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Title: The Role of the Partitioning and Coset Algorithm Quotient Group Partial Meanings in a First Isomorphism Theorem and Proof Comprehension Task

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Abstract

In many advanced mathematics courses, comprehending theorems and proofs is an essential activity for both students and mathematicians. Such activity requires readers to draw on relevant meanings for the concepts involved; however, the ways that concept meaning may shape comprehension activity is currently undertheorized. In this paper, we share a study of student activity as they work to comprehend the First Isomorphism Theorem and its proof. We analyze, using an onto-semiotic lens, the ways that students' meanings for quotient group both support and constrain their comprehension activity. Further, we suggest that the relationship between understanding concepts and proof comprehension can be reflexive: understanding of concepts not only influences comprehension activity but engaging with theorems and proofs can serve to support students in generating more sophisticated understanding of the concepts involved.

Keywords

Theorem comprehension, proof comprehension, quotient groups

1 Introduction

Comprehending mathematical theorems and proofs is an essential activity in many advanced mathematics courses. Researchers in recent years have studied how mathematicians engage in comprehending (e.g., Weber & Mejía-Ramos, 2011, Wilkerson-Jerde & Wilensky, 2011), operationalized proof comprehension for the purpose of assessment (Mejía-Ramos et al., 2012) and developed strategies to support students in engaging in comprehension (Samkoff & Weber, 2015). Concepts play an essential role in each of these studies ranging from instantiating examples (e.g., Weber & Mejía-Ramos, 2011) and drawing on definitions (e.g., Wilkerson-Jerde & Wilensky, 2011) to using definitions of concepts in a theorem to anticipate proof structure (e.g., Samkoff & Weber, 2015). Further, being able to identify accurate meanings for terms involved in the theorem and proof is correlated with other proof comprehension dimensions (Hodds et al., 2014).

Understanding a concept is more complex than just a definition; indeed, modern mathematical definitions are shaped by proving (Lakatos, 1976), though we have little research evidence about how proofs aide students in concept formation. Several researchers have illustrated the important role that formal and informal meanings play in students' reasoning when constructing proofs (e.g., Edwards & Ward, 2004; Moore, 1994). Dawkins and Karunakaran (2016) have argued for the "necessity of attending to mathematical meaning in proving research if we are to explain the emergence of student reasoning in situ" (p. 67). Research is still needed to identify what makes such ways of reasoning more and less productive for proof comprehension.

Methodologically, studies of student proof comprehension tend to focus solely on student responses to isolated prompts for definitions, meanings, or examples. While a student's response to any such question is vital, we argue that the meanings students evoke shape more of their activity in theorems and proofs than isolated prompts can uncover. Theorem and proof comprehension provide a specific context that is likely to influence what aspects of a relevant concept are attended to. By studying students' meanings in context, we gain insight into the interaction between conceptual understanding and proof texts and techniques. We focus this study on a particular context: The First Isomorphism Theorem (FIT) in abstract algebra. We selected this theorem due to its essential role in an introductory course and its complexity in relating numerous concepts in abstract algebra. In this paper, we focus specifically on students' meanings for quotient groups and how these meanings support or constrain student activity in this context. Further, we consider how the context of theorem and proof comprehension may provide opportunities for students to build more sophisticated meanings for quotient groups. We ask:

- What partial meanings for quotient groups do students evoke in the context of comprehending the First Isomorphism Theorem and its proof? And how do these meanings shape their comprehension practice?
- How does engaging with the concept of quotient group within the FIT and its proof occasion opportunity for students to develop more sophisticated meanings for quotient groups?

2 Theoretical Background and Assumptions

In proof-based mathematics, students encounter objects that are defined formally and abstractly. These concepts are then represented using signs that "are not the mathematical objects themselves but stand for them in some way" (Presmeg et al., 2016, p. 9). Theories related to signs, semiotic theories, can support analysis of how students engage with mathematical objects. Godino et al.'s (2007) *onto-semiotics* provides a way to study students' mathematical practice grounded in the essential role of signs and meanings. Our underlying assumption is that signs mediate meaning and meaning occurs in context and within a semiotic system. "A semiotic system is the system formed by the configuration of intervening and emerging objects in a system of practices, along with the interpretation processes that are established between the same (that is to say, including the network of semiotic functions that relate the constituent objects of the configuration)" (Godino et al., 2011, p. 255). Researchers have begun to take an onto-semiotics approach in abstract contexts such as group theory (Sepúlveda-Delgado et al., 2021) and in the practice of proof construction (Molina et al., 2021). We focus on the discursive mathematical practice of comprehending a mathematical theorem and proof.

Meaning plays an essential role in mathematical practice that Godino and colleagues attribute to two distinct and complementary aspects: (1) the meaning attributed to a sign (say, in definition or explanation) and (2) meaning in "usage" where a mathematical objects' meaning relates to what can be done with it in the course of problem-solving. The meanings evoked shape the practice and the practices are reflexively constituent to meaning. At one level there are "representations (language), definitions, propositions, procedures, problems and arguments (primary objects)" (Font Moll et al., 2016, p. 112) that emerge in activity. At a higher level, there is one single, unified object that "can (1) be associated with different representations, (2) has several equivalent definitions and (3) has properties, etc." (p.112). The emergence of (what the reasoner perceives as) a unified object is a key feature of onto-semiotics because the theory intends to explain how mathematical objects can emerge in social activity without a fundamental ontological status. Onto-semiotics provides rich tools for documenting how students can, through activity, construct a coherent and sophisticated understanding of an abstract object such as a quotient group. Such a stance aligns with other theories such as knowledge-in-pieces (diSessa, 2018) where knowledge is not correct or incorrect, but rather as a set of partial meanings evoked in different contexts that can build in coherence. Throughout this manuscript, we use the term *partial meanings* to emphasize both a non-deficit view of student meanings and to acknowledge the multi-faceted and practice-based nature of mathematical concepts.

While we see partial meanings as serving an overarching role in mathematical practice, we also appeal to semiotic object dualities to further operationalize students' mathematical practice. Font et al. (2013) argue that mathematical objects in practice can be "grouped into dual facets or dimensions" (p. 111):

- the expression and content (which we refer to as the *sign* and *referent* throughout this document)
- *ostensive* (observable) and *non-ostensive* (imagined)
- *personal* (to the individual) and *institutional* (which can refer to the local community of students or a larger community)
- *unitary* (a single object/sign to be acted on in its totality) or *systematic* (decomposable into a system where its constituent parts/meanings can be acted on individually)
- *type* (a general class) or *example* (a specific member of a class) alternatively referred to as *intensive* or *extensive.*

Any mathematical object is subject to these dualities, and we suggest these dualities can provide a means to develop a more sophisticated understanding of mathematical objects. By this we mean, more connected, adaptive, and consistent with institutional meanings. They also provide opportunity for semiotic *conflict*, "disparity or difference of interpretation between the meanings ascribed to an expression by two subjects, being either persons or institutions" (Godino et al., 2007, p. 133). These conflicts may or may not be observed by the participants. Conflicts can interfere with student communication with each other or the text, but also can serve as a space for students to negotiate meanings and progress in their coordination of partial meanings. For instance, the referent for an ostensive sign may be non-ostensive and imagined differently by various interlocutors. By producing some other ostensive representation of the referent of the original sign, students might more clearly negotiate its meaning. We use the term *shift* to capture movement between various dualities where successful shifts may reflect students making mathematical progress and/or developing more sophisticated and self-consistent meanings.

Background: The First Isomorphism Theorem and Quotient Groups

We situate our exploration in the context of The First Isomorphism Theorem and its proof (Figure 1). We selected this theorem and the concept of quotient groups due to their complexity.

The theorem involves both a homomorphism (given) and isomorphism map (needed) to establish that the quotient group and image of the homomorphism are isomorphic.

(The First Isomorphism Theorem). *If $\phi : G \to H$ is a group homomorphism, then*

$$\frac{G}{\ker \phi} \cong \phi(G).$$

Figure 1. Statement of the FIT used in our study.

A panel of experts in abstract algebra instruction identified both the FIT and quotient groups as two of the most important topics in introductory abstract algebra and the most difficult by a sizable margin (Melhuish, 2019). Nardi (2000) illustrated that students may struggle to coordinate the roles of the homomorphism and isomorphism maps in the theorem. Further, Nardi points to issues retrieving relevant information and moving beyond recalled facts to productively engage with the theorem and produce its proof. More recently, Mena-Lorca and Parraguez (2016) found that students who used a more generalized theorem (equivalence relations partition a set) had more sophisticated conceptions of the theorem than students who relied on abstract algebra notions exclusively. Both studies point to rather substantial coordination of abstract objects to productively engage with the theorem. This coordination is apparent in studies of mathematicians whose meaning for homomorphism and quotient groups are fundamentally linked (Rupnow, 2021). The results from these studies suggest: (1) the FIT is a context worthy of exploring theorem and proof comprehension related to concepts and (2) there is potential for students to develop more sophisticated meaning for quotient groups that incorporates notions of homomorphism.

We can then turn to the literature on quotient groups to anticipate some of the partial meanings that may be relevant to their comprehension activity. The coset algorithm often serves as students' primary tool when working with quotient groups (Asiala et. al, 1997). This is where one takes a subgroup H and an element from the original group, g, and calculates $g+H$. This is repeated with each element from the original group. So long as the subgroup was normal, the distinct cosets form a quotient group. Hazzan (1999) suggests that relying on such a process could be an attempt to create a less abstract environment for students. Reliance on the algorithm may hide the group structure of quotient groups. Further, many students may lack mental imagery reasoning with cosets which can intercede with their ability to produce proofs (Nardi, 2000; Ioannou & Iannone, 2011).

Quotient groups also require a complex coordination where a coset is simultaneously an element in a group (unitary) and a set itself (systematic) (Brenton & Edwards, 2003). Large-scale results suggest many students have not made this coordination (Melhuish, 2019). Siebert and Williams (2003) suggest there are three interpretations for cosets: sets, element set combinations, and representative elements. Each meaning is tied to a different ostensive: $\{a, b, c, \ldots\}$, $a + H$, and a, respectively. While their study was situated in the context of $\mathbb{Z}_n$, their implications spanned quotient groups more generally suggesting students need to be able to coordinate all three meanings.

Larsen and Lockwood (2013) attempted to counter this reliance on the algorithm by having students begin their quotient group activity via partitioning. Students working from this perspective developed different approaches to quotient groups, notably attending to an operation being well-defined rather than relying on normality when determining if a subgroup is a group (Larsen et al., 2013). Due to the variety of partial meanings found in the literature, we anticipate that students may draw on a multitude of meanings for quotient group when engaged with the FIT.

Onto-semiotic Analysis of the First Isomorphism Theorem and its proof

To orient the reader to the proof our study participants analyzed, we present some analysis of the proof (Figure 2) using onto-semiotic tools.

Lemma 1. *Let $\phi : G \to H$ be a group homomorphism. Then the kernel of ϕ is a normal subgroup of G.*

1 *Proof.* Let $\phi : G \to H$ be a group homomorphism and suppose that e is the identity of H.
2 First, we show that the kernel is a subgroup of G. The kernel is non-empty because $\phi(e_G) = e$
3 where e_G is the identity in G. Let $x, y \in \ker\phi$, then:

$$\phi(xy^{-1}) = \phi(x)\phi(y^{-1}) = \phi(x)(\phi(y))^{-1} = ee^{-1} = e.$$

4 So $xy^{-1} \in \ker\phi$ and thus $\ker\phi$ is a subgroup.
Let $k \in \ker\phi$, then consider $\phi(aka^{-1})$ where $a \in G$. Then

$$\phi(aka^{-1}) = \phi(a)\phi(k)\phi(a^{-1}) = \phi(a)e\phi(a)^{-1} = \phi(a)\phi(a)^{-1} = e.$$

5 Therefore, $aka^{-1} \in \ker\phi$ so $\ker\phi$ is normal. □

Theorem 1 (The First Isomorphism Theorem). *If $\phi : G \to H$ is a group homomorphism, then*

$$\frac{G}{\ker\phi} \cong \phi(G).$$

6 *Proof.* First, we note that the kernel, $K = \ker\phi$ is normal in G.

7 Define $\beta : \frac{G}{K} \to \phi(G)$ by $\beta(gK) = \phi(g)$. We first show that β is a well-defined map. If
8 $g_1K = g_2K$, then for some $k \in K, g_1k = g_2$; consequently,

$$\beta(g_1K) = \phi(g_1) = \phi(g_1)\phi(k) = \phi(g_1k) = \phi(g_2) = \beta(g_2K).$$

9 Thus, β does not depend on the choice of coset representatives, and the map $\beta : G/K \to \phi(G)$
10 is well-defined.
11 We must also show that β is a homomorphism:

$$\begin{aligned}\beta(g_1Kg_2K) &= \beta(g_1g_2K) \\ &= \phi(g_1g_2) \\ &= \phi(g_1)\phi(g_2) \\ &= \beta(g_1K)\beta(g_2K).\end{aligned}$$

12 Clearly, β is onto $\phi(G)$. To show that β is one-to-one, suppose that $\beta(g_1K) = \beta(g_2K)$.
13 Then $\phi(g_1) = \phi(g_2)$. This implies that $\phi(g_1^{-1}g_2) = e$, or $g_1^{-1}g_2$ is in the kernel of ϕ; hence,
14 $g_1^{-1}g_2K = K$; that is, $g_1K = g_2K$.

□

Figure 2. The Proof of the First Isomorphism Theorem and Lemma Needed for the Proof adapted from Judson (2018)

There are a range of challenges inherent in the notation of the theorem and proof, which relate directly to the *sign/referent* dichotomy and the *ostensive/non-ostensive* dichotomy. These dichotomies are related but distinct, since a sign's referent may also be, at least partially, ostensive, as when $K = ker(\phi)$ [line 6]. That is, the sign *K* refers to the kernel of ϕ and information about that referent is made ostensive. An ostensive/non-ostensive duality may also be drawn upon without navigating the sign/referent dichotomy. For example, $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ [line 11] can result from manipulating the ostensive (applying the homomorphism property syntactically) without any semantic connection to the underlying concepts or contexts. In many cases though, there is a strong relationship between the *sign/referent* duality and *ostensive/non-ostensive* duality within proofs. The referent of an ostensive (observable expressions treated as signs) is most often non-ostensive, and thus comprehension would involve navigating both dualities simultaneously. One challenge we note in the expressions in the theorem and proof is that many objects play non-obvious roles. Groups (including subgroups) are more often denoted by single capital letters, meaning that expressions like $\phi(G)$ and $ker(\phi)$ are uncommon group notation. The group $ker(\phi)$ is later renamed to fit the convention [line 6]. This renaming appears helpful since it renders the cosets of the kernel into a more familiar expression for a coset (gK rather than $gker(\phi)$); however, the new ostensive less obviously refers to a kernel.

The isomorphism β maps elements of the quotient group to outputs of the homomorphism ϕ. Taking cosets as inputs plays on the *unitary/systematic* dichotomy since sets of elements are treated as singular objects, which is further complicated by the coset convention of working with the ostensive representatives of the set to stand for the equivalence class. To comprehend the theorem and proof, students at various times need to see cosets as unitary objects, but also decompose them into individual elements such as arriving at the implication from coset equivalence: $g_1 k = g_2$ [line 8]. Further, like almost any proof in abstract algebra, the proof proceeds in terms of the properties stipulated to the various, general objects (*types*) leaving the more particular objects that are members of that general class (*examples*) replaced by arbitrary, ostensive placeholders (e.g., G, H, ϕ).

We provide one last example related more to practices than mere relationships of reference that portrays the role of the *personal/institutional* dichotomy. As discussed in the literature above, the coset algorithm implicitly forms a strong part of students' personal meanings for cosets. This proof defies carrying out this algorithm as the arbitrary nature of the homomorphism and groups involved does not provide a particular list of kernel elements or an operation to use. Rather, the cosets are constructed using pre-images of the homomorphism, which induces a partition of the group G. Further, the proof relies on the normality of the subgroup to argue that this structure is in fact a quotient group. These institutional meanings have the potential to conflict with students' personal meanings, if they are rooted in the coset algorithm. These examples all portray some of the many ways we see this theorem/proof as a rich opportunity to explore students' proof comprehension and meanings for group-theoretical objects through the onto-semiotics lens

Methods

This study is part of a larger project aimed at engaging students in authentic proving activity in abstract algebra (Melhuish et al., 2022). The project used a design-based research approach where tasks are developed based on hypotheses around how teacher actions and task choices can engender student engagement in a range of mathematical activities related to proof. The tasks and hypotheses were then modified through cycles of testing, first in a task-based interview setting and then in a classroom setting. For this paper, we focus on task-based interview data.

Data and Setting

We draw on two cycles of enacting a task with small groups of students who were guided to comprehend the FIT and its proof. The task-based interview was developed using Mejia-Ramos et al.'s (2012) comprehension assessment framework. The components of the framework were converted into a series of tasks that included activities such as explaining the meaning of terms in the theorem, connecting the theorem to specific examples, anticipating the proof approach, summarizing the proof, connecting the proof to specific examples, and warranting lines within the proof.

All study participants had recently completed an introductory abstract algebra course that focused primarily on introducing group theory concepts including quotient groups and typically spent one lesson (or one part of a lesson) talking about the FIT, including its proof. We worked with all students who volunteered for the study. In cycle 1, we met with two undergraduate students (Elena[1], Elsa) for three hours split between two sessions, and in cycle 2, four undergraduate students (Jasmine, Eric, Andy, Miguel) in one two-hour session. Prior to the first cycle, we conducted a pre-interview finding that neither student recalled details of the FIT. For

[1] All names are pseudonyms.

both groups of students, they engaged in a task that involved creating and analyzing cosets with $\mathbb{Z}_n$ groups in the task-based interview prior to the FIT interview, so we were confident that students had basic knowledge of constructing the elements of quotient groups and comfort working with the $\mathbb{Z}_n$ groups. Of our participants, five had a traditional introduction on quotient groups building from equivalence classes and coset generation and one (Elsa) had an instructor who used an inquiry-oriented curriculum where students partitioned groups first and introduced the coset generation algorithm later. Most had formal definitions provided with examples while one student was introduced to examples first and then encountered the formal definition.

Across both sessions, the first author of this paper served as primary instructor-researcher, and another member of the research team served as a second instructor-researcher asking additional questions. Both sessions were observed by members of the project team who took field notes. Additionally, the sessions were video-recorded and transcribed, and student work was collected.

Methods of Analysis

We began analysis with cycle 2. A member of our research team reviewed the video and transcript identifying a series of episodes where there appeared to be a personal/institutional conflict related to quotient groups or functions. Three members of the research team then analyzed these episodes in terms of semiotic conflicts, resolutions, dualities, and representations. After this initial pass, the team met to resolve any discrepancies and to develop more sophisticated operationalizations of the analytic tools. This afforded refined interpretations of the students' practice. Due to the complexities involved, we focused specifically on quotient groups for the next round of analysis.

We reviewed the data to identify any moments when students discussed quotient groups inclusive of cosets, the canonical homomorphism, or quotient groups, themselves. Episodes started just before the concepts came up, for context, and ended once the students moved on to a different topic. All episodes (across both cycles) were then analyzed independently by two researchers to arrive at a series of analytic memos documenting evidence of (1) partial meanings evoked, (2) dualities, (3) semiotic conflicts, and (4) semiotic shifts. A partial meaning was ascribed at any point in which a student was in activity with the focal mathematical object: quotient groups. Any relevant dualities were documented at this point along with any semiotic conflicts. Finally, we also documented semiotic shifts at any place where students navigated a duality (and noted if instructional prompts led to shifts to better understand the context). The independent coders met, and coding discrepancies were resolved via discussion. From this analysis, we developed a narrative of both groups' activity.

Results

In this section, we provide a description of our two cycles of implementation focusing first on how divergent partial meanings shaped the comprehension activity across the two groups of students. We begin by sharing an overview of the many partial meanings for quotient groups we documented in the students' practice (Table 1), which we elaborate in the episodes that follow. Though we present the meanings separately, they do not operate as such. For example, students may evoke the coset algorithm while also attending to group structure. In fact, we argue that coordinating multiple meanings is evidence of greater sophistication in students' practice around quotient groups. In the next subsections, we provide an overview of cycles where students drew on partial meanings related to the coset algorithm and partitioning, respectively. We focus primarily on theorem comprehension in these narratives, then conclude with several additional themes related to proof comprehension.

Table 1. Partial meetings related to quotient groups

		Description	**Related Partial Meanings**
Quotient Group (QG) Creation	Coset Algorithm	The standard approach to creating the coset elements in quotient groups via the definition: $a+H$ where H is a subgroup and a is a group element. This algorithm can be proceduralized and may be divorced from group structure as it produces a list of cosets (Hazzan, 1999).	QG as coset algorithm
			QG as list of cosets
	Partitioning	Cosets induce a partition of the original group. Partitioning G into sets that form a new group can serve as an alternate procedure for creating a quotient group (Larsen & Lockwood, 2013).	QG as partitioning G
			QG cosets composed of elements from G
			QG cosets are distinct sets

	Homomorphism Preimage	A quotient group can be thought of as the pre-images of a homomorphism and correspondingly created by identifying pre-images	QG as homomorphism pre-image sets
	Factoring Out	The language of “factor group” may evoke a notion that something is being removed – the normal subgroup	QG as all G excluding the identity subgroup
The Structure of Quotient Group and its Elements	Normal Subgroups	The cosets of a normal subgroup form a quotient group. The normality requirement is often part of formal definitions and is necessary for proofs.	QG are induced by normal subgroup
	Group Structure	Students may or may not attend to QG as a group itself (Asiala et al., 1997 Hazzan, 1999; Melhuish, 2019).	QG is a group
	Coset Duality (Set and Element)	Cosets are sets of elements from the original QG. Students may sometimes foreground cosets as sets rather than elements (e.g., Melhuish, 2019) or as individual elements rather than sets (Asiala, et. al., 1997). Coordinating both meanings is important.	Cosets as sets Cosets as elements
	Cosets and Representative Element Relationships	The representative element of a coset can serve as a proxy for the coset itself and is often accessible either as just the element or in the form element + set (Siebert & Williams, 2003). In context, necessary meanings involve coordinated its membership and arbitrary nature with the coset	Cosets representatives are not unique Coset representatives are members of that coset

The Coset Algorithm Partial Meaning and FIT Comprehension

Our cycle 2 group of students primarily drew on the coset algorithm partial meaning when engaging with the tasks. We anticipated such deference based on our literature review. The session began with the students identifying important concepts involved in the statement of the FIT (sign→referent[2]) noting “factor group” as one of the key concepts involved. When the instructor-researcher asked what is meant by factor group or quotient group, the students’ responses included “a list of cosets” or “it’s one of the cosets, right or left, depending.” The instructor-researcher responded by writing a generic example of a group and subgroup ($G = \{e, g_1, g_2, \dots, g_5\}$; $H = \{e, h_1, h_2\}$) on the board prompting type→example and non-

[2] For ease we use this notation to indicate our codes of semiotic shifts that either occurred or were invited by the instructor.

ostensive→ostensive shifts. Eric explained the resulting quotient group saying, “You take some element of G and either depending if we’re building left or right whatever the operation is you do that operation on H.” At this point, none of the students raised the partial meaning of group structure, reminiscent of the common personal/institutional conflict from the literature. After the instructor-researcher resolicited the meaning of the quotient group from the instructor-researcher, Miguel and Eric again drew on coset algorithm meanings. Miguel suggested that “the set of all the cosets, that’s G” and Eric stated they do not remember and need to write out the kernel of ϕ. In contrast, Andy and Jasmine focused on the term “factor” with Andy suggesting, “I thought it was all the elements excluding those that had been factored out. I could be wrong though.”

The instructor-researcher recognized the personal/institutional conflicts and prompted the students to attend to the group structure by focusing on the *group* part of the name (“We’re calling it quotient group so hopefully this thing we’re building is a group of some sort”) and returned to the generic example to ask what elements of the quotient group would look like (non-ostensive→ostensive). Miguel expressed hesitation stating, “I can’t think of what they look like.” Andy and Eric draw on the group structure partial meaning to suggest, “the identity will be included.” The instructor-researcher asked what the identity would be, and Eric suggested the element “e” before Andy asked whether H was a subgroup. This prompted Eric to reconsider and suggest that “$[H]$ would just be e then.”

At this point, we observe complexities introduced by the naming conventions in group theory (signs). Eric’s initial response of e may reflect a partial meaning where a coset is a set of elements rather than a single element (systematic rather than unitary). However, when discussion focused on the subgroup $H,$ Eric shifted from e as serving an example to a type providing the label for the identity coset (the sign shifted referent). The students then continued to build cosets from the generic example, providing signs. We conjecture that the sign, “e" (ostensively available) played a substantial role in the meanings students evoked.

In the next portion of the task, students worked in pairs to connect the FIT to specific examples (type→example). Jasmine and Eric worked on the example of $\phi\colon \mathbb{Z}_{12} \to \mathbb{Z}_3$ where $\phi(x) = x_{mod\ 3}$, and Andy and Miguel worked on the example $\phi\colon \mathbb{Z} \to \mathbb{Z}_4$ where $\phi(x) = x_{mod\ 4}$. The task included a prompt to create a function diagram, an ostensive that can provide insight into the theorem. In both pairs, the students created a function diagram encoding the relevant homomorphism. The students also used the coset algorithm to create cosets using the kernel.

Figure 3 reflects the two distinct ostensives: the function diagram (top) and cosets (bottom). Miguel began coordinating with the mapping noting that “There’s an isomorphism with the 1,2,3 [from the image]. I don’t know how to explain that.” He questioned, “So what is our quotient group? This is our quotient group [pointing to the three non-identity cosets]?” Andy hesitantly asked in response, “Yes. So, then our quotient group only has three elements then why not this one [the identity coset]?” reminiscent of the “factoring” partial meaning conflict from earlier in the discussion. The need for an isomorphism appeared to support Andy in recognizing the conflict with the factoring partial meeting. Jasmine and Eric had a similar set of cosets and a function diagram (Figure 4), but questioned where the “1-1” portion was. This may indicate they were not yet seeing the cosets as single elements (systematic rather than unitary).

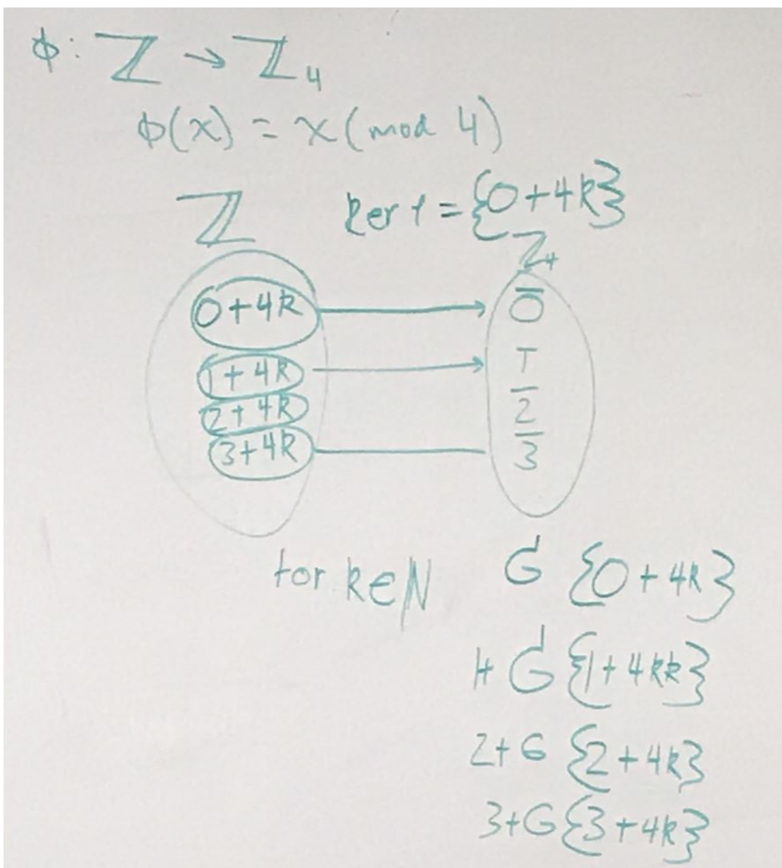

Figure 3: Function diagram from Andy and Miguel

Noticing ongoing conflicts in moving from type to example, the instructor-researcher decided to initiate a whole group discussion about one example. Jasmine and Eric explained their function diagram (Figure 4) identifying where each element mapped. Andy's explanation suggests the ostensive supported a semiotic shift:

> Andy: Although with this, at least I'm starting to really see the coset groups forming individual elements.
>
> Instructor-researcher: So you're saying you split up where are you seeing these coset groups forming in here?
>
> Andy: Where each map to... each elements that map to each individual element in $\mathbb{Z}_3$.

Grouping elements by "where each map to" is the first evidence that students began developing a partial meaning for quotient groups related to homomorphism (the idea that every homomorphism induces a quotient group on the domain is central to the FIT).

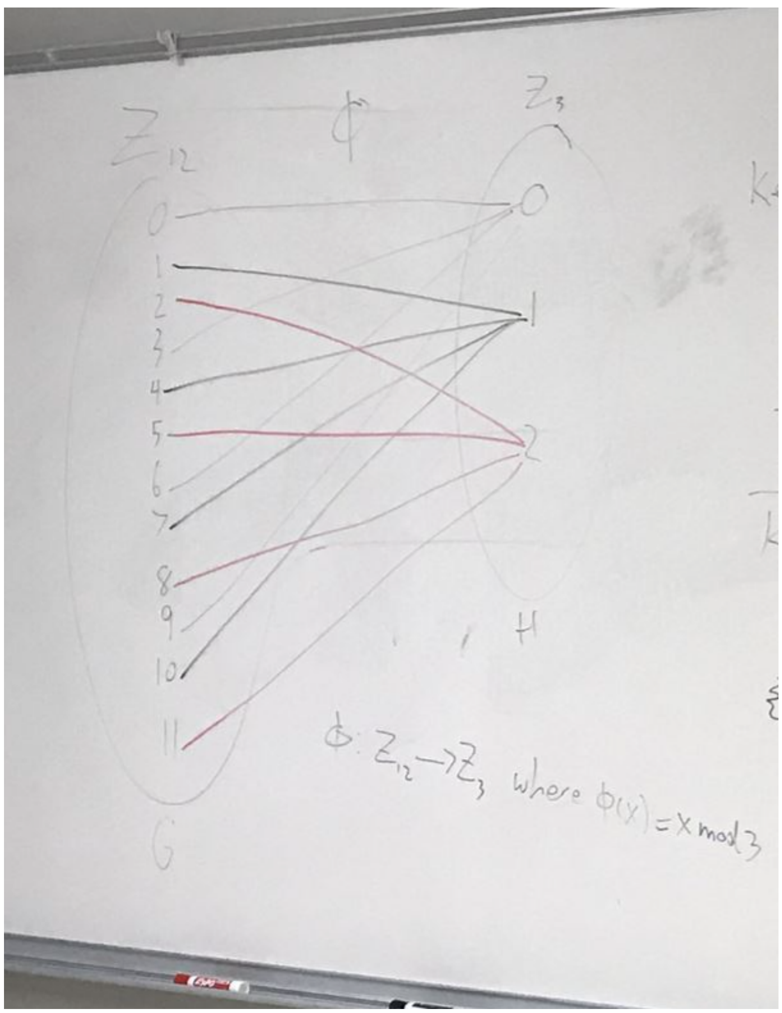

Figure 4: Function diagram from Jasmine and Eric

The instructor-researcher asked the students to identify the parts of the theorem in the examples. The students articulated that they need the kernel, whose referent Eric and Jasmine identified as the elements 0, 3, 6, and 9, and Miguel identified as the "purple lines." The instructor-researcher asked, "We've had homomorphism, you pulled out a kernel. Where is our isomorphism here? Where is our quotient group here?" Miguel responded, "I feel like you need the factor group first to see the isomorphism." This suggested Miguel was not reasoning about the quotient group as preimages of the homomorphism (as Andy did). Rather, Miguel returned to the coset algorithm partial meaning:

> So, I just started with what we have here, kernel of ϕ. Just going to rewrite it. And then what we said from the left I added the next operation of our H, $1 + ker(\phi)$ yield 1, 4, 7, 10 and then the last one, 2. So $2 + ker(\phi)$, ... 2, 5, 8, 11 and then that's how I build the cosets with that one.

The instructor-researcher re-prompted the students to identify the isomorphism (sign→referent), to which Miguel noted that the quotient group and image have the same order. The instructor-researcher further invited the students to use their diagram to identify cosets (systematic→unitary):

> Instructor-researcher: And can we actually see where these cosets somewhere over here in this picture?
>
> Miguel: If you got all the purple lines, how he drew that-
>
> Eric: They would regroup it like that
>
> Miguel: Yeah. You would have a purple little bubble for $\mathbb{Z}_{12}$.

The students grouped the elements based on their image (represented by line color). Eric explained, "So you put the factor group like an element. You put the 0, 3, 6, 9 bracketed off and that becomes the one element." This provides evidence of an emerging homomorphism-induced quotient group partial meaning. Instantiating the theorem promoted attention to the unitary/systematic duality where cosets are both sets and individual elements. Figure 5 shows the final ostensive created to express the grouping by image. Miguel and Eric noted that the red circles would be "your factor group."

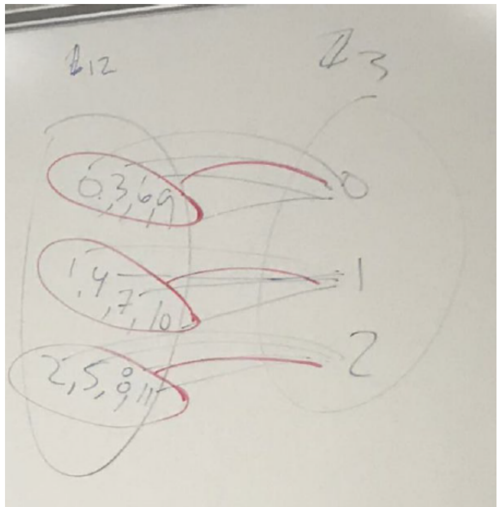

Figure 5: Function diagram with isomorphism present

The coset algorithm was the most consistent partial meaning drawn on throughout the task. At some points, the partial meaning appeared to constrain activity, especially when other meanings for quotient group were backgrounded. However, the algorithm meaning was quite productive in creating examples and ostensives that aided in comprehending and ultimately coordinating with the mappings involved in the theorem – allowing for opportunity to develop a homomorphism pre-image meaning.

The Partitioning Partial Meaning and FIT Comprehension

Unlike the students in cycle 2 who relied on the coset algorithm, the students from cycle 1 drew substantially on the partitioning and group structure partial meanings. When asked to share their meaning for "quotient group," Elena explained:

> It has to do with distinct cosets. Where if we had, let's say, G has 12 elements and our denominator, I guess of this fraction looking thing. That is groups that are made up of elements from a larger [group]. And so, the magnitude of those groups divided themselves. I don't know – that it would be another group itself.

We note two semiotic features of this explanation. First, the meanings articulated align with structural features of cosets (distinct, made up of elements from a group) that are naturally congruent with partitioning, which is reflected in the language "divided." Elena also introduced a specific order to share her meaning (type→example). The theorem's implied relationship of order likely prompted her attention to magnitude.

At this point, the instructor-researcher explicitly asked what the elements in $G/ker(\phi)$ would look like (non-ostensive→ostensive) in order to better understand the division language. After an initial voicing of the same personal/institutional conflict as the cycle 2 students (all elements not in the kernel – *factoring out meaning*), the students quickly returned to partitioning language like "sort[ing] into cosets," "main group divided by the identity H and that will give us the other elements," and "you split it into different cosets." We conjecture that use of dividing rather than factoring language may have supported a quick resolution to this conflict.

This partitioning meaning appeared quite productive for interpreting the FIT, as evidenced when the students were invited to create a diagram (ostensive) for the homomorphism $\phi: \mathbb{Z}_6 \to \mathbb{Z}_2$ defined by $\phi(x) = x_{mod\ 2}$ (type→example). The students first identified the kernel, then attended to the orders:

> Elsa: So then it would be $\mathbb{Z}_6$ divided by that group, would give us what's left, which is just
>
> Elena: Yeah. I think if we go back to the absolute value ... 0, 2, 4, we wind up with six elements over three elements
>
> Elsa: Has to have two
>
> Elena: Right, and that would make sense, because there are three elements here and there's three elements left, so there's two cosets. I don't know if it has a relation. You know there's only two relationships really? One that goes to the identity and the other that goes to one.

We can observe several features of the meanings at play. First, the idea of "divid[ing] up the group" indicates a partitioning meaning. Elena also coordinates this meaning with the homomorphism map where there are only two "relations" or "relationships" because there are only two elements to map to. Such coordination of meanings reveals sophistication in reasoning about the concepts in the FIT. This is further evidenced by their shift to an example and ostensive diagram relating the cosets and the homomorphic image, as well as the orders in the theorem ("absolute value"). We claim that a homomorphism-induced coset meaning was emerging. This meaning is drawn on throughout the discussion such as when the students identified the isomorphism in their example (type→example).

Overall, we observe the partitioning partial meaning appeared to support the cycle 1 students in quickly coordinating with a homomorphism-induced partition partial meaning. In general, these students had less personal/institutional semiotic conflict in the FIT context. It afforded them more sophistication in coordinating meanings for quotient group, supporting the emergence of a single perceived structure as described in the theorem/proof.

How Partial Meanings Play out within Proof Comprehension

We noted above how certain partial meanings better supported sophisticated reasoning about the concept of quotient group in the FIT. Since these meanings were emergent, we claim this reveals how comprehending of such theorems and proofs contributes to concept formation (the emergence of quotient groups as objects from a homomorphism). However, different partial meanings influenced proof comprehension in various ways. Samkoff and Weber (2015) explained how discussing the definition of concepts in a theorem can support proof comprehension, if the definition anticipates the proof structure. While we saw similar evidence, we also found partial meanings influence 1) perceived proof structure, 2) warranting statements in proofs, and 3) the emergence of semiotic conflicts; these are all potential learning opportunities. Our semiotic lens allows us to go beyond Samkoff and Weber's goal that students comprehend the proof (personal/institutional) to understand how proof comprehension can provide opportunities for students to develop more sophisticated understandings for concepts like quotient group.

Comparing across the two groups, we can see how the normal subgroup partial meaning of quotient group was essential to the proof structure. When reading the signs in the lemma statement (the kernel is normal) and its proof, the students in cycle 1 successfully drew on the normal subgroup partial meaning to articulate why the lemma was needed. Elena explained, "So, that factor group, we have to first establish that the kernel is normal to the group to be able to have a group in the first place." In contrast, cycle 2 students focused on the relationship between normality and coset algorithm. When asked why normality was needed, Andy shared:

> Because first of all for this subgroup I don't know about normal yet, for a subgroup it has to be a subgroup so we can properly draw out cosets then map 1-1 on this mapping that we have normal because the right and left, no matter which way you apply it, you need to be able to achieve the same cosets.

We interpret Andy's reference to drawing cosets as a reference to the coset algorithm. Eric elaborated, "Because it's normal. Or we could have put the 1 and the 2 on the other side and would have still given us the same thing" referencing the $1 + ker(\phi)$ and $2 + ker(\phi)$

ostensives from their earlier example. This focus on the coset algorithm superseded any explicit focus on the role of normality in forming a quotient group. The evocation of a meaning that aligns or does not align with proof structure is reminiscent of Samkoff and Weber's (2015) findings.

We also observed the evoked meanings influence warranting. For example, when addressing the line, $\beta(g_1Kg_2K) = \beta(g_1g_2K)$, Elena explained, "Well, if we know that the cosets g_1K and g_2K are part of the factor group G/K, then these products should also be, I guess cause it's a group." We see the students identifying what g_1K and g_2K are (sign→referent) and evoking the group structure meaning (unitary→systematic as students deconstructed the factor group structure to reason about a specific property) to provide the warrant for closure. In contrast, the cycle 2 students did not evoke a group structure meaning at any point during proof comprehension.

We conclude by examining the discussion in both groups around the line including $g_1K = g_2K$ to illustrate how the relationship between meanings and proof comprehension is reflexive (each supports the other). When the students encountered this line, the instructor-researcher asked what type of object g_1K was (sign→referent) and what it means for two cosets to be equivalent (unitary→systematic to reason about the cosets in terms of their members). In cycle 1, Elena and Elsa shared different partial meanings with Elena responding, "their elements are equal" (set equality), and Elsa, "That they map to the same thing in our $\phi(G)$ group" (homomorphism partitioning). Both partial meanings are valuable in this context and can play complementary roles in coordinating general ideas about sets and their new emerging meaning related to the homomorphism-induced partition. These meanings later supported shifts between the proof and their specific examples.

In contrast, the cycle 2 students continued to rely on their coset algorithm partial meanings; however, in this case the meaning supported them in their comprehension activity. When asked what $g_1K = g_2K$ meant, Miguel and Eric voiced conflicting personal meanings, the sets are the "same" (set-wise) or "match" (element-wise), respectively. This conflict likely reflects an important aspect of the meaning of sets: order does not matter. The instructor-researcher then prompted, "What does g_1K look like, what would that coset look like generically?" (non-ostensive→ostensive). Andy explained, "It would look like the kernel with some operation applied to it. Would g_1 operation apply to it." They used the coset algorithm to create a generic coset: $g_2K = \{g_2, g_2k_1, g_2k_2, ...\}$. This ostensive supported explication of the conflict between how two generated cosets should relate. Miguel advocated for the elements to match (g_1 and g_2 being the same as g_1k_1 and g_2k_1, respectively). We conjecture that Miguel was grappling with the unitary/systematic duality by focusing on equivalent individual elements rather than the coset as a single object. Eric countered that the elements would be the same, but "not necessarily those... all the elements in g_2K there's an element that matches them somewhere in g_1K." Jasmine revoiced this idea, "So somewhere along the line there would be g_2 in the g_1K function." The group of students appeared in agreement, and they quickly returned to the proof line to justify the claim that g_2 had a matching element. Engaging with this line of the proof provided opportunity for a conflict to become explicit, and for the students to use their coset algorithm meaning to create an ostensive that supported resolution.

Conclusion and Discussion

The FIT provides a context where students can develop additional meaning for quotient groups in terms of their relationship with homomorphisms. While much of the literature on quotient groups focuses on meanings connected to coset generation and partitioning to form a group, Rupnow's (2021) interviews with mathematicians suggest that homomorphisms inducing equivalence classes (and thus a quotient group) is an additional, essential partial meaning. Further, this meaning is reflected in textbooks such as Pinter (2010) who explained that "the notions of homomorphic image and of quotient group are interchangeable" (p. 157). Our findings echo these claims that homomorphism inducing a partition is a productive meaning for quotient groups (and their creation) that coordinates with other meanings. Further, our study suggests that students' pre-existing partial meanings for quotient groups may anticipate (partitioning) or lead to conflicts (coset algorithm) with the homomorphism partial meaning reflected in the FIT. Cycle 1 students appealed to the partitioning meaning consistently and quickly integrated the homomorphism meaning. In contrast, the cycle 2 students relied primarily on the coset algorithm meaning, initially treating the quotient group and homomorphism map as independent objects. It was through reorganizing the ostensives (mapping diagrams, Figures 4 and 5) in the examples that they coordinated the objects to develop a common referent.

The contributions of this paper are twofold. First, we documented a series of partial meanings for quotient groups emerging in proof comprehension. While some partial meanings had been identified separately before, we observed how they interacted with each and how meanings and comprehension activity co-emerge as students engage the FIT. We also documented partial meanings not found in the literature (such as “factoring out”) and how the partial meanings related to the FIT and signs used. For example, “dividing” seemed to connect to a partitioning meaning more than “factoring.” Furthermore, having to identify an isomorphism surfaced the personal/institutional conflict between the factoring partial meaning (removing the identity) and the normative quotient group structure. The FIT and its proof rely on the homomorphism preimages to induce a quotient group and alternates between unitary and systemic interpretations of cosets. Such proofs incentivize and support the emergence of rich and varied partial meanings.

The previous point reveals the inherent value of our use of the onto-semiotics lens. That framework intends to explain how mathematical objects can emerge through practice while having no ontological status. We have used it rather to document how students construct a single object (homomorphism-induced quotient group) through their semiotic activity around a theorem and proof (as well as the diagrams and examples prompted by the instructor-researcher). The formal mathematics involved in theorems and proofs involves a lexical density and degree of abstraction that lends itself well to an analysis of how students construct meaning and manage semiotic dualities.

Prior researchers have suggested that illustrating the theorem or proof with examples or trying to anticipate the proof are productive (Mejía-Ramos et al., 2012; Samkoff & Weber, 2015), our study illustrates in greater depth how exploring examples can support proof comprehension and how this can be targeted in instructional design. In our case, those activities supported the construction and coordination of partial meanings, which contributed to proof and theorem comprehension. Furthermore, we can reflect on ways the instructor-researcher introduced or amplified semiotic conflicts to provide space for resolution. By engaging students with specific homomorphisms, they were positioned to transition from example to type. The instructor-researcher played a role in prompting students to move between the more general statements and the specifics of their examples throughout the activity. We also see the instructor-researcher attending to conflicts between personal and institutional meanings such as the group structure component of quotient groups and engaging students in conversation about these ideas. That is, the instructor-researcher introduced conflicts by setting up a task where students needed to traverse between example and type and relying entirely on the coset algorithm was insufficient. They also amplified conflicts related to institutional meanings as they arose. Often, the instructor-researcher then prompted engagement with and creation of ostensives as a concrete means to surface and resolve conflicts and engage students in making important transitions.

We also wish to note that semiotic conflicts and resolutions are informed by contexts beyond the scope of our task-based interview study. That is, the students' prior experiences with instructors and textbooks may have primed certain conflicts, and both the students and instructor-researcher served to unearth a particular set. For example, Font and Contreras (2008) illustrated how the authors of textbooks leave the reader to establish semiotic functions that are key to understanding, which may account for the conflicts we observed in our study. Further, our students came from different introductory abstract algebra courses, and their instructor presentations likely influenced their activity in our study. We suggest additional research into the students' broader learning ecology to understand the role of textbooks, instructors, and semiotic functions in relation to proof comprehension.

The final point we make about our findings relates to the reflexive relationship between concepts and proving. Lakatos (1976) described certain definitions as expressing proof-generated concepts. This suggests that proofs contribute to concept formation. Our example of how reading the FIT (and our instructional supports thereof) built upon and connected partial meanings (for cycle 1 students) or helped foster the emergence of new partial meanings (for cycle 2 students). This is a practical illustration of a proof-generated concept in student experience, though in practice it co-exists and interacts with other partial meanings in complex ways.

Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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