on its subsequent strategy.

1) Revision Times in the Original Framework: In the original framework, the agents are uniquely associated with independent and identically distributed (i.i.d.) Poisson processes (clocks) with rate $\lambda > 0$ and the revision times of an agent are the instants when a jump occurs in its clock. Hence, the inter-revision intervals are i.i.d. exponential random variables [1]. The framework in [4] also considers exponentially distributed inter-revision intervals, but allows an agent's revision rate to depend explicitly on its strategy.

[2]Although our results hold in the case of multiple populations (see [1] for the setting with multiple populations), for ease of exposition, we assume that there is a single population.

2) Revision Protocols: The revision protocol of the population is a Lipschitz continuous function $\mathcal{T} : \Delta \times \mathfrak{P} \to \mathbb{R}^{n\times n}_{\geq 0}$ that satisfies $\sum_{j=1}^n \mathcal{T}_{i,j}(\xi, \pi) = \lambda$ for all $i \in \{1, \dots, n\}$, $\pi \in \mathfrak{P}$ and $\xi \in \Delta$. Intuitively, for any $i, j \in \{1, \dots, n\}$, $\mathcal{T}_{i,j}$ gives the rate with which agents playing strategy i switch to strategy j. An important quantity that appears in the stability analysis in §V is

$$c := \max_{i \in \{1,\dots,n\},\ \xi \in \Delta} \sum_{j=1,\ j \neq i}^{n} \mathcal{T}_{i,j}(\xi, \mathcal{F}(\xi)), \tag{1}$$

which is a measure of the maximum rate of strategy switching that omits "self-switches".

More precisely, the revision protocol determines the subsequent strategy of a revising agent according to the following description. Assume that an agent receives a revision opportunity at time $\bar{t}$. It follows from the definition of revision times that, with probability 1, there is a t^* (strictly) between $\bar{t}$ and the previous revision time of the population. Denoting the strategy at t^* of the revising agent as i, the probability of its subsequent strategy being j is assumed to be $\mathcal{T}_{i,j}\big(\bar{X}(t^*), P(t^*)\big)/\lambda$ for any $j \in \{1, \dots, n\}$. Then, the realization of this strategy is assigned to be the revising agent's strategy at time $\bar{t}$.

D. The Evolutionary Dynamics Model (EDM)

As a result of the revision mechanism in §II-C, the population state $\bar{X}$ is a pure jump Markov process [1, Chapter 11].

Subsequently, an important question is whether $\bar{X}$ converges to $\mathbb{NE}(\mathcal{F})$. As explained in [2, Section 5] and [1, Appendix 12.B], a deterministic (mean field) approximation provides a methodology to answer this question. Namely, provided that N is large, from [13] and [1, Appendix 12.B], it follows that the convergence with high probability of $\bar{X}$ to a neighborhood of $\mathbb{NE}(\mathcal{F})$ can be concluded by verifying that $\mathbb{NE}(\mathcal{F})$ is globally attractive under a deterministic dynamical system. This dynamical system is referred to as the Evolutionary Dynamics Model (EDM) [2], where in this paper we call it the standard EDM to distinguish it from its Erlang counterpart introduced in §III.

III. Erlang Evolutionary Dynamics

In this section, we propose a generalization of the population games and evolutionary dynamics framework, outlined in §II, by allowing Erlang inter-revision intervals.

A. Erlang Inter-Revision Intervals

To introduce Erlang clocks, we follow a construction similar to that in §II-C.1, with the difference that an agent's inter-revision intervals are i.i.d. Erlang random variables with rate $\lambda > 0$ and parameter $m \in \mathbb{N}$. With this construction, the resulting population state $\bar{X}$ is a pure jump stochastic process, but not necessarily a Markov process when $m \geq 2$. This is undesirable because, to the best of our knowledge, deterministic approximation results similar to that in [13] or [1, Appendix 12.B] do not exist for pure jump processes with Erlang distributed waiting times. So, it is not directly evident how the deterministic approach summarized in §II-D

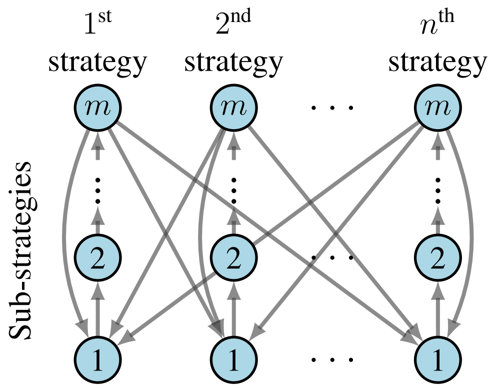

Fig. 1: Possible sub-strategy transitions.

can be altered to fit the framework with Erlang distributed inter-revision intervals. Therefore, in the following part, we present an alternative way in which the results in [13] and [1, Appendix 12.B] can be leveraged.

B. *Erlang Evolutionary Dynamics*

In what follows, we first characterize the population state in terms of a pure jump Markov process that conforms to the assumptions in [13] and [1, Appendix 12.B]. Then, we apply the aforementioned deterministic approximation results to this process and derive a deterministic dynamical system, which we utilize in the upcoming sections to ascertain the convergence properties of $\bar{X}$.

1) The Population State in Terms of a Markov Process: Given a strategy $i \in \{1, \ldots, n\}$, let us define (i, j) to be the j-th sub-strategy of i for $j \in \{1, \ldots, m\}$. Now, consider that an agent who chooses strategy i starts playing $(i, 1)$. Suppose that, for any $j \in \{1, \ldots, m-1\}$, after playing (i, j) for a period of time the agent transitions to playing $(i, j+1)$. Let the time that the agent spends playing sub-strategy (i, j), for any $j \in \{1, \ldots, m\}$, be distributed exponentially with rate λ. Furthermore, assume that the agent is given a revision opportunity after it is finished playing sub-strategy (i, m) and that, when it is given an opportunity, the agent chooses its subsequent strategy according to the procedure in §II-C.2. Finally, assume that the times spent by agents playing the sub-strategies are i.i.d. We illustrate the possible sub-strategy transitions in Fig.1.

Notice that the inter-revision intervals arising from the description above are i.i.d. Erlang random variables with rate λ and parameter m. Let us denote the proportion of agents playing sub-strategy (i, j) by $X_{i,j}$ and define

$$X := \begin{bmatrix} X_{1,1} & \ldots & X_{1,m} & \ldots & X_{n,1} & \ldots & X_{n,m} \end{bmatrix}^T.$$

Then, X is a pure jump Markov process, to which the results in [13] and [1, Appendix 12.B] can be applied. Moreover, given any $i \in \{1, \ldots, n\}$, $\sum_{j=1}^{m} X_{i,j}$ and $\bar{X}_i$ have the same distribution. Therefore, we can infer the long term behavior of $\bar{X}$ by analyzing X.

2) The Deterministic Approximation: Considering that the number of agents is large, we adopt a deterministic approximation x structured as

$$x := \begin{bmatrix} x_{1,1} & \ldots & x_{1,m} & \ldots & x_{n,1} & \ldots & x_{n,m} \end{bmatrix}^T,$$

and obtained as the unique solution of the initial value problem, with $x(0) = X(0)$, of the following system of differential equations:

$$\dot{x}_{i,1} = \sum_{j=1}^{n} x_{j,m} \mathcal{T}_{j,i}(\bar{x}, p) - \lambda x_{i,1}, \ 1 \leq i \leq n, \quad \text{(EEDMa)}$$

with the additional dynamics below present when $m \geq 2$:

$$\dot{x}_{i,l} = \lambda(x_{i,l-1} - x_{i,l}), \ 2 \leq l \leq m, \ 1 \leq i \leq n. \quad \text{(EEDMb)}$$

The input of (EEDM) is p, and is generated as

$$p := \mathcal{F}(\bar{x}), \ \bar{x} := \begin{bmatrix} \bar{x}_1 & \ldots & \bar{x}_n \end{bmatrix}^T, \ \bar{x}_i := \sum_{l=1}^{m} x_{i,l}.$$

We refer to $\bar{x}$ as the mean population state, x as the extended mean population state, p as the deterministic payoff and the dynamical system given by (EEDM) as the Erlang Evolutionary Dynamics Model (Erlang EDM). For all $t \geq 0$, we have $\bar{x}(t) \in \Delta$ and $x(t) \in \mathbb{X}$, where

$$\mathbb{X} := \left\{ \xi \in \mathbb{R}^{nm}_{\geq 0} \ \middle| \ \sum_{i=1}^{n} \sum_{l=1}^{m} \xi_{i,l} = 1 \right\}.$$

In the remainder of the paper, given $\xi \in \mathbb{R}^{nm}$, we denote $\xi = \begin{bmatrix} \xi_{1,1} & \ldots & \xi_{1,m} & \ldots & \xi_{n,1} & \ldots & \xi_{n,m} \end{bmatrix}^T$ and $\bar{\xi}_i := \sum_{l=1}^{m} \xi_{i,l}$ for any $i \in \{1, \ldots, n\}$.

From the results in [13], [2, Section V] and [1, Appendix 12.B] we have for any $T > 0$ and $\epsilon > 0$ that

$$\lim_{N \to \infty} \mathbb{P}\left(\sup_{t \in [0,T]} \|X(t) - x(t)\| < \epsilon \right) = 1,$$

Importantly, the discussions in [2, Section V] and [1, Appendix 12.B] indicate that, if a set $\mathbb{S}$ is globally attractive under the Erlang EDM, then the stationary distributions of X concentrate near $\mathbb{S}$ as the number of agents tends to infinity. This result and the assumption that N is large legitimizes the stability analysis carried out in the subsequent sections.

Note that, if $m = 1$, then $\bar{x} = x$ and the Erlang EDM reduces to the standard EDM [1], [2]. This agrees with the fact that, when $m = 1$, the constructions of the revision times in §II-C.1 and §III-A coincide. Furthermore, the Erlang EDM conforms to the higher order evolutionary dynamics format, which requires the number of states to be greater than the number of strategies. Instances of such dynamics have been analyzed in [18], [19], although the results therein do not address the dynamics that we investigate in this paper.

C. *Prelude to Stability Analysis*

In the upcoming sections, we analyze the stability properties of the Erlang EDM. However, to have a meaningful analysis, we need further structure on the revision protocols and the game.

1) Pairwise Comparison Protocols: An important class of protocols that induce preferable stability results is the pairwise comparison class [14].

Definition 1: A protocol $\mathcal{T}$ is said to belong to the pairwise comparison (PC) class if for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n\} \setminus \{i\}$, $\bar{\xi} \in \Delta$ and $\pi \in \mathfrak{P}$ it can be written as

$$\mathcal{T}_{i,j}(\bar{\xi}, \pi) = \phi_{i,j}(\pi),$$

where $\phi_{i,j} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfies sign preservation in the sense that $\phi_{i,j}(\pi) > 0$ if $\pi_j > \pi_i$ and $\phi_{i,j}(\pi) = 0$ if $\pi_j \leq \pi_i$.

Essentially, an agent following a PC protocol can only switch to strategies with payoffs that are greater than the payoff of its current strategy. Their desirable incentive properties [14, §2.5] and inherently fully decentralized operation result in the applicability of the PC class in many engineering problems. For instance, the Smith protocol [20], which belongs to the PC class, has been widely used to study traffic problems.

Thus, in the remainder of this paper, we consider the Erlang EDM under the assumption that $\mathcal{T}$ is a PC protocol. We refer to the resulting dynamics as the Erlang Pairwise Comparison EDM (Erlang PC-EDM).

Confining the protocol to be of the PC class readily yields a desirable characteristic. Namely, leveraging the so-called Nash stationarity of PC protocols [14], we identify the equilibria of the Erlang PC-EDM as

$$\mathbb{ENE}(\mathcal{F}) := \left\{ \xi \in \mathbb{X} \;\middle|\; \bar{\xi} \in \mathbb{NE}(\mathcal{F}),\ \xi_{i,l} = \tfrac{1}{m}\bar{\xi}_i \right\},$$

which implies that $\bar{\xi} \in \mathbb{NE}(\mathcal{F})$ for all $\xi \in \mathbb{ENE}(\mathcal{F})$. Notably, if $m = 1$ then $\mathbb{NE}(\mathcal{F}) = \mathbb{ENE}(\mathcal{F})$.

2) Potential and Contractive Game: We proceed to ascertain global convergence in the following senses.

Definition 2: We say that $\bar{x}$ converges to $\mathbb{NE}(\mathcal{F})$ when

$$\lim_{t\to\infty} \inf_{\bar{\xi}\in\mathbb{NE}(\mathcal{F})} \|\bar{x}(t) - \bar{\xi}\| = 0, \quad x(0) \in \mathbb{X}. \tag{2}$$

Definition 3: We say that x converges to $\mathbb{ENE}(\mathcal{F})$ when

$$\lim_{t\to\infty} \inf_{\xi\in\mathbb{ENE}(\mathcal{F})} \|x(t) - \xi\| = 0, \quad x(0) \in \mathbb{X}. \tag{3}$$

Remark 1: If x converges to $\mathbb{ENE}(\mathcal{F})$ then $\bar{x}$ will also converge to $\mathbb{NE}(\mathcal{F})$, but not necessarily the other way around. Hence, the former criterion is more informative.

Our analysis will focus[3] on potential [15], [21] and strictly contractive games [16] defined as follows.

Definition 4: A game $\mathcal{F}$ is said to be a potential game if there is a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $\nabla f = \mathcal{F}$. We refer to f as the game's potential.

Definition 5: A game $\mathcal{F}$ is said to be contractive if $\eta^T D\mathcal{F}(\bar{\xi})\eta \leq 0$ for all $\eta \in T\Delta := \{\nu \in \mathbb{R}^n \mid \sum_{i=1}^n \nu_i = 0\}$ and $\bar{\xi} \in \Delta$, where $D\mathcal{F}$ denotes the Jacobian of $\mathcal{F}$. Moreover, $\mathcal{F}$ is said to be strictly contractive if $\eta^T D\mathcal{F}(\bar{\xi})\eta < 0$, in which case we define

$$\underline{\gamma} := -\max_{\bar{\xi}\in\Delta,\eta\in T\Delta} \eta^T D\mathcal{F}(\bar{\xi})\eta,$$

$$\bar{\gamma} := -\min_{\bar{\xi}\in\Delta,\eta\in T\Delta} \eta^T D\mathcal{F}(\bar{\xi})\eta.$$

[3]Assumptions on the game's structure are necessary to ascertain stability for PC protocols even in the original framework [1, Chapter 9].

We note that the class of potential and strictly contractive games do not contain one another. For instance, the 123-coordination game [1, Example 3.1.5] is potential, but not contractive, and the "good" rock-paper-scissors (RPS) game [1, Example 3.3.2] is strictly contractive, but not potential.

IV. Convergence For Potential Games

In this section, we assume that $\mathcal{F}$ is a potential game, $\mathcal{T}$ is a PC protocol and show that $\bar{x}$ converges to $\mathbb{NE}(\mathcal{F})$. We also discuss the relevance of this result to distributed optimization.

A. Stability Analysis

When $\mathcal{F}$ is a potential game, we follow a similar approach to that in [15] to ascertain the convergence properties of $\bar{x}$, which is to investigate the potential of $\mathcal{F}$ evaluated along the trajectories of $\bar{x}$.

Theorem 1: If $\mathcal{F}$ is a potential game and $\mathcal{T}$ is a PC protocol, then (2) holds.

Proof: Since $\mathcal{F}$ is a potential game, it has a potential f as specified in Definition 4. Let us define $\mathcal{L} : \mathbb{R}^{nm} \to \mathbb{R}$ by $\mathcal{L}(\xi) = -f(\bar{\xi})$. Taking the time-derivative of $\mathcal{L}$ along the trajectories of the Erlang PC-EDM yields

$$\begin{aligned} -\frac{d}{dt}f(\bar{x}) &= -(\nabla f(\bar{x}))^T\dot{\bar{x}} = -\sum_{i=1}^n p_i \sum_{l=1}^m \dot{x}_{i,l} \\ &= -\sum_{i=1}^n p_i \left(\sum_{j=1}^n x_{j,m}\mathcal{T}_{j,i}(\bar{x}, p) - \sum_{j=1}^n x_{i,m}\mathcal{T}_{i,j}(\bar{x}, p) \right) \\ &= -\sum_{i=1}^n \sum_{j=1}^n x_{i,m}\phi_{i,j}(p)(p_j - p_i) \leq 0, \end{aligned} \tag{4}$$

where the inequality in (4) follows from the sign-preservation property of PC protocols. Moreover, the inequality in (4) holds with equality if and only if, whenever $i, j \in \{1, \dots, n\}$ and $t \geq 0$ satisfies $p_j(t) > p_i(t)$, we have $x_{i,m}(t) = 0$. Therefore, noting that $\mathbb{X}$ is compact and positively invariant under (EEDM), it follows from LaSalle's invariance principle [22, Theorem 3.4] that x converges to the largest subset of $\mathbb{E}$ that is invariant under the Erlang PC-EDM, where

$$\mathbb{E} := \left\{ \xi \in \mathbb{X} \;\middle|\; \xi_{i,m} > 0 \implies i \in \arg\max_{j\in\{1,\dots,n\}} \mathcal{F}_j(\bar{\xi}) \right\}.$$

Now, we show that such largest invariant subset of $\mathbb{E}$ is $\mathbb{M} := \{\xi \in \mathbb{X} \mid \bar{\xi} \in \mathbb{NE}(\mathcal{F})\}$.

To begin with, notice that $\mathbb{M} \subseteq \mathbb{E}$. Furthermore, observe that for all $t \geq 0$ satisfying $x(t) \in \mathbb{E}$, we have $\dot{\bar{x}}(t) = 0$. This implies that $\mathbb{M}$ is invariant under the Erlang PC-EDM.

To show that $\mathbb{M}$ is the largest of the invariant subsets of $\mathbb{E}$ under the Erlang PC-EDM, we proceed by contradiction. So assume that there is a set $\hat{\mathbb{M}} \subseteq \mathbb{E}$ satisfying $\mathbb{M} \subset \hat{\mathbb{M}}$ and $\hat{\mathbb{M}}$ is invariant under the Erlang PC-EDM. Then, there is $\xi \in \hat{\mathbb{M}}$ such that $\bar{\xi} \notin \mathbb{NE}(\mathcal{F})$ and x with $x(0) = \xi$ remains in $\hat{\mathbb{M}}$ for all $t \geq 0$. In what follows, we assume that $x(0) = \xi$ and arrive at a contradiction by showing that the resulting x leaves $\mathbb{E}$ (therefore leaves $\hat{\mathbb{M}}$). Since

$\bar{\xi} \notin \mathbb{NE}(\mathcal{F})$, there exists $i^* \in \{1,\dots,n\}$ for which $\bar{\xi}_{i^*} > 0$ and $i^* \notin \arg\max_{j\in\{1,\dots,n\}} \mathcal{F}_j(\bar{\xi})$. In other words, there is a strategy i^* such that $\bar{\xi}_{i^*} > 0$ and i^* is sub-optimal at $t = 0$. Observe from (EEDMb) that $\bar{\xi}_{i^*} > 0$ implies $x_{i^*,m}(t) > 0$ for all $t > 0$. Moreover, define $\mathcal{G} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as $\mathcal{G}(t) = \max_{j\in\{1,\dots,n\}} \mathcal{F}_j(\bar{x}(t)) - \mathcal{F}_{i^*}(\bar{x}(t))$. Because $\mathcal{G}$ is continuous and $\mathcal{G}(0) > 0$, there exists η such that $\mathcal{G}(t) > 0$ for all $t \in [0, \eta]$. Hence, within the time interval $[0, \eta]$, the strategy i^* stays sub-optimal. However, $x_{i^*,m}(t) > 0$ for all $t \in [0, \eta]$, meaning that x leaves $\mathbb{E}$. As a result, $\hat{\mathbb{M}}$ is not invariant under the Erlang PC-EDM. ∎

When the game $\mathcal{F}$ has a strictly concave potential, Theorem 1 can be augmented to arrive at the following corollary.

Corollary 1: If $\mathcal{F}$ is a potential game with a strictly concave potential and $\mathcal{T}$ is a PC protocol, then (3) holds.

Proof: If $\mathcal{F}$ has a strictly concave potential f, then $\mathbb{NE}(\mathcal{F}) = \{\bar{\xi}^*\}$, where $\bar{\xi}^*$ is the unique maximizer of f over Δ [1, Corollary 3.1.4]. From Theorem 1, it follows that $\lim_{t\to\infty} \bar{x}(t) = \bar{\xi}^*$. Moreover, $\dot{\bar{x}}$ is uniformly continuous, because $\mathcal{T}$ and $\mathcal{F}$ are Lipschitz continuous and x takes values in a compact set. Hence, leveraging Barbalat's lemma, we obtain $\lim_{t\to\infty} \dot{\bar{x}}(t) = 0$.

Now, consider the dynamics (7) of the auxiliary state $\tilde{x}$ defined in Appendix A. From the matrix A (given in (6)) being Hurwitz and $\lim_{t\to\infty} \dot{\bar{x}}(t) = 0$, we have $\lim_{t\to\infty} \tilde{x}(t) = 0$. Consequently, for all $i \in \{1,\dots,n\}$ and $l \in \{1,\dots,m\}$, $\lim_{t\to\infty} |x_{i,l}(t) - x_{i,m}(t)| = 0$. This and $\lim_{t\to\infty} \bar{x}(t) = \bar{\xi}^*$ imply that $\lim_{t\to\infty} \inf_{\xi\in\mathbb{ENE}(\mathcal{F})} \|x(t) - \xi\| = 0$. ∎

B. Potential Games and Distributed Optimization

In §VI-A, we will describe a congestion game, which is a common example of a potential game, and use it to illustrate how to employ Theorem 1. For this and other applications [15, §2.2], potential games are often associated with various forms of optimality, as we proceed to explain. The equivalence in [15, Proposition 3.1] establishes that for a game $\mathcal{F}$ with potential f, $\mathbb{NE}(\mathcal{F})$ is identical to the subset of Δ satisfying the Karush–Kuhn–Tucker conditions [23] for the problem of maximizing $f(\bar{\xi})$ subject to $\bar{\xi} \in \Delta$. Consequently, Theorem 1 guarantees for any $m \geq 1$ that $\bar{x}$ will converge to the global maxima set of f when f is concave. Hence, concave potential games induce an emergent behavior in the population that tends to maximize f. When $\mathcal{F}$ is efficient [15, §5] the maxima of f are also social optima. Interestingly, isoelastic congestion games are efficient (see [15, Lemma 5.2]). Potential games in strategic form [21], which may not be tractable for large populations, have also been used in the context of distributed optimization [24].

V. Convergence For Strictly Contractive Games

For $\mathcal{F}$ strictly contractive, we proceed to present a condition ensuring the convergence of x to $\mathbb{ENE}(\mathcal{F})$.

It is known that, even when $m = 1$, the PC-EDM may not exhibit stable behavior (see [1, Exercise 7.2.10]) for strictly contractive $\mathcal{F}$. Nonetheless, stability for $m = 1$ is ensured when the protocol is in the following impartial subclass [14].

Definition 6: A protocol $\mathcal{T}$ is said to be of the impartial pairwise comparison (IPC) class if for i, j in $\{1,\dots,n\}$, with $i \neq j$, $\bar{\xi} \in \Delta$ and $\pi \in \mathfrak{P}$ it can be written as $\mathcal{T}_{i,j}(\bar{\xi}, \pi) = \phi_j(\pi_j - \pi_i)$, where $\phi_j : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is sign preserving (meaning $\phi_j(\pi_j - \pi_i) > 0$ if and only if $\pi_j > \pi_i$).

Consequently, we seek to generalize to $m \geq 2$ the results in [14] for IPC protocols. To do so, we will need the following time-scale separation constant:

$$\underline{\lambda} := 2c\bar{\sigma}\left(\frac{n\bar{\gamma}}{(m+1)\underline{\gamma}}\right)^{1/2}. \tag{5}$$

Here, $\bar{\gamma}, \underline{\gamma}$ are given in Definition 5, c is specified by (1), and $\bar{\sigma} := \sup_{\omega\in[0,\infty)} \sigma_{\max}((j\omega - A)^{-1}B)$, where $\sigma_{\max}$ denotes the maximum singular value, and A, B are given by

$$A := \begin{bmatrix} -1 & 0 & \dots & 0 & -1 \\ 1 & -1 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -2 \end{bmatrix} \otimes \mathbf{I}_n, \;\; B := \mathbf{e}_1 \otimes \mathbf{I}_n, \tag{6}$$

in which $\otimes$ is the Kronecker product, $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{e}_1$ is the first standard basis vector in $\mathbb{R}^{m-1}$.

Remark 2: We note that $\bar{\sigma}$ is the H_∞-norm of the linear system specified by $\dot{z} = Az + Bu$ (with input u and output z). For the case when $m \leq 4$, we can compute $\bar{\sigma}$ simply as $\bar{\sigma} = ((2m^2 - 3m + 1)/(6m))^{1/2}$. As for the $m > 4$ case, computation of $\bar{\sigma}$ is more challenging, yet can be done numerically via the bisection H_∞-norm computation algorithm [25].

Having defined IPC protocols and introduced the constant $\underline{\lambda}$, we are now ready to state the following theorem.

Theorem 2: If $\mathcal{F}$ is strictly contractive, the protocol is IPC and $\lambda > \underline{\lambda}$, then (3) holds, i.e., x converges to $\mathbb{ENE}(\mathcal{F})$.

We present a proof of Theorem 2 in Appendix B, which follows mainly from the two time-scale structure of the Erlang EDM. Namely, when λ is large in comparison to c, the dynamics associated with the sub-strategies gives the "fast" part of (EEDM), whereas the dynamics of $\bar{x}$ gives its "slow" part. Thus, for any $i \in \{1,\dots,n\}$, $x_{i,1},\dots,x_{i,m}$ rapidly equalize and closely track $\bar{x}_i/m$. Thereafter, (EEDM) approximates the standard EDM, and global attractivity of $\mathbb{ENE}(\mathcal{F})$ ensues from the stability properties of the standard IPC-EDM [2], [26].

VI. Numerical Examples

We proceed to illustrate our results for two examples using the Smith protocol [20], meaning that in (EEDMa) we employ $\mathcal{T}_{i,j}(\bar{\xi}, \pi) = \max\{\pi_j - \pi_i, 0\}$, for all $\bar{\xi} \in \Delta$, $\pi \in \mathfrak{P}$ and $i, j \in \{1,\dots,n\}$ such that $i \neq j$.

A. A Congestion Game Example

We consider the congestion game in [1, Chapter 2.2.2], characterized by the graph in Fig. 2(a). Here, O denotes the origin, D denotes the destination, the links represent roads and the arrows on links represent the direction in which an agent choosing the link travels. As depicted in Fig. 2(b), the agents can choose to go from the origin to the destination via

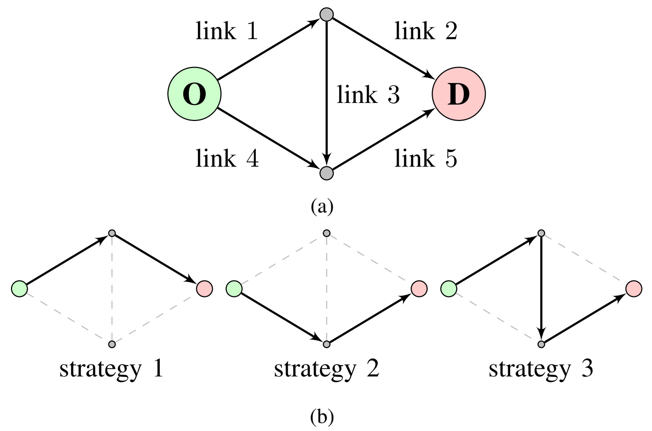

Fig. 2: Congestion game example with one origin/destination pair: (a) depicts the topology of the links and (b) illustrates the 3 strategies representing all possible routes from the origin to the destination.

one of the three available routes forming the strategy set. To each link $l \in \{1, \ldots, 5\}$, we assign a utilization-dependent cost given by $c_l \sum_{\{i \;|\; i \in \Omega_l\}} \bar{\xi}_i$, where Ω_l comprises the strategies using link l, $\bar{\xi}_i$ is the percentage of agents using route i and c_l is a positive constant quantifying the impact of utilization (congestion) on link l. Hence, noting that Fig. 2(b) displays which route corresponds to which strategy, the payoffs of using the routes under the population state value $\bar{\xi} \in \Delta$ are

$$\mathcal{F}^{Con}(\bar{\xi}) = -\begin{bmatrix} c_1 + c_2 & 0 & c_1 \\ 0 & c_4 + c_5 & c_5 \\ c_1 & c_5 & c_1 + c_3 + c_5 \end{bmatrix} \bar{\xi}.$$

Suppose the parameters in $\mathcal{F}^{Con}$ are $c_1 = 2.5$, $c_2 = 1.5$, $c_3 = 0.5$, $c_4 = 2.5$, and $c_5 = 0.7$. Since congestion games are potential games [1, Example 3.1.2] and the Smith protocol belongs to the PC class, we can invoke Theorem 1 to conclude that $\bar{x}$ converges to $\mathbb{NE}(\mathcal{F}^{Con})$.

We display in Fig. 3 a trajectory of $\bar{x}$ for $m = 3$ and $\lambda = 5$ obtained via simulation initialized with $x_{1,3}(0) = x_{2,1}(0) = 0.2$, $x_{3,1}(0) = 0.6$ and $x_{i,l}(0) = 0$ for all other $i, l \in \{1, 2, 3\}$. Observe from Fig. 3 that $\bar{x}$ indeed converges to $\mathbb{NE}(\mathcal{F}^{Con})$, which is $\{(0.349, 0.513, 0.137)\}$. Fig. 3 also presents the trajectory of $\bar{x}$ for $m = 1$, while keeping the other parameters unchanged. Recall from §III-B.2 that when $m = 1$ we are back to the standard case (see §II-C.1). As expected, the trajectories of $\bar{x}$ differ for $m = 3$ and $m = 1$.

B. A Rock-Paper-Scissors (RPS) Game Example

As noted in §III-C.2, the class of potential games and strictly contractive games do not contain one another, and the good RPS game [1, Example 3.3.2] is an example of a strictly contractive game that is not potential.

We consider $m = 4$ and specify the good RPS game by

$$\mathcal{F}^{RPS}(\bar{\xi}) = \begin{bmatrix} 0 & -2 & 3 \\ 3 & 0 & -2 \\ -2 & 3 & 0 \end{bmatrix} \bar{\xi}, \quad \bar{\xi} \in \Delta.$$

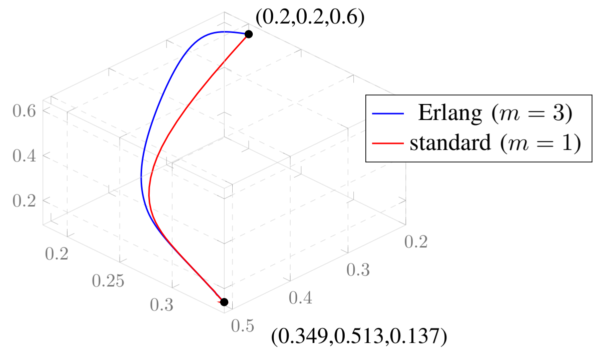

Fig. 3: Trajectories of $\bar{x}$ for the congestion game example.

Since $\mathcal{F}^{RPS}$ is strictly contractive but not potential, Theorem 1 can't be utilized and we have to resort to Theorem 2. We proceed by computing $(\bar{\gamma}, \underline{\gamma}, c)$, and, as stated in §V, when $m \leq 4$ we have $\bar{\sigma} = ((2m^2 - 3m - 1)/(6m))^{1/2}$, meaning that for $m = 4$ the value of $\bar{\sigma}$ is 0.9354. Hence, we obtain $\underline{\lambda} = 5.7965$ and it follows from Theorem 2 that, if $\lambda > 5.7965$, then x converges to $\mathbb{ENE}(\mathcal{F}^{RPS})$.

We performed a simulation for $\lambda = 5.8$ initialized with $x_{1,4}(0) = x_{2,1}(0) = 0.2$, $x_{3,1}(0) = 0.6$ and $x_{i,l}(0) = 0$ for all other $i \in \{1, 2, 3\}$, $l \in \{1, 2, 3, 4\}$. The resulting trajectories of $\bar{x}$ for $m = 4$ and $m = 1$ are displayed in Fig. 4. We can verify from Fig. 4 that the trajectories are surprisingly close and that in both cases $\bar{x}$ converges to $\mathbb{NE}(\mathcal{F}^{RPS}) = \{(1/3, 1/3, 1/3)\}$.

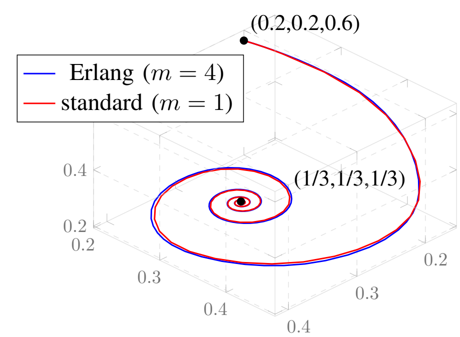

Fig. 4: Trajectories of $\bar{x}$ for the RPS game example.

VII. Conclusion and Future Directions

In this paper, we present an extension of the population games and evolutionary dynamics paradigm by allowing the agents' inter-revision intervals to be i.i.d. Erlang random variables with rate λ. We show that the long term behavior of the population state resulting from this generalization can be inferred by analyzing, what we call, the Erlang EDM. Then, we confine our focus to PC revision protocols and consider the Erlang PC-EDM. When the game $\mathcal{F}$ is potential, we show that the mean population state converges to $\mathbb{NE}(\mathcal{F})$ for any revision rate and number of sub-strategies. Similarly, when $\mathcal{F}$ is strictly contractive, we show that $\mathbb{ENE}(\mathcal{F})$ is globally attractive under the Erlang PC-EDM provided that the protocol is impartial and λ satisfies a bound condition.

The work presented in this paper also raises questions for future research. For instance, despite the results in §V, it is still unclear whether global attractivity of $\mathbb{ENE}(\mathcal{F})$ under the Erlang PC-EDM induced by an impartial protocol and strictly contractive game is guaranteed for any revision rate. Moreover, [2], [11], [12] generalizes the class of admissible payoff mechanisms to so-called payoff dynamics models; however we only consider static games. Hence, it can be investigated whether the analysis in §V can be altered to fit the δ-passivity [2] or δ-dissipativity [12] framework, which would broaden the global attractivity results therein to accommodate more general payoff structures.

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Appendix

A. Auxiliary Notation and Analysis

In this section we introduce the auxiliary states used in the proofs of Corollary 1 and Theorem 2, and characterize the dynamics of these states.

Given any $l \in \{1, \ldots, m\}$, let E_l be the $n \times nm$ matrix

$$E_l := \mathbf{I}_n \otimes \mathbf{e}_l^T,$$

where $\mathbf{e}_l$ is the l-th standard basis vector in $\mathbb{R}^m$. Moreover, given $\xi \in \mathbb{R}^{nm}$ we denote

$$\tilde{\xi} := \begin{bmatrix}(E_1 - E_m)\xi & \ldots & (E_{m-1} - E_m)\xi\end{bmatrix}^T.$$

Now, let us introduce the auxiliary states y, δ and $\tilde{x}$:

$$y := \begin{bmatrix} y_{1,1} & \ldots & y_{1,m-1} & \ldots & y_{n,1} & \ldots & y_{n,m-1}\end{bmatrix}^T,$$
$$\delta := \begin{bmatrix}\delta_1^T & \ldots & \delta_{m-1}^T\end{bmatrix}^T, \quad \tilde{x} := \begin{bmatrix}\tilde{x}_1 & \ldots & \tilde{x}_{m-1}\end{bmatrix}^T,$$

where, for any $i \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, m-1\}$ we set $\delta_l := \begin{bmatrix} y_{1,l} & \ldots & y_{n,l}\end{bmatrix}^T$, $\tilde{x}_l := (E_l - E_m)x$, and define $y_{i,l}$ to be a solution of $\dot{y}_{i,l} = \sum_{j=1}^n \phi_i(p_i - p_j)x_{j,l} - \lambda x_{i,l}$.

Notice that, from the definition of $\tilde{x}$ we have $\dot{\tilde{x}}_l = (E_l - E_m)\dot{x}$. Consequently,

$$\dot{\tilde{x}} = \lambda A\tilde{x} + B\dot{\tilde{x}}, \tag{7}$$

where we remind that A and B are given in (6).

Moreover, let us define $\Phi : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ as the matrix-valued function given for all $\pi \in \mathbb{R}^n$ by

$$\Phi_{ij}(\pi) = \begin{cases}\phi_i(\pi_i - \pi_j), & \text{if } i \neq j,\\ \sum_{j=1}^n \phi_j(\pi_j - \pi_i), & \text{if } i = j.\end{cases}$$

Then, for all $l \in \{1, \ldots, m-1\}$, we can write

$$\dot{\delta}_l = \Phi(p)E_l x - \Phi(p)E_m x = \Phi(p)\tilde{x}_l. \tag{8}$$

B. Proof of Theorem 2

We proceed to present a proof of Theorem 2 and the discussion that leads up to it. Our approach is based on analyzing the function $\mathcal{L}_\alpha : \mathbb{R}^{nm} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ inspired by the Lyapunov function for the standard IPC-EDM [16]. Namely, based on a modification of that in [16, Theorem 7.1], we set

$$\mathcal{L}_\alpha(\xi, \pi) := \sum_{i=1}^n \bar{\xi}_i \sum_{j=1}^n \Psi_j(\pi_j - \pi_i) + \alpha\tilde{\xi}^T M\tilde{\xi}, \tag{9}$$

where M is the solution of the Lyapunov equation $A^T M + MA = -I$ (since A is Hurwitz, such M exists and is symmetric positive-definite), α is a positive constant satisfying $\alpha < (m+1)\underline{\gamma}/(2\|MB\|_2^2)$, and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\Psi_j(\pi_j - \pi_i) := \int_0^{\pi_j - \pi_i} \phi_j(s) ds.$$

From a procedure similar to that in [16, Appendix A.4], we obtain the following time-derivative of $\mathcal{L}_\alpha$ along the trajectories of a solution of the Erlang PC-EDM and the deterministic payoff:

$$\tfrac{d}{dt}\mathcal{L}_\alpha(x,p) = -\mathcal{P}(x,p) + \mathcal{Q}(x,p), \tag{10}$$

where $\mathcal{P}(x,p)$ and $\mathcal{Q}(x,p)$ are specified as

$$\begin{aligned}\mathcal{P}(x,p) &:= -\alpha\lambda\tilde{x}^T\tilde{x} \\ &+ \sum_{i,j=1}^{n} \phi_i(p_i - p_j)x_{j,m} \sum_{k=1}^{n} \Psi_k(p_k - p_i) - \Psi_k(p_k - p_j), \\ \mathcal{Q}(x,p) &:= \Big(m\dot{\bar{x}}^T\dot{p} + \begin{bmatrix}\dot{p}^T & \dots & \dot{p}^T\end{bmatrix}\dot{\delta} + \alpha 2\tilde{x}^T MB\dot{\bar{x}}\Big).\end{aligned}$$

The argument in [16, Appendix A.4] can be readily adapted to prove the following proposition.

Proposition 1: If $\mathcal{F}$ is contractive, then for all $\xi \in \mathbb{X}$ and $\pi \in \mathbb{R}^n$ we have $\mathcal{P}(\xi, \pi) \geq 0$.

We now focus on the $\mathcal{Q}$ term. The following proposition is a key step in proving Theorem 2.

Proposition 2: Assume that $\mathcal{F}$ is strictly contractive and λ satisfies

$$\lambda \geq \left(\frac{2\bar{\sigma}(\alpha + 2\bar{\gamma}nc^2)}{(m+1)\underline{\gamma} - 2\alpha\|MB\|_2^2}\right)^{1/2}, \tag{11}$$

where $\bar{\gamma}, \underline{\gamma}$ are given in Definition 5, c is specified by (1) and $\bar{\sigma}$ is the supremum of the maximum singular value of $((j\omega - A)^{-1}B)$ over $\omega \in [0, \infty)$. Then, the following holds for all $t \geq 0$:

$$\int_0^t \mathcal{Q}(x(\tau), p(\tau))d\tau \leq \left(\alpha + 2\bar{\gamma}nc^2\right)\|e^{\lambda At}\tilde{x}(0)\|_2^2. \tag{12}$$

Proof: We begin by deriving a bound on $\int_0^t \|\tilde{x}(\tau)\|_2^2 d\tau$. Observe from (7) that

$$\tilde{x}(t) = e^{\lambda At}\tilde{x}(0) + \int_0^t e^{\lambda A(t-\tau)}B\dot{\bar{x}}(\tau)d\tau.$$

Thus, utilizing Parseval's theorem, we get

$$\int_0^t \|\tilde{x}(\tau)\|_2^2 d\tau \leq \|e^{-\lambda At}\tilde{x}(0)\|_2^2 + \frac{\bar{\sigma}}{\lambda^2}\int_0^t \|\dot{\bar{x}}(\tau)\|_2^2 d\tau. \tag{13}$$

We proceed by deriving a bound on $\|\dot{\delta}(t)\|_2$. Notice from (8) that for all $l \in \{1, \dots, m-1\}$ we have

$$\begin{aligned}\|\dot{\delta}_l(t)\|_2^2 &\leq \|\Phi(p(t))\|_2^2\|\tilde{x}_l(t)\|_2^2 \\ &\leq n\|\Phi(p(t))\|_1^2\|\tilde{x}_l(t)\|_2^2 = 4nc^2\|\tilde{x}_l(t)\|_2^2,\end{aligned} \tag{14}$$

where $c = \max_{\bar{\xi}\in\Delta}\sum_{j=1}^{n}\phi_j(\mathcal{F}_j(\bar{\xi}) - \mathcal{F}_i(\bar{\xi}))$ exists, since ϕ and $\mathcal{F}$ are Lipschitz continuous, and Δ is compact.

Now, we leverage (13) and (14) to obtain a condition that guarantees (12). Negative definiteness of $D\mathcal{F}(\bar{x})$ with respect to $T\Delta$ implies for all $l \in \{1, \dots, m-1\}$ that

$$\frac{1}{2}(-\dot{\delta}_l^T D\mathcal{F}(\bar{x})\dot{\delta}_l - \dot{\bar{x}}^T D\mathcal{F}(\bar{x})\dot{\bar{x}}) \leq |\dot{\delta}_l^T D\mathcal{F}(\bar{x})\dot{\bar{x}}|. \tag{15}$$

From (15), with $2|\tilde{x}^T MB\dot{\bar{x}}| \leq \|\tilde{x}\|_2^2 + \|MB\|_2^2\|\dot{\bar{x}}\|_2^2$ and negative definiteness of $D\mathcal{F}(\bar{x})$ with respect to $T\Delta$, we get

$$\begin{aligned}&\int_0^t m\dot{\bar{x}}(\tau)^T\dot{p}(\tau) + \sum_{l=1}^{m-1}\dot{\delta}_l(\tau)^T\dot{p}(\tau) + \alpha 2\tilde{x}(\tau)^T MB\dot{\bar{x}}(\tau)d\tau \\ &\leq \int_0^t -\frac{m+1}{2}\underline{\gamma}\|\dot{\bar{x}}(\tau)\|_2^2 + \frac{1}{2}\bar{\gamma}\|\dot{\delta}(\tau)\|_2^2 \\ &\qquad + \alpha\|\tilde{x}(\tau)\|_2^2 + \alpha\|MB\|_2^2\|\dot{\bar{x}}(\tau)\|_2^2 d\tau.\end{aligned} \tag{16}$$

Finally, combining (16), (13) and (14), it follows that

$$\begin{aligned}&\int_0^t m\dot{\bar{x}}(\tau)^T\dot{p}(\tau) + \sum_{l=1}^{m-1}\dot{\delta}_l(\tau)^T\dot{p}(\tau) + \alpha 2\tilde{x}(\tau)^T MB\dot{\bar{x}}(\tau)d\tau \\ &\leq \left(\alpha + 2\bar{\gamma}nc^2\right)\|e^{\lambda At}\tilde{x}(0)\|_2^2 + \int_0^t \Bigg(-\frac{m+1}{2}\underline{\gamma} \\ &\quad + \alpha\|MB\|_2^2 + (\alpha + 2\bar{\gamma}nc^2)\frac{\bar{\sigma}}{\lambda^2}\Bigg)\|\dot{\bar{x}}(\tau)\|_2^2 d\tau.\end{aligned} \tag{17}$$

As a result, if (11) holds, then

$$\int_0^t \mathcal{Q}(x(\tau), p(\tau))d\tau \leq \left(\alpha + 2\bar{\gamma}nc^2\right)\|e^{\lambda At}\tilde{x}(0)\|_2^2. \tag{18}$$

■

Now, we are ready to present a proof of Theorem 2, which is a direct consequence of Propositions 1 and 2, and Barbalat's lemma.

Proof: Assume that $\lambda > \underline{\lambda}$, where $\underline{\lambda}$ is specified in (5). Then, there exists $\alpha^* > 0$ satisfying $\alpha^* < (m+1)\underline{\gamma}/(2\|MB\|_2^2)$ such that (11) holds with $\alpha = \alpha^*$. Thus, we can leverage Propositions 1 and 2 to arrive at

$$\begin{aligned}&\int_0^t |\mathcal{P}(x(\tau), p(\tau))|d\tau \\ &\leq -\mathcal{L}_{\alpha^*}(x(t), p(t)) + \mathcal{L}_{\alpha^*}(x(0), p(0)) \\ &\qquad + \left(\alpha^* + 2\bar{\gamma}nc^2\right)\|e^{\lambda At}\tilde{x}(0)\|_2^2 \\ &\leq \mathcal{L}_{\alpha^*}(x(0), p(0)) + \left(\alpha^* + 2\bar{\gamma}nc^2\right)\|e^{\lambda At}\tilde{x}(0)\|_2^2\end{aligned} \tag{19}$$

for all $t \geq 0$. Combining (19) with the fact that A is Hurwitz, we get

$$\lim_{t\to\infty}\int_0^t |\mathcal{P}(x(\tau), p(\tau))|d\tau < \infty. \tag{20}$$

Since $\int_0^t |\mathcal{P}(x(\tau), p(\tau))|d\tau$ is increasing in t, it follows from (20) that $\int_0^t |\mathcal{P}(x(\tau), p(\tau))|d\tau$ has a finite limit as $t \to \infty$. Additionaly, $\mathcal{F}$ and $\mathcal{T}$ are Lipschitz continuous and x takes values in a compact set. Therefore x and p are uniformly continuous, meaning that $\mathcal{P}(x,p)$ is uniformly continuous. As a result, we can invoke Barbalat's lemma to conclude that $\mathcal{P}(x(t), p(t)) \to 0$ as $t \to \infty$. Finally, combining $\lim_{t\to\infty}\mathcal{P}(x(t), p(t)) = 0$ with $\mathcal{P}(\xi, \mathcal{F}(\bar{\xi})) = 0$ if and only if $\xi \in \mathbb{ENE}(\mathcal{F})$, we get $\lim_{t\to\infty}\inf_{\xi\in\mathbb{ENE}(\mathcal{F})}\|x(t) - \xi\| = 0$.

■