Quasistatic Control of Dynamical Systems

Arash Komaee

***Abstract*— This paper investigates a control strategy in which the state of a dynamical system is driven slowly along a trajectory of stable equilibria. This trajectory is a continuum set of points in the state space, each one representing a stable equilibrium of the system under some constant control input. Along the continuous trajectory of such constant control inputs, a slowly varying control is then applied to the system, aimed to create a stable quasistatic equilibrium that slowly moves along the trajectory of equilibria. As a stable equilibrium attracts the state of system within its vicinity, by moving the equilibrium slowly along the trajectory of equilibria, the state of system travels near this trajectory alongside the equilibrium. Despite the disadvantage of being slow, this control strategy is attractive for certain applications, as it can be implemented based only on partial knowledge of the system dynamics. This feature is in particular important for the complex systems for which detailed dynamical models are not available.**

I. Introduction

The identifier *quasistatic* in this paper is borrowed from thermodynamics referring to thermal processes which evolve in time sufficiently slow to allow them staying at equilibrium during the course of evolution. Then, *quasistatic control* is defined in this paper as a control that slowly drives the state of a dynamical system along a trajectory of stable equilibria. A closely related concept, *adiabatic control*, has been studied in control of quantum systems [1]–[3]. This paper focuses on general dynamical systems described by state-space models.

Consider a dynamical system that admits stable equilibria under constant (static) controls in its control space. Further, consider a continuum set of such constant controls living on a continuous trajectory in the control space, for which there is a corresponding continuous trajectory of stable equilibria in the state space. Suppose the system is driven by a quasistatic control that varies slowly along the trajectory in the control space. This control creates a quasistatic stable equilibrium moving slowly along the trajectory of equilibria in the state space. Such a stable equilibrium attracts the state of system in its vicinity, and as a result, if it moves at a slow enough pace, the state will closely track it along the trajectory of equilibria.

A remarkable feature of the quasistatic control is that it can be implemented based on partial knowledge of the system dynamics rather than its detailed mathematical model. The necessary information to construct this control includes only the static (steady-state) relationship between the control and state of the system to characterize the trajectory of equilibria, and some lower bound on the rate of temporal evolution of the system to identify the pace of control. Of course, such an incomplete information set may result in a conservatively slow control, yet that slow control can be an attractive choice for complex systems for which complete dynamical models are not available.

This work was supported by the National Science Foundation under Grant ECCS-1941944.

The author is with the School of Electrical, Computer, and Biomedical Engineering, Southern Illinois University, Carbondale, IL, 62901 USA email: akomaee@siu.edu.

In [4], we investigated the concept of quasistatic control in a stochastic framework. The particular goal in that work was to design proper controls for directed self-assembly, in which a number of charged nanoparticles form a desired geometry in space as a result of their mutual interactions and controlled electric fields generated by a set of electrodes fixed in space. Each stable equilibrium of this system associates to a certain geometry, and the system admits a large number of equilibria, out of which, only one represents the desired geometry. Then, the purpose of quasistatic control is to transition the system near a trajectory of equilibria from an initial geometry toward a desired final geometry. By maintaining the system near the trajectory of equilibria, this control minimizes the likelihood of being trapped by undesired equilibria as the system state is perturbed by intrinsic disturbances modeled stochastically.

This paper further investigates an output tracking scenario based on quasistatic control. The control goal in this scenario is to steer a system output along a reference trajectory within the output space. It is shown in the paper that achieving this goal based on quasistatic control only needs a fraction of the system model, which in particular is efficient for the systems with a large state space but a few input and output variables.

Examples of such systems are found in macroeconomics, which usually deals with high dimensional dynamics caused by interactions among a large number of players, while few control variables such as interest rate are available to control few output variable like inflation. The relationship between these variables at equilibrium are often known via the theory of macroeconomics and econometrics methods, but complete knowledge of the system dynamics is difficult to acquire due to its complexity and lack of enough empirical data.

II. Quasistatic Control

This section introduces the concept of quasistatic control of dynamical systems. To facilitate discussion, the concept is established first for linear systems in Section II-A, and then is generalized to nonlinear systems in Section II-B.

A. *Quasistatic Control of Linear Systems*

Consider the stable linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, and $y(t) \in \mathbb{R}^k$ are the state, control, and output vectors, respectively. It is assumed that the matrices A, B, C, and D have suitable dimensions, and that the static input-output gain

$$K = -CA^{-1}B + D \tag{2}$$

is invertible. This section explains the concept of quasistatic control for this linear system as an open-loop control with certain construction and properties studied in the paper.

Assume that $U(\cdot) : [0,1] \to \mathbb{R}^k$ is a continuous trajectory in the control space of (1a) and construct $X(\cdot) : [0,1] \to \mathbb{R}^n$ as a continuous trajectory in its state space according to

$$X(s) = -A^{-1}BU(s), \quad s \in [0,1]. \tag{3}$$

Then, for each fixed $s \in [0,1]$, $X(s)$ is an equilibrium point of the linear system (1a) under the constant control $U(s)$. Consequently, the hodograph of $X(\cdot)$ can be imagined as a continuous trajectory of equilibria in the state space of (1a).

Assume that at $t = 0$, the initial state $x(0)$ of (1a) is at the equilibrium point $X(0)$. Let $t_f > 0$ be a fixed final time and consider controlling the linear system (1a) under the control

$$u(t) = \begin{cases} U(t/t_f) & 0 \leqslant t \leqslant t_f \\ U(1) & t > t_f. \end{cases} \tag{4}$$

For a large t_f, this control varies slowly in time, and as result, the state of (1a) can closely track the trajectory $X(t/t_f)$ of the quasistatic stable equilibria during $t \in [0, t_f]$. Moreover, shorty after t_f, the system state settles at the final equilibrium point $X(1)$.

Denote the deviation of the state vector $x(t)$ from the quasistatic equilibrium $X(t/t_f)$ by

$$e(t) = x(t) - X(t/t_f).$$

Then, assuming that $X(\cdot)$ is differentiable, $e(t)$ evolves in time according to

$$\dot{e}(t) = Ae(t) - \frac{1}{t_f} X'(t/t_f), \tag{5}$$

where $X'(\cdot)$ denotes the derivative of $X(\cdot)$. Suppose that the state $x(t)$ of (1a) is at the equilibrium $X(0)$ at $t = 0$, and as a result, $e(0) = 0$. Then, it is concluded from (5) that the deviation $e(t)$ tends to 0 as $t_f \to \infty$, which implies

$$x(t) \to X(t/t_f), \quad t \geqslant 0$$

as $t_f \to \infty$, i.e., the system state stays near equilibrium over the course of control. A control with such property is called quasistatic in this paper, following a similar concept in thermodynamics.

Consider the control scenario of driving the output $y(t)$ of the linear system (1) along some desired reference trajectory $Y(\cdot) : [0,1] \to \mathbb{R}^k$ at an arbitrary pace over $t \in [0, t_f]$ (i.e., the hodographs of $y(t)$, $t \in [0, t_f]$ and $Y(s)$, $s \in [0,1]$ must be identical). As the pace of control can be taken arbitrarily, the reference output is generated by $y_r(t) = Y(t/t_f)$ for some $t_f > 0$. Then, the control (4) is constructed by choosing

$$U(s) = K^{-1}Y(s), \quad s \in [0,1],$$

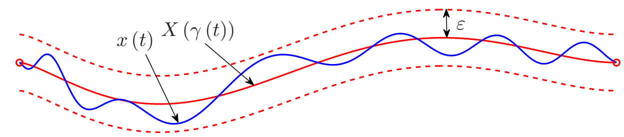

Fig. 1. The state of a dynamical system travels inside a tube of radius ε around the trajectory of quasistatic equilibria.

where K is the static gain matrix (2) of the linear system (1) and $Y(\cdot)$ is a differentiable trajectory in the output space. Under this control, the deviation of output from the desired reference is given by

$$y(t) - Y(t/t_f) = Ce(t), \tag{6}$$

which again tends to 0 as $t_f \to \infty$.

A remarkable feature of this control is that it is constructed on the basis of partial knowledge of the system, rather than its complete model. Specifically, only the static input-output relationship (i.e., K) and the stability of system are assumed to develop the control. Of course, the price for this advantage is the requirement for an infinite control horizon, which is not affordable in practice. Yet, the control horizon can be reduced to finite by accepting some deviations of the output from its reference (or the state from equilibrium) and incorporating additional information of the system into the control design process. To that end, an extension of (4) is introduced first.

Assume $t_f > 0$ and let $\gamma(\cdot) : [0, t_f] \to [0,1]$ be a strictly increasing differentiable function that satisfies $\gamma(0) = 0$ and $\gamma(t_f) = 1$. Using $\gamma(\cdot)$, the control (4) is generalized to

$$u(t) = U(\gamma(t)), \quad t \in [0, t_f] \tag{7}$$

and $u(t) = U(1)$ for $t > t_f$. Clearly, (4) is a special case of this control for $\gamma(t) = t/t_f$. Correspondingly, the deviation of state from the quasistatic equilibrium is redefined as

$$e(t) = x(t) - X(\gamma(t)) \tag{8}$$

and its dynamics is modified into

$$\dot{e}(t) = Ae(t) - \dot{\gamma}(t) X'(\gamma(t)).$$

The goal here is to determine $\gamma(\cdot)$ and t_f in such a manner that the deviation $e(t)$ stays within an acceptable bound over the course of control as the control (7) is applied to the linear system (1). The specific objective is to maintain $\|e(t)\| \leqslant \varepsilon$ on $t \in [0, t_f]$, provided that $\|e(0)\| \leqslant \varepsilon$, where $\varepsilon > 0$ is the maximum tolerable error and $\|\cdot\|$ denotes the Euclidean norm of vectors. This ensures that the state $x(t)$ of (1a) travels inside a tube of radius ε around the trajectory of quasistatic equilibria $X(\gamma(t))$, $t \in [0, t_f]$, as symbolically shown in Fig. 1. The following proposition constructs a suitable $\gamma(\cdot)$ to achieve this goal.

Proposition 1: Assume that $-\frac{1}{2}\left(A + A^T\right)$ is a positive definite matrix with eigenvalues lower bounded by $\lambda_m > 0$. Construct $X(\cdot)$ via (3) in terms of a differentiable $U(\cdot)$ and assume that $\|X'(s)\| \neq 0$ for all $s \in [0,1]$. Take $\varepsilon > 0$ and determine the final time t_f from

$$t_f = \frac{1}{\varepsilon} \int_0^1 \frac{\|X'(s)\|}{\lambda_m} \, ds. \tag{9}$$

Construct $\gamma(\cdot)$ by solving the differential equation

$$\dot{\gamma}(t) = \frac{\varepsilon \lambda_m}{\|X'(\gamma(t))\|} \tag{10}$$

on $t \in [0, t_f]$ with the initial condition $\gamma(0) = 0$ (assuming the solution exists). Then, under the control (7) and an initial state that holds $\|x(0) - X(0)\| \leqslant \varepsilon$, the state of the linear system (1a) satisfies

$$\|x(t) - X(\gamma(t))\| \leqslant \varepsilon, \quad t \in [0, t_f]. \tag{11}$$

Moreover, $\gamma(\cdot)$ is strictly increasing and satisfies $\gamma(t_f) = 1$.

Proof: This is a special case of Proposition 2 proven in Section II-C. ■

Remark 1: The control introduced in Proposition 1 holds

$$\left\| \tfrac{d}{dt} X(\gamma(t)) \right\| = \varepsilon \lambda_m$$

which means that under this control, the equilibrium moves at a constant speed. To maintain such a constant speed, $\gamma(\cdot)$ is constructed via (10) to move the equilibrium faster along the hodograph of $X(s)$, $s \in [0, 1\}$ wherever it is more straight, and move slower at points with higher curvature.

Remark 2: Proposition 1 identically holds if $\gamma(\cdot)$ instead of (10) is generated by an alternative differential equation

$$\dot{\gamma}(t) = \frac{\varepsilon \lambda_m}{\|A^{-1}B\| \cdot \|U'(\gamma(t))\|} \tag{12}$$

defined in terms of the differentiable control trajectory $U(\cdot)$ and the induced 2-norm $\|A^{-1}B\|$ of $A^{-1}B$. The advantage of this equation over (10) is that it generates $\gamma(\cdot)$ relying only on an scalar $\|A^{-1}B\|$ rather than the complete knowledge of matrix $A^{-1}B$. Of course, the control derived from (12) is more conservative with a longer control time

$$t_f = \frac{1}{\varepsilon} \int_0^1 \frac{\|A^{-1}B\| \cdot \|U'(s)\|}{\lambda_m} \, ds.$$

The expression (9) in Proposition 1 indicates that the final time is inversely proportional to ε, and therefore, the control period can be shortened by accepting a larger deviation from the trajectory of quasistatic equilibria. Vice versa, the system state can be maintained closer to this trajectory by accepting a slower pace of control over a longer control horizon.

Another observation from (9) is that the control period depends on the arc length of $X(s)$, $s \in [0, 1]$ rather than its detailed geometry. Then, in a control scenario aimed solely at transitioning a system from an initial equilibrium X_0 to a final equilibrium X_f, the shortest control time is achieved by the straight line $X(s) = (1 - s) X_0 + s X_f$ connecting X_0 to X_f. In this case, $\gamma(\cdot)$ is simply given by $\gamma(t) = t/t_f$ with $t_f = \|X_f - X_0\| / (\varepsilon \lambda_m)$.

For the purpose of output tracking, the control

$$u(t) = K^{-1} Y(\gamma(t)), \quad t \in [0, t_f] \tag{13}$$

can be determined in such a manner that the output $y(t)$ of the linear system (1) tracks the reference trajectory $Y(\gamma(t))$ within a distance not exceeding $\tilde{\varepsilon} > 0$, that is

$$\|y(t) - Y(\gamma(t))\| \leqslant \tilde{\varepsilon}, \quad t \in [0, t_f]. \tag{14}$$

For this purpose, ε in Proposition 1 is chosen as $\varepsilon = \tilde{\varepsilon} / \|C\|$, where $\|C\|$ is the induced 2-norm of C. Then, (14) is implied by (6) and (11).

The attractive feature of the control (13) is that it can be implemented using only partial knowledge of the dynamical system (1), consisting of the parameters K, λ_m, $\|A^{-1}B\|$, and $\|C\|$. This feature is particularly important for complex systems with a high dimensional state space, but relatively few input and output variables. Construction of a complete model for such systems can be infeasible or at least difficult, while estimating a few parameters from theory or empirical data can be affordable.

B. Quasistatic Control of Nonlinear Systems

This section extends the core idea of quasistatic control to nonlinear systems, albeit cautiously. The reason for such caution is the potential differences between the equilibria of linear and nonlinear systems. In opposition to the (stable) linear systems that always admit a unique equilibrium for any constant control input, nonlinear systems may not admit any equilibrium for certain controls, or conversely, may admit multiple equilibria. Analysis of this issue is postponed to Section III, while the goal in this section is to generalize the concept of quasistatic control to nonlinear systems by minimal involvement of the topology of equilibria.

Let $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $g(\cdot) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ be differentiable vector functions and consider the nonlinear state-space model

$$\dot{x}(t) = f(x(t), u(t)) \tag{15a}$$

$$y(t) = g(x(t), u(t)). \tag{15b}$$

Here, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, and $y(t) \in \mathbb{R}^k$ are the state, control, and output vectors, respectively. Let $U(s)$, $s \in [0, 1]$ be a continuous trajectory in the control space and assume that a continuous trajectory $X(s)$, $s \in [0, 1]$ in the state space exists to hold

$$f(X(s), U(s)) = 0, \quad s \in [0, 1]. \tag{16}$$

Following a procedure similar to the case of linear systems, the goal here is to construct a suitable $\gamma(\cdot)$ such that under the control (7), the state $x(t)$ of the nonlinear system (15a) travels at a distance not exceeding ε from the quasistatic equilibrium $X(\gamma(t))$ over the course of control $t \in [0, t_f]$, as mathematically expressed by (11).

Denote the deviation of state from equilibrium by $e(t)$ as given in (8) and note that its dynamics is governed by

$$\dot{e}(t) = f(X(\gamma(t)) + e(t), U(\gamma(t))) - \dot{\gamma}(t) X'(\gamma(t)),$$

provided that $X(\cdot)$ is differentiable. Assuming $\|e(t)\| \leqslant \varepsilon$ is maintained on $t \in [0, t_f]$ with a small enough ε, this equation can be approximated as

$$\dot{e}(t) \doteq F_x(X(\gamma(t)), U(\gamma(t))) e(t) - \dot{\gamma}(t) X'(\gamma(t)), \tag{17}$$

where $F_x(\cdot)$ denotes the Jacobian matrix of $f(\cdot)$ with respect to its first argument. Proposition 2 below constructs $\gamma(\cdot)$ in such a manner that $\|e(0)\| \leqslant \varepsilon$ leads to $\|e(t)\| \leqslant \varepsilon$ for the entire $t \in [0, t_f]$. For this $\gamma(\cdot)$, application of the control (7) to the nonlinear system (15a) drives its state $x(t)$ inside the tube (11) at a maximum distance ε from the quasistatic equilibrium $X(\gamma(t))$.

Proposition 2: Suppose that $X(\cdot)$ and $U(\cdot)$ are a pair of differentiable functions holding (16) and $\|X'(s)\| \neq 0$ for all $s \in [0, 1]$, and assume that

$$-\frac{1}{2}\left(F_x(X(s), U(s)) + F_x^T(X(s), U(s))\right) \tag{18}$$

is a strictly positive definite matrix for all $s \in [0, 1]$. Assume further that $\lambda(\cdot) : \mathbb{R}^n \times \mathbb{R}^k \to (0, \infty)$ is a positive-valued smooth function such that $\lambda(X(s), U(s))$ lower bounds the eigenvalues of this matrix for every $s \in [0, 1]$. Take $\varepsilon > 0$ and determine the final time t_f from

$$t_f = \frac{1}{\varepsilon}\int_0^1 \frac{\|X'(s)\|}{\lambda(X(s), U(s))}\, ds. \tag{19}$$

Construct $\gamma(\cdot)$ by solving the differential equation

$$\dot{\gamma}(t) = \frac{\varepsilon\lambda(X(\gamma(t)), U(\gamma(t)))}{\|X'(\gamma(t))\|} \tag{20}$$

on $t \in [0, t_f]$ with the initial condition $\gamma(0) = 0$ (assuming the solution exists). Then, starting from an initial state that holds $\|e(0)\| \leqslant \varepsilon$, the state of (17) satisfies

$$\|e(t)\| \leqslant \varepsilon, \quad t \in [0, t_f].$$

Moreover, $\gamma(\cdot)$ is strictly increasing and satisfies $\gamma(t_f) = 1$.

Proof: See Section II-C ■

Remark 3: The lower bound on the eigenvalues of (18) can be taken as a constant λ_m independent of $X(\cdot)$ and $U(\cdot)$, if the complete knowledge of $\lambda(\cdot)$ is not available. Since this constant necessarily holds $\lambda_m \leqslant \min_{s\in[0,1]} \lambda(X(s), U(s))$, its associated control time will be longer than (19).

The output tracking control of Section II-A is extended to the nonlinear system (15) as follows. Let $Y(s)$, $s \in [0, 1]$ be a differentiable trajectory in the output space of (15) and assume that $X(\cdot)$ and $U(\cdot)$ jointly solve

$$f(X(s), U(s)) = 0 \tag{21a}$$

$$g(X(s), U(s)) = Y(s) \tag{21b}$$

for every $s \in [0, 1]$. The objective is to develop a control of the form (7) under which the output $y(t)$ of the nonlinear system (15) closely tracks the reference trajectory $Y(\gamma(t))$. More precisely, $y(t)$ must satisfy (14) for some given $\tilde{\varepsilon} > 0$, provided that the initial state $x(0)$ is at a distance from $X(0)$ not exceeding ε.

To that end, let $G_x(\cdot)$ be the Jacobian matrix of $g(\cdot)$ with respect to its first argument and define the constant

$$c = \max_{s\in[0,1]} \|G_x(X(s), U(s))\|.$$

Then, by taking $\varepsilon = \tilde{\varepsilon}/c$ in Proposition 2, the resulting $\gamma(\cdot)$ leads to a state error holding $\|e(t)\| \leqslant \tilde{\varepsilon}/c$. It is shown next that with this state error, the output error satisfies the desired condition (14).

The deviation of $y(t)$ from $Y(\gamma(t))$ is given by

$$y(t) - Y(\gamma(t)) = g(X(\gamma(t)) + e(t), U(\gamma(t))) - Y(\gamma(t))$$

which can be approximated as

$$y(t) - Y(\gamma(t)) \doteq G_x(X(\gamma(t)), U(\gamma(t)))\, e(t)$$

when $\|e(t)\| \leqslant \tilde{\varepsilon}/c$ holds with a small enough $\tilde{\varepsilon}/c$. It is then concluded that

$$\begin{aligned}\|y(t) - Y(\gamma(t))\| &\leqslant \|G_x(X(\gamma(t)), U(\gamma(t)))\| \cdot \|e(t)\| \\ &\leqslant \tilde{\varepsilon}.\end{aligned}$$

It worth mentioning that the nature of output tracking for nonlinear systems can be different from linear systems if they admit multiple equilibrium points. For a linear system or a nonlinear system with a unique equilibrium, if it is known that the system is initially at rest at $x(0) = X(0)$, it can be immediately concluded that under the control (7), the system output closely tracks its reference. However, this conclusion cannot be similarly made for nonlinear systems with multiple equilibria. The difficulty here is the possibility for existence of a trajectory $\bar{X}(s)$, $s \in [0, 1]$ with the property

$$\begin{aligned}&f\left(\bar{X}(s), U(s)\right) = 0\\ &g\left(\bar{X}(s), U(s)\right) \neq Y(s).\end{aligned}$$

In that case, if the system is initially at rest at $x(0) = \bar{X}(0)$, its output obviously can not track the desired reference.

C. Proof of Proposition 2

Multiplying both sides of (17) by $e^T(t)$ results in

$$\begin{aligned}\frac{1}{2}\cdot\frac{d}{dt}\|e(t)\|^2 = {}& e^T(t)\, F_x(X(\gamma(t)), U(\gamma(t)))\, e(t)\\ &- \dot{\gamma}(t)\, e^T(t)\, X'(\gamma(t)).\end{aligned}$$

Based on assumption that $\lambda(X(s), U(s))$ lower bounds the eigenvalues of (18) and using the Cauchy-Schwarz inequality with $\dot{\gamma}(t) > 0$, this equation yields the differential inequality

$$\begin{aligned}\frac{1}{2}\cdot\frac{d}{dt}\|e(t)\|^2 \leqslant {}& -\lambda(X(\gamma(t)), U(\gamma(t)))\, \|e(t)\|^2\\ &+ \dot{\gamma}(t)\, \|e(t)\| \cdot \|X'(\gamma(t))\|.\end{aligned}$$

Dividing both sides of this inequality by $\|e(t)\|$ and using the shorthand $\ell(t) = \lambda(X(\gamma(t)), U(\gamma(t)))$, it is expressed as

$$\frac{d}{dt}\|e(t)\| \leqslant -\ell(t)\, \|e(t)\| + \dot{\gamma}(t)\, \|X'(\gamma(t))\|.$$

This differential inequality is next solved on $t \geqslant 0$ for

$$\begin{aligned}\|e(t)\| \leqslant {}& \exp\left(-\textstyle\int_0^t \ell(\tau)\, d\tau\right) \|e(0)\|\\ &+ \int_0^t \exp\left(-\textstyle\int_\tau^t \ell(\xi)\, d\xi\right) \dot{\gamma}(\tau)\, \|X'(\gamma(\tau))\|\, d\tau.\end{aligned}$$

Substituting $\dot{\gamma}(\cdot)$ from (20) into the right-hand side of this inequality and applying the assumption $\|e(0)\| \leqslant \varepsilon$ lead to

$$\begin{aligned}\|e(t)\| &\leqslant \varepsilon \exp\left(-\textstyle\int_0^t \ell(\tau)\, d\tau\right)\\ &\quad + \varepsilon \int_0^t \exp\left(-\textstyle\int_\tau^t \ell(\xi)\, d\xi\right) \ell(\tau)\, d\tau\\ &= \varepsilon, \quad t \in [0, t_f].\end{aligned}$$

Since $\dot{\gamma}(\cdot)$ is positive by (20), $\gamma(\cdot)$ is strictly increasing. To show that $\gamma(t_f) = 1$ holds for t_f in (19), the differential equation (20) is rearranged and integrated on $[0, t_f]$ to obtain

$$t_f = \frac{1}{\varepsilon} \int_0^{t_f} \frac{\|X'(\gamma(t))\|}{\lambda(X(\gamma(t)), U(\gamma(t)))} \dot{\gamma}(t)\, dt.$$

By the change of integration variable $s = \gamma(t)$ and noting that $\gamma(0) = 0$, this expression can be rewritten as

$$t_f = \frac{1}{\varepsilon} \int_0^{\gamma(t_f)} \frac{\|X'(s)\|}{\lambda(X(s), U(s))}\, ds.$$

Comparing this result with (19) and noting that the integrands in both expressions are strictly positive imply $\gamma(t_f) = 1$.

III. Trajectory of Equilibria

This section covers three topics on construction of the trajectory of equilibria. First in Section III-A, application of the *homotopy continuation* techniques [5] in construction of this trajectory is considered. Section III-B establishes an optimal control framework for optimization of the trajectory of equilibria. Finally, Section III-C briefly discusses the case in which a nonlinear system admits multiple equilibria under the same constant control.

A. Application of Homotopy Continuation

A challenging step in implementation of quasistatic control for nonlinear systems is to obtain the trajectory of equilibria (and control in the case of output tracking) from the algebraic equations (16) or (21). Computation of this trajectory needs to repeat solving these nonlinear equations for all values of the parameter s varying in the continuum set $s \in [0, 1]$. Such heavy computation can be drastically simplified by means of homotopy continuation. Using this continuation technique the trajectory of equilibria can be constructed through solving certain differential equations with a boundary condition that solves (16) or (21) either at $s = 0$ or $s = 1$. Then, generating the entire trajectory requires solving (16) or (21) only once.

The homotopy continuation technique relies on a simple observation: if $X(\cdot)$ and $U(\cdot)$ jointly satisfy

$$\frac{d}{ds} f(X(s), U(s)) = 0, \quad s \in [0, 1] \tag{22a}$$

$$f(X(0), U(0)) = 0, \tag{22b}$$

it can be concluded that

$$f(X(s), U(s)) = f(X(0), U(0)) = 0, \quad s \in [0, 1].$$

Using the chain rule of differentiation, (22a) is written as

$$F_x(X(s), U(s)) X'(s) + F_u(X(s), U(s)) U'(s) = 0,$$

where $F_x(\cdot)$ and $F_u(\cdot)$ denote the Jacobian matrices of $f(\cdot)$ with respect to its first and second arguments, respectively. This linear algebraic equation is solved with respect to $X'(s)$ to obtain the differential equation

$$X'(s) = -F_x^{-1}(X(s), U(s)) F_u(X(s), U(s)) U'(s), \tag{23}$$

which can be solved on $s \in [0, 1]$ with the initial state (22b) in order to construct the trajectory of equilibria.

A major concern in homotopy continuation is the existence of solutions for the differential equation (23). A necessary condition for existence of a solution to this equation is that the Jacobian matrix $F_x(\cdot)$ must stay nonsingular along the entire trajectory of $(X(s), U(s))$, $s \in [0, 1]$. This necessary condition always holds under the assumption of Proposition 2 that requires (18) to be positive definite. Then, under a mild Lipschitz continuity assumption, (23) will admit a unique solution [6, Thm. 3.2].

For output tracking control, application of the homotopy continuation technique to the set of algebraic equations (21) results in a differential equation given in the compact form

$$Z'(s) = H(Z(s)) Y'(s), \tag{24}$$

where $Z(s) = (X(s), U(s))$ and $H(\cdot)$ is defined as

$$H(Z) = \begin{bmatrix} F_x(Z) & F_u(Z) \\ G_x(Z) & G_u(Z) \end{bmatrix}^{-1} \begin{bmatrix} 0_{n\times k} \\ I_{k\times k} \end{bmatrix}. \tag{25}$$

Here, $G_x(\cdot)$ and $G_u(\cdot)$ denote the Jacobian matrices of $g(\cdot)$ with respect to its first and second arguments, respectively. The solution to (24) for an initial state Z_0 solving $f(Z_0) = 0$ and $g(Z_0) = Y(0)$ generates both trajectories of equilibria and control over $s \in [0, 1]$. Certainly, a necessary condition for existence of this solution is that the inverse matrix in (25) must exist along the entire trajectory of $Z(s)$, $s \in [0, 1]$.

Construction of $Z(s)$, $s \in [0, 1]$ via solving (24) is not a real-time computation. The following proposition explains how (24) can be combined with (20) in Proposition 2 in order to generate the control (7) in real time.

Proposition 3: Let $Y(s)$, $s \in [0, 1]$ be any differentiable trajectory in the output space of (15) and assume that $X(\cdot)$ and $U(\cdot)$ solve (21) on $s \in [0, 1]$. Take $U(\cdot)$ and construct the control $u(t)$ in (7) based on $\gamma(\cdot)$ in Proposition 2. Then, this control can be computed in real time by solving the state-space equations (assuming the solution exists)

$$\dot{z}(t) = \varepsilon \lambda(z(t)) \frac{H(z(t)) Y'(\gamma(t))}{\|E_1 H(z(t)) Y'(\gamma(t))\|} \tag{26a}$$

$$\dot{\gamma}(t) = \frac{\varepsilon \lambda(z(t))}{\|E_1 H(z(t)) Y'(\gamma(t))\|} \tag{26b}$$

$$u(t) = E_2 z(t) \tag{26c}$$

on $t \in [0, t_f]$ with the initial state $\gamma(0) = 0$ and $z(0)$ solving the algebraic equations $f(z(0)) = 0$ and $g(z(0)) = Y(0)$. Here, E_1 and E_2 are matrices defined as

$$E_1 = \begin{bmatrix} I_{n\times n} & 0_{n\times k} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0_{k\times n} & I_{k\times k} \end{bmatrix}.$$

Proof: Define the state vector $z(t) = (X(\gamma(t)), u(t))$ and note that $z(t) = Z(\gamma(t))$, where $Z(\cdot)$ is the solution of (24) with the initial state $Z(0)$ that solves $f(Z(0)) = 0$ and $g(Z(0)) = Y(0)$. Replacing $s = \gamma(t)$ in (24) and multiplying its both sides by $\dot{\gamma}(t)$ result in

$$\dot{z}(t) = \dot{\gamma}(t) H(z(t)) Y'(\gamma(t)). \tag{27}$$

Observing from (24) that $X'(s) = E_1 H(Z(s)) Y'(s)$, (20) in Proposition 2 can be expressed as (26b), which is then substituted into (27) to obtain (26a). ■

B. Trajectory Optimization

Again consider the control of a nonlinear system aimed to transition its state from an initial equilibrium (X_0, U_0) to a final equilibrium (X_f, U_f). The problem then is to construct a continuous trajectory $(X(s), U(s))$, $s \in [0,1]$ to connect (X_0, U_0) to (X_f, U_f), and simultaneously, minimize the final time (19) in Proposition 2. The solution to this problem was given in Section II-A for linear systems as a straight line that connects (X_0, U_0) to (X_f, U_f). For nonlinear systems, the problem is formulated as the optimal control problem below.

In the differential equation (23), take $V(s) = U'(s)$ as a control input and $(X(s), U(s))$ as the state vector to rewrite it in the form of state-space equations

$$X'(s) = -F_x^{-1}(X(s), U(s)) F_u(X(s), U(s)) V(s)$$
$$U'(s) = V(s).$$

Then, subject to the boundary conditions

$$(X(0), U(0)) = (X_0, U_0), \quad (X(1), U(1)) = (X_f, U_f),$$

the goal is to obtain an optimal control $V(\cdot)$ that minimizes the cost functional

$$J = \int_0^1 \frac{\left\| F_x^{-1}(X(s), U(s)) F_u(X(s), U(s)) V(s) \right\|}{\lambda(X(s), U(s))} ds.$$

This cost functional clearly represents t_f in (19).

C. Systems with Multiple Equilibria

Suppose that the pairs $Z_0 = (X_0, U_0)$ and $Z_f = (X_f, U_f)$ hold $f(Z_0) = 0$ and $f(Z_f) = 0$ and assume that $x = X_f$ is the unique solution to the algebraic equation $f(x, U_f) = 0$. Consider the problem of constructing a continuous trajectory $Z(s)$, $s \in [0,1]$ to connect Z_0 to Z_f and hold $f(Z(s)) = 0$ on $s \in [0,1]$. This problem can be tackled by solving (23) with the initial state $X(0) = X_0$ and any differentiable $U(\cdot)$ that holds $U(0) = U_0$ and $U(1) = U_f$.

On the other hand, if the equation $f(x, U_f) = 0$ admits multiple solutions, say $x = \bar{X}_f$, an arbitrarily taken $U(\cdot)$ can generate a trajectory of equilibria that connects X_0 to $\bar{X}_f$. To ensure the trajectory indeed hits the desired equilibrium point X_f, a smart procedure for selection of $U(\cdot)$ is required. This section presents the core idea of an approach to develop such procedure, which of course is still at an early stage and needs more work to become operational.

Let $\phi(\cdot): \mathbb{R}^{n+k} \times [0,1] \to \mathbb{R}$ be a scalar function holding $\phi(Z_0, 0) = 0$ and $\phi(Z_f, 1) = 0$, and also $\phi(\bar{X}_f, U_f, 1) \neq 0$ for any $\bar{X}_f \neq X_f$ that solves $f(x, U_f) = 0$. Then, the set of equations consisting of $f(x, U_f) = 0$ and $\phi(x, U_f, 1) = 0$ admits a unique solution at $x = X_f$. Note that the algebraic equation (16) is underdetermined in the sense that it consists of n scalar equations satisfied by $n+k$ unknowns. Therefore, by appending a new equation to (16), the system of equations

$$f(Z(s)) = 0 \tag{28a}$$
$$\phi(Z(s), s) = 0 \tag{28b}$$

still admits solutions, albeit within a smaller solution family. This extended system of equations can be solved on $s \in [0,1]$ by means of homotopy continuation to construct a continuous trajectory connecting Z_0 to Z_f. Of course, the main question of how to construct $\phi(\cdot)$ is yet under investigation.

The system of equations (28) is typically underdetermined (when $k > 1$) with $n+k$ unknowns and only $n+1$ equations. Hence, it admits a parametric family of solutions as discussed next. Differentiating the equations in (28) with respect to s yields the underdetermined system of linear equations

$$F_z(Z(s)) Z'(s) = 0$$
$$\Phi_z(Z(s), s) Z'(s) = -\Phi_s(Z(s), s),$$

where $\Phi_z(\cdot)$ and $\Phi_s(\cdot)$ are respectively the Jacobian matrix of $\phi(\cdot)$ with respect to its first argument, and its partial derivative with respect to the second argument. This system of linear equations admits the family of solutions

$$Z'(s) = -\begin{bmatrix} F_z(Z(s)) \\ \Phi_z(Z(s), s) \end{bmatrix}^{\dagger} \begin{bmatrix} 0_{n\times 1} \\ \Phi_s(Z(s), s) \end{bmatrix} + q(s)$$

parameterized by the $(n+k) \times 1$ vector $q(s)$ constrained to satisfy $n+1$ constraints

$$\begin{bmatrix} F_z(Z(s)) \\ \Phi_z(Z(s), s) \end{bmatrix} q(s) = 0_{(n+1)\times 1}, \quad s \in [0,1].$$

Here, the superscript † denotes the Moore-Penrose inverse of non-square matrices.

IV. Conclusion

A control strategy was investigated under which dynamical systems evolve in time near an equilibrium moving slowly along a trajectory of equilibria. Application of this strategy was examined for two control scenarios: first, transitioning a system from an initial state to a targeted final state in shortest time, and second, output tracking control aimed at steering a system output along a reference trajectory within the output space. It was shown how these controls can be implemented based on partial knowledge of the system dynamics involving fewer parameters than the complete model of the system. For nonlinear systems, application of homotopy continuation in construction and optimization of the trajectory of equilibria was examined and concerns around the existence of multiple equilibria in these systems were partially addressed.

References

[1] N. Augier, U. Boscain, and M. Sigalotti, "Effective adiabatic control of a decoupled Hamiltonian obtained by rotating wave approximation," *Automatica*, vol. 136, p. 110034, 2022.

[2] N. Augier, *Adiabatic control of quantum systems*. PhD thesis, Université Paris Saclay, 2019.

[3] U. V. Boscain, F. Chittaro, P. Mason, and M. Sigalotti, "Adiabatic control of the Schrödinger equation via conical intersections of the eigenvalues," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 1970–1983, 2012.

[4] A. Komaee and P. I. Barton, "Potential canals for control of nonlinear stochastic systems in the absence of state measurements," *IEEE Transactions on Control Systems Technology*, vol. 25, no. 1, pp. 161–174, 2017.

[5] E. L. Allgower and K. Georg, *Introduction to Numerical Continuation Methods*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2003.

[6] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall, 3rd ed., 2002.