

ON UNIFORM AND COARSE RIGIDITY OF $L^p([0, 1])$

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ABSTRACT. If X is an almost transitive Banach space with amenable isometry group (for example, if $X = L^p([0, 1])$ with $1 \leq p < \infty$) and X admits a uniformly continuous map $X \xrightarrow{\phi} E$ into a Banach space E satisfying

$$\inf_{\|x-y\|=r} \|\phi(x) - \phi(y)\| > 0$$

for some $r > 0$ (that is, ϕ is *almost uncollapsed*), then X admits a simultaneously uniform and coarse embedding into a Banach space V that is finitely representable in $L^2(E)$.

The aim of the present paper is to prove a rigidity result regarding a priori very weak notions of embeddings between Banach spaces assuming that the domain space satisfies additional analytical assumptions. The general motivating problem for our study is the following still unresolved question.¹

Question 1. *Are the following two properties equivalent for all Banach spaces X and E ?*

- (a) X uniformly embeds into E ,
- (b) X coarsely embeds into E .

Let us recall that a map $X \xrightarrow{\phi} E$ is a *uniform embedding* if, for all sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in X , we have

$$\lim_n \|x_n - y_n\| = 0 \Leftrightarrow \lim_n \|\phi(x_n) - \phi(y_n)\| = 0,$$

whereas it is a *coarse embedding* if, for all sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$, we have

$$\lim_n \|x_n - y_n\| = \infty \Leftrightarrow \lim_n \|\phi(x_n) - \phi(y_n)\| = \infty.$$

Alternatively, these notions may be expressed in terms of certain moduli of the map ϕ . Namely, for a map $X \xrightarrow{\phi} E$, let us define its *compression modulus* κ_ϕ by

$$\kappa_\phi(r) = \inf_{\|x-y\| \geq r} \|\phi(x) - \phi(y)\|$$

and its *expansion modulus* θ_ϕ by

$$\theta_\phi(r) = \sup_{\|x-y\| \leq r} \|\phi(x) - \phi(y)\|.$$

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¹The origin of the question is not quite clear, but the lacking understanding of the relationship between uniform and coarse embeddings of Banach spaces was already pointed out in N. J. Kalton's survey paper [6].

Then ϕ is a uniform embedding if and only if $\lim_{r \rightarrow 0+} \theta_\phi(r) = 0$ and $\kappa_\phi(r) > 0$ for all $r > 0$. Similarly, ϕ is a coarse embedding if and only if $\lim_{r \rightarrow \infty} \kappa_\phi(r) = \infty$ and $\theta_\phi(r) < \infty$ for all $r < \infty$.²

As noted, Question 1 remains open in general, but the strongest result to date is that (a) implies (b) provided that $E \oplus E$ is isomorphic to a closed subspace of E [11] (see also [10, 2] for related results). In fact, in [11] a stronger result is obtained, which states that, with the same assumption on E , if $X \xrightarrow{\phi} E$ is a uniformly continuous map satisfying just that

$$(1) \quad \kappa_\phi(r) > 0$$

for some $r < \infty$, then X admits a simultaneously coarse and uniform embedding into E . Maps satisfying (1) for some $r < \infty$ are called *uncollapsed* in [10] and that paper also introduces the even weaker property that

$$(2) \quad \inf_{\|x-y\|=r} \|\phi(x) - \phi(y)\| > 0$$

for some $r < \infty$, which, in turn, is termed *almost uncollapsed* in [3].³ Observe that both coarse and uniform embeddings are uncollapsed and a fortiori almost uncollapsed. Note also that, for example, the exponential map

$$t \in \mathbb{R} \mapsto e^{it} \in \mathbb{C}$$

is almost uncollapsed, but not uncollapsed. Furthermore, building on work of Kalton [5], Corollary 11 of [10] states that there is no uniformly continuous almost uncollapsed map from c_0 into a reflexive Banach space. In [3], B. Braga takes these issues further and shows that, if $X \xrightarrow{\phi} E$ is a uniformly continuous almost uncollapsed map and E has nontrivial type, then $q_X \leq q_E$, where

$$q_X = \inf\{q \in [2, \infty[\mid X \text{ has cotype } q\}$$

and similarly for E (see Theorem 1.3 [3]). Braga's theorem is a variation of earlier breakthrough results by M. Mendel and A. Naor (Theorem 1.9 and Theorem 1.11 [7]) giving the same conclusion provided that X embeds either uniformly or coarsely in E .

Our main result, Theorem 2, instead aims more directly at Question 1 by producing a simultaneously uniform and coarse embedding of X into a new space with local properties similar to those of E . For example, note that cotype is preserved under both finite representability and under the passage from E to $L^2(E)$. However, this comes at the cost of imposing a significant restriction on X that is satisfied, for example, by $X = L^p([0, 1])$, $1 \leq p < \infty$, and $G = \text{Isom}(X)$.

Theorem 2. *Suppose X is a Banach space admitting a strongly continuous linear isometric action $G \curvearrowright X$ by an amenable topological group so that G induces a dense orbit on the unit sphere S_X . Assume also there is a uniformly continuous map $X \xrightarrow{\phi} E$ into a Banach space E so that*

$$\inf_{\|x-y\|=r} \|\phi(x) - \phi(y)\| > 0$$

²Observe that the condition $\lim_{r \rightarrow 0+} \theta_\phi(r) = 0$ simply expresses that ϕ is uniformly continuous, in which case θ_ϕ is also known as the modulus of uniform continuity.

³The modulus $\tilde{\kappa}_\phi(r) = \inf_{\|x-y\|=r} \|\phi(x) - \phi(y)\|$ is called the *exact compression modulus* in [10]. Thus, uncollapsed maps have non-trivial compression modulus, whereas almost uncollapsed maps have non-trivial exact compression modulus.

for some $r > 0$. Then X admits a simultaneously uniform and coarse embedding into a Banach space V that is finitely representable in $L^2(E)$.

Proof. Without loss of generality, we may assume that $\dim X \geq 2$. By Proposition 7 of [10], from ϕ we can construct another uniformly continuous map $X \xrightarrow{\psi} \ell^2(E)$, which, moreover, is *solvent*, i.e., so that, for some sequence $(R_n)_{n \in \mathbb{N}}$ of positive constants and all $x, y \in X$, we have

$$(3) \quad R_n \leq \|x - y\| \leq R_n + n \Rightarrow \|\psi(x) - \psi(y)\| \geq n.$$

Consider now the semidirect product group $X \rtimes G$ arising from the linear action of G on X . For clarity of notation, elements $x \in X$ are denoted by τ_x and the group operation is written multiplicatively. Thus every element of $X \rtimes G$ can be written uniquely in the form $\tau_x g$ for some $x \in X$ and $g \in G$. Moreover, for all $x, y \in X$ and $g \in G$, we have

$$\tau_x \tau_y = \tau_{x+y} \quad \text{and} \quad g \tau_x = \tau_{g(x)} g.$$

Define a pseudometric d on $X \rtimes G$ by $d(\tau_x g, \tau_y f) = \|x - y\|$ and note that d is left-invariant. Indeed, for all $x, y, z \in X$ and $g, f, h \in G$, we have

$$\begin{aligned} d(\tau_z h \cdot \tau_x g, \tau_z h \cdot \tau_y f) &= d(\tau_{z+h(x)} h g, \tau_{z+h(y)} h f) \\ &= \|(z + h(x)) - (z + h(y))\| \\ &= \|h(x) - h(y)\| \\ &= \|x - y\| \\ &= d(\tau_x g, \tau_y f), \end{aligned}$$

showing left-invariance. Note also that, if $X \rtimes G$ is given the topology induced by the identification with the cartesian product $X \times G$ via $(x, g) \mapsto \tau_x g$, then d is continuous. In other words, d is a continuous left-invariant pseudometric on the topological semidirect product $X \rtimes G$.

Define now a map $X \rtimes G \xrightarrow{\Psi} \ell^2(E)$ by setting $\Psi(\tau_x g) = \psi(x)$ and note that Ψ is uniformly continuous with respect to the pseudometric d . For $t > 0$, let also

$$\tilde{\kappa}(t) = \inf_{d(\tau_x g, \tau_y f) = t} \|\Psi(\tau_x g) - \Psi(\tau_y f)\| = \inf_{\|x - y\| = t} \|\psi(x) - \psi(y)\|$$

and

$$\theta(t) = \sup_{d(\tau_x g, \tau_y f) \leq t} \|\Psi(\tau_x g) - \Psi(\tau_y f)\| = \sup_{\|x - y\| \leq t} \|\psi(x) - \psi(y)\|.$$

As X is abelian and G amenable, also $X \rtimes G$ is amenable (see, for example, Theorem G.2.1 and Proposition G.2.2.(ii) in [1]). So, by Theorem 6.1 of [12], which improves Theorem 16 of [10], there is a Banach space V , that is finitely representable in $L^2(\ell^2(E))$ and thus also in $L^2(E)$, and a continuous linear isometric action $X \rtimes G \curvearrowright^\pi V$ with an associated continuous cocycle $X \rtimes G \xrightarrow{b} V$ satisfying the bounds

$$(4) \quad \tilde{\kappa}(d(\tau_x g, \tau_y f)) \leq \|b(\tau_x g) - b(\tau_y f)\| \leq \theta(d(\tau_x g, \tau_y f)),$$

that is,

$$\tilde{\kappa}(\|x - y\|) \leq \|b(\tau_x g) - b(\tau_y f)\| \leq \theta(\|x - y\|)$$

for all $x, y \in X$ and $g, f \in G$. Here, that b is a cocycle simply means that it satisfies the equation

$$(5) \quad b(\sigma\gamma) = \pi(\sigma)b(\gamma) + b(\sigma)$$

for all $\sigma, \gamma \in X \rtimes G$.

Observe that, by the solvency condition (3) on ψ , we have $\tilde{\kappa}(t) \geq n$ whenever $R_n \leq t \leq R_n + n$. It thus follows that

$$R_n \leq \|x - y\| \leq R_n + n \Rightarrow \|b(\tau_x g) - b(\tau_y f)\| \geq n$$

for all $n \geq 1$, $x, y \in X$ and $g, f \in G$.

Note also that, by the cocycle equation (5), we have for $\sigma, \gamma \in X \rtimes G$ that

$$b(\gamma) = b(\sigma\sigma^{-1}\gamma) = \pi(\sigma)b(\sigma^{-1}\gamma) + b(\sigma),$$

whereby

$$\|b(\gamma) - b(\sigma)\| = \|\pi(\sigma)b(\sigma^{-1}\gamma)\| = \|b(\sigma^{-1}\gamma)\|.$$

Observe also that, because $\tau_0 = 1 = \tau_0 1$ is the identity in $X \rtimes G$, we have $b(\tau_0 1) = 0$ and therefore, for all $g \in G$,

$$\|b(g)\| = \|b(\tau_0 g) - b(\tau_0 1)\| \leq \theta(\|0 - 0\|) = 0,$$

that is $b(g) = 0$.

Suppose now that x and y are two elements of X with $\|x\| = \|y\|$. Then, because some and hence every G -orbit on S_X is dense, there are $g_n \in G$ so that $g_n(x) \xrightarrow[n \rightarrow \infty]{} y$. It thus follows from continuity of b and the cocycle equation that

$$\begin{aligned} \|b(\tau_y)\| &= \lim_{n \rightarrow \infty} \|b(\tau_{g_n(x)})\| \\ &= \lim_{n \rightarrow \infty} \|b(g_n \tau_x g_n^{-1})\| \\ &= \lim_{n \rightarrow \infty} \|\pi(g_n \tau_x) b(g_n^{-1}) + \pi(g_n) b(\tau_x) + b(g_n)\| \\ &= \lim_{n \rightarrow \infty} \|\pi(g_n \tau_x) 0 + \pi(g_n) b(\tau_x) + 0\| \\ &= \lim_{n \rightarrow \infty} \|b(\tau_x)\| \\ &= \|b(\tau_x)\|. \end{aligned}$$

In other words, $\|b(\tau_x)\| = \|b(\tau_y)\|$ whenever $\|x\| = \|y\|$.

Let $x \in X$ be any element with $\alpha = \|x\| \geq R_n$. We claim that $\|b(\tau_x)\| \geq \frac{n}{2}$. To see this, pick any $y \in X$ with $\|y\| = R_n$. Because $\dim X \geq 2$, $y + \alpha S_X$ is a path connected set containing both points of norm $\leq \alpha$ and points of norm $\geq \alpha$. Therefore, $y + \alpha S_X$ must intersect αS_X and thus $y = z + u$ for some z, u with $\|z\| = \|u\| = \alpha = \|x\|$. Because $R_n \leq \|y - 0\| \leq R_n + n$, it follows that

$$\begin{aligned} n &\leq \|b(\tau_y) - b(\tau_0)\| \\ &= \|b(\tau_y)\| \\ &= \|b(\tau_z \tau_u)\| \\ &= \|\pi(\tau_z) b(\tau_u) + b(\tau_z)\| \\ &\leq \|\pi(\tau_z) b(\tau_u)\| + \|b(\tau_z)\| \\ &= \|b(\tau_u)\| + \|b(\tau_z)\| \\ &= 2\|b(\tau_x)\|, \end{aligned}$$

and so $\|b(\tau_x)\| \geq \frac{n}{2}$, as claimed.

From this it follows that, for all $n \geq 1$ and $x, y \in X$,

$$\|x - y\| \geq R_n \Rightarrow \|b(\tau_x) - b(\tau_y)\| = \|b(\tau_y^{-1}\tau_x)\| = \|b(\tau_{x-y})\| \geq \frac{n}{2}.$$

Thus, if we restrict $X \rtimes G \xrightarrow{b} V$ to the factor X , we obtain a continuous cocycle $X \xrightarrow{b} V$ for the action $X \curvearrowright^\pi V$ satisfying also

$$\lim_n \|x_n - y_n\| = \infty \Rightarrow \lim_n \|b(x_n) - b(y_n)\| = \infty.$$

Applying Proposition 1 [10], we see that $X \xrightarrow{b} V$ is uniformly continuous and that there are constants $c, C > 0$ so that, for all $x, y \in X$,

$$c \cdot \min\{\|x - y\|, 1\} \leq \|b(x) - b(y)\| \leq C\|x - y\| + C.$$

From this it follows that b is simultaneously a uniform and coarse embedding of X into V . $\square$

For $1 \leq p < \infty$, the Banach space $L^p([0, 1])$ is *almost transitive* (Theorem 9.6.3 and Theorem 9.6.4 [9]), that is, has a dense orbit on the unit sphere under the action of its linear isometry group $\text{Isom}(L^p([0, 1]))$. Furthermore, $\text{Isom}(L^p([0, 1]))$ is amenable when equipped with the strong operator topology (in fact, by Theorem 6.6 of [4], it is extremely amenable). The following corollary is therefore immediate.

Corollary 3. *Let $1 \leq p < \infty$ and suppose that there is a uniformly continuous map $L^p([0, 1]) \xrightarrow{\phi} E$ into a Banach space so that*

$$\inf_{\|x-y\|=r} \|\phi(x) - \phi(y)\| > 0$$

for some $r > 0$. Then $L^p([0, 1])$ admits a simultaneously uniform and coarse embedding into a Banach space V that is finitely representable in $L^2(E)$.

This result also supports the view that uniform continuity is the most significant requirement with regards to uniform embeddings, whereas uniform continuity of the inverse is easier to obtain. In this connection, A. Naor [8] gives an example of a bornologous map between two separable Banach spaces that is not close to any uniformly continuous map.⁴

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⁴A map $M \xrightarrow{\phi} N$ between two metric spaces is said to be *bornologous* provided that $\sup_{d(x,y) \leq t} d(\phi(x), \phi(y)) < \infty$ for all $t < \infty$. Moreover, two maps $M \xrightarrow{\phi, \psi} N$ are *close* provided that $\sup_x d(\phi(x), \psi(x)) < \infty$.

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