RESEARCH ARTICLE

On a topological Erdős similarity problem

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Abstract
A pattern is called universal in another collection of sets, when every set in the collection contains some linear and translated copy of the original pattern. Paul Erdős proposed a conjecture that no infinite set is universal in the collection of sets with positive measure. This paper explores an analogous problem in the topological setting. Instead of sets with positive measure, we investigate the collection of dense G_δ sets and the collection of generic sets (dense G_δ and complement has Lebesgue measure zero). We refer to such pattern as topologically universal and generically universal, respectively. It is easy to show that any countable set is topologically universal, while any set containing an interior cannot be topologically universal. In this paper, we will show that Cantor sets on $\mathbb{R}^d$ are not topologically universal and Cantor sets with positive Newhouse thickness on $\mathbb{R}^1$ are not generically universal. This gives a positive partial answer to a question by Svetic concerning the Erdős similarity problem on Cantor sets. Moreover, we also obtain a higher dimensional generalization of the generic universality problem.

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1 | INTRODUCTION

A question that frequently arises has the following generic form: Does every 'large' (or unstructured) set possess a 'copy' of a 'small' (or structured) set? For example, Erdős and Turán

conjectured that every $X \subset \mathbb{N}$ of positive density (large, unstructured) contains a copy of the set $\{1, 2, \ldots, n\}$ (small, structured) in the form of an arithmetic progression. The conjecture was famously proven true by Szemerédi [20, 21]. In a similar vein, Steinhaus proved that the difference set of a set of positive measure in $\mathbb{R}$ (large) contains a scaled copy of the interval $(-1, 1)$ (small) [18].

The words 'large', 'small', and 'copy' can take on multiple forms, so we begin by defining some of our terms.

Definition 1.1. Let $E \subset \mathbb{R}^d$ be a set and let $\mathcal{X}$ be a collection of subsets in $\mathbb{R}^d$.

(1) An **affine copy** of E is a copy of the form $t + T(E)$ where $t \in \mathbb{R}^d$ and T is an invertible linear transformation on $\mathbb{R}^d$. A **similar copy** of E is an affine copy such that $T = \lambda O$ where $\lambda > 0$ is a scalar and O is an orthogonal transformation.
(2) We say that E is **universal in** $\mathcal{X}$ if for every $K \in \mathcal{X}$, there exists an affine copy of E, $t + T(E)$, such that $t + T(E) \subset K$.
(3) We say that E is **measure-universal** if E is universal in $\mathcal{X}$, where $\mathcal{X}$ is taken to be the collection of all Lebesgue measurable set with positive Lebesgue measure.

In one dimension, affine copies and similar copies coincide and they are of the form $t + \lambda E$ where $t \in \mathbb{R}$ and $\lambda \neq 0$. Many problems in mathematics can be formulated in terms of universality. Szemerédi's theorem then can be stated as: the set $\{1, 2, \ldots, n\}$ is universal in the collection of sets of positive density in $\mathbb{N}$. The Toeplitz square peg problem asserted that every Jordan curve admits four points on the curve forming a square. Formulated in our notation and interpreting universality in terms of similarity copy, it means that the unit square corners $\{(0,0), (1,0), (0,1), (1,1)\}$ are universal in the collection of all Jordan curves. The problem was recently solved for smooth Jordan curves [11].

Our notion of universality was first introduced by Kolountzakis [13], in which the goal was to study the famous Erdős similarity conjecture.

Conjecture (Erdős): There is no set of infinite cardinality that is measure-universal.

Steinhaus [18] first showed that finite sets are measure-universal. This motivated Paul Erdős to pose the conjecture back in 1974 and he offered \$100 for solving this problem. The conjecture is still open; for a survey of the problem, one can refer to [19]. Let us summarize some progress here. With a simple observation, we can see that the conjecture can be resolved in its full generality if we can show that all positive decreasing sequences whose limit is zero are not measure-universal. Falconer [7] made a substantial progress by showing that slowly decaying sequences $\{x_n : n \in \mathbb{N}\}$ in the sense that

$$\liminf_{n \to \infty} \frac{x_n}{x_{n+1}} = 1$$

are not measure-universal. Bourgain [2] demonstrated that the sum-set $S_1 + S_2 + S_3$ of any three infinite sets S_1, S_2, S_3 cannot be measure-universal. Kolountzakis [13] demonstrated using probabilistic arguments that certain set with large gaps cannot be measure-universal. Currently it is still an open question whether or not exponential decaying sequences such as $\{2^{-n}\}$ are measure-universal. Cruz, the second-named author, and Pramanik recently constructed a Cantor set K such that the set of Erdős points in K, that is,

$$\{x \in K : \forall \delta \neq 0, \ x + \delta\{2^{-n}\} \not\subset K\},$$

has Hausdorff dimension 1. If one could show that the above set could be of positive Lebesgue measure, the Erdős similarity problem will be solved for $\{2^{-n}\}$. Their result also works on sequences which do not reach super-exponential decay [4].

1.1 | Main results

The main purpose of this paper is to study a topological version of the Erdős similarity problem. If we regard a set of positive Lebesgue measure as measure-theoretically large, then a dense G_δ set will be regarded as topologically large. Recall that a G_δ set is a set G that can be written as countable intersection of open sets. If each open set is dense in $\mathbb{R}^d$, then the well-known Baire category theorem shows that G is a dense and uncountable set. There is no relation between sets with positive Lebesgue measure and dense G_δ sets. A fat Cantor set has positive Lebesgue measure, but is nowhere dense. On the other hand, the set of all Liouville's numbers is a dense G_δ but with Lebesgue measure zero (Hausdorff dimension zero indeed).

Definition 1.2.

(1) We say that a set $E \subset \mathbb{R}^d$ is **topologically universal** if E is universal in the collection of all dense G_δ sets in $\mathbb{R}^d$.
(2) We say that a set $E \subset \mathbb{R}^d$ is **generically universal** if E is universal in the collection of all dense G_δ sets G such that $m(\mathbb{R}^d \setminus G) = 0$ in $\mathbb{R}^d$ *(Here m denotes the Lebesgue measure).*

In the first definition, we are interested in what set is universal for topologically large sets. In the second definition, we notice that $m(G) = \infty$, so we are interested in what set is universal for both measure-theoretically and topologically large sets (such sets are sometimes referred to as *generic sets*, which is the reason why we choose this definition).

It is a simple observation from the Baire Category theorem that all countable sets are topologically universal. On the other hand, a set containing an interior point cannot be topologically or generically universal because there are dense G_δ sets with full Lebesgue measures with empty interior. As any affine copy of a set with interior must have interior, a dense G_δ set with empty interior cannot contain any such affine copy. Hence, our focus will be on whether nowhere dense sets are topologically universal. Let us first make precise the meaning of Cantor set in our setting.

Definition 1.3. E is a Cantor set in $\mathbb{R}^d$ if it is a totally disconnected, perfect, and compact subset of $\mathbb{R}^d$.

Because of the existence of G_δ sets that have Hausdorff dimension zero, by monotonicity of Hausdorff measures, all sets of positive Hausdorff dimension cannot be contained inside such a G_δ set, and hence sets of positive Hausdorff dimension are topologically non-universal. Our first theorem is to show that by considering arbitrary dimension functions, no Cantor sets are topologically universal.

Theorem 1.4. *For any Cantor set $E \subset \mathbb{R}^d$, there exists a dense G_δ set such that it does not contain an affine copy of E. Consequently, there do not exist any topologically universal Cantor sets on $\mathbb{R}^d$.*

We now turn to study generic universality of Cantor sets. Because $m(\mathbb{R}^d \setminus G) = 0$ for any generic G_δ set, $m(G)$ must be infinity and thus those arbitrarily small-dimensional dense G_δ sets no longer exist. Generic universality is also a closer analog to the Erdős similarity conjecture because we now require the sets of interest having positive Lebesgue measure.

We first focus on Cantor sets in $\mathbb{R}^1$. In addition to Hausdorff dimension, Newhouse thickness of a Cantor set (see Section 3 for the precise definition) has been another useful quantity to describe the size of Cantor sets. In particular, the gap lemma provides a natural sufficient condition for two thick Cantor sets to intersect. Our main theorem is the following.

Theorem 1.5. *There exists a dense G_δ set G with $m(\mathbb{R} \setminus G) = 0$ such that for all Cantor sets J with positive Newhouse thickness, G does not contain an affine copy of J.*

As a consequence, Cantor sets on $\mathbb{R}^1$ with positive Newhouse thickness are not generically universal.

Our theorem also tells us something about the measure non-universality of Cantor sets. Although the Erdős similarity conjecture can be resolved if we can show that all decreasing sequence are not universal, it is not even an easy question to show that a Cantor set is measure non-universal. Indeed, Svetic [19] proposed the following stronger question in this regard. *'Is it true that for every uncountably infinite set, E, of real numbers, there exists $S \subset [0, 1]$ of full measure that does not contain an affine copy of E?'* Notice that if a set is generically non-universal, then it must be measure non-universal.

Our Theorem 1.5 now answers Svetic's question in a very strong way. There exists a fixed set, namely, $S = G \cap [0, 1]$ where G is defined in Theorem 1.5, of full Lebesgue measure in [0,1] which does not contain affine copies of any Cantor sets with positive thickness.

We now consider higher dimensions. First, one can show that a set containing a path-connected component cannot be generically universal (see Proposition 5.1). Therefore, our main interest will be focused on totally disconnected Cantor set. There has been recent work on generalizing the gap lemma into high dimension (see, for example, [8]). However, their results do not seem to adjust into our situation. Instead, we consider the projection of the Cantor set onto the one-dimensional coordinate axis. Newhouse thickness for any compact sets can be defined easily (see Section 3). We have the following definition.

Definition 1.6. Let E be a Cantor set on $\mathbb{R}^d$. We say that E is **Newhouse projectively thick** if for all invertible linear transformations T, the orthogonal projection of $T(E)$ onto the x_1-axis has positive Newhouse thickness.

We now have the following theorem.

Theorem 1.7. *Let E be a Cantor set on $\mathbb{R}^d$ that is Newhouse projectively thick. Then E is not generically universal.*

This theorem covers many examples of Cantor sets. We will show that every self-similar set on $\mathbb{R}^d$, not lying on a hyperplane, whose linear parts are rotation-free, will be Newhouse projectively thick. We note that there has been intensive research about the dimensional properties of projections of Cantor sets (for a survey, see, for example, [9]), but the properties of Newhouse thickness along orthogonal projections that we present here appears to be new. We conjecture that all self-similar or self-affine sets, not lying on a hyperplane, are Newhouse projectively thick.

1.2 | Some discussion and open problems

Let us summarize our results and other known results in the following table.

	Measure universal	Topologically universal	Generically universal
Finite sets	Yes	Yes	Yes
Countably infinite sets	Unknown	Yes	Yes
Cantor sets on $\mathbb{R}^1$	Unknown	No — Theorem 1.4	No* — Theorem 1.5
Cantor sets on $\mathbb{R}^d$, $d > 1$	Unknown	No — Theorem 1.4	No* — Theorem 1.7
Sets with interior	No	No	No

In the table, No* indicates a partial result established in this paper. Theorems 1.5 and 1.7 refer to Cantor sets with positive Newhouse thickness on $\mathbb{R}^1$ and Newhouse projectively thick Cantor sets on $\mathbb{R}^d$ are not generically universal. It provides evidence that Cantor sets are unlikely to be generically universal. We believe that the following may be true, which draws an analogue of the Erdős similarity conjecture for generic sets.

Conjecture 1.8. *There are no generically universal Cantor sets on $\mathbb{R}^d$.*

It is also reasonable that the following conjecture draws a parallel analogy of the Erdős similarity conjecture in the purely topological non-measure-theoretic setting.

Conjecture 1.9. *There are no uncountable topologically universal sets on $\mathbb{R}^d$.*

Unfortunately, Theorem 1.4 does not imply the validity of Conjecture 1.9. This is because in the realm of descriptive set theory, it is known that with the axiom of choice, one can construct a so-called *Bernstein set* [12, p. 48], in which neither the set nor its complement contain a perfect set. That is, the set contains no perfect subset and is uncountable. This means that we cannot use Theorem 1.4 to conclude that Bernstein sets are topologically non-universal. It is unclear to us whether Conjecture 1.9 is even decidable within the ZFC (Zermelo-Fraenkel wth Choice) axioms of set theory. Nonetheless, despite such a pathological example, every uncountable Borel set (or more generally analytic set) of $\mathbb{R}^d$ contains a perfect subset (see [12, p. 85, 88]), so they will not be topologically universal.

The paper is organized as follows. We prove Theorem 1.4 in Section 2. We will define Newhouse thickness for compact sets of $\mathbb{R}^1$ in Section 3. We will prove our theorems on $\mathbb{R}^1$ in Section 4 and then theorems on $\mathbb{R}^d$ in Section 5.

2 | TOPOLOGICAL NON-UNIVERSALITY OF CANTOR SETS

A function h is called a **dimension function/gauge function** if $h : [0, 1] \to [0, \infty)$ is non-decreasing, continuous and $h(0) = 0$. The h-Hausdorff measure is the translation-invariant Borel measure such that

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} h(|U_i|) : E \subset \bigcup_{i=1}^{\infty} U_i, \; |U_i| \leqslant \delta \right\},$$

where $|U|$ denotes the diameter of U. If $h(x) = x^s$, then $\mathcal{H}^h$ is the standard s-dimensional Hausdorff measure.

Proposition 2.1. *For any dimension function h, there always exist a dense G_δ set G on $\mathbb{R}^d$ such that $\mathcal{H}^h(G) = 0$.*

Proof. For any dimension function h, h^{-1} may not exist since h may not be strictly increasing. However, we define

$$W(s) = \inf\{t > 0 : h(t) > s\} = \sup\{t > 0 : h(t) \leqslant s\}.$$

Then we have $h(W(s)) = s$. Moreover, W is strictly increasing whence $W(s) > 0$ for all $s > 0$. Let us now enumerate the rationals $\mathbb{Q}^d = \{r_1, r_2, ...\}$. Consider the following dense G_δ set

$$G = \bigcap_{k=1}^{\infty} \left(\bigcup_{i=1}^{\infty} \left(r_i - \frac{W(2^{-(i+k)})}{2\sqrt{d}}, r_i + \frac{W(2^{-(i+k)})}{2\sqrt{d}} \right)^d \right).$$

Then the diameter of the open squares inside the union is $W(2^{-(i+k)})$, so

$$\mathcal{H}^h(G) \leqslant \sum_{i=1}^{\infty} h(W(2^{-(i+k)})) = \sum_{i=1}^{\infty} 2^{-(i+k)} = 2^{-k}.$$

As k is arbitrary, $\mathcal{H}^h(G) = 0$. The proof is complete. $\qquad\square$

We need a result from Rogers [16, p.67].

Proposition 2.2. *Let Ω be an uncountable complete separable metric space. Then there exists a compact perfect set C and a dimension function h such that*

$$0 < \mathcal{H}^h(C) < \infty.$$

Consequently, suppose that a compact set K in a metric space satisfies $\mathcal{H}^h(K) = 0$ for all dimension functions h. Then K is a countable set.

We remark that in [16], dimension functions were defined to be right-continuous, but if we inspect the proof on page 65 in the book carefully, it is clear that we can construct h to be continuous for the first statement. For the second statement, we note that if K is uncountable and compact, then K contains a perfect subset Ω. Applying the first statement, we have a perfect set $C \subset \Omega \subset K$ with $\mathcal{H}^h(C) > 0$ for some dimension function h which leads to a contradiction.

We should also remark, on the other hand, that there exists a Cantor set K on $\mathbb{R}^1$ such that for all dimension functions h, either $\mathcal{H}^h(K) = 0$ or ∞ [5]. See also [3] for a recent survey. Nonetheless, in this case, we can still extract a sub-Cantor set of K with finite positive Hausdorff measure for some gauge function h by Proposition 2.2. It means that $\mathcal{H}^h(K) = \infty$.

Heuristically, to prove Theorem 1.4, we just take a suitable gauge function and a dense G_δ set according to Proposition 2.1. Then the monotonicity of measure immediately leads to a contradic-

tion. However, for general dimension functions, we do not have a dilation formula for all invertible linear transformations. Therefore, we need the following lemma.

Recall that for any invertible linear transformation T, $\|T\|$ denotes the operator norm of T : $\mathbb{R}^d \to \mathbb{R}^d$ with $\mathbb{R}^d$ endowed with the Euclidean norm, that is,

$$\|Tx\| \leqslant \|T\|\|x\|$$

holds for all $x \in \mathbb{R}^d$.

Lemma 2.3. *Let $E \subset \mathbb{R}^d$ be a Borel set, h a dimension function, and let $c > 0$. Then the dimension function $h_c = h(cx)$ satisfies*

$$\mathcal{H}^{h_c}(T(E)) \geqslant \mathcal{H}^h(E)$$

for all T such that $\|T^{-1}\| \leqslant c$.

Proof. First, from a direct observation we see that $h_c(x) = h(cx)$ is a dimension function. Let T such that $\|T^{-1}\| \leqslant c$. We note that any covering $\bigcup_{i=1}^{\infty} V_i$ of $T(E)$ implies that $\bigcup_{i=1}^{\infty} T^{-1}(V_i)$ is a covering of E, so

$$\mathcal{H}^h(E) \leqslant \sum_{i=1}^{\infty} h(|T^{-1}V_i|).$$

But from the definition of $\|T^{-1}\|$, the diameters satisfy

$$|T^{-1}V_i| \leqslant \|T^{-1}\||V_i| \leqslant c|V_i|.$$

Hence,

$$\mathcal{H}^h(E) \leqslant \sum_{i=1}^{\infty} h(|T^{-1}V_i|) \leqslant \sum_{i=1}^{\infty} h(c|V_i|) = \sum_{i=1}^{\infty} h_c(|V_i|).$$

We now take infimum among all covers and obtain our desired conclusion. $\square$

Proof of Theorem 1.4. Let E be a Cantor set on $\mathbb{R}^d$. By Proposition 2.2, we can find a dimension function h such that $\mathcal{H}^h(E) > 0$. For each $n \in \mathbb{N}$, let us take the dimension function h_n in Lemma 2.3 such that $\mathcal{H}^{h_n}(T(E)) \geqslant \mathcal{H}^h(E) > 0$ whenever $\|T^{-1}\| \leqslant n$.

Now using Proposition 2.1, we can find a dense G_δ set G_n such that $\mathcal{H}^{h_n}(G_n) = 0$. By the Baire Category theorem, $G = \bigcap_{n=1}^{\infty} G_n$ is a dense G_δ set. We now claim that this G cannot contain any affine copy of the Cantor set E. Indeed, suppose that $t + T(E)$ is contained in G. Let $n \in \mathbb{N}$ be such that $\|T^{-1}\| \leqslant n$, then $t + T(E) \subset G_n$. By taking $\mathcal{H}^{h_n}$ Hausdorff measure, we find a contradiction since $\mathcal{H}^{h_n}(G_n) = 0$, but

$$\mathcal{H}^{h_n}(t + T(E)) = \mathcal{H}^{h_n}(T(E)) > 0$$
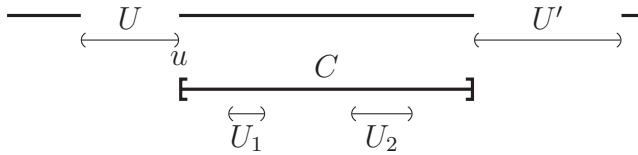
by Lemma 2.3. $\square$

FIGURE 1 At point u, we move to the right until we hit another gap of longer length. The interval traveled is the bridge C. Note that the Bridge contains gaps of smaller length than U such as U_1 and U_2 in the figure.

3 | PRELIMINARIES ON NEWHOUSE THICKNESS

The proof of our theorems on generic universality relies on the Newhouse gap lemma. The purpose of this section is to define the thickness and state the gap lemma that is necessary for our proof. The definition of thickness and the gap lemma we use were first introduced by Newhouse [14]. Our definition below is taken from the book of Palis and Takens [15]. We first need to define the gaps and bridges of Cantor sets in order to define Newhouse thickness.

Definition 3.1 (Gap). Let K be a Cantor set on $\mathbb{R}^1$. A **gap** of K is a connected component of $\mathbb{R} \setminus K$. A **bounded gap** is a bounded connected component of $\mathbb{R} \setminus K$.

We now define the bridge of C of Cantor set K. $|I|$ denotes the length of the interval I.

Definition 3.2 (Bridge, cf. [15]). Let K be a Cantor set on $\mathbb{R}^1$ and $U = (u', u)$ be a bounded gap of K with boundary point u. The **bridge** C of K at u is the maximal interval on the right-hand side of u such that:

- u is a boundary point of C,
- C contains no point of a gap U' whose length $|U'| \geqslant |U|$.

We can define analogously the bridge for u' by considering the maximal interval on the left-hand side of u' with the same property.

For clarity, Figure 1 shows that there may be smaller bounded gaps contained in C.

We use this notion to define the Newhouse Thickness. Intuitively the thickness of a Cantor set can be thought of as the infimum of ratios between the bounded gaps and the bridges.

Definition 3.3 (Newhouse thickness for Cantor sets [15]). The **Newhouse Thickness** of K at u is defined as

$$\tau(K, u) = \frac{|C|}{|U|}.$$

Moreover, let $\mathcal{V}$ be the set of all boundary points of bounded gaps in the Cantor set, the thickness of the entire Cantor set is

$$\tau(K) = \inf_{u \in \mathcal{V}} \tau(K, u)$$

We will consider projections of Cantor sets in higher dimension onto the x_1-axis. Such projections may not be perfect or may contain intervals, so we need to define the Newhouse thickness for general compact sets of $\mathbb{R}^1$.

We first recall some terminologies in point set topology [17]. Let $K \subset \mathbb{R}^1$ be a compact set; $x \in K$ is called a **condensation point** of K if every open neighborhood of x contains uncountably many points of K. It is known that the set of all condensation points of K is a perfect set inside K. We call the set of all condensation points of K the **perfect part** of K.

Definition 3.4 (Newhouse thickness for general compact sets). Let K be a compact set on $\mathbb{R}^1$ and let P_K be the perfect part of K. We now define

$$\tau(K) = \begin{cases} 0 & \text{if } P_K = \emptyset \\ \infty & \text{if } P_K \text{ contains an interval} \\ \tau(P_K) & \text{otherwise.} \end{cases}$$

Example 3.5 (Newhouse thickness of the N-digit expansion Cantor set). Let $N \geqslant 2$ and let $j \in \{1, \dots, N-2\}$. Define K to be the self-similar Cantor set by dividing $[0,1]$ into N intervals of equal length, deleting the interval $[\frac{j}{N}, \frac{j+1}{N}]$, and repeating the process. Then it is well known that K consists of all real numbers whose N-adic expansion omit the digit j:

$$K = \left\{ \sum_{k=1}^{\infty} \frac{d_k}{N^k} : d_k \in \{0, 1, \dots, N-1\} \setminus \{j\} \right\}.$$

Now, each gap at the nth iteration is of length N^{-n}. The Newhouse thickness is equal to $\min\{j, N - j - 1\}$.

We notice an important fact that Newhouse thickness is invariant under any invertible affine transformation, $x \mapsto t + \lambda x$ where $\lambda \neq 0$, on $\mathbb{R}^1$. The following lemma is now commonly referred to as the newhouse gap lemma.

Lemma 3.6. *(Newhouse gap lemma) Let $K_1, K_2, \subset \mathbb{R}$ be Cantor sets with Newhouse thickness τ_1 and τ_2, respectively, and $\tau_1 \cdot \tau_2 \geqslant 1$. Suppose that K_1 is not contained in one of the gaps of K_2 and K_2 is not contained in one of the gaps of K_1. Then $K_1 \cap K_2 \neq \emptyset$.*

For additional information about the intersection in the above gap lemma, one can refer to [1]. We are now ready to prove our main results.

4 | GENERIC NON-UNIVERSALITY OF CANTOR SETS ON $\mathbb{R}^1$

We first prove our main theorems on $\mathbb{R}^1$. The construction of the F_σ set in Equation (1) in the proof below was motivated from [6], in which the authors constructed wavelets on a real line analogue of Cantor sets. The set in Equation (1) is exactly the set they used.

Proof of Theorem 1.5. We will establish the following claim.

Claim: Given an $\epsilon_0 > 0$, there exists a dense G_δ set G with $m(\mathbb{R} \setminus G) = 0$ such that for any Cantor set J with Newhouse thickness $\tau(J) \geqslant \epsilon_0$, G contains no affine copy of J.

Assuming the claim, we construct a dense G_δ set G_n of $m(\mathbb{R} \setminus G_n) = 0$ with the property that it does not contain affine copies of Cantor sets with Newhouse thickness at least $1/n$. Then we consider

$$G = \bigcap_{n=1}^{\infty} G_n.$$

Baire category theorem ensures that G is a dense G_δ set. This G will not contain any affine copy of any Cantor sets with positive Newhouse thickness. Moreover, by the subadditivity of measure, it is easy to see that $m(\mathbb{R} \setminus G) = 0$. This will complete the proof.

We now justify the claim. Let $\epsilon_0 > 0$ be given. Consider the Cantor sets K defined by contraction ratio $1/N$ and digits $\{0, 1, \dots, N-1\} \setminus \{(N-1)/2\}$ and N is odd as in Example 3.5, we know that $\tau(K) = \frac{N-1}{2}$. Therefore, we can find a sufficiently large N so that $\tau(K) > \epsilon_0^{-1}$.

Using the Cantor set K, we now define X such that

$$X = \bigcup_{n \in \mathbb{Z}} \bigcup_{\ell \in \mathbb{Z}} N^n(K + \ell), \tag{1}$$

creating an F_σ set. Now consider X^c. Because K^c is open and dense and so is its translated and dilated copies, $G = X^c$ is a dense G_δ and $m(\mathbb{R} \setminus G) = m(X) = 0$ as the Cantor set K we constructed is of Lebesgue measure zero. We now show that for any Cantor set J with $\tau(J) \geqslant \epsilon_0$, $G = X^c$ contains no affine copy of J.

Suppose that we have some Cantor set J with Newhouse thickness $\tau(J) \geqslant \epsilon_0$. Without loss of generality, by rescaling and translation, we can assume that the convex hull of J is equal to $[0,1]$. We now fix any affine copy $t + \lambda J$ where $t \in \mathbb{R}$ and $\lambda \neq 0$. There exists a unique n such that

$$|\lambda| \in (N^{n-1}, N^n]. \tag{2}$$

Similarly there exists a unique ℓ such that

$$t \in (\ell N^n, (\ell + 1)N^n]. \tag{3}$$

Let

$$K_1 = N^n(K + \ell) \text{ and } K_2 = t + \lambda J.$$

The convex hull of K_1, is $[\ell N^n, (\ell + 1)N^n]$. So, by our choice of t, we know that K_2 is not in the unbounded gap of K_1 and vice versa.

Now we will check the construction of our Cantor sets such that each is not contained in the bounded gaps of the other. For $i = 1, 2$, we define O_i to be the largest open bounded gap in K_i and I_i be the convex hull of K_i. For K_1, we have $|O_1| = N^{n-1}$ and $|I_1| = N^n$. For K_2, we recall that the convex hull of J is $[0,1]$. Therefore, we have

$$|O_2| = |\lambda| \cdot |O_J| \leqslant |\lambda| \text{ and } |I_2| = |\lambda|,$$

where O_J is the largest open-bounded gap interval in J. Therefore by our construction in (2), the following two inequalities hold:

$$|O_1| \leqslant |I_2| \text{ and } |O_2| \leqslant |I_1|.$$

The inequalities imply that K_1 is not fully contained in the bounded gaps of K_2 and K_2 is not fully contained in the bounded gaps of K_1.

Since Newhouse thickness is invariant under affine transformation on $\mathbb{R}^1$, by our choice of K we have that

$$\tau(K_1)\tau(K_2) = \tau(K)\tau(J) \geqslant \epsilon_0^{-1} \cdot \epsilon_0 = 1.$$

Therefore, the gap lemma in Lemma 3.6 implies that $K_1 \cap K_2$ is non-empty and hence $K_2 = t + \lambda J$ intersects with one of the unions in X in (1). It implies that $t + \lambda J$ cannot be fully contained in the G_δ set $G = X^c$. This establishes the claim, and therefore, we conclude that J is not topologically universal. $\square$

Remark 4.1. We would like to remark that Bourgain proved that a Minkowski sum of three infinite sets cannot be measured universal. We can use this result to deduce that some Cantor sets of zero Newhouse thickness cannot be measure universal. Let $N_j \geqslant 2$ be integers and $\mathcal{D}_j \subset \{0, 1, \dots, N_j - 1\}$ be subsets of cardinality at least 2. Define

$$C = \left\{ \sum_{j=1}^{\infty} \frac{d_j}{N_1 \dots N_j} : d_j \in \mathcal{D}_j \right\}. \tag{4}$$

Then C is not measure universal. Indeed, for $k = 0, 1, 2$, let

$$S_k = \left\{ \sum_{j \equiv k (\text{mod } 3)} \frac{d_j}{N_1 \dots N_j} : d_j \in \mathcal{D}_j \right\}.$$

By the result of Bourgain, $C = S_0 + S_1 + S_2$ is a sum of three infinite sets and hence is not measure universal. Moreover, if $N_j \to \infty$, then the Cantor set C above has zero Newhouse thickness.

On the other hand, our Theorem 1.5 is independent from Bourgain's result in the sense that our construction of the avoiding set is explicit and of full Lebesgue measure, while the set constructed by Bourgain was not explicit and the Lebesgue measure is not known. Therefore, we still cannot determine if all above Cantor sets are generically universal if we merely use Bourgain's result.

5 | GENERIC NON-UNIVERSALITY OF CANTOR SETS ON $\mathbb{R}^d$

We now turn to our results in higher dimensions. Our first goal is to show that some obvious examples cannot be generically universal. They include a set with a path-connected component and embedding a lower dimensional generically non-universal set into higher dimensions.

Proposition 5.1. *If $X \subset \mathbb{R}^d$ contains a path-connected component, then X is not generically universal.*

Proof. Let us consider the dense G_δ set that removes all the hyperplanes parallel that correspond to the coordinate hyperplanes shifted by rationals:

$$G = \bigcap_{i=1}^{d} \bigcap_{r \in \mathbb{Q}} \mathbb{R}^d \setminus \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d \, : \, x_i = r \right\}.$$

This is clearly a dense G_δ set and $m(\mathbb{R}^d \setminus G) = 0$ since there are only countably many hyperplanes and hyperplanes have d-dimensional Lebesgue measure zero. Consider any affine copy of X. Then this affine copy must contain a path L. The projection of L onto the coordinate axes will be non-degenerate on some interval for at least one of the axes. Call this the ith axis. This interval will contain a rational number r. Therefore L will intersect with the coordinate plane, $x_i = r$. In other words this dense G_δ cannot contain L. Thus, X cannot be topologically universal. ∎

The following simple lemma is needed in the following proofs.

Lemma 5.2. *Let G_1 and G_2 be two dense G_δ sets in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Then $G_1 \times G_2$ is a dense G_δ set in $\mathbb{R}^{d_1 + d_2}$.*

Proof. Suppose that we write $G_1 = \bigcap_{n=1}^{\infty} O_n$ and $G_2 = \bigcap_{n=1}^{\infty} O_n'$ where O_n and O_n' are open dense sets in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. The lemma follows immediately by observing that

$$G_1 \times G_2 = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} O_n \times O_m'.$$

∎

Proposition 5.3. *Let $0 < k < d$ be two positive integers. Suppose that $E \subset \mathbb{R}^k$ is generically non-universal in $\mathbb{R}^k$. Then $E \times \{0\}$ cannot be generically universal in $\mathbb{R}^d$ (0 here is the $d - k$-dimensional zero vector).*

Proof. Let $\mathbf{e}_i$ be the canonical coordinate basis in $\mathbb{R}^d$ and let $W = \mathbb{R}^k \times \{0\}$. By our assumption, we can find a dense G_δ set $G_0 \subset \mathbb{R}^k$ such that it does not contain k-dimensional affine copies of E. Let G_0' be any dense G_δ set in $\mathbb{R}^{d-k}$ with $m(\mathbb{R}^{d-k} \setminus G_0') = 0$. Then $G_0 \times G_0'$ is a dense G_δ in $\mathbb{R}^d$. By Fubini's theorem, $m((\mathbb{R}^k \setminus G_0) \times \mathbb{R}^{d-k}) = 0$, so is the other union. We let $\Pi_{d,k}$ be the collection of all k-dimensional coordinate planes in $\mathbb{R}^d$. There are $\binom{d}{k}$ such planes. For each $P \in \Pi_{d,k}$, there exists a permutation matrix σ_P such that

$$P = \sigma_P(W).$$

We now define

$$G = \bigcap_{P \in \Pi_{d,k}} \sigma_P(G_0 \times G_0').$$

Note that

$$\mathbb{R}^d \setminus (G_0 \times G_0') = \left((\mathbb{R}^k \setminus G_0) \times \mathbb{R}^{d-k}\right) \cup \left(\mathbb{R}^k \times (\mathbb{R}^{d-k} \setminus G_0')\right).$$

By Fubini's theorem, $m((\mathbb{R}^k \setminus G_0) \times \mathbb{R}^{d-k}) = 0$, so is the other set in the above union. As σ_P has unit determinant, we obtain that $m(\mathbb{R}^d \setminus G) = 0$.

To finish the proof, our next step is to show that G cannot contain any affine copies of $E \times \{0\}$. To see this, we argue by contradiction. Suppose that there exists an invertible linear transformation T on $\mathbb{R}^d$ such that $t + T(E) \subset G$. Then the subspace

$$T(W) = \text{span}\{T\mathbf{e}_1, \dots, T\mathbf{e}_k\}$$

is k-dimensional and $\{T\mathbf{e}_1, \dots, T\mathbf{e}_k\}$ forms a basis for $T(W)$, putting T in matrix representation under the canonical basis. The matrix

$$A = \begin{pmatrix} | & \cdots & | \\ T\mathbf{e}_1 & \cdots & T\mathbf{e}_k \\ | & \cdots & | \end{pmatrix}$$

is of column rank k. Hence, it has row rank k as well. Therefore, there exists k-linearly independent row vectors. Let $\mathcal{I} = \{i_1, \dots, i_k\}$ be the position of the row vectors of A for which they are linearly independent. Let $A_{\mathcal{I}}$ be the square matrix whose rows are exactly the rows of A at positions in $\mathcal{I}$. Then $A_{\mathcal{I}}$ is invertible on $\mathbb{R}^k$. Moreover, if we consider the k-dimensional coordinate plane P at those $x_{i_1}, \dots, x_{i_k}$ axes and denote by $P_{\mathcal{I}}$ the orthogonal projection onto P, then we have

$$P_{\mathcal{I}}(t + T(E)) = P_{\mathcal{I}}(t) + A_{\mathcal{I}}(E)$$

and

$$P_{\mathcal{I}}(\sigma_P(G_0 \times G_0')) = G_0.$$

By the construction of G, $t + T(E) \subset \sigma_P(G_0 \times G_0')$, meaning that $P(t) + A_{\mathcal{I}}(E) \subset G_0$. As $A_{\mathcal{I}}$ is invertible, we find an affine copy of E inside G_0, which is a contradiction. This completes the proof. □

As we know already that the middle-third Cantor set is not generically universal, the above proposition shows that it cannot be embedded to become generically universal in higher dimensions either. Notice also that such an embedding of a Cantor set will never be Newhouse projectively thick since the projection will always be a singleton in the orthogonal complement. We are now ready to prove our main theorem on $\mathbb{R}^d$ stated in the introduction.

Proof of Theorem 1.7. Suppose that we have a Newhouse projectively thick Cantor set J on $\mathbb{R}^d$. We now take G_0 in Theorem 1.5 and construct

$$G = G_0 \underbrace{\times \cdots \times}_{d\text{-times}} G_0.$$

Applying Lemma 5.2, $G_0 \times \cdots \times G_0$ is a dense G_δ set in $\mathbb{R}^d$ and therefore G is also a dense G_δ set. With Fubini's theorem, it is not difficult to show that $\mathbb{R}^d \setminus G$ has zero Lebesgue measure.

It remains to prove that G has no affine copy of J. Assume to the contrary that G contains an affine copy of J and denote it by $t + T(J)$. Then

$$t + T(J) \subset \underbrace{G_0 \times \cdots \times G_0}_{d\text{-times}}.$$

Denote by P the orthogonal projection onto the x_1-axis. We have $P[t + T(J)] \subset G_0$. By linearity we can express the orthogonal projection $P[t + T(J)]$ as $P(t) + P[T(J)]$. We have that G_0 contains an affine copy of $P[T(J)]$. But J is Newhouse projectively thick which implies that $\tau(P[T(J)]) > 0$. We obtain a contradiction since, by Theorem 1.5, G_0 cannot contain any affine copies of $P[T(J)]$. This completes the proof. $\qquad\square$

To conclude this paper, we consider a class of self-similar sets that are Newhouse projectively thick. Recall that if we are given finitely many contractive similarity maps $\phi_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \ldots, N$, such that

$$\phi_i(x) = \rho_i O_i x + b_i,$$

where $0 < \rho_i < 1$, O_i is an orthogonal transformation and $b_i \in \mathbb{R}^d$, $\Phi = \{\phi_i : i = 1, \ldots, N\}$ forms an **iterated function system (IFS)** and there exists a unique non-empty compact set $K = K_\Phi$ such that

$$K = \bigcup_{i=1}^{N} \phi_i(K).$$

We say that the IFS is **rotation-free** if all O_i are identity transformations. We also say that a self-similar set is **non-degenerate** if it is not contained in any hyperplane of $\mathbb{R}^d$.

Example 5.4. All non-degenerate self-similar sets on $\mathbb{R}^d$ generated by rotation-free IFS must be Newhouse projectively thick.

Proof. Let P be the orthogonal projection onto the x_1-axis and let T be any invertible linear transformation. We note that for a rotation-free IFS, the set $PT(K)$ is still generated by a self-similar IFS on $\mathbb{R}^1$ with maps

$$\widetilde{\phi}_i(x) = \rho_i x + PT(b_i).$$

Notice that the self-similar set is non-degenerate, meaning that $PT(K)$ is not a singleton. The self-similar set $PT(K)$ is a compact perfect set. In Feng and Wu [10, Lemma 3.5], the authors showed that all self-similar sets not lying on a hyperplane have a positive thickness τ_{FW} defined in [10, Definition 1.1]. On the other hand, it was claimed without proof in the paragraph after Definition 1.1 in [10] that if $d = 1$, then $\tau_{FW}(E) > 0$ if and only if the Newhouse thickness $\tau(E) > 0$. This would have implied that $\tau(PT(K)) > 0$.

For the self-containment of this paper, we justify the direction required in this proof in the following claim.

Claim: If $d = 1$, then $\tau_{FW}(E) > 0$ implies that $\tau(E) > 0$.

To see this claim, [10, Definition 1.1] states that

$$\tau_{FW}(E) = \sup\{c \geqslant 0 : \forall x \in E, \forall r \in (0, |E|], \exists\, y \in \mathbb{R} \text{ s.t. } \operatorname{conv}(B(x, r) \cap E) \supset B(y, cr)\}.$$

Here, $|E|$ is the diameter of E, $\operatorname{conv}(K)$ means the convex hull of a set K, and $B(x, r)$ denotes the Euclidean ball centered at x of radius r. For each fixed $x \in E$ and $r \in (0, |E|]$, we define

$$\tau_{FW}(E, x, r) = \sup\{c \geqslant 0 : \exists\, y \in \mathbb{R} \text{ s.t. } \operatorname{conv}(B(x, r) \cap E) \supset B(y, cr)\}.$$

Then $\tau_{FW}(E) = \inf\limits_{x \in E} \inf\limits_{r \in (0, |E|]} \tau_{FW}(E, x, r)$.

Suppose that $\tau_{FW}(E) > 0$. Consider $u \in \mathcal{U}$ (using the notation as in Definition 3.3) where u is a boundary point of a bounded gap U. Consider the open interval $(u - |U|, u + |U|)$. Then one of the endpoints of $\operatorname{conv}(B(u, |U|) \cap E)$ is u. Let C be the bridge of u and without loss of generality assume that C is on the right-hand side of U.

If $u + |U| \in C$, then $|C| \geqslant |U|$ and $\tau(E, u) \geqslant 1$. If, however, $u + |U| \notin C$, then because of the definition of the bridge, $u + |U|$ is in the first gap whose length is larger than $|U|$. Hence, $\operatorname{conv}(B(u, |U|) \cap E) = C = B(z, \frac{|C|}{2|U|}|U|)$ for some center z. This means that $\tau_{FW}(E, u, |U|) = \frac{|C|}{2|U|}$. Then

$$\tau(E, u) = \frac{|C|}{|U|} = 2 \cdot \tau_{FW}(E, u, |U|) \geqslant 2 \cdot \tau_{FW}(E) > 0.$$

Taking infimum among all $u \in \mathcal{U}$, we show that $\tau(E) \geqslant \min\{2\tau_{FW}(E), 1\} > 0$. This completes the proof of the claim.

Coming back to the proof, we now know that all self-similar sets, not a singleton, on $\mathbb{R}^1$ must have a positive Newhouse thickness. So the self-similar set $PT(K)$ has a positive Newhouse thickness. This shows that all non-degenerate self-similar sets generated by rotation-free IFS must be Newhouse projectively thick. □

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