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Note

Distinct angles in general position ☆

Henry L. Fleischmann [a], Sergei V. Konyagin [b], Steven J. Miller [c], Eyvindur A. Palsson [d,*], Ethan Pesikoff [e], Charles Wolf [f]

[a] *Department of Mathematics, University of Michigan, Ann Arbor, 48109, United States of America*
[b] *Steklov Institute of Mathematics, 8 Gubkin Street, Moscow, 119991, Russia*
[c] *Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, United States of America*
[d] *Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, United States of America*
[e] *Department of Mathematics, Yale University, New Haven, CT 06511, United States of America*
[f] *Department of Mathematics, Rochester, NY, 14627, United States of America*

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A B S T R A C T

The Erdős distinct distance problem is a ubiquitous problem in discrete geometry. Somewhat less well known is the Erdős distinct angle problem, the problem of finding the minimum number of distinct angles between n non-collinear points in the plane. Recent work has introduced bounds on a wide array of variants of this problem, inspired by analogous questions in the distance setting.

In this short note, we improve the best known upper bound for the minimum number of distinct angles formed by n points in general position from $O(n^{\log_2(7)})$ to $O(n^2)$. We consider a point-set to be in general position if no three points lie on a common line and no four lie on a common circle. Before this work, similar bounds relied on projections onto a generic plane from higher dimensional space. In this paper, we introduce a construction employing the geometric properties of a logarithmic spiral, sidestepping the need for a projection.

We also apply this configuration to reduce the upper bound on the largest integer such that any set of n points in general position has a subset of that size with all distinct angles. This bound is decreased from $O(n^{\log_2(7)/3})$ to $O(n^{1/2})$.

1. Introduction

Erdős introduced the distinct distance problem in his 1946 paper “On sets of distances of n points,” in which he investigated the minimum number of distinct distances formed by n points in the plane. He conjectured a solution of $\Theta(n/\sqrt{\log n})$, the number of distances formed by points in the $\sqrt{n} \times \sqrt{n}$ integer lattice [2]. This problem, while simple to state, proved challenging. In 2015, Guth and Katz finally proved a nearly matching lower bound of $\Omega(n/\log n)$ on the minimal number of

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* Corresponding author.
E-mail addresses: henryfl@umich.edu (H.L. Fleischmann), konyagin23@gmail.com (S.V. Konyagin), sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu (S.J. Miller), palsson@vt.edu (E.A. Palsson), ethan.pesikoff@yale.edu (E. Pesikoff), charles.wolf@rochester.edu (C. Wolf).

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distinct distances [7]. Since 1946, numerous variants of the problem have been considered, including the minimum number of distinct distances on restricted point sets.

There is an analogous, far less studied problem for angles introduced by Erdős and Purdy [5]. What is $A(n)$, the minimum number of distinct angles formed by n non-collinear points on the plane? Corredi, Erdős, and Hajnal conjectured that regular n-gons are optimal configurations [5].

Recent work introduced new bounds on a variety of variants of the distinct angle problem [6]. In particular, $A_{\text{gen}}(n)$, the minimum number of distinct angles formed by n points in general position (with no three points on a line and no four on a circle) was shown to be $\Omega(n)$ and $O(n^{\log_2(7)})$. In this paper, we first show that the constructions in [6] can be extended to provide a bound of $O(n^2 2^{O(\sqrt{\log n})})$. We discuss this proof in Section 2. Then, by a new construction which avoids projections altogether and chooses a configuration of points on a logarithmic spiral, we have the following.

Theorem 1.1. *We have* $A_{gen}(n) = O(n^2)$.

Theorem 1.1 is proved in Section 3.

In Section 4 we consider a related variant of this distinct angle problem also considered in [6]. We call a point-set a *distinct-angle* point-set if it contains no repeated angles. We define $R_{\text{gen}}(n)$ to be the largest integer m such that any planar point-set of n points contains a distinct-angle subset of size m. In other words, this is the minimum—taken over all general position point-sets S of size n—of the maximum size of a distinct-angle subset of S. In [6] $R_{\text{gen}}(n)$ is shown to be $O(n^{\log_2(7)/3})$ and $\Omega(n^{1/5})$. As an application of the logarithmic spiral configuration we show the following.

Theorem 1.2. *We have* $R_{gen}(n) = O(\sqrt{n})$.

2. Discussion of methods

In [6], the bound $A_{\text{gen}}(n) = O(n^{\log_2(7)})$ is proved by projecting the vertices of a d-dimensional hypercube onto a generic plane. The argument relies closely on an observation from a paper of Erdős, Hickerson, and Pach [3]. Given an orthogonal projection T and points p_1, p_2, p_3, and p_4,

$$p_1 - p_2 = p_3 - p_4 \implies d(T(p_1), T(p_2)) = d(T(p_3), T(p_4)). \tag{2.1}$$

This follows from orthogonal projections being idempotent and self-adjoint. In [6], this observation is extended. Two (congruent) triangles with edges composed of the same difference vectors are mapped to congruent triangles under orthogonal projections. Hence, it suffices to count the number of classes of translation equivalent triangles to asymptotically bound the number of distinct angles in the configuration.

It turns out that a similar argument can be used to show that $A_{\text{gen}}(n) = O(n^2 2^{O(\sqrt{\log n})})$. It is easy to orthogonally project a high-dimensional point set onto the plane such that no four projected points lie on a circle. However, since we choose the projection to be injective, points on a line are projected onto a line. Hence, the original high-dimensional configuration must not have three points on a line.

In [6] this is avoided by drawing the points from a hypercube. However, in the paper of Erdős, Füredi, Pach, and Ruzsa showing the best known bound for the distance problem in general position, the points are instead drawn from a lattice [4]. The potential obstruction of three points on a line is avoided by taking a subset of the lattice points intersecting with a hypersphere. We outline a similar argument below to get an improved bound to illustrate how this projection technique may be extended. We take inspiration from a paper of Behrend [1].

Proposition 2.1. *We have* $A_{gen}(n) = O(n^2 2^{2 2\sqrt{\log_2 n}})$.

Proof. Consider a grid $G_{r,d} = \{0, \dots, r\}^d$.

The triples of points (a, b, c) and (a', b', c') are *equivalent* if the second triple can be obtained from the first triple by translation. Note that this is equivalent to requiring $a - b = a' - b'$, $a - c = a' - c'$, and $b - c = b' - c'$. Let p_i denote the ith coordinate of a point p. If we have any triple (a, b, c), then for $i = 1, \dots, d$ we can replace the triple of integers (a_i, b_i, c_i) by $(a_i - m_i, b_i - m_i, c_i - m_i)$ where $m_i = \min(a_i, b_i, c_i)$. If we do this for all i, we get an equivalent triple (a', b', c') satisfying $\min(a'_i, b'_i, c'_i) = 0$ for all i. The number of triples (a'_i, b'_i, c'_i) with $a'_i, b'_i, c'_i \in \{0, \dots, r\}$ and $\min(a'_i, b'_i, c'_i) = 0$ is $(r+1)^3 - r^3$. We call such a triple *reduced*. Thus, the number of reduced triples (a', b', c') is $N_{r,d} = ((r+1)^3 - r^3)^d$. Hence, the number of angles formed by points from $G_{r,d}$ is at most $N_{r,d}/2$, since our triples are ordered.

For $r > 1$ the points in $G_{r,d}$ are not in general position: there are many lines containing three or more points. For $a \in G_{r,d}$ we define $f(a) = \sum_{i=1}^{d} a_i^2$. We have $0 \le f(a) \le dr^2$. For $l = 0, \dots, dr^2$ we define $G_{r,d,l} = \{a \in G_{r,d} : f(a) = l\}$. We can take l so that $|G_{r,d,l}| \ge (r+1)^d (dr^2 + 1)^{-1}$, this quantity being the mean of the number of points at each radius $0, \dots, dr^2$. No three points from $G_{r,d,l}$ are on a line, as they lie on a common sphere. Taking a subset of the points of $G_{r,d,l}$, there is a set of $M := (r+1)^d (dr^2+1)^{-1}$ points with no three on a line.

Now, let $r = 2^d$ and assume for simplicity that $M = \lfloor 2^{d(d-2)}/d \rfloor$. For large enough n, there exists d such that $2^{(d-1)(d-3)}/(d-1) < n \le M$. Then, from the above, there exists some l such that a subset of $G_{r,d,l}$ has n points. This subset has no three points on a line, so the configuration can be projected onto a planar configuration in general position. So, it suffices to bound the number of translation equivalent triples by $N_{r,d}$ to yield a bound on $A_{\text{gen}}(n)$.

Now, note that, for $d \ge 17$, $d^2 \ge 16d + 4\log_2 d$. Then,

$$\begin{aligned} dn &\ge 2^{(d-1)(d-3)} \implies \\ \log_2 n &\ge (d-1)(d-3) - \log_2 d \implies \\ 4\log_2 n &\ge 3d^2 + d^2 - 16d + 12 - 4\log_2 d \ge d^2 \implies \\ 2\sqrt{\log_2 n} &\ge d. \end{aligned}$$

Now, we have

$$N_{r,d} = (3r^2 + 3r + 1)^d \le (4r^2)^d = 2^{2(d+1)d} < n^2 2^{11d} \le n^2 2^{22\sqrt{\log_2 n}},$$

yielding the desired result. □

3. An improved bound on $A_{\text{gen}}(n)$

In the previous section, the extra factor of $2^{O(\sqrt{\log n})}$ arises from taking a subset of the lattice without three points on a line. We can remove such a factor by avoiding projections altogether. In this section, we describe a configuration of points on a logarithmic spiral yielding $A_{\text{gen}}(n) = O(n^2)$.

Let the logarithmic spiral S be given by the polar equation $r = e^\theta$ for $\theta \in (-\infty, \infty)$. Note that there is a set of mappings $S \to S$ given by

$$F_\alpha(r, \theta) = (e^\alpha r, \theta + \alpha).$$

Scaling by e^α is a dilation, which maps triangles to similar triangles. Rotating by α also maps triangles to similar triangles. Hence, mapping via F_α preserves angles.

We now prove Theorem 1.1 that $A_{\text{gen}}(n) = O(n^2)$.

Proof. Let S be given by the polar equation $r = e^\theta$ for $\theta \in (-\infty, \infty)$. Then, consider the collection of points $\mathcal{P} = \{(e^{j\beta}, j\beta) : j \in [n]\}$ on S. First, note that, for sufficiently small β, $\mathcal{P}$ lies within a small arc S' of S. As this arc S' forms part of the boundary of its own convex hull C, any line ℓ intersecting C has at most two intersections with S'. Consequently no three $p \in \mathcal{P}$ lie on a common line. Likewise, since the curvature of S is strictly monotone, β can be chosen small enough such that no four points of $\mathcal{P}$ are on a common circle.

Now we show that the number of distinct angles formed by the points in $\mathcal{P}$, $A(\mathcal{P})$, is at most $3\binom{n-1}{2}$. Given a triple of distinct points $t = ((e^{j_1\beta}, j_1\beta), (e^{j_2\beta}, j_2\beta), (e^{j_3\beta}, j_3\beta)) \in \mathcal{P}^3$, let $m = \min\{j_1, j_2, j_3\}$. Then, the map $f_t := F_{(1-m)\beta}$ maps this triple to another forming the same angles, now with one of the points as (e^β, β).

Hence, each of the distinct angles formed by points in $\mathcal{P}$ is formed by a triple with one point (e^β, β). Observe that there are $\binom{n-1}{2}$ ways to choose the other two points in the triple, and each triple can yield at most three distinct angles. Then the number of distinct angles $A(\mathcal{P})$ formed by the points in $\mathcal{P}$ is at most $3\binom{n-1}{2}$, yielding $A_{\text{gen}}(n) = O(n^2)$, as desired. □

4. An improved bound on $R_{\text{gen}}(n)$

The fact that this configuration introduces no three points on a line and no four on a circle yields an improved upper bound for $R_{\text{gen}}(n)$. (Recall that $R_{\text{gen}}(n)$ is defined to be the largest integer m such that any planar point-set of n points contains a distinct-angle subset of size m.) The current best known upper bound on this quantity is $O(n^{\log_2(7)/3})$ from [6].

Letting $x_i, y_i \in [n]$ for $1 \le i \le 3$, we say that two triples $(x_1, x_2, x_3), (y_1, y_2, y_3)$ are equivalent if $x_1 - y_1 = x_2 - y_2 = x_3 - y_3$. We then have the following lemma.

Lemma 4.1. *Let $R \subseteq [n]$ such that $|R| = m$. If $\binom{m}{2} \ge 2n - 1$, then R contains a pair of distinct but equivalent triples.*

Proof. The number of pairs $(x, y) \in R^2$ such that $x > y$ is $\binom{m}{2}$, and the maximum number of possible differences is $n - 1$ (ranging from 1 to $n - 1$). Then the condition $\binom{m}{2} \ge 2n - 1$ ensures by the pigeonhole principle that there are three pairs with the same difference and hence a pair of equivalent triples. □

We now prove Theorem 1.2.

Proof. Let $\mathcal{P}$ be the logarithmic spiral point configuration as in Theorem 1.1. Let $\mathcal{P}' \subseteq \mathcal{P}$ with $|\mathcal{P}'| = m$, again assuming $\binom{m}{2} \geq 2n-1$. Define $Q \subseteq [n]$ such that $\mathcal{P}' = \{(e^{j\beta}, j\beta) : j \in Q\}$. By Lemma 4.1, Q contains a pair of equivalent triples $s = (x_1, x_2, x_3)$ and $t = (y_1, y_2, y_3)$. Therefore the triples of points in $\mathcal{P}'$ corresponding to s and t define repeated angles. This is because the triple of points corresponding to s are mapped to those corresponding to t by $F_{((y_1-x_1)\beta)}$.

Now, note that $m \geq 2n^{1/2} + 1/2$ implies $\binom{m}{2} \geq 2n-1$. Then $R_{\text{gen}}(n) = O(\sqrt{n})$, as desired. $\square$

5. Future work

While this paper substantially improves the state of the art upper bound for A_{gen} to $O(n^2)$, we still only have $A_{\text{gen}}(n) = \Omega(n)$. Lessening or even eliminating this gap would be interesting for future research. Additionally, this paper significantly improves the upper bound of $R_{\text{gen}}(n)$ to $O(\sqrt{n})$ from $O(n^{\log_2(7)/3})$ in [6]. Nonetheless, reducing the gap with the current lower bound of $\Omega(n^{1/5})$ (also from [6]) is an open problem.

The logarithmic spiral configuration may also have applications in other angle problems, such as repeated angle problems and angle chain problems appearing in the literature. For example, see Palsson, Senger, and Wolf's work on angle chains in [8].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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