A sharp lower bound for the spectral radius in K_4-saturated graphs

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Abstract

For a graph H, a graph G is H-saturated if G does not contain H as a subgraph but for any $e \in E(\overline{G})$, $G+e$ contains H. In this note, we prove that if G is an n-vertex K_{r+1}-saturated graph such that for each vertex $v \in V(G)$,

$$\sum_{s\in N(v)} d_G(w) \geq (r-2)d(v) + (r-1)(n-r+1),$$

then $\rho(G) \geq \rho(S_{n,r})$, where $S_{n,r}$ is the graph obtained from a copy of K_{r-1} with vertex set S by adding $n-r+1$ vertices, each of which has neighborhood S. With this result, for $r = 2, 3$, we prove a sharp lower bound for the spectral radius in an n-vertex K_{r+1}-saturated graph; for $r = 2$, equality holds only when G is $S_{n,2}$ or a Moore graph, and for $r = 3$, equality holds only when G is $S_{n,3}$.

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1 Introduction

We consider finite undirected graphs with no loops or multiple edges. For a graph H, a graph G is *H-saturated* if H is not a subgraph of G but for any $e \in E(\overline{G})$, H is a subgraph of $G+e$. For a positive integer n and a graph H, the *extremal number*, written $ex(n,H)$, is the maximum number of edges in an n-vertex graph not containing H. On the

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other hand, the *saturation number of H*, written $sat(n,H)$, is the least number of edges in an n-vertex H-saturated graph. Clearly, if G is an n-vertex H-saturated graph, then $sat(n,H) \le |E(G)| \le ex(n,H)$.

In 1941, Turán [13] determined the extremal number $ex(n, K_{r+1})$ initiating the study of extremal graph theory. He also proved that there is a unique extremal graph, $T_{n,r}$, the n-vertex complete r-partite graph whose partite sets differ in size by at most one. The first result on saturation numbers was proved in 1964 [4]. Let $S_{n,r}$ denote the n-vertex graph obtained from a copy of K_{r-1} with vertex set S by adding $n-r+1$ vertices, each of which has neighborhood S. In particular, $S_{n,2}$ is an n-vertex star. Erdős, Hajnal and Moon [4] determined the saturation number of K_{r+1} and described the extremal graphs.

Theorem A [4]. *If $2 \le r < n$, then $sat(n, K_{r+1}) = (r-1)(n-r+1) + \binom{r-1}{2}$. The only n-vertex K_{r+1}-saturated graph with $sat(n, K_{r+1})$ edges is the graph $S_{n,r}$.*

We refer the reader to Faudree, Faudree, and Schmitt [5] for an excellent survey on saturation numbers.

For a graph G, let $A(G)$ denote its adjacency matrix, and *spectral radius* $\rho(G)$ be the spectral radius of $A(G)$, that is, $\rho(G) = \max\{|\lambda_i| : 1 \le i \le n\}$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A(G)$. Since $A(G)$ is real-valued and symmetric, all λ_is are real numbers, so we may assume $\lambda_1(G) \ge \cdots \ge \lambda_n(G)$. By the Perron-Frobenius Theorem (see [7, 8]), we have $\rho(G) = \lambda_1(G)$.

Nikiforov [11] proved that if G is an n-vertex a K_{r+1}-free graph, then $\rho(G) \le \rho(T_{n,r})$. Since each K_{r+1}-saturated graph is K_{r+1}-free, his theorem implies the following.

Theorem B [10]. *If G is a K_{r+1}-saturated graph with n vertices, then*

$$\rho(G) \le \rho(T_{n,r}).$$

The Perron-Frobenius Theorem implies that for a connected graph G and $e \in \overline{G}$, we have $\rho(G+e) > \rho(G)$. Thus it is natural to ask what can be said about a lower bound for $\rho(G)$ if we replace "K_{r+1}-free" in Theorem B by "K_{r+1}-saturated". Kim, Kim, Kostochka, and O [9] gave a new lower bound for the spectral radius of an n-vertex K_{r+1}-saturated graph. The bound is tight for $r=2$, but not for $r \ge 3$. For $r=2$ the result is as follows.

Theorem C [9]. *If G is an n-vertex K_3-saturated graph, then $\rho(G) \ge \rho(S_{n,2})$; equality holds only when G is $S_{n,2}$ or a Moore graph.*

In this note, we prove that if G is an n-vertex K_{r+1}-saturated graph such that for each vertex $v \in V(G)$, $\sum_{w \in N(v)} d(w) \ge (r-2)d(v) + (r-1)(n-r+1)$, then $\rho(G) \ge \rho(S_{n,r})$. By using this result, we give a simpler proof of Theorem C and prove a new sharp lower bound for $\rho(G)$ in an n-vertex K_4-saturated graph G.

For undefined terms of graph theory, see West [14]. For basic properties of spectral graph theory, see Brouwer and Haemers [2] or Godsil and Royle [7].

2 Results and proofs

We first prove Theorem 2.2. Note that the spectral radius of $S_{n,r}$ is as follows.

Proposition 2.1. [6, 9, 12] *For integers $2 \le r < n$,*

$$\rho(S_{n,r}) = \frac{r-2+\sqrt{(r-2)^2+4(r-1)(n-r+1)}}{2}.$$

Theorem 2.2. *If G is an n-vertex K_{r+1}-saturated graph such that for each vertex $v \in V(G)$, $\sum_{w\in N(v)} d(w) \ge (r-2)d(v) + (r-1)(n-r+1)$, then $\rho(G) \ge \rho(S_{n,r})$.*

Proof. Let A be the adjacency matrix of G and let x be the perron vector corresponding to the spectral radius of G, say ρ. Note that x has all positive entries. Without loss of generality, we may assume that $\sum_{i=1}^{n} x_i = 1$. Suppose that $p(x) = x^2 - (r-2)x - (r-1)(n-r-1)$. Then we have

$$p(A)x = \left[A^2 - (r-2)A - (r-1)(n-r+1)I\right] x = p(\rho)x. \tag{1}$$

Thus we have

$$\begin{aligned} p(\rho) = p(\rho)\left(\sum_{v\in V(G)} x_v\right) &= \sum_{v\in V(G)} p(\rho)x_v = \sum_{v\in V(G)}\sum_{u\in V(G)} p(A)_{vu}x_u = \sum_{v\in V(G)} x_v \sum_{u\in V(G)} p(A)_{vu} \\ &\ge \min_{v\in V(G)} \sum_{u\in V(G)} p(A)_{uv} = \min_{v\in V(G)} \sum_{u\in V(G)} (A^2-(r-2)A-(r-1)(n-r+1)I)_{uv} \\ &= \min_{v\in V(G)}\left[\left(\sum_{w\in N(v)} d(w)\right) - (r-2)d(v) - (r-1)(n-r+1)\right] \ge 0, \end{aligned} \tag{2}$$

which yields that $\rho(G) \ge \rho(S_{n,r})$. ☐

With Theorem 2.2, we now give a simpler proof of Theorem C.

Proof of Theorem C. By Theorem 2.2, it suffices to show that for each vertex $v \in V(G)$,

$$\sum_{w\in N(v)} d(w) \ge n-1.$$

Since the diameter of G equals 2 and G is K_3-free, the Breadth First Search yields $\sum_{w\in N(v)} d(w) = d(v)+(n-1-d(v)) \ge n-1$. Equality in the bound holds only when for every vertex $v \in V(G)$, and every vertex $x \in V(G)\backslash N[v]$, we have $|N(v)\cap N(x)| = 1$. If $V(G)\backslash N[v] = \emptyset$, then G is $S_{n,2}$. Otherwise, the girth of G is at least 5, which implies that G is a Moore graph. ☐

Next, we prove a sharp lower bound for the spectral radius in an n-vertex K_4-saturated graph.

Theorem 2.3. *If G is an n-vertex K_4-saturated graph, then $\rho(G) \geq \rho(S_{n,3})$; equality holds only when G is $S_{n,3}$.*

Proof. By Theorem 2.2, it suffices to show that for each vertex $v \in V(G)$,

$$\sum_{w \in N(v)} d(w) \geq d(v) + 2(n-2).$$

Let v be an arbitrary vertex in $V(G)$.

Case 1. The graph induced by the closed neighborhood $N[v]$ is K_4-saturated. Since $N[v]$ is K_4-saturated, $N(v)$ is K_3-saturated. Thus by Theorem A,

$$\begin{aligned}\sum_{w \in N(v)} d(w) &\geq d(v) + 2|E(G[N(v)])| + 2(n - d(v) - 1) \\ &\geq d(v) + |E(G[N(v)])| + (d(v)-1) + 2(n-d(v)-1) \qquad (3) \\ &= 2(n-2) + 1 + |E(G[N(v)])| \geq 2(n-2) + 1 + (d(v)-1) = d(v) + 2(n-2).\end{aligned}$$

Case 2. The graph induced by the closed neighborhood $N[v]$ is not K_4-saturated. In this case, we may assume that $\Delta(G) \leq n-2$, which implies that $\delta(G) \geq 4$. Also, since G is K_4-saturated, for each vertex $x \in V(G) - N[v] \neq \emptyset$, we have $|N(x) \cap N(v)| \geq 2$ and also $E(G[N(v)]) \neq \emptyset$. Thus if $|E(G[N(v)])| \geq d(v) - 1$, then we have

$$\begin{aligned}\sum_{w \in N(v)} d(w) &\geq d(v) + 2|E(G[N(v)])| + 2(n - d(v) - 1) \\ &\geq d(v) + 2(d(v)-1) + 2(n-d(v)-1) = d(v) + 2(n-2). \qquad (4)\end{aligned}$$

Now we may assume that $|E(G[N(v)])| \leq d(v) - 2$. Let $G_1, \ldots, G_t$ be the nontrivial components in $G[N(v)]$ and let $N_1 = V(G_1 \cup \cdots G_t)$, $N_2 = N(v) - N_1$, and $N_3 = V(G) - N[v]$. Note that $N(v) = N_1 \cup N_2$ and $G[N_2]$ is a non-empty trivial graph. Since $|E(G[N(v)])| \geq 1$, we have $t \geq 1$.

Let $e = u_1v_1 \in E(G[N_1])$. If two different vertices u_3, v_3 in N_3 are adjacent to both u_1 and v_1, then they are not adjacent. For a vertex v_2 in N_2, by adding an edge u_1v_2 or v_1v_2, we have a copy of K_4 in $G[\{u_1, v_2\} \cup N_3]$ or $G[\{v_1, v_2\} \cup N_3]$, respectively. Thus we can say that $|[N_1, N_3]| \geq 2(n - d(v) - 1) + 2t$, so we have

$$\begin{aligned}\sum_{w \in N(v)} d(w) &= d(v) + 2|E(G[N_1])| + |[N_1, N_3]| + |[N_2, N_3]| \\ &\geq d(v) + 2(d(v) - |N_2| - t) + 2(n - d(v) - 1) + 2t + 3|N_2| \\ &= d(v) + |N_2| + 2(n-1) > d(v) + 2(n-1).\end{aligned}$$

Equality in the bound requires equality in each step of the computation. To have equality in (3), we must have $G[N[v]] = S_{d(v)+1,3}$ and every vertex in $V(G) - N[v]$ is adjacent to exactly two vertices. Since $G[N[v]] = S_{d(v)+1,3}$, we must have a vertex w in $N(v)$, which is adjacent to all other vertices in $N[v]$. For any edge $e \in G[N(v)]$, exactly two vertices in N_3 are adjacent to the end-vertices of the edge, so we have $\Delta(G) = n - 1$. Thus we have $d(v) = n - 1$, which implies $G = S_{n,3}$.

To have equality in (4), we must have $d(v) \leq n - 2$, $|E[G[N(v)]]| = d(v) - 1$, and $|[N(v), N_3]| = 2(n - d(v) - 1)$ for each vertex $v \in V(G)$. Thus for each vertex $x \in N_3$, we have $|N(x) \cap N(v)| = 2$, and $G[N(v)]$ is a tree by following the proof of $t \geq 2$. If $G[N(v)]$ contains a copy of P_4 as a subgraph, then by adding an edge between the two end-vertices of the path, we must have a copy of K_4, which implies that some vertex in N_3 must have three neighbors in $N(v)$. This is s a contradiction. Thus we may assume that $G[N(v)]$ is a star, which implies that $G[N[v]]$ is a star. For each edge $e \in G[N(v)]$, every vertex in N_3 is adjacent to the end-vertices of e. Thus $\Delta(G) = n - 1$, which is a contradiction. □

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