Accepted for Publication in the *Journal of Mathematical Behavior*
Published version may contain minor edits.

Operationalizing Authentic Mathematical Proof Activity Using Disciplinary Tools

Many educators advocate for students to engage in authentic mathematical activity – activity reflective of the discipline. In our respective design studies, we had a common goal of engaging students in authentic mathematical activity in relation to proof. However, we identified a need to better operationalize "authentic" and characterize student activity in relationship to the discipline in the undergraduate proof setting. We introduce the Authentic Mathematical Proof Activity (AMPA) framework as a theoretical tool for researchers interested in taking a multi-dimensional approach to documenting authenticity in students' proof-related activity. The framework provides both a means to deconstruct activity systems in terms of tools and objectives of the professional mathematician community and a set of dimensions (agency, authority, alignment, complexity, variety, and accuracy) to account for differing elements of authenticity related to both student and disciplinary aims.

Keywords: authentic mathematical proof activity, advanced undergraduate, disciplinary practice

1 Introduction

Learning and achievement in mathematics are often characterized in two ways (1) acquiring cognitive skills and beliefs and (2) participating in authentic mathematical activity (Sfard, 1998). The majority of research literature on students learning proof takes an acquisition lens (Stylianides et al., 2017) painting a rather deficit-view of students' engagement with proof, showcasing that students have not yet acquired many of these skills and beliefs. While there are some studies and scholarly reflections that take a participation lens on learning proof via attention to design heuristics (e.g., Larsen & Zandieh, 2008), discursive shifts (e.g., Nardi et al., 2014), or communities of inquiry (e.g., Biza et al., 2014), mathematics education researchers have only begun to understand how students participate in authentic mathematical proof activity and instruction that promotes it.

At the same time, mathematics education reform emphasizes a shift towards authentic mathematical activity that reflects disciplinary practice (e.g., Lampert, 1992). In the case of proof-based undergraduate classrooms, we see this shift as focusing on apprenticing mathematics majors into the work of professional mathematicians.[1] However, the common definition-theorem-proof paradigm in many proof classes is generally seen as rather *in*authentic to mathematical practice. When students are given a definition, theorem, then formal proof of the theorem, students are positioned to see proof as "a formal necessity required by the teacher" (Alibert & Thomas, 2002, p. 216) rather than a crucial part of a scientific process. Further, proofs presented following a step-by-step deductive argument may hide the non-linear formal and informal exploration and argumentation that often supports the eventual construction of a formal proof. Advocates of authentic mathematical activity focus on goals beyond just formal deductive proof (e.g., Lampert, 1992) such as Lakatos's (1976) analytical processes or Polya's (1954) problem-solving. There have been efforts (including our own) to reform proof-based courses to foster student participation in such authentic mathematical activity that becomes more sophisticated over time. Yet, existing approaches to analyzing in-the-moment classroom activity primarily foreground cognitive analogs (e.g., Wawro's (2014) use of taken-as-shared practices) or argumentation (e.g., Inglis et al.'s (2007) use of Toulmin's argumentation scheme).

We see research on students' engagement in, and instruction that promotes, authentic mathematical proof activity as constrained by a lack of specification of what it means to engage in authentic mathematical proof activity; how can we determine if students are participating in authentic mathematical activity without a framework to characterize it? We argue that such a framework would serve research on students learning proof in several ways. A framework would allow researchers to identify aspects of student proof activity that are authentic and link it to instructional moves that fostered this activity. This could in turn move towards engendering an asset-view of students' engagement with proof reflecting a multitude of competencies found in disciplinary activity. It could also inform instructional design to encourage authentic mathematical proof activity as well as offer a systematic way to assess the quality of such

[1] When referencing "mathematicians" in this manuscript, we refer to professional mathematicians who engage in mathematical research.

interventions. Finally, a framework that characterizes authentic mathematical proof activity would provide the field with common language that would afford opportunities for research to build on prior work in a cumulative manner.

The purpose of this paper is to introduce the *Authentic Mathematical Proof Activity* (AMPA) framework in order to analyze and describe authenticity of activity in the context of undergraduate proof-based settings. We leverage studies of mathematician activity (e.g., Inglis & Alcock, 2012; Mejía-Ramos & Weber, 2014; Weber, 2008) and design-research incorporating heuristics aligned with authentic mathematical activity (e.g., Dawkins, 2015; Larsen, 2013) to identify the activities of professional mathematicians that can be adapted in the undergraduate setting. Using activity theory, we decompose this activity into goal-directed actions in which *tools* serve to mediate the activity as subjects work towards *objectives* (motive-object pairings). Tools and their use reflect an "accumulation and transmission of social knowledge" (Kaptelinin et al., 1999, p. 32) from the historical context of their development. As such, we position tools as the lynchpin to connect student activity with the activity of professional mathematicians.

In the remainder of this paper, we first provide an overview on our conceptualization of authenticity, connecting it to other uses of the term (section 2.1) and then introduce how activity theory guided our work (section 2.2). In section 3, we introduce the Authentic Mathematical Proof Activity (AMPA) framework, which provides a means for two layers of analysis. First, the AMPA framework can be used to deconstruct the activity systems of students' and instructors' (as well as mathematicians') proof-related activity in terms of motives, objects, and tools. We selected the disciplinary tools and objectives from a careful review of relevant literature (discussed in section 3.1). Second, the AMPA framework can be used to analyze these activity systems through the lens of authenticity via the constructs of validity, complexity, accuracy, agency, authority, and alignment (elaborated in section 3.2). In section 4 we present two illustrations of using the framework to analyze data from two different contexts (real analysis and modern algebra) with students who have different mathematical backgrounds. We conclude the paper in Section 5, discussing how the AMPA framework provides insight into students' activity in proof-based settings and provides a means to make claims about student participation in authentic mathematical proof activity.

2 Background and Theoretical Orientation

We take the perspective that in higher education courses students are to be apprenticed into a community of practice (Lave & Wenger, 1991) reflective of the discipline. In our projects, we aim to promote advanced undergraduate mathematics classrooms that enculture students into the practices of mathematicians[2] (e.g., Gueudet, 2008; Selden, 2012) by engaging students in activities using disciplinary tools (cultural artifacts including concepts and processes) towards disciplinary objectives. Broadly, we term students engaging in disciplinary activity as "authentic." We take the stance that authentic experiences do not need to be perfect replicas of professional settings, but rather share "cognitive realism" (Herrington et al., 2014; Radović et al., 2021). That is, as articulated by Tochon (2000), "[c]lusters of features derived from disciplinary genres may be assembled in premises that may provide the basis for genuine disciplinary experiences" (p. 357). It is not the contexts that must reflect the discipline, but rather the intellectual work. In this section, we provide a literature-grounded overview of authentic activity and introduce activity theory as a lens for analyzing participatory learning from this viewpoint.

2.1 What is Authentic Mathematical Activity?

Across research paradigms within undergraduate proof settings, we found that while researchers invoke the language of authenticity,[3] the construct itself is frequently left implicit. Howell and Mikeska (2021) argued not only that there is a need to better attend to the term "authenticity" in the literature as authenticity "is unlikely to function as a simple continuous descriptor ranging from less to more authentic" (p. 17). While the literature on authenticity is rather substantial, we focus this discussion specifically on scholarly work that provides insight into what might be observable in student-teacher activity and relevant to the proof-based context. We exclude discussion of task planning and evaluating the impact of authentic learning environments (not observable student-teacher activity) or relationships to real-world contexts (not relevant to proof). In the remainder of this section, we will elaborate on the set of distinctions between (1) discipline and student and (2) content and practice.

2.1.1 The Discipline and the Student. The general commonality across the authentic learning literature base is an emphasis on students engaging in activity that reflects professional practice in the discipline (Herrington et al., 2014). Radović et al. (2021) elaborated, "authentic

[2] Also referred to as disciplinary practice; see Rasmussen et al. (2015).

[3] e.g., *anthropological theory of the didactic* (ATD): "authentic mathematical praxeologies", (Winsløw et al., 2014, p. 108); *inquiry-oriented instruction* (IOI): "authentic mathematical activity", (Kuster et al., 2019, p. 187); *inquiry-based learning* (IBL): "authentic mathematical inquiry" (Dawkins et al., 2019, p. 316).

learning happens when learners use professional tools, knowledge and skills, when [learners] imitate behaviour of experts and develop relevant outputs" (p. 2711). Such an approach emphasizes professional practice as the defining element of authenticity and is common within the mathematics subject-area drawing on the activity of research mathematicians (e.g., Watson, 2008; Weber et al., 2020). While this provides one means to discuss authenticity, we suggest it under emphasizes the interaction between the students and the disciplinary activity.

In contrast, another group of scholars have emphasized both a discipline dimension and student dimension of authenticity. Tochon (2000) argued that planned disciplinary activity only materializes into authentic experiences for students through intersections between the discipline and the students' situated and prior experiences. Stein et al. (2004) took a similar stance, focusing on both personal meaning and purpose in combination with authenticity to a professional community of practice.

In the context of mathematics, we can coarsely divide two authenticity goals:

- Staying authentic to mathematical disciplinary activity (discipline)
- Staying authentic to students' mathematical communication, activity, and thinking (student)

The dual authenticity goals can lead to tensions for teachers attempting to support authentic learning experiences. In Lampert's (1992) reflection, she identified this tension as one between "being authentic (that is, meaningful and important) to the immediate participants and being authentic in its reflection of a wider mathematical culture" (p. 310). Herbst (2002) refers to the tension between students engaging in authentic mathematical activity and the need to progress in normative ways while teaching proof as a *double bind*. In the undergraduate proof setting, others have expanded similar double binds such as "supporting success for all students and authentic mathematical activity" (Dawkins et al., 2019, p. 331). The tension between staying authentic to students and to the discipline may be amplified by the demands of the formal proof that require substantial shifts in language and argumentation practices (Weber & Melhuish, 2022).

2.1.2 The Content and Practice. A number of scholars have treated multidimensionality as more nuanced than just the discipline and student divide. In this section, we share an additional divide within both student and disciplinary authenticity that was relevant and insightful for explaining instructional tensions: content and practice. We first elaborate on this distinction relative to the discipline (a distinction found in Weiss et al.'s (2009) study of teachers). Mathematical content can be more or less accurate in terms of “definitions, language, concepts, and assumptions” of the discipline (Chazan & Ball, 1999, p. 7). From this viewpoint, the more accurate and reflective of the discipline the mathematical content is, the more authentic. In contrast, the practice dimension reflects the degree to which the “ practices and habits of working mathematicians” (Weiss et al., p. 277) are reflected in the classroom. That is, it is not about the accuracy of statements or definitions, but rather that the ways “for testing ideas, for establishing the validity of a proposition, for challenging an assertion” (Chazan and Ball, p. 7) align with those of the research mathematician. The focus on disciplinary practice is more heavily emphasized in the authenticity literature; however, accuracy of content is often a major consideration when attending to student contributions.

Across the literature, we similarly observed many ways that the student component of authenticity can be conceptualized, ranging from emerging professional identity (e.g., Sutherland & Markauskaite, 2012), alignment with purposes (e.g., Stein et al., 2004), or relevance to student lives (e.g., Strobel et al., 2013). For our exploration, we identify a parallel practice-content distinction relative to students: the degree to which students are engaged with disciplinary practice and the degree to which content in a classroom reflects student contributions. Authentic student practice would mean that students primarily make decisions and shape the mathematical explorations. Such authenticity necessitates open and collaborative tasks, deemed as an essential component of authentic learning experiences by a number of scholars (Herrington et al., 2014; Strobel et al., 2013). Authentic student content would then be reflected by the centrality of student-contributed mathematical content (e.g., definitions, propositions, proofs) in a lesson. Such student-generated content may or may not be accurate to disciplinary content and lead to fundamental tensions in how to proceed in instruction (see Ball, 1993 for such an instance).

2.1.3 Summary. We focused on these distinctions – that of students and discipline, and that of content and practice – because they were salient to student-teacher activity and particularly the tensions at play related to authenticity. In the next section, we use activity theory and literature on mathematician activity to operationalize an analytic framework to account for discipline, student, content, and practice elements of authenticity.

2.2 Mathematician and Student Activity Systems: A Participatory Learning Perspective

To operationalize the socially constituted activity of mathematicians and students and situate our larger view on learning, we turn to cultural-historical activity theory (Engeström, 2000). Both professional mathematicians and students operate within activity systems, which relate an individual's goal-driven actions to how communities work together when they share a common objective. Goal-driven actions can be deconstructed into subjects, objects, tools, norms, community, division of labor, and outcomes. The object, with an embedded motive, underscores the objective of the particular action made by the subject. Because researchers diverge on whether objects and motives are distinct constructs (Blunden, 2009), we use the term objective to capture the object/motive pairing. Mathematicians share a common objective that can be decomposed into object (mathematical theory) and motive (building this theory, see Bass, 2017). Then more specifically, a mathematician (subject) may be working with a proof (object) for a particular motive -- perhaps engaging with a proof for the motive of comprehending it. Tools then mediate the relationship between the individual (subject) and object, including material tools, mental concepts, procedures, or other culturally situated means for a subject to transform an object. Finally, goal-driven actions are shaped by the community's norms and rules and the division of labor between community members as they work towards a goal, resulting in an outcome.

Participatory learning can be documented through changes or expansions (Greeno & Engeström, 2014) and appropriation of tools (Grossman et al., 1999; Nelson & Kim, 2001) in activity systems. Expansion involves increased variety and complexity of tool use as new configurations become acceptable in the community and former goal-directed action outcomes become tools for continued mathematical activity (e.g., Nelson & Kim, 2001). Students' objectives (and activity systems) can expand to be more reflective of the discipline, such as proof construction objectives that are not limited to the application of syntactic strategies (e.g., Weber & Alcock, 2004). Tool appropriation (Grossman et al., 1999; Nelson & Kim, 2001) reflects students' increased role in the division of labor through selecting, generating, and using tools. Thus, participatory learning can be conceptualized as students engaging in activity that resembles mathematicians' activity, and learning occurs over time as students appropriate tools, expand objectives, and use outcomes from prior activity as tools towards new objectives.

3 The Authentic Mathematical Proof Activity Framework

Table 1. Authentic Mathematical Proof activity (AMPA) Framework

Tools	Objects	Motives

• Analyzing/ Refining • Formalizing • Deformalizing • Warranting • Analogizing/ Transferring	• Examples • Diagrams • Logic Structures/ Frameworks • Existent PSC Objects	• Proofs • Statements • Concepts	• Explore • Test • Construct
Authenticity in Tool Use Characteristics			
Discipline Tool Use Characteristics (Practice)	Discipline Tools and Outcomes (Content)	Student Role in Division of Labor (Practice)	Student Tools and Outcomes (Content)
• Variety • Complexity	• Accuracy	• Agency • Authority	• Alignment

In this section, we introduce the Authentic Mathematical Proof Activity (AMPA) framework. Our operationalization of authentic mathematical proof activity hinges on disciplinary tools and objectives and the framework can be decomposed into two levels: a set of tools (and objectives), and an examination of tool use in activity systems to address authenticity dimensions. The first theoretical contribution of this paper is the identification of AMPA tools and objectives (the top portion of the framework in table 1). These are tools that can be both empirically linked to the activity of mathematicians and have the potential to be used with undergraduate students in proof-based classrooms.

There have been recent concerns about mathematics educators' characterization of mathematician activity (Weber et al., 2020), including relying on mathematicians' possibly idiosyncratic reflections (Hanna & Larvor, 2020) or potentially artificial task-based interviews (Mejía-Ramos & Weber, 2020). Addressing such concerns, we conducted a thorough literature search to identify tools that have been linked with multiple empirical studies of working mathematicians' practice. We began this process with a full text search, using Google Scholar to search texts of high- and very high-quality mathematics education journals (Williams & Leatham, 2017) and the *International Journal of Research in Undergraduate Mathematics Education*, for studies that included mathematicians, "authentic mathematical activity" or common paradigms aimed at engaging students in authentic mathematical activity ("inquiry-oriented instruction" and "inquiry-based learning") in combination with the word "proof." This process led to the initial creation of a set of themes found in literature on both mathematicians and student activity. We then conducted an exhaustive search for empirical research on mathematicians by broadly searching EBSCO for articles with "mathematician" in the title and abstracts that included our mathematical objects (proofs, definitions, propositional statements). This led to identifying work in journals of practice and philosophy such as *Synthese* and *Philosophia Mathematica*, resulting in the identification of 48 relevant papers. From these papers, we identified objects, motives, and tools used by the mathematicians in their work. We also expanded our search parameters for student literature to include "cognitive unity"[4] and "anthropological theory of didactics"[5] to account for international paradigms. We worked reflexively between literature on mathematicians and literature on undergraduate students purported to be engaged in authentic disciplinary activity to identify a set of objectives (objects, motives) and tools found across both bases of literature.

To focus our efforts, our final set of tools satisfy three criteria:

1. documented in empirical studies of mathematician practice,
2. used in relation to at least two objects and all three types of motives, and
3. linked to student activity in undergraduate mathematics.

3.1 Disciplinary Tools and Objectives

Our framework includes three objects: *proofs* (proofs and arguments purported as proofs), *concepts* (mathematical concepts including, but not limited to, definitions), and *statements* (theorems/propositions that relate mathematical concepts) (cf. Dawkins, 2015). These objects can then be paired with the following motives:

- *Exploring* involves making sense of a mathematical object. This corresponds to comprehending a proof (e.g., Mejía-Ramos & Weber, 2014), understanding a definition/concept (e.g., Parameswaran, 2010), or understanding a statement/proposition (e.g., Lockwood et al., 2016). The observable outcome would be an interpretation.
- *Testing* involves determining the validity of a mathematical object. This corresponds to validating a proof (e.g., Weber, 2008), validating a definition (e.g., Ouvrier-Buffet,

[4] Cognitive unity is derived from the work of mathematicians where informal argumentation and exploration is often aligned in the proving process (see Garuti et al., 1998).
[5] Researchers in this paradigm often focus on the transposition of the discipline into the classrooms with attention to how the university institution imposes constraints (see Winsløw et al., 2014).

2015), or testing the truth of a statement/proposition (e.g., Lockwood et al., 2016). The observable outcome would be an evaluation.

- *Constructing* involves creating a new mathematical object. This corresponds to constructing a proof (e.g., Savic, 2015), creating a new definition/concept (e.g., Martín-Molina et al., 2018), or conjecturing a new statement/proposition (e.g., Smith & Hungwe, 1998). The observable outcome would be a new object.

The ten tools that mathematicians used to meet such objectives can be found in Table 1. A single goal-directed action can be decomposed into a triad of *tool(s), object*, *motive* such as: [formalizing] an [example] (tools) in service of *constructing* (motive) a *statement* (object) to achieve an outcome (theorem, lemma, or other propositional statement). Note, the set of tools in Table 2 is not exhaustive; rather it is representative of tools we identified as fundamental and transferrable to undergraduate proof-based settings based on our literature review criteria.

Table 2. Mediating Tools in Mathematician Activity

Tool	Description	Example
Analyzing/ Refining	A process of analyzing and/or refining an object via attention to the strength and consequence of assumptions.	Starting with a statement and changing assumptions to build a stronger statement (e.g., Fernández-León et al., 2020).
Formalizing	A process of translating informal ideas into symbolic or formal rhetoric form.	Syntactifying a noticing from a diagram (e.g., Samkoff et al., 2012).
Deformalizing	A process of translating an object from formal rhetoric form to informal form.	Summarizing the point of a portion of a proof (e.g., Fang & Chapman, 2020).
Warranting	A process of inferring why a particular claim is true based on the provided premises.	Determining why a particular line in a proof is valid (e.g., Weber, & Mejía-Ramos, 2011).
Analogizing/ Transferring	A process of importing an object across domains and adapting to the new setting.	Transferring a proof technique to prove a new statement (e.g., Mejía-Ramos et al., 2012).
Examples	A specific, concrete instantiation of an object representing a class of objects.	Using examples to test whether a statement is true (e.g., Lynch & Lockwood, 2017).
Diagrams	A visual representation of an object that captures structural features.	Creating a graphical representation to find a key idea for a proof (e.g., Samkoff et al., 2012).

Logic	The rules of logic which allow for precisely quantified statements. and deductive arguments.	Creating a deductive subproof while validating (e.g., Weber, 2008).
Structures/ Frameworks	A top-level structure for a proof (or modular section of a proof) which is determined by statements to be proven.	Determining if a proof approach is valid by testing the alignment of a statement and proof (e.g., Weber, 2008).
Existent Objects	Objects that are accepted as valid in the community.	Using an existent definition of a concept to understand a new statement (e.g., Wilkerson-Jerde & Wilensky, 2011).

Beyond identifying a set of tools, we also explored this literature to identify themes in how mathematicians use these tools with particular attention to how this might differ from how novice provers use these tools. We share a summary of Fernández-León et al.'s (2020) case study of a mathematician's conjecturing and proof activity in order to illustrate our three overarching themes (complexity, variety, accuracy) that emerged from the literature related to mathematicians' activity systems. We emphasize our [tools] and *objectives* throughout the following instantiation and note that this is a high-level summary of the article's contents.

> A mathematician (and their colleagues) began with a statement [existent object] found in a publication, "all complete CAT[6](0) spaces satisfy the (Q4) condition" (p. 773) with the objective of developing a new theorem, *constructing a statement*. They [analyzed/refined] the statement through the process of exploring [examples] to arrive at a new, stronger statement [formalized] as: "every CAT(0) space has property (Q4)" (p. 774). They then *tested the statement* with additional [examples] aided by a [diagram] which led to rejecting the statement, and [analyzing/refining] to *construct a new statement* with altered assumptions: "any CAT(0) space with constant curvature satisfies the (Q4) condition" (p. 12). The mathematicians then *tested this statement* in a new [example] producing a proof of the Q4 condition being met (using [logic] and [framework]) in a specific context, spheres. This proof then served as a generic example

[6] A CAT(0) space "is a geodesic space for which each geodesic triangle is at least as 'thin' as its comparison triangle in the Euclidean plane" (Kirk & Panyanak, 2008, p. 3689.)

that could be [analogized] to the more general statement in service of *constructing the proof*.

We now turn to the three literature themes exemplified in the example. First, mathematicians use a *variety* of types of tools to meet objectives. In the above case, the mathematicians did not exclusively work with formal tools as they constructed and proved a new statement. Rather, they used informal tools (examples), translating tools (moving between formal and informal) and generating tools (weakening conditions of a statement or analogizing a proof to produce a new object). This type of tool use contrasts findings about typical student activity in several ways. Unlike novice students[7] who may hold empirical proof schemes (Harel & Sowder, 1998) or limit their proof production to syntactic manipulation (e.g., Weber & Alcock, 2004), mathematicians often use informal tools in the process of arriving at a formal outcome (e.g., Karunakaran, 2018; Lockwood et al., 2016; Samkoff et al., 2012; Weber & Alcock, 2004). Further, different tools including those to translate and generate are used in spontaneous ways not found in typical student activity (e.g., Lynch & Lockwood, 2017). Therefore, we suggest the existence and use of a broad variety of tools characterizes more disciplinary mathematical activity.

Second, mathematicians use tools in *complex* ways – in conjunction and succession towards an objective. For instance, the mathematicians in the above case study used [analyzing/refining] with [examples] to detect patterns in needed assumptions (tools in conjunction). Then, they generated additional [examples] and accompanying [diagrams] as they *tested the statement* (tools in succession). Further, when mathematicians arrived at an outcome, it became a tool for additional mathematization. That is, prior objects become tools to construct, test, or understand new objects. In the above case, the mathematicians analogized their [previous proof] to *create a new proof* (any CAT(0) space with constant curvature satisfies the (Q4) condition). In contrast, novice students often proceduralize proofs or focus on using one tool towards their objectives. For example, Zazkis et al. (2016) found most students did not use a diagram in their proof approaches, and those who did struggled to translate between the diagrams and a formal proof. Lynch and Lockwood's (2017) investigation of examples echoes similar results as students were less likely to use examples with other tools, such as the logic of statements, when compared to mathematicians. Karunakaran's (2018) expert-novice study

[7] See Weber (2010) for a challenge on the prevalence of empirical proof schemes.

similarly highlighted novice students' adherence to linear deduction. We posit that disciplinary mathematical activity may be characterized by the use of multiple types of tools towards a single objective and shifting objectives so that prior outcomes become tools for continued mathematical work.

Finally, we discuss the *accuracy* of the tools and outcomes in mathematicians' activity, a theme often left implicit in the reviewed papers. An overarching goal of mathematicians' activity is to discover, understand, and prove what is mathematically valid according to the standards of the mathematical community, that is, to arrive at valid outcomes. Dawkins and Weber (2017) noted that, "developing and justifying theories" includes "definitions, theorems, proofs, examples, and algorithms, in addition to the shared conceptual tools that mathematicians use to understand, discuss, and reason about these concepts" (p. 124). The objects and tools used by mathematicians come from a shared body of propositional statements, theorems, proofs, and practices. In the example above, the mathematicians began with a statement established in the literature, then did substantial work to create and verify a new statement by using tools and objects acceptable to the larger mathematical community. Accuracy is an expected and well-documented (e.g., Selden & Selden, 2003) point of divergence between novice and expert provers.

3.2 Authenticity Dimensions

In this section, we describe how dimensions of authenticity can be discerned in goal-directed actions through tool use. We focus on tools and objectives as they have the potential to exist across disciplinary and educational systems (e.g., Nolen et al., 2020). These dimensions can be broadly tied to the student/discipline and content/practice divides previously discussed (sections 2.1.1-2). We moved from these coarse divides to the six dimensions via a reflexive process working with data from our respective studies (described in section 4) and the literature. Of note, we found documenting students' engagement in tool use largely insufficient to make claims about students engaging in disciplinary practice. This led to revisiting the mathematician literature to identify the themes in how they used tools: complexity and variety. Similarly, we found that broadly citing student use of tools did not sufficiently account for qualitative differences in our data. Returning to the activity theory framing, we considered elements of division of labor in order to make some qualitative distinctions (between who generates and who evaluates tools and outcomes). We elaborate each of the six dimensions below.

We propose three complementary aspects of teacher-student activity systems related to discipline authenticity: the types of tools used in proof-related objectives (variety), how tools connect with each other and objectives (complexity), and the accuracy of the tools and outcomes in relation to disciplinary standards (accuracy). The first two are elements of practice that we documented across the mathematician literature base, and the latter is an element of content. Variety reflects the use of different types of tools. High variety would include tools that are informal (such as examples), formal (such as logic), tools that might translate between them (such as summarizing), and tools that lead to generating new mathematics (such as analogizing). Complexity reflects how tools are used together to achieve objectives, rather than tools being used in isolation. Complexity is also reflected in the ways that prior objects (such as a theorem or proof) can become a tool used towards a new objective. Accuracy refers to the degree tools and objects in an activity system would be considered conventionally correct in the discipline.

We can then consider activity in terms of student authenticity. First, authenticity of students' practice can be reflected in the division of labor (agency and authority). By agency we mean who generates and uses various tools and who makes decisions in that process. This parallels the degrees of freedom in a task. If students are provided an objective and otherwise left to their own devices, they would have high agency (many degrees of freedom) to generate and use tools. If an instructor generates and uses all the tools, then agency would be low (no degrees of freedom). However, much activity falls between these extremes with the instructor perhaps prompting for a type of tool (e.g., create some examples) while giving students agency in their creation, or providing a tool and allowing students to decide when and how to use it. The other salient characteristic in division of labor is authority. By authority we mean who evaluates tools and objectives to determine whether an objective is, or will be, met. This could involve attention to whether this is an appropriate tool type to meet aims (e.g., how this tool helps us get to our objective) or whether a particular tool or outcome is accurate. Finally, we can consider student authenticity in terms of content (alignment). By alignment we mean the degree to which tools and objects that are taken up in the activity system reflect student contributions. This could range from complete adherence to a student contribution to instructor-researcher contributions being the only ones endorsed. As in other categories, there is substantial room for middle ground such as a student contribution being centered, but an instructor modifying or formalizing it.

These six interconnected components – variety, complexity, accuracy, agency, authority, and alignment – account for differing dimensions of authenticity and promote a shift away from a binary or continuum view of authenticity. Our intention is not to suggest that proof-based classrooms should include activity systems that maximize authenticity in all dimensions at all times. Typically, the division of labor in traditional undergraduate courses positions students as responsible for taking notes and answering largely closed-form questions while the instructor presents definitions, theorems, and formal proofs accompanied by informal verbal explanations, with the instructor maintaining agency and authority over tools and their use (e.g., Artemeva & Fox, 2011; Paoletti et al., 2018; Weber, 2004). We argue that an activity system more authentic to students would include students having more agency and authority in the division of labor (see, David & Tomaz, 2012; González & DeJarnette, 2012). However, we emphasize that the instructor maintains a vital role managing dimensions of authenticity with their pedagogical knowledge and knowledge of the social and historic use of disciplinary tools.

4 Analysis of Authentic Mathematical Proof activity in Teacher-Student Activity Systems

Our work developing authenticity dimensions combined with identifying tools mediating mathematician activity culminated in the framework found in Table 1. The framework is intended as an analytic tool to document activity and address dimensions of authenticity. In this section, we illustrate how this framework provides insight into activity and authenticity by drawing on two diverse research projects with a common goal of engaging students in authentic mathematical proof activity. The focal episodes are situated in real analysis and algebra contexts, with students engaged in defining prior to proving and analyzing/refining a statement after proofs were constructed, respectively. We share these episodes because they not only illustrated diverse settings, they also exemplified differing authenticity characteristics and important moments in which authenticity changed. For each setting, one author classified student activity according to the framework and another served as an additional reader to challenge or endorse interpretations. The analysis process was two-fold: first attending to the tools and objectives, then attending to the six authenticity dimensions. Through discussion, the researchers arrived at the shared interpretations provided in this section. Throughout these illustrations we continue to emphasize [tools] and *objectives*.

4.1 Real Analysis Illustration

The following illustration comes from a pilot study for the *Advancing Students' Proof Practices in Mathematics through Inquiry, Reinvention, and Engagement* (ASPIRE) project which draws on the instructional design theory of Realistic Mathematics Education. The main goal of the study was to refine an instructional theory for supporting students in reinventing concepts in real analysis. The two participants, Chloe and Gabe (pseudonyms), were recruited from the last two courses of an undergraduate calculus sequence and had yet to take any proof-based courses. The data presented here comes from approximately one hour of a teaching experiment session[8]. The overarching goal of this session was for students to prove $\{1/2^n\}$ converges to 0. A substep in this process was to *construct a proof* that the sequence $\{2^n\}$ tends to infinity. The selected episodes describe the instructor-researcher and student activity as they worked to *construct a definition* for a sequence tending to infinity needed to make progress on proving $\{2^n\}$ tends to infinity. The instructor-researcher launched this portion of the activity by explaining students would propose defining properties and generate [examples and non-examples] that would later serve the objective of *testing their definition* by [analyzing/refining] whether the defining properties were of sufficient strength, ultimately [formalizing] their definition.

We selected the episodes for this illustration because they demonstrate how the instructor-researcher balanced student authenticity and disciplinary authenticity. The instructor-researcher took on more authority and some agency to promote authenticity to the discipline checking for accuracy in student contributions and prompting students to use a variety of tools in conjunction toward their larger *construct the concept definition* objective. At the same time, the activity was mainly focused on the students' contributions, reflecting high alignment. Further, there was evidence of expansion as prior objects became tools for new objectives. We note that the student authenticity dimensions remained steady throughout these excerpts while the discipline dimensions (variety and complexity) increased.

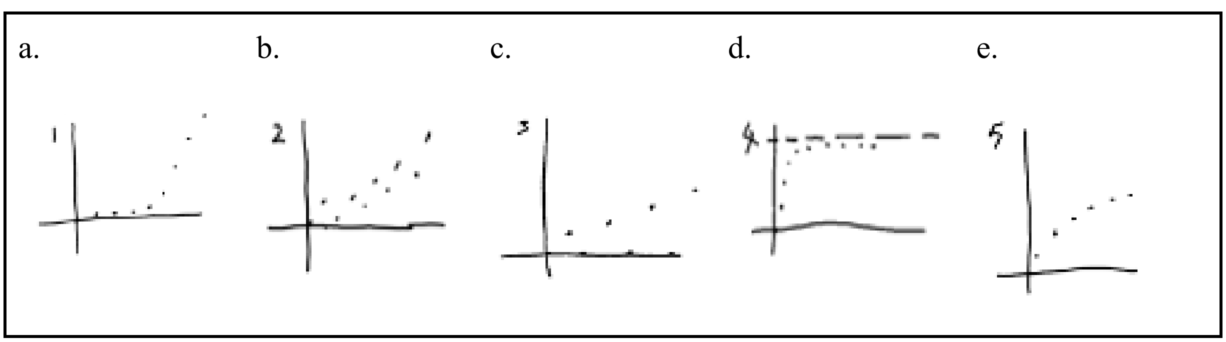
4.1.1 Using Examples/Diagrams to Explore a Concept: Sequence Tending to Infinity. In this episode, to engage students in *constructing a definition* for a sequence tending to infinity, the instructor-researcher prompted the students to first *explore the concept* to develop a set of [examples and non-examples] of sequences that tend to infinity (see Figure 3).

---

[8] See Vroom (2020) for additional information about this study.

While Chloe and Gabe generated [examples and non-examples], the instructor-researcher attended to inaccuracies and amplified certain contributions. For instance, as Chloe created a continuous function defined on the real numbers that increased without bound, the instructor-researcher questioned, “it definitely looks like a thing that goes to infinity, but is it a sequence?” Gabe responded, using his [existent concept] of sequence that it should have “discrete inputs” and altered the [diagram[9]] accordingly. Gabe initially sketched a sequence that was unbounded above with a constant subsequence at 0 (Figure 3c), but erased it, explaining he did not know how to deal with that yet. The instructor-researcher endorsed this [non-example] by commenting on the utility of cases “close to the boundary.” The division of labor shifted slightly as the students continued *testing the concept* by debating and eventually coming to an agreement that it “didn’t really approach anything” even after the instructor-researcher evaluated it as a non-example. Upon the instructor-researcher’s request, the students continued to *explore the concept* by considering their collection of [examples and non-examples] to generate additional “borderline” [examples and non-examples].

Throughout this episode, authenticity was relatively low regarding students’ use of tools which were informal and relatively isolated (variety and complexity), yet high with respect to centering student contributions (alignment). Other student authenticity dimensions were moderate as the students created a collection of examples and non-examples with instructor-researcher prompting (agency) and the instructor-researcher explained how their collection would be used to achieve the objective (authority). There was some indication of shared authority as the students continued to debate whether a particular non-example was in fact a non-example since both students and the instructor-researcher were evaluating tools in relation to the objective. However, the instructor-researcher maintained the role of attending to the accuracy of student contributions, which constrained student authority to some extent.



[9] A visual example is a diagram tool when usage reflects attention to structure embedded in the visual.

Figure 3. Students' collection of examples and non-examples of a sequence that tends to infinity

4.1.2 Using Examples, Formalizing, and Prior Definition to Construct Definitions: Tends to infinity and No Upper Bound. In this episode, the instructor-researcher shifted the objective to *constructing a definition* of a sequence that tends to infinity by prompting the students to define a "condition in which all the [examples] meet but the [non-examples] don't" and emphasized that they would [refine] their condition using their previously generated [examples and non-examples] (Figure 3). The following activity between the students and instructor-researcher reflects the instructor-researcher maintaining authority while promoting some student agency as the students proposed several defining properties which the instructor-researcher challenged for the sake of accuracy.

After the instructor-researcher focused the students' attention on a [non-example] (Figure 3d) and [example] (Figure 3e), the students explained that the [example] "has no upper bound" while the [non-example] does. With prompting, Gabe continued to display agency by using another [non-example] (Figure 3c) to *test the condition* 'no upper bound,' suggesting that it was a sequence that did not have an upper bound and did not tend to infinity. The instructor-researcher suggested they *construct a definition* for 'no upper bound'[10] to then [analyze/refine] the definition to "rule out" the [non-example]. After some prompting, Gabe used Figure 3d to explain that if a sequence had an upper bound then "there would be some constant that [it] would be less than or equal to." The instructor-researcher requested they [formalize] the property, which led the students to produce Figure 4a. After the instructor-researcher requested they use their [existent definition] to define 'no upper bound,' Gabe explained the definition would be "the opposite of that statement [Figure 4a], a_n is greater than K." He then [formalized] the property with support from the instructor-researcher who followed-up with questions to address language usage that violated the larger mathematics communities norms, thereby increasing the accuracy of the outcome (eventually producing Figure 4b).

This episode evidenced that the authenticity of the students' activity increased relative to the disciplinary practice dimensions compared to the activity described in the first episode. The

[10] This instructional decision is similar to Chorlay's (2019) approach to supporting students in defining the infinite limit for sequences.

students used both informal and translating tools (variety) together to create a definition of upper bound (complexity). While their upper bound definition had some ambiguity (e.g., *n* was not quantified), they were able to use this object as a tool to construct a new definition for ‘no upper bound’ (complexity). The instructor-researcher played an integral role in increasing authenticity to disciplinary practice by taking on some authority to link the tools to the objectives (e.g., use the upper bound definition to construct a no upper bound definition) and guided the students to use and generate the tools (agency). At the same time, their contributions remained central to the activity (alignment).

For the remainder of the session, the students continued to reflexively use [examples] and [diagrams] to [analyze/refine] their prior properties as [existent objects] (Figure 4b and 4c) to specify sequences that tend to infinity. The instructor-researcher maintained authority while continuing to share agency by encouraging students to make refinements and challenging contributions as needed to address inaccuracies. By the end of the session, they constructed the following definition: “$\{a_n\}$ converges to infinity if for any real number K there exists an a_m such that $a_m > K$ and $a_{m+n} > K$ where n is any positive integer”.

a.	has upper bound $a_n \leq K$ where k is some constant real number	has upper bound $a_n \leq K$ where K is some constant real number
b.	no upper bound for any real number, K there is an a_n that is greater	no upper bound for any real number, K there is an a_n that is greater
c.	[for any real number, K] if $a_x > K$ then $a_{x+n} > K$ where n= any positive integer	[for any real number, K] If $a_x > K$ then $a_{x+n} > K$ where n=any positive integer

Figure 4. Students’ defining properties

4.2 Modern Algebra Illustration

The second episode comes from a cycle of a designed-based research project, *Orchestrating Discussion Around Proof (ODAP)*, which focuses on using K-12 pedagogical practices to promote student engagement in proof activities. The episode corresponds to the middle thirty minutes of a two-hour task-based interview session. The four participating students, Miguel, Andy, Jasmine, and Eric, were recruited based on their recent completion of an introductory undergraduate abstract algebra class. The session centered the theorem: *Let G and H be isomorphic groups. If G is abelian, then H is abelian.*[11] This episode focuses on a portion of the task session with the overarching goal of *constructing a statement* from the initial theorem that remains valid, but does not contain any unnecessary assumptions (in this case, the groups need not be isomorphic, rather they need an onto homomorphism between them to preserve the commutativity).

We selected this illustration for two reasons. First, unlike in the analysis illustration, the student authenticity dimensions shifted dramatically at different points. Second, it is an episode that illustrates an appropriation of a tool which can be used to support participatory claims. Prior to this point in the lesson, the students had been prompted to explore the theorem (including identifying assumptions and conclusions, and stating definitions of key terms such as isomorphic), begin proof construction, then compare two student proof approaches to identify similarities and differences. The comparison task led students to engage with a variety of tools to *explore the proofs*. The students were then prompted to decide whether all of the assumptions in the statement were needed [analyzing/refining] by using the proof. The students had suggested replacing "isomorphic" with "homomorphic." With instructor-researcher guidance, they *constructed a new statement.* At this point, the student suggestions were the focal object (alignment), and students were primarily involved in generating tools (agency) and evaluating tools and objects (authority).

4.2.1 Testing the New Statement with Examples. With the new statement, *Suppose there exists a homomorphism from G to H. Then if G is Abelian, H is Abelian*, the instructor-researcher prompted students to use a specific tool: *test the statement* with [examples] or produce a counter[example]. The students quickly decided the statement was false and tried generating a

[11] See Melhuish et al. (2019) for a survey of common proof approaches to this theorem. See Melhuish et al. (2020) for a practitioner view on the overall lesson.

counterexample. Students shared their strategies to generate [examples] such as, "... so, since we lost one-to-one and onto, maybe think of some element in *H* that doesn't have a pre-image." This comment also reflected the use of the [existent definition] of isomorphism (one-to-one and onto), [deformalizing] to the idea of no pre-image, and [logic] since they used the statement structure to determine counterexample properties. The students continued generating [examples] which were evaluated by the students such as Andy's suggestion, "integers under addition [mapped] to all real numbers, under multiplication" which Eric countered, "that would just be the onto part, wouldn't it?" The students continued making suggestions, but were not constructing a valid counterexample.

Identifying the lack of accuracy in the students' counterexamples, the division of labor shifted as the instructor-researcher scaffolded the tool generation, asking pointed questions about what needed to be true about *G, phi,* and *H*. Throughout the scaffolding, students suggested several abelian and non-abelian groups [examples] which the instructor-researcher challenged when inaccurate, prompting students to use [existent definitions] to determine if their [examples] met the needed criteria (sub-objective: *test concept*). After some discussion, the student and instructor endorsed the [example] groups, {1, -1} under multiplication (Abelian) and D_8 (non-Abelian), suggested by the students.

The students expressed uncertainty for how to construct a homomorphism that was not an isomorphism between the groups. The instructor-researcher responded by writing the domain and codomain groups as a function [diagram] on the board and had the students suggest labels and requested an "easy" homomorphism. After a hesitant student map suggestion ("x=y") [example] that the student then retracted "nevermind," the instructor-researcher asked, "If we have a homomorphism, where do we know the identity has to go?" Eric answered, "the identity" and asked if the other element, -1, should be mapped to a different element in the codomain. The instructor-researcher questioned if it was necessary to choose a different element, to which Miguel noted that "it's not 1-1 or onto" (implicitly using the [existent definition] of homomorphism/isomorphism). Endorsing this idea that a map need not be one-to-one nor onto, the instructor-researcher drew the identity map.

The instructor-researcher then prompted students to use the [diagram] and link the counter[example] to the objective of *testing the statement*. Andy explained, "Because we have an abelian group that maps to a non-abelian group, therefore *H* does not always have to be abelian."

This contribution evidenced the students identified the [diagram] (in conjunction with the [logic] of the statement) as a tool to meet the objective of *testing the statement.*

At the onset of this episode, authenticity was high in terms of centering student contributions, their conjectured statement and examples (alignment), as well as variety (formal, informal, some translating between the informal examples and formal statement) and complexity (tools were used together towards an objective), but other authentic student practice dimensions were middling. The instructor-researcher prompted what tool to use, but in the beginning the students did create the examples (agency). The instructor-researcher also endorsed the appropriateness of tool use, the examples, towards the objective, testing the statement, by relating them in the prompt; however, there was some evidence of student authority with Eric evaluating Andy's tool. At the beginning of the episode, accuracy was quite low as neither the statement being tested was accurate, nor were the counterexamples generated by the students.

The low accuracy of the counterexamples appeared to shift the division of labor. The students' agency was reduced in terms of creating the homomorphism where the instructor-researcher introduced the diagram and asked a series of scaffolding questions to arrive at the identity homomorphism. Student contributions, examples and statements, were the focus for most of the episode, but the instructor-researcher also recognized inaccuracies and more substantially contributed to the final counterexample, lowering alignment. Authority was similarly mixed as the instructor-researcher prompted students to use definitions to determine if (non-accurate) examples, suggested groups, met the required definitions. However, the students ultimately made those evaluations. The shift in the system to the instructor-researcher taking on more agency in developing the counterexample aligned with higher disciplinary authenticity, especially accuracy. We note after the construction of the counterexample, we see a shift to students taking on more authority as they are the ones involved in evaluating whether the counterexample met the testing objective.

4.2.2 Analyzing and Refining (Reappropriating the Diagram) to Construct a New Statement. For the sake of brevity, we conclude this illustration by sharing some of the initial student activity in the next portion of the task. Once there was agreement on a counterexample, the instructor-researcher prompted for continued [analysis/refinement] based on their prior outcome (the constructed statement was invalid) to *construct a new version of the statement*: "So, potentially we need something more than just homomorphism. And I guess that question becomes, ‘What other things do we need to be able to make this argument?’” The instructor-researcher suggested the students use whatever tools they would like such as the prior [existent proofs] or [examples]. The students spontaneously began using the [diagram] from the prior exploration, with Andy explaining “I mean either way, if you just say one-to-one, then this example automatically goes away” with Eric and Jasmine elaborating, “So, yeah it would still be wrong” and “Yeah, it contradicts.” They continued using the [diagram] (Figure 5) to [analyze/refine] their conjecture moving from just homomorphism to conjecturing that 1-1 and onto are also necessary.

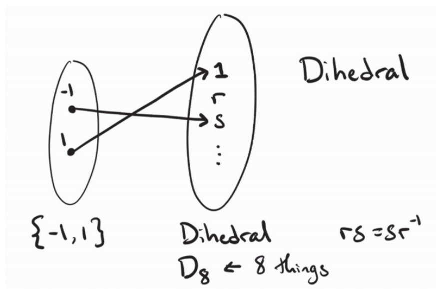


Figure 5. Function diagram

This episode illustrated increased authenticity to the students compared to the prior episode. The students spontaneously re-appropriated a diagram (high agency), connected the tools to objectives explicating that their example contradicts a version of the statement with only 1-1 (high authority), and the student contributions remained centered (high alignment). Although their conjectured refinement was not completely accurate, there was some variety and complexity in tool use. Over the next thirty minutes, the students (with some instructor-researcher questions) continued to debate about needed assumptions using both the [diagram] and the [existent proofs] along with [warranting, logic, definitions, and structure] towards the *statement construction* objective. Overall, this episode suggests that temporarily decreasing student authenticity can shift to increased student authenticity. Furthermore, the students began to appropriate tools without needing explicit prompting from the instructor-researcher.

5 Discussion

In this paper, we contribute to the larger literature base on authentic mathematical activity and its dimensions by focusing on ways that observable student (and instructor) activity can be analyzed rather than forefronting task design (Herrington et al., 2014; Strobel et al., 2013), assessment (Gulikers et al., 2004; Koh, 2017), or outcomes after learning sessions (Radović et al., 2021; Sutherland & Markauskaite, 2012). Furthermore, we are meeting a need to operationalize activity specific to the advanced undergraduate setting in which the formal proof and the practice of professional mathematicians play a more salient role. We join other researchers in taking a participatory stance to learning at this level (e.g., Biza et al., 2014; Nardi et al., 2014; Rasmussen et al., 2015), and contribute a framework serving a complementary purpose: providing a concrete way to analyze authenticity in student-teacher activity systems. That is, we focus on the in-the-moment, observable activity with emphasis on elements of professional mathematician activity related to the formal proof system. Our perspectives are aligned with broader authenticity requirements set out by other scholars (Stylianides et al., 2022) interested in promoting student engagement in goal-directed actions of professional mathematicians in the context of proof. We directly reflect Stylianides et al.'s criteria: aligned with the activity of the professional mathematician community, containing features that are essential to activity and not features that are antithetical, and classroom goals aligned with mathematician goals. We note they suggest one additional criterion, "The classroom activity and goals were pedagogical and developmentally appropriate for the classroom" (p. 5) which while not directly observable with our frameworks, can be evidenced by students having agency and authority in division labor. Thus, we position the AMPA framework as fulfilling a distinct need in this setting to document, deconstruct, and make claims about students' progressive engagement in authentic mathematical proof activity.

We argue that analyzing activity systems, with particular attention to tool use, can support making participatory claims about authentic activity, and provide nuance into the multi-dimensionality of authenticity. Authentic mathematical activity has many conceptualizations and often hinges on the tension between discipline authenticity and student authenticity (e.g., Lampert, 1992; Weiss et al., 2009). Our claims of engagement in authentic mathematical proof activity draw on six tool-use dimensions: disciplinary practice (variety and complexity of tool use), disciplinary content (accuracy of tools and outcomes), student practice (agency to generate and use tools, authority to evaluate and link tool use), and student content (tools and outcomes align with student contributions).

The AMPA framework provides a means for close-in analysis of student activity, an important aspect of authentic learning environments. Throughout the authenticity literature, the instructor’s role is salient (e.g., Herbst & Chazan, 2020; Lampert, 1992; Stein et al., 2004; Tochon, 2000), and we suggest the AMPA dimensions provide a lens, particularly in the proof-based setting, for capturing instructional tensions and better understanding the role of the instructor in student-teacher activity systems. Furthermore, analyzing such activity systems over time can provide a means for participatory learning claims (expanded in the next section). Such analyses can be complementary to more holistic approaches such as Stein et al.’s (2004) framework, which situates authentic learning as not just disciplinary activity, but also part of a system that includes instruction, assessments, learning goals, and the specific needs of students relative to classes.

5.1 Summary and Discussion of the Illustrations

To evidence the flexibility and usability of the AMPA framework, we shared illustrative analyses of teacher-student activity systems across two distinct research projects. While the projects differed in content area, student experience with proof, and design heuristics, our authenticity analyses were productive for describing complex mathematical activity and exploring authenticity. We suggest our theoretical exploration supports a multidimensional view of authenticity and provides ways that such analyses can lead to participatory learning claims.

5.1.2 Activity Systems, Authenticity, and the Instructor’s Role. At the onset of the algebra episode, student authenticity dimensions were quite high, reflecting student agency in generating and using tools, their contributions being centered, and students linking their tools to the objectives. These dimensions ebbed as students made less accurate contributions and struggled generating a counterexample. The instructor-researcher assumed more authority and agency by prompting students to evaluate their contributions (authority) and scaffolding a counterexample that would be considered valid in the mathematician community (agency). By scaffolding the counterexample, the alignment with student content dimension lowered as the example was largely structured by the instructor-researcher. This temporary decrease in student authenticity was followed by increases in all student authenticity dimensions when the students spontaneously appropriated the diagram to make new conjectures.

In the real analysis illustration, we see a slightly different management of authenticity tensions. Rather than an ebb-and-flow in the student-authenticity dimensions, the instructor-researcher played a consistent role: prompting for students to generate and use specific tools such as examples (agency), while explicating the complex manner students would use a variety of tools toward an objective as well as evaluating the accuracy of student contributions, (authority). This authority (and to some degree agency) seemed to be in service of increasing discipline dimensions such as variety of types of tools, complexity in using tools together and shifting prior objects to tools for more usage, and accuracy of the tools and outcomes involved. In contrast to the algebra example, alignment never decreased. That is, throughout the episode the tools and objects involved were created solely by the students. Fluctuations and balances in authenticity dimensions emphasize the instructor's role in managing tensions between the discipline and students in ways that can position students to appropriate new tools and expand viable objectives.

5.1.2 Activity Systems, Authenticity, and Participatory Learning Claims. Finally, we can consider how we might evidence claims of learning from a participatory perspective by identifying ways that classroom activity systems more closely resemble mathematician activities as evidenced by expansions in activity systems (Greeno & Engeström, 2014) and appropriation of tool use (Grossman et al., 1999; Nelson & Kim, 2001). Expansion involves increases in the variety of tools in use and complexity in their usage as new configurations become acceptable in the community and former goal-directed action outcomes become tools for continued mathematization (discipline authenticity dimensions). Tool appropriation is reflected in students having an increased role in the division of labor where they become the ones to increasingly select, generate, and use tools (student authenticity dimensions).

As the episodes we shared come from single teaching sessions, we are hesitant to make over-time participatory learning claims; however, we can point to areas that show promising evidence of expansion in the activity systems and appropriation of tools. In the real analysis example, there were several points when prior objects shifted to the role of tools, evidencing potential for expansion. As one example, the students used tools to construct an 'upper bound' definition, and then used their upper bound definition (with other tools) to construct a 'no upper bound' definition. In the algebra example, we shared some evidence of appropriation of a diagram tool. At one extreme, the students struggled to construct a counterexample with the instructor-researcher ultimately introducing a diagram to help co-construct a valid counterexample. At the other extreme, students spontaneously used this diagram and modified it towards their objective of constructing a new statement. Future studies can use the AMPA framework to attend to potential tool appropriation and expansion in a similar way but over multiple teaching sessions to evidence claims of students' participatory learning.

5.2 Limitations

Our goal was to provide insight into authentic mathematical proof activity via attention to both the discipline and students. Our claims focus on student activity that can more closely resemble or be seen as analogous (Herbst & Chazan, 2020) to mathematicians' work. Designing for such activity involves recognizing which elements of professional practice are appropriate for the particular context and students. As argued by Sutherland and Markauskaite (2012), "Authentic learning experiences are activities, which motivate and support students learning by providing them with experiences that give students 'real-world' experiences but protect them from harmful or irrelevant elements" (p. 749). In our framework, we have not provided insight into how to select or emphasize elements of professional mathematician practice. We note we have limited our selection of tools and objectives in ways that remain focused on epistemic aims and are considered relevant elements in the classroom (in alignment with Weber and Melhuish's (2022) suggestions for promoting authentic mathematical proof activity without preserving harmful elements of professional practice).

We also acknowledge that our own theoretical assumptions impact how we interpret activity and authenticity. We took a broad approach to proof-related activity which may not reflect other views on formal proof.[12] Further, by focusing on tools, we provide one image of authentic mathematical proof activity; however, researchers with a different perspective may reasonably attend to different aspects of teacher and student activity. For example, others may see authenticity of tasks (e.g., Herrington et al., 2014; Weiss et al., 2009) – the degree the tasks are genuine – as an essential component of such activity. We relied on the assumption that teachers provide tasks that evoke authentic mathematical proof activity, yet such tasks may be contrived within the bounds of classroom settings. For example, in the abstract algebra episodes, students compared across two proof approaches. This is a task that may be somewhat artificial to the discipline; however, we would argue that the task provided an opportunity to engage in authentic mathematical proof activity.

---

[12] See Balacheff (2008) for a discussion of approaches to proof.

Additionally, there are likely important facets of mathematicians' work that have not been addressed in existing research on mathematicians. Relying on a literature synthesis may reproduce problematic inferences by relying on artificial task-based interviews or idiosyncratic mathematicians' reflections (as cautioned in Mejía-Ramos & Weber, 2020). We attempted to minimize such biases by triangulating sources and requiring evidence from empirical studies of multiple mathematicians before including a tool in our framework. Further, motives (e.g., evaluating aesthetic) or tools (e.g., computers) were excluded if not prominent in the student literature. We suggest that the AMPA framework could be adapted to incorporate additional objectives and tools as needed.

Finally, we want to acknowledge the scope of our framing. Our goal was to provide insight into the authenticity of activity via attention to both the discipline and students. This approach does not call attention to other constraints in the classroom setting such as those imposed by the institution. It is artificial to presume that all instructor actions are tied to authenticity dimensions, and a researcher wanting to more deeply analyze instructional practice may consider other tensions such as those elaborated in Herbst and Chazan (2020).

5.3 Implications and Future Research

In our own work as design-based researchers, we have found the AMPA framework useful for both designing tasks and analyzing the activity when implementing such tasks. In both of our projects, we designed tasks in which students would be actively involved in the use of tools with a particular focus on how informal tools could be integrated into students' activity in a proof context. By operationalizing tools, objects, and motives at this grain-size, we can (re)design task sequences around what tools students could be prompted to use and for what motives. Additionally, we can intentionally plan for expansion of activity systems and how we might support students in appropriating tools.

We also see this framework as useful for researchers who are looking to analyze and document authentic mathematical activity in the proof-based setting. Our analytic lens can aid in making claims about authenticity in relation to students and the discipline across a number of essential dimensions. Shifts in tool use can point to evidence of learning via appropriation and expansion of what tools are permissible in undergraduate activity systems. A natural expansion of this work would be using this framework in conjunction with other analytic tools that focus more on content, argumentation or managing layers of analysis between the individual and activity system (such as the approach advocated for by Rasmussen et al., 2015).

Lastly, we argue for the potential of our framework in expanding what competency can look like in the undergraduate mathematics classroom. Herbst and Chazan (2020) recently argued, “Images of mathematical practice, from history and from the present may support a teacher in more generously and generatively interpreting what students do as being similar to mathematicians’ work” (p. 1156). We suggest our framework can serve as a basis for broadening views of competence in undergraduate proof-based classrooms. We focus not on *whether* students are engaged in disciplinary activity, but *in what ways* and *to what degree.* As noted by Hanna (1991) “competence in mathematics might readily be misperceived as synonymous with the ability to create the form, a rigorous proof” (p. 60). The activity of mathematicians is not limited to the formal rhetoric system, and students can use, with some guidance, a similar array of disciplinary tools in sophisticated ways. By offering some common language, the AMPA framework allows us to articulate and document the ways in which students do so.

References

Alibert, D., & Thomas, M. (2002). Research on mathematical proof. In *Advanced mathematical thinking* (pp. 215-230). Springer, Dordrecht.

Artemeva, N., & Fox, J. (2011). The writing's on the board: The global and the local in teaching undergraduate mathematics through chalk talk. Written Communication, 28(4), 345-379.

Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: an essay on the case of proof. *ZDM*, 40(3), 501-512.

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal,* 93(4), 373–397.

Bass, H. (2017). Designing opportunities to learn mathematics theory-building practices. *Educational Studies in Mathematics*, *95*(3), 229-244.

Chorlay, R. (2019). A pathway to a student-worded definition of limits at the secondary-tertiary transition. *International Journal of Research in Undergraduate Mathematics Education*, *5*(3), 267-314.

Biza, I., Jaworski, B., & Hemmi, K. (2014). Communities in university mathematics. *Research in Mathematics Education*, 16(2), 161-176.

Chazan, D., & Ball, D. (1999). Beyond being told not to tell. *For the learning of mathematics*, 19(2), 2-10.

David, M. M., & Tomaz, V. S. (2012). The role of visual representations for structuring classroom mathematical activity. *Educational Studies in Mathematics,* 80(3), 413-431.

Dawkins, P. C. (2015). Explication as a lens for the formalization of mathematical theory through guided reinvention. *The Journal of Mathematical Behavior*, *37*, 63-82.

Dawkins, P. C., Oehrtman, M., & Mahavier, W. T. (2019). Professor goals and student experiences in traditional IBL Real Analysis: A case study. *International Journal of Research in Undergraduate Mathematics Education*, *5*(3), 315-336.

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, *95*(2), 123-142.

Engeström, Y. (2000). Activity theory as a framework for analyzing and redesigning work. *Ergonomics*, 43(7), 960-974.

Fang, Z., & Chapman, S. (2020). Disciplinary literacy in mathematics: One mathematician's reading practices. *The Journal of Mathematical Behavior*, *59*, 1-15.

Fernández-León, A., Gavilán-Izquierdo, J. M., & Toscano, R. (2020). A case study of the practices of conjecturing and proving of research mathematicians. *International Journal of Mathematical Education in Science and Technology*, 1-15.

González, G., & DeJarnette, A. F. (2012). Agency in a geometry review lesson: A linguistic view on teacher and student division of labor. *Linguistics and Education,* 23(2), 182-199.

Greeno, J. G., & Engeström, Y. (2014). Learning in activity. In *The cambridge handbook of the learning sciences* (pp. 128-150).

Grossman, P. L., Smagorinsky, P., & Valencia, S. (1999). Appropriating tools for teaching English: A theoretical framework for research on learning to teach. *American journal of Education*, 108(1), 1-29.

Gueudet, G. (2008). Investigating the secondary–tertiary transition. *Educational Studies in Mathematics*, *67*(3), 237-254.

Gulikers, J., Bastiaens, T. J., & Kirschner, P. A. (2004). A five-dimensional framework for authentic assessment. *Educational technology research and development*, *52*(3), 67-86.

Hanna, G. (1991) Mathematical proof. In D. Tall (Ed.) *Advanced mathematical thinking* (pp. 54-61). Springer

Hanna, G., & Larvor, B. (2020). As Thurston says? On using quotations from famous mathematicians to make points about philosophy and education. *ZDM*, *52*, 1137–1147. Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society,* 7, 234-283.

Herbst, P. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education*, 33(3), 176-203.

Herbst, P., & Chazan, D. (2020). Mathematics teaching has its own imperatives: mathematical practice and the work of mathematics instruction. *ZDM*, 52(6), 1149-1162.

Herrington, J., Reeves, T. C., & Oliver, R. (2014). Authentic learning environments. *Handbook of research on educational communications and technology*, 401-412.

Howell, H., & Mikeska, J. N. (2021). Approximations of practice as a framework for understanding authenticity in simulations of teaching. *Journal of Research on Technology in Education*, 53(1), 8-20.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*(4), 358-390.

Inglis, M., Mejía-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*(1), 3-21.

Kaptelinin, V., Nardi, B., & Macaulay, C. (1999). Methods & tools: The activity checklist. *Methods*.

Karunakaran, S. S. (2018). The need for "linearity" of deductive logic: An examination of expert and novice proving processes. In *Advances in Mathematics Education Research on Proof and Proving* (pp. 171-183). Springer, Cham.

Kirk, W. A., & Panyanak, B. (2008). A concept of convergence in geodesic spaces. *Nonlinear analysis: theory, methods & applications*, 68(12), 3689-3696.

Koh, K. H. (2017). Authentic assessment. In *Oxford research encyclopedia of education*.

Kuster, G., Johnson, E., Rupnow, R., & Wilhelm, A. G. (2019). The inquiry-oriented instructional measure. *International Journal of Research in Undergraduate Mathematics Education*, *5*(2), 183-204.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge University Press.

Lampert, M. (1992). Practices and problems in teaching authentic mathematics. In F. Oser, A. Dick, & J. L. Patry (Eds.), *Effective and responsible teaching: The new synthesis* (pp. 295–313). Jossey-Bass.

Larsen, S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712-725. Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, *67*(3), 205-216.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge university press.

Lockwood, E., Ellis, A. B., & Lynch, A. G. (2016). Mathematicians' example-related activity when exploring and proving conjectures. *International Journal of Research in Undergraduate Mathematics Education*, *2*(2), 165-196.

Lynch, A. G., & Lockwood, E. (2017). A comparison between mathematicians' and students' use of examples for conjecturing and proving. *The Journal of Mathematical Behavior*, *53*, 323-338.

Martín-Molina, V., González-Regaña, A. J., & Gavilán-Izquierdo, J. M. (2018). Researching how professional mathematicians construct new mathematical definitions: A case study.

International Journal of Mathematical Education in Science and Technology, 49(7), 1069-1082.

Mejía-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*(1), 3-18.

Mejía-Ramos, J. P., & Weber, K. (2014). Why and how mathematicians read proofs: Further evidence from a survey study. *Educational Studies in Mathematics*, *85*(2), 161-173.

Mejía-Ramos, J. P., & Weber, K. (2020). Using task-based interviews to generate hypotheses about mathematical practice: Mathematics education research on mathematicians' use of examples in proof-related activities. *ZDM*, *52*(6), 1099–1112.

Melhuish, K., Larsen, S., & Cook, S. (2019). When students prove a theorem without explicitly using a necessary condition: Digging into a subtle problem from practice. *International Journal of Research in Undergraduate Mathematics Education*, *5*(2), 205-227.

Melhuish, K., Lew, K., & Hicks, M. (2020). Comparing Student Proofs to Explore a Structural Property in Abstract Algebra. *PRIMUS*, 1-17.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education*, 16(2), 182-198.

Nelson, C. P., & Kim, M. K. (2001). Contradictions, Appropriation, and Transformation: An Activity Theory Approach to L2 Writing and Classroom Practices. *Texas papers in foreign language education,* 6(1), 37-62.

Nolen, S. B., Wetzstein, L., & Goodell, A. (2020). Designing Material Tools to Mediate Disciplinary Engagement in Environmental Science. *Cognition and Instruction*, *38*(2), 179-223.

Ouvrier-Buffet, C. (2015). A model of mathematicians' approach to the defining processes. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education (CERME9, 4-8 February 2015)* (pp. 2214–2220). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

Paoletti, T., Krupnik, V., Papadopoulos, D., Olsen, J., Fukawa-Connelly, T., & Weber, K. (2018). Teacher questioning and invitations to participate in advanced mathematics lectures. *Educational Studies in Mathematics*, 98(1), 1-17

Parameswaran, R. (2010). Expert Mathematics' Approach to Understanding Definitions. *Mathematics Educator*, *20*(1), 43-51.

Polya, G. (1954). *Mathematics and Plausible Reasoning: Induction and Analogy in Mathematics*. Princeton University Press.

Radović, S., Firssova, O., Hummel, H. G., & Vermeulen, M. (2021). Strengthening the ties between theory and practice in higher education: an investigation into different levels of authenticity and processes of re-and de-contextualisation. *Studies in Higher Education*, *46*(12), 2710-2725.

Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics*, *88*(2), 259-281.

Samkoff, A., Lai, Y., & Weber, K. (2012). On the different ways that mathematicians use diagrams in proof construction. *Research in Mathematics Education*, *14*(1), 49-67.

Savic, M. (2015). On similarities and differences between proving and problem solving. *Journal of Humanistic Mathematics*, *5*(2), 60-89.

Selden, A. (2012). Transitions and proof and proving at tertiary level. In *Proof and proving in mathematics education* (pp. 391-420). Springer, Dordrecht.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational studies in mathematics*, *29*(2), 123-151.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational researcher*, *27*(2), 4-13.

Smith, J. P., & Hungwe, K. (1998). Conjecture and verification in research and teaching: conversations with young mathematicians. *For the Learning of Mathematics*, *18*(3), 40-46.

Sutherland, L., & Markauskaite, L. (2012). Examining the role of authenticity in supporting the development of professional identity: an example from teacher education. *Higher Education*, *64*(6), 747-766.

Stein, S. J., Isaacs, G., & Andrews, T. (2004). Incorporating authentic learning experiences within a university course. Studies in Higher Education, 29(2), 239-258.

Strobel, J., Wang, J., Weber, N. R., & Dyehouse, M. (2013). The role of authenticity in design-based learning environments: The case of engineering education. *Computers & Education*, *64*, 143-152.

Stylianides, A. J., Komatsu, K., Weber, K., Stylianides, G.J. (2022). Teaching and Learning Authentic Mathematics: The Case of Proving. In: Danesi, M. (eds) *Handbook of Cognitive Mathematics*. Springer, Cham. https://doi.org/10.1007/978-3-030-44982-7_9-1

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237-266). National Council of Teachers of Mathematics.

Tochon, F. V. (2000). When authentic experiences are "enminded" into disciplinary genres: Crossing biographic and situated knowledge. *Learning and Instruction*, 10(4), 331-359.

Vroom, K. (2020). Guided reinvention as a context for investigating students' thinking about mathematical language and for supporting students in gaining fluency (Doctoral dissertation). *Dissertations and theses*.

Watson, A. (2008). School mathematics as a special kind of mathematics. *For the Learning of Mathematics*, *28*(3), 3-7.

Wawro, M. (2014). Student reasoning about the invertible matrix theorem in linear algebra. *ZDM*, *46*(3), 389-406.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior*, *23*(2), 115-133.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education*, *39*(4), 431-459.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*(2-3), 209-234.

Weber, K., Dawkins, P., & Mejía-Ramos, J. P. (2020). The relationship between mathematical practice and mathematics pedagogy in mathematics education research. *ZDM*, 1-12.

Weber, K., & Mejía-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics*, *76*(3), 329-344.

Weber, K., & Melhuish, K. (2022). Can we engage students in authentic mathematical activity while embracing critical pedagogy? A commentary on the tensions between disciplinary activity and critical education. *Canadian Journal of Science, Mathematics and Technology Education*, 22(2), 305-314.

Weiss, M., Herbst, P., & Chen, C. (2009). Teachers' perspectives on "authentic mathematics" and the two-column proof form. *Educational Studies in Mathematics*, 70(3), 275-293.

Wilkerson-Jerde, M. H., & Wilensky, U. J. (2011). How do mathematicians learn math?: Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics*, *78*(1), 21.

Williams, S. R., & Leatham, K. R. (2017). Journal quality in mathematics education. *Journal for Research in Mathematics Education*, *48*(4), 369-396.

Winsløw, C., Barquero, B., De Vleeschouwer, M., & Hardy, N. (2014). An institutional approach to university mathematics education: From dual vector spaces to questioning the world. *Research in Mathematics Educatio*n, *16*(2), 95-111.

Zazkis, D., Weber, K., & Mejía-Ramos, J. P. (2016). Bridging the gap between graphical arguments and verbal-symbolic proofs in a real analysis context. *Educational Studies in Mathematics*, *93*(2), 155-173.