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Affinely representable lattices, stable matchings, and choice functions

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Abstract

Birkhoff's representation theorem (Birkhoff, Duke Math J 3(3):443–454, 1937) defines a bijection between elements of a distributive lattice and the family of upper sets of an associated poset. Although not used explicitly, this result is at the backbone of the combinatorial algorithm by Irving et al. (J ACM 34(3):532-543, 1987) for maximizing a linear function over the set of stable matchings in Gale and Shapley's stable marriage model (Gale and Shapley, Am Math Monthly 69(1):9–15 1962). In this paper, we introduce a property of distributive lattices, which we term as affine representability, and show its role in efficiently solving linear optimization problems over the elements of a distributive lattice, as well as describing the convex hull of the characteristic vectors of the lattice elements. We apply this concept to the stable matching model with path-independent quota-filling choice functions, thus giving efficient algorithms and a compact polyhedral description for this model. To the best of our knowledge, this model generalizes all those for which similar results were known, and our paper is the first that proposes efficient algorithms for stable matchings with choice functions, beyond classical extensions of the Deferred Acceptance algorithm.

Keywords Stable matching · Choice function · Distributive lattice · Birkhoff's representation theorem · Extended formulations

Mathematics Subject Classification Primary: 91B68 (Matching models) · Secondary: 52B12 (Special polytopes (linear programming, centrally symmetric, etc.))

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1 Introduction

Since Gale and Shapley's seminal publication [21], the concept of stability in matching markets has been widely studied by the optimization community. With minor modifications, the one-to-many version of Gale and Shapley's original stable *marriage* model is currently employed in the National Resident Matching Program [36], which assigns medical residents to hospitals in the US, and for assigning eighth-graders to public high schools in many major cities in the US [2].

In this paper, matching markets have two sides, which we call firms F and workers W. In the marriage model, every agent from $F \cup W$ has a *strict preference list* that ranks agents in the opposite side of the market. The problem asks for a *stable matching*, which is a matching where no pair of agents prefer each other to their assigned partners. A stable matching can be found efficiently via the Deferred Acceptance (DA) algorithm [21]. Although successful, the marriage model does not capture features that have become of crucial importance both inside and outside academia. For instance, there is growing attention to models that can increase diversity in school cohorts [34, 43]. Such constraints cannot be represented in the original model, or its one-to-many or many-to-many generalizations, since admission decisions with diversity concerns cannot be captured by a strict preference list.

To model these and other markets, instead of ranking individual potential partners, each agent $a \in F \cup W$ is endowed with a *choice function* $\mathcal{C}_a$ that picks a team she prefers the best from a given set of potential partners. See, e.g., [8, 18, 28] for more applications of models with choice functions. Models with choice functions were first studied in [29, 37]. *Mutatis mutandis*, one can define a concept of stability in this model as well (for this and the other technical definition mentioned below, see Sect. 2). Two classical assumptions on choices functions are *substitutability* and *consistency*, under which the existence of stable matchings is guaranteed [7, 25]. Clearly, existence results are not enough for applications (and for optimizers). Interestingly, little is known about efficient algorithms in models with choice functions. Only extensions of the classical Deferred Acceptance algorithm for finding the one-side optimal matching have been studied for this model [14, 37].

The goal of this paper is to study algorithms for optimizing a linear function w over the set of stable matchings in models with choice functions, where w is defined over firm-worker pairs. Such questions are classical in combinatorial optimization, see, e.g., [41] (and [31] for problems on matching markets), and having efficient algorithms for such questions allows one to find the optimal stable matching for various linear objectives, such as profit-maximal and egalitarian (i.e., fair for both sides of the market). We focus on two models. The first model (CM- MODEL) assumes that all choice functions are substitutable, consistent, and cardinal monotone. The second model (CM- QF- MODEL) additionally assumes that for one side of the market, choice functions are also *quota-filling*. Before proceeding, let us give some concrete examples of such models.

In the school choice problem, many mechanisms that combine stability with affirmative actions can be viewed as modeling schools' decisions via choice functions. We here describe two popular mechanisms, which are called *majority quota* [2] and *minority reserve* [24]. Under majority quota, a school's choice function first chooses the top students one by one until the number of majority students chosen is at its majority quota, and then chooses the top minority students until the total quota is met or until there are no more minority students. Under minority reserve, a school' choice function first chooses the top minority students up to its reservation quota, and then chooses the remaining top students (both majority and minority) until the total quota is met or until there are no more students to choose from. Both mechanisms have students simply ranking the schools in order of preference, as in the classical model, and thus both fall under the CM- QF- MODEL.

Both the (CM- QF- MODEL) and the (CM- MODEL) generalize all classical models where agents have strict preference lists, on which results for the question above were known. For these models, Alkan [5] has shown that stable matchings form a distributive lattice. As we argue next, this is a fundamental property that allows us to solve our optimization problem efficiently.

1.1 Our contributions and techniques

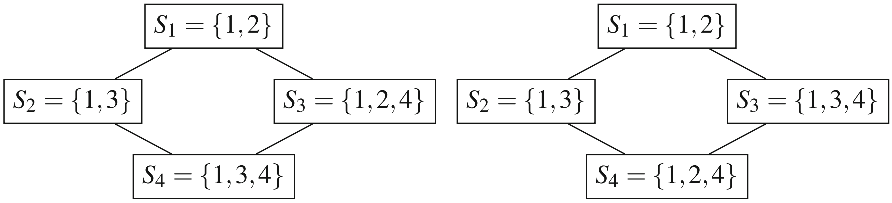
We give here a high-level description of our approach and results. For the standard notions of posets, distributive lattices, and related definitions see Sect. 2. All sets considered in this paper are finite.

Let $\mathcal{L} = (\mathcal{X}, \succeq)$ be a distributive lattice[1], where the elements of $\mathcal{X}$ are distinct subsets of a base set E and $\succeq$ is a partial order on $\mathcal{X}$. We refer to $S \in \mathcal{X}$ as an *element* (of the lattice). Birkhoff's theorem [12] implies that we can associate[2] to $\mathcal{L}$ a poset $\mathcal{B} = (Y, \succeq^\star)$ such that there is a bijection $\psi : \mathcal{X} \to \mathcal{U}(\mathcal{B})$, where $\mathcal{U}(\mathcal{B})$ is the family of *upper sets* of $\mathcal{B}$. $U \subseteq Y$ is an upper set of $\mathcal{B}$ if $y \in U$ and $y' \succeq^\star y$ for some $y' \in Y$ implies $y' \in U$. We say therefore that $\mathcal{B}$ is a *representation poset* for $\mathcal{L}$ with the *representation function* ψ. See Example 1 below. $\mathcal{B}$ may contain much fewer elements than the lattice $\mathcal{L}$ it represents, thus giving a possibly "compact" description of $\mathcal{L}$. The representation poset $\mathcal{B}$ and the representation function ψ are univocally defined per Birkhoff's theorem. Moreover, the representation function ψ satisfies that for $S, S' \in \mathcal{X}$, $S \succeq S'$ if and only if $\psi(S) \subseteq \psi(S')$. Although $\mathcal{B}$ explains how elements of $\mathcal{X}$ are related to each other with respect to $\succeq$, it does not contain any information on which items from E are contained in each lattice element. We introduce therefore Definition 1. For $S \in \mathcal{X}$ and $U \in \mathcal{U}(\mathcal{B})$, we write $\chi^S \in \{0,1\}^E$ and $\chi^U \in \{0,1\}^Y$ to denote their characteristic vectors, respectively.

Definition 1 Let $\mathcal{L} = (\mathcal{X}, \succeq)$ be a distributive lattice on a base set E and $\mathcal{B} = (Y, \succeq^\star)$ be a representation poset for $\mathcal{L}$ with representation function ψ. $\mathcal{B}$ is an *affine representation* of $\mathcal{L}$ if there exists an affine function $g : \mathbb{R}^Y \to \mathbb{R}^E$ such that $g(\chi^U) = \chi^{\psi^{-1}(U)}$, for all $U \in \mathcal{U}(\mathcal{B})$. In this case, we also say that $\mathcal{B}$ *affinely represents* $\mathcal{L}$ via function g and that $\mathcal{L}$ is *affinely representable*.

[1] Note that the ordering of lattices used in this paper is opposite to the standard notations in lattice theory, where lattices are usually represented by $\mathcal{L} = (\mathcal{X}, \preceq)$. Our notation, however, follows classical stable matching literature.

[2] The result proved by Birkhoff is actually a bijection between the families of lattices and posets, but in this paper we shall not need it in full generality.



(a) Lattice affinely representable **(b)** Lattice not affinely representable

Fig. 1 Lattices for Example 1

We observe that, in Definition 1, we can always assume $g(u) = Au + x^0$, where $A \in \{0, \pm 1\}^{E \times Y}$ and x^0 is the characteristic vector of the maximal element of $\mathcal{L}$. Indeed, $g(\chi^{\emptyset}) = x^0$. Moreover, for every $y \in \mathcal{B}$, there is $U, U' \in \mathcal{U}(\mathcal{B})$ such that $U' = U \setminus \{y\}$. Hence, letting a^y be the column of A corresponding to y, we have

$$a^y = g(\chi^U) - g(\chi^{U'}) = \chi^{\psi^{-1}(U)} - \chi^{\psi^{-1}(U')} \in \{0, \pm 1\}^E.$$

Example 1 Consider first the distributive lattice $\mathcal{L} = (\mathcal{X}, \succeq)$ whose Hasse diagram is given in the Fig. 1a, with base set $E = \{1, 2, 3, 4\}$.

The representation poset $\mathcal{B} = (Y, \succeq^\star)$ of $\mathcal{L}$ contains two non-comparable elements, y_1 and y_2. The representation function ψ maps S_i to U_i for $i \in [4]$ with $U_1 = \emptyset$, $U_2 = \{y_1\}$, $U_3 = \{y_2\}$, and $U_4 = \{y_1, y_2\}$. That is, $\mathcal{U}(\mathcal{B}) = \{U_1, U_2, U_3, U_4\}$. One can think of y_1 as the operation of adding $\{3\}$ and removing $\{2\}$, and y_2 as the operation of adding $\{4\}$. $\mathcal{B}$ affinely represents $\mathcal{L}$ via the function $g(\chi^U) = A\chi^U + \chi^{S_1}$ where

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ as } \begin{array}{l} g(\chi^{U_1})^\intercal = (0, 0, 0, 0) \quad + (1, 1, 0, 0) = (1, 1, 0, 0) = (\chi^{S_1})^\intercal; \\ g(\chi^{U_2})^\intercal = (0, -1, 1, 0) + (1, 1, 0, 0) = (1, 0, 1, 0) = (\chi^{S_2})^\intercal; \\ g(\chi^{U_3})^\intercal = (0, 0, 0, 1) \quad + (1, 1, 0, 0) = (1, 1, 0, 1) = (\chi^{S_3})^\intercal; \\ g(\chi^{U_4})^\intercal = (0, -1, 1, 1) + (1, 1, 0, 0) = (1, 0, 1, 1) = (\chi^{S_4})^\intercal. \end{array}$$

Next consider the distributive lattice $\mathcal{L}'$ whose Hasse diagram is presented in Fig. 1b. Note that the same poset $\mathcal{B}$ represents $\mathcal{L}'$ with the same representation function ψ. Nevertheless, $\mathcal{L}'$ is not affinely representable. If it is and such a function $g(\chi^U) = A\chi^U + \chi^{S_1}$ exists, then we must have

$$\chi^{S_1} + \chi^{S_4} = (\chi^{S_1} + A\chi^{U_1}) + (\chi^{S_1} + A\chi^{U_4}) = (\chi^{S_1} + A\chi^{U_2}) + (\chi^{S_1} + A\chi^{U_3}) = \chi^{S_2} + \chi^{S_3},$$

since $(\chi^{U_1} + \chi^{U_4})^\intercal = (1, 1) = (\chi^{U_2} + \chi^{U_3})^\intercal$. However, this is clearly not the case as $(\chi^{S_1} + \chi^{S_4})^\intercal = (2, 2, 0, 1)$ but $(\chi^{S_2} + \chi^{S_3})^\intercal = (2, 0, 2, 1)$.

As we show next, affine representability allows one to efficiently solve linear optimization problems over elements of a distributive lattice. In particular, it generalizes properties that are at the backbone of algorithms for optimizing a linear function over the set of stable matchings in the marriage model and its one-to-many and many-to-many generalizations (see, e.g., [11, 27]). For instance, in the marriage model, the base set E is the set of potential pairs of agents from two sides of the market, $\mathcal{X}$ is the set of stable matchings, and for $S, S' \in \mathcal{X}$, we have $S \succeq S'$ if every firm prefers its partner in S to its partner in S'. Elements of its representation poset are certain (trading) cycles, called *rotations*.

Lemma 1 *Suppose we are given a poset $\mathcal{B} = (Y, \succeq^\star)$ that affinely represents a lattice $\mathcal{L} = (\mathcal{X}, \succeq)$ with representation function ψ. Let $w : E \to \mathbb{R}$ be a linear function over the base set E of $\mathcal{L}$. Then the problem $\max\{w^\intercal \chi^S : S \in \mathcal{X}\}$ can be solved in time $\texttt{min-cut}(|Y| + 2)$, where $\texttt{min-cut}(k)$ is the time complexity required to solve a minimum $s - t$ cut problem with nonnegative weights in a digraph with k nodes.*

Proof Let $g(u) = Au + x^0$ be the affine function from the definition of affine representability. We have:

$$\max_{S \in \mathcal{X}} w^\intercal \chi^S = \max_{U \in \mathcal{U}(\mathcal{B})} w^\intercal g(\chi^U) = \max_{U \in \mathcal{U}(\mathcal{B})} w^\intercal (A\chi^U + x^0) = w^\intercal x^0 + \max_{U \in \mathcal{U}(\mathcal{B})} (w^\intercal A)\chi^U.$$

Our problem boils down therefore to the optimization of a linear function over the upper sets of $\mathcal{B}$. It is well-known that the latter problem is equivalent to computing a minimum $s - t$ cut in a digraph with $|Y| + 2$ nodes [35]. □

We want to apply Lemma 1 to the CM- QF- MODEL. Observe that a choice function may be defined on all the (exponentially many) subsets of agents from the opposite side. We avoid this computational concern by modeling choice functions via an oracle model. That is, choice functions can be thought of as agents' private information. The complexity of our algorithms will therefore be expressed in terms of $|F|$, $|W|$, and the time required to compute the choice function $\mathcal{C}_a(X)$ of an agent $a \in F \cup W$, where the set X is in the domain of $\mathcal{C}_a$. The latter running time is denoted by `oracle-call` and we assume it to be independent of a and X. Our first result is the following.

Theorem 1 *The distributive lattice $(\mathcal{S}, \succeq)$ of stable matchings in the* CM- MODEL *is affinely representable. Its representation poset $(\Pi, \succeq^\star)$ has $O(|F||W|)$ elements. This representation poset, as well as its representation function ψ and affine function $g(u) = Au + x^0$, can be computed in time $O(|F|^3|W|^3 \texttt{oracle-call})$ for the* CM- QF- MODEL. *Moreover, matrix A has full column rank.*

In Theorem 1, we assumed that operations such as comparing two sets and obtaining an entry from their set difference take constant time. If this is not the case, a factor mildly polynomial in $|F| \cdot |W|$ needs to be added to the running time. Theorem 1 is the union of two statements. First, the distributive lattice of stable matchings in the CM- MODEL is affinely representable. Second, this representation and the corresponding functions ψ and g can be found efficiently for the CM- QF- MODEL. Those results are proved in Sect. 3 and Sect. 4, respectively. Combining Theorem 1, Lemma 1 and algorithms for $\texttt{min-cut}(\cdot)$, we obtain the following.

Corollary 1 *The problem of optimizing a linear function over the set of stable matchings in the* CM- QF- MODEL *can be solved in time $O(|F|^3|W|^3 \texttt{oracle-call})$.*

Since algorithms for solving `min-cut`(k) in time sub-cubic in k are known (see, e.g., [15]), the bottleneck in the running time of Corollary 1 is given by the operations that construct the poset. As a consequence of studying a distributive lattice via the poset that affinely represents it, one immediately obtains a linear description of the convex hull of the characteristic vectors of elements of the lattice (see Sect. 5). In contrast, most stable matching literature has focused on deducing linear descriptions for special cases of our model via ad-hoc proofs, independently of the lattice structure.

Theorem 2 *Let $\mathcal{L} = (\mathcal{X}, \succeq)$ be a distributive lattice and $\mathcal{B} = (Y, \succeq^\star)$ be a poset that affinely represents it via function $g(u) = Au + x^0$. Then the extension complexity of* $\mathrm{conv}(\mathcal{X})$:=$\mathrm{conv}\{\chi^S : S \in \mathcal{X}\}$ *is $O(|Y|^2)$. If moreover A has full column rank, then* $\mathrm{conv}(\mathcal{X})$ *has $O(|Y|^2)$ facets.*

Theorems 1 and 2 imply the following description of the stable matching polytope, i.e., the convex hull of the characteristic vectors of stable matchings.

Corollary 2 $\mathrm{conv}(\mathcal{S})$ *has $O(|F|^2|W|^2)$ facets in the* CM- MODEL.

Lastly, in Sect. 6, we discuss alternative ways to represent choice functions, dropping the oracle-model assumption. Interestingly, we show that choice functions in the CM- MODEL (i.e., substitutable, consistent, and cardinal monotone) do not have polynomial-size representation because the number of possible choice functions in such a model is doubly-exponential in the size of acceptable partners.

For examples and extended discussions, we refer to the arXiv version of the paper[3] and to the forthcoming Ph.D. thesis of the second author [45].

1.2 Relationship with the literature

Gale and Shapley [21] introduced the one-to-one stable marriage (SM- MODEL) and the one-to-many stable admission model (SA- MODEL), and presented an algorithm which finds a stable matching. McVitie and Wilson [33] proposed the break-marriage procedure that allows us to find the full set of stable matchings. Irving et al. [27] presented an efficient algorithm for the maximum-weighted stable matching problem with weights over pairs of agents, utilizing the fact stable matchings form a distributive lattice [30] and that its representation poset – an affine representation following our terminology – can be constructed efficiently via the concept of rotations [26]. The above-mentioned structural and algorithm results were shown for its many-to-many generalization (MM- MODEL) by Baïou and Balinski [9], and Bansal et al. [11]. A complete survey of results on these models can be found, e.g., in [23, 31].

For models with substitutable and consistent choice functions, Roth [37] proved that stable matchings always exist by generalizing the algorithm presented in [21]. Blair [13] proved that stable matchings form a lattice, although not necessarily distributive. Alkan [4] showed that if choice functions are further assumed to be quota-filling, the lattice is distributive. Results on (non-efficient) enumeration algorithms for certain choice functions appeared in [32].

[3] https://arxiv.org/abs/2011.06763.

It is then natural to investigate whether algorithms from [11, 26, 33] can be directly extended to, e.g., construct the representation poset in the CM- QF- Model or the more general CM- Model. However, their definition of rotation and techniques rely on the fact that there is a strict ordering of partners, which is not available with choice functions. This, for instance, leads to the fact that the symmetric difference of two stable matchings that are adjacent in the Hasse Diagram of the lattice is a simple cycle, which is not always true in the CM- Model. We take then a more fundamental approach by showing a carefully defined ring of sets is isomorphic to the set of stable matchings, and thus we can construct the rotation poset following a maximal chain of the stable matching lattice. This approach conceptually follows the one by Gusfield and Irving [23] for the SM- Model and leads to a generalization of the break-marriage procedure from [33]. Again, proofs in [23, 33] heavily rely on the strict ordering of partners, while we need to tackle the challenge of not having one.

Besides the combinatorial perspective, another line of research focuses on the polyhedral aspects. Linear descriptions of the convex hull of the characteristic vectors of stable matchings are provided for the SM- Model [39, 40, 44], the SA- Model [10], and the MM- Model [20]. In this paper, we provide a polyhedral description for the CM- Model, by drawing connections between the order polytope (i.e., the convex hull of the characteristic vectors of upper sets of a poset) and Birkhoff's representation theorem of distributive lattices. A similar approach has been proposed in [6]: their result can be seen as a specialization of Theorem 1 to the SM- Model.

2 Basics

2.1 Posets, lattices, and distributivity

A set X endowed with a partial order relation $\geq$, denoted as $(X, \geq)$, is called a *partially ordered set (poset)*. When the partial order $\geq$ is clear from context, we often times simply use X to denote the poset $(X, \geq)$. Let $a, a' \in X$, if $a' > a$, we say a' is a *predecessor* of a in poset $(X, \geq)$, and a is a *descendant* of a' in poset $(X, \geq)$. If moreover, there is no $b \in X$ such that $a' > b > a$, we say that a' an *immediate predecessor* of a in poset $(X, \geq)$ and that a is an *immediate descendant* of a' in poset $(X, \geq)$. If $a \not\geq a'$ and $a' \not\geq a$, we say a and a' are *incomparable*.

For a subset $S \subseteq X$, an element $a \in X$ is said to be an *upper bound* (resp. *lower bound*) of S if for all $b \in S$, $a \geq b$ (resp. $b \geq a$). An upper bound (resp. lower bound) a' of S is said to be its *least upper bound* or *join* (resp. *greatest lower bound* or *meet*), if $a \geq a'$ (resp. $a' \geq a$) for each upper bound (resp. lower bound) a of S.

A *lattice* is a poset for which every pair of elements has a join and a meet and for every pair those are unique by definition. Thus, two binary operations are defined over a lattice: join and meet. A lattice is *distributive* where the operations of join and meet distribute over each other. Two lattices are said to be *isomorphic* if there is a structure-preserving mapping between them that can be reversed by an inverse mapping. Such a structure-preserving mapping is called an *isomorphism* between the two lattices. For $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \cdots, n\}$.

2.2 The firm-worker models

Let F and W denote two disjoint finite sets of agents, say firms and workers respectively. Associated with each firm $f \in F$ is a *choice function* $\mathcal{C}_f : 2^{W(f)} \to 2^{W(f)}$ where $W(f) \subseteq W$ is the set of *acceptable* partners of f and $\mathcal{C}_f$ satisfies the property that for every $S \subseteq W(f)$, $\mathcal{C}_f(S) \subseteq S$. Similarly, a choice function $\mathcal{C}_w : 2^{F(w)} \to 2^{F(w)}$ is associated to each worker w. We assume that for every firm-worker pair (f, w), $f \in F(w)$ if and only if $w \in W(f)$. We let $\mathcal{C}_W$ and $\mathcal{C}_F$ denote the collection of firms' and workers' choice functions respectively. A *matching market* (or an instance) is a tuple $(F, W, \mathcal{C}_F, \mathcal{C}_W)$. Following [5], we define below the properties of *substitutability*, *consistency*, and *cardinal monotonicity (law of aggregate demand)* for choice function $\mathcal{C}_a$ of an agent a.

Definition 2 *(Substitutability)* An agent a's choice function $\mathcal{C}_a$ is substitutable if for any set of partners S, $b \in \mathcal{C}_a(S)$ implies that for all $T \subseteq S$, $b \in \mathcal{C}_a(T \cup \{b\})$.

Definition 3 *(Consistency)* An agent a's choice function $\mathcal{C}_a$ is consistent if for any sets of partners S and T, $\mathcal{C}_a(S) \subseteq T \subseteq S$ implies $\mathcal{C}_a(S) = \mathcal{C}_a(T)$.

Definition 4 *(Cardinal monotonicity)* An agent a's choice function $\mathcal{C}_a$ is cardinal monotone if for all sets of partners $S \subseteq T$, we have $|\mathcal{C}_a(S)| \leq |\mathcal{C}_a(T)|$.

Intuitively, substitutability implies that if an agent is selected from a set of candidates, she will also be selected from a smaller subset; consistency is also called "irrelevance of rejected contracts"; and cardinal monotonicity implies that the size of the image of the choice function is monotone with respect to set inclusion.

Aizerman and Malishevski [3] showed that a choice function is substitutable and consistent if and only if it is *path-independent*.

Definition 5 *(Path-independence)* An agent a's choice function $\mathcal{C}_a$ is path-independent if for any sets of partners S and T, $\mathcal{C}_a(S \cup T) = \mathcal{C}_a\big(\mathcal{C}_a(S) \cup T\big)$.

We next prove a few properties of path-independent choice functions.

Lemma 2 *Let $\mathcal{C} : 2^A \to 2^A$ be a path-independent choice function and let $A_1, A_2 \subseteq A$. If $\mathcal{C}(A_1 \cup \{a\}) = \mathcal{C}(A_1)$ for every $a \in A_2 \setminus A_1$, then $\mathcal{C}(A_1 \cup A_2) = \mathcal{C}(A_1)$.*

Proof Assume $A_2 \setminus A_1 = \{a_1, a_2, \cdots, a_t\}$. Then, by repeated application of the path independence property,

$$\begin{aligned}\mathcal{C}(A_1 \cup A_2) &= \mathcal{C}(A_1 \cup \{a_1, a_2, \cdots, a_t\}) = \mathcal{C}(\mathcal{C}(A_1 \cup \{a_1\}) \cup \{a_2, \cdots, a_t\}) \\ &= \mathcal{C}(\mathcal{C}(A_1) \cup \{a_2, \cdots, a_t\}) = \mathcal{C}(A_1 \cup \{a_2, a_3, \cdots, a_t\}) = \cdots = \mathcal{C}(A_1).\end{aligned}$$

□

Corollary 3 *Let $\mathcal{C} : 2^A \to 2^A$ be a path-independent choice function and let $A_1, A_2 \subseteq A$. If $a \notin \mathcal{C}(A_1 \cup \{a\})$ for every $a \in A_2 \setminus A_1$, then $\mathcal{C}(A_1 \cup A_2) = \mathcal{C}(A_1)$.*

Proof By the consistency property of $\mathcal{C}$, $a \notin \mathcal{C}(A_1 \cup \{a\})$ implies $\mathcal{C}(A_1 \cup \{a\}) = \mathcal{C}(A_1)$. Lemma 2 then applies directly. □

Lemma 3 *Let $\mathcal{C} : 2^A \to 2^A$ be a path-independent choice function and let $A_1, A_2 \subseteq A$, $a \in A$. Assume $\mathcal{C}(A_1 \cup A_2) = A_1$ and $a \in \mathcal{C}(A_1 \cup \{a\})$. Then, $a \in \mathcal{C}(A_2 \cup \{a\})$.*

Proof By path-independence, we have that $\mathcal{C}(A_1 \cup A_2 \cup \{a\}) = \mathcal{C}(\mathcal{C}(A_1 \cup A_2) \cup \{a\}) = \mathcal{C}(A_1 \cup \{a\})$ and thus $a \in \mathcal{C}(A_1 \cup A_2 \cup \{a\})$. Also, by path-independence, we have $\mathcal{C}(A_1 \cup A_2 \cup \{a\}) = \mathcal{C}\big(\mathcal{C}(A_1 \setminus \{a\}) \cup \mathcal{C}(A_2 \cup \{a\})\big)$. Since $a \notin \mathcal{C}(A_1 \setminus \{a\})$, it must be that $a \in \mathcal{C}(A_2 \cup \{a\})$. □

A *matching* μ is a mapping from $F \cup W$ to $2^{F \cup W}$ such that for all $w \in W$ and for all $f \in F$, (1) $\mu(w) \subseteq F(w)$; (2) $\mu(f) \subseteq W(f)$; and (3) $w \in \mu(f)$ if and only if $f \in \mu(w)$. A matching can also be viewed as a collection of firm-worker pairs. That is, $\mu \equiv \{(f, w) : f \in F, w \in \mu(f)\}$. Thus, we use $(f, w) \in \mu$, $w \in \mu(f)$, and $f \in \mu(w)$ interchangeably. We say a matching μ is *individually rational* if for every agent a, $\mathcal{C}_a(\mu(a)) = \mu(a)$. An acceptable firm-worker pair $(f, w) \notin \mu$ is called a *blocking pair* if $w \in \mathcal{C}_f(\mu(f) \cup \{w\})$ and $f \in \mathcal{C}_w(\mu(w) \cup \{f\})$, and when such pair exists, we say μ is *blocked by* the pair or the pair blocks μ. A matching μ is *stable* if it is individually rational and it admits no blocking pairs. If f is matched to w in some stable matching, we say that (f, w) is a *stable pair* and that f (resp. w) is a *stable partner* of w (resp. f). We denote by $\mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$ the set of stable matchings in the market $(F, W, \mathcal{C}_F, \mathcal{C}_W)$, and when the market is clear from the context we abbreviate $\mathcal{S} := \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$.

Alkan [5] showed the following.

Theorem 3 ([5]) *Consider a matching market $(F, W, \mathcal{C}_F, \mathcal{C}_W)$ and assume $\mathcal{C}_F$ and $\mathcal{C}_W$ are substitutable, consistent, and cardinal monotone. Then $\mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$ is a distributive lattice under the partial order relation $\succeq$ where $\mu_1 \succeq \mu_2$ iff for all $f \in F$, $\mathcal{C}_f(\mu_1(f) \cup \mu_2(f)) = \mu_1(f)$. The join (denoted by $\vee$) and meet (denoted by $\wedge$) operations of the lattice are defined component-wise. That is, for all $f \in F$:*

$$
\begin{aligned}
(\mu_1 \vee \mu_2)(f) &:= \mu_1(f) \vee \mu_2(f) := \mathcal{C}_f(\mu_1(f) \cup \mu_2(f)),\\
(\mu_1 \wedge \mu_2)(f) &:= \mu_1(f) \wedge \mu_2(f)\\
&:= \big[\big(\mu_1(f) \cup \mu_2(f)\big) \setminus (\mu_1 \vee \mu_2)(f)\big] \cup \big(\mu_1(f) \cap \mu_2(f)\big).
\end{aligned}
$$

Moreover, $\mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$ satisfies the polarity *property: $\mu_1 \succeq \mu_2$ if and only if for every worker $w \in W$, $\mathcal{C}_w(\mu_1(w)) \cup \mu_2(w)) = \mu_2(w)$.*

Because of the lattice structure, the firm- and worker-optimal stable matchings are well-defined, and we denote them respectively by μ_F and μ_W. In addition, Alkan [5] showed two properties, *concordance* (Proposition 7, [5]) and *equal-quota* (Proposition 6, [5]), satisfied by the family of sets of partners under all stable matchings for every agent a. Let $\Phi_a := \{\mu(a) : \mu \in \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)\}$. Then for all $S, T \in \Phi_a$,

$$S \cap T \subseteq S \vee T \qquad \text{(concordance)}$$

and

$$|S| = |T| =: \overline{q}_a. \qquad \text{(equal-quota)}$$

Instead of cardinal monotonicity, an earlier paper of Alkan [4] considers a more restrictive property of choice functions, called *quota-filling*.

Definition 6 *(Quota-filling)* An agent a's choice function $\mathcal{C}_a$ is quota-filling if there exists $q_a \in \mathbb{N}$ such that for any set of partners S, $|\mathcal{C}_a(S)| = \min(q_a, |S|)$. We call q_a the *quota* of agent a.

Intuitively, quota-filling means that an agent has a number of positions and she tries to fill as many of these positions as possible. Note that quota-filling implies cardinal monotonicity. Let q_a denote the quota of each agent $a \in F \cup W$.

Our results from Sect. 3 assume path-independence (i.e., substitutability and consistency) and cardinal monotonicity. In Sect. 4, we will restrict our model by replacing cardinal monotonicity with quota-filling for one side of the market. These two models are what we call the CM- Model and the CM- QF- Model, respectively.

3 Affine representability of the stable matching lattice

In this section, we show that the distributive lattice of stable matchings in the model by [5] is affinely representable. An algorithm to construct an affine representation is given in Sect. 4 where we additionally impose the quota-filling property upon choice functions of agents in one side of the markets. The proof of this section proceeds as follows. First, we show in Sect. 3.1 that the lattice of stable matchings $(\mathcal{S}, \succeq)$ is isomorphic to a lattice $(\mathcal{P}, \subseteq)$ belonging to a special class, that is called *rings of sets*. In Sect. 3.2, we then show that rings of sets are always affinely representable. In Sect. 3.3, we show a poset $(\Pi, \succeq^\star)$ representing $(\mathcal{S}, \succeq)$. Lastly, in Sect. 3.4, we show how to combine all those results and "translate" the affine representability of $(\mathcal{P}, \subseteq)$ to the affine representability of $(\mathcal{S}, \succeq)$, concluding the proof.

3.1 Isomorphism between the stable matching lattice and a ring of sets

A family $\mathcal{H} = \{H_1, H_2, \cdots, H_k\}$ of subsets of a *base set* B is a *ring of sets* over B if $\mathcal{H}$ is closed under set union and set intersection [12]. Note that a ring of sets is a distributive lattice with the partial order relation $\subseteq$, and the join and meet operations corresponds to set intersection and set union, respectively.

In this and the following section, we fix a matching market $(F, W, \mathcal{C}_F, \mathcal{C}_W)$ and assume that $\mathcal{C}_F$ and $\mathcal{C}_W$ are path-independent and cardinal monotone (i.e., the framework of [5]). Let $\phi(a)$ denote the set of stable partners of agent a. That is, $\phi(a) := \{b : b \in \mu(a) \text{ for some } \mu \in \mathcal{S}\}$. For $\mu \in \mathcal{S}$, let $P_f(\mu) := \{w \in \phi(f) : w \in \mathcal{C}_f(\mu(f) \cup \{w\})\}$, and define the *P-set* of μ as $P(\mu) := \{(f, w) : f \in F, w \in P_f(\mu)\}$.

The goal of this section is to show the following theorem, which gives a representation of the stable matching lattice as a ring of sets. Let $\mathcal{P}(\mathcal{C}_F, \mathcal{C}_W)$ denote the set $\{P(\mu) : \mu \in \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)\}$, and we often abbreviate $\mathcal{P} := \mathcal{P}(\mathcal{C}_F, \mathcal{C}_W)$.

Theorem 4 *Assume $\mathcal{C}_F$ and $\mathcal{C}_W$ are path-independent and cardinal monotone. Then,*

(i) the mapping $P : \mathcal{S} \to \mathcal{P}$ is a bijection;
(ii) $(\mathcal{P}, \subseteq)$ is isomorphic to $(\mathcal{S}, \succeq)$. That is, for two stable matchings $\mu_1, \mu_2 \in \mathcal{S}$, we have $\mu_2 \succeq \mu_1$ if and only if $P(\mu_2) \subseteq P(\mu_1)$. Moreover, $P(\mu_1 \vee \mu_2) = P(\mu_1) \cap P(\mu_2)$ and $P(\mu_1 \wedge \mu_2) = P(\mu_1) \cup P(\mu_2)$. In particular, $(\mathcal{P}, \subseteq)$ is a ring of sets over the base set $\{(f, w) : f \in F, w \in \phi(f)\}$.

Remark 1 An isomorphism between the lattice of stable matchings and a ring of sets (also called P-set) is proved in the SM- MODEL by Gusfield and Irving [23] as well. However, they define $P(\mu) := \{(f, w) : f \in F, w \geq_f \mu(f)\}$, hence including firm-worker pairs that are not stable. As a consequence, while in their model the construction of the P-set for a given stable matching is immediate, in ours it is not, since we need to know first which pairs are stable.

Lemma 4 *Let μ_1 and μ_2 be two stable matchings such that $\mu_2 \succeq \mu_1$. Then, $P_f(\mu_2) \subseteq P_f(\mu_1)$ for every firm f.*

Proof Since $\mu_2 \succeq \mu_1$, we have that $\mathcal{C}_f(\mu_2(f) \cup \mu_1(f)) = \mu_2(f)$. The claim then follows from Lemma 3. □

Lemma 5 *Let μ_1 be a stable matching such that $w \in P_f(\mu_1)$ for some firm f and worker w. Then, there exists a stable matching μ_2 such that $\mu_2 \succeq \mu_1$ and $w \in \mu_2(f)$.*

Proof By definition of $P_f(\mu_1)$, we know there exists a stable matching μ_1' such that $w \in \mu_1'(f)$. Let $\mu_2 := \mu_1 \vee \mu_1'$. We want to show that $w \in \mu_2(f)$. If $w \in \mu_1(f)$, then the claim follows due to the concordance property. So assume $w \notin \mu_1(f)$ and also assume by contradiction that $w \notin \mu_2(f)$. Then, we must have $w \in (\mu_1 \wedge \mu_1')(f)$ by definition of the meet. Since $\mu_1 \succeq \mu_1 \wedge \mu_1'$, we have $\mathcal{C}_f(\mu_1(f) \cup (\mu_1 \wedge \mu_1')(f)) = \mu_1(f)$. However, applying path-independence and consistency, we have

$$\begin{aligned}\mathcal{C}_f(\mu_1(f) \cup (\mu_1 \wedge \mu_1')(f)) &= \mathcal{C}_f\big(\mathcal{C}_f(\mu_1(f) \cup (\mu_1 \wedge \mu_1')(f) \setminus \{w\}) \cup \{w\}\big) \\ &= \mathcal{C}_f(\mu_1(f) \cup \{w\}) \neq \mu_1(f),\end{aligned}$$

which is a contradiction. □

Lemma 6 *Let μ_1 and μ_2 be two stable matchings such that $\mu_2 \succeq \mu_1$. Assume $w \in P_f(\mu_1) \setminus P_f(\mu_2)$ for some firm f. Then, there exists a stable matching $\overline{\mu}_1$ with $\mu_2 \succeq \overline{\mu}_1 \succeq \mu_1$ such that $w \in \overline{\mu}_1(f)$.*

Proof By Lemma 5, there exists a stable matching $\overline{\mu}_2 \succeq \mu_1$ such that $w \in \overline{\mu}_2(f)$. Let $\overline{\mu}_1 := \overline{\mu}_2 \wedge \mu_2$ and we claim that $\overline{\mu}_1$ is the desired matching. First, by definition of meet, we have $\mu_2 \succeq \overline{\mu}_1 \succeq \mu_1$. Since $w \notin P_f(\mu_2)$, by the contrapositive of the substitutability property, we have $w \notin \mathcal{C}_f(\mu_2(f) \cup \overline{\mu}_2(f))$, which implies that $w \notin (\mu_2 \vee \overline{\mu}_2)(f)$. Therefore, $w \in \overline{\mu}_1(f)$, again by the definition of meet. □

Lemma 7 *Let μ_1 and μ_2 be two stable matchings. Then,*

$$P(\mu_1 \vee \mu_2) = P(\mu_1) \cap P(\mu_2) \quad \text{and} \quad P(\mu_1 \wedge \mu_2) = P(\mu_1) \cup P(\mu_2).$$

Proof Fix a firm f, and we want to show $P_f(\mu_1 \vee \mu_2) = P_f(\mu_1) \cap P_f(\mu_2)$ and $P_f(\mu_1 \wedge \mu_2) = P_f(\mu_1) \cup P_f(\mu_2)$. If $\mu_1(f) = \mu_2(f)$, then the claim is obviously true. Thus, for the following, we assume $\mu_1(f) \neq \mu_2(f)$. We first show that $P_f(\mu_1 \vee \mu_2) \subseteq P_f(\mu_1) \cap P_f(\mu_2)$. Since $\mu_1 \vee \mu_2 \succeq \mu_1, \mu_2$, the claim follows from Lemma 4. Next, we show that $P_f(\mu_1 \vee \mu_2) \supseteq P_f(\mu_1) \cap P_f(\mu_2)$. If $P_f(\mu_1) \cap P_f(\mu_2) = \emptyset$, then the claim follows trivially. So we assume $P_f(\mu_1) \cap P_f(\mu_2) \neq \emptyset$ and let $w \in P_f(\mu_1) \cap P_f(\mu_2)$. By Lemma 5, there exists a stable matching $\overline{\mu}_1$ such that $\overline{\mu}_1 \succeq \mu_1$ and $w \in \overline{\mu}_1(f)$. Similarly, there exists a stable matching $\overline{\mu}_2$ such that $\overline{\mu}_2 \succeq \mu_2$ and $w \in \overline{\mu}_2(f)$. Consider the stable matching $\overline{\mu}_1 \vee \overline{\mu}_2$. Because of the concordance property, $w \in (\overline{\mu}_1 \vee \overline{\mu}_2)(f)$. As $\overline{\mu}_1 \vee \overline{\mu}_2 \succeq \mu_1, \mu_2$ by transitivity of $\succeq$, we have $\overline{\mu}_1 \vee \overline{\mu}_2 \succeq \mu_1 \vee \mu_2$ by minimality of $\mu_1 \vee \mu_2$. Hence, by Lemma 4, $w \in P_f(\mu_1 \vee \mu_2)$. This proves the first part of the thesis.

For the second part, we first show $P_f(\mu_1 \wedge \mu_2) \subseteq P_f(\mu_1) \cup P_f(\mu_2)$. Let $w \notin P_f(\mu_1) \cup P_f(\mu_2)$ and assume by contradiction that $w \in P_f(\mu_1 \wedge \mu_2)$. $w \notin P_f(\mu_1) \cup P_f(\mu_2)$ implies $w \notin \mu_1(f)$ and $w \notin \mu_2(f)$ and thus, $w \notin (\mu_1 \wedge \mu_2)(f)$. By Lemma 6, for both $i \in \{1, 2\}$, there exists a stable matching $\overline{\mu}_i$ such that $\mu_i \succeq \overline{\mu}_i \succeq \mu_1 \wedge \mu_2$ and $w \in \overline{\mu}_i(f)$. Note that $\mu_1 \wedge \mu_2 \succeq \overline{\mu}_1 \wedge \overline{\mu}_2 \succeq \mu_1 \wedge \mu_2$, where the first relation holds because $\mu_i \succeq \overline{\mu}_i$ for both $i \in \{1, 2\}$, and the second relation holds because $\overline{\mu}_1, \overline{\mu}_2 \succeq \mu_1 \wedge \mu_2$. Hence, $\overline{\mu}_1 \wedge \overline{\mu}_2 = \mu_1 \wedge \mu_2$. However, by applying the meet operator $\wedge$ over $\overline{\mu}_1$ and $\overline{\mu}_2$, we have $w \in (\overline{\mu}_1 \wedge \overline{\mu}_2)(f) = (\mu_1 \wedge \mu_2)(f)$, which is a contradiction.

Lastly, we show $P_f(\mu_1 \wedge \mu_2) \supseteq P_f(\mu_1) \cup P_f(\mu_2)$. Let $w \in P_f(\mu_1) \cup P_f(\mu_2)$ and wlog assume $w \in P_f(\mu_1)$. Since $\mu_1 \succeq \mu_1 \wedge \mu_2$, by Lemma 4, $w \in P_f(\mu_1 \wedge \mu_2)$. □

Lemma 8 *Let μ_1 and μ_2 be two stable matchings such that $\mu_2 \succ \mu_1$ and assume that $\mu_1(f) \neq \mu_2(f)$ for some $f \in F$. Then, $P_f(\mu_1) \neq P_f(\mu_2)$.*

Proof Assume by contradiction that $P_f(\mu_1) = P_f(\mu_2)$. Let $w \in \mu_1(f) \setminus \mu_2(f)$, which exists because of the equal-quota property. Since the stable matching lattice $(\mathcal{S}, \succeq)$ has the polarity property as shown in Theorem 3, we have that $\mathcal{C}_w(\mu_1(w) \cup \mu_2(w)) = \mu_1(w)$ and thus, by substitutability, we have $f \in \mathcal{C}_w(\mu_2(w) \cup \{f\})$. On the other hand, $w \in \mu_1(f)$ implies that $w \in P_f(\mu_1) = P_f(\mu_2)$. Since $w \notin \mu_2(f)$, this means that (f, w) is a blocking pair of μ_2, which contradicts the stability assumption. □

Lemma 9 *Let μ_1 and μ_2 be two distinct stable matchings and assume that $\mu_1(f) \neq \mu_2(f)$ for some $f \in F$. Then, $P_f(\mu_1) \neq P_f(\mu_2)$.*

Proof Assume by contradiction that $P_f(\mu_1) = P_f(\mu_2)$. Then, we have $P_f(\mu_1 \vee \mu_2) = P_f(\mu_1 \wedge \mu_2)$ by Lemma 7. However, $\mu_1(f) \neq \mu_2(f)$ implies that $(\mu_1 \vee \mu_2)(f) \neq (\mu_1 \wedge \mu_2)(f)$, which contradicts Lemma 8 since $\mu_1 \vee \mu_2 \succ \mu_1 \wedge \mu_2$. □

Proof of Theorem 4 For (i), note that the mapping P is onto by definition. It is therefore a bijection since it is also injective as shown in Lemma 9. Next, we show (ii). One direction of the first statement is shown in Lemma 4. Conversely, if $P(\mu_2) \subseteq P(\mu_1)$, then by Lemma 7, $P(\mu_1 \vee \mu_2) = P(\mu_1) \cap P(\mu_2) = P(\mu_2)$. Hence, by Lemma 9, we have $\mu_1 \vee \mu_2 = \mu_2$ and thus, $\mu_2 \succeq \mu_1$. The second statement of (ii) follows from Lemma 7. The third follows from the second and the fact that stable matchings form a distributive lattice (Theorem 3). □

3.2 Affine representability of rings of sets via the posets of minimal differences

We now describe (mostly known) facts about posets representing rings of sets, and observe that the *affine* representability of rings of sets easily follows from those.

Fix a ring of sets $(\mathcal{H}, \subseteq)$ over a base set B, and let H_0 and H_z denote respectively the unique minimal and maximal elements of $\mathcal{H}$. That is, for all $H \in \mathcal{H}$, we have $H_0 \subseteq H \subseteq H_z$. For $a \in H_z$, let $H(a)$ denote the unique inclusion-wise minimal set among all sets in $\mathcal{H}$ that contain a, where uniqueness follows from the fact that $\mathcal{H}$ is closed under set intersection. That is,

$$H(a) := \bigcap \{H \in \mathcal{H} : a \in H\}.$$

In addition, define the set $\mathcal{I}(\mathcal{H})$ of the *irreducible* elements of $\mathcal{H}$ as follows

$$\mathcal{I}(\mathcal{H}) := \{H \in \mathcal{H} : \exists\, a \in H_z \text{ s.t. } H = H(a)\}.$$

Since $\mathcal{I}(\mathcal{H})$ is a subset of $\mathcal{H}$, we can view $\mathcal{I}(\mathcal{H})$ as a poset under the set containment relation. For $H \in \mathcal{I}(\mathcal{H})$, let $K(H) := \{a \in H_z : H(a) = H\}$ denote the *centers* of H. Note that $K(H_0) = H_0$. Define $\mathcal{D}(\mathcal{H})$ as the set of centers of irreducible elements of $\mathcal{H}$ without the set H_0. Formally,

$$\mathcal{D}(\mathcal{H}) := \{K(H) : H \in \mathcal{I}(\mathcal{H}), H \neq H_0\}.$$

Immediately from the definition of centers, we obtain the following.

Lemma 10 *Let $a \in B$. There is at most one $K_1 \in \mathcal{D}(\mathcal{H})$ such that $a \in K_1$. In particular, $|\mathcal{D}(\mathcal{H})| = O(|B|)$.*

For $K_1 \in \mathcal{D}(\mathcal{H})$, let $I(K_1)$ denote the irreducible element from $\mathcal{I}(\mathcal{H})$ such that $K(I(K_1)) = K_1$. Let $\sqsupseteq$ be a partial order over the set $\mathcal{D}(\mathcal{H})$ that is inherited from the set containment relation of the poset $\mathcal{I}(\mathcal{H})$. That is, for $K_1, K_2 \in \mathcal{D}(\mathcal{H})$, we have $K_1 \sqsupseteq K_2$ if and only if $I(K_1) \subseteq I(K_2)$.

Theorem 5 ([12]) *Let $(\mathcal{H}, \subseteq)$ be a ring of sets. Then, $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$ is a representation poset for $(\mathcal{H}, \subseteq)$ with representation function $\psi_{\mathcal{H}}$, where $\psi_{\mathcal{H}}^{-1}(\overline{\mathcal{D}}) = \bigcup\{K_1 : K_1 \in \overline{\mathcal{D}}\} \cup H_0$ for any upper set $\overline{\mathcal{D}}$ of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$, and H_0 is the minimal element of $\mathcal{H}$.*

Lemma 10 and Theorem 5 directly imply the following.

Theorem 6 *Let $(\mathcal{H}, \subseteq)$ be a ring of sets over base set B. Then, $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$ affinely represents $(\mathcal{H}, \subseteq)$ via affine function $g(u) = Au + x^0$, where x^0 is the characteristic vector of the minimal element of $\mathcal{H}$, and $A \in \{0,1\}^{B \times \mathcal{D}(\mathcal{H})}$ has columns χ^{K_1} for each $K_1 \in \mathcal{D}(\mathcal{H})$. Moreover, A has full column rank.*

Proof Because of the representation function $\psi_{\mathcal{H}}$ given in Theorem 5, it is clear that $g(\chi^U) = \chi^{\psi_{\mathcal{H}}^{-1}(U)}$ for every upper set $U \in \mathcal{U}((\mathcal{D}(\mathcal{H}), \sqsupseteq))$. Note that every row of A has at most one non-zero entry due to Lemma 10, and every column of A contains at least one non-zero entry by definition. Therefore, A has full column rank. $\square$

Lemma 11 *Let $(\mathcal{H}, \subseteq)$ be a ring of sets with minimal element H_0, and let $H \in \mathcal{H}$. If $H = \bigcup\{K_1 : K_1 \in \overline{\mathcal{D}}\} \cup H_0$ for some subset $\overline{\mathcal{D}}$ of $\mathcal{D}(\mathcal{H})$, then $\overline{\mathcal{D}}$ is an upper set of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$.*

Proof By Lemma 10, there is at most one subset of $\mathcal{D}(\mathcal{H})$ whose union of the elements together with H_0 gives H. On the other hand, Theorem 5 implies that there exists one such subset which is also an upper set of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$. The claim follows thereafter. $\square$

Alternatively, one can view $\mathcal{D}(\mathcal{H})$ as the set of *minimal differences* between elements of $\mathcal{H}$. The following lemma is established directly from Lemma 2.4.3 and Corollary 2.4.1 of [23].

Lemma 12 $\mathcal{D}(\mathcal{H}) = \{H \setminus H' : H'$ *is an immediate predecessor of H in $(\mathcal{H}, \subseteq)$*$\}$.

A direct consequence of Lemma 10 and Lemma 12 is the following.

Lemma 13 *Let $H', H \in \mathcal{H}$. If $H' \subseteq H$ and $H \setminus H' \in \mathcal{D}(\mathcal{H})$, then H' is an immediate predecessor of H in $(\mathcal{H}, \subseteq)$.*

Proof Let $K_1 := H \setminus H'$. Assume by contradiction that there exists $\overline{H} \in \mathcal{H}$ with $H' \subsetneq \overline{H} \subsetneq H$. Then, because of Lemma 12, there exists a center $K_2 \in \mathcal{D}(\mathcal{H})$ such that $\emptyset \neq K_2 \subsetneq K_1$. However, this contradicts Lemma 10. $\square$

3.3 Representation of $(\mathcal{S}, \succeq)$ via the poset of rotations

As discussed in Sect. 3.2, the poset $(\mathcal{D}(\mathcal{P}), \sqsupseteq)$ associated with $(\mathcal{P}, \subseteq)$ provides a compact representation of $(\mathcal{P}, \subseteq)$ and can be used to *reconstruct* $\mathcal{P}$ via Theorem 6. In this section, we show how to associate with $(\mathcal{S}, \succeq)$ a poset that is isomorphic to $(\mathcal{D}(\mathcal{P}), \sqsupseteq)$, which can be used to reconstruct $\mathcal{S}$. The precise statement is given in Theorem 7 below.

For $\mu, \mu' \in \mathcal{S}$, with μ' being an immediate predecessor of μ in the stable matching lattice, let

$$\rho^+(\mu', \mu) = \{(f, w) : f \in F, w \in \mu(f) \setminus \mu'(f)\}$$

and

$$\rho^-(\mu', \mu) = \{(f, w) : f \in F, w \in \mu'(f) \setminus \mu(f)\}.$$

Note that by definition,

$$\mu = \mu' \triangle \rho^-(\mu', \mu) \triangle \rho^+(\mu', \mu) = \mu' \setminus \rho^-(\mu', \mu) \cup \rho^+(\mu', \mu). \tag{1}$$

We call $\rho(\mu', \mu) := (\rho^+(\mu', \mu), \rho^-(\mu', \mu))$ a *rotation* of $(\mathcal{S}, \succeq)$. Let $\Pi(\mathcal{S})$ denote the set of rotations of $(\mathcal{S}, \succeq)$. That is,

$$\Pi(\mathcal{S}) := \{\rho(\mu', \mu) : \mu' \text{ is an immediate predecessor of } \mu \text{ in } (\mathcal{S}, \succeq)\}.$$

Remark 2 While in the MM- Model, rotations are simple cycles in the associated bipartite graph of agents [9], this may not be the case for our model.

In the following, we focus on proving a bijection between $\mathcal{D}(\mathcal{P})$ and $\Pi(\mathcal{S})$, and we often abbreviate $\Pi := \Pi(\mathcal{S})$ and $\mathcal{D} := \mathcal{D}(\mathcal{P})$. In particular, we show the following.

Theorem 7 *Assume $\mathcal{C}_F$ and $\mathcal{C}_W$ are path-independent and cardinal monotone. Then,*

(i) the mapping $Q : \Pi \to \mathcal{D}$, with $Q(\rho) = \rho^+$, is a bijection;
(ii) $(\mathcal{D}, \sqsupseteq)$ is isomorphic to the rotation poset $(\Pi, \succeq^\star)$, *where for two rotations $\rho_1, \rho_2 \in \Pi$, $\rho_1 \succeq^\star \rho_2$ if $Q(\rho_1) \sqsupseteq Q(\rho_2)$;*
(iii) $(\Pi, \succeq^\star)$ is a representation poset for $(\mathcal{S}, \succeq)$ with representation function $\psi_{\mathcal{S}}$ such that for any upper set $\overline{\Pi}$ of $(\Pi, \succeq^\star)$, $P(\psi_{\mathcal{S}}^{-1}(\overline{\Pi})) = \psi_{\mathcal{P}}^{-1}(\{Q(\rho) : \rho \in \overline{\Pi}\})$ where $\psi_{\mathcal{P}}$ is the representation function of $(\mathcal{P}, \subseteq)$ per Theorem 5; and $\psi_{\mathcal{S}}^{-1}(\overline{\Pi}) = \big(\triangle_{\rho\in\overline{\Pi}}(\rho^-\triangle\rho^+)\big)\triangle\mu_F$, where $\triangle$ is the symmetric difference operator. Equivalently, we have $\psi_{\mathcal{S}}^{-1}(\overline{\Pi}) = \mu_F \cup (\bigcup_{\rho\in\overline{\Pi}} \rho^+) \setminus (\bigcup_{\rho\in\overline{\Pi}} \rho^-)$.

Lemma 14 *Let $\mu, \mu' \in \mathcal{S}$ such that $\mu' \succ \mu$. If $w \in \mu(f) \setminus \mu'(f)$ for some f, then $w \notin P_f(\mu')$.*

Proof Since $\mu' \succ \mu$, we have $\mathcal{C}_f(\mu'(f) \cup \mu(f)) = \mu'(f)$. By path-independence and consistency, we have

$$w \notin \mu'(f) = \mathcal{C}_f(\mu'(f) \cup \mu(f)) = \mathcal{C}_f(\mathcal{C}_f(\mu'(f) \cup \mu(f) \setminus \{w\}) \cup \{w\}) = \mathcal{C}_f(\mu'(f) \cup \{w\}).$$

Therefore, $w \notin P_f(\mu')$, concluding the proof. □

Lemma 15 *Let $\mu, \mu' \in \mathcal{S}$ such that μ' is an immediate predecessor of μ in the stable matching lattice. Then, $\mu(f) \setminus \mu'(f) = P_f(\mu) \setminus P_f(\mu')$ for all $f \in F$. In particular, $P(\mu) \setminus P(\mu') = \rho^+(\mu', \mu)$.*

Proof Fix a firm f. $\mu(f) \setminus \mu'(f) \subseteq P_f(\mu) \setminus P_f(\mu')$ follows by definition and from Lemma 14. For the reverse direction, assume by contradiction that there exists $w \in P_f(\mu) \setminus P_f(\mu')$ but $w \notin \mu(f) \setminus \mu'(f)$. Since $w \notin P_f(\mu')$ implies that $w \notin \mu'(f)$ by definition of $P_f(\cdot)$, we also have $w \notin \mu(f)$. By Lemma 6, there exists a stable matching $\overline{\mu}$ such that $\mu' \succeq \overline{\mu} \succeq \mu$ and $w \in \overline{\mu}(f)$. However, since μ' is an immediate predecessor of μ in the stable matching lattice, we either have $\overline{\mu} = \mu$ or $\overline{\mu} = \mu'$. However, both are impossible since we deduced $w \notin \mu(f) \cup \mu'(f)$. □

Lemma 16 *Let $\mu_1, \mu_2, \mu_3 \in \mathcal{S}$ such that $\mu_1 \succ \mu_2 \succ \mu_3$. If $w \in \mu_1(f) \setminus \mu_2(f)$ for some firm f, then $w \notin \mu_3(f)$.*

Proof First, note that $\mu_1 \succ \mu_2$ implies $w \in \mu_1(f) = \mathcal{C}_f(\mu_1(f) \cup \mu_2(f))$. Thus, by substitutability, we have $w \in \mathcal{C}_f(\mu_2(f) \cup \{w\})$. Assume by contradiction that $w \in \mu_3(f)$. Then, applying Lemma 14 on μ_2 and μ_3, we have that $w \notin P_f(\mu_2)$, which is a contradiction. □

Lemma 17 *Let $\mu_1, \mu_1', \mu_2, \mu_2' \in \mathcal{S}$ and assume that μ_1', μ_2' are immediate predecessors of μ_1, μ_2 in the stable matching lattice, respectively. In addition, assume that $\mu_1 \succ \mu_2$. If $P(\mu_1) \setminus P(\mu_1') = P(\mu_2) \setminus P(\mu_2')$, then $\mu_1'(f) \setminus \mu_1(f) = \mu_2'(f) \setminus \mu_2(f)$ for all firms $f \in F$.*

Proof Fix a firm f. Due to Lemma 15, we know $\mu_1(f) \setminus \mu_1'(f) = \mu_2(f) \setminus \mu_2'(f)$. By the equal-quota property, we have $|\mu_1(f)| = |\mu_1'(f)|$ and $|\mu_2(f)| = |\mu_2'(f)|$. Thus, $|\mu_1'(f) \setminus \mu_1(f)| = |\mu_2'(f) \setminus \mu_2(f)|$ ($\natural$). If $\mu_1'(f) \setminus \mu_1(f) = \emptyset$, the claim follows immediately, and thus, in the following, we assume $\mu_1'(f) \setminus \mu_1(f) \neq \emptyset$. Assume by contradiction that there exists $w \in \mu_1'(f) \setminus \mu_1(f)$ but $w \notin \mu_2'(f) \setminus \mu_2(f)$. Since $\mu_1 \succ \mu_2$ and $\mu_i' \succ \mu_i$ for $i \in \{1, 2\}$, by Theorem 4, we have $P(\mu_1) \subsetneq P(\mu_2)$ and $P(\mu_i') \subsetneq P(\mu_i)$ for $i \in \{1, 2\}$. Therefore, $P(\mu_1') \subsetneq P(\mu_2')$ due to the assumption that $P(\mu_1) \setminus P(\mu_1') = P(\mu_2) \setminus P(\mu_2')$. Again by Theorem 4, we have $\mu_1' \succ \mu_2'$. Hence, $\mu_1 \vee \mu_2' = \mu_1'$ and we must have $w \in \mu_2'(f)$ and thus, $w \in \mu_2(f)$. However, since $\mu_1' \succ \mu_1 \succ \mu_2$ and $w \in \mu_1'(f) \setminus \mu_1(f)$, we can apply Lemma 16 and conclude that $w \notin \mu_2(f)$, which is a contradiction. This shows $\mu_1'(f) \setminus \mu_1(f) \subseteq \mu_2'(f) \setminus \mu_2(f)$. Together with ($\natural$), we have $\mu_1'(f) \setminus \mu_1(f) = \mu_2'(f) \setminus \mu_2(f)$. □

Lemma 18 *Let A, B, A', B' be sets such that $A \subseteq A'$ and $B \subseteq B'$. In addition, assume that $A' \setminus A = B' \setminus B$. Then, $(A' \cap B') \setminus (A \cap B) = A' \setminus A$.*

Proof Let $X := A' \setminus A = B' \setminus B$. Notice that $A' = A \sqcup X$ and $B' = B \sqcup X$, where $\sqcup$ is the disjoint union operator. Therefore, we have $A \cap B = (A' \setminus X) \cap (B' \setminus X) = (A' \cap B') \setminus X$ and the claim follows. □

Lemma 19 *Let $\mu_1, \mu_1', \mu_2, \mu_2' \in \mathcal{S}$ and assume that μ_1', μ_2' are immediate predecessors of μ_1, μ_2 in the stable matching lattice, respectively. If $P(\mu_1) \setminus P(\mu_1') = P(\mu_2) \setminus P(\mu_2')$, then $\mu_1'(f) \setminus \mu_1(f) = \mu_2'(f) \setminus \mu_2(f)$ for every firm f. In particular, $\rho^-(\mu_1', \mu_1) = \rho^-(\mu_2', \mu_2)$.*

Proof We first consider the case where $\mu_1 = \mu_2$. By Lemma 4, we have $P(\mu_i') \subseteq P(\mu_i)$ for $i \in \{1, 2\}$. Therefore, $P(\mu_1') = P(\mu_1) \setminus (P(\mu_1) \setminus P(\mu_1')) = P(\mu_2) \setminus (P(\mu_2) \setminus P(\mu_2')) = P(\mu_2')$, where the second equality is due to our assumptions that $\mu_1 = \mu_2$ and $P(\mu_1) \setminus P(\mu_1') = P(\mu_2) \setminus P(\mu_2')$. Thus, $\mu_1' = \mu_2'$ because of Theorem 4, and the thesis then follows. Since the cases when $\mu_1 \succ \mu_2$ or $\mu_2 \succ \mu_1$ have already been considered in Lemma 17, for the following, we assume that μ_1 and μ_2 are not comparable. Let $\mu_3 := \mu_1 \vee \mu_2$ and $\mu_3' := \mu_1' \vee \mu_2'$. Note that $\mu_3' \succeq \mu_3$. Then, by Lemma 7 and Lemma 18, we have $P(\mu_3) \setminus P(\mu_3') = (P(\mu_1) \cap P(\mu_2)) \setminus (P(\mu_1') \cap P(\mu_2')) = P(\mu_1) \setminus P(\mu_1')$.

By Theorem 4, Lemma 12 and Lemma 13, we also have that μ_3' is an immediate predecessor of μ_3 in the stable matching lattice. Note that by construction, we have $\mu_3 \succ \mu_1$ and $\mu_3 \succ \mu_2$ since μ_1 and μ_2 are incomparable. Applying Lemma 17 on μ_1 and μ_3 as well as on μ_2 and μ_3, we have $\mu_1'(f) \setminus \mu_1(f) = \mu_3'(f) \setminus \mu_3(f) = \mu_2'(f) \setminus \mu_2(f)$ for all firms $f \in F$, as desired. □

Theorem 8 *Let $\mu_1, \mu_1', \mu_2, \mu_2' \in \mathcal{S}$ and assume that μ_1', μ_2' are immediate predecessors of μ_1, μ_2 in the stable matching lattice, respectively. Then, $P(\mu_1) \setminus P(\mu_1') = P(\mu_2) \setminus P(\mu_2')$ if and only if $\rho(\mu_1', \mu_1) = \rho(\mu_2', \mu_2)$.*

Proof For the "only if" direction, assume $P(\mu_1) \setminus P(\mu_1') = P(\mu_2) \setminus P(\mu_2')$. Then, $\rho^+(\mu_1', \mu_1) = \rho^+(\mu_2', \mu_2)$ by Lemma 15 and $\rho^-(\mu_1', \mu_1) = \rho^-(\mu_2', \mu_2)$ by Lemma 19. Thus, $\rho(\mu_1', \mu_1) = \rho(\mu_2', \mu_2)$. For the "if" direction, assume $\rho(\mu_1', \mu_1) = \rho(\mu_2', \mu_2)$. Then, immediately from Lemma 15, we have that $P(\mu_1) \setminus P(\mu_1') = \rho^+(\mu_1', \mu_1) = \rho^+(\mu_2', \mu_2) = P(\mu_2) \setminus P(\mu_2')$. ◻

Remark 3 In the SM- Model with P-sets defined as by Gusfield and Irving [23] stated in Remark 1, Theorem 8 immediately follows from the definition of P-set. In fact, one can explicitly and uniquely construct $\rho(\mu', \mu)$ from $P(\mu) \setminus P(\mu')$. In particular, $\rho^+(\mu', \mu)$ is the set of edges (f, w) such that $P_f(\mu) \neq P_f(\mu')$ and w is the least preferred partner of f among $P_f(\mu) \setminus P_f(\mu')$, and $\rho^-(\mu', \mu)$ is the set of edges (f, w) such that $P_f(\mu) \neq P_f(\mu')$ and w is the partner that, in the preference list $\geq_f$, is immediately before the most preferred partner of f among $P_f(\mu) \setminus P_f(\mu')$.

Proof of Theorem 7 Because of Theorem 4 and Lemma 15, for every $K_1 \in \mathcal{D}$, there exist stable matchings μ' and μ with μ' being an immediate predecessor of μ such that $K_1 = P(\mu) \setminus P(\mu') = \rho^+(\mu', \mu)$. Thus, the mapping Q is onto. Theorem 8 further implies that Q is injective. Hence, the mapping Q is a bijection. This bijection and the definition of $\succeq^\star$ immediately imply that $(\mathcal{D}, \sqsupseteq)$ is isomorphic to $(\Pi, \succeq^\star)$. Together with the isomorphism between $(\mathcal{S}, \succeq)$ and $(\mathcal{P}, \subseteq)$, and the fact that $(\mathcal{D}, \sqsupseteq)$ is a representation poset of $(\mathcal{P}, \subseteq)$, we deduce a bijection between elements of $(\mathcal{S}, \succeq)$ and upper sets of $(\Pi, \succeq^\star)$. That is, $(\Pi, \succeq^\star)$ is a representation poset of $(\mathcal{S}, \succeq)$ and its representation function $\psi_{\mathcal{S}}$ satisfies that for every $\mu \in \mathcal{S}$, $\{Q(\rho) : \rho \in \psi_{\mathcal{S}}(\mu)\} = \psi_{\mathcal{P}}(P(\mu))$. It remains to show that the formula for the inverse of $\psi_{\mathcal{S}}$ given in the statement of the theorem is correct. Let $\mu \in \mathcal{S}$ and let $\mu_0, \mu_1, \cdots, \mu_k$ be a sequence of stable matchings such that μ_{i-1} is an immediate predecessor of μ_i in $(\mathcal{S}, \succeq)$ for all $i \in [k]$, $\mu_0 = \mu_F$ and $\mu_k = \mu$. In addition, let $\rho_i = \rho(\mu_{i-1}, \mu_i)$ for all $i \in [k]$. Note that $\mu = \mu_F \Delta(\rho_1^- \Delta \rho_1^+) \Delta(\rho_2^- \Delta \rho_2^+) \Delta \cdots \Delta(\rho_k^- \Delta \rho_k^+)$ ($\natural$). By Theorem 4, $P(\mu_0) \subseteq P(\mu_1) \subseteq \cdots \subseteq P(\mu_k)$, and thus,

$$P(\mu) = P(\mu_0) \cup \big(P(\mu_1) \setminus P(\mu_0)\big) \cup \big(P(\mu_2) \setminus P(\mu_1)\big) \cup \cdots \cup \big(P(\mu_k) \setminus P(\mu_{k-1})\big).$$

Therefore, by Lemma 15, $P(\mu) = P(\mu_F) \cup Q(\rho_1) \cup \cdots \cup Q(\rho_k)$. By Lemma 11, we know that $\{Q(\rho_i) : i \in [k]\}$ is an upper set of $\mathcal{D}$ and thus, $\psi_{\mathcal{P}}(P(\mu)) = \{Q(\rho_i) : i \in [k]\}$ due to Theorem 5. Hence, $\psi_{\mathcal{S}}(\mu) = \{\rho_i : i \in [k]\}$. The inverse of $\psi_{\mathcal{S}}$ must be as in the first definition in the thesis so that ($\natural$) holds.

Let (f, w) be a firm-worker pair. If $(f, w) \in \rho_i^-$ for some $i \in [k]$, then $(f, w) \notin \mu$ due to Lemma 16. In addition, because of Lemma 10 and the bijection Q, μ_F, ρ_1^+, $\rho_2^+, \cdots, \rho_i^+$ are disjoint. Hence, if $(f, w) \in \mu_F \cup (\bigcup\{\rho_i^+ : i \in [k]\})$ but $(f, w) \notin \bigcup\{\rho_i^- : i \in [k]\}$, then $(f, w) \in \mu$. The second definition of $\psi_{\mathcal{S}}$ from the thesis follows immediately from these facts and the previous definition. ◻

3.4 Concluding the proof for the first part of Theorem 1

Because of Theorem 7, part (iii), we know that poset $(\Pi, \succeq^\star)$ represents lattice $(\mathcal{S}, \succeq)$. Let $\psi_{\mathcal{S}}$ be the representation function as defined in Theorem 7. We denote by $E \subseteq F \times W$ the set of acceptable firm-worker pairs. Hence, E is the base set of lattice $(\mathcal{S}, \succeq)$. We deduce the following, proving the structural statement from Theorem 1.

Lemma 20 *Let $\overline{\Pi}_1, \overline{\Pi}_2$ be two upper sets of $(\Pi, \succeq^\star)$ and let $\mu_i = \psi_{\mathcal{S}}^{-1}(\overline{\Pi}_i)$ for $i \in \{1, 2\}$. If $\overline{\Pi}_1 \subseteq \overline{\Pi}_2$, then $\mu_1 \succeq \mu_2$.*

Proof Let $\overline{\mathcal{D}}_i := \{Q(\rho) : \rho \in \overline{\Pi}_i\}$ and let $P_i := \psi_{\mathcal{P}}^{-1}(\overline{\mathcal{D}}_i)$ for $i \in \{1, 2\}$. Since $\overline{\Pi}_1 \subseteq \overline{\Pi}_2$, we have $\overline{\mathcal{D}}_1 \subseteq \overline{\mathcal{D}}_2$ and thus subsequently $P_1 \subseteq P_2$. Since $\psi_{\mathcal{P}}^{-1}(\overline{\mathcal{D}}_i) = P(\psi_{\mathcal{S}}^{-1}(\overline{\Pi}_i))$ by Theorem 7, $P_i = P(\mu_i)$ for both $i = 1, 2$. Therefore, by Theorem 4, $\mu_1 \succeq \mu_2$. $\square$

Lemma 21 *Let $\rho_1, \rho_2 \in \Pi$. If $\rho_1^+ \cap \rho_2^- \neq \emptyset$, then $\rho_1 \succ^\star \rho_2$.*

Proof Assume by contradiction that $\rho_1 \not\succ^\star \rho_2$, that is, either $\rho_2 \succ^\star \rho_1$ or that they are not comparable. Let $\overline{\Pi}_1 := \{\rho \in \Pi : \rho \succeq \rho_2\}$ be the inclusion-wise smallest upper set of Π that contains ρ_2, let $\overline{\Pi}_0 := \overline{\Pi}_1 \setminus \{\rho_2\}$, and let $\overline{\Pi}_2 := \{\rho \in \Pi : \rho \succeq \rho_1 \text{ or } \rho \succeq \rho_2\}$ be the inclusion-wise smallest upper set of Π that contains both ρ_1 and ρ_2. Note that $\overline{\Pi}_0 \subsetneq \overline{\Pi}_1 \subsetneq \overline{\Pi}_2$, where the second strict containment is due to our assumption that $\rho_1 \not\succ^\star \rho_2$ and thus $\rho_1 \notin \overline{\Pi}_1$. For $i \in \{0, 1, 2\}$, let $\mu_i := (\triangle_{\rho \in \overline{\Pi}_i} (\rho^- \triangle \rho^+)) \triangle \mu_F$. Since $\overline{\Pi}_i$ is an upper set of $(\Pi, \succeq^\star)$, μ_i is a stable matching by Theorem 7. Moreover, $\mu_0 \succ \mu_1 \succ \mu_2$ by Lemma 20. Let $(f, w) \in \rho_1^+ \cap \rho_2^-$. Since $\rho(\mu_0, \mu_1) = \rho_2$, we have $(f, w) \in \mu_0 \setminus \mu_1$. Since ρ_1 is a $\succeq$-minimal element in $\overline{\Pi}_2$, $\overline{\Pi}_2 \setminus \{\rho_1\}$ is also an upper set of Π. Then, $\mu_2' := (\triangle_{\rho \in \overline{\Pi}_2 \setminus \{\rho_1\}} (\rho^- \triangle \rho^+)) \triangle \mu_F$ is a stable matching by Theorem 7, and $\mu_2 = \mu_2' \setminus \rho_1^- \cup \rho_1^+$ by (1). Thus, we have $(f, w) \in \mu_2$. Together, we have $w \in (\mu_0(f) \cap \mu_2(f)) \setminus \mu_1(f)$. However, this contradicts Lemma 16. $\square$

Theorem 9 *The rotation poset $(\Pi, \succeq^\star)$ affinely represents the stable matching lattice $(\mathcal{S}, \succeq)$ with affine function $g(u) = Au + \chi^{\mu_F}$, where $A \in \{0, \pm 1\}^{E \times \Pi}$ is matrix with columns $\chi^{\rho^+} - \chi^{\rho^-}$ for each $\rho \in \Pi$. Moreover, $|\Pi| = O(|F||W|)$ and matrix A has full column rank.*

Proof The first claim follows immediately because by Theorem 7, part (iii), $\chi^\mu = A\chi^{\psi_{\mathcal{S}}(\mu)} + \chi^{\mu_F}$, for any stable matching μ. Because of Theorem 7, $|\Pi| = |\mathcal{D}|$. In addition, by Lemma 10, we have $|\mathcal{D}| = |E| = O(|F||W|)$. Thus, $|\Pi| = O(|F||W|)$. Finally, we show that matrix A has full column rank. Assume by contradiction that there is a non-zero vector $\lambda \in \mathbb{R}^\Pi$ such that $\sum_{\rho \in \Pi} \lambda_\rho (\chi^{\rho^+} - \chi^{\rho^-}) = \mathbf{0}$. Let $\widetilde{\Pi} := \{\rho \in \Pi : \lambda_\rho \neq 0\}$ denote the set of rotations whose corresponding coefficients in λ are non-zero. Let ρ_1 be a minimal rotation (w.r.t. $\succeq^\star$) in $\widetilde{\Pi}$ and let (f, w) be a firm-worker pair in ρ_1^+. Because of Lemma 10 and the bijection Q, there is no rotation $\rho \neq \rho_1$ such that $(f, w) \in \rho^+$. Therefore, there must exist a rotation $\rho_2 \in \widetilde{\Pi}$ with $(f, w) \in \rho_2^-$. Note that we must have $\rho_1 \succ^\star \rho_2$ due to Lemma 21. However, this contradicts the choice of ρ_1. $\square$

4 Algorithms

Because of Theorem 9, in order to conclude the proof of Theorem 1, we are left to explicitly construct $(\Pi, \succeq^\star)$. That is, we need to find elements of Π, and how they relate to each other via $\succeq^\star$. We fix an instance $(F, W, \mathcal{C}_F, \mathcal{C}_W)$ and abbreviate $\mathcal{S} := \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$.

In this section, we further assume workers' choice functions to be quota-filling. Under this additional assumption, for each worker $w \in W$, the family of sets of partners w is assigned to under all stable matchings (denoted as Φ_w) satisfies an additional property, which we call the *full-quota*[4] property (see Lemma 22). Recall that q_w denote the quota of worker w and $\overline{q}_w$ is the number of firms matched to w under every stable matching, which is constant due to the equal-quota property (i.e., $|S| = \overline{q}_w$ for all $S \in \Phi_w$).

Lemma 22 *For every worker $w \in W$, if $\overline{q}_w < q_w$, then w is matched to the same set of firms in all stable matchings. That is,*

$$\overline{q}_w < q_w \implies |\Phi_w| = 1. \tag{full-quota}$$

Proof Assume by contradiction that $\overline{q}_w < q_w$ but $|\Phi_w| > 1$. Let S_1, S_2 be two distinct elements from Φ_w and let μ_i be the matching such that $\mu_i(w) = S_i$ for $i = 1, 2$. Note that due to the equal-quota property, we have $|S_1| = |S_2| = \overline{q}_w$. Consider the stable matching $\mu := \mu_1 \wedge \mu_2$. Then,

$$|\mu(w)| = |\mathcal{C}_w(\mu_1(w) \cup \mu_2(w))| = |\mathcal{C}_w(S_1 \cup S_2)| = \min(|S_1 \cup S_2|, q_w|) > \overline{q}_w,$$

where the first equality is by Theorem 3 and the last two relations are by quota-filling. However, this contradicts the equal-quota property since μ is a stable matching. $\square$

Our approach to construct $(\Pi, \succeq^\star)$ is as follows. First, we recall Roth's adaptation of the Deferred Acceptance algorithm to find a firm- or worker-optimal stable matching (Sect. 4.1). Second, we feed the output of Roth's algorithm to an algorithm that produces a maximal chain $C_1, C_2, \ldots, C_k$ of $(\mathcal{S}, \succeq)$ and the set Π (Sect. 4.2). In Sect. 4.3, we give an algorithm that, given a maximal chain of a ring of sets, constructs the partial order of the poset of minimal differences. This and previous facts are then exploited in Sect. 4.4 to construct the partial order $\succeq^\star$ on elements of Π. We sum up our algorithm in Sect. 4.5, where we show that the overall running time is $O(|F|^3|W|^3\texttt{oracle-call})$.

We start with a definition and properties which will be used in later algorithms. For a matching μ, let

$$\overline{X}_f(\mu) := \{w \in W(f) : \mathcal{C}_f(\mu(f) \cup \{w\}) = \mu(f)\},$$

and define the *closure* of μ, denoted by $\overline{X}(\mu)$, as the collection of sets $\{\overline{X}_f(\mu) : f \in F\}$. Note that $\mu(f) \subseteq \overline{X}_f(\mu)$ for every firm f and individually rational matching μ.

[4] Note that the full-quota property is analogous to the *Rural Hospital Theorem* [38] in the SA- Model where agents have preferences over individual partners instead of over sets of partners.

Lemma 23 *Let μ be an individually rational matching. Then, for every firm f, we have $\mathcal{C}_f(\overline{X}_f(\mu)) = \mu(f)$.*

Proof Fix a firm f. Since μ is individually rational, $\mathcal{C}_f(\mu(f)) = \mu(f)$. The claim then follows from Lemma 2 with $A_1 = \mu(f)$ and $A_2 = \overline{X}_f(\mu)$. $\square$

Lemma 24 *Let $\mu_1, \mu_2 \in \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$ such that $\mu_1 \succeq \mu_2$. Then, for every firm f, $\mu_2(f) \subseteq \overline{X}_f(\mu_1)$.*

Proof Since $\mu_1 \succeq \mu_2$, we have $\mathcal{C}_f(\mu_1(f) \cup \mu_2(f)) = \mu_1(f)$ for every firm f. Thus, by the consistency property of $\mathcal{C}_f$, for every $w \in \mu_2(f)$, we have $\mathcal{C}_f(\mu_1(f) \cup \{w\}) = \mu_1(f)$. The claim follows. $\square$

Lemma 25 *In the following, we give the running time of three operations.*

(i) given a matching μ, computing its closure $\overline{X}(\mu)$ can be performed in time $O(|F||W|\texttt{oracle-call})$;
(ii) given a matching μ, deciding whether it is stable can be performed in time $O(|F||W|\texttt{oracle-call})$;
(iii) given stable matchings $\mu, \mu' \in \mathcal{S}$, deciding whether $\mu' \succeq \mu$ can be performed in time $O(|F|\texttt{oracle-call})$.

Proof (i). For any firm f, computing $\overline{X}_f(\mu)$ requires $O(|W|)$ `oracle-calls` by definition and thus, computing the closure of μ takes $O(|F||W|)$ `oracle-calls`. (ii). To check if a matching μ is stable, we need to check first if it is individually rational, which takes $O(|F|+|W|)$ `oracle-calls`, and then to check if it admits any blocking pair, which takes $O(|F||W|)$ `oracle-calls`. (iii). To decide if $\mu' \succeq \mu$, one need to check if for every firm $f \in F, \mathcal{C}_f(\mu'(f) \cup \mu(f)) = \mu'(f)$, and this takes $O(|F|)$ `oracle-calls`. $\square$

4.1 Deferred acceptance algorithm

The deferred acceptance algorithm introduced in [37][5] can be seen as a generalization of the algorithm proposed in [21]. For the following, we assume that firms are the proposing side. Initially, for each firm f, let $X_f{:=}W(f)$, i.e., the set of acceptable workers of f. At every step, every firm f *proposes* to workers in $\mathcal{C}_f(X_f)$. Then, every worker w considers the set of firms X_w who made a proposal to w, *temporarily accepts* $Y_w{:=}\mathcal{C}_w(X_w)$, and *rejects* the rest. Afterwards, each firm f removes from X_f all workers that rejected f. The *firm-proposing* algorithm iterates until there is no rejection. Hence, throughout the algorithm, X_f denotes the set of acceptable workers of f that have not rejected f. A formal description is given in Algorithm 1.

Note that for every step s other than the final step, there exists a firm $f \in F$ such that $X_f^{(s)} \subsetneq X_f^{(s-1)}$. Therefore, the algorithm terminates, since there is a finite number of firms and workers. Moreover, the output has interesting properties.

[5] The model considered in [37] is more general than our setting here, where choice functions are only assumed to be substitutable and consistent, not necessarily quota-filling.

Algorithm 1 Firm-proposing DA algorithm for an instance $(F, W, \mathcal{C}_F, \mathcal{C}_W)$.

initialize the step count $s \leftarrow 0$
for each firm f **do** initialize $X_f^{(s)} \leftarrow W(f)$ **end for**
repeat
  for each worker w **do**
    $X_w^{(s)} \leftarrow \{f \in F : w \in \mathcal{C}_f(X_f^{(s)})\}$
    $Y_w^{(s)} \leftarrow \mathcal{C}_w(X_w^{(s)})$
  end for
  for each firm f **do**
    update $X_f^{(s+1)} \leftarrow X_f^{(s)} \setminus \{w \in W : f \in X_w^{(s)} \setminus Y_w^{(s)}\}$
  end for
  update the step count $s \leftarrow s + 1$
until $X_f^{(s)} = X_f^{(s-1)}$ for every firm f
Output: matching $\overline{\mu}$ with $\overline{\mu}(w) = Y_w^{(s-1)}$ for every worker w

Theorem 10 (Theorem 2, [37]) *Let $\overline{\mu}$ be the output of Algorithm 1 over a matching market $(F, W, \mathcal{C}_F, \mathcal{C}_W)$ assuming $\mathcal{C}_F, \mathcal{C}_W$ are path-independent. Then, $\overline{\mu} = \mu_F$.*

Due to the symmetry between firms and workers in a market where the only assumption on choice functions is path-independence, swapping the role of firms and workers in Algorithm 1, we have the *worker-proposing* deferred acceptance algorithm, which outputs μ_W.

4.2 Constructing Π via a maximal chain of $(\mathcal{S}, \succeq)$

Let $(\mathcal{H}, \subseteq)$ be a ring of sets. A *chain* $C_0, \cdots, C_k$ in $(\mathcal{H}, \subseteq)$ is an ordered subset of $\mathcal{H}$ such that C_{i-1} is a predecessor of C_i in $(\mathcal{H}, \subseteq)$ for all $i \in [k]$. The chain is *complete* if moreover C_{i-1} is an immediate predecessor of C_i for all $i \in [k]$; it is *maximal* if it is complete, $C_0 = H_0$ and $C_k = H_z$. Consider $K \in \mathcal{D}(\mathcal{H})$. If $K = C_i \setminus C_{i-1}$ for some $i \in [k]$, then we say that the chain *contains* the minimal difference K. We start with the theorem below, where it is shown that the set $\mathcal{D}(\mathcal{H})$ can be obtained by following any maximal chain of $(\mathcal{H}, \subseteq)$.

Theorem 11 (Theorem 2.4.2, [23]) *Let $H', H \in \mathcal{H}$ such that $H' \subseteq H$. Then, there exists a complete chain from H' to H in $(\mathcal{H}, \subseteq)$, and every such chain contains exactly the same set of minimal differences. In particular, for any maximal chain $(C_0, \cdots, C_k)$ in $(\mathcal{H}, \subseteq)$, we have $\{C_i \setminus C_{i-1} : i \in [k]\} = \mathcal{D}(\mathcal{H})$ and $k = |\mathcal{D}(\mathcal{H})|$.*

In this section, we present Algorithm 3 that, on inputs μ', outputs a stable matching μ that is an immediate descendant of μ' in $(\mathcal{S}, \succeq)$. Then, using Algorithm 3 as a subroutine, Algorithm 4 gives a maximal chain of $(\mathcal{S}, \succeq)$.

We start by extending to our setting the *break-marriage* idea proposed by McVitie and Wilson [33] for finding the full set of stable matchings in the one-to-one stable marriage model. Given a stable matching μ' and a firm-worker pair $(f', w') \in \mu' \setminus \mu_W$, the break-marriage procedure, denoted as `break-marriage`(μ', f', w'), works as follows. We first initialize X_f to be $\overline{X}_f(\mu')$ for every firm $f \neq f'$, while we set $X_{f'} = \overline{X}_{f'}(\mu') \setminus \{w'\}$. We then restart the deferred acceptance process. The algorithm continues in iterations as in the **repeat** loop of Algorithm 1, with the exception that worker w' temporarily accepts $Y_{w'} := \mathcal{C}_{w'}(X_{w'} \cup \{f'\}) \setminus \{f'\}$. As an intuitive explanation, this acceptance rule of w' ensures that for the output matching $\overline{\mu}$, we have $\mathcal{C}_{w'}(\overline{\mu}(w') \cup \mu'(w')) = \overline{\mu}(w')$, as we show in Lemma 27. The formal break-marriage procedure is summarized in Algorithm 2. Note that by choice of the pair (f', w'), we have $|\mu'(w')| = q_{w'}$.

Algorithm 2 `break-marriage`(μ', f', w'), with $(f', w') \in \mu' \setminus \mu_W$ and $\mu' \in \mathcal{S}$

for each firm $f \neq f'$ **do** initialize $X_f^{(0)} \leftarrow \overline{X}_f(\mu')$ **end for**
initialize $X_{f'}^{(0)} \leftarrow \overline{X}_{f'}(\mu') \setminus \{w'\}$
initialize the step count $s \leftarrow 0$
repeat
    for each worker w **do**
        $X_w^{(s)} \leftarrow \{f \in F : w \in \mathcal{C}_f(X_f^{(s)})\}$
        if $w \neq w'$ **then** $Y_w^{(s)} \leftarrow \mathcal{C}_w(X_w^{(s)})$ **else** $Y_w^{(s)} \leftarrow \mathcal{C}_w(X_w^{(s)} \cup \{f'\}) \setminus \{f'\}$
    end for
    for each firm f **do**
        update $X_f^{(s+1)} \leftarrow X_f^{(s)} \setminus \{w \in W : f \in X_w^{(s)} \setminus Y_w^{(s)}\}$
    end for
    update the step count $s \leftarrow s + 1$
until $X_f^{(s-1)} = X_f^{(s)}$ for every firm f
Output: matching $\overline{\mu}$ with $\overline{\mu}(w) = Y_w^{(s-1)}$ for every worker w

---

With the same reasoning as for the DA algorithm, the `break-marriage`(μ', f', w') procedure is guaranteed to terminate. Let $s^\star$ be the value of step count s at the end of the algorithm. Note that, for every firm $f \in F$, we have

$$\overline{X}_f(\mu') \supseteq X_f^{(0)} \supseteq X_f^{(1)} \supseteq \cdots \supseteq X_f^{(s^\star)}, \tag{2}$$

where the first containment is an equality unless $f = f'$. In particular, (2) implies that $f' \notin X_{w'}^{(s)}$ for all $s \in \{0, 1, \cdots, s^\star\}$. Also note that the termination condition implies

$$\overline{\mu}(f) = \mathcal{C}_f(X_f^{(s^\star)}) = \mathcal{C}_f(X_f^{(s^\star-1)}) \tag{3}$$

for every firm f, while for every worker $w \neq w'$ it implies that

$$\overline{\mu}(w) = Y_w^{(s^\star-1)} = \mathcal{C}_w(X_w^{(s^\star-1)}) = X_w^{(s^\star-1)}. \tag{4}$$

Let $(f, w) \in F \times W$, we say f is *rejected by w at step s* if $f \in X_w^{(s)} \setminus Y_w^{(s)}$, and we say f is *rejected by* w if f is rejected by w at some step during the break-marriage procedure. Note that a firm f is rejected by all and only the workers in $X_f^{(0)} \setminus X_f^{(s^\star)}$.

In the following, we prove Theorem 12.

Theorem 12 *Let $\mu', \mu \in \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$ and assume μ' is an immediate predecessor of μ in the stable matching lattice. Pick $(f', w') \in \mu' \setminus \mu$ and let $\overline{\mu}$ be the output matching of* `break-marriage`(μ', f', w'). *Then, $\overline{\mu} = \mu$.*

We start by outlining the proof steps of Theorem 12. We first show in Lemma 26 that the output matching $\overline{\mu}$ of `break-marriage`(μ', f', w') is individually rational. We then show in Lemma 30 that under a certain condition (i.e., the break-marriage operation being *successful*), $\overline{\mu}$ is a stable matching and $\mu' \succ \overline{\mu}$. Lastly, we show that under the assumptions in the statement of Theorem 12, the above-mentioned condition is satisfied and $\overline{\mu} \succeq \mu$.

Lemma 26 *Let $\mu' \in \mathcal{S}$ be a stable matching that is not the worker-optimal stable matching μ_W and let $(f', w') \in \mu' \backslash \mu_W$. Consider the* `break-marriage`$(\mu', f', w')$ *procedure with output $\overline{\mu}$. Then, $\overline{\mu}$ is individually rational.*

Proof By (3) and (4), for every agent $a \in F \cup W \setminus \{w'\}$, $\overline{\mu}(a) = \mathcal{C}_a(X_a^{(s^\star-1)})$ and thus, $\mathcal{C}_a(\overline{\mu}(a)) = \mathcal{C}_a(\mathcal{C}_a(X_a^{(s^\star-1)})) = \mathcal{C}_a(X_a^{(s^\star-1)}) = \overline{\mu}(a)$, where the second equality is due to path-independence. For worker w', note that $X_{w'}^{(s^\star-1)} = Y_{w'}^{(s^\star-1)} = \overline{\mu}(w') = \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\}) \setminus \{f'\}$, where the first equality is due to the termination criterion. Then, by the substitutability property, with $T = X_{w'}^{(s^\star-1)}$ and $S = X_{w'}^{(s^\star-1)} \cup \{f'\}$, we have that for every firm $f \in \overline{\mu}(w')$, $f \in \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)})$ holds. Thus, $\overline{\mu}(w') \subseteq \mathcal{C}_{w'}(\overline{\mu}(w'))$. Since $\mathcal{C}_{w'}(X) \subseteq X$ for any X in the domain of $\mathcal{C}_{w'}$, we have $\overline{\mu}(w') = \mathcal{C}_{w'}(\overline{\mu}(w'))$. Therefore, $\overline{\mu}$ is individually rational. $\square$

Lemma 27 *Consider the* `break-marriage`(μ', f', w') *procedure with output matching $\overline{\mu}$. Then, for every firm f, $\mathcal{C}_f(\overline{\mu}(f) \cup \mu'(f)) = \mu'(f)$.*

Proof For a firm f, we have

$$\begin{aligned}
\mathcal{C}_f(\overline{\mu}(f) \cup \mu'(f)) &= \mathcal{C}_f(\mathcal{C}_f(X_f^{(s^\star)}) \cup \mathcal{C}_f(\overline{X}_f(\mu')) \\
&= \mathcal{C}_f(X_f^{(s^\star)} \cup \overline{X}_f(\mu')) = \mathcal{C}_f(\overline{X}_f(\mu')) = \mu'(f),
\end{aligned}$$

where the first and last equality hold since $\mu'(f) = C_f(\overline{X}_f(\mu'))$ by Lemma 23 and $\overline{\mu}(f) = C_f(X_f^{(s^\star)})$ by (3), the second equality is by path-independence, and the third equality is due to $X_f^{(s^\star)} \subseteq X_f^{(0)} \subseteq \overline{X}_f(\mu')$ by (2). $\square$

The following two properties of the break-marriage procedure are direct consequences of the path-independence assumption imposed on choice functions. These properties are also true for the deferred acceptance algorithm, as shown in [37]. Let $f \in F$ and $w \in W$ be an arbitrary firm and worker. Lemma 28 states that once f

proposes to w in some step of the algorithm, it will keep proposing to w in future steps until w rejects f. Lemma 29 states that once w rejects f, w would never accept f in later steps even if the proposal is offered again.

Lemma 28 *For all $s \in [s^\star - 1]$ and $w \in W$, we have $Y_w^{(s-1)} \subseteq X_w^{(s)}$.*

Proof Let $f \in Y_w^{(s-1)}$. By construction, we have $w \in \mathcal{C}_f(X_f^{(s-1)}) \cap X_f^{(s)}$. Since $X_f^{(s)} \subseteq X_f^{(s-1)}$ by (2), we have $w \in \mathcal{C}_f(X_f^{(s)})$ by substitutability. Hence, $f \in X_w^{(s)}$. □

Lemma 29 *Let $s \in [s^\star - 1]$, $f \in F$, and $w \in W$. Assume $f \in X_w^{(s-1)} \setminus Y_w^{(s-1)}$, i.e., f is rejected by w at step $s-1$. If $w \neq w'$, then for every step $s' \geq s$, $f \notin \mathcal{C}_w(X_w^{(s')} \cup \{f\})$; and if $w = w'$, then for every step $s' \geq s$, $f \notin \mathcal{C}_w(X_w^{(s')} \cup \{f'\} \cup \{f\})$.*

Proof By construction, $w \notin X_f^{(s)}$. Hence, $f \notin X_w^{(s')}$ for all $s' \geq s$ because of (2) and the definition of $X_w^{(s')}$. Fix a value of $s' \geq s$. First consider the case when $w \neq w'$. By repeated application of the path-independence property of $\mathcal{C}_w$ and Lemma 28, we have

$$
\begin{aligned}
\mathcal{C}_w(X_w^{(s')} \cup \{f\}) &= \mathcal{C}_w(X_w^{(s')} \cup Y_w^{(s'-1)} \cup \{f\}) = \mathcal{C}_w(X_w^{(s')} \cup \mathcal{C}_w(Y_w^{(s'-1)} \cup \{f\})) \\
&= \mathcal{C}_w(X_w^{(s')} \cup \mathcal{C}_w(\mathcal{C}_w(X_w^{(s'-1)}) \cup \{f\})) \\
&= \mathcal{C}_w(X_w^{(s')} \cup \mathcal{C}_w(X_w^{(s'-1)} \cup \{f\})) \\
&= \cdots \\
&= \mathcal{C}_w(\underbrace{X_w^{(s')}}_{\not\ni f} \cup \underbrace{X_w^{(s'-1)}}_{\not\ni f} \cup \cdots \cup \underbrace{\mathcal{C}_w(X_w^{(s-1)} \cup \{f\})}_{=\mathcal{C}_w(X_w^{(s-1)}) = Y_w^{(s-1)} \not\ni f}).
\end{aligned}
$$

Therefore, $f \notin \mathcal{C}_w(X_w^{(s')} \cup \{f\})$ as desired. We next consider the case where $w = w'$. Since $w \notin X_{f'}^{(0)}$ by construction, we have $w \notin X_{f'}^{(s-1)}$ by (2), which then implies $f' \notin X_w^{(s-1)}$ by definition. Thus, we have $f \neq f'$. Again, by repeated application of the path-independence property of $\mathcal{C}_w$ and Lemma 28, we have

$$
\begin{aligned}
\mathcal{C}_w(X_w^{(s')} \cup \{f'\} \cup \{f\}) &= \mathcal{C}_w(X_w^{(s')} \cup Y_w^{(s'-1)} \cup \{f'\} \cup \{f\}) \\
&= \mathcal{C}_w(X_w^{(s')} \cup \{f'\} \cup \mathcal{C}_w(Y_w^{(s'-1)} \cup \{f'\} \cup \{f\})) \\
&= \mathcal{C}_w(X_w^{(s')} \cup \{f'\} \cup \mathcal{C}_w((\mathcal{C}_w(X_w^{(s'-1)} \cup \{f'\}) \setminus \{f'\}) \cup \{f'\} \cup \{f\})) \\
&= \mathcal{C}_w(X_w^{(s')} \cup \{f'\} \cup \mathcal{C}_w(X_w^{(s'-1)} \cup \{f'\} \cup \{f\})) \\
&= \cdots \\
&= \mathcal{C}_w(\underbrace{X_w^{(s')}}_{\not\ni f} \cup \underbrace{X_w^{(s'-1)}}_{\not\ni f} \cup \cdots \cup \{f'\} \cup \underbrace{\mathcal{C}_w(X_w^{(s-1)} \cup \{f'\} \cup \{f\})}_{=\mathcal{C}_w(X_w^{(s-1)} \cup \{f'\}) \not\ni f}).
\end{aligned}
$$

Therefore, $f \notin \mathcal{C}_w(X_w^{(s')} \cup \{f'\} \cup \{f\})$ as desired in this case as well. □

We say the break-marriage procedure `break-marriage`(μ', f', w') is *successful* if $f' \notin \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\})$. We next show that when the procedure is successful, the output matching is stable.

Lemma 30 *If* `break-marriage`(μ', f', w') *is successful, then the output matching* $\overline{\mu}$ *is stable. Moreover,* $\mu' \succ \overline{\mu}$.

Proof Since `break-marriage`(μ', f', w') is successful, applying the consistency property with $T = X_{w'}^{(s^\star-1)}$ and $S = T \cup \{f'\}$, we have $\mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\}) = \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)})$ and thus, $Y_{w'}^{(s^\star-1)} = \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)})$. In addition, by the termination condition, $Y_{w'}^{(s^\star-1)} = X_{w'}^{(s^\star-1)}$. Therefore, we have the following identity (similar to (4) for other workers)

$$\overline{\mu}(w') = Y_{w'}^{(s^\star-1)} = X_{w'}^{(s^\star-1)} = \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)}) = \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\}). \tag{5}$$

Claim 1 Let $(f, w) \in F \times W$. If f is rejected by w during the break-marriage procedure, then $f \notin \mathcal{C}_w(\overline{\mu}(w) \cup \{f\})$.

Proof If $w \neq w'$, then by Lemma 29, $f \notin \mathcal{C}_w(X_w^{(s^\star-1)} \cup \{f\}) = \mathcal{C}_w(\overline{\mu}(w) \cup \{f\})$ where the equality is due to (4). This is also true if $w = w'$ because again by Lemma 29,

$$f \notin \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\} \cup \{f\}) = \mathcal{C}_{w'}(\mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\}) \cup \{f\}) = \mathcal{C}_{w'}(\overline{\mu}(w') \cup \{f\}),$$

where the first equality is by path-independence, and the second equality by (5). $\square$

Claim 2 $\mathcal{C}_w(\mu'(w) \cup \overline{\mu}(w)) = \overline{\mu}(w)$ for all $w \in W$.

Proof Let $f \in \mu'(w) \setminus \overline{\mu}(w)$, and suppose first $(f, w) \neq (f', w')$. Because of Lemma 23 and Lemma 28, f must be rejected by w during the break-marriage procedure since otherwise $f \in X_w^{(s)}$ for all $s \in [s^\star] \cup \{0\}$, which in particular implies $w \in \overline{\mu}(f)$ due to (4). Then, by Claim 1, $f \notin \mathcal{C}_w(\overline{\mu}(w) \cup \{f\})$. Next assume $(f, w) = (f', w')$. By (5), we know that $X_w^{(s^\star-1)} = \overline{\mu}(w)$. Since `break-marriage`$(\mu', f', w')$ is successful, we have $f' \notin \mathcal{C}_w(X_w^{(s^\star-1)} \cup \{f'\}) = \mathcal{C}_w(\overline{\mu}(w) \cup \{f'\})$. We conclude that in both cases, $\mathcal{C}_w(\overline{\mu}(w) \cup \{f\}) = \mathcal{C}_w(\overline{\mu}(w))$ by consistency. Thus, we can apply Lemma 2 with $A_1 = \overline{\mu}(w)$ and $A_2 = \mu'(w)$ and conclude that $\mathcal{C}_w(\mu'(w) \cup \overline{\mu}(w)) = \mathcal{C}_w(\overline{\mu}(w))$. The claim then follows from Lemma 26. $\square$

Fix an acceptable firm-worker pair $(f, w) \notin \overline{\mu}$. We show that (f, w) does not block $\overline{\mu}$. Assume by contradiction that $f \in \mathcal{C}_w(\overline{\mu}(w) \cup \{f\})$ (†) and $w \in \mathcal{C}_f(\overline{\mu}(f) \cup \{w\})$ (‡). We claim that $(f, w) \notin \mu'$. If this is not the case, the consistency property of $\mathcal{C}_w$, with $S = \mu'(w) \cup \overline{\mu}(w)$ and $T = \overline{\mu}(w) \cup \{f\}$, implies $\mathcal{C}_w(\overline{\mu}(w) \cup \{f\}) = \mathcal{C}_w(\mu'(w) \cup \overline{\mu}(w)) = \overline{\mu}(w)$, where the last equality is by Claim 2. Thus, $f \notin \mathcal{C}_w(\overline{\mu}(w) \cup \{f\})$, which contradicts our assumption (†). Thus, $(f, w) \notin \mu'$. Note that in particular, $(f, w) \neq (f', w')$. By Lemma 3 and Claim 2, (†) implies $f \in \mathcal{C}_w(\mu'(w) \cup \{f\})$. Hence, we must have $w \notin \mathcal{C}_f(\mu'(f) \cup \{w\})$ since μ' is stable, i.e., not blocked by

(f, w). This implies $\mathcal{C}_f(\mu'(f) \cup \{w\}) = \mathcal{C}_f(\mu'(f)) = \mu'(f)$ due to the consistency property or $\mathcal{C}_f$ and the fact that μ' is individually rational. Thus, $w \in \overline{X}_f(\mu') = X_f^{(0)}$ since $f \neq f'$.

Suppose first $w \notin X_f^{(s^\star)}$. Then, worker w rejected firm f during the break-marriage procedure. This implies $f \notin \mathcal{C}_w(\overline{\mu}(w) \cup \{f\})$ by Claim 1, contradicting assumption (†). Suppose next $w \in X_f^{(s^\star)}$. Since $(f, w) \notin \overline{\mu}$, we have $w \notin \overline{\mu}(f) = \mathcal{C}_f(X_f^{(s^\star)})$, where the equality is due to (3). Then by the consistency property, with $S = X_f^{(s^\star)}$ and $T = \overline{\mu}(f) \cup \{w\}$, we have that $w \notin \mathcal{C}_f(\overline{\mu}(f) \cup \{w\})$. However, this contradicts (‡). Therefore, $\overline{\mu}$ must be stable. By Lemma 27, $\mu' \succeq \overline{\mu}$. Moreover, we have $\mu' \neq \overline{\mu}$ since $f' \in \mu'(w') \setminus \overline{\mu}(w')$. Hence, $\mu' \succ \mu$ as desired. □

We now give the proof of Theorem 12.

Proof of Theorem 12 Note that by Lemma 24, $\mu(f) \subseteq \overline{X}_f(\mu')$ for every $f \in F$. We start by showing that during the break-marriage procedure, for every firm f, no worker in $\mu(f)$ rejects f. Assume by contradiction that this is not true. Let s' be the first step where such a rejection happens, with firm f_1 being rejected by worker $w_1 \in \mu(f_1)$. Hence, $f_1 \in X_{w_1}^{(s')} \setminus Y_{w_1}^{(s')}$.

Claim 3 There exists a firm $f_2 \in Y_{w_1}^{(s')} \setminus \mu(w_1)$ such that $f_2 \in \mathcal{C}_{w_1}(\mu(w_1) \cup \{f_2\})$.

Proof Assume by contradiction that such a firm f_2 does not exist. We first consider the case when $w_1 \neq w'$. By Corollary 3 with $A_1 = \mu(w_1)$ and $A_2 = Y_{w_1}^{(s')}$, we have $\mathcal{C}_{w_1}(\mu(w_1) \cup Y_{w_1}^{(s')}) = \mathcal{C}_{w_1}(\mu(w_1)) = \mu(w_1)$, where the last equality is because μ is individually rational. Hence, $f_1 \in \mathcal{C}_{w_1}(\mu(w_1) \cup Y_{w_1}^{(s')})$, and using substitutability, we deduce $f_1 \in \mathcal{C}_{w_1}(Y_{w_1}^{(s')} \cup \{f_1\})$. However, using consistency, with $T = Y_{w_1}^{(s')} \cup \{f_1\}$ and $S = X_{w_1}^{(s')}$, we conclude $\mathcal{C}_{w_1}(Y_{w_1}^{(s')} \cup \{f_1\}) = \mathcal{C}_{w_1}(X_{w_1}^{(s')}) = Y_{w_1}^{(s')} \not\ni f_1$, a contradiction.

We next consider the case when $w_1 = w'$. Note that $f_1 \neq f'$, because $(f', w') \notin \mu$ by choice of (f', w'). Since $\mu' \succeq \mu$, by Theorem 3, $\mathcal{C}_{w'}(\mu'(w') \cup \mu(w')) = \mu(w')$. Thus, by the consistency property, with $S = \mu'(w') \cup \mu(w')$ and $T = \mu(w') \cup \{f'\}$, we have $\mathcal{C}_{w'}(\mu(w') \cup \{f'\}) = \mu(w') \not\ni f'$. As in the case $w_1 \neq w'$, by Corollary 3 with $A_1 = \mu(w')$ and $A_2 = Y_{w_1}^{(s')} \cup \{f'\}$ and the fact that μ is individually rational, $\mu(w') = \mathcal{C}_{w'}(\mu(w')) = \mathcal{C}_{w'}(\mu(w') \cup \{f'\} \cup Y_{w'}^{(s')})$. Then, since $f_1 \in \mu(w') \cap X_{w'}^{(s')}$, by substitutability and path independence, we have:

$$
\begin{aligned}
f_1 &\in \mathcal{C}_{w'}(Y_{w'}^{(s')} \cup \{f'\} \cup \{f_1\}) \\
&= \mathcal{C}_{w'}(\mathcal{C}_{w'}(X_{w'}^{(s')} \cup \{f'\}) \setminus \{f'\} \cup \{f'\} \cup \{f_1\}) \\
&= \mathcal{C}_{w'}(X_{w'}^{(s')} \cup \{f'\}).
\end{aligned}
$$

However, since $f_1 \notin Y_{w'}^{(s')}$ by our choice and $f_1 \neq f'$, we should have $f_1 \notin \mathcal{C}_{w'}(X_{w'}^{(s')} \cup \{f'\})$, which is again a contradiction. □

Now let f_2 be the firm whose existence is guaranteed by Claim 3. In particular, $f_2 \in Y_{w_1}^{(s')}$ implies $w_1 \in \mathcal{C}_{f_2}(X_{f_2}^{(s')}) \subseteq X_{f_2}^{(s')}$. Note that by our choice of f_1, $\mu(f_2) \subseteq X_{f_2}^{(s')}$.

Therefore, by substitutability and $w_1 \in \mathcal{C}_{f_2}(X_{f_2}^{(s')})$, we have $w_1 \in \mathcal{C}_{f_2}(\mu(f_2) \cup \{w_1\})$. However, this means that (f_2, w_1) is a blocking pair of μ, which contradicts stability of μ. Thus, for every firm $f \in F$, no worker in $\mu(f)$ rejects f during the break-marriage procedure as we claimed, which, together with the fact that $\mu(f) \subseteq \overline{X}_f(\mu')$, implies $\mu(f) \subseteq X_f^{(s^\star)}$. By path-independence and (3), we have that for every firm f:

$$\begin{aligned}\mathcal{C}_f(\overline{\mu}(f) \cup \mu(f)) &= \mathcal{C}_f(\mathcal{C}_f(X_f^{(s^\star)}) \cup \mu(f)) = \mathcal{C}_f(X_f^{(s^\star)} \cup \mu(f)) \\ &= \mathcal{C}_f(X_f^{(s^\star)}) = \overline{\mu}(f).\end{aligned} \tag{6}$$

Moreover,

$$|\mu(f)| = |\mu'(f)| = |\overline{\mu}(f)|, \ \forall f \in F \tag{7}$$

because

$$\begin{aligned}|\mu(f)| = |\mu'(f)| &= |\mathcal{C}_f(\overline{\mu}(f) \cup \mu'(f))| \geq |\mathcal{C}_f(\overline{\mu}(f))| = |\overline{\mu}(f)| \\ &= |\mathcal{C}_f(\overline{\mu}(f) \cup \mu(f))| \geq |\mathcal{C}_f(\mu(f))| = |\mu(f)|,\end{aligned}$$

where the first equality is due to the equal-quota property, the second and the fourth equalities are by Lemma 27 and (6) respectively, the remaining two equalities are due to the fact that $\overline{\mu}$ and μ are individually rational, and the two inequalities hold because of cardinal monotonicity.

We next show that the break-marriage procedure is successful. Consider the following two cases for a worker $w \neq w'$. The first is when $|\mu'(w)| < q_w$. By the full-quota property, w has the same set of partners in all stable matchings. In particular, $\mu'(w) = \mu(w)$. We claim that only firms from $\mu(w)$ propose to w during the break-marriage procedure. Assume by contradiction that a firm $f \notin \mu(w)$ proposes to w at step s (i.e., $w \in \mathcal{C}_f(X_f^{(s)})$). Then, since $\overline{\mu}(f) = \mathcal{C}_f(X_f^{(s^\star)}) \subseteq X_f^{(s^\star)} \subseteq X_f^{(s)}$ due to (2) and (3), by substitutability, we have $w \in \mathcal{C}_f(\overline{\mu}(f) \cup \{w\})$ and thus, $w \in C_f(\mu(f) \cup \{w\})$ because of (6) and Lemma 3. Since $|\mu(w)| < q_w$, we also have that $f \in C_w(\mu(w) \cup \{f\})$ by the quota-filling property of $\mathcal{C}_w$. However, this contradicts stability of μ. Therefore, $Y_w^{(s)} = X_w^{(s)} = \mu'(w)$ for all $s \in \{0, 1, \cdots, s^\star\}$ by Lemma 23 and Lemma 28. Hence, $\overline{\mu}(w) = \mu'(w)$ by (4).

We next consider the second case for worker $w \neq w'$, which is when $|\mu'(w)| = q_w$, and we claim that $|Y_w^{(s)}| = q_w$ for all $s \in \{0\} \cup [s^\star]$. We will show this by induction. For the base case with $s = 0$, we want to show that $X_w^{(0)} \supseteq \mu'(w)$ because then we have $|X_w^{(0)}| \geq q_w$ and thus $|Y_w^{(0)}| = q_w$ by quota-filling. Let $f \in \mu'(w)$. If $f \neq f'$, then by Lemma 23, we have $w \in \mathcal{C}_f(X_f^{(0)})$; and if $f = f'$, by substitutability of $\mathcal{C}_{f'}$, we also have $w \in \mathcal{C}_f(X_f^{(0)})$ since $w \neq w'$. Hence, $f \in X_w^{(0)}$ by definition of $X_w^{(0)}$. For the inductive step, assume that $|Y_w^{(s-1)}| = q_w$ and we want to show that $|Y_w^{(s)}| = q_w$. Because of Lemma 28, $X_w^{(s)} \supseteq Y_w^{(s-1)}$. Hence, similar to the base case, we have $|X_w^{(s)}| \geq q_w$ and subsequently $|Y_w^{(s)}| = q_w$ by quota-filling. Therefore, $|\overline{\mu}(w)| = |\mu'(w)|$ by (4).

Combining both cases, we have $|\overline{\mu}(w)| = |\mu'(w)|$ for every worker $w \neq w'$. Together with (7), we have:

$$\begin{aligned}
&\sum_{w \in W \setminus \{w'\}} |\overline{\mu}(w)| + |\overline{\mu}(w')| \\
&= \sum_{w \in W} |\overline{\mu}(w)| = \sum_{f \in F} |\overline{\mu}(f)| = \sum_{f \in F} |\mu(f)| \\
&= \sum_{w \in W} |\mu(w)| = \sum_{w \in W \setminus \{w'\}} |\mu(w)| + |\mu(w')|.
\end{aligned}$$

Hence, we must also have $|\overline{\mu}(w')| = |\mu'(w')| = q_{w'}$. Therefore, $f' \notin \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\})$ because otherwise $|\overline{\mu}(w')| = |\mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\}) \setminus \{f'\}| \leq q_{w'} - 1$, where the inequality is by quota-filling. Hence, the break-marriage procedure is successful.

Finally, by Lemma 30, we have $\overline{\mu} \in \mathcal{S}$ and $\mu' \succ \overline{\mu}$. We also have $\overline{\mu} \succeq \mu$ by (6). Therefore, it must be that $\overline{\mu} = \mu$ since μ is an immediate descendant of μ' in $\mathcal{S}$. □

We are now ready to present the algorithm that finds an immediate descendant for any given stable matching, using the break-marriage procedure. The details of the algorithm are presented in Algorithm 3.

Algorithm 3 Immediate descendant of stable matching $\mu' \neq \mu_W$

Input: μ', μ_W

1: initialize $\mathcal{T} \leftarrow \emptyset$
2: **for each** $(f', w') \in \mu' \setminus \mu_W$ **do**
3: run the `break-marriage(`μ', f', w'`)` procedure
4: **if** the procedure is successful **then** add the output matching $\overline{\mu}$ to $\mathcal{T}$
5: **end for**
6: let μ^* be a matching in $\mathcal{T}$
7: **for each** $\mu \in \mathcal{T} \setminus \{\mu^*\}$ **do**
8: **if** $\mu \succeq \mu^*$ **then** update $\mu^* \leftarrow \mu$
9: **end for** ▷ μ^* is a maximal matching from $\mathcal{T}$

Output: μ^*

Theorem 13 *The output μ^* of Algorithm 3 is an immediate descendant of μ' in the stable matching lattice $(\mathcal{S}, \succeq)$.*

Proof First note that due to Lemma 30, all matchings in the set $\mathcal{T}$ constructed by Algorithm 3 are stable matchings and $\mu' \succeq \mu$ for all $\mu \in \mathcal{T}$. Moreover, we claim that $\mathcal{T} \neq \emptyset$. Let $\mu_1 \in \mathcal{S}$ such that μ' is an immediate predecessor of μ_1 in $(\mathcal{S}, \succeq^\star)$. Such a stable matching μ_1 exists because $\mu' \neq \mu_W$. Because of Lemma 16, we have $\mu' \setminus \mu_1 \subseteq \mu' \setminus \mu_W$ and thus by Theorem 12, we have $\mu_1 \in \mathcal{T}$. Hence, $\mathcal{T} \neq \emptyset$ as desired. Now, to prove the theorem, assume by contradiction that the output matching μ^* is not an immediate descendant of μ' in $(\mathcal{S}, \succeq)$. Then, there exists a stable matching

μ such that $\mu' \succ \mu \succ \mu^*$. By Lemma 16, for every firm-worker pair $(f', w') \in \mu' \setminus \mu$, we also have $(f', w') \notin \mu_W$. Thus, $\mu \in \mathcal{T}$ due to Theorem 12. However, this means that μ^* is not a maximal matching from $\mathcal{T}$, which is a contradiction. □

Finally, putting everything together, Algorithm 4 finds a maximal chain of the stable matching lattice, as well as the set of rotations. Its correctness follows from Theorems 13, 11, and 7.

Algorithm 4 A maximal chain of $(\mathcal{S}, \succeq)$ and the set of rotations Π

Input: μ_F and μ_W
1: initialize counter $k \leftarrow 0$ and $C_k \leftarrow \mu_F$
2: **while** $C_k \neq \mu_W$ **do**
3: run Algorithm 3 with input C_k and μ_W, and let μ^* be its output
4: update counter $k \leftarrow k + 1$ and $C_k \leftarrow \mu^*$
5: **end while**
Output: maximal chain $C_0, C_1, \cdots, C_k$; and $\Pi = \{\rho_i{:=}\rho(C_{i-1}, C_i) : i \in [k]\}$.

4.3 Finding irreducible elements via maximal chains

The goal of this section is to prove the following. Note that the result below holds for any ring of sets.

Theorem 14 *Consider a ring of sets $(\mathcal{H}, \subseteq)$ with base set B. Let $C_0, C_1, \cdots, C_k$ be a maximal chain of $(\mathcal{H}, \subseteq)$ and let $K_i{:=}C_i \setminus C_{i-1}$ for all $i \in [k]$. For $H \subseteq B$, let* `ros-membership` *denote the running time of an algorithm that decides if $H \in \mathcal{H}$. There exists an algorithm with running time $O(k^2$*`ros-membership`*$)$ that takes $C_0, C_1, \cdots, C_k$ as input and outputs, for each minimal difference K_i, a set of indices $\Lambda(K_i)$ such that $I(K_i) = \bigcup\{K_j : j \in \Lambda(K_i)\} \cup C_0$. In particular, this algorithm can be used to obtain the partial order $\sqsupseteq$ over $\mathcal{D}(\mathcal{H})$.*

The theorem below gives an alternative definition of the partial order $\sqsupseteq$.

Theorem 15 (Theorem 2.4.4, [23]) *Let $K_1, K_2 \in \mathcal{D}(\mathcal{H})$. Then, $K_1 \sqsupseteq K_2$ if and only if K_1 appears before K_2 on every maximal chain in $(\mathcal{H}, \subseteq)$.*

We now present the algorithm stated in Theorem 14 in Algorithm 5. The idea is as follows. In order to find $I(K_i)$ (i.e., the minimal element in $\mathcal{H}$ that contains K_i), the algorithm tries to remove from the set C_i as many items as possible, while keeping $C_i \in \mathcal{H}$. That is, the algorithm removes from C_i all minimal differences $K \in \{K_1, K_2, \cdots, K_i\}$ such that $K \not\sqsupseteq K_i$. As we show in the proof of Theorem 14, the resulting set is $I(K_i)$.

We now give the proof of Theorem 14.

Proof of Theorem 14 It is clear that the running time of Algorithm 5 is $O(k^2$`ros-membership`$)$. Suppose first the output of Algorithm 5 is correct, that

Algorithm 5

Input: A maximal chain $C_0, C_1, \cdots, C_k$ of $(\mathcal{H}, \subseteq)$.
1: **for** $i = 1, 2, \cdots, k$ **do**
2: define $K_i \leftarrow C_i \setminus C_{i-1}$
3: initialize $H \leftarrow C_i$ and $\Lambda(K_i) \leftarrow \{1, 2, \cdots, i\}$
4: **for** $j = i-1, i-2, \cdots, 1$ **do**
5: **if** $H \setminus K_j \in \mathcal{H}$ **then**
6: update $H \leftarrow H \setminus K_j$ and $\Lambda(K_i) \leftarrow \Lambda(K_i) \setminus \{j\}$
7: **end if**
8: **end for**
9: **end for**
Output: $\Lambda(K_i)$ for all $i \in [k]$

is, $I(K_i) = \bigcup\{K_j : j \in \Lambda(K_i)\} \cup C_0$. Then, for two minimal differences $K_{i_1}, K_{i_2} \in \mathcal{D}(\mathcal{H})$, $K_{i_1} \sqsupseteq K_{i_2}$ if and only if $\Lambda(K_{i_1}) \subseteq \Lambda(K_{i_2})$ by definition of $\sqsupseteq$. Hence, the partial order $\sqsupseteq$ can be obtained in time $O(k^2)$ from the output of Algorithm 5. It remains to show the correctness of Algorithm 5. Fix a value of $i \in [k]$ and for the following, consider the i^{th} iteration of the outer **for** loop of the algorithm. Let $\{j_1, j_2, \cdots, j_M\}$ be an enumeration of $\Lambda(K_i)$ at the end of the iteration such that $j_1 < j_2 < \cdots < j_M$. Note that $j_M = i$. We start by showing the following claim.

Claim 4 For all $m \in [M-1]$, $K_{j_m} \sqsupseteq K_i$.

Proof We prove this by induction on m, where the base case is $m = M-1$. We start with the base case. Note that j_m is the first index for which the **if** statement at Line 5 is evaluated to be false. That is, $(\bigcup_{\ell=1}^{j_m} K_\ell) \cup K_i \cup C_0 \in \mathcal{H}$ but $(\bigcup_{\ell=1}^{j_m-1} K_\ell) \cup K_i \cup C_0 \notin \mathcal{H}$. By Lemma 11, $\{K_1, K_2, \cdots, K_{j_m}, K_i\}$ is an upper set of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$, and by Theorem 5, $\{K_1, K_2, \cdots, K_{j_m-1}, K_i\}$ is not an upper set of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$. Since for all $j' < j_m$, $K_{j_m} \not\sqsupseteq K_{j'}$ because of Theorem 15, the reason why $\{K_1, K_2, \cdots, K_{j_m-1}, K_i\}$ is not an upper set of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$ must be that $K_{j_m} \sqsupseteq K_i$. For the inductive step, assume the claim is true for all $m' > m$ and we want to show that $K_{j_m} \sqsupseteq K_i$. Note that again by Theorem 5, $\{K_1, K_2, \cdots, K_{j_m}, K_{j_{m+1}}, K_{j_{m+2}}, \cdots, K_i\}$ is an upper set of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$ but $\{K_1, K_2, \cdots, K_{j_m-1}, K_{j_{m+1}}, K_{j_{m+2}}, \cdots, K_i\}$ is not an upper set of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$. With the same argument as in the base case, since for all $j' < j_m$, $K_{j_m} \not\sqsupseteq K_{j'}$ by Theorem 15, it must be that $K_{j_m} \sqsupseteq K_{j_{m'}}$ for some $m' > m$. Therefore, applying the inductive hypothesis, we have $K_{j_m} \sqsupseteq K_i$ as desired. □

Let H^* be set H at the end of i^{th} iteration of the outer **for** loop. Note that $H^* = \bigcup\{K_j : j \in \Lambda(K_i)\} \cup C_0$ by construction. Since $K_i \subseteq H^*$, we have $I(K_i) \subseteq C_i \subseteq H^*$ by definition. Also note that by definition, $I(K_i) \in \mathcal{H}$. Assume by contradiction that $H^* \neq I(K_i)$ (i.e., $H^* \nsubseteq I(K_i)$). Consider a complete chain from the minimal element H_0 of $(\mathcal{H}, \subseteq)$ to $I(K_i)$ in $(\mathcal{H}, \subseteq)$, whose existence is guaranteed by Theorem 11. Then, at least one minimal difference from $\{K_j : j \in \Lambda(K_i) \setminus \{i\}\}$, call it K', is not contained in this complete chain. However, this means $K' \not\sqsupseteq K_i$ due to Theorem 11, which contradicts Claim 4. Therefore, we must have $I(K_i) = H^*$. □

4.4 Partial order $\succeq^\star$ over Π

In this section, we show how to obtain the partial order $\succeq^\star$ over the rotation poset Π. Recall that as stated in Theorem 14 of the previous section, there exists an algorithm that finds the partial order $\sqsupseteq$ over $\mathcal{D}:=\mathcal{D}(\mathcal{P})$ when given as input a maximal chain of $\mathcal{P}$. Employing the isomorphism between $\mathcal{S}$ and $\mathcal{P}$ shown in Theorem 4 and that between $\mathcal{D}$ and Π shown in Theorem 7, we adapt the algorithm so that from a maximal chain of $\mathcal{S}$, we obtain the partial order $\succeq^\star$ over Π.

Algorithm 6

Input: outputs of Algorithm 4 – maximal chain $C_0, \cdots, C_k$ of $(\mathcal{S}, \succeq)$ and the set of rotations $\Pi = \{\rho_i := \rho(C_{i-1}, C_i) : i \in [k]\}$

for $i = 1, 2, \cdots, k$ **do**
    initialize $\mu \leftarrow C_i$ and $\Lambda(\rho_i) \leftarrow \{1, 2, \cdots, i\}$
    for $j = i-1, i-2, \cdots, 1$ **do**
        if $\mu \triangle \rho_j^- \triangle \rho_j^+ \in \mathcal{S}$ **then**
            update $\mu \leftarrow \mu \triangle \rho_j^- \triangle \rho_j^+$ and $\Lambda(\rho_i) \leftarrow \Lambda(\rho_i) \setminus \{j\}$
        end if
    end for
end for

Output: $\Lambda(\rho_i)$ for all $i \in [k]$

Theorem 16 *Let $\Lambda(\rho)$ and $\Lambda(\rho')$ be the outputs of Algorithm 6 for rotations $\rho, \rho' \in \Pi$, respectively. Then, $\rho \succeq^\star \rho'$ if and only if $\Lambda(\rho) \subseteq \Lambda(\rho')$.*

Proof To distinguish between the inputs of Algorithm 6 and Algorithm 5, we let $\mu_0, \mu_1, \cdots, \mu_k$ denote the maximal chain in the input of Algorithm 6. Consider the outputs of Algorithm 5 with inputs $C_i = P(\mu_i)$ for all $i \in [k] \cup \{0\}$. Then, because of the isomorphism between $(\mathcal{S}, \succeq)$ and $(\mathcal{P}, \subseteq)$ and the isomorphism between $(\Pi, \succeq^\star)$ and $(\mathcal{D}, \sqsupseteq)$ stated respectively in Theorem 4 and Theorem 7, $K_i = Q(\rho_i)$ and $\Lambda(\rho_i) = \Lambda(K_i)$ for all $i \in [k]$, where $K_i = C_i \setminus C_{i-1}$ as defined in Algorithm 5. Thus, together with Theorem 14,

$$\rho \succeq^\star \rho' \Leftrightarrow Q(\rho) \sqsupseteq Q(\rho') \Leftrightarrow \Lambda(Q(\rho)) \subseteq \Lambda(Q(\rho')) \Leftrightarrow \Lambda(\rho) \subseteq \Lambda(\rho'),$$

concluding the proof. □

4.5 Summary and time complexity analysis

The complete procedure to build the rotation poset is summarized in Algorithm 7.

The time complexity analysis of Algorithm 7 now follows easily from the results deduced above and is deferred to Appendix A.

Theorem 17 *Algorithm 7 runs in time $|W|^3|F|^3$`oracle-call`.*

Algorithm 7 Construction of the rotation poset $(\Pi, \succeq^\star)$

1: Run Algorithm 1's, firm-proposing and worker-proposing, to obtain μ_F and μ_W.
2: Run Algorithm 4 to obtain a maximal chain $C_0, C_1, \cdots, C_k$ of the stable matching lattice $(\mathcal{S}, \succeq)$, and the set of rotations $\Pi \equiv \{\rho_1, \rho_2, \cdots, \rho_k\}$.
3: Run Algorithm 6 to obtain the sets $\Lambda(\rho_i)$ for each rotation $\rho_i \in \Pi$.
4: Define the partial order relation $\succeq^\star$: for $\rho_i, \rho_j \in \Pi$, $\rho_i \succeq^\star \rho_j \Leftrightarrow \Lambda(\rho_i) \subseteq \Lambda(\rho_j)$.

5 The convex hull of lattice elements

Consider a poset $(Y, \succeq^\star)$. Its associated *order polytope* is defined as

$$\mathcal{O}(Y, \succeq^\star) := \{y \in [0, 1]^Y : y_i \geq y_j, \ \forall i, j \in Y \text{ s.t. } i \succeq^\star j\}.$$

A characterization of vertices and facets of $\mathcal{O}(X, \succeq^\star)$ is given in [42].

Theorem 18 ([42]) *The vertices of $\mathcal{O}(Y, \succeq^\star)$ are the characteristic vectors of upper sets of Y. The facets of $\mathcal{O}(Y, \succeq^\star)$ are all and only the following: $y_i \geq 0$ if i is a minimal element of the poset; $y_i \leq 1$ if i is a maximal element of the poset; $y_i \geq y_j$ if i is an immediate predecessor of j.*

Proof of Theorem 2 Let $(Y, \succeq^\star)$ affinely represent $(X, \succeq)$ via functions ψ and $g(u) = Au + x^0$. We claim that

$$\begin{aligned} \text{conv}(\mathcal{X}) := \text{conv}(\{\chi^\mu : \mu \in X\}) &= \{x^0\} \oplus A \cdot \mathcal{O}(Y, \succeq^\star) \\ &= \{x \in \mathbb{R}^X : x = x^0 + Ay, y \in \mathcal{O}(Y, \succeq^\star)\}, \end{aligned} \tag{8}$$

where $\oplus$ denotes the Minkowski sum operator. Indeed, by definition of affine representation and the fact that both polytopes, $\text{conv}(\mathcal{X})$ and $\mathcal{O}(Y, \succeq^\star)$, have $0/1$ vertices, g defines a bijection between vertices of these two polytopes. Convexity then implies (8). As $\mathcal{O}(Y, \succeq^\star)$ has $O(|Y|^2)$ facets shown in Theorem 18, we conclude the first statement from Theorem 2.

Now suppose that A has full column rank. This implies that $\text{conv}(\mathcal{X})$ is affinely isomorphic to $\mathcal{O}(Y, \succeq^\star)$. Hence, there is a one-to-one correspondence between facets of $\mathcal{O}(Y, \succeq^\star)$ and facets of $\text{conv}(\mathcal{X})$, concluding the proof. □

Notice that, when A as in the statement of Theorem 2 does not have full column rank, the extended formulation given in (8) may not give any information on the number of facets of $\text{conv}(\mathcal{X})$. For instance, the vertices of any polytope can be ordered as to form a chain. In this case, Theorem 2 simply implies the well-known fact that $\text{conv}(\mathcal{X})$ can be written as the projection of a simplex with as many vertices as that of $\text{conv}(\mathcal{X})$.

6 Representations of choice functions and algorithms

Recall that a choice function is defined on all the (exponentially many) subsets of agents from the opposite side. The oracle model bypasses the computational concerns of representing choice functions explicitly. However, one drawback of this model is that it requires multiple rounds of communication between the "central planner" and each agent in the market. This, from an application point of view, is time-consuming: one of the major improvements brought about by the implementations of the Deferred Acceptance algorithm when applied, e.g., to the New York City school system, lies in the fact that it does not require multiple rounds of communication between the agents and the central planner [1].

This observation leads to the following practically relevant and theoretically intriguing questions: is there a way to represent choice functions "compactly", and do our algorithms perform efficiently in such a model? A natural starting point is the MC-representation [3] which we introduce next. We show that, in the model where choice functions are given through their MC-representations, the time complexity of our algorithms is polynomial in the input size (where now the input includes the MC-representations). However, the number of preference relations required by the MC-representation of a choice function may be exponential in the number of agents (see Remark 4).

It is therefore interesting to investigate whether there are other ways to represent choice functions that is of size polynomial in the number of agents. Via a counting argument, we give a negative answer to this question for choice functions that are substitutable, consistent, and cardinal monotone (see Theorem 21 and Remark 5). We remark that our argument leaves it open whether a similar result holds if we replace cardinal monotonicity with quota-filling.

6.1 MC-representation for path-independent choice functions

We now introduce an alternative, equivalent description of path-independent choice functions. Aizerman and Malishevski [3] showed that a choice function $\mathcal{C}_a$ is path-independent if and only if there exists a finite sequence of $p(\mathcal{C}_a) \in \mathbb{N}$ preference relations over acceptable partners, denoted as $\{\geq_{a,i}\}_{i\in[p(\mathcal{C}_a)]}$ indexed by i, such that for every subset of acceptable partners S, $\mathcal{C}_a(S) = \cup_{i\in[p(\mathcal{C}_a)]}\{x^*_{a,i}\}$, where $x^*_{a,i} = \max(S, \geq_{a,i})$ is the maximum element[6] of S according to $\geq_{a,i}$. We call this sequence of preference relations the *Maximizer-Collecting representation* (MC-representation) of choice function $\mathcal{C}_a$. Note that for distinct $i_1, i_2 \in [p(\mathcal{C}_a)]$, it is possible to have $x^*_{a,i_1} = x^*_{a,i_2}$.

Conceptually, one can view the MC-representation as follows: a firm is a collection of *positions*, each of which has its own preference relation; a worker is a collection of *personas*, each of whom also has his or her own preference relation. Each firm hires the best candidate for each position, and the same candidate can be hired for two positions if (s)he is the best for both. A symmetric statement holds for workers and personas.

[6] If $S = \emptyset$, then $\max(S, \geq_{a,i})$ is defined to be $\emptyset$.

6.2 Algorithms with MC-representation

We now show how to modify the algorithms and analyze their time complexities when agents' choice functions are explicitly given via the MC-representations.

In Algorithm 1 and Algorithm 2, instead of relying on an oracle model, we need to compute the outcomes of choice functions $\mathcal{C}_a(S)$ for agent $a \in F \cup W$ and subset of acceptable partners S. Recall that $\mathcal{C}_a(S)$ can be obtained as a set of maximizers: $\{\max(S, \geq_{a,i}) : i \in [p(\mathcal{C}_a)]\}$. Since each $\max(S, \geq_{a,i})$ requires $O(\max(|F|, |W|)$ time to compute, the time-complexity for obtaining $\mathcal{C}_a(S)$ is $O(\max(|F|, |W|)p(\mathcal{C}_a))$. Thus, for all previous results in terms of time complexity, one can simply replace $O(\texttt{oracle-calls})$ with $O(\max(|F|, |W|) \max_{a\in F\cup W} p(\mathcal{C}_a))$. Note that this time complexity bound is polynomial in the input size, but could be exponential in the number of agents, since $\max_{a\in F\cup W} p(\mathcal{C}_a)$ maybe exponential in the number of the agents as discussed in Remark 4.

Remark 4 Doğan et al. [16] constructed strict preference lists with quotas (i.e., choice functions for the MM- MODEL) whose MC-representation needs exponentially many preference relations. Since such choice functions are a special case of the quota-filling choice functions, in general the MC-representation of quota-filling choice functions is not polynomial in the number of agents.

6.3 On the number of substitutable, consistent, and cardinal monotone choice functions

In the following, the domain of all choice functions is the family of subsets of X, with $|X| = n$. The simplest choice functions $\mathcal{C}$ appears in the SM- MODEL, where there is a single underlying strict preference list. The number of such choice functions is

$$\sum_{i=0}^{n} \binom{n}{i} i! = \sum_{i=0}^{n} \frac{n!}{(n-i)!} = \sum_{i=0}^{n} \frac{n!}{(n-i)!} = n! \sum_{i=0}^{n} \frac{1}{i!} \leq en!,$$

hence, singly exponential in n. On the other extreme, the number of all choice functions is doubly-exponential in n (see, e.g., [17]).

Theorem 19 *The number of choice functions on subsets of X with $|X| = n$ is $2^{n2^{n-1}}$.*

It has also been shown by Echenique [17] that when choice functions are assumed to be substitutable and consistent (i.e., path-independent), the number of choice functions remains doubly exponential in n.

Theorem 20 ([17]) *The number of substitutable and consistent choice functions on subsets of X with $|X| = n$ is $2^{\Omega\left(\frac{2^{n-1}}{\sqrt{n-1}}\right)}$.*

In the rest of the section, we show the following. The proof idea follows from [17].

Theorem 21 *The number of substitutable, consistent, and cardinal monotone choice functions on subsets of X with $|X| = n$ is $2^{\Omega\left(\frac{2^{n-1}}{\sqrt{n-1}}\right)}$.*

Remark 5 Because of Theorem 21, in order to encode all substitutable, consistent, and cardinal monotone choice function in binary strings, we need a number of strings that is super-polynomial in n, i.e., the number of agents in the market.

A family of subsets $\mathcal{A} \subseteq 2^X$ is an *antichain* of $(2^X, \subseteq)$ if for any subsets $A, B \in \mathcal{A}$, they are not comparable, i.e., $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. A family of subsets $\mathcal{F} \subseteq 2^X$ is a *filter* (i.e., *lower set*) if for all $F \in \mathcal{F}$, $F' \supseteq F$ implies $F' \in \mathcal{F}$. Moreover, we say filter $\mathcal{F}$ is a *filter at* x if for all $F \in \mathcal{F}$, we have $x \in F$. Note that $\emptyset$ is a filter at x.

Theorem 22 ([17]) *There is an injective function mapping collections of antichains* $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_x : x \in X\}$ *where each* $\mathcal{A}_x$ *is an antichain of the poset* $(2^{X\setminus\{x\}}, \subseteq)$ *to substitutable choice functions. The image of* $\boldsymbol{\mathcal{A}}$ *is defined as follows: for all* $S \subseteq X$,

$$\mathcal{C}(S){:=}\{x \in S : S \notin \mathcal{T}_x\},$$

where

$$\mathcal{T}_x{:=}\{B \subseteq X : A \cup \{x\} \subseteq B \textit{ for some } A \in \mathcal{A}_x\}.$$

Moreover, $\mathcal{T}_x$ *is a filter at* x *for all* $x \in X$.

Let $\mathcal{C}[\boldsymbol{\mathcal{A}}]$ denote the substitutable choice function corresponding to the collection of antichains $\boldsymbol{\mathcal{A}}$ constructed as in the statement of Theorem 22.

Lemma 31 *Let* (Y, W) *be a partition of* X *with* $W = \{w\}$. *Let* $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_x : x \in X\}$ *be a collection of antichains such that (i) for all* $x \in Y$, $\mathcal{A}_x = \emptyset$ *and (ii)* $\mathcal{A}_w$ *is an antichain of* $(2^Y, \subseteq)$. *Then* $\mathcal{C}[\boldsymbol{\mathcal{A}}]$ *is consistent and cardinal monotone.*

Proof We abbreviate $\mathcal{C} := \mathcal{C}[\boldsymbol{\mathcal{A}}]$. Let $\mathcal{T}_x$ be as defined in the statement of Theorem 22. That is, $\mathcal{T}_x = \emptyset$ for all $x \in Y$ and $\mathcal{T}_w$ is a filter at w. Hence, note that $S \cap Y \subseteq \mathcal{C}(S)$ for all $S \subseteq X$ ($\sharp$).

Let $T \subseteq X$. We consider first the case when $w \notin T$. Then, $\mathcal{C}(T) = T$ because of ($\sharp$). Let $S \subseteq X$ be such that $\mathcal{C}(T) \subseteq S \subseteq T$. Then it must be that $S = T$ and it follows immediately that $\mathcal{C}(T) = \mathcal{C}(S)$. In addition, for all $S \subseteq T$, we also have $S \subseteq Y$ and thus, using ($\sharp$) again, $|\mathcal{C}(S)| = |S| \leq |T| = |\mathcal{C}(T)|$.

We next consider the case when $w \in T$. Then, either $\mathcal{C}(T) = T$ or $\mathcal{C}(T) = T \setminus \{w\}$, again because of ($\sharp$). We start with the consistency property. Assume we are in the former case, and let $S \subseteq X$ be such that $\mathcal{C}(T) \subseteq S \subseteq T$. Since $T = \mathcal{C}(T)$, we have $S = T$ and thus $\mathcal{C}(T) = \mathcal{C}(S)$. Now assume we are in the latter case: $\mathcal{C}(T) = T \setminus \{w\}$. If $S \subseteq X$ satisfies $\mathcal{C}(T) \subseteq S \subseteq T$, we either have $S = T$ or $S = T \setminus \{w\}$. Regardless, we have $\mathcal{C}(S) = \mathcal{C}(T)$. Lastly, we show the cardinality monotonicity property, and we consider both cases at once. For all $S \subsetneq T$, we either have $\mathcal{C}(S) = S$ or $\mathcal{C}(S) = S \setminus \{w\}$ due to ($\sharp$). Either way, $|\mathcal{C}(S)| \leq |S| \leq |T| - 1 \leq |\mathcal{C}(T)|$. Hence, $\mathcal{C}$ is both consistent and cardinal monotone, concluding the proof. □

Thus, a lower bound to the number of substitutable, consistent, and cardinal monotone choice functions can be obtained by counting the number of antichains. The problem of counting the number of antichains of a poset is called the *Dedekind's* *problem*. Let $\mathcal{N}(k)$ denote the collection of antichains of poset $(2^{[k]}, \subseteq)$. The following result is well-known and we include the proof for completeness.

Lemma 32 $|\mathcal{N}(k)| \geq 2^{\binom{k}{\lfloor k/2 \rfloor}} = 2^{\Theta(2^k/\sqrt{k})}$.

Proof Consider any two distinct subsets $A, B \subseteq X$ with $|A| = |B|$, then it must be that $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Thus, a collection of subsets, each with the same size, is an antichain of $(2^{[k]}, \subseteq)$. Therefore, the number of antichains of $(2^{[k]}, \subseteq)$ is at least the number of subsets of $\{A \subseteq X : |A| = \lfloor k/2 \rfloor\}$, which is exactly $2^{\binom{k}{\lfloor k/2 \rfloor}}$ since there are $\binom{k}{\lfloor k/2 \rfloor}$ subsets of X with size $\lfloor k/2 \rfloor$. The last equality follows from Stirling's approximation. □

We now present the proof for Theorem 21.

Proof of Theorem 21 Let (Y, W) be a partition of X with $|Y| = n - 1$ and $|W| = 1$, as in the statement of Lemma 31. By Lemma 32, the possible choices of antichains $\mathcal{A}_x$ for $x \in W$ is at least $\mathcal{N}(n-1)$. Hence, the number of $\boldsymbol{\mathcal{A}}$ (i.e., collection of antichains) in the statement of Lemma 31 is also at least $\mathcal{N}(n-1)$. Finally, together with Theorem 22, we have that the number of substitutable, consistent, and cardinal monotone choice functions is again at least $\mathcal{N}(n-1)$. □

7 Concluding remarks

Our results show that approaching stable matching problems by regarding their feasible regions as a distributive lattice leads to efficient optimization algorithms and a polyhedral description of the associated convex sets. Our study leaves some questions open and it poses research directions which we think are worth exploring.

First, it is not clear if algorithms from Sect. 4 extend to the CM- Model—or even beyond—and if conversely the lower bound from Sect. 6 extends to choice functions that are quota-filling. Second, there has been some recent work showing how feasible regions of certain problems in combinatorial optimization can be seen as a distributive lattice [22]. This fact, combined with our approach, may lead to (known or new) efficient algorithms for optimizing linear functions over the associated polytopes.

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A Proof of Theorem 17

DA algorithm (Algorithm 1)

Because of Lemma 28 and Lemma 29, Algorithm 1 can be implemented as in Algorithm 8 to reduce the number of `oracle-calls`. In particular, during each **repeat** loop, only firms that are rejected in the previous step (i.e., in $\overline{F}$) and only workers who receive new proposals (i.e., in $\overline{W}$) need to invoke their choice functions. Therefore, the **for** loop at Line 5 is entered at most $|F||W|$ times, and similarly, the **for** loop at Line 13 is entered at most $|F||W|$ times. That is, the total number of `oracle-calls` is $O(|F||W|)$. Moreover, and for each firm-worker pair (f, w), w is removed from X_f at most once and f is added to X_w at most once. That is, Line 8 (resp. Line 16) is repeated at most $|F||W|$ times. Therefore, the running time of the DA algorithm is $O(|F||W|$`oracle-call`$)$.

Algorithm 8 Efficient implementation of Algorithm 1

1: set $\overline{F} \leftarrow F$ and $\overline{W} \leftarrow \emptyset$
2: **for each** firm f **do** initialize $X_f \leftarrow W(f)$ and $Y_f^{\text{prev}} \leftarrow \emptyset$ **end for**
3: **for each** worker w **do** initialize $X_w \leftarrow \emptyset$ and $Y_w^{\text{prev}} \leftarrow \emptyset$ **end for**
4: **repeat**
5:     **for each** firm $f \in \overline{F}$ **do**
6:         $A_f \leftarrow \mathcal{C}_f(X_f)$
7:         **for each** worker $w \in A_f \setminus Y_f^{\text{prev}}$ **do**
8:             update $X_w \leftarrow X_w \cup \{f\}$ and $\overline{W} \leftarrow \overline{W} \cup \{w\}$
9:         **end for**
10:         update $Y_f^{\text{prev}} \leftarrow A_f$
11:     **end for**
12:     re-set $\overline{F} \leftarrow \emptyset$
13:     **for each** worker $w \in \overline{W}$ **do**
14:         $X_w \leftarrow \mathcal{C}(X_w)$
15:         **for each** firm $f \in Y_w^{\text{prev}} \setminus X_w$ **do**
16:             update $X_f \leftarrow X_f \setminus \{w\}$ and $\overline{F} \leftarrow \overline{F} \cup \{f\}$
17:         **end for**
18:         update $Y_w^{\text{prev}} \leftarrow X_w$
19:     **end for**
20:     re-set $\overline{W} \leftarrow \emptyset$
21: **until** $\overline{F} = \emptyset$
Output: matching $\overline{\mu}$ with $\overline{\mu}(w) = Y_w^{\text{prev}}$ for every worker w; closure $\widetilde{X}(\overline{\mu})$ with $\widetilde{X}_f(\overline{\mu}) = X_f$ for every firm f

Break-marriage procedure (Algorithm 2)

Since the core steps (i.e., the loops) of the break-marriage procedure is the same as that of the DA algorithm, the running time of the break-marriage procedure is $O(|F||W|$`oracle-call`$)$, with the same arguments as above.

Immediate descendant (Algorithm 3)

Recall that $\overline{q}_f$ denotes the number of workers matched to firm f under any stable matching (see the equal-quota property). Let $\Upsilon := \sum_{f \in F} \overline{q}_f$ denote the number of worker-firm pairs in any stable matching. Then, Algorithm 2 is run for at most Υ times. In addition, finding one maximal element μ^* from $\mathcal{T}$ requires at most Υ comparisons of pairs of stable matchings, each of which requires $|F|$ `oracle-calls` by Part (iii) of Lemma 25. All together, since $\Upsilon \leq |F||W|$, the running time of Algorithm 3 is $O(|F|^2|W|^2 \texttt{oracle-call})$.

Maximal chain (Algorithm 4)

Since the length of a maximal chain of $\mathcal{P}$, and equivalently of $\mathcal{S}$ due to Theorem 4, is at most the size of its base set due to Lemma 10 and Theorem 11, Algorithm 3 is repeated for at most $|F||W|$ times. Thus, the running time of Algorithm 4 is $O(|F|^3|W|^3 \texttt{oracle-call})$.

Partial order $\succeq^\star$ (Algorithm 6)

Recall that checking if a matching is stable requires $O(|F||W|)$ `oracle-calls` by Part (ii) of Lemma 25. Thus, `ros-membership` is $O(|F||W| \texttt{oracle-call})$. Since k is at most $|F||W|$ as argued above, the running time of Algorithm 6 is $O(|F|^3|W|^3 \texttt{oracle-call})$.

Rotation poset $(\Pi, \succeq^\star)$ (Algorithm 7)

Summing up the time of running Algorithm 1 twice, then Algorithm 4, and lastly Algorithm 6, the time complexity for building the rotation poset is $O(|F|^3|W|^3 \texttt{oracle-call})$.

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