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THE COMPLEXITY THRESHOLD FOR THE EMERGENCE OF KAKUTANI INEQUIVALENCE*

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We dedicate this paper to Benjy Weiss on the occasion of his 80^{th} birthday

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ABSTRACT

We show that linear complexity is the threshold for the emergence of Kakutani inequivalence for measurable systems supported on a minimal subshift. In particular, we show that there are minimal subshifts of arbitrarily low superlinear complexity that admit both loosely Bernoulli and non-loosely Bernoulli ergodic measures and that no minimal subshift with linear complexity can admit inequivalent measures.

1. Complexity and Kakutani equivalence

1.1. Block complexity and constraints on the system. The growth rate of the complexity function of a symbolic dynamical system gives rise to combinatorial invariants that allow for a finer classification of zero-entropy systems and can be an obstruction for realizing certain dynamic properties. For example, the Morse–Hedlund Theorem [16] says that if the number of n blocks in the language of a subshift grows more slowly than $n+1$, then the subshift is periodic, putting a lower bound on the growth rate for the emergence of interesting behavior. Boshernitzan showed [1] that the complexity bound on a minimal subshift with linear complexity constrains the number of ergodic measures supported by the subshift and provided precise bounds related to the linear growth rate of the complexity function (two of the authors [2] lifted the assumption of minimality when counting nonatomic measures). Ferenczi [7] showed that any two ergodic measure preserving systems supported on a minimal subshift with linear complexity are even Kakutani equivalent and more specifically are loosely Bernoulli (see Section 2 for definitions).

For each such result, it is natural to explore the complexity threshold where the constraint is no longer present. It was shown in [2] that linear complexity is the threshold for Bozhernitzan's result. In particular, given arbitrarily low superlinear complexity there exists a minimal subshift with at most that complexity that supports uncountably many invariant ergodic measures. Here we show that linear complexity growth is also the threshold for Ferenczi's result: we show that there are minimal subshifts with arbitrarily slow superlinear complexity growth which support Kakutani inequivalent measures.

1.2. Even Kakutani equivalence and loosely Bernoulli. Recall that an orbit equivalence between two measurable systems (X, T, μ) and (Y, S, ν) is a bi-measurable, measure preserving map $\phi\colon X \to Y$ that maps orbits to

orbits. Ornstein's seminal result [17] states that two Bernoulli systems are measure theoretically isomorphic if and only if they have equal entropy. On the opposite end of the spectrum, Dye's theorem [4, 5] states that any two measurable systems are measurably orbit equivalent if and only if they are ergodic. These are the two extremes of orbit equivalence, with the first an isomorphism preserving order on orbits and the second permuting points in an orbit without any restriction beyond measurability. Even Kakutani equivalence lies in between: an orbit equivalence between two ergodic systems (X, T, μ) and (Y, S, ν) is an **even Kakutani equivalence** if there exist sets of equal positive measure $A \subset X$ and $B \subset Y$ such that the measure preserving map $\phi\colon X \to Y$ is order preserving on A. Namely, if and only if the induced transformations T_A and S_B are measurably isomorphic.

Feldman, in [6], was the first to use the term Kakutani equivalence to describe this equivalence relation given by inducing. In that paper he introduced the property of loosely Bernoulli, which he showed to be an invariant for Kakutani equivalence. He also constructed the first example of a zero-entropy non-loosely Bernoulli system, which he used to build the first example of a K and not Bernoulli automorphism which is not Kakutani equivalent to any Bernoulli. He defined the loosely Bernoulli property by introducing a new metric to use in the definition of very weak Bernoulli, a key ingredient in Ornstein theory. This new metric, called $\overline{f}$, weakens the $\overline{d}$ metric to capture the effect of an even Kakutani equivalence on orbits.

Feldman's work was extended by Ornstein, Rudolph and Weiss [18] who developed the general equivalence theory for this metric. They showed that the loosely Bernoulli transformations play the role for even Kakutani equivalence that the Bernoulli transformations play in the isomorphism theory. In particular, two loosely Bernoulli transformations are even Kakutani equivalent if and only if they have equal entropy. This result, and the work in [18], is the motivating example for the more general theory of restricted orbit equivalence developed by Rudolph in [19]. He showed that if an orbit equivalence satisfies certain regularity conditions then there is always a distinguished family of transformations playing the role of Bernoulli transformations for the associated equivalence relation. Finally, we note that Katok and Sataev [15] and Katok [13, 14] independently defined the $\overline{f}$ metric and proved the equivalence theorem in the zero-entropy category, using the term **standard** to describe the loosely Bernoulli family of transformations.

The loosely Bernoulli class of systems contains all Bernoulli transformations but is strictly larger, even in the positive-entropy category. Here we focus our attention on the zero-entropy transformations. The simplest characterization of zero-entropy loosely Bernoulli systems is that they are the family of transformations that induce rotations. For this reason, they are sometimes referred to as loosely Kroenecker systems. Examples of zero-entropy loosely Bernoulli systems include rotations and, more generally, all finite rank systems [18]. Many more examples exist, and the study of the loosely Bernoulli property and the role of the $\overline{f}$ metric continues to be an active area of research; see for example [9, 10, 12, 11].

1.3. Complexity and loosely Bernoulli. Turning to complexity and symbolic systems, for a subshift (X, σ), let $P(n)$ denote the block complexity of X, meaning the number of words of length n that occur in any $x \in X$ (see Section 2 for precise definitions). Ferenczi [7, Proposition 4] showed that if a minimal subshift has low complexity, namely $P_X(n) = O(n)$, then it has finite rank. More generally, essentially using Ferenczi's proof, we check (see Appendix A) that the same result holds under the milder assumption that $\liminf_{n\to\infty} \frac{P_X(n)}{n} < \infty$. As finite rank transformations are loosely Bernoulli, we can rephrase Ferenczi's result in the language of Kakutani equivalence: if a minimal subshift has linear complexity, then all invariant ergodic measures on the subshift give rise to measurable systems that are even Kakutani equivalent.

We show that linear complexity is the threshold for which this result holds. In particular, our main result shows that there are minimal subshifts of arbitrary low superlinear complexity that admit both loosely Bernoulli and non-loosely Bernoulli ergodic measures:

Theorem 1.1: *Let $(p_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of integers satisfying*

$$\liminf_{n\to\infty} \frac{p_n}{n} = \infty \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\log p_n}{n} = 0. \tag{1}$$

Then there exists a zero-entropy minimal subshift (X, σ) satisfying

$$\liminf_{n\to\infty} \frac{P_X(n)}{p_n} = 0$$

which supports two ergodic measures, μ and ν, such that (X, σ, μ) is loosely Bernoulli while (X, σ, ν) is not loosely Bernoulli.

Our construction builds on Feldman's classical example from [6], but requires significant modification in order to both find a loosely Bernoulli system supported on the subshift and to control its block complexity.

By Ferenczi's result (see the version in the Appendix), it follows immediately that the complexity of the subshift (X, σ) that we construct is constrained. Namely, any subshift satisfying the conclusions of Theorem 1.1 satisfies the superlinear growth condition

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} = \infty.$$

1.4. Guide to the paper. In Section 2, we give a short summary of the definitions and background results and in Section 3 we build the system (X, σ) used to prove Theorem 1.1. Sections 4 and 5 are devoted to proving the existence of the measures μ and ν such that the systems (X, σ, μ) and (X, σ, ν) are Kakutani inequivalent. All the results are sewn together in Section 6 to prove Theorem 1.1. In the Appendix, we review Ferenczi's result showing that linear complexity implies finite rank.

2. Background

2.1. Symbolic systems. Let $\mathcal{A}$ be a finite alphabet, and denote $x \in \mathcal{A}^{\mathbb{Z}}$ as $x = (x_n)_{n\in\mathbb{Z}}$. We endow $\mathcal{A}^{\mathbb{Z}}$ with the topology induced by the metric

$$d(x, y) = 1/2^k$$

where

$$k = \inf\{|i| \colon x_i \neq y_i\}.$$

The **left shift** $\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by

$$(\sigma x)_n = x_{n+1}$$

for all $n \in \mathbb{Z}$. If $X \subset \mathcal{A}^{\mathbb{Z}}$ is closed and σ-invariant, then (X, σ) is a **subshift**.

The set $\mathcal{A}^n$ consists of all words of length n and we denote $w \in \mathcal{A}^n$ by $w_0 w_1 \cdots w_{n-1} = w_{[0,n-1]}$. Define the **cylinder set** determined by w to be the set

$$[w] = \{x \in X \colon x_j = w_j \text{ for } j = 0, \ldots, n-1\}.$$

Definition 2.1: Given $\omega \in \mathcal{A}^n$, we define $[[\omega]]$ to be the union of cylinder sets associated with all words of length n that appear in the $2n$-length block $\omega\omega$.

If $F = \{u_1, u_2, \ldots, u_{|F|}\}$ is a collection of words u_i (possibly of varying lengths), we similarly define

$$[F] = \bigcup_{i=1}^{|F|} [u_i] \quad \text{and} \quad [[F]] = \bigcup_{i=1}^{|F|} [[u_i]].$$

Associated to the subshift (X, σ) is the set of all shift-invariant probability measures, $\mathcal{M}(X, \sigma)$, defined on the Borel σ-algebra generated by the cylinder sets. We denote the subset of ergodic measures by $\mathcal{M}_e(X, \sigma)$. Standard results from topological dynamics tell us that both of these sets are nonempty.

2.2. Complexity. If (X, σ) is a subshift and $n \in \mathbb{N}$, the **words** $\mathcal{L}_n(X)$ of **length** n are defined to be the collection of all $w \in \mathcal{A}^n$ such that $[w] \neq \emptyset$. We denote the length of a word w by $|w|$. The **language** $\mathcal{L}(X)$ of the subshift X is the union of all its words:

$$\mathcal{L}(X) = \bigcup_{n=1}^{\infty} \mathcal{L}_n(X).$$

If $w \in \mathcal{L}(X)$ is a word, we say that $u \in \mathcal{L}(X)$ is a **subword** of w if $w = w_1 u w_2$ for some (possibly empty) words $w_1, w_2 \in \mathcal{L}(X)$.

For a subshift (X, σ), the **word complexity** $P_X \colon \mathbb{N} \to \mathbb{N}$ is defined to be the number of words of length n in the language:

$$P_X(n) = |\mathcal{L}_n(X)|.$$

Thus $P_X(1)$ is the size of the alphabet, meaning that $P_X(1) = |\mathcal{A}|$.

We say that (X, σ) has **linear complexity** if

$$\liminf_{n\to\infty} P_X(n)/n < \infty.$$

One can also consider the stronger condition, $\limsup_{n\to\infty} P_X(n)/n < \infty$. This is a distinct condition from the above, as there exist systems satisfying the first condition but not satisfying the second (see [3], Example 4.1). In this paper we restrict our attention to the lim inf condition.

2.3. Loosely Bernoulli. Let $x, y \in \mathcal{L}(A)$ be finite words, written $x = x_1 x_2 \cdots x_m$ and $y = y_1 y_2 \cdots y_n$. We define a **match** between x and y to be an order preserving bijection $\pi\colon \mathcal{I}_x^\pi \to I_y^\pi$ where $\mathcal{I}_x^\pi \subset \{1, \ldots, m\}$ and $\mathcal{I}_y^\pi \subset \{1, \ldots, n\}$ with the property that $x_i = y_{\pi(i)}$. We then say that the indices i and $j = \pi(i)$ are **matched**. The **size** of the match is defined to be

$$|\pi| = |\mathcal{I}_x^\pi| + |\mathcal{I}_y^\pi|,$$

and the **best fit** between the two words x and y is

$$\overline{f}^c(x, y) = \frac{\max\{|\pi|\colon \pi \text{ is a match between } x \text{ and } y\}}{m + n}.$$

We can then define the $\overline{f}$-distance between x and y to be

$$\overline{f}(x, y) = 1 - \overline{f}^c(x, y) = 1 - \frac{\max\{|\pi|\colon \pi \text{ is a match between } x \text{ and } y\}}{m + n}.$$

This distance measures the proportion of letters such that, once they are deleted, the remaining words are identical. See for example [6] or [18] for more details and properties of this metric.

Let $\mathcal{B}$ be the Borel σ-algebra generated by the cylinder sets and $\mu \in \mathcal{M}(X, \sigma)$. Then $(X, \mathcal{B}, \mu)$ is a Lebesgue probability space: together with σ it is a measurable dynamical system. We abbreviate this as (X, σ, μ).

Now assume that $\mu \in \mathcal{M}_e(X, \sigma)$ is such that (X, σ, μ) is zero-entropy. In this case, we can define loosely Bernoulli as follows:

Definition 2.2: The zero-entropy ergodic subshift (X, σ, μ) is **loosely Bernoulli** if for all $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$, there exists $W \subset \mathcal{L}_n(X)$ with

- $\mu(W) > 1 - \varepsilon$; and
- for any pair $\omega, \omega' \in W$, $\overline{f}(\omega, \omega') < \varepsilon$.

3. The construction

3.1. Feldman words. The core symbolic structure of the subshift we build is closely related to the first example of a zero-entropy non-loosely Bernoulli system that was given by Feldman [6]. We begin by describing a slight modification of his example, where changes are introduced to accommodate the additional requirements our subshift must satisfy.

Let $\{n_k\}$ be an increasing sequence of integers with $n_0 \geq 2$. We inductively define sets of words of increasing sizes. Define the 0th stage alphabet

$$\mathcal{A}_0 := \{a_{0,1}, a_{0,2}, \ldots, a_{0,n_0}\}.$$

For stage one of the construction we define words

$$\mathcal{A}_1 := \{a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}\}$$

each of length $n_0^{4n_1+3}$ by setting

$$a_{1,i} := (a_{0,1}^{n_0^{2(i+n_1)}} a_{0,2}^{n_0^{2(i+n_1)}} \cdots a_{0,n_0}^{n_0^{2(i+n_1)}})^{n_0^{2(n_1-i+1)}} \quad \text{for } 1 \leq i \leq n_1.$$

For $k \geq 1$, given a set of n_k distinct words

$$\mathcal{A}_k = \{a_{k,1}, a_{k,2}, \ldots, a_{k,n_k}\}$$

each of length $|a_{k-1,1}|n_{k-1}^{4n_k+3}$, we define a set of n_{k+1} new words

$$\mathcal{A}_{k+1} = \{a_{k+1,1}, a_{k+1,2}, \ldots, a_{k+1,n_{k+1}}\}$$

each of length $|a_{k,1}|n_k^{4n_{k+1}+3}$ by setting

$$a_{k+1,i} = (a_{k,1}^{n_k^{2(i+n_{k+1})}} a_{k,2}^{n_k^{2(i+n_{k+1})}} \cdots a_{k,n_k}^{n_k^{2(i+n_{k+1})}})^{n_k^{2(n_{k+1}-i+1)}} \quad \text{for all } 1 \leq i \leq n_{k+1}.$$

In what follows we refer to the words $a_{n,k}$ as **Feldman words**. Throughout our construction we introduce conditions on the growth rate of the sequence $\{n_k\}$ that guarantee that it grows rapidly enough for our subshift to have the necessary properties.

3.2. Extended Feldman words. The sets of words used in our construction are extensions of the words described above. We begin with a base-case alphabet of size $n_0 + 1$ to be $\mathcal{A}_0$ with an additional symbol c_0:

$$\mathcal{B}_0 := \{a_{0,1}, a_{0,2}, \ldots, a_{0,n_0}, c_0\}.$$

We then define n_1 words of length $L_1 = 1 + n_0^{4n_1+3}$ by setting

$$b_{1,i} := (a_{0,1}^{n_0^{2(i+n_1)}} a_{0,2}^{n_0^{2(i+n_1)}} \cdots a_{0,n_0}^{n_0^{2(i+n_1)}})^{(n_0)^{2(n_1-i+1)}} c_0 \quad \text{for } 1 \leq i \leq n_1.$$

In addition, we define a new word, also of length L_1, but with a different combinatorial structure:

$$c_1 := c_0^{L_1-n_0} a_{0,1} a_{0,2} \cdots a_{0,n_0}.$$

Finally, we denote the collection of these stage one words by

$$\mathcal{B}_1 := \{b_{1,1}, b_{1,2}, \ldots, b_{1,n_1}, c_1\}.$$

Note that every letter in $\mathcal{B}_0$ appears at least once in every word in $\mathcal{B}_1$, similar to how every letter in $\mathcal{A}_0$ appears at least once in every Feldman word in $\mathcal{A}_1$.

We now proceed inductively. Fix $k \geq 1$ and suppose we are given the set

$$\mathcal{B}_k = \{b_{k,1}, b_{k,2}, \ldots, b_{k,n_k}, c_k\}$$

comprised of $n_k + 1$ many words, all of length $L_k = (1 + n_{k-1}^{4n_k+3})L_{k-1}$, written in the stage-$(k-1)$ words of $\mathcal{B}_{k-1}$, meaning each word in $\mathcal{B}_k$ is a concatenation of words from $\mathcal{B}_{k-1}$. We then define n_{k+1} words, all of which have length

$$L_{k+1} = (1 + n_k^{4n_{k+1}+3})L_k, \tag{2}$$

by setting

$$b_{k+1,i} = (b_{k,1}^{n_k^{2(i+n_{k+1})}} b_{k,2}^{n_k^{2(i+n_{k+1})}} \cdots b_{k,n_k}^{n_k^{2(i+n_{k+1})}})^{n_k^{2(n_{k+1}-i+1)}} c_k. \tag{3}$$

We define a new word, also of length L_{k+1} but with a combinatorial structure similar to the words c_k constructed at previous levels, by setting

$$c_{k+1} := c_k^{(L_{k+1}/L_k)-n_k} b_{k,1} b_{k,2} \cdots b_{k,n_k}. \tag{4}$$

Finally, we define the collection of $(k+1)$-words to be

$$\mathcal{B}_{k+1} := \{b_{k+1,1}, b_{k+1,2}, \ldots, b_{k+1,n_{k+1}}, c_{k+1}\}.$$

Note that every element of $\mathcal{B}_k$ appears at least once in each element of $\mathcal{B}_{k+1}$ (and similarly with $\mathcal{A}_k$ and $\mathcal{A}_{k+1}$). It also follows that

$$\lim_{k\to\infty} L_k = \infty \quad \text{and} \quad \frac{L_k}{L_{k+1}} < \frac{1}{n_k}. \tag{5}$$

In what follows we often distinguish between words of the type $b_{k,i}$ and the word c_k. As the words $b_{k,i}$ are similar in form to the Feldman words, we refer to them as **extended Feldman words**, and we denote the set of extended Feldman words at stage k by

$$\mathcal{B}_k^F = \{\omega \in \mathcal{B}_k \colon \omega \neq c_k\} = \mathcal{B}_k \backslash \{c_k\}.$$

3.3. Constructing the subshift. We use the extended Feldman words to construct the subshift X. We emphasize that for the construction and the properties we prove about the system, all of the results only rely on choosing the sequence $\{n_k\}$ with sufficiently rapid growth. As they become necessary, we introduce new growth conditions on this sequence, clarifying when each new condition is needed. For the preliminary properties and construction of the space X, we only require that the sequence satisfies $n_0 \geq 2$ and $n_k \to \infty$.

Given such a sequence, by the inductive procedure described above we obtain, for each $k \geq 1$, a set $\mathcal{B}_k$ comprised of words of equal length, each of which is a concatenation of words from $\mathcal{B}_{k-1}$, and every word in $\mathcal{B}_{k-1}$ appears at least once in every word in $\mathcal{B}_k$. Moreover note that, by construction, $b_{k,1}$ is the leftmost subword of length L_k in the word $b_{k+1,1}$ for all $k \geq 0$. Therefore we can define a one-sided infinite word $b_{\infty,1}$ as the unique word whose leftmost subword of length L_k is $b_{k,1}$ for all $k \geq 1$. We note that we use the less standard term of **leftmost** (or **rightmost**) word, rather than prefix (or suffix), to emphasize that the core part of the word is not the focus and to highlight the visual placement of these subwords as part of the intuition behind the construction.

We choose a new symbol, denoted $*$, that was not included in $\mathcal{B}_0$ and we define a $\{b_{1,1}, b_{1,2}, \ldots, b_{1,n_1}, c_1, *\}$-coloring of $\mathbb{Z}$ by coloring $\mathbb{N}$ with the word $b_{\infty,1}$ and coloring the set $\{\ldots, -2, -1, 0\}$ with $*$; call this $\mathbb{Z}$-coloring α. Finally let X be the space

$$(6)\quad \{x \in \{b_{1,1}, b_{1,2}, \ldots, b_{1,n_1}, c_1, *\}^{\mathbb{Z}} \colon \forall i > 0, \exists j > i \text{ such that } d(x, \sigma^j \alpha) < 2^{-i}\},$$

where σ and d are the left-shift and the metric as defined in Section 2.1. Note that $*$ does not appear in any $\mathbb{Z}$-coloring that can be obtained by taking larger and larger left-shifts of α, and so the elements of X are actually written in the letters $\mathcal{B}_0$. Thus we make no abuse of notation by referring to X as a subshift of $\mathcal{B}_0^{\mathbb{Z}}$.

PROPOSITION 3.1: *The system (X, σ), where X is the space defined in* (6) *and $\sigma \colon X \to X$ is the left-shift, is minimal.*

Proof. Let $u \in \mathcal{L}(X)$. Then by the construction of X, u occurs as a subword of $b_{\infty,1}$ and therefore there exists some n such that u occurs as a subword of $b_{n,1}$. Note that every word in $\mathcal{B}_{n+1}$ contains $b_{n,1}$ as a subword (in fact every word in $\mathcal{B}_{n+1}$ contains every word in $\mathcal{B}_n$ as a subword). We also have that for all $k > 1$, $b_{n+k,1} \in \mathcal{B}_{n+k}$ can be written as a concatenation of words in $\mathcal{B}_{n+1}$. Thus any subword of length at least $2L_{n+1}$ in $b_{n+k,1}$ has u as a subword. Therefore any subword of length at least $2L_{n+1}$ in $b_{\infty,1}$ has u as a subword. We conclude that if $v \in \mathcal{L}(X)$ is a word satisfying

$$|v| \geq 2L_{n+1},$$

then u occurs as a subword of v. This means that u occurs syndetically in every element of the subshift X and the maximum gap between consecutive occurrences of u is at most $2L_{n+1}$.

Since u was arbitrary, we conclude that every word in $\mathcal{L}(X)$ occurs syndetically in every element of X and, for any fixed word in $\mathcal{L}(X)$, the maximum gap between consecutive occurrences is uniform throughout X (but may depend on the word itself). Therefore X is minimal. ∎

4. Non-loosely Bernoulli

4.1. Overview of the existence of a non-loosely Bernoulli measure. The goal of this section is to show that if the sequence $\{n_k\}$ used in constructing the space X grows sufficiently rapidly, then there is a non-loosely Bernoulli measurable system supported on X. For ease of exposition, in what follows, we assume that the sequence grows sufficiently rapidly and we defer defining the explicit growth condition to later in the section where we provide proofs of the key results.

Theorem 4.1: *If the sequence $\{n_k\}$ used in constructing the space X grows sufficiently rapidly, then there exists an ergodic measure $\nu \in \mathcal{M}_e(X)$ such that the system (X, σ, ν) is not loosely Bernoulli.*

The proof follows from several propositions, which we now state, deferring their more technical proofs until after the proof of the theorem. The first proposition shows that extensions of different Feldman words from the same stage of the construction do not match well in the $\overline{f}$ metric:

Proposition 4.2: *If the sequence $\{n_k\}$ grows sufficiently rapidly, then for all integers $r, s, k \geq 0$, if $i \neq j$ then*

$$\overline{f}(b_{k,i}^r, b_{k,j}^s) \geq \frac{5}{8}. \tag{7}$$

The next two results establish necessary properties of ergodic measures of extended Feldman words.

Proposition 4.3: *If the sequence $\{n_k\}$ grows sufficiently rapidly, then there exists a measure $\nu \in \mathcal{M}_e(X)$ such that*

$$\lim_{k\to\infty} \nu([[\mathcal{B}_k^F]]) = 1.$$

PROPOSITION 4.4: *Given any $\xi \in \mathcal{M}_e(X)$ and $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$ and $b_{k,m}, b_{k,j} \in \mathcal{B}_k$, we have*

$$|\xi([[b_{k,m}]]) - \xi([[b_{k,j}]])| < \varepsilon.$$

With these results in hand, the proof of the theorem follows quickly:

Proof of Theorem 4.1. We prove the result by contradiction. Assume that the sequence $\{n_k\}$ grows sufficiently rapidly such that Propositions 4.2, 4.3 and 4.4 hold, and assume the constructed system (X, σ, ν) is loosely Bernoulli. Fix $\varepsilon > 0$.

Using Propositions 4.3 and 4.4 and the fact that $|\mathcal{B}_k| \to \infty$, we can choose K such that for all $k \geq K$ we have $\frac{1}{|\mathcal{B}_k|} < \varepsilon$, $\nu([[\mathcal{B}_k^F]]) > 1 - \varepsilon$, and

$$|\nu([[b_{k,m}]]) - \nu([[b_{k,j}]])| < \varepsilon \quad \text{for all } m, j.$$

Then for any set of words $W \subset \mathcal{L}_{L_k}(X)$ with $\nu([W]) > 1 - \varepsilon$, we have

$$\nu([W] \cap [[B_k^F]]) > 1 - 2\varepsilon. \tag{8}$$

Since there are more than $1/\varepsilon$ sets of the form $[[b_{k,i}]]$, each with similar measure, there must be distinct words $u, v \in W$ that are elements of $[[b_{k,m}]]$ and $[[b_{k,j}]]$, respectively, for some $m \neq j$. Since $|u| = |v| = L_k$, the words u and v must cover exactly half of $b_{k,m}^2$ and $b_{k,j}^2$, respectively. Suppose $\overline{f}(u, v) < \varepsilon$ and thus $\overline{f}^c(u, v) > 1 - \varepsilon$. Extending the match that realizes this value to all of $b_{k,m}^2$ and $b_{k,j}^2$ gives that $\overline{f}^c(b_{k,m}^2, b_{k,j}^2) \geq \frac{1}{2}(1 - \varepsilon)$. Equivalently, this means that

$$\overline{f}(b_{k,m}^2, b_{k,j}^2) < \frac{1}{2}(1 + \varepsilon).$$

But for sufficiently small ε, this is a contradiction of Proposition 4.2. ∎

The remainder of this section is devoted to the proofs of the three propositions.

4.2. PROOF OF PROPOSITION 4.2: BAD $\overline{f}$ MATCH OF EXTENSIONS OF FELDMAN WORDS. We begin by proving that the Feldman words themselves do not match well in $\overline{f}$, and then use the fact that their extensions add only a small proportion of symbols to obtain our result.

The first lemma is essentially Feldman's original argument in [6]. We include it here for the sake of completeness.

LEMMA 4.5: *If $\{n_k\}$ increases sufficiently rapidly, then for all integers $r, s, k \geq 0$, if $i \neq j$, then*

$$\overline{f}(a_{k,i}^r, a_{k,j}^s) \geq \frac{7}{8}. \tag{9}$$

Proof. Let $\{n_k\}$ be an increasing sequence with the property that

$$\prod_{k=0}^{\infty} \frac{n_k}{n_k - 2} \leq 2 \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{2}{n_k} \leq \frac{1}{32}. \tag{10}$$

We construct a sequence $\{\Gamma_k\}$ by setting $\Gamma_0 = 0$ and for $k \geq 1$, set

$$\Gamma_k = \sum_{i=0}^{k-1} \left(\prod_{j=i}^{k-1} \frac{n_j}{n_j - 2} \right)^2 \frac{2}{n_i}. \tag{11}$$

We prove the lemma by showing that for all $k \geq 0$ and $i \neq j$, the match

$$\overline{f}^c(a_{k,i}^r, a_{k,j}^s) \leq \Gamma_k. \tag{12}$$

Clearly (12) holds for $k = 0$ and so assume that (12) holds for some $k \geq 0$. Our goal is to show that

$$\overline{f}^c(a_{k+1,i}^r, a_{k+1,j}^s) \leq \Gamma_{k+1}, \tag{13}$$

and then the statement follows since $\Gamma_k \leq \frac{1}{8}$ for all $k \geq 0$.

Assume that $j = i + m$, with $m \geq 1$. Let $\alpha_h = a_{k,h}^{n_k^{2(i+n_{k+1})}}$ denote the building blocks of the word $a_{k+1,i}$. With this notation, we can rewrite

$$a_{k+1,i}^r = (\alpha_1 \cdots \alpha_{n_k})^{n_k^{2(n_{k+1}-i+1)} \cdot r}$$
$$a_{k+1,j}^s = (\alpha_1^{n_k^{2m}} \cdots \alpha_{n_k}^{n_k^{2m}})^{n_k^{2(n_{k+1}-j+1)} \cdot s}.$$

Consider an arbitrary match π between these two words and take the restriction of this match to each subword $\alpha_h^{n_k^{2m}}$ of $a_{k+1,j}^s$. Using the restriction, we can partition $a_{k+1,i}^r$ into disjoint subwords that contain indices all of which are matched to a unique $\alpha_h^{n_k^{2m}}$. Each such subword must have the form $\beta(\alpha_1 \cdots \alpha_{n_k})^t \gamma$, where β and γ are substrings from the beginning and end, respectively, of $(\alpha_1 \cdots \alpha_{n_k})$. Thus to prove (13) it suffices to show that for all h

$$\overline{f}^c(\alpha_h^{n_k^{2m}}, \beta(\alpha_1 \cdots \alpha_{n_k})^t \gamma) \leq \Gamma_{k+1}. \tag{14}$$

Instead, we consider the quantity

$$\overline{f}^c(\alpha_h^{n_k^{2m}}, (\alpha_1 \cdots \alpha_{n_k})^{t+2}) \tag{15}$$

where we have added at most $2|\alpha_h|n_k$ symbols to $\beta(\alpha_1 \cdots \alpha_{n_k})^t\gamma$ and the worst fit would be if none of these additional symbols improved the match between the original pair of strings. Therefore, letting

$$\ell_{\mathcal{O}} = |\alpha_h^{n_k^{2m}}| + |\beta(\alpha_1 \cdots \alpha_{n_k})^t\gamma|$$

denote the lengths of the original two strings being matched, we have

$$\begin{aligned}\overline{f}^c(\alpha_h^{n_k^{2m}}, (\alpha_1 \cdots \alpha_{n_k})^{t+2}) &\geq \frac{|\pi|}{\ell_{\mathcal{O}} + 2|\alpha_h|n_k} \\ &\geq \overline{f}^c(\alpha_h^{n_k^{2m}}, \beta(\alpha_1 \cdots \alpha_{n_k})^t\gamma)\Big(1 - \frac{2|\alpha_h|n_k}{|\alpha_h|n_k^{2m}}\Big) \\ &\geq \overline{f}^c(\alpha_h^{n_k^{2m}}, \beta(\alpha_1 \cdots \alpha_{n_k})^t\gamma)\Big(1 - \frac{2}{n_k}\Big).\end{aligned}$$

Thus to prove (13), it suffices to show that

$$\frac{n_k}{n_k - 2}\overline{f}^c(\alpha_h^{n_k^{2m}}, (\alpha_1 \cdots \alpha_{n_k})^{t+2}) \leq \Gamma_{k+1}. \tag{16}$$

For ease of notation, define $\omega = \alpha_h^{n_k^{2m}}$ and $\omega' = (\alpha_1 \cdots \alpha_{n_k})^{t+2}$. Consider the partition of ω into disjoint subwords $\omega_{u,v}$ corresponding to contiguous subblocks that contain (but do not necessarily consist of) indices matched by π to a symbol in the v-th occurrence of α_u in ω'. Formally we define $\omega_{u,v}$ to be the subblock of ω corresponding to the indices in the interval $\omega_{[i_*,i^*]}$, where

$$i_* = \min\{i\colon \pi(i) \text{ lies in the } v\text{-th occurrence of } \alpha_u \text{ in } \omega'\}$$

and

$$i^* = \min\{i \geq i_*\colon i \in \mathcal{I}_\omega^\pi, \text{ but } \pi(i) \text{ does not lie in the } v\text{-th occurrence of } \alpha_u\} - 1,$$

recalling that the notation $\mathcal{I}_\omega^\pi$ was introduced at the beginning of Section 2.3. Since π is order preserving, these blocks are disjoint and contiguous. In order to guarantee that $\omega_{u,v}$ be a partition of ω, we add any initial (respectively, final) indices in ω that are not matched to anything in ω' to $\omega_{1,1}$ (respectively, $\omega_{n_k,t+2}$). Note that it is possible for some $\omega_{u,v}$ to be empty.

Adopting this notation, we have

$$\begin{aligned}&\overline{f}^c(\omega,\omega')\\ =&\frac{2(\#\text{ of total indices in }\omega'\text{ that are matched by }\pi)}{|\omega|+|\omega'|}\\ =&\frac{2}{|\omega|+|\omega'|}\sum_{u=1}^{n_k}(\#\text{ of indices in }\omega'\text{ matched by }\pi\text{ lying in an occurrence of }\alpha_u).\end{aligned}$$

Note that if $u = h$, then there is the possibility that π provided a perfect match for every possible occurrence of α_h in ω'. Recall that

$$|\omega| = |\alpha_h| n_k^{2m} \quad \text{and} \quad |\omega'| = |\alpha_h|(t+2)n_k. \tag{17}$$

So

$$\begin{aligned}&\frac{2}{|\omega|+|\omega'|}(\#\text{ of indices in }\omega'\text{ matched by }\pi\text{ lying in an occurrence of }\alpha_h)\\ &\qquad\le \frac{2|\alpha_h|(t+2)}{|\alpha_h|n_k^{2m}+|\alpha_h|(t+2)n_k}\le\frac{2}{n_k}.\end{aligned}$$

We now turn to the matches between α_u and $\omega_{u,v}$ where $u \neq h$, namely the other summands:

$$\begin{aligned}&\frac{2}{|\omega|+|\omega'|}\sum_{u=1,u\neq h}^{n_k}\sum_{v=1}^{t+2}(\#\text{ of indices matched in }\omega'\\ &\qquad\qquad\text{lying in the }v\text{-th occurrence of }\alpha_u)\\ &\qquad=\frac{1}{|\omega|+|\omega'|}\sum_{u=1,u\neq h}^{n_k}\sum_{v=1}^{t+2}\overline{f}^c(\omega_{u,v},\alpha_u)(|\alpha_u|+|\omega_{u,v}|).\end{aligned} \tag{18}$$

Recall that $\alpha_u = a_{k,u}^p$ and $\omega_{u,v} = ba_{k,h}^{p'}c$ for some $p, p' \in \mathbb{N}$ and where c and d are the end and beginning substrings of $a_{k,h}$, respectively. As before we complete each $\omega_{u,v}$ to $a_{k,h}^{p'+2}$, adding at most $2|a_{k,h}|$ symbols, obtaining a match between strings where our inductive hypothesis holds. Therefore, for each u, v we have

$$\Gamma_k \ge \overline{f}^c(a_{k,h}^{p'+2},\alpha_u)\ge\overline{f}^c(\omega_{u,v},\alpha_u)\Big(1-\frac{2}{n_k}\Big). \tag{19}$$

Then we have that the quantity in (18) is less than or equal to

$$\frac{1}{|\omega|+|\omega'|}\sum_{u=1,u\neq h}^{n_k}\sum_{v=1}^{t+2}\Big(\frac{n_k}{n_k-2}\Big)\Gamma_k(|\alpha_u|+|\omega_{u,v}|).$$

Recall that $|\alpha_u| = |\alpha_h|$ for all u and so we have that this last quantity is equal to

$$\frac{1}{|\omega|+|\omega'|}\Big(\frac{n_k}{n_k-2}\Big)\Gamma_k\Big[n_k(t+2)|\alpha_h| + \sum_{u=1,u\neq h}^{n_k}\sum_{v=1}^{t+2}|\omega_{u,v}|\Big].$$

By (17) and the fact that the $\omega_{u,v}$ form a partition of ω, this is

$$\leq \frac{1}{|\omega|+|\omega'|}\Big(\frac{n_k}{n_k-2}\Big)\Gamma_k(|\omega'|+|\omega|) = \Big(\frac{n_k}{n_k-2}\Big)\Gamma_k.$$

Putting all this together with (11), we see that (16) is satisfied:

$$\frac{n_k}{n_k-2}\overline{f}^c(\alpha_h^{n_k^{2m}}, (\alpha_1\cdots\alpha_{n_k})^{t+2}) \leq \Big(\frac{n_k}{n_k-2}\Big)^2\Gamma_k + \Big(\frac{n_k}{n_k-2}\Big)\frac{2}{n_k} \leq \Gamma_{k+1}. \qquad \blacksquare$$

The following property of the $\overline{f}$ metric (see for example [10, Property 2.4]) is used in the proof of Proposition 4.2:

LEMMA 4.6: *Suppose b_1 and b_2 are strings of symbols of length n and m, respectively, from an alphabet $\mathcal{A}$. If a_1 and a_2 are strings of symbols obtained by deleting at most $\lfloor\rho(n+m)\rfloor$ terms from b_1 and b_2 altogether, where $0<\rho<1$, then*

$$\overline{f}(b_1,b_2) \geq \overline{f}(a_1,a_2) - 2\rho. \tag{20}$$

Proof of Proposition 4.2. Suppose the sequence $\{n_k\}$ grows sufficiently rapidly such that both (10) is satisfied and

$$\prod_{j=0}^{\infty}\frac{n_j^{4n_{j+1}+3}}{n_j^{4n_{j+1}+3}+1} > \frac{7}{8}. \tag{21}$$

We then have

$$\frac{|a_{k,i}|}{|b_{k,i}|} = \prod_{j=0}^{k-1}\frac{n_j^{4n_{j+1}+3}}{n_j^{4n_{j+1}+3}+1} > \frac{7}{8}$$

and thus any two Feldman words $a_{k,j}$ and $a_{k,i}$ are obtained from the extended words $b_{k,j}$ and $b_{k,i}$ by eliminating at most $\frac{1}{8}(L_k+L_k)$ symbols. Therefore we can apply Lemma 4.6 with $\rho=\frac{1}{8}$ and obtain that $\overline{f}(b_{k,i}^r, b_{k,j}^s) \geq \overline{f}(a_{k,i}^r, a_{k,j}^s) - \frac{1}{4}$. Using the result of Lemma 4.5, we can conclude that $\overline{f}(b_{k,i}^r, b_{k,j}^s) \geq \frac{5}{8}$. $\blacksquare$

4.3. Proofs of Propositions 4.3 and 4.4: properties of ergodic measures on (X, σ). We start with the proof of Proposition 4.3, which depends on the following lemma:

Lemma 4.7: *Let (X, σ, μ) be a measure preserving system and let $\{b_m\}_{m=1}^{\infty}$ be a sequence of measurable sets satisfying $\mu(b_m) > 1 - \frac{1}{4^m}$ for all $m \geq 1$. Then there exists an ergodic measure ν, supported on X, which satisfies $\nu(b_m) > 1 - \frac{1}{2^m}$ for all $m \geq 1$.*

Proof. If μ is ergodic, then we are done by setting $\nu = \mu$. Otherwise recall that μ has an ergodic decomposition, meaning there is a measurable map, $x \mapsto \mu_x$, from X to the space of probability measures on X satisfying with the property that μ_x is ergodic for μ-almost everywhere $x \in X$, and for any measurable set M we have

$$\mu(M) = \int_X \mu_x(M) d\mu(x).$$

For each $m \geq 1$, define the measurable set

$$a_m := \Big\{ x \in X : \mu_x(b_m) > 1 - \frac{1}{2^m} \Big\}.$$

Then for any fixed m, we have

$$\begin{aligned} 1 - \frac{1}{4^m} < \mu(b_m) &= \int_X \mu_x(b_m) d\mu(x) = \int_{a_m} \mu_x(b_m) d\mu(x) + \int_{X \setminus a_m} \mu_x(b_m) d\mu(x) \\ &\leq \mu(a_m) + \Big(1 - \frac{1}{2^m}\Big)(1 - \mu(a_m)). \end{aligned}$$

Therefore $\mu(a_m) > 1 - \frac{1}{2^m}$ or, equivalently, $\mu(X \setminus a_m) < \frac{1}{2^m}$. This means that

$$\mu\Big(\bigcap_{m=1}^{\infty} a_m \Big) = 1 - \mu\Big(\bigcup_{m=1}^{\infty} (X \setminus a_m) \Big) \geq 1 - \sum_{m=1}^{\infty} \mu(X \setminus a_m)$$

which gives us $\mu(\bigcap_{m=1}^{\infty} a_m) > 0$. But for μ-almost every $x \in X$, the measure μ_x is ergodic and so there exists $x \in \bigcap_{m=1}^{\infty} a_m$ such that μ_x is ergodic. Pick such an x and define $\nu := \mu_x$. Then, since $x \in a_m$ for all $m \geq 1$, we have $\nu(b_m) = \mu_x(b_m) > 1 - \frac{1}{2^m}$. ∎

We are now ready to prove Proposition 4.3, which we recall states that there is an ergodic measure on X that gives large measure to the sets $[[\mathcal{B}_m^F]]$ for sufficiently large m.

Proof of Proposition 4.3. By Lemma 4.7, it suffices to show that there is a σ-invariant (but not necessarily ergodic) measure μ with the property that for all $m \geq 1$, we have

$$\mu([[\mathcal{B}_m^F]]) > 1 - \frac{1}{4^m}.$$

Suppose the sequence $\{n_k\}$ grows sufficiently rapidly such that for all $m \geq 1$,

$$\prod_{i=m}^{\infty} \left(1 - \frac{1}{n_i}\right) > 1 - \frac{1}{4^{m+1}}. \tag{22}$$

Fix an arbitrary $m \geq 1$ and choose $x \in X$ such that the restriction of x to its first L_k symbols is exactly the word $b_{k,1}$ for every $k \geq 1$. Let μ be a weak* accumulation point of measures of the form

$$\frac{1}{L_k} \sum_{i=0}^{L_k - 1} \delta_{\sigma^i x}.$$

Note that this means there is a subsequence k_j and a value $J = J(m)$ such that for all $j \geq J$,

$$\left| \mu([[\mathcal{B}_m^F]]) - \frac{1}{L_{k_j}} \sum_{i=0}^{L_{k_j} - 1} \delta_{\sigma^i x}([[\mathcal{B}_m^F]]) \right| < \frac{1}{4^{m+1}}. \tag{23}$$

We show below that for all large k,

$$\frac{1}{L_k} \sum_{i=0}^{L_k - 1} \delta_{\sigma^i x}([[\mathcal{B}_m^F]]) > 1 - \frac{1}{4^{m+1}}, \tag{24}$$

where as usual δ denotes the Dirac measure. Together the two inequalities (24) and (23) yield that $\mu([[\mathcal{B}_m^F]]) > 1 - \frac{1}{4^m}$, as wanted.

It only remains to show that (24) holds. By the definition of x, it suffices to count the number of L_m-sized subwords in $b_{k,1}$ that lie in $[[\mathcal{B}_m^F]]$. Note that $b_{k,1}$ can be thought of as the concatenation of blocks from $\mathcal{B}_{m+1}$, meaning the concatenation of blocks of the form c_{m+1} and $b_{m+1,j}$, $j = 1, \ldots, n_{m+1}$. Thus, we can obtain a lower bound on the number of these L_m-sized subwords in $b_{k,1}$ by counting the number of L_m-sized subwords in $b_{k,1}$ that are within one of the $\mathcal{B}_{m+1}$-words that lie in $[[\mathcal{B}_m^F]]$. This quantity is in turn bounded below by the product of the interior count I_m and the multiplicity count M_m, where I_m is the number of L_m-sized subwords within one $b_{m+1,j}$-block that lie in $[[\mathcal{B}_m^F]]$ and M_m is the number of $b_{m+1,j}$-blocks in $b_{k,1}$. We count each of these separately.

For the interior count I_m, consider one $b_{m+1,j}$-block and consider all of the L_m-sized windows that lie in $[[\mathcal{B}_m^F]]$. Such a window yields a subword that is not in $[[\mathcal{B}_m^F]]$ exactly when it straddles two words of the form $b_{m,i}$ and $b_{m,i+1}$, b_{m,n_m} and $b_{m,1}$, or b_{m,n_m} and c_m. The number of such subwords is at most $L_m n_m n_m^{2(n_{m+1}+1)}$. Since there are exactly $L_{m+1} - (L_m - 1)$ subwords of length L_m in any $b_{m+1,j}$ we have

$$I_m \geq L_{m+1} - (L_m - 1) - L_m n_m^{2n_{m+1}+3} \geq L_{m+1} - L_m - L_m n_m^{2n_{m+1}+3}.$$

For the multiplicity count M_m, we again view $b_{k,1}$ as the concatenation of blocks from $\mathcal{B}_{m+1}$ and count the number that are of the form $b_{m+1,j}$ for some j. We do this in steps: first view $b_{k,1}$ as the concatenation of blocks from $\mathcal{B}_{k-1}$ and note that there must be L_k/L_{k-1} such blocks and all but one is of the form $b_{k-1,i}$ for some i. Each of these $b_{k-1,i}$, in turn, can be thought of as the concatenation of blocks from $\mathcal{B}_{k-2}$. There are L_{k-1}/L_{k-2} such blocks and all but one is of the form $b_{k-2,i}$ for some i. We continue in this vein, and after $k-(m+1)$ steps, we have that the number of the $\mathcal{B}_{m+1}$-blocks of the form $b_{m+1,i}$ for some i is bounded below by

$$\prod_{i=m+2}^{k} \Big(\frac{L_i}{L_{i-1}} - 1\Big) = \prod_{i=m+2}^{k} \frac{L_i}{L_{i-1}}\Big(1 - \frac{L_{i-1}}{L_i}\Big).$$

Combining these two estimates, we have that

$$\frac{1}{L_k} \sum_{i=0}^{L_k-1} \delta_{\sigma^i x}([[\mathcal{B}_m^F]]) \geq \frac{1}{L_k}(L_{m+1} - L_m - L_m n_m^{2n_{m+1}+3}) \prod_{i=m+2}^{k} \frac{L_i}{L_{i-1}}\Big(1 - \frac{L_{i-1}}{L_i}\Big).$$

Since $L_k = \frac{L_k}{L_{m+1}} L_{m+1}$, we can rewrite the right hand side as

$$\frac{1}{L_{m+1}}(L_{m+1} - L_m - L_m n_m^{2n_{m+1}+3}) \frac{\prod_{i=m+2}^{k} \frac{L_i}{L_{i-1}}(1 - \frac{L_{i-1}}{L_i})}{\prod_{i=m+2}^{k} \frac{L_i}{L_{i-1}}}.$$

Then by (2) and (5), this last quantity is greater than or equal to

$$\Big(1 - \frac{1}{n_m}\Big) \prod_{i=m+2}^{k} \Big(1 - \frac{1}{n_{i-1}}\Big) = \prod_{i=m}^{k-1} \Big(1 - \frac{1}{n_i}\Big) \geq \prod_{i=m}^{\infty} \Big(1 - \frac{1}{n_i}\Big),$$

which by condition (22) shows that $\frac{1}{L_k} \sum_{i=0}^{L_k-1} \delta_{\sigma^i x}([[\mathcal{B}_m^F]]) > 1 - \frac{1}{4^{m+1}}$. ∎

We end this section by proving Proposition 4.4 which assures us that all ergodic measures on X for fixed k give approximately the same measure to sets of the form $[[b_{k,m}]]$.

Proof of Proposition 4.4. Fix $\varepsilon > 0$. Since $n_k \to \infty$, we can choose $K \in \mathbb{N}$ such that for all $k \geq K$ we have $\frac{2}{n_k} < \frac{\varepsilon}{4}$. Choose such a k and consider any pair $b_{k,m}, b_{k,j} \in \mathcal{B}_k$. By the ergodicity of ξ, we can find a point $x \in X$ such that

$$\xi([[b_{k,m}]]) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{[[b_{k,m}]]}(\sigma^i x)$$

and

$$\xi([[b_{k,j}]]) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{[[b_{k,j}]]}(\sigma^i x).$$

We first show that it is enough to look at the frequency of the sets $[[b_{k,m}]]$ and $[[b_{k,j}]]$ in a certain subword of the point x.

We rewrite the difference:

$$\begin{aligned} |\xi([[b_{k,m}]]) - \xi([[b_{k,j}]])| \leq & \left|\xi([[b_{k,m}]]) - \frac{1}{N} \sum_{i=0}^{N-1} 1_{[[b_{k,m}]]}(\sigma^i x)\right| \\ & + \left|\xi([[b_{k,j}]]) - \frac{1}{N} \sum_{i=0}^{N-1} 1_{[[b_{k,j}]]}(\sigma^i x)\right| \\ & + \left|\frac{1}{N} \sum_{i=0}^{N-1} 1_{[[b_{k,m}]]}(\sigma^i x) - \frac{1}{N} \sum_{i=0}^{N-1} 1_{[[b_{k,j}]]}(\sigma^i x)\right|. \end{aligned}$$

Choosing N large enough, we can assume that the first two terms are each bounded by $\frac{\varepsilon}{4}$. The last term is the difference between the number of times $\sigma^i x$ lands in the set $[[b_{k,m}]]$ as compared to $[[b_{k,j}]]$, for $i = 0$ to $N-1$. In other words, this last term simply gives the difference between the number of L_k-length-subblocks of $x_{[0,N-1]}$ that are in $[[b_{k,m}]]$ as compared to $[[b_{k,j}]]$.

Since x can be written as a concatenation of words from $\mathcal{B}_{k+1}$, there is a subword z of $x_{[0,N-1]}$ whose length is at least $N - 2L_{k+1}$ that can be written exactly as a concatenation of words from $\mathcal{B}_{k+1}$. We restrict our attention to this subword z, and let D_N denote the difference between the number of L_k-length-subblocks of z that are in $[[b_{k,m}]]$ as compared to $[[b_{k,j}]]$. Then choosing N such that $\frac{2L_{k+1}}{N} < \varepsilon/4$, we have

$$\frac{1}{N}\left|\sum_{i=0}^{N-1} 1_{[[b_{k,m}]]}(\sigma^i x) - \sum_{i=0}^{N-1} 1_{[[b_{k,j}]]}(\sigma^i x)\right| \leq \varepsilon/4 + \frac{1}{N} D_N.$$

We thus have

$$|\xi([[b_{k,m}]]) - \xi([[b_{k,j}]])| \leq 3\varepsilon/4 + \frac{1}{N} D_N.$$

We are then left with showing that D_N/N is sufficiently small. Consider the subword z and divide it into disjoint subblocks of $(k+1)$-words from $\mathcal{B}_{k+1}$. As we look at subblocks of z of length L_k, these are either entirely contained in one of these $(k+1)$-words or partially overlapping two adjacent $(k+1)$-words.

Let us first consider the L_k-length-subblocks of the second type, those overlapping two adjacent $(k+1)$-words in z. Since the number of ways an L_k-length-subblock can overlap two specific $(k+1)$-words is $L_k - 1$, and the number of adjacent $(k+1)$-words in z is bounded by $|z|/L_{k+1} \leq N/L_{k+1}$, we can bound the difference between these L_k-length-subblocks that are in $[[b_{k,m}]]$ as compared to $[[b_{k,j}]]$ by

$$L_k(N/L_{k+1}) = N(L_k/L_{k+1}) \leq N(1/n_k).$$

We next consider the L_k-length-subblocks of the first type, the ones entirely contained in one of the $(k+1)$-words from $\mathcal{B}_{k+1}$. If this $(k+1)$-word is an extended Feldman word (see (3)), then we see that blocks of length L_k either lie within a repeated k-word, $b_{k,\ell}^{n_k^{2(n_{k+1}+i)}}$, or an overlap between two k-words, $b_{k,\ell}b_{k,\ell+1}$. Note that the number of occurrences of $[[b_{k,m}]]$ and $[[b_{k,j}]]$ in $b_{k,\ell}^{n_k^{2(n_{k+1}+i)}}$, as ℓ ranges from 1 to n_k, are exactly the same. Thus we can bound the difference by the number of L_k-length-subblocks that overlap a subblock of the form $b_{k,\ell}b_{k,\ell+1}$. This is bounded by $L_k n_k n_k^{2(n_{k+1}-i+1)}$ for one $(k+1)$-word.

In the case that the $(k+1)$-word has the form of (4), then the only possible occurrences of $[[b_{k,m}]]$ and $[[b_{k,j}]]$ occur at the end, when c_{k+1} cycles through the various $b_{k,\ell}$. We can thus bound the difference between the occurrences of these sets by $L_k n_k$, which is less than the bound used above.

Thus altogether we have that

$$D_N \leq N(1/n_k) + L_k n_k n_k^{2(n_{k+1}-i+1)}(\text{number of } (k+1)\text{-words in } z).$$

Using that $N \geq (\text{number of } (k+1)\text{-words in } z)L_{k+1}$, we have

$$\frac{D_N}{N} \leq \frac{1}{n_k} + \frac{L_k n_k n_k^{2(n_{k+1}-i+1)}}{L_{k+1}} < \frac{1}{n_k} + \frac{1}{n_k} = \frac{2}{n_k}.$$

It then follows that

$$|\xi([[b_{k,m}]]) - \xi([[b_{k,j}]])| \leq \varepsilon. \qquad \blacksquare$$

5. Loosely Bernoulli

In Section 4, we made use of the words $b_{k,i}$ to find a measure that yielded a non-loosely Bernoulli system. Now we make use of the words c_k to find a loosely Bernoulli system. We begin with a result that is analogous to Proposition 4.3 in that it shows that there is an ergodic measure that gives large measure to the sets $[[c_m]]$ for sufficiently large m.

PROPOSITION 5.1: *If the sequence $\{n_k\}$ grows sufficiently rapidly, then there exists an ergodic measure ξ supported on X with the property that*

$$\lim_{m\to\infty} \xi([[c_m]]) = 1. \tag{25}$$

Proof. This proof is very similar to the proof of Proposition 4.3. Suppose $\{n_k\}$ satisfies (22). We begin by noting that, because of Lemma 4.7, it suffices to show that there is a σ-invariant (but not necessarily ergodic) measure μ which satisfies

$$\mu([[c_m]]) > 1 - \frac{1}{4^m} \quad \text{for all } m = 1, 2, \ldots.$$

We find μ by choosing $y \in X$ such that the restriction of y to its first L_k symbols is exactly the word c_k for every k. Just as in Proposition 4.3, we need only show that

$$\frac{1}{L_k} \sum_{i=0}^{L_k-1} \delta_{\sigma^i y}([[c_m]]) > 1 - \frac{1}{4^{m+1}}. \tag{26}$$

Thus it suffices to count the number of L_m-sized subwords of c_k that lie in $[[c_m]]$, which is bounded below by the number of L_m-sized subwords of c_k that lie both in $[[c_m]]$ and within one of the c_{m+1} words which makes up c_k. We bound this last quantity with the product of the interior count I_m and the multiplicity count M_m, where I_m is the number of L_m-sized subwords within one c_{m+1}-block that lie in $[[c_m]]$ and M_m is the number of c_{m+1}-blocks one has when c_k is thought of as the concatenation of blocks from $\mathcal{B}_{m+1}$. We count each of these separately.

For the interior count I_m, consider one c_{m+1}-block and consider all of the L_m-sized windows that lie in $[[c_m]]$. Given the structure of c_{m+1} (see (4)), every L_m-sized subword that lies within the $c_m^{(L_{m+1}/L_m)-n_m}$ portion of c_{m+1} lies in $[[c_m]]$. There are $L_{m+1} - L_m n_m - L_m$ such subwords and thus

$$I_m \geq L_{m+1} - L_m(n_m + 1).$$

For the multiplicity count M_m, we again view c_k as the concatenation of blocks from $\mathcal{B}_{m+1}$ and count the number that are equal to c_{m+1}. We do this in steps: first view c_k as the concatenation of blocks from $\mathcal{B}_{k-1}$ and note that there must be L_k/L_{k-1} such blocks and all but n_{k-1} are c_{k-1}'s. Each of these c_{k-1}, in turn, can be thought of as the concatenation of blocks from $\mathcal{B}_{k-2}$. There are L_{k-1}/L_{k-2} such blocks and all but n_{k-2} are c_{k-2}'s. Continue in this vein, and after $k-(m+1)$ steps, we have that the number of the $\mathcal{B}_{m+1}$-blocks that are c_{m+1} is exactly

$$M_m = \Big(\frac{L_k}{L_{k-1}} - n_{k-1}\Big)\Big(\frac{L_{k-1}}{L_{k-2}} - n_{k-2}\Big)\dots\Big(\frac{L_{m+2}}{L_{m+1}} - n_{m+1}\Big).$$

Combining these two estimates, we have that $\frac{1}{L_k}\sum_{i=0}^{L_k-1}\delta_{\sigma^i y}([[c_m]])$ is bounded below by

$$\frac{1}{L_k}(L_{m+1} - L_m(n_m+1))\prod_{i=m+2}^{k}\Big(\frac{L_i}{L_{i-1}} - n_{i-1}\Big).$$

Since $L_k = \frac{L_k}{L_{m+1}}L_{m+1}$, we can write this as

$$\frac{1}{L_{m+1}}(L_{m+1}-L_m(n_m+1))\frac{\prod_{i=m+2}^{k}(\frac{L_i}{L_{i-1}}-n_{i-1})}{\prod_{i=m+2}^{k}\frac{L_i}{L_{i-1}}}$$
$$=\frac{1}{L_{m+1}}(L_{m+1}-L_m(n_m+1))\prod_{i=m+2}^{k}\Big(1-n_{i-1}\frac{L_{i-1}}{L_i}\Big).$$

Since $L_i = L_{i-1}(1+n_{i-1}^{4n_i+3})$, we have $L_{i-1}/L_i = 1/(1+n_{i-1}^{4n_i+3})$ and thus

$$(n_{i-1}L_{i-1})/L_i \le 1/n_{i-1}.$$

Similarly,

$$(n_m+1)L_m/L_{m+1} \le (n_m+1)/n_m^{4n_{m+1}+3} \le 1/n_m.$$

We thus have

$$\frac{1}{L_k}\sum_{i=0}^{L_k-1}\delta_{\sigma^i y}([[c_m]]) \ge \Big(1-\frac{1}{n_m}\Big)\prod_{i=m+2}^{k}\Big(1-\frac{1}{n_{i-1}}\Big).$$

It then follows from (22) that $\frac{1}{L_k}\sum_{i=0}^{L_k-1}\delta_{\sigma^i y}([[c_m]]) \ge 1-\frac{1}{4^{m+1}}$. ∎

THEOREM 5.2: *If the sequence $\{n_k\}$ grows sufficiently rapidly, then X carries a measure ξ such that (X, σ, ξ) is loosely Bernoulli.*

Proof. Let $\{n_k\}$ be a sequence that satisfies (22). We apply Proposition 5.1 to obtain a measure ξ satisfying (25).

Let K be large enough such that both $\frac{1}{n_{K-1}} < \varepsilon$ and $\xi([[c_K]]) > 1 - \varepsilon$. Take $k \geq K$. Let $x, y \in \mathcal{L}_{L_k}(X)$ be two words that occur as subwords of $c_k c_k$, so each is a word of length L_k which looks like the end of a c_k concatenated with the beginning of a c_k.

Recall that c_k is a concatenation of many copies of c_{k-1} followed by the block $b_{k-1,1}, \ldots, b_{k-1,n_{k-1}}$. Therefore, by eliminating at most $L_{k-1}(n_{k-1}+2)$ indices from both x and y, we can remove any indices corresponding to the extended Feldman words and any partial copies of c_{k-1}. What is left now are words of the form c_{k-1}^r and c_{k-1}^s, where r and s could differ as different choices of x and y might necessitate removal of a different number of indices. However, $\frac{L_k}{L_{k-1}} - n_{k-1} - 2 \leq r, s \leq \frac{L_k}{L_{k-1}} - n_{k-1}$, i.e. $|r - s| \leq 2$. Thus by throwing out at most another $2L_{k-1}$ indices, we obtain identical strings.

This means that

$$\overline{f}(x, y) \leq \frac{1}{2L_k}(L_{k-1}(n_{k-1}+2) + 2L_{k-1}) = \frac{L_{k-1}(n_{k-1}+2)}{L_k} < \frac{1}{n_{k-1}} < \varepsilon.$$

Taking W and N in Definition 2.2 to be the set $[[c_K]]$ and our choice of K respectively yields the result. ∎

6. Proof of Theorem 1.1

We are now ready to prove our main result, showing that the system (X, σ) we constructed carries both a loosely Bernoulli measure and a non-loosely Bernoulli measure.

Proof of Theorem 1.1. We fix a non-decreasing sequence $\{p_n\}_{n \in \mathbb{N}}$ satisfying

$$\liminf_{n \to \infty} \frac{p_n}{n} = \infty \quad \text{and} \quad \limsup_{n \to \infty} \frac{\log p_n}{n} = 0.$$

We choose a sequence $\{n_k\}_{k \geq 0}$ satisfying all of the growth conditions needed to apply our arguments. More precisely, we choose this sequence such that $n_0 \geq 2$ and $n_k \to \infty$, and the growth conditions corresponding to (10), (21) and (22)

are all satisfied, meaning that

$$\prod_{k=0}^{\infty} \frac{n_k}{n_k - 2} \leq 2; \quad \sum_{k=0}^{\infty} \frac{2}{n_k} \leq \frac{1}{32}; \quad \prod_{k=0}^{\infty} \frac{n_k^{4n_{k+1}+3}}{n_k^{4n_{k+1}+3} + 1} > \frac{7}{8};$$
$$\prod_{k=m}^{\infty} \left(1 - \frac{1}{n_k}\right) > 1 - \frac{1}{4^{m+1}}.$$

We remark that while it is possible to simplify the assumptions on the growth, as some of these conditions imply others, for clarity in how they are used, we keep each of them. As these are all growth conditions on the sequence, they are clearly compatible. For example, the sequence $n_k = 4^{k+4}$ suffices.

We inductively modify the sequence $\{n_k\}_{k\geq 0}$, again only possibly increasing the growth rate. Given n_k, choose $n_{k+1} > n_k$ such that, in addition to the growth conditions already satisfied (meaning conditions (10), (21) and (22)), we also have that if $m = L_k n_k^{2n_{k+1}+2}$, then

$$p_m > k(6 + 3n_k)m. \tag{27}$$

Now construct a subshift X as described in Section 3.3. By Proposition 3.1, Theorem 4.1, and Theorem 5.2 this subshift X is minimal, and there exist $\nu, \xi \in \mathcal{M}_e(X)$ such that (X, σ, ν) is not loosely Bernoulli and (X, σ, ξ) is loosely Bernoulli.

We now check that X satisfies the complexity condition

$$\liminf_{n\to\infty} \frac{P_X(n)}{p_n} = 0. \tag{28}$$

Fix k and consider the words of length $m\,(= L_k n_k^{2n_{k+1}+2})$. Since any point $x \in X$ can be written as a bi-infinite concatenation of elements of $\mathcal{B}_{k+1}$, we can count the number of words of length m by counting the number found entirely within an element in $\mathcal{B}_{k+1}$ or overlapping two concatenated elements of $\mathcal{B}_{k+1}$.

To make this count, recall the forms of the words $b_{k+1,i}$ and c_{k+1} given in (3) and (4). The word $b_{k+1,1}$ has each $b_{k,j}$ repeated exactly $n_k^{2(1+n_{k+1})}$ times, meaning that this portion of $b_{k+1,1}$ has length $L_k n_k^{2(1+n_{k+1})} = m$. The other $b_{k+1,i}$ have even longer lengths of repeated k-words.

We first count the number of words of length m that are subwords of some $b_{k+1,i}$. Note that for each choice of i, one of the following occurs:

(i) The word of length m is a subword of a repeated k-word $b_{k,j}^{n_k^{2(i+n_{k+1})}}$ for $j = 1, \ldots, n_k$. The repetition of $b_{k,j}$ means the number of distinct words of length m of this type for a specific j is just the length of $b_{k,j}$ and thus altogether we have $L_k n_k$ number of such words of length m.

(ii) The word of length m overlaps two consecutive repeated k-words and thus is a subword of $b_{k,j}^{n_k^{2(i+n_{k+1})}} b_{k,j+1}^{n_k^{2(i+n_{k+1})}}$. For a specific j, we can count the number of distinct words of length m by counting the number of locations within the word at which it switches from a $b_{k,j}$ to a $b_{k,j+1}$: this number is m. Since j ranges from 1 to n_k, this yields a total of mn_k distinct words.

(iii) The word of length m lies towards the end of $b_{k+1,i}$ and thus, for all i, is a subword of $b_{k,n_k}^{n_k^{2(i+n_{k+1})}} c_k$. The number of distinct words of length m can be counted by simply noting the number of places at which c_k can begin in the word, which is L_k.

To count the number of words of length m that are subwords of c_{k+1}, note that one of two situations occur:

(i) The word of length m is a subword of the repeated c_k's. There are L_k distinct such subwords.

(ii) The word of length m lies towards the end of c_{k+1} and thus is a subword of $c_k \cdots c_k b_{k,1} b_{k,2} \cdots b_{k,n_k}$. We can count this by counting the number of locations within the word at which it switches from the c_k to $b_{k,1}$, which is $n_k L_k$.

Thus the number of words of length m that are subwords of some element $\mathcal{B}_{k+1}$ is $2L_k n_k + mn_k + 2L_k$.

We next count the number of words of length m that overlap two concatenated elements of $\mathcal{B}_{k+1}$. There are four possible combinations for such concatenations:

$$\text{(i) } b_{k+1,i} b_{k+1,j}, \quad \text{(ii) } c_{k+1} b_{k+1,j}, \quad \text{(iii) } b_{k+1,j} c_{k+1}, \quad \text{(iv) } c_{k+1} c_{k+1}.$$

To count the number of combinations for the first situation, we note that for all choices of i and j, the word of length m is a subword of $b_{k,n_k} \cdots b_{k,n_k} c_k b_{k,1} \cdots b_{k,1}$. We count the number of distinct words of length m by noting that there are exactly m locations where we can see the beginning of c_k within it, which leads

to a count of m distinct words. The three other possible combinations can be counted in a similar manner, with each yielding at most m distinct words. Thus we obtain a total of $4m$ words of this type.

Combining these counts and using that $L_k \leq m$, we have that

$$P_X(m) \leq m(3n_k + 6).$$

But by (27), we have that $mk(3n_k + 6) < p_m$. Therefore $P_X(m) \leq p_m/k$. But recall that $m = L_k n_k^{2n_{k+1}+2}$ so this shows that there is a sequence, indexed by k, showing that (28) is satisfied.

Finally, to see that (X, σ) has zero topological entropy note that by (28) there are infinitely many n for which $P_X(n) \leq p_n$. Combining this fact with the subexponential growth condition on the sequence $\{p_n\}$ given in (1), we have that

$$\liminf_{n\to\infty} \frac{\log P_X(n)}{n} \leq \limsup_{n\to\infty} \frac{\log p_n}{n} = 0.$$

Since the limit defining topological entropy exists, it follows that

$$h_{top}(X) = \lim_{n\to\infty} \frac{\log P_X(n)}{n} = \liminf_{n\to\infty} \frac{\log P_X(n)}{n} = 0. \qquad \blacksquare$$

Appendix A. Ferenczi's theorem on the rank of systems of linear complexity

We would like to conclude from our work here that a minimal system that has a non-loosely Bernoulli measure must have $\liminf_{n\to\infty} P_X(n)/n = \infty$. If we knew that a minimal subshift whose complexity function satisfies $\liminf_{n\to\infty} P_X(n)/n < \infty$ has finite rank, then as discussed previously we would have that all measures supported on such a subshift are loosely Bernoulli and our conclusion would follow. While Ferenczi [7] shows that a minimal subshift whose complexity function satisfies $P_X(n) = O(n)$ has finite rank, essentially the same proof can be used to show that the result holds under the slightly weaker hypothesis $\liminf_{n\to\infty} P_X(n)/n < \infty$. As we were not able to find the liminf version of this result in the literature, for completeness we include the proof which demonstrates how to use Ferenczi's argument to get this result.

First we recall what it means for a system to have finite rank, taking the definition directly from [8, Definition 8]:

Definition: A subshift (X, σ) and invariant measure μ has **rank at most** r if for every partition $P = (A_1, A_2, \dots, A_{|P|})$ and every $\varepsilon > 0$, there exist r many subsets $F_i \subseteq X$, r many positive integers h_i, and a partition $P' = (B_1, B_2, \dots, B_{|P'|})$ such that

(1) all sets of the form $T^k F_i$, where $1 \le i \le r$ and $0 \le k < h_i$, are pairwise disjoint;

(2)
$$d(P, P') := \min\left\{ \sum_{i=1}^{\max\{|P|,|P'|\}} \mu(A_i \triangle B_{\sigma(i)}) \colon \sigma \in \mathrm{Sym}(\max\{|P|, |P'|\}) \right\} < \varepsilon$$
where P or P' have been padded with null sets to give them the same number of elements;

(3) the elements of P' can be expressed as unions of elements of the partition consisting of all sets of the form $T^k F_i$, where $1 \le i \le r$ and $0 \le k < h_i$, as well as $X \setminus \bigcup_i \bigcup_k T^k F_i$.

The subshift (X, σ) has **finite rank** if it has rank at most r for some $r \in \mathbb{N}$.

While the following lemma is slightly stronger than [7, Proposition 4], it readily follows from the proof given there. For completeness we include the argument.

LEMMA: *Let (X, σ) be a subshift and let μ be an invariant measure supported on X. If*
$$\liminf_{n\to\infty} \frac{P_X(n)}{n} < \infty, \tag{29}$$
then (X, σ, μ) has finite rank.

Proof. By (29), there is an integer R and an increasing sequence $\{n_j\}_{j=1}^{\infty}$ such that
$$P_X(n_j + 1) - P_X(n_j) \le R \tag{30}$$
for all $j \ge 1$. By (30), there are at most R many elements of $\mathcal{L}_{n_j+1}(X)$ whose leftmost subword of length n_j is a word that fails to extend uniquely to its right. Passing to a subsequence if necessary, we can assume there exists $r \le R$ such that for all j there are exactly r many elements of $\mathcal{L}_{n_j}(X)$ that appear as the

rightmost subword of length n_j in a word in $\mathcal{L}_{n_j+1}(X)$ whose leftmost subword of length n_j fails to extend uniquely to its right. For each j enumerate these words as

$$F_1^j, F_2^j, \ldots, F_r^j \in \mathcal{L}_{n_j}(X).$$

Now for each $1 \leq k \leq r$ let $h_k^j \in \mathbb{N}$ be the smallest positive integer for which there exists $1 \leq i \leq r$ such that $[F_i^j] \cap \sigma^{h_k^j}[F_k^j] \neq \emptyset$.

Note that all sets of the form $\sigma^k[F_i^j]$, where $1 \leq i \leq r$ and $0 \leq k < h_i^j$, are pairwise disjoint by definition of h_i^j. (These sets are actually a partition of X and there is no need to add the complement of their union.) Also note that the elements of the partition Q_j of X into cylinder sets of length n_j are unions of elements of the partition given by sets of the form $\sigma^k[F_i^j]$, where $1 \leq i \leq r$ and $0 \leq k < h_i^j$, meaning that Q_j is a coarser partition. Finally, note that any partition P of X can be approximated arbitrarily well by partitions coarser than the partitions into cylinder sets of increasing length. Therefore for any P and any $\varepsilon > 0$ there exists j such that some partition, P', coarser than Q_j satisfies $d(P, P') < \varepsilon/2$ and such that

$$d(Q, \{\sigma^k[F_i^j] \colon 1 \leq i \leq r, 0 \leq k < h_i^j\}) < \varepsilon/2.$$

Thus (X, σ, μ) has rank at most r, and in particular it has finite rank. ∎

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