AGT

Algebraic & Geometric Topology

Volume 22 (2022)

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Algebraic & Geometric Topology 22 (2022) 2267–2291

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A striking result of McDuff and Schlenk asserts that in determining when a four-dimensional symplectic ellipsoid can be symplectically embedded into a four-dimensional symplectic ball, the answer is governed by an "infinite staircase" determined by the odd-index Fibonacci numbers and the golden mean.

There has recently been considerable interest in better understanding this phenomenon for more general embedding problems. Here we study embeddings of one four-dimensional symplectic ellipsoid into another, and we show that if the target is rational, then the infinite staircase phenomenon found by McDuff and Schlenk can be characterized completely. Specifically, in the rational case, we show that there is an infinite staircase in precisely three cases: when the target has "eccentricity" $1, 2$, or $\frac{3}{2}$. In each of these cases, work of Casals and Vianna shows that the corresponding embeddings can be constructed explicitly using polytope mutation; meanwhile, for all other eccentricities, the embedding function is given by the classical volume obstruction, except on finitely many compact intervals, on which it is linear.

Our work verifies in the special case of ellipsoids a conjecture by Holm, Mandini, Pires and the author. The case where the target is the ellipsoid $E(4,3)$ is also interesting from the point of view of this Cristofaro-Gardiner–Holm–Mandini–Pires work: the "staircase obstruction" introduced in that work vanishes for this target, but nevertheless a staircase does not exist. To prove this, we introduce a new combinatorial technique for understanding the obstruction coming from embedded contact homology which is applicable in other situations where the staircase obstruction vanishes, and so is potentially of independent interest.

53D05; 57R58

Dedicated to my father on the occasion of his 85th birthday

1 Introduction

1.1 The main theorem

A *symplectic embedding* of one symplectic manifold (M_1, ω_1) into another (M_2, ω_2) is a smooth embedding

$$\Psi\colon M_1 \to M_2$$

Published: 25 October 2022 DOI: 10.2140/agt.2022.22.2267

such that $\Psi^*\omega_2 = \omega_1$. Determining whether or not a symplectic embedding exists can be very subtle, even in simple examples. For example, define the (open) *symplectic ellipsoid*

$$E(a_1, \dots, a_n) := \left\{ \pi \frac{|z_1|^2}{a_1} + \dots + \pi \frac{|z_n|^2}{a_n} < 1 \right\} \subset \mathbb{C}^n = \mathbb{R}^{2n}.$$

This inherits a symplectic form by restricting the symplectic form on $\mathbb{R}^{2n}$. Define the *symplectic ball*

$$B^{2n}(\lambda) := E(\lambda, \dots, \lambda).$$

In [16], McDuff and Schlenk determined exactly when a four-dimensional symplectic ellipsoid can be symplectically embedded into a four-dimensional symplectic ball. Specifically, they computed the function

$$c(a) := \min\{\lambda \mid E(1, a) \to B^4(\lambda)\}$$

for $a \geq 1$, where here and below the arrow denotes a symplectic embedding. They found that the function $c(a)$, which they show is continuous, has a surprisingly rich structure:

Theorem 1.1 [16]

- *For $1 \leq a \leq \tau^4$, the function $c(a)$ is given by an "infinite staircase" determined by the odd-index Fibonacci numbers.*
- *For $a \geq \left(\frac{17}{6}\right)^2$, we have $c(a) = \sqrt{a}$; in other words, the only obstruction to the embedding problem is the classical volume obstruction.*
- *For $\tau^4 \leq a \leq \left(\frac{17}{6}\right)^2$, we have $c(a) = \sqrt{a}$, except on finitely many intervals on which it is linear.*

In fact, they compute the function precisely — see [16] — but we do not need their exact result here. To explain the nomenclature in the first bullet point in Theorem 1.1, the authors show that the interval $[1, \tau^4]$ can be decomposed into an infinite sequence of intervals on which the graph of c alternates from lying on the line through the origin to being horizontal. In particular, the graph of c qualitatively can be described an infinite staircase.

The McDuff–Schlenk result mentioned above has sparked considerable interest in better understanding this staircase phenomenon. To elaborate, we say here and below that a continuous, nondecreasing, real-valued function on $[1, \infty)$ *has an infinite staircase* if it has infinitely many *singular points*, namely points where the function is not differentiable. For example, define the *four-dimensional polydisc* $P(a, b) := D^2(a) \times D^2(b)$.

Frenkel and Müller [11] studied embeddings of an ellipsoid into $P(1,1)$, and found an infinite staircase determined by the Pell numbers for the analogue of the function $c(a)$ with polydisc target. Usher [17] studied embeddings into other polydiscs and found infinitely many infinite staircases.

In a different direction, Casals and Vianna [4] found a connection between certain infinite staircases and polytope mutation, which we will further discuss below. Bertozzi, Holm, Maw, McDuff, Mwakyoma, Pires and Weiler [2] studied embeddings into Hirzebruch surfaces and found infinitely many infinite staircases, with singular points converging from above. The author and Kleinman [10] studied embeddings into $E(1,2)$ and $E(2,3)$ and found connections with the theory of Ehrhart functions; the author and Hind [7] found a connection between the McDuff–Schlenk staircase and certain higher-dimensional embedding problems.

Despite all the above results, we still do not have a good sense of how characteristic this staircase phenomenon actually is for four-dimensional symplectic embedding problems. The aim of this work is to answer this question completely when the target is a rational ellipsoid. It turns out that, in this case, infinite staircases are in fact quite rare and can be understood completely; it seems plausible that this holds more generally.

To make all this precise, for fixed $b \geq 1$, define the function

$$c_b(a) = \min\{\lambda \mid E(1,a) \to E(\lambda, \lambda b)\}. \tag{1}$$

Then the function $c_1(a)$ is precisely the McDuff–Schlenk function considered above. The function $c_b(a)$ is a continuous function, for example by [10, Lemma 5.1], but is not in general C^1, as seen for example by the McDuff–Schlenk result above. Following for example McDuff [15], we call b the *eccentricity* of the ellipsoid $E(1,b)$.

We can now state our main result:

Theorem 1.2 *Fix a rational $b \geq 1$. Then, unless $b \in \{1, 2, \frac{3}{2}\}$, we have $c_b(a) = \sqrt{a/b}$, except for finitely many compact intervals, on which it is linear.*

Note that the quantity $\sqrt{a/b}$ represents the classical volume obstruction here. (Symplectic embeddings must preserve volume.)

In view of Theorem 1.2, it is natural to ask what is known about $c_b(a)$ when $b \in \{1, 2, \frac{3}{2}\}$. In fact, it was previously shown [10] that, in each of these cases, the function $c_b(a)$ starts with an infinite staircase, determined by an infinite sequence that generalizes the odd-index Fibonacci numbers. So, from the point of view of infinite staircases for embeddings into rational ellipsoids, Theorem 1.2 is an optimal result.[1] These infinite staircases have the same form as the McDuff–Schlenk one; that is, they consist of intervals on which the function alternates between being linear, with a graph lying on the line through the origin, and being constant.

We also emphasize that, recently, Casals and Vianna [4] have introduced a beautiful method for explicitly constructing the embeddings required for the infinite staircase in the $b \in \{1, 2, \frac{3}{2}\}$ case using polytope mutation and almost toric fibrations. Thus, in view of Theorem 1.2, the embeddings required for all infinite staircases arise this way, in the case of ellipsoids. It seems worth exploring whether the other infinite staircases mentioned above can be constructed in a similar way.

In view of Theorem 1.2, it is natural to ask the following:

Question 1.3 Are there irrational numbers b for which $c_b(a)$ has an infinite staircase?

The results of Bertozzi, Holm, Maw, McDuff, Mwakyoma, Pires, Weiler and Usher mentioned above give Question 1.3 additional intrigue.

1.2 Reflexive polygons

Theorem 1.2 verifies in a special case a recent conjecture of the author and Holm, Mandini and Pires.

To explain this in more detail, and to partly explain the motivation for this conjecture, we need to recall some terminology from (for example) Choi, Cristofaro-Gardiner, Frenkel, Hutchings and Ramos [5] and our [6]. Let $\Omega \subset \mathbb{R}^2$ be a region in the first quadrant. We define the *toric domain* corresponding to Ω to be the subset

$$X_\Omega = \{(z_1, z_2) \mid (\pi|z_1|^2, \pi|z_2|^2) \in \Omega\} \subset \mathbb{C}^n = \mathbb{R}^{2n},$$

with the symplectic form inherited from the standard from on $\mathbb{R}^{2n}$. For example, when Ω is a triangle with legs on the axes, X_Ω is an ellipsoid; when Ω is a rectangle with legs on the axes, X_Ω is a polydisc.

A toric domain X_Ω is called a *convex toric domain* if Ω is a convex, connected, open subset of the first quadrant containing the origin, and is called *rational* if Ω has rational

[1] In fact, Theorem 1.2 was originally conjectured in [10, Conjecture 1.8].

vertices. We can define the ellipsoid embedding function $c_\Omega(a)$ for any convex toric domain analogously to the definition of $c_b(a)$. For example, the author with Holm, Mandini and Pires [9, Theorem 1.2] showed that the ellipsoid embedding function for any closed symplectic toric four-manifold agrees with the embedding function of a rational convex toric domain.

As mentioned above, it seems natural to try to understand how characteristic infinite staircases are for embedding problems. When the target is a rational convex toric domain, conjecturally there is a complete characterization. Namely, recall that a convex polygon with integral vertices is called *reflexive* if its dual polygon is also integral. It is known that this is equivalent to the polytope having one interior lattice point.

We can now state the conjecture introduced at the beginning of this section:

Conjecture 1.4 [9] *The embedding function $c_\Omega(a)$ of a rational convex toric domain has infinitely many singular points only if some scaling of Ω is reflexive.*

An integral triangle with vertices $(m,0)$, $(0,0)$ and $(0,n)$ with $m \geq n$ is reflexive if and only if

$$(m,n) \in \{(3,2),(4,2),(3,3)\}.$$

Indeed, if $n = 1$, then the triangle has no interior lattice points at all; if $n \geq 3$, then the triangle contains the $(3,3)$ triangle, which has exactly one interior lattice point, so there are too many interior lattice points unless $n = m = 3$; and if $n = 2$, there are no interior lattice points if $m = 2$, and too many if $m > 4$.

In particular, our main Theorem 1.2 therefore implies the following corollary:

Corollary 1.5 *Conjecture 1.4 holds for four-dimensional ellipsoids.*

Acknowledgements

This paper is an offshoot of my joint work with Tara Holm, Alessia Mandini and Ana Rita Pires [9], part of which was summarized in Section 1.2, which is aimed at understanding the phenomenon of infinite staircases in considerable generality. I thank my wonderful collaborators for many stimulating discussions. I also thank Roger Casals and Renato Vianna for explaining their beautiful work [4] to me, and I thank the anonymous referee for very useful comments. I also thank the Institute for Advanced

Study, the Minerva Research Foundation, and the NSF, under agreement DMS 1711976, for their support.

This paper is dedicated to my dad, Martin Gardiner, on the occasion of his 85th birthday. He has shared his love of mathematics with me from as early as I can remember and I am infinitely grateful for this.

2 Proof of the main theorem

We now explain the proof of the main theorem.

2.1 Outline of the argument

We begin by explaining the basic idea behind the argument.

It is already known that, for fixed b, if a is sufficiently large then the function $c_b(a)$ is given by the volume obstruction, by [3, Theorem 1.3]. It was also recently proved as a special case of [9, Proposition 2.1] that if the function $c_b(a)$ has finitely many singular points, then it is piecewise linear except on intervals where it coincides with the volume curve. So, we only have to understand whether or not infinitely many singular points can occur.

In Section 2.2, we apply a recent theorem by the author, Holm, Mandini and Pires to find a unique point a_0, determined by b, where singular points must accumulate if infinitely many of them exists. Next, we show in Sections 2.3 and 2.4 that, for all but four values of b, this number a_0 is small enough that one can understand enough about $c_b(a)$ for $1 \le a \le a_0 + \varepsilon$ to rule out the possibility of infinitely many singular points around a_0. The part of the argument in Section 2.4 uses the theory of "embedded contact homology" (ECH) capacities, which we explain there, while the part of the argument in Section 2.3 is completely elementary.

Three of the four possible values for b from above correspond to the $1, 2$ and $\frac{3}{2}$ cases, where an infinite staircase in fact exists. The fourth value corresponds to $b = \frac{4}{3}$; this turns out to be a delicate and interesting case, since the "staircase obstruction" — see Section 2.2 — vanishes. We treat this case separately in Section 2.5; our proof here also uses ECH capacities, together with a powerful theorem by McDuff [14], stating that these capacities give sharp obstructions to ellipsoid embeddings; the method we introduce here is potentially of independent interest and will be used in an update of [9] in other situations where the staircase obstruction vanishes; see Remark 2.13. The proof of Theorem 1.2 is then given in Section 2.6.

2.2 Computing the accumulation point

In [9], the author and collaborators show that, for a large class of symplectic 4–manifolds, any infinite staircase must accumulate at a unique point characterized as a solution to a certain quadratic equation. We will want to use these results here, to find this accumulation point. We begin by summarizing the relevant mathematics, in the special case of ellipsoids.

Any rational ellipsoid $E(1, p/q)$ has a *negative weight sequence*

$$(w; w_1, \dots, w_k),$$

defined by the procedure in [6, Section 2]. The weights can be read off from the triangle $\Delta_{1,p/q}$, with vertices $(0,0)$, $(1,0)$ and $(0, p/q)$; this should be regarded as the "moment polytope" of the ellipsoid.

More precisely, the number w is the smallest real number such that $\Delta_{1,p/q} \subset \Delta_{w,w}$: so, in this case, we have $w = p/q$. To find the w_i, we look at the complement of $\Delta_{1,p/q}$ in $\Delta_{p/q,p/q}$. This is itself a triangle, which is affine equivalent to a right triangle $\Delta^{(1)}$ with legs on the axes. The w_i are then given as follows. We take w_1 to be the largest number such that $\Delta_{w_1,w_1} \subset \Delta^{(1)}$; then, if this inclusion is not surjective, we look at the complement of Δ_{w_1,w_1} in $\Delta^{(1)}$, which is itself a triangle affine equivalent to a right triangle $\Delta^{(2)}$ with legs on the axes; we then take w_2 to be the largest number such that $\Delta_{w_2,w_2} \subset \Delta^{(2)}$ and iterate until the complement of Δ_{w_k,w_k} in $\Delta^{(k)}$ is empty. For the details, see [6].

We remark that the $w_1, \dots, w_k$ as described above are also called the *weight sequence* of the triangle $\Delta^{(1)}$.

We now define

$$\operatorname{per}(E(1, p/q)) = 3w - \sum w_i, \quad \operatorname{vol}(E(1, p/q)) = w^2 - \sum w_i^2,$$

where $(w; w_1, \dots, w_k)$ is the negative weight sequence. The term $\operatorname{vol}(E(1, p/q))$, which we denote by vol for short, is the volume of $E(1, p/q)$, appropriately normalized; the term $\operatorname{per}(E(1, p/q))$, which we denote by per, should be regarded as the perimeter.

We now have the following, from [9]:

Theorem 2.1 ([9, Theorem 1.10], in the special case of an ellipsoid) *Let b be a rational number. Then, if the ellipsoid embedding function $c_b(a)$ has infinitely many* *singular points, they must accumulate at a_0, the unique solution to*

$$a^2 - \left(\frac{\mathrm{per}^2}{\mathrm{vol}} - 2\right)a + 1 = 0, \tag{2}$$

with $a_0 \geq 1$. Moreover, $c_b(a) = \sqrt{a/b}$.

We note that Theorem 2.1 provides an obstruction to the existence of infinite staircases: if one exists, then we must have $c_b(a_0) = \sqrt{a_0/b}$; Theorem 1.10 of [9] produces a similar obstruction for any convex toric domain target of finite type. For future reference, we will call this obstruction the *staircase obstruction*; this obstruction is a central part of [9].

Theorem 2.1 can be used here to prove the following key lemma. Recall that the a values for the Fibonacci staircase terminated at $a = \tau^4$. We now define an analogue of τ^4 that varies with b. Assume now that $b = k/l$. The analogue of τ^4 is defined implicitly by the following lemma:

Lemma 2.2 *Fix $b = k/l$. Then, if the graph of $c(a,b)$ has infinitely many nonsmooth points, they must accumulate at*

$$a_0 = \frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2 - 4kl}}{2k}\right)^2,$$

and $c_b(a_0) = \sqrt{a_0/b}$.

For the benefit of the reader, we connect with the Fibonacci staircase by noting that if $k = l = 1$, then

$$\varphi := \frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2 - 4kl}}{2k}\right)^2 = \tau^4.$$

We will call φ the *accumulation point*.

Proof By Theorem 2.1, the accumulation point a_0 must occur at the unique solution to (2) that is at least 1, and we must have $c_b(a_0) = \sqrt{a_0/b}$.

To compute a_0 explicitly, we need to compute the terms per and vol. We already computed above that $w = k/l$. Next, we compute

$$\Delta^{(1)} = \Delta_{k/l-1,k/l}.$$

As mentioned above, the remaining weights w_i can be interpreted as the weight sequence for $\Delta_{k/l-1,k/l}$. We now apply a result of McDuff and Schlenk from [16]; specifically, in [16, Lemma 1.2.6], it is shown that, for any $\Delta_{1,p/q}$ with p/q in lowest terms, the weight sequence $(a_1,\dots,a_k)$ satisfies

$$\sum_i a_i = \frac{p}{q} + 1 - \frac{1}{q}, \quad \sum_i a_i^2 = \frac{p}{q}.$$

In the present situation, then, we find

$$\mathrm{per} = \frac{3k}{l} - \left(\frac{k}{l} - 1\right)\left(\frac{k}{k-l} + 1 - \frac{1}{k-l}\right) = \frac{k+l+1}{l}$$

and

$$\mathrm{vol} = \left(\frac{k}{l}\right)^2 - \left(\frac{k}{l} - 1\right)^2 \frac{k/l}{k/l-1} = \frac{k}{l}.$$

It is now convenient to use another version of (2). That is, it is shown in [9] that the solutions to (2) are the same as the solutions to

$$a + 1 - \sqrt{a \cdot \frac{\mathrm{per}^2}{\mathrm{vol}}} = 0.$$

Plugging in for per and vol from above, we therefore get

$$a + 1 - (k + l + 1)\sqrt{\frac{a}{kl}} = 0.$$

Thus, we see that $a' = (l/k)a$ satisfies

$$ka' - (k + l + 1)\sqrt{a'} + l = 0,$$

hence the result. □

2.3 The accumulation point is usually small

We now collect some elementary arguments to show that, for most k and l,

$$\frac{k}{l}\left(\frac{k + l + 1 + \sqrt{(k + l + 1)^2 - 4kl}}{2k}\right)^2,$$

is quite small.

Lemma 2.3 *Assume that $l \neq 1$ and that $(k,l) \notin \{(3,2),(5,2),(4,3),(5,3),(5,4)\}$. Then*

$$\frac{k}{l}\left(\frac{k + l + 1 + \sqrt{(k + l + 1)^2 - 4kl}}{2k}\right)^2 < \frac{k+l+1}{l}.$$

Proof We proceed in five steps.

Step 1 Here we prove the following claim:

Claim 2.4 *If $l \geq 7$ and $k \neq l$, then $(k+l+1)^2 - 4kl \leq \bigl(k - \frac{1}{4}l - \frac{2}{5}\bigr)^2$.*

Proof We know that

$$\bigl(k - \tfrac{1}{4}l - \tfrac{2}{5}\bigr)^2 = k^2 - \tfrac{1}{2}kl - \tfrac{4}{5}k + \tfrac{1}{5}l + \tfrac{1}{16}l^2 + \tfrac{4}{25}.$$

We also know that

$$(k+l+1)^2 - 4kl = k^2 + l^2 + 1 - 2kl + 2k + 2l.$$

Hence, the claim is true if and only if

$$\tfrac{15}{16}l^2 - k\bigl(\tfrac{3}{2}l - \tfrac{14}{5}\bigr) + \tfrac{9}{5}l + \tfrac{21}{25} \leq 0.$$

We know that $\frac{3}{2}l - \frac{14}{5} > 0$ (since $l \geq 2$). We also know that $k \geq l+1$. Hence, we know that

$$\begin{aligned} \tfrac{15}{16}l^2 - k\bigl(\tfrac{3}{2}l - \tfrac{14}{5}\bigr) + \tfrac{9}{5}l + \tfrac{21}{25} &\leq \tfrac{15}{16}l^2 - (l+1)\bigl(\tfrac{3}{2}l - \tfrac{14}{5}\bigr) + \tfrac{9}{5}l + \tfrac{21}{25} \\ &= -\tfrac{9}{16}l^2 + \tfrac{31}{10}l + \tfrac{91}{25}. \end{aligned}$$

The larger of the two roots of $-\frac{9}{16}l^2 + \frac{31}{10}l + \frac{91}{25}$ is smaller than 7. So, since

$$-\tfrac{9}{16}l^2 + \tfrac{31}{10}l + \tfrac{91}{25} < 0$$

if $l \geq 7$, the result follows. □

Step 2 Claim 2.4 is very useful when $l \geq 7$. We need a slightly different version of this claim to handle most of the other l.

Claim 2.5 *If $l \geq 3$ and $k \geq l+6$, then $(k+l+1)^2 - 4kl \leq \bigl(k - \frac{1}{4}l - \frac{2}{5}\bigr)^2$.*

Proof From the proof of Claim 2.4, we know that Claim 2.5 is true if and only if

$$\tfrac{15}{16}l^2 - k\bigl(\tfrac{3}{2}l - \tfrac{14}{5}\bigr) + \tfrac{9}{5}l + \tfrac{21}{25} \leq 0.$$

We know that $\frac{3}{2}l - \frac{14}{5} > 0$. We also know that $k \geq l+6$. Hence,

$$\begin{aligned} \tfrac{15}{16}l^2 - k\bigl(\tfrac{3}{2}l - \tfrac{14}{5}\bigr) + \tfrac{9}{5}l + \tfrac{21}{25} &\leq \tfrac{15}{16}l^2 - (l+6)\bigl(\tfrac{3}{2}l - \tfrac{14}{5}\bigr) + \tfrac{9}{5}l + \tfrac{21}{25} \\ &= \tfrac{-1}{400}(225l^2 + 1760l - 7056). \end{aligned}$$

Since

$$225l^2 + 1760l - 7056 > 0$$

if $l \geq 3$, the result follows. □

Step 3 Using these two claims, we can now take care of almost every case.

More precisely, in this step, assume that either $l \geq 7$, or $l \geq 3$ and $k \geq l+6$. Then, by Claims 2.4 and 2.5, we know that

$$\begin{aligned}\frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k}\right)^2 &\leq \frac{k}{l}\left(\frac{k+l+1+\sqrt{\left(k-\frac{1}{4}l-\frac{2}{5}\right)^2}}{2k}\right)^2 \\ &= \frac{\left(2k+\frac{3}{4}l+\frac{3}{5}\right)^2}{4kl} \\ &= \frac{4k^2+3kl+\frac{12}{5}k+\frac{9}{16}l^2+\frac{9}{25}+\frac{9}{10}l}{4kl} \\ &= \frac{1}{l}\left(k+\frac{3}{4}l+\frac{12}{20}+\frac{9}{64}\frac{l}{k}l+\frac{9}{60k}+\frac{9l}{40k}\right).\end{aligned}$$

We know that $k \geq 1$ and $l/k \leq 1$. Hence, we know that

$$\left(k+\tfrac{3}{4}l+\tfrac{12}{20}+\frac{9}{64}\frac{l}{k}l+\frac{9}{60k}+\frac{9l}{40k}\right) \leq k+l+1.$$

This completes the proof of Lemma 2.3 in the case where $l \geq 7$, or $l \geq 3$ and $k \geq l+6$.

Step 4 Now assume that $l = 2$ and let $k \geq 8$; we will now prove Lemma 2.3 in this case.

As $k \geq 6$, we know that

$$k^2-2k+9 < \left(k-\tfrac{1}{4}\right)^2.$$

We therefore know that

$$\begin{aligned}\frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k}\right)^2 &= \frac{k}{l}\left(\frac{k+3+\sqrt{(k+3)^2-8k}}{2k}\right)^2 \\ &\leq \frac{1}{l}\frac{\left(k+3+k-\frac{1}{4}\right)^2}{4k} \\ &= \frac{1}{l}\frac{4k^2+11k+\frac{121}{16}}{4k} \\ &= \frac{1}{l}\left(k+\frac{11}{4}+\frac{121}{64k}\right) \\ &\leq \frac{1}{l}(k+3),\end{aligned}$$

where, in the last inequality, we have used the fact that $k \geq 8$. Thus, Lemma 2.3 holds in this case as well.

Step 5 The previous steps have proved Lemma 2.3 under the assumption that $l \geq 7$, or $l \geq 3$ and $k \geq l+6$, or $l = 2$ and $k \geq 8$.

Thus, it remains to check Lemma 2.3 in the following cases:

$$(k,l) \in \{(7,2),(7,3),(8,3),(7,4),(9,4),(6,5),(7,5),(8,5),(9,5),(7,6),(11,6)\}.$$

We can compute directly that Lemma 2.3 holds for these as well. □

2.3.1 Rounding up the nonintegral stragglers We can deal with the $(5,2)$, $(5,3)$ and $(5,4)$ cases by using the following simple fact:

Claim 2.6 *If $b = (k,l) \in \{(5,2),(5,3),(5,4)\}$, then*

$$\frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k}\right)^2 < \frac{b(\lfloor b\rfloor+2)^2}{(\lfloor b\rfloor+1)^2}.$$

Proof This is verified by direct computation. □

2.3.2 The integral case It is easy to see that Lemma 2.3 is not true when $l = 1$. However, the following is true:

Lemma 2.7 *Let $k \geq 3$ and let $l = 1$. Then*

$$\frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k}\right)^2 < \frac{k(k+3)^2}{(k+1)^2}.$$

Proof First, if $k \geq 4$, then

$$(k+2)^2 - 4k < \left(k+\tfrac{1}{2}\right)^2.$$

We now show that this implies Lemma 2.7 for $k \geq 4$. Indeed, in this case we have

$$\begin{aligned}\frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k}\right)^2 &= k\left(\frac{k+2+\sqrt{(k+2)^2-4k}}{2k}\right)^2\\ &< k\left(\frac{k+2+k+\frac{1}{2}}{2k}\right)^2\\ &= k\left(\frac{k+\frac{5}{4}}{k}\right)^2.\end{aligned}$$

Since

$$\frac{k+\frac{5}{4}}{k} < \frac{k+3}{k+1}$$

if $k \geq 4$ (in fact, even if $k \geq 2$), the result follows in this case.

Thus, we need only consider the case where $k = 3$. But this can be verified by direct computation. □

2.4 Bounding the graph of $c_b(a)$ from below

The aim of this section is to prove the following lemma, which will make use of the estimates on the accumulation point from Section 2.3.

Lemma 2.8 *Assume that* $(k,l) \notin \{(1,1),(2,1),(3,2),(4,3)\}$. *Let*

$$a \le \frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k}\right)^2$$

and assume that $c_b(a)$ *is equal to the volume obstruction. Then*

$$c_b(x) \ge c_b(a) \tag{3}$$

for $x \le a$ *sufficiently close to* a, *and*

$$c_b(x) \ge \frac{x}{a} c_b(a) \tag{4}$$

for $a \le x$ *close to* a.

To motivate for the reader why this lemma will be useful for us, we remark that we will later show that the inequalities (3) and (4) can be upgraded to very useful equalities under the assumptions of the lemma, using some general properties of the function $c_b(a)$; we defer this short argument to later in the paper, focusing on the obstructive theory in this section.

The proof of Lemma 2.8 will use the theory of "ECH capacities", defined in [12]. The ECH capacities of a symplectic 4–manifold (X,ω) are a sequence of nonnegative real numbers

$$0 \le c_0(X,\omega) \le \cdots \le c_k(X,\omega) \le \cdots \le \infty$$

that are monotone with respect to symplectic embeddings. That is, if there is a symplectic embedding

$$(X_1,\omega_1) \to (X_2,\omega_2),$$

then we must have

$$c_k(X_1,\omega_1) \le c_k(X_2,\omega_2) \tag{5}$$

for all k. Hence, ECH capacities are obstructions to the existence of a symplectic embedding. ECH capacities are defined using "embedded contact homology"; for more, see for example the survey article [13].

In the case of ellipsoids, the ECH capacities have been computed in [12]. The result is that $c_k(E(a,b))$ is the $(k+1)^{\text{st}}$ smallest element in the matrix

$$(ma+nb)_{(m,n)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}}.$$

Using ECH capacities, we can now prove the following lower bound, which we will then use to prove Lemma 2.8:

Lemma 2.9 *Fix any real number $b\geq 1$. Then:*

- $c_b(a)=1$ *for* $1\leq a\leq b$.
- $c_b(a)\geq a/b$ *for* $b\leq a\leq \lfloor b\rfloor+1$.
- $c_b(a)\geq (\lfloor b\rfloor+1)/b$ *for* $\lfloor b\rfloor+1\leq a\leq (\lfloor b\rfloor+1)^2/b$.
- $c_b(a)\geq a/(\lfloor b\rfloor+1)$ *for* $(\lfloor b\rfloor+1)^2/b\leq a\leq \lfloor b\rfloor+2$.
- $c_b(a)\geq (\lfloor b\rfloor+2)/(\lfloor b\rfloor+1)$ *for* $\lfloor b\rfloor+2\leq a\leq b(\lfloor b\rfloor+2)^2/(\lfloor b\rfloor+1)^2$.
- *If b is an integer, then* $c_b(a)\geq a/(b+1)$ *for* $(b+1)^2/b\leq a\leq b+3$.
- *If b is an integer, then* $c_b(a)\geq (b+3)/(b+1)$ *for* $b+3\leq a\leq b(b+3)^2/(b+1)^2$.

Concerning the statement of the lemma, we remark, for example, that it might be the case that various bullets points are vacuously true — for example, for $b=1.2$, $(\lfloor b\rfloor+1)^2/b>\lfloor b\rfloor+2$.

Proof To prove the first bullet point, we note that $E(1,a)$ includes into $E(1,b)$ for $a\leq b$; since $c_1(E(1,a))=c_1(E(1,b))=1$, this inclusion is optimal by (5), in the sense that no larger scaling of $E(1,a)$ also embeds, so the bullet point holds.

To prove the second and third bullet points, we note first that $c_{\lfloor b\rfloor+1}(E(1,b))=b$. Then, with $b\leq a\leq\lfloor b\rfloor+1$, we have $c_{\lfloor b\rfloor+1}(E(1,a))=a$, so that the second bullet point follows by (5); and with $a\geq\lfloor b\rfloor+1$, we have $c_{\lfloor b\rfloor+1}(E(1,a))=\lfloor b\rfloor+1$, hence the third bullet point follows by (5).

To prove the fourth and fifth bullet points, we note first that $c_{\lfloor b\rfloor+2}(E(1,b))=\lfloor b\rfloor+1$. Then, if $a\geq\lfloor b\rfloor+2$, we have $c_{\lfloor b\rfloor+2}(E(1,a))=\lfloor b\rfloor+2$, hence the fifth bullet point follows by (5). If $(\lfloor b\rfloor+1)^2/b\leq a\leq\lfloor b\rfloor+2$, then, as $(\lfloor b\rfloor+1)^2/b\geq\lfloor b\rfloor+1$, we must have $c_{\lfloor b\rfloor+2}(E(1,a))=a$, hence the fourth bullet point follows by (5).

To prove the sixth and seventh bullet points, we note that if b is an integer, then $c_{b+3}(E(1,b))=b+1$. Then, if $b+2\leq a\leq b+3$, we have $c_{b+3}(E(1,a))=a$, hence the sixth bullet point follows by (5), since, for a in the domain of the sixth bullet point, $b+2 \le a \le b+3$. If $a \ge b+3$, we have $c_{b+3}(E(1,a)) = b+3$, hence the seventh bullet point follows by (5). □

We can now prove the main result of this section.

Proof of Lemma 2.8 Recall that the volume obstruction is given by $\sqrt{a/b}$. We can find the point a_i on the domain of the i^{th} bullet point of Lemma 2.9, where the volume obstruction agrees with the lower bound given by each bullet point by setting this lower bound equal to the volume obstruction, and solving for the point a_i. Doing this gives

$$a_1 = a_2 = b, \qquad a_3 = \frac{(\lfloor b\rfloor+1)^2}{b}, \quad a_4 = \frac{(\lfloor b\rfloor+1)^2}{b},$$
$$a_5 = b\left(\frac{\lfloor b\rfloor+2}{\lfloor b\rfloor+1}\right)^2, \quad a_6 = \frac{(b+1)^2}{b}, \qquad a_7 = b\left(\frac{b+3}{b+1}\right)^2.$$

We also compute that, on each of these intervals, away from a_i the lower bound given by Lemma 2.9 is strictly larger than the volume bound.

We next observe that, for $a_1, \dots, a_4, a_6$, it follows from Lemma 2.9 that the bounds (3) and (4) required by Lemma 2.8 hold. More precisely, (3) and (4) for a_1 and a_2 follow from the first two bullet points; (3) and (4) for a_3 and a_4 follow from the third and fourth; for a_6, this follows from the third and sixth.

Now assume first that (k,l) is such that the assumptions of Lemma 2.3 hold. Then the accumulation point a_0 is bounded by $k/l + 1/l + 1$, which in turn is less than or equal to $\lfloor k/l\rfloor + 2$. Thus, if $a \le a_0$, then a is in the domain of one of the first four bullet points of Lemma 2.9. Thus, if $c_b(a) = \sqrt{a/b}$, then $a \in \{a_1, \dots, a_4\}$, since, at any other point in the i^{th} interval, c_b is bounded from below by a function that is strictly larger than the volume bound. So the conclusions of Lemma 2.3 hold in this case by the analysis in the previous paragraph.

Next, assume that $(k,l) \in \{(5,2),(5,3),(5,4)\}$. Then, by Claim 2.6, a_0 is strictly bounded from above by $b(\lfloor b\rfloor+2)^2/(\lfloor b\rfloor+1)^2$ for $b = k/l$. Thus, if $a \le a_0$, then a is in the domain of one of the first five bullet points, but is not the right endpoint point of the fifth and in particular must be strictly smaller than a_5; hence, if $c_b(a) = \sqrt{a/b}$, then, as in the previous paragraph, $a \in \{a_1, \dots, a_4\}$, so that the conclusions of Lemma 2.3 hold in this case as well.

Finally, assume that $l = 1$ and $k \ge 3$. Then $b = k/l$ is an integer. By Lemma 2.7, the accumulation point a_0 is strictly bounded from above by $b(b+3)^2/(b+1)^2$. Hence, if $a \le a_0$, then a is in the domain of either the first three bullet points, or the sixth or seventh; moreover, it is not the right endpoint of the seventh and in particular must be strictly smaller than a_7. It follows that if $c_b(a) = \sqrt{a/b}$, then $a \in \{a_1, \dots, a_3, a_6\}$, and so, just as in the previous paragraphs, the conclusions of Lemma 2.3 hold as well. □

2.5 The $E\left(1, \frac{4}{3}\right)$ case

To deal with the case where $b = \frac{4}{3}$, the staircase obstruction from Theorem 1.2 is not sufficient, since in this case $a_0 = 3$ and we will see in Proposition 2.10 below that $c_{4/3}(3) = \sqrt{a/\frac{4}{3}} = \frac{3}{2}$. We need to prove the following:

Proposition 2.10 *For ε sufficiently small,*

$$c_{4/3}(a) = \tfrac{1}{4}(a+3)$$

if $3 \le a \le 3+\varepsilon$, and

$$c_{4/3}(a) = \tfrac{3}{2},$$

if $3-\varepsilon \le a \le 3$.

The proof of Proposition 2.10 is rather delicate, and will be the topic of this section. The result itself is loosely analogous to the difficult [16, Theorem 1.1.2.ii], although we use a different method in our proof; it is not clear how to generalize the method in [16] to our situation, since it exploits the convergents of τ^4, while in our case the analogous number, $a_0 = 3$, does not have any interesting convergents at all. The main challenging fact that we need to prove is the following:

Proposition 2.11 *We have*

$$c_{4/3}(a) \le \tfrac{1}{4}(a+3) \tag{6}$$

for $a \ge 3$.

Proof To prove (6), we want to show that there exists a symplectic embedding

$$E(1,a) \to \tfrac{1}{4}(a+3)E\left(1, \tfrac{4}{3}\right).$$

By rescaling, it is equivalent to find an embedding

$$E\left(\frac{12}{a+3}, \frac{12a}{a+3}\right) \to E(3,4). \tag{7}$$

Since $c_{4/3}$ is continuous in a, we can in addition assume that a is irrational, which is convenient for some of the arguments below.

To find this embedding, we use in general terms a technique first introduced in [10; 12; 16; 14].

Namely, McDuff showed in [14] that the obstruction coming from ECH capacities is in fact sharp for four-dimensional ellipsoid embeddings. In other words, the existence of embeddings like (7) can be approached through purely combinatorial considerations. In principle, since there are infinitely many ECH capacities c_k, this requires checking infinitely many potential obstructions. However, in [10], this was rephrased in rational cases in terms of "Ehrhart functions", defined below. Ehrhart functions are a classical object of study in enumerative combinatorics which are often amendable to computations.

More precisely, we can apply [10, Lemma 5.2] to conclude that an embedding (7) exists if and only if

$$L_{\mathcal{T}_{(a+3)/12,(a+3)/12a}}(t) \geq L_{\mathcal{T}_{1/3,1/4}}(t) \tag{8}$$

for all positive integers t. Here, $\mathcal{T}_{u,v}$ denotes[2] the triangle with vertices $(u,0)$ and $(0,v)$, and L denotes its *Ehrhart function*

$$L_{\mathcal{T}_{u,v}}(t) = \#\{\mathbb{Z}^2 \cap \mathcal{T}_{tu,tv}\}.$$

Our method is now loosely inspired by the proof in [8, Lemma 3.2.3] (see also [8, Remark 3.2.6]), although there is a new idea needed here, which we will comment on below.

As in the proof in [8, Lemma 3.2.3], we will first observe that (8) holds when $a = 3$. In fact, strict inequality holds in (8), as we will see below. The idea is now to vary a and see how $L_{\mathcal{T}_{(a+3)/12,(a+3)/12a}}(t)$ changes. As in [8, Lemma 3.2.3], we do this by decomposing the region between $\mathcal{T}_{(a+3)/12,(a+3)/12a}$ for some a and $\mathcal{T}_{(3+3)/12,(3+3)/(12\cdot 3)}$ into two regions R_U and R_D, and comparing the number of lattice points U and D.

More precisely, for a positive integer t, we let R_U be the region bounded by the y–axis, the line L_1 given by the equation $(12/(a+3))x + (12a/(a+3))y = t$ and the line L_2 given by the equation $2x + 6y = t$. Let R_D be the region bounded by these two lines and the x–axis. Let U denote the number of lattice points in R_U and D the number of lattice points in R_D. We note that the lines L_1 and L_2 intersect at the point $\left(\frac{1}{4}t, \frac{1}{12}t\right)$. We have illustrated the setup in Figure 1.

[2] The paper [10] actually uses the convention that $\mathcal{T}_{u,v}$ denotes the triangle with vertices $(0,u)$, $(v,0)$ and $(0,0)$, but this triangle has the same number of lattice points as the triangle defined using the conventions in this paper.

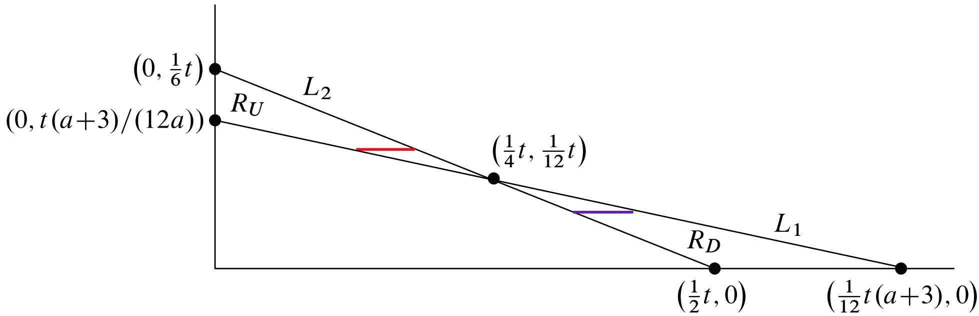

Figure 1: The regions R_U and R_D that we want to compare. The lines L_1 and L_2 are also labeled for the convenience of the reader. The two line segments correspond to the kind of horizontal slices that we make in order to compare lattice point counts.

We now have the following key lemma:

Lemma 2.12

- *$U \le D$.*
- *If t is congruent to 4 modulo 12, then $U \le D - 1$.*

Up to this point, our method in this section has been mostly parallel to the method in [8] described above. However, at this point, the ideas in [8] no longer seem to work, and something new is needed. The new technique we introduce here is to compare the lattice points in R_U and R_D by comparing the number of lattice points on horizontal slices at integer height; see Figure 1. It turns out that we can get the inequality we need by establishing the analogous inequality for each slice individually, which is a priori considerably stronger than what is required.

The details are as follows:

Proof We begin with the proof of the first bullet point.

Let $\frac{1}{6}t \ge y_0 \ge \frac{1}{12}t$ be an integer. We define

$$y_1 = \lfloor \tfrac{1}{6}t \rfloor - y_0.$$

Then $0 \le y_1 \le \frac{1}{12}t$. We will show that, for each y_0, the number of lattice points in R_U with y–coordinate y_0 is no more than the number of lattice points in R_D with y–coordinate y_1, which will imply the first bullet point of the lemma.

In other words, if we define

$$x_1 := \tfrac{1}{12}(t(a+3) - 12ay_0), \quad x_2 := \tfrac{1}{2}(t - 6y_0)$$

and

$$x_3 := \tfrac{1}{2}(t - 6y_1), \quad x_4 := \tfrac{1}{12}(t(a+3) - 12ay_1),$$

then we need to show that

$$\lfloor x_2 \rfloor - \lceil \max(0, x_1) \rceil + 1 \le \lfloor x_4 \rfloor - \lceil x_3 \rceil + 1. \tag{9}$$

We now explain why (9) holds.

Our argument will be as follows. Assume that x_3 is not an integer; note that x_1 is never an integer, since a is irrational. Then we will show below that

$$\lfloor x_2 \rfloor - \lfloor x_1 \rfloor \le \lfloor x_4 \rfloor - \lfloor x_3 \rfloor. \tag{10}$$

Next, in the case where x_3 is an integer, we will show that

$$\lfloor x_2 \rfloor - \lfloor x_1 \rfloor \le \lfloor x_4 \rfloor - \lfloor x_3 \rfloor + 1. \tag{11}$$

The equations (10) and (11) will imply (9), since $\lfloor x_2 \rfloor - \lceil \max(0, x_1) \rceil + 1 \le \lfloor x_2 \rfloor - \lfloor x_1 \rfloor$.

We now explain why (10) and (11) hold. We know that

$$\lfloor x_2 \rfloor - \lfloor x_1 \rfloor = \tfrac{1}{2}t - \left\{\tfrac{1}{2}t\right\} - 3y_0 - \tfrac{1}{12}t(a+3) + ay_0 + \left\{\tfrac{1}{12}t(a+3) - ay_0\right\}.$$

Here, $\{\cdot\}$ denotes the fractional part function, defined by $\{z\} = z - \lfloor z \rfloor$. We also know that

$$\lfloor x_4 \rfloor - \lfloor x_3 \rfloor = \tfrac{1}{12}t(a+3) - ay_1 - \left\{\tfrac{1}{12}t(a+3) - ay_1\right\} - \tfrac{1}{2}t + \left\{\tfrac{1}{2}t\right\} + 3y_1.$$

We first prove (10) in the case where x_3 is not an integer, which is the heart of the argument.

To do this, we want to show, in view of combining the previous two equations, that

$$t - 2\left\{\tfrac{1}{2}t\right\} - 3(y_0+y_1) - \tfrac{1}{6}t(a+3) + a(y_0+y_1) + \left\{\tfrac{1}{12}t(a+3) - ay_1\right\} + \left\{\tfrac{1}{12}t(a+3) - ay_0\right\} \le 0.$$

Substituting for y_1, we have that the above expression is equal to

$$t - 2\left\{\tfrac{1}{2}t\right\} - 3\left\lfloor \tfrac{1}{6}t \right\rfloor - \tfrac{1}{6}t(a+3) + a\left\lfloor \tfrac{1}{6}t \right\rfloor + \delta,$$

where

$$\delta := \left\{\tfrac{1}{12}t(a+3) - ay_0\right\} + \left\{\tfrac{1}{12}t(a+3) + ay_0 - a\left\lfloor \tfrac{1}{6}t \right\rfloor\right\}.$$

So, collecting $\left\lfloor \tfrac{1}{6}t \right\rfloor$ terms, we want to show that

$$t - 2\left\{\tfrac{1}{2}t\right\} + (a-3)\left\lfloor \tfrac{1}{6}t \right\rfloor - \tfrac{1}{6}t(a+3) + \delta \le 0.$$

Equivalently, we want to show that

$$-2\left\{\tfrac{1}{2}t\right\}-(a-3)\left\{\tfrac{1}{6}t\right\}+\delta\leq 0. \tag{12}$$

Since, for any two numbers m and n, we have[3] $\{m\}+\{n\}\leq\{m+n\}+1$, we know that

$$\delta\leq\left\{\tfrac{1}{2}t+a\left\{\tfrac{1}{6}t\right\}\right\}+1. \tag{13}$$

The terms $\left\{\frac{1}{2}t\right\}$, $\left\{\frac{1}{6}t\right\}$ and $\left\{\frac{1}{2}t+a\left\{\frac{1}{6}t\right\}\right\}$ only depend on the equivalence class of t, modulo 6. So, to bound the left-hand side of (12) using the above bound for δ, we can assume $t\in\{0,\dots,5\}$. With this additional assumption, we then have

$$\left\{\tfrac{1}{2}t+a\left\{\tfrac{1}{6}t\right\}\right\}=\left\{\tfrac{1}{2}t+a\tfrac{1}{6}t\right\}=\left\{\tfrac{1}{6}(a-3)t\right\}\leq\tfrac{1}{6}(a-3)t=(a-3)\left\{\tfrac{1}{6}t\right\}. \tag{14}$$

Combining (13) and (14), we thus have that

$$-2\left\{\tfrac{1}{2}t\right\}-(a-3)\left\{\tfrac{1}{6}t\right\}+\delta\leq-2\left\{\tfrac{1}{2}t\right\}+1=0, \tag{15}$$

where, for the very last equality, we have used the fact that x_3 is not an integer, so that t is odd. This proves (12), and hence (10).

When x_3 is an integer, all of the proof of (10) holds, except that, in the very last line, $\left\{\frac{1}{2}t\right\}=0$, so that the very last equation (15) must be replaced by the bound

$$-2\left\{\tfrac{1}{2}t\right\}-(a-3)\left\{\tfrac{1}{6}t\right\}+\delta\leq 1,$$

hence the weaker bound (11).

We now explain the proof of the second bullet point.

The argument for the first bullet point still holds to imply that $U\leq D$. To get the sharper bound, we show that, under the assumption that t is congruent to 4 modulo 12, as y_0 above ranges over all integers between $\frac{1}{6}t$ and $\frac{1}{12}t$, the corresponding y_1 is never $y'=\left\lfloor\frac{1}{12}t\right\rfloor$.

Indeed, the y_1 corresponding to y_0 is maximized for $y_0=\left\lceil\frac{1}{12}t\right\rceil$, so, in this case, $y_1=\left\lfloor\frac{1}{6}t\right\rfloor-\left\lceil\frac{1}{12}t\right\rceil$. Now,

$$\left\lfloor\tfrac{1}{6}t\right\rfloor-\left\lceil\tfrac{1}{12}t\right\rceil=\left\lfloor\tfrac{1}{12}t\right\rfloor-1,$$

since t is congruent to 4 modulo 12, which is strictly less than y'.

[3]Indeed, the equation is invariant under adding integers to m or n, so we can assume $0\leq m,n<1$, in which case it is immediate.

Thus, since $\left(\frac{1}{2}t-6y', y'\right)$ is a lattice point in R_D, not accounted for by the counts in the proof of the first bullet point, the sharper estimate asserted by the second bullet points holds. □

We now explain how to use the lemma to prove the proposition. Continue to assume as above that a is irrational.

We first observe that

$$L_{\mathcal{T}_{(a+3)/12,(a+3)/(12a)}}(t) = L_{\mathcal{T}_{1/2,1/6}}(t) + D - U - d, \tag{16}$$

where d is the number of lattice points on the left boundary of D, not including the possible lattice point $\left(\frac{1}{4}t, \frac{1}{12}t\right)$ defined above.

We can solve for d explicitly. Namely, assume that there is a lattice point (m,n) satisfying

$$2m+6n = t.$$

Then it follows that t must be an even integer. Conversely, assume that t is an even integer, and (x, y) is on the line L_2. Then we have

$$x = \tfrac{1}{2}(t-6y).$$

In particular, for any integer $y < \frac{1}{12}t$ such that (x, y) is on the line L_2, x must be an integer as well. It follows that

$$d = \left\lceil \tfrac{1}{12}t \right\rceil \tag{17}$$

when t is even; if t is odd then we have $d = 0$.

We know from Lemma 2.12 that $D \geq U$; however, the $-d$ term is not in general nonnegative, so, to prove (8), we need to compute the difference

$$L_{\mathcal{T}_{1/2,1/6}}(t) - L_{\mathcal{T}_{1/3,1/4}}(t).$$

Each of the two terms in the above expression are Ehrhart functions of rational triangles, so they are readily computed. In particular, using the formulas in [1, Theorem 2.10 and Exercise 2.34], each is a periodic polynomial of degree 2, with leading order term $\frac{1}{24}t^2$. The linear term for $L_{\mathcal{T}_{1/3,1/4}}(t)$, by [1, Theorem 2.10] is $\frac{1}{3}t$. The linear term for $L_{\mathcal{T}_{1/2,1/6}}(t)$ is $\frac{5}{12}t$ when t is even, and $\frac{1}{3}t$ when t is odd, by [1, Exercise 2.34].

To compute the constant terms, we use the fact that the period of $L_{\mathcal{T}_{1/2,1/6}}(t)$ divides 6, and the period of $L_{\mathcal{T}_{1/3,1/4}}(t)$ divides 12; indeed, the basic structure theorem for Ehrhart functions (see for example [1, Theorem 3.23]) states that the period for a rational convex polytope divides the least common multiple of the denominators of the vertices.

More precisely, we first compute the constant terms for $L_{\mathcal{T}_{1/2,1/6}}(t)$. We begin by computing

$$\begin{aligned} L_{\mathcal{T}_{1/2,1/6}}(0) &= L_{\mathcal{T}_{1/2,1/6}}(1) = 1, \\ L_{\mathcal{T}_{1/2,1/6}}(2) &= L_{\mathcal{T}_{1/2,1/6}}(3) = 2, \\ L_{\mathcal{T}_{1/2,1/6}}(4) &= L_{\mathcal{T}_{1/2,1/6}}(5) = 3. \end{aligned}$$

Now, since $L_{\mathcal{T}_{1/2,1/6}}(t)$ is a periodic polynomial, with period dividing 6, we can define $C_0, \dots, C_5$ to be the constant terms for this periodic polynomial, ie C_i is the constant term when t is congruent to i modulo 6. We can then compute the constant terms by using the computations above, namely

$$\begin{aligned} C_0 &= 1 - \tfrac{1}{24}(0)^2 - \tfrac{5}{12}(0) = 1, & C_3 &= 2 - \tfrac{1}{24}(3)^2 - \tfrac{1}{3}(3) = \tfrac{5}{8}, \\ C_1 &= 1 - \tfrac{1}{24}(1)^2 - \tfrac{1}{3}(1) = \tfrac{5}{8}, & C_4 &= 3 - \tfrac{1}{24}(4)^2 - \tfrac{5}{12}(4) = \tfrac{2}{3}, \\ C_2 &= 2 - \tfrac{1}{24}(2)^2 - \tfrac{5}{12}(2) = 1, & C_5 &= 3 - \tfrac{1}{24}(5)^2 - \tfrac{1}{3}(5) = \tfrac{7}{24}. \end{aligned}$$

We can compute the constant terms $C'_0, \dots, C'_{11}$ by the same method. We omit the details, which are analogous to above, for brevity, only giving the result:

$$\begin{aligned} &C'_0 = 1, \quad C'_1 = \tfrac{5}{8}, \quad C'_2 = \tfrac{1}{6}, \quad C'_3 = \tfrac{5}{8}, \quad C'_4 = 1, \quad C'_5 = \tfrac{7}{24}, \\ &C'_6 = \tfrac{1}{2}, \quad C'_7 = \tfrac{5}{8}, \quad C'_8 = \tfrac{2}{3}, \quad C'_9 = \tfrac{5}{8}, \quad C'_{10} = \tfrac{1}{2}, \quad C'_{11} = \tfrac{7}{24}. \end{aligned}$$

Having computed both Ehrhart functions explicitly, and applying the formula (17) for d, we now see that

$$L_{\mathcal{T}_{1/2,1/6}}(t) - L_{\mathcal{T}_{1/3,1/4}}(t) = d,$$

except when t is congruent to 4 mod 12, in which case the difference in the Ehrhart functions is $d-1$. The proposition now follows from (16), in combination with Lemma 2.12. □

We can now prove Proposition 2.10:

Proof We just showed that $c_{4/3}(a) \le \frac{1}{4}(a+3)$ for $a \ge 3$. In particular, as an immediate consequence, $c_{4/3}(3) = \frac{3}{2}$, since a symplectic embedding must be volume-preserving, and then

$$c_{4/3}(a) \le \tfrac{3}{2} \tag{18}$$

for $a \le 3$, since $E(1,a) \subset E(1,3)$ for a in this range.

To find the lower bounds needed to prove the proposition, we again use the theory of ECH capacities.

More precisely, we first compute

$$c_{10}\big(E\big(1,\tfrac{4}{3}\big)\big) = 4, \quad c_{10}(E(1,a)) = a+3$$

for $3 \le a \le 4$. Hence, $c_{4/3}(a) \ge \frac{1}{4}(a+3)$ for $3 \le a \le 4$, by (5). Combining this with the matching upper bound (6) then implies that $c_{4/3}(a) = \frac{1}{4}(a+3)$ for a in this range.

We next compute

$$c_2\big(E\big(1,\tfrac{4}{3}\big)\big) = \tfrac{4}{3}, \quad c_2(E(1,a)) = 2$$

for $a \ge 2$. Hence, $c_{4/3}(a) \ge \frac{3}{2}$ for $2 \le a \le 3$, by (5). Combining this with the matching upper bound (18) then implies that $c_{4/3}(a) = \frac{3}{2}$ for a in this range. □

Remark 2.13 The above method is useful for other staircase analysis. In particular, in an update of [9], we plan to use it to show that there is no infinite staircase for three particular reflexive polytopes for which the staircase obstruction does not vanish, but there is still no infinite staircase. The case of reflexive polytopes is important to this work in view of Conjecture 1.4.

2.6 Completing the proof of Theorem 1.2

We can now complete the proof of our main theorem:

Proof of Theorem 1.2 As explained in Section 2.1, it follows from known results that $c_b(a) = \sqrt{a/b}$ for a sufficiently large with respect to b; it also follows from known results that the function $c_b(a)$ is piecewise linear away from the limit of distinct singular points. So we just have to analyze the case of infinitely many distinct singular points.

Let $b = k/l$, where k and l are relatively prime, and recall the number

$$a_0 = \frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k}\right)^2$$

from Lemma 2.2.

Assume that there are infinitely many singular points s_i. Then, by Lemma 2.2, the s_i must accumulate at a_0, and $c_b(a_0)$ must equal the volume obstruction. We now argue that there is a contradiction if $b \notin \{1, 2, \frac{3}{2}\}$.

Namely, if $b \notin \{1, 2, \frac{3}{2}, \frac{4}{3}\}$, then we know from Lemma 2.8 that (3) and (4) hold at a_0. We now claim that this implies that the graph of $c_b(a_0)$ would locally be given by these lines near a_0 — which is an absurdity, since a_0 was the limit of distinct singular points.

To see why this final claim is true, we need the following two properties for the function $c_b(a)$:

- **Monotonicity** $c_b(x_1) \le c_b(x_2)$ if $x_1 \le x_2$.
- **Subscaling** $c_b(\ell x_1) \le \ell c_b(x_1)$ for any $\ell \ge 1$.

The first bullet point is immediate, since $E(1, x_1) \subset E(1, x_2)$ if $x_1 \le x_2$. The second follows by a short scaling argument; see for example [9, Proposition 2.1] for the details.

With these two properties, we can now verify the final claim — in view of the lower bounds (3) and (4), monotonicity would then imply that (3) is an equality for $x \le a_0$ close to a_0, and subscaling would then imply that (4) is an equality for $x \ge a_0$ close to a_0.

If $b = \frac{4}{3}$, then it follows from Proposition 2.10 that $c_b(a_0)$ has a unique singular point near a_0, namely a_0 itself. Thus, in this case a_0 also can not be the limit of distinct singular points. □

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Received: 23 June 2020 Revised: 7 January 2021

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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nonprofit scientific publishing

Algebraic & Geometric Topology

Volume 22 Issue 5 (pages 2007–2532) 2022