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Global F-splitting ratio of modules

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Techniques are developed to extend the notions of F-splitting ratios to modules over rings of prime characteristic, which are not assumed to be local. We first develop the local theory for F-splitting ratio of modules over local rings, and then extend it to the global setting. We also prove that strong F-regularity of a pair $(R, \mathscr{D})$, where $\mathscr{D}$ is a Cartier algebra, is equivalent to the positivity of the global F-signature $s(R, \mathscr{D})$ of the pair. This extends a result previously proved by these authors, by removing an extra assumption on the Cartier algebra.

1. Introduction

This article is focused on extending the notion of *F-splitting ratio* of a local ring in two directions: from the local to the global setting, and from the ring to all finitely

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generated R-modules. The F-splitting ratio of a local ring, denoted $r_F(R)$, is a measure of the asymptotic free-rank of the modules F^e_*R. More specifically, if $(R, \mathfrak{m}, k)$ is a local ring with perfect residue field, for each $e \in \mathbb{Z}_{>0}$ we write $F^e_*R = R^{\oplus a_e(R)} \bigoplus M_e$, where M_e has no free summands. It is easy to see that, under these assumptions, the integers $a_e(R)$ do not depend on the chosen direct sum decomposition. The *F-splitting ratio of* R is defined as

$$r_F(R) = \lim_{e\to\infty} \frac{a_e(R)}{p^{e\,\mathrm{sdim}(R)}},$$

where $\mathrm{sdim}(R)$ is the splitting dimension of R (see [1], and Section 2). The F-splitting ratio is always positive for F-pure rings. Its existence as a limit was first proved by Tucker for local rings [20], while its positivity for F-pure local rings was established in [2].

Observe that $r_F(R)$ is defined similarly to the F-signature of R; in fact, the two definitions coincide if and only if $\mathrm{sdim}(R) = \dim(R)$. However, $r_F(R)$ is always positive for an F-pure ring, while $\mathrm{s}(R)$ is non-zero only for strongly F-regular rings.

The splitting dimension and splitting numbers can naturally be reinterpreted for a finitely generated module M over an F-finite ring which is not necessarily local (see Section 4). We call our generalization of the splitting dimension the *splitting rate of* M, and we denote it by $\mathrm{sr}(M)$. When $\mathrm{sr}(M) \geqslant 0$, the F-splitting ratio of M is defined as

$$r_F(M) = \lim_{e\to\infty} \frac{a_e(M)}{p^{e\,\mathrm{sr}(M)}},$$

provided the limit exists.

Our first main result provides strong uniform bounds local splitting numbers of a module. An immediate consequence of this is the existence of the F-splitting ratio of a module in the local case.

Theorem A *(see Theorem 4.6). Let R be an F-finite ring, and M be a finitely generated R-module. Then the local F-splitting ratios $r_F(M_Q)$ exist for all $Q \in \mathrm{Spec}(R)$. Furthermore, there exists a constant C such that for all $Q \in \mathrm{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$,*

$$\left| a_e(M_Q) - p^{e\,\mathrm{sr}(M_Q)} r_F(M_Q) \right| \leqslant C p^{e(\mathrm{sr}(M_Q)-1)}.$$

In particular, if $(R, \mathfrak{m}, k)$ is local and M is a finitely generated R-module, then $r_F(M)$ exists as a limit.

We provide an example to show that, in general, lower semi-continuity may not hold on the whole spectrum of a ring, even in the case when the ring is a domain (Example 5.4). This is in contrast with the behavior of several other invariants: Hilbert-Kunz multiplicity and F-signature [17,13], Frobenius Betti numbers and Frobenius Euler characteristics [4].

Using Theorem A and some partial lower semi-continuity results for the F-splitting ratio (see Theorem 5.1), we prove the existence of the F-splitting ratio of a module over a ring which is not necessarily local. We also relate both the splitting rate and the F-splitting ratio of a module to the respective invariants in the localizations at prime ideals. This last fact allows us to relate the positivity of $r_F(R)$ to the F-purity of R.

Theorem B *(see Theorem 5.6). Let R be an F-finite domain of prime characteristic $p > 0$. Then*

(1) The limit $r_F(R)$ exists.
(2) We have equalities

$$\mathrm{sr}(R) = \min\{\mathrm{sr}(R_P) \mid P \in \mathrm{Spec}(R)\}$$

and

$$r_F(R) = \min\{r_F(R_P) \mid \mathrm{sr}(R) = \mathrm{sr}(R_P)\}.$$

(3) $r_F(R) > 0$ if and only if R is F-pure.

Theorem B is here stated only for global F-splitting ratio of the ring R, under the additional assumption that it is a domain. We refer the reader to 5 for more general results on finitely generated R-modules. We point out that (3) is an important property of F-splitting ratios that mimics an important property of F-signature; $\mathrm{s}(R) > 0$ if and only if R is strongly F-regular. Item (3) follows by item (2) and [2, Corollary 4.3].

Among other properties of F-splitting ratios, we prove that if R is a positively graded algebra over a local ring $(R_0, \mathfrak{m}_0)$, then $\mathrm{sr}(R) = \mathrm{sr}(R_\mathfrak{m})$ and $r_F(R) = r_F(R_\mathfrak{m})$, where $\mathfrak{m} = \mathfrak{m}_0 + R_{>0}$ (see Proposition 5.7). This result gives an analogous statement for the global F-signature (see Corollary 5.8).

In the final section of this article, we positively answer [5, Question 4.24]. In the local case, it was proved in [2] that the F-signature of a Cartier algebra $\mathscr{D}$ on R is positive if and only if the pair $(R, \mathscr{D})$ is strongly F-regular. These authors were able to recover the same result in the global setting, provided the Cartier algebra $\mathscr{D}$ satisfies certain additional assumptions [5, Theorem 2.24]. We are able to remove these extra conditions:

Theorem C. *Let R be an F-finite domain, and $\mathscr{D}$ be a Cartier algebra on R. Then $(R, \mathscr{D})$ is strongly F-regular if and only if* $\mathrm{s}(R, \mathscr{D}) > 0$.

2. Background on F-splitting ratio of local rings

Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Aberbach and Enescu introduced the concepts of splitting prime and F-splitting ratio of a local F-finite ring in [1]. Assume that R is F-pure, that is, the Frobenius map is pure as a map of rings. In our assumptions, this is the same as requiring that R is F-split [9, Corollary 5.3]. For a finitely generated R-module M, we let $\mathrm{frk}_R(M)$ be the maximal rank of a free summand of M. Equivalently, $\mathrm{frk}_R(M)$ is the maximal rank of a free module G for which there is a surjection $M \to G \to 0$. For all $e \in \mathbb{Z}_{>0}$, we let $a_e(R) = \mathrm{frk}_R(F^e_* R)$ be the *e-th splitting number of R*. Let $\alpha(\mathfrak{m}) = \log_p[F_* k : k]$. The *splitting dimension of R* is

$$\mathrm{sdim}(R) := \sup \left\{ \ell \in \mathbb{Z}_{\geqslant 0} \;\middle|\; \liminf_{e\to\infty} \frac{a_e(R)}{p^{e(\ell+\alpha(\mathfrak{m}))}} > 0 \right\}.$$

The *F-splitting ratio of R* is defined to be the limit

$$r_F(R) := \lim_{e\to\infty} \frac{a_e(R)}{p^{e(\mathrm{sdim}(R)+\alpha(\mathfrak{m}))}},$$

which always exists [20, Theorem 4.9] and is always positive for F-pure rings by work of Blickle, Schwede, and Tucker [2, Corollary 4.3].

Remark 2.1. Observe that, when $\mathrm{sdim}(R) = \dim(R)$, the F-splitting ratio is equal to the F-signature of R.

Continue to let $(R, \mathfrak{m}, k)$ denote an F-finite and F-pure local ring of prime characteristic $p > 0$. For each $e \in \mathbb{Z}_{>0}$ let $I_e = \{r \in R \mid R \xrightarrow{\cdot F^e_*(r)} F^e_* R \text{ is not pure}\}$ be the e-th splitting ideal of R. Aberbach and Enescu show in [1] that $\mathcal{P} := \bigcap_{e\in\mathbb{Z}_{>0}} I_e$ is a prime ideal of R and $R/\mathcal{P}$ is a strongly F-regular local ring. The ideal $\mathcal{P}$ is called the splitting prime of the local ring R. Moreover, it is shown in [2] that the splitting dimension of R is the Krull dimension of the local ring $R/\mathcal{P}$.

We recall that a graded $\mathbb{F}_p$-subalgebra $\mathscr{D}$ of $\bigoplus_{e\in\mathbb{Z}_{\geqslant 0}} \mathrm{Hom}_R(F^e_* R, R)$, with $\mathscr{D}_0 = \mathrm{Hom}_R(R, R)$ and multiplication $\varphi \bullet \psi = \varphi \circ F^e_* \psi \in \mathscr{D}_{e+e'}$ for all $\varphi \in \mathscr{D}_e$ and $\psi \in \mathscr{D}_{e'}$, is called a Cartier algebra. If $\mathscr{D}_e = \mathrm{Hom}_R(F^e_* R, R)$ for all e, we refer to $\mathscr{D}$ as the full Cartier algebra on R. See [2] for more details on Cartier algebras.

If $I \subseteq R$ is an ideal, then we let $\mathscr{D}_{R/I}$ be the Cartier algebra on R/I whose e-th graded component is denoted by $\mathscr{D}_{R/I,e}$ and consists of R/I-linear maps $\varphi : F^e_*(R/I) \to R/I$ which can be factored through an R-linear map $\phi : F^e_* R \to R$. That is, there exists commutative diagram of R-modules of the form

$$\begin{tikzcd} F^e_*(R/I) \arrow[r, "\varphi"] & R/I \\ F^e_* R \arrow[u] \arrow[r, dashed, "\exists \phi"] & R \arrow[u] \end{tikzcd}$$

Observe that the construction of this Cartier algebra did not require R to be local. Moreover, if P is a prime ideal of R which contains I, then the localized Cartier algebra $(\mathscr{D}_{R/I})_P$ agrees with $\mathscr{D}_{R_P/IR_P}$.

We now recall the definition of splitting numbers of a pair $(R, \mathscr{D})$ in the local case. Let $(R, \mathfrak{m}, k)$ be a local F-finite and F-pure ring of prime characteristic $p > 0$, and $\mathscr{D}$ be a

Cartier algebra. We let $a_e(R,\mathscr{D})$ be the largest rank of a free $\mathscr{D}$-summand of F^e_*R. More explicitly, we look at the largest rank of a free R-module $G \cong \bigoplus R$ for which there is a surjection $F^e_*R \xrightarrow{\varphi} G \to 0$, with φ that is a direct sum of elements in $\mathscr{D}_e$ when viewed as an element of $\operatorname{Hom}_R(F^e_*R, G) \cong \bigoplus \operatorname{Hom}_R(F^e_*R, R)$. It was proved in [2] that, if $\mathscr{D}$ is the full Cartier algebra on R, and $\mathcal{P}$ is the splitting prime of R, one has

$$a_e(R) = a_e(R/\mathcal{P}, \mathscr{D}_{R/\mathcal{P}}).$$

We record the following theorem of Blickle, Schwede, and Tucker for future reference.

Theorem 2.2 *([2]). Let $(R,\mathfrak{m},k)$ be a local F-finite and F-pure ring of prime characteristic $p>0$. Let $\mathscr{D}$ be the full Cartier algebra on R, and $\mathcal{P}$ be the splitting prime of R. Then $a_e(R) = a_e(R/\mathcal{P}, \mathscr{D}_{R/\mathcal{P}})$ for all $e \in \mathbb{Z}_{>0}$, and thus $r_F(R) = \mathrm{s}(R/\mathcal{P}, \mathscr{D}_{R/\mathcal{P}}) = r_F(R/\mathcal{P}, \mathscr{D}_{R/\mathcal{P}})$. In particular, the F-splitting ratio of R is strictly positive.*

3. Uniform bounds for splitting numbers

With the goal in mind of extending the theory of F-splitting ratios to modules over rings which are not necessarily local, we must first discuss and understand properties of centers of F-purity, i.e., compatibly split subvarieties, whose properties are developed by Schwede in [15] and [16], and by Kumar and Mehta in [10].

Let R be an F-finite ring of prime characteristic $p>0$, not necessarily local, and M be a finitely generated R-module. For $e \in \mathbb{Z}_{>0}$ we let $a_e(M) = \operatorname{frk}_R(F^e_*M)$, and assume that $a_e(M)>0$ for some e. Under these assumptions, we make the following definition.

Definition 3.1. We define the *F-splitting rate of M* to be

$$\operatorname{sr}(M) := \sup\left\{ \ell \in \mathbb{Z}_{\geqslant 0} \;\middle|\; \liminf_{e\to\infty} \frac{a_e(M)}{p^{e\ell}} > 0 \right\}.$$

If $(R,\mathfrak{m},k)$ is local, then $\operatorname{sr}(R) = \operatorname{sdim}(R) + \alpha(\mathfrak{m})$. Moreover, if $\mathcal{P}$ the splitting prime of $(R,\mathfrak{m},k)$, then $\operatorname{sr}(R) = \gamma(R/\mathcal{P})$ by [1, Theorem 1.1] and [2, Corollary 4.3]. When $a_e(M)=0$ for all $e \in \mathbb{Z}_{>0}$ we set $\operatorname{sr}(M) = -1$.

Now assume that R is F-finite and F-pure, that is, $a_e(R)>0$ for some (equivalently, for all) $e \in \mathbb{Z}_{>0}$. An ideal $P \in \operatorname{Spec}(R)$ is called a center of F-purity if for every $x \in P$ and every $e \in \mathbb{Z}_{>0}$ the map

$$R_P \xrightarrow{\cdot F^e_* x} F^e_*(R_P)$$

is not pure as a map of R_P-modules. If R is local and $\mathcal{P}$ the splitting prime of R then $\mathcal{P}$ is the unique maximal center of F-purity of R, [16, Remark 4.4]. An important property enjoyed by all F-finite F-pure rings is that they only admit finitely many centers of F-purity [15, Theorem C].

Also crucial to our proof of existence of global F-splitting ratio will be that Cartier algebras of the form $\mathscr{D}_{R/I}$ described above satisfy the following technical condition.

Condition 3.2. Let R be an F-finite ring and $\mathscr{D}$ a Cartier algebra. We say that $\mathscr{D}$ satisfies condition $(*)$ if we require that for each $\varphi \in \mathscr{D}_{e+1}$ that the natural map $i \circ \varphi \in \mathscr{D}_e$ where $i : F^e_* R \to F^{e+1}_* R$ is the Frobenius.

Lemma 3.3. *Let R be an F-finite ring of prime characteristic $p > 0$ and $I \subseteq R$ be an ideal. Assume that the Cartier algebra $\mathscr{D}$ on R satisfies $(*)$. Then the Cartier algebra $\mathscr{D}_{R/I}$ on R/I satisfies condition $(*)$ as well.*

Proof. Let $\varphi \in \mathscr{D}_{R/I,e+1}$, and $i : F^e_*(R/I) \to F^{e+1}_*(R/I)$ be the Frobenius map on $F^e_*(R/I)$. We are assuming there exists a commutative diagram of R-modules of the form

$$\begin{tikzcd} F^e_*(R/I) \arrow[r, "i"] & F^{e+1}_*(R/I) \arrow[r, "\varphi"] & R/I \\ F^e_* R \arrow[u] \arrow[r, dashed] & F^{e+1}_* R \arrow[u] \arrow[r, "\phi"] & R \arrow[u] \end{tikzcd}$$

The Frobenius map on $F^e_*(R/I)$ can be lifted by the Frobenius map on $F^e_* R$. Therefore the above commutative diagram can be filled in, and it follows that $\varphi \circ i \in \mathscr{D}_{R/I,e}$. $\square$

We use the following notation: given a prime $P \in \operatorname{Spec}(R)$ we let $\alpha(P) = \log_p[F_*\kappa(P) : \kappa(P)]$ and $\gamma(R) = \max\{\alpha(P) \mid P \in \operatorname{Spec}(R)\}$. Moreover, given a pair $(R, \mathscr{D})$, $P \in \operatorname{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$, we let $a_e(R_P, \mathscr{D}_P)$ be the maximal rank of a free $\mathscr{D}_P$-summand of $F^e_*(R_P)$. In the case when $\mathscr{D} = \operatorname{Hom}_R(F^e_* R, R)$ is the full Cartier algebra, we simply write $a_e(R_P)$, which is also equal to $\operatorname{frk}_{R_P}(F^e_* R_P)$. We are almost ready to prove a uniform bound result for the localized splitting numbers $a_e(R_P)$ of an F-finite ring R, but first we recall a uniform bound found in [13]. In the proof of [13, Theorem 6.4] it is shown that if $\mathscr{D}$ is a Cartier algebra satisfying condition $(*)$ then there exists a constant C such that

$$\left| \frac{a_e(R_P, \mathscr{D}_P)}{p^{e\gamma(R_P)}} - \mathrm{s}(R_P, \mathscr{D}_P) \right| \leqslant \frac{C}{p^e}$$

for all $e \in \mathbb{N}$ and $P \in \operatorname{Spec}(R)$. We record this uniform bound for future reference.

Theorem 3.4 *([13, Proof of Theorem 6.4]). Let R be an F-finite ring, and $\mathscr{D}$ be a Cartier algebra satisfying condition $(*)$. There exists a constant C such that for all $P \in \operatorname{Spec}(R)$ and all $e \in \mathbb{Z}_{>0}$*

$$\left| a_e(R_P, \mathscr{D}_P) - p^{e\gamma(R_P)}\, \mathrm{s}(R_P, \mathscr{D}_P) \right| \leqslant C p^{e(\gamma(R_P)-1)}.$$

Using this, we obtain uniform bounds for the difference of localized splitting numbers of an F-finite F-pure ring and the corresponding F-splitting ratios.

Theorem 3.5. *Let R be an F-finite ring and F-pure ring. There is a constant $C \in \mathbb{R}$ such that for all $P \in \operatorname{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$*

$$\left| a_e(R_P) - p^{e\,\mathrm{sr}(R_P)} r_F(R_P) \right| \leqslant C p^{e(\mathrm{sr}(R_P)-1)}.$$

Proof. Let $Y = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_N\}$ be the finitely many centers of F-purity of $\operatorname{Spec}(R)$, and $\mathscr{D}$ be the full Cartier algebra on R. Observe that $\mathscr{D}$ trivially satisfies condition $(*)$. For each $\mathfrak{p}_i$, let C_i be a constant as in Theorem 3.4 for the pair $(R/\mathfrak{p}_i, \mathscr{D}_{R/\mathfrak{p}_i})$. We claim that we can choose $C = \max\{C_1, \ldots, C_N\}$. In fact, given $P \in \operatorname{Spec}(R)$, there is a unique $\mathfrak{p}_i \in Y$ such that $\mathfrak{p}_i R_P$ is the splitting prime of R_P. If we let $S = R/\mathfrak{p}_i$, by Theorem 2.2 we have that $a_e(R_P) = a_e(S_P, \mathscr{D}_{S_P})$ and $r_F(R_P) = r_F(S_P, \mathscr{D}_{S_P})$. As the Cartier algebra $\mathscr{D}_S$ still satisfies condition $(*)$, it then follows from Theorem 3.4 that

$$\begin{aligned} \left| a_e(R_P) - p^{e\,\mathrm{sr}(R_P)} r_F(R_P) \right| &= \left| a_e(S_P, \mathscr{D}_{S_P}) - p^{e\,\mathrm{sr}(R_P)} r_F(S_P, \mathscr{D}_{S_P}) \right| \\ &\leqslant C_i p^{e(\gamma(S_P)-1)} \leqslant C p^{e(\mathrm{sr}(R_P)-1)}. \qquad \square \end{aligned}$$

A consequence of Theorem 3.5 is the following:

Corollary 3.6. *Let R be an F-finite and F-pure ring of prime characteristic $p > 0$. Then the normalized splitting number functions $\tilde{a}_e : \operatorname{Spec}(R) \to \mathbb{R}$ mapping $P \mapsto a_e(R_P)/p^{e\,\mathrm{sr}(R_P)}$ converge uniformly as $e \to \infty$ to the F-splitting ratio function $r_F : \operatorname{Spec}(R) \to \mathbb{R}$ mapping $P \mapsto r_F(R_P)$.*

4. F-splitting ratio of modules over local rings

The theory of splitting ratios over a local ring developed in [1] and [2] only concerns itself with the Frobenius splitting numbers $a_e(R)$ of a local ring $(R, \mathfrak{m}, k)$. In this section we extend the local theory by studying the Frobenius splitting numbers of finitely generated modules. We first make a more general definition.

Definition 4.1. Let R be an F-finite ring of prime characteristic $p > 0$, and M be a finitely generated R-module. If $a_e(M) > 0$ for some $e \in \mathbb{Z}_{>0}$, we let

$$r_F(M) = \lim_{e \to \infty} \frac{a_e(M)}{p^{e\,\mathrm{sr}(M)}},$$

provided the limit exists. If $a_e(M) = 0$ for all $e \in \mathbb{Z}_{>0}$, we let $r_F(M) = 0$.

The goal of this section is to prove the existence of the limit when R is assumed to be local.

We begin with the following observation.

Lemma 4.2. *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of prime characteristic $p > 0$ and let M be a finitely generated R-module. If $a_{e_0}(M) > 0$ for some $e_0 \in \mathbb{Z}_{>0}$ then* $\operatorname{sr}(M) = \operatorname{sr}(R)$.

Proof. Choose an onto R-linear map $R^{\oplus n} \to M$. Then $a_e(M) \leqslant n a_e(R)$ and it follows that $\operatorname{sr}(M) \leqslant \operatorname{sr}(R)$. If $F_*^{e_0} M \cong R \oplus M_{e_0}$ for some e_0 then $F_*^{e+e_0} M \cong F_*^e R \oplus F_*^e M_{e_0}$ for each $e \in \mathbb{Z}_{>0}$. Therefore $a_e(R) \leqslant a_{e+e_0}(M)$ for each $e \in \mathbb{Z}_{>0}$ and $\operatorname{sr}(R) \leqslant \operatorname{sr}(M)$. $\square$

In what follows, it will be useful to keep track of the primes P for which the splitting rate of M is non-negative. We make the following definition.

Definition 4.3. Let R be an F-finite ring and M a finitely generated R-module. The *F-split locus of M* is $\operatorname{fs}(M) = \{P \in \operatorname{Spec}(R) \mid F_*^e(M_P) \text{ has a free summand for some } e > 0\}$.

Observe that, if $F_*^e(M_P)$ has a free summand, then so does $F_*^e(R_P)$. Therefore $\operatorname{fs}(M) \subseteq \operatorname{fs}(R)$. Moreover, Lemma 4.2 proves that, if $P \in \operatorname{fs}(M)$, then the splitting rates of M_P and R_P agree. Our next lemma establishes the existence of the F-splitting ratio of a finitely generated module over a local ring $(R, \mathfrak{m}, k)$ under the assumption that $\mathfrak{m}$ is the splitting prime ideal of R.

Lemma 4.4. *Let $(R, \mathfrak{m}, k)$ be an F-finite and F-pure local ring, with $\mathfrak{m}$ being its splitting prime. Let $\gamma = \gamma(R/\mathfrak{m})$. For every $e \geqslant 0$, write $F_*^e M \cong R^{\oplus a_e(M)} \oplus M_e$. Then*

(1) The sequence $\{a_e(R)/p^{e\gamma}\}$ is the constant sequence $\{1\}$. In particular $r_F(R) = 1$.
(2) The sequence $\{a_e(M)/p^{e\gamma}\}_{e \geqslant 0}$ is a bounded non-decreasing sequence of integers, and therefore eventually constant. In particular, the F-splitting ratio $r_F(M)$ exists. Moreover, $\operatorname{sr}(M) = \gamma \iff r_F(M) > 0 \iff \mathfrak{m} \in \operatorname{fs}(M)$.
(3) If $a_e(M)/p^{e\gamma} = r_F(M)$ then $a_{e'}(M_e) = 0$ for all $e' \geqslant 0$.

Proof. If we let $I_e = \{r \in R \mid R \xrightarrow{\cdot F_*^e r} F_*^e R \text{ does not split}\}$ then I_e is an $\mathfrak{m}$-primary ideal such that $\lambda(R/I_e) = \frac{a_e(R)}{p^{e\gamma}}$, and $\bigcap_{e \in \mathbb{Z}_{>0}} I_e$ is the splitting prime of R, see [1, Corollary 2.8 and Theorem 3.3]. Hence $I_e = \mathfrak{m}$ for each $e \in \mathbb{Z}_{>0}$ and therefore $\lambda(R/I_e) = \frac{a_e(R)}{p^{e\gamma}} = 1$ for each $e \in \mathbb{Z}_{>0}$.

Given finitely generated module M we let $I_e(M) = \{m \in M \mid R \xrightarrow{\cdot F_*^e m} F_*^e M \text{ does not split}\}$. It is known, and easy to prove, that $I_e(M)$ is a submodule of M containing $\mathfrak{m}^{[p^e]} M$ and $\lambda(M/I_e(M)) = \frac{a_e(M)}{p^{e\gamma}}$ is an integer.

As M is a homomorphic image of $R^{\oplus n}$ for some integer $n \geqslant 0$, we see that

$$\frac{a_e(M)}{p^{e\gamma}} \leqslant \frac{a_e(R^{\oplus n})}{p^{e\gamma}} = \frac{a_e(R)n}{p^{e\gamma}} = n.$$

Also observe that for all $e' \geqslant 0$, we have $a_{e+e'}(M) = a_e(M)a_{e'}(R) + a_{e'}(M_e) = a_e(M)p^{e'\gamma} + a_{e'}(M_e)$ and hence

$$\frac{a_{e+e'}(M)}{p^{(e+e')\gamma}} = \frac{a_e(M)}{p^{e\gamma}} + \frac{a_{e'}(M_e)}{p^{(e+e')\gamma}} \geqslant \frac{a_e(M)}{p^{e\gamma}}.$$

In summary, $\{a_e(M)/p^{e\gamma}\}_{e\geqslant 0}$ is a non-decreasing sequence of integer values with an upper bound. So it is eventually constant. All remaining claims follow immediately. $\square$

Let $(R, \mathfrak{m}, k)$ be a local ring, not necessarily of prime characteristic, and M a finitely generated R-module. Similar to the above, we define $I(M) = \{m \in M \mid R \xrightarrow{\cdot m} M \text{ does not split}\}$. Then $I(M) \subseteq M$ is a submodule of M satisfying $\mathfrak{m}M \subseteq I(M)$ and $\lambda(M/I(M)) = \operatorname{frk}(M)$. We refer to $I(M)$ as the non-split submodule of M. Notice that $I(F^e_* M) = F^e_* I_e(M)$. Our next lemma studies the behavior of non-split submodules under R-linear maps.

Lemma 4.5. *Let $(R, \mathfrak{m}, k)$ be a local ring (of any characteristic), let M, N and K be finitely generated R-modules, $f \in \operatorname{Hom}_R(M, N)$ and $g \in \operatorname{Hom}_R(N, K)$. Let $I(M)$, $I(N)$ and $I(K)$ be the non-split submodules of M, N and K respectively.*

(1) We have $\operatorname{frk}(N) \geqslant \lambda(M/(g \circ f)^{-1}(I(K)))$.
(2) Further assume that R is an F-finite ring of prime characteristic p, $M = K$ and $g \circ f = c1_M$ for some $c \in R$. Then, for all $e \geqslant 0$,

$$a_e(N) \geqslant a_e(M) - \lambda(M/(I_e(M) + cM))p^{e\gamma(\mathfrak{m})}.$$

Proof. For (1) first observe that $g(I(N)) \subseteq I(K)$. Else, if there exists $n \in I(N)$ such that $g(n) \notin I(K)$ then there is $\varphi : K \to R$ such that $\varphi(g(n)) = 1$ contradicting the assumption $n \in I(N)$. Therefore $g(f(f^{-1}(I(N)))) \subseteq g(I(N)) \subseteq I(K)$. In particular, $f^{-1}(I(N)) \subseteq (g \circ f)^{-1}(I(K))$ and hence

$$\operatorname{frk}(N) = \lambda(N/I(N))) \geqslant \lambda(M/f^{-1}(I(N))) \geqslant \lambda(M/(g \circ f)^{-1}(I(K))).$$

We now prove part (2). Suppose $(R, \mathfrak{m}, k)$ is an F-finite ring of prime characteristic $p > 0$. For each $e \geqslant 0$, the induced maps $F^e_* f$ and $F^e_* g$ satisfy $F^e_* g \circ F^e_* f = (F^e_* c)1_{F^e_* M}$. So $(F^e_* g \circ F^e_* f)^{-1}(I(F^e_* M)) = (I(F^e_* M) :_{F^e_* M} F^e_* c) = F^e_*(I_e(M) :_M c)$. By (1), we see

$$\begin{aligned} a_e(N) = \operatorname{frk}(F^e_* N) &\geqslant \lambda(F^e_* M/F^e_*(I_e(M) :_M c)) = \lambda(M/(I_e(M) :_M c))p^{e\alpha(\mathfrak{m})} \\ &= [\lambda(M/I_e(M)) - \lambda(M/(I(M) + cM))]p^{e\gamma(\mathfrak{m})} \\ &= \lambda(M/I_e(M))p^{e\alpha(\mathfrak{m})} - \lambda(M/(I(M) + cM))p^{e\alpha(\mathfrak{m})} \\ &= a_e(M) - \lambda(M/(I_e(M) + cM))p^{e\alpha(\mathfrak{m})}. \end{aligned}$$

The equation $\lambda(M/(I_e(M) :_M c)) = \lambda(M/I_e(M)) - \lambda(M/(I_e(M) + cM))$ follows since length is additive and there is short exact sequence

$$0 \to M/(I_e(M) :_M c) \to M/I_e(M) \to M/(I_e(M) + cM) \to 0. \quad \square$$

We are now ready to accomplish two tasks simultaneously: proving the existence of the F-splitting ratio of a finitely generated module over a local ring, and a uniform convergence result which extends Theorem 3.5 to finitely generated modules.

Theorem 4.6. *Let R be an F-finite ring, M a finitely generated R-module, and for each prime ideal $Q \in \mathrm{fs}(R)$ let $\mathcal{P}(Q)$ be the splitting prime ideal of R_Q. Then $r_F(M_Q) = r_F(M_{\mathcal{P}(Q)})r_F(R_Q)$ and $\mathrm{sr}(M_Q) = \mathrm{sr}(M_{\mathcal{P}(Q)})$ for all $Q \in \mathrm{fs}(R)$. Moreover, there exists a constant C such that for all $Q \in \mathrm{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$,*

$$\left| a_e(M_Q) - p^{e\,\mathrm{sr}(M_Q)} r_F(M_Q) \right| \leqslant C p^{e(\mathrm{sr}(M_Q)-1)}.$$

Proof. If $Q \notin \mathrm{fs}(R)$ then $a_e(M_Q) = a_e(R_Q) = 0$ for all $e \in \mathbb{Z}_{>0}$ and any choice of constant $C \geqslant 0$ satisfies the desired inequality for all such prime ideals. Furthermore, as the F-pure locus $\mathrm{fs}(R)$ is open, we can write $\mathrm{fs}(R) = \mathrm{Spec}(R) \smallsetminus V(f_1, \ldots, f_n) = D(f_1) \cup \cdots \cup D(f_n)$ where $f_1, \ldots, f_n$ generate the defining ideal of the non-F-pure locus of R. Therefore $\mathrm{fs}(R)$ is covered by finitely many principal open sets of the form $\mathrm{Spec}(R_f)$ with each R_f being F-pure. Thus we may prove the theorem for each of these pieces of the affine cover and assume for the remainder of the proof that R is an F-pure ring. In particular, R has only finitely many centers of F-purity (see [15, Theorem C] and [10, Theorem 1.1]).

Our approach is to stratify $\mathrm{Spec}(R)$ as a finite union of locally closed sets of the form $V(\mathcal{P}) \cap D(s)$ where $\mathcal{P}$ is the unique maximal center of F-purity of $D(s)$. We then provide a uniform constant C for which the desired inequality holds for each of piece of the stratification. For each center of F-purity $\mathcal{P}$, let $\mathcal{Q}(\mathcal{P}) = \{Q \in \mathrm{Spec}(R) \mid \mathcal{P}(Q) = \mathcal{P}R_Q\}$. If $Q \in \mathrm{Spec}(R)$ then $\mathcal{P}(Q) = \mathcal{P}R_Q$ if and only if $\mathcal{P}R_Q$ is the splitting prime ideal of R_Q, i.e., the maximal center of F-purity of R_Q. Let $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ be all the centers of F-purity that are not subsets of $\mathcal{P}$, and let $\cap_{i=1}^{\ell} \mathcal{P}_i = (s_1, \ldots, s_t)$. We may assume that $s_j \notin \mathcal{P}$ for all $j = 1, \ldots, t$. In fact, $\cap_{i=1}^{\ell} \mathcal{P}_i \nsubseteq \mathcal{P}$, and we can assume $s_1 \notin \mathcal{P}$; if $s_j \in \mathcal{P}$ for some $j > 1$, then we can replace s_j by $s_1 + s_j$. We have that $Q \in \mathcal{Q}(\mathcal{P})$ if and only if $Q \in V(\mathcal{P}) \smallsetminus V(\cap_{i=1}^{\ell} \mathcal{P}_i)$, which is equivalent to $Q \in \cup_{j=1}^{t} \big(V(\mathcal{P}) \cap D(s_j)\big)$. Note that, for each $j = 1, \ldots, t$, the centers of F-purity of $\mathrm{Spec}(R)$ contained in $D(s_j)$ are subsets of $\mathcal{P}$, so $\mathcal{P}$ is the unique maximal center of F-purity in $D(s_j)$. Because there are only finitely many centers of F-purity $\mathcal{P} \in \mathrm{Spec}(R)$, we can realize $\mathrm{Spec}(R)$ as a finite union of locally closed sets of the form $V(\mathcal{P}) \cap D(s)$ where $\mathcal{P}$ is the unique maximal center of F-purity of $D(s)$. If Q is in one such $V(\mathcal{P}) \cap D(s)$ then we replace R by R_s and may assume that R has a unique maximal center of F-purity $\mathcal{P}$ and $Q \in \mathcal{Q}(\mathcal{P}) = V(\mathcal{P}) \subseteq \operatorname{Spec}(R)$.[2]

If $r_F(M_\mathcal{P}) = 0$ then $r_F(M_Q) = 0$ and the conclusion holds for all $Q \in \mathcal{Q}(\mathcal{P})$. So we assume $r_F(M_\mathcal{P}) > 0$ for the rest of proof. Let $\gamma = \gamma(\mathcal{P}) = \operatorname{sr}(M_\mathcal{P})$. By Lemma 4.4, there exists e_0 such that $a_{e_0}(M_\mathcal{P})/p^{e_0\gamma} = r_F(M_\mathcal{P})$. Let $a = a_{e_0}(M_\mathcal{P})$. Then $F_*^{e_0} M_\mathcal{P} \cong R_\mathcal{P}^{\oplus a} \oplus (M_{e_0})_\mathcal{P}$ over $R_\mathcal{P}$, for some finitely generated R-module M_{e_0}. Lifting to R, we obtain R-linear maps

$$R^{\oplus a} \to F_*^{e_0} M \to R^{\oplus a} \quad \text{and} \quad F_*^{e_0} M \to R^{\oplus a} \oplus M_{e_0} \to F_*^{e_0} M$$

such that both compositions are multiplication by some $c \in R \setminus \mathcal{P}$. Applying Lemma 4.5 to the composition map $R^{\oplus a} \to F_*^{e_0} M \to R^{\oplus a}$, we see that for all $Q \in \mathcal{Q}(\mathcal{P})$ and $e \geqslant 0$,

$$a_{e_0+e}(M_Q) \geqslant a \cdot (a_e(R_Q) - \lambda(R_Q/(I_e(R_Q) + cR_Q))p^{e\gamma(Q)}).$$

Therefore

$$\begin{aligned}
\frac{a_{e_0+e}(M_Q)}{p^{(e_0+e)\gamma}} &\geqslant \frac{a \cdot (a_e(R_Q) - \lambda(R_Q/(I_e(R_Q) + cR_Q))q^{\gamma(Q)})}{p^{(e_0+e)\gamma}} \\
&= \frac{a}{p^{e_0\gamma}} \left(\frac{a_e(R_Q)}{p^{e\gamma}} - \frac{\lambda(R_Q/(I_e(R_Q) + cR_Q))}{p^{e \dim(R_Q/\mathcal{P}R_Q)}} \right) \\
&\geqslant r_F(M_\mathcal{P}) \left(\frac{a_e(R_Q)}{p^{e\gamma}} - \frac{\lambda(R_Q/((Q^{[p^e]} + \mathcal{P} + cR)R_Q)}{p^{e \dim(R_Q/\mathcal{P}R_Q)}} \right).
\end{aligned}$$

The last inequality comes from the observation that $(Q^{[p^e]} + \mathcal{P} + cR)R_Q \subseteq I_e(R_Q) + cR_Q$ for all $e \in \mathbb{N}$. Indeed, $Q^{[p^e]}R_Q \subseteq I_e(R_Q)$ for all e, see [20, Lemma 4.4], and $\mathcal{P}R_Q \subseteq I_e(R_Q)$ for all e since $\mathcal{P}R_Q = \cap_{e\in\mathbb{N}} I_e(R_Q)$ by [1] and [16, Remark 4.4].

By Theorem 3.5 there exists a constant C_1, independent of e and $Q \in \mathcal{Q}(\mathcal{P})$, such that $\frac{a_e(R_Q)}{p^{e\gamma}} \geqslant r_F(R_Q) - \frac{C_1}{p^e}$, where $\gamma = \operatorname{sr}(M_\mathcal{P})$ as above. This is because, by Lemma 4.2, we have $\operatorname{sr}(M_\mathcal{P}) = \operatorname{sr}(R_\mathcal{P})$. Moreover, since $\operatorname{sdim}(R_Q) = \dim(R_Q/\mathcal{P}R_Q)$, we have $\operatorname{sr}(R_\mathcal{P}) = \operatorname{sr}(R_Q)$ for all $Q \in \mathcal{Q}(\mathcal{P})$. Thus, $\gamma = \operatorname{sr}(M_\mathcal{P}) = \operatorname{sr}(R_\mathcal{P}) = \operatorname{sr}(R_Q)$ for all $Q \in \mathcal{Q}(\mathcal{P})$. By [13, Proposition 3.3], there exists a constant C_2, independent of e and $Q \in \mathcal{Q}(\mathcal{P})$, such that $\frac{\lambda(R_Q/(Q^{[p^e]}+\mathcal{P}+cR)_Q)}{p^{e \dim(R_Q/\mathcal{P}R_Q)}} \leqslant \frac{C_2}{p^e}$. Therefore the constant $C = r_F(M_\mathcal{P})p^{e_0}(C_1 + C_2)$, which is independent of e and $Q \in \mathcal{Q}(\mathcal{P})$, is such that

$$\frac{a_{e_0+e}(M_Q)}{p^{(e_0+e)\gamma}} \geqslant r_F(M_\mathcal{P})r_F(R_Q) - \frac{C}{p^{e+e_0}}.$$

[2] We thank the anonymous referee for pointing out this simpler approach: With $\mathcal{P}$ ranging over the finitely many centers of F-purity of R, the subsets $\mathcal{Q}(\mathcal{P})$ give rise to a partition of $\operatorname{Spec}(R)$. So it suffices to pick a center of F-purity $\mathcal{P}$ and prove the claims (including the existence of a constant C) for all $Q \in \mathcal{Q}(\mathcal{P})$. The rest of the proof works verbatim, without replacing R by its localization R_s.

An argument similar to the above, applied to the composition of maps $F_*^{e_0} M \to R^{\oplus a} \oplus M_{e_0} \to F_*^{e_0} M$, will provide the existence of a constant C', independent of $Q \in \mathcal{Q}(\mathcal{P})$ and e, such that

$$\frac{a_{e_0+e}(M_Q)}{p^{(e_0+e)\gamma}} \leqslant r_F(M_{\mathcal{P}}) r_F(R_Q) + \frac{C'}{p^{e+e_0}}.$$

This shows, in particular, that $\frac{a_e(M_Q)}{p^{e\gamma}}$ converges uniformly to $r_F(M_{\mathcal{P}})r_F(R_Q) > 0$, and thus $\gamma = \operatorname{sr}(M_{\mathcal{P}}) = \operatorname{sr}(M_Q)$, for all $Q \in \mathcal{Q}(\mathcal{P})$. All assertions of the theorem now follow. $\square$

5. Lower semi-continuity and global F-splitting ratio

The main purpose of this section is to prove existence of the global F-splitting ratio of a finitely generated module M.

We first need to establish some lower semi-continuity results for F-splitting ratios. We will see that, unlike the F-signature, the F-splitting ratio of an F-finite domain may not be a lower semi-continuous function.

Let R be an F-finite ring and M a finitely generated R-module. For each $-1 \leqslant \ell \leqslant \gamma(R)$ we set $W_\ell(M) = \{P \in \operatorname{Spec}(R) \mid \operatorname{sr}(M_P) = \ell\}$. From Lemma 4.2 and subsequent observations, we have that $W_\ell(M) = W_\ell(R) \cap \operatorname{fs}(M)$ for all $\ell \geqslant 0$.

Theorem 5.1. *Let R be an F-finite and F-pure ring of prime characteristic $p > 0$, set $X = \operatorname{Spec}(R)$ and let M be a finitely generated R-module. Then there is a finite stratification of X into locally closed quasi-compact subsets such that the restriction of the F-splitting ratio function on each subset is lower semi-continuous. Specifically, $X = \bigcup_{i=-1}^{\gamma(R)} W_i(M)$, $W_i(M) \cap W_j(M) = \emptyset$ whenever $i \neq j$, the sets $W_i(M)$ are locally closed and quasi-compact, and the function $r_F : \operatorname{Spec}(R) \to \mathbb{R}$ mapping $P \mapsto r_F(M_P)$ is lower semi-continuous when restricted to each $W_i(M)$.*

Proof. The functions $a_e : \operatorname{Spec}(R) \to \mathbb{R}$ mapping $P \mapsto a_e(M_P)$ are easily checked to be lower semi-continuous, see [6, Proposition 2.2]. The normalized functions $\tilde{a}_e$ mapping $P \mapsto a_e(M_P)/p^{e \operatorname{sr}(M_P)}$ are therefore lower semi-continuous when restricted to each of the subsets $W_\ell(M)$. It follows that the function r_F is lower semi-continuous when restricted to each $W_\ell(M)$ as it is realized as the uniform limit of lower semi-continuous functions by Theorem 4.6. It is also easy to see that the sets $W_{-1}(M), W_0(M), \dots, W_{\gamma(R)}(M)$ are disjoint and $X = \bigcup_{i=-1}^{\gamma(R)} W_i(M)$. It remains to show each of the sets $W_\ell(M)$ are locally closed and quasi-compact.

We adopt the convention that $W_i(M) = \emptyset$ if $i < -1$, and we let $\mathcal{P}(Q)$ denote the splitting prime of R_Q. For every $Q \in \operatorname{Spec}(R)$, and every $-1 \leqslant \ell \leqslant \gamma(R)$, Theorem 4.6 shows that $\operatorname{sr}(M_Q) = \operatorname{sr}(M_{\mathcal{P}(Q)})$, and hence $Q \in W_\ell(M)$ if and only if $\mathcal{P}(Q) \in W_\ell(M)$.

Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_N\}$ be the finitely many centers of F-purity of R that are contained in $\mathrm{fs}(M)$. Relabeling if necessary, we may assume $\gamma(R/\mathfrak{p}_j) = \ell$ if and only if $1 \leqslant j \leqslant i$, and $\gamma(R/\mathfrak{p}_j) < \ell$ if and only if $i+1 \leqslant j \leqslant t$. Observe that

$$W_\ell(M) = \left(\bigcup_{j=1}^{i} V(\mathfrak{p}_j)\right) \smallsetminus \left(\bigcup_{j=i+1}^{t} V(\mathfrak{p}_j)\right) = \left(\bigcup_{j=1}^{i} V(\mathfrak{p}_j)\right) \cap \left(X \smallsetminus \bigcup_{j=i+1}^{t} V(\mathfrak{p}_j)\right),$$

hence it is a locally closed set. Finally, note that every locally closed set of $\mathrm{Spec}(R)$, with R Noetherian, is quasi-compact. $\square$

Corollary 5.2. *Let R be an F-finite and F-pure ring and let M be a finitely generated R-module. For $\ell \geqslant 0$, if $W_\ell(M) \neq \emptyset$, then the F-splitting ratio function defined by $r_F : \mathrm{Spec}(R) \to \mathbb{R}$ mapping $P \mapsto r_F(M_P)$ has a nonzero minimum value when restricted to $W_\ell(M)$.*

Proof. The function r_F is lower semi-continuous when restricted to the non-empty quasi-compact set $W_\ell(M)$ and therefore attains a minimum value. $\square$

The F-splitting ratio function is generally not a lower semi-continuous function when viewed as a function on the spectrum of a ring. We provide an example of such a ring, but first we need a lemma.

Lemma 5.3. *Let $(R, \mathfrak{m}, k)$ be an F-finite and F-pure ring satisfying the following:*

(1) R is F-pure;
(2) R is not strongly F-regular;
(3) R_P is strongly F-regular for all $P \neq \mathfrak{m}$;
(4) R_P is not regular for some $P \neq \mathfrak{m}$.

Then the F-splitting ratio function $r_F : \mathrm{Spec}(R) \to \mathbb{R}$ is not lower semi-continuous.

Proof. For each $e \in \mathbb{Z}_{>0}$ let $I_e = \{r \in R \mid R \xrightarrow{\cdot F^e_*(r)} F^e_* R \text{ is not pure}\}$ be the eth splitting ideal of R and set $\mathcal{P} = \bigcap_{e \in \mathbb{Z}_{>0}} I_e$. Recall that $\mathcal{P}$ is referred to as the splitting prime of R, and since R is assumed to be not strongly F-regular, the closed set $V(\mathcal{P})$ is contained in the non-strongly F-regular locus of R. Therefore $\mathcal{P} = \mathfrak{m}$ and it is straightforward to check that $a_e(R) = a_e(R/\mathfrak{m}) = [k^{1/p^e} : k]$ for all e. In particular, $a_e(R)/p^{e\,\mathrm{sr}(R)} = 1$ for all e and therefore $r_F(R) = 1$. However, localizing at a prime $P \neq \mathfrak{m}$ for which R_P is not regular it follows R_P is strongly F-regular by assumption but not regular and therefore $r_F(R_P) = \mathrm{s}(R_P) < 1$ by [8, Corollary 16] and therefore the F-splitting function is not lower semi-continuous. $\square$

Theorem 5.4. *There exist an F-finite ring R for which the F-splitting ratio function is not lower semi-continuous as a function from* $\operatorname{Spec}(R)$ *to* $\mathbb{R}$.

Proof. Let k be a perfect field of prime characteristic p and let A be a non-regular strongly F-regular ring of finite type over k, write $A = k[x_1, \dots, x_n]/I$, with $I \subseteq (x_1, \dots, x_n)$, and assume $A_{(x_1,\dots,x_n)}$ is a non-regular local ring. Let $B = A[v]$ and $R = \{f \in B \mid f(0, \dots, 0, 0) = f(0, \dots, 0, 1)\}$ localized at the maximal ideal $R \cap (x_1, \dots, x_n, v)B = R \cap (x_1, \dots, x_n, v-1)B$. The ring R is realized as a fiber product, i.e. a gluing of the local rings $B_{(x_1,\dots,x_n,v)}$ and $B_{(x_1,\dots,x_n,v-1)}$ at their maximal ideals. It readily follows that the conductor ideal of R inside its normalization is the unique maximal ideal of R and R is isomorphic to localizations of B on the punctured spectrum. Hence R is strongly F-regular on the punctured spectrum, but not an isolated singularity. Moreover, R is an F-pure ring since there exist splittings of $B_{(x_1,\dots,x_n,v)}$ and $B_{(x_1,\dots,x_n,v-1)}$ compatible at the residue field level of these rings. Moreover, the conductor ideal is compatible under all R-linear maps $F^e_* R \to R$ by [12, Lemma 3.1]. Therefore R is not strongly F-regular and satisfies all hypotheses of Lemma 5.3.[3] $\square$

For convenience of the reader, we recall the following:

Theorem 5.5 *([19,3]). Let R be a Noetherian ring of Krull dimension $d < \infty$ and M a finitely generated R-module. If $\operatorname{frk}_{R_P}(M_P) \geqslant \dim(R/P) + k$ for all $P \in \operatorname{Spec}(R)$, then $\operatorname{frk}_R(M) \geqslant k$. In particular, $\operatorname{frk}_R(M) \geqslant \min\{\operatorname{frk}_{R_P}(M_P) \mid P \in \operatorname{Spec}(R)\} - d$.*

We are finally ready to show the existence of the global F-splitting ratio of modules, and relate it to the F-splitting ratio of the localization at prime ideals.

Theorem 5.6. *Let R be an F-finite ring of prime characteristic $p > 0$ and M a finitely generated module. Then*

(1) We have $\operatorname{sr}(M) = \min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\}$.
(2) The limit $r_F(M) = \lim\limits_{e\to\infty} \dfrac{a_e(M)}{p^{e \operatorname{sr}(M)}}$ exists, and it is positive if $\operatorname{sr}(M) \geqslant 0$.
(3) We have $r_F(M) = \min\{r_F(M_P) \mid \operatorname{sr}(M_P) = \operatorname{sr}(M)\}$. In particular, $r_F(M)$ is positive whenever there exists $e \in \mathbb{Z}_{>0}$ and onto R-linear map $F^e_* M \to R$.
(4) If $\operatorname{sr}(R) = 0$, then the sequence $\{a_e(R)\}$ is the constant sequence $\{1\}$. Therefore, we have $r_F(R) = 1$.
(5) If $\operatorname{sr}(M) = 0$ then the sequence $\{a_e(M)\}$ is a non-decreasing sequence of eventually positive integers bounded from above, hence is eventually the constant sequence $\{r_F(M)\}$.

[3] If the reader was interested in finding an example of normal ring whose F-splitting ratio function is not lower semi-continuous, then one could instead consider the cone of a singular Calabi-Yau 3-fold and show such a ring localized at the homogeneous maximal ideal satisfies the hypotheses of Lemma 5.3.

Proof. If there exists a prime ideal $P \in \operatorname{Spec}(R)$ such that $a_e(M_P) = 0$ for all $e \in \mathbb{Z}_{>0}$, i.e., if $W_{-1}(M) \neq \emptyset$, then all statements of the theorem trivially follow, and we have $r_F(M) = 0$.

For the remainder of the proof, we assume that $W_{-1}(M) = \emptyset$. Since $a_e(M) \leqslant a_e(M_P)$ for all $P \in \operatorname{Spec}(R)$, it easily follows that $\operatorname{sr}(M) \leqslant \min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\}$.

First, assume that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} > 0$. For each $e \in \mathbb{Z}_{>0}$, we let $P_e \in \operatorname{Spec}(R)$ be such that $a_e(M_{P_e}) = \min\{a_e(M_P) \mid P \in \operatorname{Spec}(R)\}$. If we set $d = \dim(R)$, it follows from Theorem 5.5 that $a_e(M) \geqslant a_e(M_{P_e}) - d$. Let C be as in Theorem 4.6, and let $r = \min\{r_F(M_P) \mid P \in \operatorname{Spec}(R)\}$. Such an r exists, and is positive by Corollary 5.2. In particular, we have

$$\begin{aligned} a_e(M) \geqslant a_e(M_{P_e}) - d &\geqslant r_F(M_{P_e})p^{e\operatorname{sr}(M_{P_e})} - Cp^{e(\operatorname{sr}(M_{P_e})-1)} - d \\ &\geqslant rp^{e\operatorname{sr}(M_{P_e})} - Cp^{e(\operatorname{sr}(M_{P_e})-1)} - d. \end{aligned}$$

Since $\operatorname{sr}(M_{P_e}) > 0$, it follows that $a_e(M) \geqslant \frac{rp^e}{2}$ for all $e \gg 0$, and therefore $\operatorname{sr}(M) > 0$. Moreover, we have that $\operatorname{sr}(M_{P_e}) > \operatorname{sr}(M)$ only for finitely many values of e. Else, from the inequalities above we would get

$$r \leqslant \frac{a_e(M)}{p^{e\operatorname{sr}(M_{P_e})}} + \frac{C}{p^e} + \frac{d}{p^{e\operatorname{sr}(M_{P_e})}} \leqslant \frac{a_e(M)}{p^{e(\operatorname{sr}(M)+1)}} + \frac{C}{p^e} + \frac{d}{p^{e(\operatorname{sr}(M)+1)}},$$

for infinitely many values of e. Because $\operatorname{sr}(M) > 0$, the expression on the right hand side can be made arbitrarily close to 0 for $e \gg 0$, contradicting the fact that $r > 0$. Therefore we have $\operatorname{sr}(M_{P_e}) = \operatorname{sr}(M)$ for all $e \gg 0$ and, in particular, this gives the reverse inequality $\operatorname{sr}(M) \geqslant \min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\}$. This finishes the proof of (1) under the assumption that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} > 0$.

Continue to assume that $\ell = \operatorname{sr}(M) > 0$, and let $P_e \in \operatorname{Spec}(R)$ be as above. We have already observed that $\operatorname{sr}(M_{P_e}) = \ell$ for all $e \gg 0$. Moreover, there are inequalities

$$\frac{a_e(M_{P_e}) - d}{p^{e\ell}} \leqslant \frac{a_e(M)}{p^{e\ell}} \leqslant \frac{a_e(M_{P_e})}{p^{e\ell}}.$$

Under the assumption that $\ell > 0$, parts (2) and (3) follow if $\lim\limits_{e\to\infty} \frac{a_e(M_{P_e})}{p^{e\ell}}$ exists and is equal to $\min\{r_F(M_P) \mid \operatorname{sr}(M_P) = \ell\}$. But this is indeed the case since the F-splitting ratio function restricted to the quasi-compact set $W_\ell(M) = \{P \in \operatorname{Spec}(R) \mid \operatorname{sr}(M_P) = \ell\}$ is the uniform limit of the lower semi-continuous functions $\frac{a_e(-)}{p^{e\ell}}$. In particular, the minimum the functions $\frac{a_e(-)}{p^{e\ell}}$ on $W_\ell(M)$ converges to the minimum of the F-splitting ratio functions on $W_\ell(M)$. This proves (2) and (3) under the assumption that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} > 0$.

Now we prove (4), so we assume $\operatorname{sr}(R) = 0$. By what we have shown above, we must necessarily have $0 = \operatorname{sr}(R) \leqslant \min\{\operatorname{sr}(R_P) \mid P \in \operatorname{Spec}(R)\} \leqslant 0$, and thus $\operatorname{sr}(R_P) = 0$ for some $P \in \operatorname{Spec}(R)$. Observe that we have $\operatorname{sr}(R_P) \geqslant \alpha(P)$, with equality if and only if

PR_P is the splitting prime of R_P. Thus, $\operatorname{sr}(R_P) = 0$ implies that PR_P is the splitting prime of R_P, and that $\kappa(P)$ is perfect. It is well-known that an F-finite ring is F-pure if and only if R_P is F-pure for all $P \in \operatorname{Spec}(R)$, see for example [18, Exercise 2.10]. It follows from Lemma 4.4 that $1 \leqslant a_e(R) \leqslant a_e(R_P) = 1$ for all $e \in \mathbb{Z}_{>0}$, and therefore $a_e(R) = 1$ for all $e \in \mathbb{Z}_{>0}$. This proves (4).

Now suppose that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} = 0$. Let $P \in \operatorname{Spec}(R)$ be such that $\operatorname{sr}(M_P) = 0$. By Lemma 4.2, this also gives $\operatorname{sr}(R_P) = \operatorname{sr}(R) = 0$. To prove (5), we choose for each $e \in \mathbb{Z}_{>0}$ a direct sum decomposition $F^e_* M \cong R^{\oplus a_e(M)} \oplus M_e$. Then

$$F^{e+1}_* M \cong F_* R^{\oplus a_e(M)} \oplus F_* M_e.$$

As R is F-pure, $F^e_* R^{\oplus a_e(M)}$ has a free summand of rank $a_e(M)$, and therefore $a_{e+1}(M) \geqslant a_e(M)$. To see that the sequence $\{a_e(M)\}$ is bounded from above, choose an onto map $R^{\oplus N} \to M$. By part (4), the condition that $\operatorname{sr}(R) = 0$ implies that $a_e(R^{\oplus N}) = N$ for each $e \in \mathbb{Z}_{>0}$, and therefore $a_e(M) \leqslant N$ for all $e \in \mathbb{Z}_{>0}$. We have now proven, under the assumption that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} = 0$, that the sequence $\{a_e(M)\}$ is a non-decreasing sequence of non-negative integers, and is therefore eventually the constant sequence $\{r_F(M)\}$.

To complete the proof it is enough to show that $r_F(M) = \min\{r_F(M_P) \mid \operatorname{sr}(M_P) = 0\}$, which concludes (5). Moreover, since $\min\{r_F(M_P) \mid \operatorname{sr}(M_P) = 0\} > 0$ by Corollary 5.2, this also implies $a_e(M) = r_F(M) > 0$ for $e \gg 0$. Hence $\operatorname{sr}(M) = 0$, which concludes the proof of parts (1), (2), and (3).

Let $\{P_1, \dots, P_s\}$ be the set of maximal objects, with respect to containment, of the set of all centers of F-purity of R. We refer to them as the maximal centers of F-purity of R. We may assume that $\operatorname{sr}(M_{P_i}) = 0$ for all $1 \leqslant i \leqslant r$, and $\operatorname{sr}(M_{P_i}) > 0$ for $r+1 \leqslant i \leqslant s$. From what shown above, we know that for all $e \gg 0$ we have $F^e_* M \cong R^{\oplus r_F(M)} \oplus M_e$, where $F^{e'}_* M_e$ does not have a free summand for all $e' \geqslant 0$. We claim that $\operatorname{frk}((M_e)_{P_i}) \geqslant 1$ for all $r+1 \leqslant i \leqslant s$. To see this, we assume by contradiction that, for some $r+1 \leqslant i \leqslant s$, we have $\operatorname{frk}((M_e)_{P_i}) = 0$ for infinitely many $e \in \mathbb{Z}_{>0}$. Then the splitting rate of M_{P_i} would be 0, and this contradicts our arrangement of the maximal centers of F-purity of R.

Suppose $r_F(M) < \min\{r_F(M_P) \mid \operatorname{sr}(M_P) = 0\}$. Then $\operatorname{frk}((M_e)_{P_i}) > 0$ for each $1 \leqslant i \leqslant r$, and $e \gg 0$. Then for each $1 \leqslant i \leqslant s$ we can find $m_i \in M_e$ and $h_i \in \operatorname{Hom}_R(M_e, R)$ such that $h_i(m_i) \notin P_i$. By prime avoidance we can find for each $1 \leqslant i \leqslant s$ an element $r_i \in \left(\bigcap_{j \neq i} P_j\right) \smallsetminus P_i$. Let $m = \sum r_i m_i$ and $h = \sum r_i h_i$. Then $x := h(m) = \sum\sum r_i r_j h_i(m_j) \notin \bigcup_{i=1}^s P_i$. Therefore the element x avoids all maximal centers of F-purity of R, hence all centers of F-purity of R. In particular, if $Q \in \operatorname{Spec}(R)$, then there exists $e_Q \in \mathbb{Z}_{>0}$ such that $R_Q \xrightarrow{\cdot F^{e'}_* x} F^{e'}_* R_Q$ splits for all $e' \geqslant e_Q$. Therefore, the union of the sets $U_{e'} := \{Q \in \operatorname{Spec}(R) \mid R_Q \xrightarrow{\cdot F^{e'}_* x} F^{e'}_* R_Q \text{ splits}\}$ is equal to $\operatorname{Spec}(R)$. Moreover, they are open sets, and they form an ascending chain [7]. By quasi-compactness of $\operatorname{Spec}(R)$, there exists $e' \in \mathbb{Z}_{>0}$ such that $U_{e'} = \operatorname{Spec}(R)$. Therefore $R \xrightarrow{F^{e'}_* x} F^{e'}_* R$ splits, since splitting of is a local condition. Suppose $\varphi : F^{e'}_* R \to R$ satisfies $\varphi(F^{e'}_* x) = 1$. Then, the composition $F^{e'}_* M_e \xrightarrow{F^{e'}_* h} F^{e'}_* R \xrightarrow{\varphi} R$ maps $F^e_* m \mapsto 1$, and this contradicts the property that $F^{e'}_* M_e$ does not have a free R-summand for all $e' \geqslant 0$. This completes the proof. $\square$

We end this section by showing that the global F-splitting ratio of a positively graded algebra is equal to the F-splitting ratio at the irrelevant maximal ideal.

Proposition 5.7. *Let $(R_0, \mathfrak{m}_0, k)$ be an F-finite local ring and let R be a positively graded algebra of finite type over R_0. Let $R_{>0}$ be the ideal of R generated by elements of positive degree and $\mathfrak{m} = \mathfrak{m}_0 + R_{>0}$. Suppose that M is a finitely generated graded R-module. We have the equality $a_e(M) = a_e(M_\mathfrak{m})$. In particular, we have $\mathrm{sr}(M) = \mathrm{sr}(M_\mathfrak{m})$, and $r_F(M) = r_F(M_\mathfrak{m})$.*

Proof. Since $a_e(M) \leqslant a_e(M_\mathfrak{m})$ always holds, it is sufficient to prove the other inequality. To this end, we observe that $F^e_* M$ is a $\mathbb{Q}$-graded module. Hence, we can find a graded isomorphism $F^e_* M \cong \bigoplus_{i=1}^{b_e} R[n_i] \oplus M_e$, where $n_i \in \mathbb{Q}$, and M_e is a $\mathbb{Q}$-graded module with no graded free summands. Here, $R[n_i]$ denotes the cyclic $\mathbb{Q}$-graded free module whose generator is in degree $-n_i$. We claim that $(M_e)_\mathfrak{m}$ has no free summands either. In fact, if it did, there would be a surjective $R_\mathfrak{m}$-linear map $(M_e)_\mathfrak{m} \to R_\mathfrak{m}$. Such a map lifts to an R-linear map $\varphi : M_e \to R$ with $\varphi(M_e) \nsubseteq \mathfrak{m}$. Since $\mathrm{Hom}_R(M_e, R)$ is a graded module, we can find a graded component ψ of φ that still satisfies $\psi(M_e) \nsubseteq \mathfrak{m}$. Such a map ψ gives rise to a graded free summand of M_e, contradicting our assumptions. This shows that $a_e(M) \geqslant b_e = a_e(M_\mathfrak{m})$, as claimed. $\square$

Corollary 5.8. *Let R and $\mathfrak{m}$ be as in Proposition 5.7. We have $\mathrm{s}(R) = \mathrm{s}(R_\mathfrak{m})$.*

Proof. In our assumptions, the ideal defining the non-strongly F-regular locus is homogeneous [11, Lemma 4.2]. If R is not strongly F-regular, then $R_\mathfrak{m}$ is also not strongly F-regular; thus, $\mathrm{s}(R) = \mathrm{s}(R_\mathfrak{m}) = 0$ in this case. Now assume R is strongly F-regular. Then $R_\mathfrak{m}$ is also strongly F-regular, and thus $\mathrm{sr}(R_\mathfrak{m}) = \gamma(R_\mathfrak{m}) = \gamma(R)$. Using Proposition 5.7, we conclude that $\mathrm{sr}(R) = \gamma(R)$, and hence $\mathrm{s}(R) = r_F(R) = r_F(R_\mathfrak{m}) = \mathrm{s}(R_\mathfrak{m})$. $\square$

6. Positivity of F-signature of Cartier algebras and strong F-regularity

This section is devoted to giving a positive answer to [5, Question 4.24]. We recall the following condition from [5]. For unexplained notation and terminology we refer to Subection 2.4 of the same article.

Condition 6.1. We say that $(R, \mathscr{D})$ satisfies condition (†) if at least one of the following conditions is satisfied:

- $\mathscr{D}$ satisfies condition $(*)$, as in 3.2.
- $\mathscr{D} = \mathcal{C}^{\mathfrak{a}^t}$ for some ideal $\mathfrak{a} \subseteq R$ and $t > 0$.
- R is normal and $\mathscr{D} = \mathcal{C}^{(R,\Delta)}$ for some effective $\mathbb{Q}$-divisor Δ.

Using the same notation as in Section 5, we now recall the definition of global F-signature of a pair $(R, \mathscr{D})$. Given an F-finite and F-pure ring R, and a Cartier algebra $\mathscr{D}$, the F-signature of $(R, \mathscr{D})$ is

$$\mathrm{s}(R, \mathscr{D}) = \lim_{e \to \infty} \frac{a_e(R, \mathscr{D})}{p^{e\gamma(R)}}.$$

When $\mathscr{D}$ is the full Cartier algebra, we simply write $\mathrm{s}(R)$ for $\mathrm{s}(R, \mathscr{D})$. In this case, if we also have $\gamma(R) = \mathrm{sr}(R)$, the global F-signature $\mathrm{s}(R)$ coincides with the global F-splitting ratio $r_F(R)$ defined in Section 5. The limit above was shown to exist in [5, Theorem 4.19]. In the same article, a global version of a result of Blickle, Schwede and Tucker [2], relating the positivity of $\mathrm{s}(R, \mathscr{D})$ to the strong F-regularity of the pair $(R, \mathscr{D})$ was established in this setup.

Theorem 6.2. *[5, Corollary 4.23] Let R be an F-finite domain, and let $\mathscr{D}$ be a Cartier algebra satisfying condition $(\dagger)$. Then $\mathrm{s}(R, \mathscr{D}) > 0$ if and only if $(R, \mathscr{D})$ is strongly F-regular*

The way Theorem 6.2 was proved in [5] was by exploiting the relation

$$\mathrm{s}(R, \mathscr{D}) = \min\{\mathrm{s}(R_P, \mathscr{D}_P) \mid P \in \mathrm{Spec}(R)\}.$$

Since the strong F-regularity of $(R, \mathscr{D})$ is equivalent to such minimum being positive, this was sufficient. However, the proof of the equality between the global F-signature of $(R, \mathscr{D})$ and the minimum of the local invariants required some semi-continuity results, that are only known to hold under the additional assumption that $(\dagger)$ holds [13,14]. The goal of this section is to show that Theorem 6.2 is true without assuming $(\dagger)$. In particular, we will provide a direct way to show that the signature of a strongly F-regular pair $(R, \mathscr{D})$ is positive, without looking at the corresponding invariants in the localizations at prime ideals.

We start with two preparatory lemmas.

Lemma 6.3. *[2, Lemma 3.13c] and [14, Lemma 4.2] Let R be an F-finite normal domain and let $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$. There exists $0 \neq z \in R$ such that for all $n \in \mathbb{Z}_{>0}$, and all $\psi \in \mathrm{Hom}_R(F^{ne}_* R, R)$, there exists $r \in R$ such that*

$$z\psi = \varphi^n(F^{ne}_* r -)$$

where $\varphi^n = \varphi \circ F^e_* \varphi \circ F^{2e}_* \varphi \circ \cdots \circ F^{(n-1)e}_* \varphi$ and $\varphi^n(F^{ne}_* r-)$ is composition of the maps

$$F_*^{ne}R \xrightarrow{\cdot F_*^{ne}r} F_*^{ne}R \xrightarrow{\varphi^n} R.$$

Lemma 6.4. *Let R be a strongly F-regular F-finite domain. Then there exists $\varepsilon > 0$ such that for all $e \in \mathbb{Z}_{>0}$, $a_e(R) \geqslant \varepsilon \operatorname{rank}(F_*^e R)$.*

Proof. As R is strongly F-regular, $\mathrm{s}(R) > 0$ by [5, Theorem 4.15]. Hence, there exists $e' \in \mathbb{Z}_{>0}$ such that for all $e > e'$, $a_e(R)/\operatorname{rank}(F_*^e R) \geqslant \mathrm{s}(R)/2$. Let

$$\varepsilon = \min\left\{\frac{a_1(R)}{\operatorname{rank}(F_*R)}, \dots, \frac{a_{e'}(R)}{\operatorname{rank}(F_*^{e'}R)}, \frac{\mathrm{s}(R)}{2}\right\}.$$

Then $a_e(R) \geqslant \varepsilon \operatorname{rank}(F_*^e R)$ for all $e \in \mathbb{Z}_{>0}$. $\square$

The following theorem extends [5, Corollary 4.20], giving a positive answer to [5, Question 4.21].

Theorem 6.5. *Let R be an F-finite domain and let $\mathscr{D}$ be a Cartier algebra. Then $(R, \mathscr{D})$ is strongly F-regular if and only if $\mathrm{s}(R, \mathscr{D}) > 0$.*

Proof. If $(R, \mathscr{D})$ is not strongly F-regular, then there exists $P \in \operatorname{Spec}(R)$ such that $(R_P, \mathscr{D}_P)$ is not strongly F-regular. Since $a_e(R, \mathscr{D}) \leqslant a_e(R_P, \mathscr{D}_P)$, we get $\mathrm{s}(R, \mathscr{D}) \leqslant \mathrm{s}(R_P, \mathscr{D}_P) = 0$.

Conversely, suppose that $(R, \mathscr{D})$ is strongly F-regular. Then R is strongly F-regular and by Lemma 6.4 there exists $\varepsilon > 0$ such that $a_e(R) \geqslant \varepsilon \operatorname{rank}(F_*^e R)$ for all $e \in \mathbb{Z}_{>0}$. Let $e_0 \in \mathbb{Z}_{>0}$ be such that $\varepsilon \geqslant \frac{1}{p^{e_0}}$. If $\operatorname{rank}(F_*^e R) = 1$ for each $e \in \mathbb{Z}_{>0}$, then R is a perfect field and there is nothing to prove. We assume R is not a perfect field so that, for all $e \geqslant e_0$, p^{e_0} divides $\operatorname{rank}(F_*^e R)$. Let $\ell_e = \operatorname{rank}(F_*^e R)/p^{e_0}$, so that $a_e(R) \geqslant \ell_e$ for each $e \in \mathbb{Z}_{>0}$.

Let $e_1 > 0$ be such that $a_{e_1}(R, \mathscr{D}) > 0$, and let $\varphi \in \mathscr{D}_{e_1}$ be a non-zero map. Let z be as in Lemma 6.3. In particular, for each $n \in \mathbb{Z}_{>0}$ and for each $\psi \in \operatorname{Hom}_R(F_*^{ne_1}R, R)$, the map $z\psi$ belongs to $\mathscr{D}_{ne_1}$. Consider integers of the form $e = ne_1 \geqslant e_0$. As $a_e(R) \geqslant \ell_e$, we can write $F_*^e R \cong R^{\oplus \ell_e} \oplus M_e$ for some R-module M_e. Let $\lambda_1, \dots, \lambda_{\ell_e} \in F_*^e R$ form a basis for the free summand $R^{\oplus \ell_e}$ of $F_*^e R$. Denote by $\tilde{\lambda}_i : F_*^e R \to R$ the R-linear map defined by $\lambda_i \mapsto 1$, $\lambda_j \mapsto 0$ for all $j \neq i$, and $x \mapsto 0$ for all $x \in M_e$.

We chose $0 \neq z \in R$ such that $z\tilde{\lambda}_i \in \mathscr{D}_e$, and $z\tilde{\lambda}_i$ maps $\lambda_i \mapsto z$ and $\lambda_j \mapsto 0$ for all $j \neq i$. As $(R, \mathscr{D})$ is strongly F-regular, there exists $e_2 \in \mathbb{Z}_{>0}$ and $\gamma \in \mathscr{D}_{e_2}$ such that $\gamma(F_*^{e_2} z) = 1$. Then the R-linear maps $\gamma_i := \gamma \circ F_*^{e_2} z\tilde{\lambda}_i : F_*^{e+e_2} R \to R$ are elements of $\mathscr{D}_{e+e_2}$ such that $F_*^{e_2}\lambda_i \mapsto 1$ and $F_*^{e_2}\lambda_j \mapsto 0$ for all $j \neq i$. Therefore, for each $e = ne_1 \geqslant e_0$, we have

$$a_{ne_1+e_2}(R, \mathscr{D}) \geqslant \ell_{ne_1} = \frac{\operatorname{rank}(F_*^{ne_1}R)}{p^{e_0}} = \frac{\operatorname{rank}(F_*^{ne_1+e_2}R)}{p^{e_0}\operatorname{rank}(F_*^{e_2}R)},$$

and thus

$$\mathrm{s}(R,\mathscr{D}) = \lim_{e' \in \Gamma_{\mathscr{D}} \to \infty} \frac{a_{e'}(R,\mathscr{D})}{\operatorname{rank}(F_*^{e'} R)} = \lim_{n\to\infty} \frac{a_{ne_1+e_2}(R,\mathscr{D})}{\operatorname{rank}(F_*^{ne_1+e_2} R)} \geqslant \frac{1}{p^{e_0} \operatorname{rank}(F_*^{e_2} R)} > 0. \quad \square$$

Remark 6.6. As pointed out above, the proof of Theorem 6.2 contained in [5] requires the extra assumption that (†) holds, because it is based on the equality $\mathrm{s}(R,\mathscr{D}) = \min\{\mathrm{s}(R_P,\mathscr{D}_P) \mid P \in \operatorname{Spec}(R)\}$. Theorem 6.5 settles the positivity of $\mathrm{s}(R,\mathscr{D})$ for strongly F-regular pairs $(R,\mathscr{D})$, but it does not indicate any progress in the direction of showing that $\mathrm{s}(R,\mathscr{D})$ is equal to the minimum of the local invariants. In particular, it does not show the existence of a prime $P \in \operatorname{Spec}(R)$ such that $\mathrm{s}(R,\mathscr{D}) = \mathrm{s}(R_P,\mathscr{D}_P)$.

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