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MANUSCRIPT

Quantitative Derivation and Scattering of the 3D Cubic NLS in the Energy Space

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Abstract
We consider the derivation of the defocusing cubic nonlinear Schrödinger equation (NLS) on $\mathbb{R}^3$ from quantum N-body dynamics. We reformat the hierarchy approach with Klainerman-Machedon theory and prove a bi-scattering theorem for the NLS to obtain convergence rate estimates under H^1 regularity. The H^1 convergence rate estimate we obtain is almost optimal for H^1 datum, and immediately improves if we have any extra regularity on the limiting initial one-particle state.

Keywords N-body quantum BBGKY hierarchy · Convergence rate · Klainerman-Machedon theory · Nonlinear scattering · Koch-Tataru U-V spaces

Mathematics Subject Classification Primary 35P25 · 35Q55 · 81V70; Secondary 35A23 · 35B45 · 81Q05

1 Introduction

The aim of this paper is to close, with a simple and short argument, the regularity gap that is currently present in the literature on the derivation of the cubic nonlinear Schrödinger equation (NLS) from quantum many-body dynamics on $\mathbb{R}^3$. Let us write the cubic NLS

$$\begin{aligned} &i\partial_t\phi = -\Delta_x\phi + b_0\,|\phi|^2\,\phi \text{ in } \mathbb{R}^{3+1} \\ &\phi(0,x) = \phi_0(x) \end{aligned} \tag{1.1}$$

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and the linear N-body Schrödinger equation

$$i\partial_t \psi_N = H_N \psi_N \text{ in } \mathbb{R}^{3N+1} \tag{1.2}$$

where the N-body Hamiltonian is

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N}\sum_{i<j} N^{3\beta} V(N^\beta (x_i - x_j)) \tag{1.3}$$

and define the marginal densities $\gamma_N^{(k)}$ associated with ψ_N in kernel form by

$$\gamma_N^{(k)}\left(t, \mathbf{x}_k, \mathbf{x}_k'\right) = \int \psi_N(t, \mathbf{x}_k, \mathbf{x}_{N-k}) \bar{\psi}_N(t, \mathbf{x}_k', \mathbf{x}_{N-k}) d\mathbf{x}_{N-k} \tag{1.4}$$

where $\mathbf{x}_k = (x_1, ..., x_k) \in \mathbb{R}^{3k}$. The main object of study in the derivation is, then, the $N \to \infty$ limit

$$\gamma_N^{(k)}(t) \to |\phi(t)\rangle \langle \phi(t)|^{\otimes k} \tag{1.5}$$

in operator form[1], where $\phi(t)$ is given by (1.1), provided that $\gamma_N^{(1)}(0) \to |\phi_0\rangle \langle \phi_0|$. Limit (1.5) was first rigorously justified in [35–37] assuming H^1 regularity via the now so-called hierarchy method that concluded convergence but with no estimates on the convergence rate as it was via a compactness argument. Later on, the work [5, 41] pioneered the study of the rate of convergence in limit (1.5) via the theory of Bogoliubov rotation / metaplectic representations and the now so-called Fock space method, but it requires at least H^4 regularity. We will explain more of these two methods later in the paper, but it is obvious that there is, at the moment, a significant regularity gap: H^1 vs H^4, between proving limit (1.5) holds and proving limit (1.5) holds with a rate.[2] It is certainly of mathematical interest to reduce the required regularity and provide an optimal result. At the same time, there are physical reasons to eliminate this gap.

The physical background of these derivational problems is the Bose-Einstein condensate, also called the fifth state of matter, first experimentally discovered in 1995 [2, 33] after the prediction by Einstein. In this context, the initial datum $\psi_N(0)$ of (1.2) represents a trapped N-particle gas cooled very close to absolute zero during the preparation phase and the dynamics $\psi_N(t)$ is the evolution of the system during the observation phase after the confinement is switched. That is, $\psi_N(0)$ is (or is very near) the ground state of a N-body Schrödinger operator with an external trapping potential and hence its smoothness fully depends on the variable coefficients inside the N-body Schrödinger operator, which is mainly the trapping potential in this case. In the original qualitative experiments [2, 33], the trap was generated by a strong magnetic field which is smooth by definition. However, since around 1997, the experiments – see [68, 73] for examples – have been instead favoring a pulse-type laser trapping, as it

1 As usual, in the notation, we do not distinguish the kernel and the operator it defines.

2 The gap could be less severe in 1D and 2D as the corresponding critical regularity drops.

produces less background noise for quantitative measurement and gives more control of the parameters of the system. However, due to the discrete / pulse nature and the complicated deployment of the technology, such an optical confinement is not very smooth and can only be approximated as harmonic $\omega_0^2 |x|^2$ when far off.[3] That is, away from the usual difficulties in measuring a high Sobolev norm of a microscopic quantum mechanical system, the initial datum $\psi_N(0)$ of (1.2) may not be very smooth at all due to the setup of the system. On the other hand, it is always safe to assume the H^1 condition as every particle in the system must have finite kinetic and potential energy which are also primary characteristics of the system. It is, therefore, of substantial physical interest to close the aforementioned regularity gap.

In this paper, we address the issue of the regularity gap using the hierarchy method in the Klainerman-Machedon theory format and refining an idea from the Fock space method. Let $S^{(\alpha,k)} = \prod_{j=1}^{k} \left\langle \nabla_{x_j} \right\rangle^{\alpha} \left\langle \nabla_{x'_j} \right\rangle^{\alpha}$ as usual, we define our master norm $\|\cdot\|_{H_Z^\alpha}$ for a hierarchy of marginal densities $\Gamma = \left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$, following [9–14], by

$$\|\Gamma\|_{H_Z^\alpha} = \sum_{k=1}^{\infty} Z^{-k} \left\| S^{(\alpha,k)} \gamma^{(k)} \right\|_{L^2}. \tag{1.6}$$

We note that this norm is guaranteed to converge for $Z > C$ provided that for all $k \geq 1$, $\|S^{(\alpha,k)}\gamma^{(k)}\|_{L^2} \leq C^k$ and we will only use the norm when the condition $\|S^{(\alpha,k)}\gamma^{(k)}\|_{L^2} \leq C^k$ is known to hold, and thus we call it a "norm". We will assume the following usual conditions for our main theorem under norm (1.6):

(a) $\psi_N(0)$ is normalized, that is $\|\psi_N(0)\|_{L^2(\mathbb{R}^{3N})} = 1$ or $\mathrm{Tr}\gamma_N^{(k)}(0) = 1$.

(b) The uniform energy bounds hold:[4]

$$\left\langle \psi_N(0), (H_N/N)^k \, \psi_N(0) \right\rangle \leq E_0^k \tag{1.7}$$

which, as shown by [35], implies that for all $k \geq 1$ and all t , $\|S^{(1,k)}\gamma_N^{(k)}\|_{L^2} \leq 2^k E_0^k$, which further implies

$$\sup_{t\in[0,T]} \|\Gamma_N(t)\|_{H_Z^1} < +\infty. \tag{1.8}$$

for any $Z > 2E_0$.

(c) For some $Z_0 > 2E_0$, and for some $\phi_0 \in H^1(\mathbb{R}^3)$, the initial condition is asymptotically factorized:

$$\lim_{N\to\infty} \left\| \Gamma_N(0) - \left\{ |\phi_0\rangle \langle\phi_0|^{\otimes k} \right\} \right\|_{H_{Z_0}^1} = 0.$$

[3] See [67] for some locally half-circle shaped or paralleled-tube shaped examples.

[4] One can use either (1.7) or (1.8) as (b).

Our main theorem is the following.

Theorem 1.1 *(Main Theorem) Assume the marginal densities $\Gamma_N = \{\gamma_N^{(k)}\}$ associated with ψ_N, the solution to the N-body dynamics (1.2) with a smooth even pair interaction $V \geq 0$, satisfy (a)-(c). Then for $T = c_1 E_0^{-2}$ and $Z = c_2 E_0$ for some specific multiples c_1, c_2, we have the estimate*

$$\begin{aligned}&\sup_{t\in[0,T]} \left\| \Gamma_N(t) - \left\{ |\phi(t)\rangle \langle \phi(t)|^{\otimes k} \right\} \right\|_{H_Z^1} \\ &\lesssim \|\Gamma_N(0) - \left\{ |\phi_0\rangle \langle \phi_0|^{\otimes k} \right\} \|_{H^1_{Z/2}} + \max(N^{\frac{5}{2}\beta-1}, N^{-\beta}(\ln N)^7)\end{aligned} \tag{1.9}$$

where $\phi(t)$ solves (1.1) with $\phi(0) = \phi_0$ and $b_0 = \int V$.

The proof of Theorem 1.1 certainly allows general datum as usual. In the context of the quantum de Finetti theorem in [8] from [60], Theorem 1.1 reads as follows.

Corollary 1.2 *(General Datum) Assume the marginal densities $\Gamma_N = \{\gamma_N^{(k)}\}$ associated with ψ_N, the solution to the N-body dynamics (1.2) with $V \geq 0$, satisfy (a), (b) and*

(c') For some $E_0 > 0$, and for some probability measure $d\mu_0$ supported on $\mathbb{S}(L^2(\mathbb{R}^3))$, we have

$$\lim_{N\to\infty} \|\Gamma_N(0) - \Gamma_\infty(0)\|_{H^1_{E_0}} = 0.$$

where

$$\Gamma_\infty(0) = \left\{ \int_{\mathbb{S}(L^2(\mathbb{R}^3))} |\phi\rangle \langle\phi|^{\otimes k} \, d\mu_0(\phi) \right\}$$

Then for $T = c_1 E_0^{-2}$ and $Z = c_2 E_0$ for some specific multiples c_1, c_2, we have the estimate

$$\sup_{t\in[0,T]} \|\Gamma_N(t) - \Gamma_\infty(t)\|_{H_Z^1} \lesssim \|\Gamma_N(0) - \Gamma_\infty(0)\|_{H^1_{Z/2}} + \max(N^{\frac{5}{2}\beta-1}, N^{-\beta}(\ln N)^7)$$

where

$$\Gamma_\infty(t) = \left\{ \int_{\mathbb{S}(L^2(\mathbb{R}^3))} |S_t\phi\rangle \langle S_t\phi|^{\otimes k} \, d\mu_0(\phi) \right\}$$

and $S_t : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$, $\forall t \in \mathbb{R}$,[5] *is the solution map of (1.1).*

For $\beta \in \left(0, \frac{2}{5}\right)$, Theorem 1.1 and Corollary 1.2 give a convergence rate estimate.[6] Also, to be precise, the concluded rate is in the H^1 norm which is stronger than the

[5] Condition (c) implies $\Gamma_\infty(0) \in H^1_{E_0}$ which implies $d\mu_0$ is supported in the subset of $\mathbb{S}(L^2(\mathbb{R}^3))$ in which $\|\phi\|_{H^1(\mathbb{R}^3)} \leq E_0$. Hence $S_t\phi$ is well-defined inside the $d\mu_0$ integral.

[6] The method does yield the optimal N^{-1} rate when $\beta = 0$ but one needs to change (1.1).

usual trace norm convergence. Of course, both are physically meaningful with one being the kinetic energy and one being the probability when restricted to $\gamma^{(1)}$. We remark that the rate $\frac{(\ln N)^7}{N^\beta}$, when $\beta \in \left(0, \frac{2}{7}\right]$, is an almost optimal in N rate[7] as the optimal in N rate is $\frac{1}{N^\beta}$ if we require both sides of the estimates to be in H^1. With a more delicate argument, the power of $\ln N$ can be reduced. We leave it at 7 for simplicity as $(\ln N)^7$ is still better than N^ε. If we assume extra regularity in the limiting initial datum ϕ_0, then our rate improves and in fact reveals more details of the story.

Corollary 1.3 *(Improved Rate with H^q, $q > 1$ Datum) In addition to the assumptions (a)-(c), assume that the limiting intial one-particle state $\phi_0 \in H^q(\mathbb{R}^3)$ for some $q > 1$, then estimate (1.10) in Theorem 1.1 can be improved to*

$$\begin{aligned}&\sup_{t\in[0,T]} \left\| \Gamma_N(t) - \left\{ |\phi(t)\rangle \langle \phi(t)|^{\otimes k} \right\} \right\|_{H^1_Z} \\ &\lesssim_{q,\|\phi_0\|_{H^q}} \|\Gamma_N(0) - \left\{ |\phi_0\rangle \langle \phi_0|^{\otimes k} \right\} \|_{H^1_{Z/2}} + \max(N^{\frac{5}{2}\beta-1}, N^{-\min(q,2)\beta}(\ln N)^7)\end{aligned} \tag{1.10}$$

Corollary 1.3 proves that we get an optimal H^1 rate if we have $H^{1+\varepsilon}$ regularity as $\frac{(\ln N)^7}{N^{(1+\varepsilon)\beta}}$ is better than $\frac{1}{N^\beta}$. On the other hand, apparently, the optimal H^1 rate improves if we have H^q, $q > 1$.

We remark that it is not too difficult to use some extra Littlewood-Paley argument to improve the $N^{\frac{5\beta}{2}-1}$ inside estimates (1.9) and (1.10) to $N^{\frac{3\beta}{2}-1}$ which concludes convergence for $\beta \in \left(0, \frac{2}{3}\right)$ and yields the optimal in N rate for $\beta \in \left(0, \frac{2}{5}\right]$. We choose not to do so in this paper as we would like to keep this paper short. The $\frac{(\ln N)^7}{N^\beta}$ and $\frac{(\ln N)^7}{N^{q\beta}}$ in estimates (1.9) and (1.10) come from the following new NLS "bi-scattering" result. To state it, let ϕ_N solve the Hartree type NLS (H-NLS) equation

$$\begin{aligned} i\partial_t \phi_N &= -\Delta\phi_N + (V_N * |\phi_N|^2)\phi_N \text{ in } \mathbb{R}^{3+1} \\ \phi_N(0,x) &= \phi_0(x) \end{aligned} \tag{1.11}$$

where $V_N(x) = N^{3\beta}V(N^\beta x)$ as in (1.3) with $V \geq 0$.

Theorem 1.4 *(bi-scattering) Let ϕ solve (1.1) and ϕ_N solve (1.11). We then have the following.*

(i) For H^1 data, both $\phi(t)$ and $\phi_N(t)$ satisfy the global-in-time bounds

$$\|\langle\nabla\rangle\phi\|_{L^2_t(\mathbb{R})L^6_x} \lesssim_{\|\phi_0\|_{H^1}} 1, \qquad \|\langle\nabla\rangle\phi_N\|_{L^2_t(\mathbb{R})L^6_x} \lesssim_{\|\phi_0\|_{H^1}} 1,$$

[7] We emphasize the "in N" aspect of the optimality here because the best "in t" growth rate is unknown. But a rate better than exponential growth has been proven to be possible in related scenarios with the second-order correction – see [17, 31, 41–44, 58, 59] for examples.

and scatter in H^1 – that is, there exist forward-in-time states ϕ_+ and $\phi_{N,+}$ such that

$$\lim_{t\to\infty} \|\phi(t) - e^{it\Delta}\phi_+\|_{H_x^1} = 0\,, \qquad \lim_{t\to\infty} \|\phi_N(t) - e^{it\Delta}\phi_{N,+}\|_{H_x^1} = 0$$

(ii) We have the global-in-time H^1 comparison estimate

$$\|\phi - \phi_N\|_{L_t^\infty(\mathbb{R})H_x^1} \lesssim_{q,\|\phi_0\|_{H^q}} \frac{(\ln N)^7}{N^{q\beta}} \tag{1.12}$$

provided that $\phi_0 \in H^q$, $q \in [1, 2]$, and the optimal rate is $N^{-q\beta}$.

We say Theorem 1.4 is a "bi-scattering" result not because, as stated in (i), both (1.1) and (1.11) scatter, which is in fact known, but because the conclusion in (ii) that the interaction potential $V_N \to \delta$ as $N \to \infty$ for two corresponding Hamiltonian evolution is called a 2-body scattering process in the context of quantum many-body dynamics. Moreover, estimate (1.12) holds globally and thus carries some $t \to \infty$ information. That is, one scattering is the usual $t \to \infty$ scattering while another scattering is the $N \to \infty$ scattering, and they happen simultaneously as in (1.12).

On the one hand, we prove Theorem 1.4 which is usually an ingredient in a Fock space approach paper. On the other hand, our proof does not have a compactness or uniqueness argument as in the standard hierarchy approach. One could, in fact, view the main proof of this paper as integrating the idea from Fock space approach that, using (1.11) as an intermediate dynamic, into the hierarchy method in the Klainerman-Machedon theory format. The fact that we close the regularity gap and prove the (almost) optimal rates with such a simple combination is exactly the main novelty of this paper. Let us now give a brief review of the two approaches.

Limit (1.5) was first established in the work of Erd ös, Schlein, and Yau [35–37] for the $\mathbb{R}^3$ defocusing cubic case around 2005.[8] They first proved (1.7) implies (1.8) as a preparation. They then proved that $\{\Gamma_N(t)\}$ is a compact sequence with respect to a suitable weak* topology on trace class operators using the fact $\Gamma_N(t)$ satisfies the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy

$$\begin{aligned} i\partial_t\gamma_N^{(k)} &= \sum_{j=1}^{k}\left[-\Delta_{x_j}, \gamma_N^{(k)}\right] + \frac{1}{N}\sum_{1\le i<j\le k}\left[V_N(x_i - x_j), \gamma_N^{(k)}\right] \\ &\quad + \frac{N-k}{N}\sum_{j=1}^{k}\mathrm{Tr}_{k+1}\left[V_N(x_j - x_{k+1}), \gamma_N^{(k+1)}\right], \end{aligned} \tag{1.13}$$

[8] See also [1] for the 1D defocusing cubic case around the same time.

and that every limit point $\Gamma_\infty(t) = \left\{\gamma_\infty^{(k)}\right\}$ of $\{\Gamma_N(t)\}$ satisfies the Gross-Pitaevskii (GP) hierarchy

$$i\partial_t \gamma_\infty^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_k}, \gamma_\infty^{(k)}\right] + b_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\left[\delta(x_j - x_{k+1}), \gamma_\infty^{(k+1)}\right]. \tag{1.14}$$

Finally, they proved delicatedly that there is a unique solution to the $\mathbb{R}^3$ cubic GP hierarchy in a H^1-type space (unconditional uniqueness) in [35] with a sophisticated Feynman graph analysis and many highly technical singular integral techniques. Because the desired limit $\{|\phi(t)\rangle \langle\phi(t)|^{\otimes k}\}$ solves hierarchy (1.14), limit (1.5) is then proved without any rate estimate. This first series of ground breaking papers have motivated a large amount of work. Moreover, in [36, 37], the weak* convergence was upgraded to strong via an elementary functional analysis theorem. This "small" weak* to strong upgrade firmly hinted that a convergence rate result is possible.

In 2007, Klainerman and Machedon [54], inspired by [35, 53], proved the uniqueness of solutions regarding (1.14) in a Strichartz-type space (conditional uniqueness). They proved a collapsing type estimate, to estimate the inhomogeneous term in (1.14), and provided a different combinatorial argument, the now so-called Klainerman-Machedon (KM) board game, to combine the inhomogeneous terms effectively reducing their numbers. At that time, it was unknown how to prove that the limits coming from (1.13) are in the Strichartz type spaces even though the target limit $\{|\phi(t)\rangle \langle\phi(t)|^{\otimes k}\}$ generated by (1.1) naturally lie in both the H^1 -type space and the Strichartz type space. Nonetheless, [54] has made the analysis of (1.14) approachable to PDE analysts and the KM board game has been used in every work involving hierarchy (1.14).

When Kirkpatrick, Schlein, and Staffilani [52] found that the KM Strichartz-type bound can be obtained via a simple trace theorem for the defocusing case in $\mathbb{R}^2$ and $\mathbb{T}^2$ in 2008, many works [10, 18, 22, 23, 39, 46, 69, 70, 75] then followed such a scheme for the uniqueness of GP hierarchies. However, how to check the KM bound in the 3D cubic case remained fully open at that time.

T. Chen and Pavlović studied the 1D and 2D defocusing quintic case and laid the foundation for the 3D quintic defocusing energy-critical case in their late 2008 work [10], in which they proved that the 2D quintic case, a case usually considered equivalent to the 3D cubic case, does satisfy the KM bound though proving it for the 3D cubic case was still open.

In [9, 11, 12], T. Chen and Pavlović generalized the problem and launched the well-posedness theory of (1.14) with general initial datum as an independent subject away from (1.2). (See also [15, 62–64, 70, 72].) Then in 2011, T. Chen and Pavlovic proved the 3D cubic KM Strichartz type bound for the defocusing $\beta < 1/4$ case in [13]. (See also [14].) The result was quickly improved to $\beta \leq 2/7$ by X.C. in [19] and to the almost optimal case, $\beta < 1$, by X.C. and J.H. in [21, 24], by lifting the $X_{1,b}$ space techniques from NLS theory into the field. Away from being the 1st work to prove the

KM bound, the work [13], in fact, hinted[9] two unforeseen research directions of the hierarchy method today.

One direction is to prove new NLS results via the more general but at the same time more complicated hierarchy (1.14). The hierarchy uniqueness theorems started to match the corresponding NLS results in [23, 30, 48, 49, 71] following the 2013 introduction of the quantum de Finetti theorem from [60] to the field by T. Chen, Hainzl, Pavlović, and Seiringer [8]. Then, recently, the previously open $\mathbb{T}^d$ NLS unconditional uniqueness problems either saw substantial progress or were solved via the analysis of the supposedly more complicated GP hierarchy. In [47], Herr and Sohinger generalized the Sobolev multilinear estimates in [8] and obtained new unconditional uniqueness results regarding $\mathbb{T}^d$ GP hierarchy and hence $\mathbb{T}^d$ NLS. In [26], by discovering the hierarchical uniform frequency localization (HUFL) property, X.C. and J.H. established, for the $\mathbb{T}^3$ quintic energy-critical GP hierarchy, a H^1-type uniqueness theorem which was neither conditional nor unconditional but implies the H^1 unconditional uniqueness for the $\mathbb{T}^3$ quintic energy-critical NLS. More recently, in [27], X.C. and J.H. worked out an extended KM board game from scratch to enable, finally, the application of dispersive norms like U-V and $X_{s,b}$ in the field and proved the H^1 unconditional uniqueness for the $\mathbb{T}^4$ cubic energy-critical NLS, an unanticipated "special" case in the $\mathbb{R}^3/\mathbb{R}^4/\mathbb{T}^3/\mathbb{T}^4$ energy-critical sequence, with the hierarchy approach. The proof in [27] went so smoothly that, X.Chen, Shen, and Zhang completely and unifiedly solved the unconditional uniqueness for $\mathbb{R}^d$ and $\mathbb{T}^d$ cubic and quintic energy-supercritical NLS in [28].

The other direction hinted in [13] is that it is possible to use the KM theory to construct a hierarchy method without the compactness argument, that is, it is possible to establish convergence rate estimates using the hierarchy approach with H^1 regularity. In other words, [13, 19, 21, 24], can be considered as premodels of this paper. However, completing the proof directly using solely the hierarchy approach in KM format will need to pass, on the road, some extra technical difficulties, like extra error terms, which charge a $N^{-\frac{\beta}{2}}$ price. On the other hand, in the Fock space approach, equation (1.11) naturally pops out 1st in the 2nd quantization argument and one always needs to compare between equations (1.1) and (1.11) to close limit (1.5), see, for example [5–7, 31, 41, 42, 59, 66].[10] We assimilate this idea into our proof.

1.1 Outline of the Proof

As the proof of Corollary 1.2 only requires adding the $d\mu_0$ integrals in suitable places, we prove only Theorem 1.1 and Corollary 1.3 in detail. In fact, we stated Corollary 1.2 not to show the generality, but to clarify a logical question that if the proof for Theorem 1.1 relies on any uniqueness theorems regarding (1.14). For Theorem 1.1 and Corollary 1.3, we indeed did not use any uniqueness results regarding (1.14) as the desired limit could be guessed in multiple ways. (The 2nd quantization argument is certainly an option.) But for the general datum case, Corollary 1.2, we are not aware

[9] Private communication in 2011.

[10] This is certainly only a fraction of all possible references as the Fock space approach is also such a vast and sophisticated subject now. Please also see the references within them and the newer ones online.

of any method to guess the desired limit away from using uniqueness results regarding (1.14) in [8].

As mentioned before, the main proof here can be understood as utilizing the hierarchy approach in KM format but put (1.11) as an intermediate dynamic. That is, we prove estimates (1.9) and (1.10) by summing two estimates. The 1st one is

$$\begin{aligned}&\sup_{t\in[0,T_0]}\left\|\Gamma_N(t)-\left\{|\phi_N(t)\rangle\langle\phi_N(t)|^{\otimes k}\right\}\right\|_{H^1_{Z_1}}\\&\lesssim_{E_0}\left\|\Gamma_N(0)-\left\{|\phi_0\rangle\langle\phi_0|^{\otimes k}\right\}\right\|_{H^1_{Z_1/2}}+\frac{1}{N^{1-\frac{5\beta}{2}}},\end{aligned}\tag{1.15}$$

while the 2nd one is

$$\sup_{t\in\mathbb{R}}\left\|\left\{|\phi_N(t)\rangle\langle\phi_N(t)|^{\otimes k}\right\}-\left\{|\phi(t)\rangle\langle\phi(t)|^{\otimes k}\right\}\right\|_{H^1_{Z_2}}\lesssim_{q,\|\phi_0\|_{H^q}}\frac{(\ln N)^7}{N^{q\beta}}.\tag{1.16}$$

Estimates (1.9) and (1.10) then follow by selecting $Z=\max(Z_1,Z_2)$.

We prove estimate (1.15) in §2 by directly taking the difference between (1.13) and the “H-NLS” hierarchy (2.2) generated by (1.11). We iterate the difference hierarchy (2.4) by coupling into the next level multiple times and group the terms into the free part, driving part, and interaction part. We can then proceed to estimate following the scheme in [19, 21, 24]. This part is new but the method is not. We comment that another option would be taking the direct difference between (1.13) and (1.14). The problem is that such a route would have produced a difference hierarchy with two options to couple to the next level, namely $\mathrm{Tr}_{k+1}\left[V_N(x_j-x_{k+1}),\gamma_N^{(k+1)}\right]$ and $\mathrm{Tr}_{k+1}\left[\delta(x_j-x_{k+1}),\gamma_\infty^{(k+1)}\right]$, compared to (2.2) in which there is only one interaction term. While iterating the hierarchy is basically the only way to obtain hierarchy estimates since the beginning of the hierarchy approach, it is evident that the “more direct” route indeed has way more error terms. Finally, one also needs to face the classical “trace vs power” technical dilemma without the intermediate dynamic. This is the technical reason that we chose to use (1.11) as an intermediate dynamic. But indeed, one can still get a $N^{-\frac{\beta}{2}}$ rate with KM theory alone.

We then prove estimate (1.16) by proving Theorem 1.4 in Sect. 3. Though (1.1) is $H^{\frac{1}{2}}$ critical instead of H^1 critical, the error estimate (1.12) yielding $N^{-q\beta}$ is critical in the sense that all spatial derivatives and space-time Hölder norms are fully absorbed in the estimates. That is, one has no choice but to use the U-V spaces for the $N\to\infty$ scattering proof. It then forces the $t\to\infty$ argument to be in the U-V spaces as well. Theorem 1.4 is the 1st bi-scattering theorem of its type and it is obvious by this paper that it has direct applications. Another highlight of the proof is that it employs the full strength of the $\mathbb{R}^d$ Schrödinger bilinear estimate. Finally, we provide the first proof of the optimality of the $N^{-\beta}$ rate, which has been mentioned multiple times with physical insight in the literature, via the method of space-time resonance in Sect. 3.1 .

Putting Sect. 2 and Sect. 3 together concludes the proof of the main theorems, Theorem 1.1 and Corollary 1.3. It is surprising that under the simple scheme in this paper, without too much extra work, the hierarchy approach yields convergence rate estimate which was obtainable, so far, only via the Fock space approach. Moreover, it eliminates the H^1 vs H^4 regularity gap by requiring H^1 or $H^{1+\varepsilon}$ regularity. It is also astonishing that the almost optimal or optimal in N convergence rate can be obtained with this easy method.[11] The discovery of this simple hierarchy approach is the main novelity of this paper.

With the help of the extended KM board game in [27] which allows the application of the U-V space-time estimates, and a $X^{\frac{1}{2}+}$ frequency localized version of the KM estimates, we expect improving (1.15) up to $\beta = 1$ once we put in the correlation structures we had in [24]. (Of course, (1.11) has to be changed as well.) We shall do so in the next (longer) paper. The main argument of this paper can also be extended to any finite time but with a $1/\ln N$ rate. (See [29].)

2 Comparing the BBGKY hierarchy and the H-NLS

The main goal in this section is to prove (1.15) which will result from Theorem 2.3. We adopt the shorthands

$$U^{(k)}(t) = e^{it\triangle_{\mathbf{x}_k}} e^{-it\triangle_{\mathbf{x}'_k}},$$
$$V_N^{(k)}\gamma_N^{(k)} = \frac{1}{N}\sum_{1\leq i<j\leq k}\left[V_N(x_i - x_j), \gamma_N^{(k)}\right],$$
$$B_N^{(k+1)}\gamma_N^{(k+1)} = \sum_{j=1}^{k} B_{N,j,k+1}\gamma_N^{(k+1)} = \sum_{j=1}^{k}\operatorname{Tr}_{k+1}[V_N(x_j - x_{k+1}), \gamma_N^{(k+1)}].$$

and assume $\int V = 1$ for convenience. We start by rewriting the 3D cubic BBGKY hierarchy (1.13) in integral form

$$\gamma_N^{(k)}(t_k) = U^{(k)}(t_k)\gamma_{N,0}^{(k)} + \int_0^{t_k} U^{(k)}(t_k - t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})dt_{k+1} + \frac{N-k}{N}\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\gamma_N^{(k+1)}(t_{k+1})dt_{k+1}. \tag{2.1}$$

where we have omitted the $(-i)$ in front of the 2nd and the 3rd term in the right side of (2.1) as usual as we are going to put everything in absolute values. In addition to (2.1), for $k = 1, \ldots, N, \ldots$, we consider the H-NLS hierarchy

$$\gamma_H^{(k)}(t_k) = U^{(k)}(t_k)\gamma_0^{(k)} + \int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\gamma_H^{(k+1)}(t_{k+1})dt_{k+1}, \tag{2.2}$$

[11] Obtaining the optimal $N^{-\beta}$ rate using the Fock space approach assuming H^4 has been done in the much harder $\beta = 1$ case. See [7].

generated by $\left\{\gamma_H^{(k)}\left(t_k, \mathbf{x}_k, \mathbf{x}_k'\right) = |\phi_N\rangle\langle\phi_N|^{\otimes k}\right\}$, the tensor products of solutions to (1.11).

The main concern in this section is the difference $\omega_{N,H}^{(k)} = \gamma_N^{(k)} - \gamma_H^{(k)}$ which solves the hierarchy

$$\begin{aligned}\omega_{N,H}^{(k)}(t_k) = U^{(k)}(t_k)\omega_{N,0}^{(k)} + \int_0^{t_k} U^{(k)}(t_k - t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})dt_{k+1} \\ -\frac{k}{N}\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\gamma_N^{(k+1)}(t_{k+1})dt_{k+1} \\ +\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\omega_{N,H}^{(k+1)}(t_{k+1})dt_{k+1}\end{aligned} \tag{2.3}$$

Of course, we are using the convention that $\gamma_N^{(k)} = 0$ if $k > N$ in (2.3).

As the error term

$$\frac{k}{N}\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\gamma_N^{(k+1)}(t_{k+1})dt_{k+1}$$

in (2.3) can be handled by adding an extra $\frac{\ln N}{N}$ to our estimates of the main terms, we can assume it drops out[12] and rewrite (2.3) as

$$\begin{aligned}\omega_{N,H}^{(k)}(t_k) = U^{(k)}(t_k)\omega_{N,0}^{(k)} + \int_0^{t_k} U^{(k)}(t_k - t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})dt_{k+1} \\ +\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\omega_{N,H}^{(k+1)}(t_{k+1})dt_{k+1}.\end{aligned} \tag{2.4}$$

Iterating hierarchy (2.4) ℓ_c times, we have

$$\begin{aligned}\omega_{N,H}^{(k)}(t_k) &= U^{(k)}(t_k)\omega_{N,0}^{(k)} + \int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}U^{(k+1)}(t_{k+1})\omega_{N,0}^{(k+1)}dt_{k+1} \\ &+ \int_0^{t_k} U^{(k)}(t_k - t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})dt_{k+1} \\ &+ \int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)} \\ &\times \int_0^{t_{k+1}} U^{(k+1)}(t_{k+1} - t_{k+2})V_N^{(k+1)}\gamma_N^{(k+1)}(t_{k+2})dt_{k+2}dt_{k+1} \\ &+ \int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)} \\ &\times \int_0^{t_{k+1}} U^{(k+1)}(t_{k+1} - t_{k+2})B_N^{(k+2)}\omega_{N,H}^{(k+2)}(t_{k+2})dt_{k+2}dt_{k+1} \\ &= \ldots\end{aligned}$$

12 Intereseted readers can see [29] for a detailed handling of this error term.

$$\equiv \mathrm{FP}^{(k,\ell_c)}(t_k) + \mathrm{DP}^{(k,\ell_c)}(t_k) + \mathrm{IP}^{(k,\ell_c)}(t_k). \tag{2.5}$$

where we have grouped the terms in $\omega_{N,H}^{(k)}(t_k)$ into three parts.

To write out the three parts of $\omega_{N,H}^{(k)}$, we define, the notation that, for $j \geq 1$,

$$\begin{aligned} &J_N^{(k,j)}(\underline{t}_{(k,j)})(f^{(k+j)}) \\ &\quad = \left(U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\right)\cdots\left(U^{(k+j-1)}(t_{k+j-1} - t_{k+j})B_N^{(k+j)}\right) f^{(k+j)}, \end{aligned}$$

and $J_N^{(k,0)}(\underline{t}_k)(f^{(k)}) = f^{(k)}(t_k)$, where $\underline{t}_{(k,j)}$ means $\left(t_{k+1}, \ldots, t_{k+j}\right)$ for $j \geq 1$ and t_k for $j = 0$. In this notation, the *free part* of $\omega_{N,H}^{(k)}$ at coupling level ℓ_c is given by

$$\begin{aligned} FP^{(k,\ell_c)} &= U^{(k)}(t_k)\omega_{N,0}^{(k)} + \sum_{j=1}^{\ell_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)} \cdots \\ &\qquad \times U^{(k+j-1)}(t_{k+j-1} - t_{k+j})B_N^{(k+j)} \left(U^{(k+j)}(t_{k+j})\omega_{N,0}^{(k+j)}\right) d\underline{t}_{(k,j)} \\ &= \sum_{j=0}^{\ell_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_N^{(k,j)}(\underline{t}_{(k,j)})(f_{FP}^{(k,j)})(t_{k+j}) d\underline{t}_{(k,j)} \end{aligned}$$

where in the $j = 0$ case, it is meant that there are no time integrals and $J^{(k,0)}$ is the identity operator, and

$$f_{FP}^{(k,j)}(t_{k+j}) = U^{(k+j)}(t_{k+j})\omega_{N,0}^{(k+j)};$$

the *driving part*, which is a forcing term for $\omega_{N,H}^{(k)}$ but a potential term for $\gamma_N^{(k)}$, is given by

$$\begin{aligned} DP^{(k,\ell_c)} &= \int_0^{t_k} U^{(k)}(t_k - t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})dt_{k+1} \\ &\quad + \sum_{j=1}^{\ell_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)} \cdots U^{(k+j-1)}(t_{k+j-1} - t_{k+j}) \\ &\quad \times B_N^{(k+j)}(\int_0^{t_{k+j}} U^{(k+j)}(t_{k+j} - t_{k+j+1})V_N^{(k+j)}\gamma_N^{(k+j)}(t_{k+j+1})dt_{k+j+1})d\underline{t}_{(k,j)} \\ &= \sum_{j=0}^{\ell_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_N^{(k,j)}(\underline{t}_{(k,j)})(f_{DP}^{(k,j)})(t_{k+j})d\underline{t}_{(k,j)}, \end{aligned}$$

where in the $j = 0$ case, it is meant that there are no time integrals and $J^{(k,0)}$ is the identity operator, and

$$f_{DP}^{(k,j)}(t_{k+j}) = \int_0^{t_{k+j}} U^{(k+j)}(t_{k+j} - t_{k+j+1}) V_N^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1}) dt_{k+j+1}; \tag{2.6}$$

and the *interaction part* is given by

$$\begin{aligned} IP^{(k,\ell_c)} &= \int_0^{t_k} \cdots \int_0^{t_{k+\ell_c}} dt_{k+1} \cdots dt_{k+\ell_c+1}\, U^{(k)}(t_k - t_{k+1}) B_N^{(k+1)} \cdots \\ &\qquad \cdots U^{(k+\ell_c)}(t_{k+\ell_c} - t_{k+\ell_c+1}) B_N^{(k+\ell_c+1)} \left(\omega_{N,H}^{(k+\ell_c+1)}(t_{k+\ell_c+1}) \right) \\ &= \int_0^{t_k} \cdots \int_0^{t_{k+\ell_c}} J_N^{(k,\ell_c+1)}(\underline{t}_{(k,\ell_c+1)}) \left(\omega_{N,H}^{(k+\ell_c+1)}(t_{k+\ell_c+1}) \right) d\underline{t}_{(k,\ell_c+1)}. \end{aligned}$$

Notice that, on the one hand, the $FP^{(k,\ell_c)}$ and $DP^{(k,\ell_c)}$ are sums while $IP^{(k,\ell_c)}$ is a single term; on the other hand, $DP^{(k,\ell_c)}$ depends solely on $\gamma_N^{(k)}$ and is independent of $\gamma_H^{(k)}$, while $FP^{(k,\ell_c)}$ and $IP^{(k,\ell_c)}$ depends on $\gamma_H^{(k)}$. We have the following estimates.

Proposition 2.1 *We have the following estimates. For the free part,*

$$\left\| S^{(1,k)} FP^{(k,\ell_c)} \right\|_{L_{t_k}^\infty [0,T] L_{x,x'}^2} \leq \sum_{j=0}^{\ell_c} 2^k (4CT^{1/2})^j \left\| S^{(1,k+j)} \omega_{N,H}^{(k+j)}(0) \right\|_{L_{x,x'}^2} \tag{2.7}$$

Provided $T \lesssim E_0^{-2}$, the driving part satisfies

$$\left\| S^{(1,k)} DP^{(k,\ell_c)} \right\|_{L_{t_k}^\infty [0,T] L_{x,x'}^2} \leq CT^{1/2} N^{\frac{5}{2}\beta - 1} (2E_0)^k k^2 \tag{2.8}$$

For the interaction part, we have

$$\left\| S^{(1,k)} IP^{(k,\ell_c)} \right\|_{L_{t_k}^\infty [0,T] L_{x,x'}^2} \tag{2.9}$$

$$\leq 2^k T^{1/2} (4CT^{1/2})^{\ell_c+1} N^{\frac{5\beta}{2}} \left\| S^{(1,k+\ell_c+1)} \omega_{N,H}^{(k+\ell_c+1)}(t) \right\|_{L_{t_k}^\infty [0,T] L_{x,x'}^2} \tag{2.10}$$

Proof See Sect. 2.1. □

The interaction part is addressed by following the method in [19] which was inspired by [13] of using $\ell_c = \ln N$ to gain a negative power of N from the power-of-T coefficient in the above estimate. Then we can use the crude bound

$$\left\| \omega_{N,H}^{(k+\ell_c+1)}(t) \right\| \leq \left\| \gamma_N^{(k+\ell_c+1)}(t) \right\| + \left\| \gamma_H^{(k+\ell_c+1)}(t) \right\|$$

that ignores the difference structure of $\omega_{N,H}$.

Lemma 2.2 *For $\ell_c = \ln N$ and provided $T \lesssim E_0^{-2}$,*

$$\left\| S^{(1,k)} I P^{(k,\ell_c)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \le C T^{1/2} (2E_0)^k N^{-2} \tag{2.11}$$

Proof By (2.9) and the energy bounds on $\gamma_N^{(k+\ell_c+1)}(t)$ and $\gamma_H^{(k+\ell_c+1)}(t)$, it suffices to show that

$$(4CE_0 T^{1/2})^{\ell_c+1} N^{\frac{5}{2}\beta} \le N^{-2}$$

We assume $T \lesssim E_0^{-2}$, specifically that T is small enough so that $4CE_0T^{1/2} \le e^{-5}$. Then

$$(4CE_0T^{1/2})^{\ell_c+1} \le e^{-5\ell_c} = e^{-5\ln N} = N^{-5}$$

□

That is, the interaction part estimate can be made into N^{-s} for any s, the limiting factor is solely the potential part which will get better once one puts in the correlation functions as in [24].

Carrying out the sum in k for the estimates in Proposition 2.1 gives us what we need in the master norm (1.6).

Theorem 2.3 *For T and Z such that $T \lesssim Z^{-2}$ and $Z \gtrsim E_0$,*

$$\begin{aligned} &\|\Gamma_N(t) - \left\{ |\phi_N(t)\rangle \langle \phi_N(t)|^{\otimes k} \right\} \|_{L^\infty_{[0,T]} H^1_Z} \\ &\le_{E_0,Z} C \|\Gamma_N(0) - \left\{ |\phi_0\rangle \langle \phi_0|^{\otimes k} \right\} \|_{H^1_{Z/2}} + C T^{1/2} N^{\frac{5}{2}\beta - 1} \end{aligned}$$

Proof See Sect. 2.2. □

2.1 Proof of Proposition 2.1

First of all, the summands inside each part can be grouped / combined together further using the KM board game argument [54], which is below, to avoid a factorial factor.

Lemma 2.4 *([21,Lemma 2.1]) For $j \ge 1$, one can express*

$$\int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_N^{(k,j)}(\underline{t}_{(k,j)})(f^{(k+j)}) d\underline{t}_{(k,j)}$$

as a sum of at most 2^{k+2j-2} terms of the form

$$\int_D J_N^{(k,j)}(\underline{t}_{(k,j)}, \mu_m)(f^{(k+j)}) d\underline{t}_{(k,j)},$$

or in other words,

$$\int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_N^{(k,j)}(\underline{t}_{(k,j)})(f^{(k+j)})d\underline{t}_{(k,j)} = \sum_m \int_D J_N^{(k,j)}(\underline{t}_{(k,j)}, \mu_m)(f^{(k+j)})d\underline{t}_{(k,j)}.$$

Here $D \subset [0, t_k]^j$, μ_m *are a set of maps from* $\{k+1, \ldots, k+j\}$ *to* $\{1, \ldots, k+j-1\}$ *and* $\mu_m(l) < l$ *for all* l, *and*

$$\begin{aligned} & J_N^{(k,j)}(\underline{t}_{(k,j)}, \mu_m)(f^{(k+j)}) \\ & = \left(U^{(k)}(t_k - t_{k+1})B_{N,\mu_m(k+1),k+1}\right)\left(U^{(k+1)}(t_{k+1} - t_{k+2})B_{N,\mu_m(k+2),k+2}\right) \cdots \\ & \quad \cdots \left(U^{(k+j-1)}(t_{k+j-1} - t_{k+j})B_{N,\mu_m(k+j),k+j}\right)(f^{(k+j)}). \end{aligned}$$

The counting 2^{k+2j-2} in Lemma 2.4 is actually an easy upper bound of a Catalan number.

Lemma 2.5 *(counting of KM reduced forms) The number of mappings*

$$\mu : \{k+1, \ldots, k+j\} \to \{1, \ldots, k+j-1\}$$

satisfying $\mu(r) < r$ *for each* $k+1 \le r \le k+j$ *that are nondecreasing* $(\mu(r) \le \mu(r+1)$ *for each* $k+1 \le r \le k+j-1)$ *is at most the Catalan number*

$$\mathcal{C}(k, j) \equiv \binom{k+2j-2}{j} \le 2^{k+2j-2} \tag{2.12}$$

Proof We can associate to every reduced map μ a sequence s

$$s(1) = \mu(k+1), \quad s(2) = \mu(k+2) + 1, \quad \ldots, \quad s(j) = \mu(k+j) + j - 1$$

Note that s is a (strictly) increasing subsequence of $\{1, \ldots, k+2j-2\}$ of length j. Moreover, this process of converting from μ to s is invertible: for any increasing subsequence of $\{1, \ldots, k+2j-2\}$ of length j, let μ be defined by

$$\mu(k+a) = s(a) - a + 1, \qquad \text{for } a = 1, \ldots, j$$

Since s necessarily satisfies $s(a) \le k+j+a-2$, it follows that $\mu(k+a) \le k+j-2$ but this condition is not strong enough to guarantee admissibility $\mu(k+a) \le k+a-1$. Thus, the count of the number of increasing subsequences of $\{1, \ldots, k+2j-2\}$ of length j, which is (2.12), is an over-count of the number of reduced admissible maps μ, but a useful upper bound. □

We can then estimate $J_N^{(k,j)}(\underline{t}_{(k,j)})(f^{(k+j)})$ via the collapsing estimate in Lemma A.1.

Claim 2.6 *For* $j \geq 1$,

$$\left\| \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} S^{(1,k)} J_N^{(k,j)}(\underline{t}_{(k,j)})(f^{(k+j)}) d\underline{t}_{(k,j)} \right\|_{L^\infty_{t_k}[0,T]L^2_{x,x'}}$$
$$\leq 2^k (4CT^{1/2})^j \left\| S^{(1,k+j-1)} B_{N,1,k+j} f^{(k+j)}(t_{k+j}) \right\|_{L^2_{t_{k+j}}[0,T]L^2_{x,x'}}$$

Proof The proof follows the same steps usually used to estimate

$$\left\| \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} S^{(1,k-1)} B_{N,1,k} J_N^{(k,j)}(\underline{t}_{(k,j)})(f^{(k+j)}) d\underline{t}_{(k,j)} \right\|_{L^1_{t_k}[0,T]L^2_{x,x'}}$$

and is well-known by now. We include the proof for completeness. We start by using Lemma 2.4,

$$\left\| \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} S^{(1,k)} J_N^{(k,j)}(\underline{t}_{(k,j)})(f^{(k+j)}) d\underline{t}_{(k,j)} \right\|_{L^\infty_{t_k}[0,T]L^2_{x,x'}}$$
$$\leq 2^k 4^j \left\| \int_D S^{(1,k)} J_N^{(k,j)}(\underline{t}_{(k,j)}, \mu_m)(f^{(k+j)}) d\underline{t}_{(k,j)} \right\|_{L^\infty_{t_k} L^2_{x,x'}}$$
$$\leq 2^k 4^j \int_{[0,T]^j} \left\| S^{(1,k)} J_N^{(k,j)}(\underline{t}_{(k,j)}, \mu_m)(f^{(k+j)}) \right\|_{L^2_{x,x'}} d\underline{t}_{(k,j)}$$

Cauchy-Schwarz at dt_{k+1}

$$\leq 2^k 4^j T^{\frac{1}{2}} \int_{[0,T]^{j-1}} d\underline{t}_{(k+1,j-1)}$$
$$\times \left\| S^{(1,k)} B_{N,\mu_m(k+1),k+1} U^{(k+1)}(t_{k+1} - t_{k+2}) \ldots \right\|_{L^2_{t_k} L^2_{x,x'}}$$

Use Lemma A.1,

$$\leq 2^k 4^j C T^{\frac{1}{2}} \int_{[0,T]^{j-1}} d\underline{t}_{(k+1,j-1)}$$
$$\times \left\| S^{(1,k+1)} B_{N,\mu_m(k+2),k+2} U^{(k+2)}(t_{k+2} - t_{k+3}) \ldots \right\|_{L^2_{x,x'}}$$

Repeating such a process gives

$$\leq 2^k 4^j C^{j-1} T^{\frac{j-1}{2}} \int_{[0,T]} \left\| S^{(1,k+j-1)} B_{N,\mu_m(k+j),k+j}(f^{(k+j)}) \right\|_{L^2_{x,x'}} dt_{k+j}$$

By symmetry,

$$= 2^k \left(4CT^{\frac{1}{2}}\right)^{j-1} \int_{[0,T]} \left\| S^{(1,k+j-1)} B_{N,1,k+j}(f^{(k+j)}) \right\|_{L^2_{x,x'}} dt_{k+j}$$

Applying Cauchy-Schwarz in time once more yields the claim. □

Starting with the formulae for $\mathrm{FP}^{(k,\ell_c)}$, $\mathrm{DP}^{(k,\ell_c)}$, and $\mathrm{IP}^{(k,\ell_c)}$, we apply Lemma 2.4 using the bound in Lemma 2.5 to reduce the number of Duhamel terms, and apply the estimate in Claim 2.6 for each term. This provides preliminary estimates for the three parts in the expansion of $\omega_{N,H}^{(k)}$.

Specifically, for the free part, this yields

$$\begin{aligned} &\left\| S^{(1,k)} \mathrm{FP}^{(k,\ell_c)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &\leq \| S^{(1,k)} f_{\mathrm{FP}}^{(k,0)}(t_k) \|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &\quad + 2^k \sum_{j=1}^{\ell_c} \left(4CT^{\frac{1}{2}}\right)^j \| S^{(1,k+j-1)} B_{N,1,j+k} f_{\mathrm{FP}}^{(k,j)}(t_{k+j}) \|_{L^2_{t_{k+j}}[0,T] L^2_{x,x'}} \end{aligned}$$

Plugging in $f_{\mathrm{FP}}^{(k,j)}$ and applying the Klainerman-Machedon trilinear estimate (Lemma A.1),

$$\leq \| S^{(1,k)} \omega_{N,H}^{(k)}(0) \|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} + 2^k \sum_{j=1}^{\ell_c} \left(4CT^{\frac{1}{2}}\right)^j \| S^{(1,k+j)} \omega_{N,H}^{(k+j)}(0) \|_{L^2_{x,x'}}$$

which completes the proof for the free part in Proposition 2.1.

For the driving part, this yields

$$\begin{aligned} \left\| S^{(1,k)} \mathrm{DP}^{(k,\ell_c)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} &\leq \| S^{(1,k)} f_{\mathrm{DP}}^{(k,0)}(t_k) \|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &\quad + 2^k \sum_{j=1}^{\ell_c} \left(4CT^{\frac{1}{2}}\right)^j \| S^{(1,k+j-1)} B_{N,1,k+j} f_{\mathrm{DP}}^{(k,j)}(t_{k+j}) \|_{L^2_{t_{k+j}}[0,T] L^2_{x,x'}} \end{aligned} \tag{2.13}$$

For the interaction part, this yields

$$\begin{aligned} &\left\| S^{(1,k)} \mathrm{IP}^{(k,\ell_c)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &\leq 2^k \left(4CT^{\frac{1}{2}}\right)^{\ell_c+1} \left\| S^{(1,k+\ell_c)} B_{N,1,k+\ell_c+1} \omega_{N,H}^{(k+\ell_c+1)}(t_{k+\ell_c+1}) \right\|_{L_{t^2_{k+\ell_c+1}} L^2_{x,x'}} \end{aligned} \tag{2.14}$$

We continue the estimates of the driving part and the interaction part separately below.

2.1.1 Estimate for the Driving Part

We complete the bound of the right side of (2.13). Using the $X^{(k+j)}_{\frac{1}{2}+} \hookrightarrow L^{\infty}_{t_{k+j}}[0,T]L^2_{x,x'}$ embedding,

$$\|f^{(k,0)}_{\rm DP}(t_k)\|_{L^{\infty}_{t_k}[0,T]L^2_{x,x'}} \leq C\|\theta(t_k)S^{(1,k)}f^{(k,0)}_{\rm DP}(t_k)\|_{X^{(k)}_{\frac{1}{2}+}}$$

where $\theta(t)$ is a smooth cutoff in time such that $\theta(t) = 1$ on $[0, T]$. For $1 \leq j \leq \ell_c$, by Lemma A.3 (a version of the Klainerman-Machedon trilinear estimate with X-norm on the right side)

$$\begin{aligned}&\|S^{(1,k+j-1)}B_{N,1,k+j}f^{(k,j)}_{\rm DP}(t_{k+j})\|_{L^2_{t_{k+j}}[0,T]L^2_{x,x'}}\\ &\leq C\|\theta(t_{k+j})S^{(1,k+j)}f^{(k,j)}_{\rm DP}(t_{k+j})\|_{X^{(k+j)}_{\frac{1}{2}+}}\end{aligned}$$

Thus to complete the bound of (2.13), it remains to estimate for $0 \leq j \leq \ell_c$,

$$\|\theta(t_{k+j})S^{(1,k+j)}f^{(k,j)}_{\rm DP}(t_{k+j})\|_{X^{(k+j)}_{\frac{1}{2}+}} \tag{2.15}$$

Referring to the definition (2.6) of $f^{(k,j)}_{\rm DP}$, insert $\tilde{\theta}(t_{k+j+1})$ inside the integrand, where $\tilde{\theta}(t)$ is a smooth cutoff in time such that $\tilde{\theta}(t) = 1$ on the support of $\theta(t)$. Applying Claim A.2,

$$\leq C\|\tilde{\theta}(t_{k+j+1})S^{(1,k+j)}V^{(k+j)}_N\gamma^{(k+j)}_N(t_{k+j+1})\|_{X^{(k+j)}_{-\frac{1}{2}+}}$$

By dual Strichartz (Lemma A.4) we complete the bound of (2.15) by

$$\begin{aligned}&\leq CN^{\frac{5}{2}\beta-1}(k+j)^2\|S^{(1,k+j)}\gamma^{(k+j)}_N(t_{k+j+1})\|_{L^2_{t_{k+j+1}}L^2_{x,x'}}\\ &\leq CT^{1/2}N^{\frac{5}{2}\beta-1}(k+j)^2E_0^{k+j}\end{aligned}$$

where, in the last step, we appealed to the energy bound and the $(k+j)^2$ factor came from the expansion of $V^{(k+j)}_N$ into component terms. Inserting this to bound of (2.15) into the right side of (2.13),

$$\begin{aligned}\|S^{(1,k)}{\rm DP}^{(k,\ell_c)}\|_{L^{\infty}_{t_k}[0,T]L^2_{x,x'}} &\leq CT^{1/2}N^{\frac{5}{2}\beta-1}2^k\sum_{j=0}^{\ell_c}(k+j)^2(4CT^{1/2})^jE_0^{k+j}\\ &\leq CT^{1/2}N^{\frac{5}{2}\beta-1}(2E_0)^kk^2\end{aligned}$$

provided T is small enough so that $4CT^{1/2}E_0 \leq \frac{1}{2}$, which completes the bound for the driving part in Proposition 2.1.

2.1.2 Estimate for the Interaction Part

From (2.14), we see that it remains to bound

$$\left\| S^{(1,k+\ell_c)} B_{N,1,k+\ell_c+1} \omega_{N,H}^{(k+\ell_c+1)} (t_{k+\ell_c+1}) \right\|_{L^2_{t_{k+\ell_c+1}} L^2_{x,x'}} \tag{2.16}$$

Take the crude estimate that burns derivatives and gains bad powers of N:

$$\begin{aligned} &\left\| S^{(1,k+\ell_c)} B_{N,1,k+\ell_c+1} \omega_{N,H}^{(k+\ell_c+1)} (t_{k+\ell_c+1}) \right\|^2_{L^2_{t_{k+\ell_c+1}} L^2_{x,x'}} \\ &\leq CTN^{2\beta} \left\| V_N' \right\|^2_{L^2} \left\| S^{(1,k+\ell_c)} \omega_{N,H}^{(k+\ell_c+1)} (t, \mathbf{x}_k, x_{k+1}, \mathbf{x}_k', x_{k+1}) \right\|^2_{L^\infty_t [0,T] L^2_{x,x'}} \end{aligned}$$

and use the trace theorem,

$$\leq CTN^{5\beta} \left\| V' \right\|^2_{L^2} \left\| S^{(1,k+\ell_c+1)} \omega_{N,H}^{(k+\ell_c+1)} \right\|^2_{L^\infty_t [0,T] L^2_{x,x'}}.$$

Inserting this estimate of (2.16) into (2.14),

$$\begin{aligned} &\left\| S^{(1,k)} \mathrm{IP}^{(k,\ell_c)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &\leq 2^k T^{1/2} (4CT^{1/2})^{\ell_c+1} N^{\frac{5\beta}{2}} \left\| S^{(1,k+\ell_c+1)} \omega_N^{(k+\ell_c+1)} (t) \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \end{aligned}$$

2.2 Summing in *k* / Proof of Theorem 2.3

Using the definition of the master norm (1.6) and the decomposition (2.5)

$$\begin{aligned} &\| \Gamma_N(t) - \left\{ |\phi_N(t)\rangle \langle \phi_N(t)|^{\otimes k} \right\} \|_{L^\infty_{[0,T]} H^1_Z} \\ &\leq \sum_{k=0}^{\infty} Z^{-k} (\| S^{(1,k)} \mathrm{FP}^{(k,\ell_c)} \|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &+ \| S^{(1,k)} \mathrm{DP}^{(k,\ell_c)} \|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} + \| S^{(1,k)} \mathrm{IP}^{(k,\ell_c)} \|_{L^\infty_{t_k}[0,T] L^2_{x,x'}}) \end{aligned}$$

Applying the bounds on each component in (2.7), (2.8), (2.11), we obtain

$$\|\Gamma_N(t) - \left\{|\phi_N(t)\rangle\langle\phi_N(t)|^{\otimes k}\right\}\|_{L^\infty_{[0,T]}H^1_Z}$$
$$\leq \sum_{k=0}^{\infty}\sum_{j=0}^{\ell_c}(2Z^{-1})^k(4CT^{1/2})^j\|S^{(1,k+j)}\omega_{N,H}^{(k+j)}(0)\|_{L^2_{x,x'}}$$
$$+CT^{1/2}N^{\frac{5}{2}\beta-1}\sum_{k=0}^{\infty}(2E_0Z^{-1})^k k^2$$

In the double sum, changing (k, j) to (m, j) where $m = k + j$, and using the discrete Fubini that $\sum_{k=0}^{\infty}\sum_{j=0}^{\ell_c} = \sum_{j=0}^{\ell_c}\sum_{k=0}^{\infty} = \sum_{j=0}^{\ell_c}\sum_{m=j}^{\infty} = \sum_{m=0}^{\infty}\sum_{j=0}^{\min(m,\ell_c)}$, we get

$$\|\Gamma_N(t) - \left\{|\phi_N(t)\rangle\langle\phi_N(t)|^{\otimes k}\right\}\|_{L^\infty_{[0,T]}H^1_Z}$$
$$\leq \sum_{m=0}^{\infty}\sum_{j=0}^{\min(m,\ell_c)}(2Z^{-1})^m(2CZT^{1/2})^j\|S^{(1,m)}\omega_{N,H}^{(m)}(0)\|_{L^2_{x,x'}}$$
$$+CT^{1/2}N^{\frac{5}{2}\beta-1}\sum_{k=0}^{\infty}(2E_0Z^{-1})^k k^2$$

Provided $Z \gtrsim E_0$ and $T \lesssim Z^{-2}$, we can carry out the j and k sums. This completes the proof of Theorem 2.3.

3 Comparing H-NLS and NLS

In this section, we give the proof of Theorem 1.4 which will be concluded after Propositions 3.6 and 3.8. The estimate (1.16) in the introduction then follow.

We need the atomic U spaces introduced by Koch & Tataru [55, 56] and the V spaces of bounded p-variation of Wiener [74]. Their properties have been further elaborated in Hadac, Herr, & Koch [45] and Koch, Tataru, & Visan [57]. Here, following [45,Definition 2.1 and Definition 2.3] (see the slight change in the erratum for that paper), we define $U^p(I; H)$ and $V^p(I; H)$, where $I = [T_1, T_2) \subset \mathbb{R}$ is a time interval and H is a Hilbert space (in x) below.

Let $\mathcal{Z}$ be the set of all finite partitions $T_1 = t_0 < t_1 < ... < t_K \leq T_2$ of I and let us use the convention that $v(T_2) = 0$ for all functions $v : I \to H$.

Definition 3.1 Let $p \in [1, \infty)$ .We call a function $a : I \to H$ a U^p-atom if it takes the form $a = \sum_{k=1}^{K} \mathbf{1}_{[t_{k-1},t_k)}\phi_{k-1}$ where $\{t_k\} \in \mathcal{Z}$ and $\{\phi_k\} \subset H$ with $\sum_{k=0}^{K-1}\|\phi_k\|_H^p = 1$. The atomic space $U^p(I; H) \subset L^\infty(I; H)$ is the space of functions $u : I \to H$ given the norm:

$$\|u\|_{U^p} = \inf\left\{\sum_{j=1}^{\infty}\left|\lambda_j\right| : u = \sum_{j=1}^{\infty}\lambda_j a,\ \lambda_j \in \mathbb{C}, a_j \text{ is a } U^p\text{-atom for all } j\right\}.$$

Definition 3.2 Let $p \in [1, \infty)$ .The space $V^p(I; H)$ is the space of of all functions $v : I \to H$ such that

$$\|v\|_{V^p} = \sup_{\{t_k\} \in \mathcal{Z}} \left(\sum_{j=1}^{\infty} \|v(t_k) - v(t_{k-1})\|_H^p \right)^{\frac{1}{p}} < +\infty$$

and the space $V^p_{\text{rc}}(I; H)$ denotes the closed subspace of all right-continuous functions $v : I \to H$ such that $v(T_1) = 0$.

We have, for $1 \le p < q < \infty$ (see Proposition 2.4, Corollary 2.6 in [45]) the continuous embeddings

$$U^p \hookrightarrow V^p_{\text{rc}} \hookrightarrow U^q \hookrightarrow L^\infty \tag{3.1}$$

We in fact work exclusively with the variants $U^p_\Delta L^2_x$, $V^p_\Delta L^2_x$ defined as the $U^p L^2_x$ and $V^p L^2_x$ norms, respectively, after pulling-back by the linear flow $e^{it\Delta}$ (as in [45,Definition 2.15]), and will denote the restriction of such norms to a time subinterval I as $U^p_{I,\Delta} L^2_x$ and $V^p_{I,\Delta} L^2_x$.

It is immediate from the definition of the $U^p_{I,\Delta} L^2_x$ norm that for any $1 \le p < \infty$,

$$\|e^{it\Delta}\phi\|_{U^p_{I,\Delta} L^2_x} \le \|\phi\|_{L^2_x}$$

From [45,Theorem 2.8, Proposition 2.10], we have the duality relationship

$$\left\| \int_0^t e^{i(t-t')\Delta} f(t')\, dt' \right\|_{U^2_{I,\Delta} L^2_x} = \sup_{\substack{g \in V^2_{I,\Delta} L^2_x \\ \|g\|_{V^2_{I,\Delta} L^2_x} \le 1}} \left| \int_I \int_x f(x,t)\, g(x,t)\, dx\, dt \right| \tag{3.2}$$

which is key to estimating Duhamel terms.

It follows from [45,Proposition 2.19] that the Strichartz estimates imply

$$\|u\|_{L^q_I L^r_x} \lesssim \|u\|_{U^q_{I,\Delta} L^2_x} \tag{3.3}$$

for admissible (q, r):

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}, \qquad 2 \le q < \infty, \quad 2 \le r \le 6$$

where we note that the q exponent appears on both the left and right. From (3.1), the larger the q, the smaller the right side (the better the resulting bound) in (3.3).

Also, from [45,Proposition 2.20], we have the following property as a substitute for the failure of the $V^2 \hookrightarrow U^2$ embedding (compare (3.1)). If T is a bilinear operator satisfying

$$\|T(u_1, u_2)\|_{L^2_I L^2_x} \le C \|u_1\|_{U^q_{I,\Delta} L^2_x} \|u_2\|_{U^q_{I,\Delta} L^2_x} \tag{3.4}$$

for some $q > 2$ and

$$\|T(u_1, u_2)\|_{L_I^2 L_x^2} \leq C_2 \|u_1\|_{U_{I,\Delta}^2 L_x^2} \|u_2\|_{U_{I,\Delta}^2 L_x^2} \tag{3.5}$$

then it follows that

$$\|T(u_1, u_2)\|_{L_I^2 L_x^2} \leq C_2 \left(\log \frac{C}{C_2} + 1 \right) \|u_1\|_{U_{I,\Delta}^2 L_x^2} \|u_2\|_{V_{I,\Delta}^2 L_x^2} \tag{3.6}$$

To present an application that we need below, first note that the following bilinear Strichartz estimate holds.

Lemma 3.3 *(blinear Strichartz [4]) For $x \in \mathbb{R}^d$,*

$$\|P_{M_1} e^{it\Delta} \phi_1 \overline{P_{M_2} e^{it\Delta} \phi_2}\|_{L^2_{[0,1]} L_x^2} \lesssim \left(\frac{\min(M_1, M_2)^{d-1}}{\max(M_1, M_2)} \right)^{1/2} \|P_{M_1} \phi_1\|_{L_x^2} \|P_{M_2} \phi_2\|_{L_x^2}$$

which is, in U-V notation,

$$\begin{aligned} &\|P_{M_1} \phi_1 \overline{P_{M_2} \phi_2}\|_{L^2_{[0,1]} L_x^2} \lesssim \left(\frac{\min(M_1, M_2)^{d-1}}{\max(M_1, M_2)} \right)^{1/2} \\ &\|P_{M_1} \phi_1\|_{U_{I,\Delta}^2 L_x^2} \|P_{M_2} \phi_2\|_{U_{I,\Delta}^2 L_x^2} \end{aligned} \tag{3.7}$$

Lemma 3.3 fits the template (3.5) with $T(u_1, u_2) = u_1 u_2$, $u_1 = P_{M_1} \phi_1$, $u_2 = P_{M_2} \phi_2$, and $C_2 = \frac{\min(M_1, M_2)}{\max(M_1, M_2)^{1/2}}$. However, by Hölder, Sobolev and Strichartz estimates, we have

$$\begin{aligned} \|P_{M_1} \phi_1 P_{M_2} \phi_2\|_{L_I^2 L_x^2} &\lesssim \|P_{M_1} \phi_1\|_{L_I^4 L_x^4} \|P_{M_1} \phi_2\|_{L_I^4 L_x^4} \\ &\lesssim M_1^{1/4} M_2^{1/4} \|\phi_1\|_{U_{I,\Delta}^4 L_x^2} \|\phi_2\|_{U_{I,\Delta}^4 L_x^2} \end{aligned}$$

which fits the template of (3.4) with $q = 4$ and $C = M_1^{1/4} M_2^{1/4}$. The conclusion (3.6) reads

$$\begin{aligned} &\|P_{M_1} \phi_1 P_{M_2} \phi_2\|_{L_I^2 L_x^2} \lesssim \frac{\min(M_1, M_2)}{\max(M_1, M_2)^{1/2}} \left(1 + \log \frac{\max(M_1, M_2)}{\min(M_1, M_2)} \right) \\ &\|P_{M_1} \phi_1\|_{U_{I,\Delta}^2 L_x^2} \|P_{M_2} \phi_2\|_{V_{I,\Delta}^2 L_x^2} \end{aligned} \tag{3.8}$$

where in fact the position of the U_Δ^2 and V_Δ^2 norms on the right can be switched. The result is that we have been able to take (3.7) and upgrade one of the norms on the right side to V^2 at the expense a logarithmic loss.

After this background, we now proceed with the proof of Theorem 1.4. Recall ϕ and ϕ_N are the solutions to (1.1) and (1.11) and let

$$\tilde{\phi} = \phi_N - \phi.$$

It follows from energy conservation and classical well-posedness theory in the Strichartz spaces that (1.1) and (1.11) in the $\mathbb{R}^3$ defocusing case satisfy the global in time bounds

$$\|\phi\|_{L_t^\infty H_x^1} \leq C_1 , \qquad \|\phi_N\|_{L_t^\infty H_x^1} \leq C_1$$

where the constant C_1 depends on the size of the initial data in H^1 . The following theorem on scattering was obtained for NLS by Ginibre & Velo [38] using a Morawetz estimate of Lin & Strauss [61]. An alternate proof using an interaction Morawetz was given by Colliander, Keel, Staffilani, Takaoka, & Tao [32]. A version in the focusing setting by Duyckaerts, Holmer, Roudenko [34] was obtained using the concentration compactness and virial rigidity method of Kenig & Merle [50]. The corresponding Hartree result was obtained by Miao, Xu, & Zhao [65].

Theorem 3.4 *(scattering) Defocusing cubic NLS (1.1) and defocusing cubic H-NLS (1.11) in $\mathbb{R}^3$ both scatter in H^1. In particular, for H^1 data, we have the global-in-time bounds*

$$\|\langle\nabla\rangle\phi\|_{L_t^2 L_x^6} \lesssim 1 , \qquad \|\langle\nabla\rangle\phi_N\|_{L_t^2 L_x^6} \lesssim 1$$

Bounds on other Strichartz norms can be obtained by interpolation. As a corollary, we have that there exists a *finite* partition of the time interval $[0, +\infty)$

$$0 = t_0 < t_1 < \cdots < t_J = \infty$$

such that on each subinterval $I = [t_j, t_{j+1})$ for $0 \leq j \leq J$, there holds

$$\|\langle\nabla\rangle\phi\|_{L_I^2 L_x^6} \leq \delta , \qquad \|\langle\nabla\rangle\phi_N\|_{L_I^2 L_x^6} \leq \delta \tag{3.9}$$

Corollary 3.5 *If $\delta > 0$ is chosen small*[13] *in terms of C_1, then for each interval I on which (3.9) holds, we have*

$$\|\phi\|_{U_{I,\Delta}^2 H_x^1} \leq 2C_1 , \qquad \|\phi_N\|_{U_{I,\Delta}^2 H_x^1} \leq 2C_1 \tag{3.10}$$

Proof The argument for ϕ (NLS) and ϕ_N (HNLS) is similar, so we will just write it for ϕ. On $I = [t_*, t^*]$, we have

$$\phi(t) = e^{i(t-t_*)\Delta}\phi(t_*) + \int_{t_*}^{t} e^{i(t-t')\Delta} \left|\phi(t')\right|^2 \phi(t')\, dt'$$

By (3.2),

$$\|\phi\|_{U_{I,\Delta}^2 H_x^1} \lesssim \|\phi(t_*)\|_{H_x^1} + \sup_{\substack{g\in V_{I,\Delta}^2 L_x^2 \\ \|g\|_{V_{I,\Delta}^2 L_x^2}\leq 1}} \left| \int_I \int_x \langle\nabla\rangle(|\phi|^2\phi)\, g\, dx\, dt \right| \tag{3.11}$$

[13] The proof shows that $\delta \lesssim \langle C_1\rangle^{-1/3}$ suffices

For a particular g, we estimate as

$$\left| \int_I \int_x \langle \nabla \rangle (|\phi|^2 \phi) \; g \, dx \, dt \right| \lesssim \| \langle \nabla \rangle \phi \|_{L_I^2 L_x^6} \| \phi \|_{L_I^4 L_x^3} \| \phi \|_{L_I^\infty L_x^6} \| g \|_{L_I^4 L_x^3}$$

By Hölder interpolation

$$\| \phi \|_{L_I^4 L_x^3} \le \| \phi \|_{L_I^2 L_x^6}^{1/2} \| \phi \|_{L_x^\infty L_x^2}^{1/2} \le \delta^{1/2} C_1^{1/2}$$

By Sobolev embedding, $\| \phi \|_{L_I^\infty L_x^6} \lesssim C_1$. And by (3.3) and (3.1),

$$\| g \|_{L_I^4 L_x^3} \lesssim \| g \|_{U_{I,\Delta}^4 L_x^2} \lesssim \| g \|_{V_{I,\Delta}^2 L_x^2}$$

Inserting these above, we obtain

$$\left| \int_I \int_x \langle \nabla \rangle (|\phi|^2 \phi) \; g \, dx \, dt \right| \lesssim \delta^{3/2} C_1^{3/2} \| g \|_{V_{I,\Delta}^2 L_x^2}$$

By (3.11), and the fact that $\| \phi(t_*) \|_{H_x^1} \le C_1$, we obtain the result. □

Now we will show that on each time interval I in the finite partition of $0 \le t < +\infty$, we obtain a bound on $\tilde{\phi}$ in terms of the initial difference for that subinterval.

Proposition 3.6 *Suppose that on a time interval I the solutions to (1.1) and (1.11) satisfy*

$$\| \phi \|_{U_{I,\Delta}^2 H_x^1} \le 2 C_1 \, , \qquad \| \phi_N \|_{U_{I,\Delta}^2 H_x^1} \le 2 C_1$$

for some constant C_1 and

$$\| \langle \nabla \rangle \phi \|_{L_I^2 L_x^6} \le \delta \, , \qquad \| \langle \nabla \rangle \phi_N \|_{L_I^2 L_x^6} \le \delta$$

Consider the difference

$$\tilde{\phi}(t) = \phi_N(t) - \phi(t)$$

with initial condition $\tilde{\phi}_0 = (\phi_N)_0 - \phi_0$ for the time interval I.

Provided $\delta > 0$ is chosen small, $\| \tilde{\phi}_0 \|_{H_x^1}$ is sufficiently small, and N is sufficiently large (all of these thresholds are expressed in terms of C_1 only), then we have

$$\| \tilde{\phi} \|_{U_{I,\Delta}^2 H_x^1} \lesssim \| \tilde{\phi}_0 \|_{H_x^1} + (\log N)^7 N^{-\beta} C_1^3 \tag{3.12}$$

Remark 3.7 This result just fails by a logarithm to obtain the optimal $N^{-\beta}$ rate at 1 derivative of regularity. With more delicate arguments, we can indeed reduce the power on the $\log N$ factor, although we do not see a way to completely eliminate the $\log N$ factor.

Proof Plug in $\phi_N = \phi + \tilde{\phi}$ into (1.11), and using that ϕ solves (1.1) to simplify, we obtain that $\tilde{\phi}$ solves

$$0 = i\partial_t\tilde{\phi} + \Delta\tilde{\phi} - (V_N * |\phi + \tilde{\phi}|^2)(\phi + \tilde{\phi}) + |\phi|^2\phi$$

Adopting the shorthand,

$$W_N(x) = N^{3\beta}V(N^\beta x) - b_0\delta(x)$$

we expand the nonlinearity

$$\begin{aligned}
0 = {} & i\partial_t\tilde{\phi} + \Delta\tilde{\phi} \\
& - (W_N * |\phi|^2)\phi && \leftarrow \text{forcing} \\
& - 2[V_N * \mathrm{Re}(\bar{\phi}\tilde{\phi})]\phi - (V_N * |\phi|^2)\tilde{\phi} && \leftarrow \text{linear in } \tilde{\phi} \\
& - 2[V_N * \mathrm{Re}(\bar{\phi}\tilde{\phi})]\tilde{\phi} - V_N * |\tilde{\phi}|^2)\phi && \leftarrow \text{quadratic in } \tilde{\phi} \\
& - (V_N * |\tilde{\phi}|^2)\tilde{\phi} && \leftarrow \text{ cubic in } \tilde{\phi}
\end{aligned}$$

By (3.2),

$$\begin{aligned}
& \|\tilde{\phi}\|_{U^2_{I,\Delta}H^1_x} \\
& \leq \|\tilde{\phi}_0\|_{H^1_x} \\
& \quad + \sup_{\substack{g \in V^2_{I,\Delta}L^2_x \\ \|g\|_{V^2_{I,\Delta}L^2_x} \leq 1}} \int_I \int \langle\nabla\rangle[(W_N * |\phi|^2)\phi]\, g\, dx\, dt && (3.13) \\
& + K\|\langle\nabla\rangle\Big([V_N * \mathrm{Re}(\bar{\phi}\tilde{\phi})]\phi\Big)\|_{L^1_I L^2_x} + K\|\langle\nabla\rangle[(V_N * |\phi|^2)\tilde{\phi}]\|_{L^1_I L^2_x} && (3.14) \\
& + K\|\langle\nabla\rangle\Big([V_N * \mathrm{Re}(\bar{\phi}\tilde{\phi})]\tilde{\phi}\Big)\|_{L^1_I L^2_x} + K\|\langle\nabla\rangle[(V_N * |\tilde{\phi}|^2)\phi]\|_{L^1_I L^2_x} && (3.15) \\
& + K\|\langle\nabla\rangle[(V_N * |\tilde{\phi}|^2)\tilde{\phi}]\|_{L^1_I L^2_x} && (3.16)
\end{aligned}$$

in which (3.13) corresponds to the forcing, (3.14) corresponds to the terms linear in $\tilde{\phi}$, (3.15) corresponds to the terms quadratic in $\tilde{\phi}$, and (3.16) corresponds to the terms cubic in $\tilde{\phi}$, for some absolute constant $K > 0$. The linear, quadratic, and cubic terms are estimated in a standard way (the $V_N *$ operator is treated the same of a delta convolution (a product pairing)), yielding a bound by

$$\begin{aligned}
\|\tilde{\phi}\|_{U^2_{I,\Delta}H^1_x} & \leq \|\tilde{\phi}_0\|_{H^1_x} + Q_N(\phi) + K\|\langle\nabla\rangle\phi\|_{L^2_I L^6_x}\|\phi\|_{L^\infty_I H^1_x}\|\langle\nabla\rangle\tilde{\phi}\|_{L^2_I L^6_x} \\
& \quad + K\|\langle\nabla\rangle\phi\|_{L^2_I L^6_x}\|\tilde{\phi}\|_{L^\infty_I H^1_x}\|\langle\nabla\rangle\tilde{\phi}\|_{L^2_I L^6_x} \\
& \quad + K\|\langle\nabla\rangle\tilde{\phi}\|_{L^2_I L^6_x}\|\tilde{\phi}\|_{L^\infty_I H^1_x}\|\langle\nabla\rangle\tilde{\phi}\|_{L^2_I L^6_x} \\
& \leq \|\tilde{\phi}_0\|_{H^1_x} + Q_{N,I}(\phi) + R(\tilde{\phi})\|\tilde{\phi}\|_{U^2_{I,\Delta}H^1_x}
\end{aligned} \tag{3.17}$$

where

$$Q_{N,I}(\phi) \stackrel{\text{def}}{=} \sup_{\substack{g \in V^2_{I,\Delta} L^2_x \\ \|g\|_{V^2_{I,\Delta} L^2_x} \le 1}} \int_I \int \langle \nabla \rangle [(W_N * |\phi|^2)\phi] \, g \, dx \, dt$$

$$R_I(\tilde{\phi}) \stackrel{\text{def}}{=} KC_1\delta + K\delta \|\tilde{\phi}\|_{U^2_{I,\Delta} H^1_x} + K\|\tilde{\phi}\|^2_{U^2_{I,\Delta} H^1_x}$$

In fact, (3.17) holds for any $I' \subset I$. Let $I' \subset I$ be the maximal subinterval on which

$$\|\tilde{\phi}\|_{U^2_{I',\Delta} H^1_x} \le 4(\|\tilde{\phi}_0\|_{H^1_x} + Q_{N,I}(\phi)) \tag{3.18}$$

(notice that it is I' on the left in $U^2_{I',\Delta} H^1_x$ but I in the right in $Q_{N,I}(\phi)$) Then, provided $3KC_1\delta \le \frac{1}{2}$, we have

$$R_{I'}(\tilde{\phi}) \le KC_1\delta + 4K\delta(\|\tilde{\phi}_0\|_{H^1_x} + Q_{N,I}(\phi)) + 16K(\|\phi_0\|_{H^1_x} + Q_{N,I}(\phi))^2 \tag{3.19}$$

by plugging (3.18) into $R_{I'}(\tilde{\phi})$. Provided N is chosen sufficiently large in terms of C_1, the estimate (3.20) below for $Q_{N,I}(\phi)$ will in particular imply

$$4(\|\tilde{\phi}_0\|_{H^1_x} + Q_{N,I}(\phi)) \le \delta$$

From this and (3.19), it follows that $R_{I'}(\tilde{\phi}) \le 3KC_1\delta \le \frac{1}{2}$. Substituting this into (3.17) (with I replaced by I'), we obtain

$$\|\tilde{\phi}\|_{U^2_{I',\Delta} H^1_x} \le \|\tilde{\phi}_0\|_{H^1_x} + Q_{N,I}(\phi) + \frac{1}{2}\|\tilde{\phi}\|_{U^2_{I',\Delta} H^1_x}$$

or, after absorbing $\frac{1}{2}\|\tilde{\phi}\|_{U^2_{I',\Delta} H^1_x}$ into the left,

$$\|\tilde{\phi}\|_{U^2_{I',\Delta} H^1_x} \le 2\|\tilde{\phi}_0\|_{H^1_x} + 2Q_{N,I}(\phi)$$

This contradicts the maximality of $I' \subset I$ satisfying (3.18) unless $I' = I$. Thus, we are able to conclude that (3.18) holds for $I' = I$, which is the desired result, once we have suitably estimated $Q_{N,I}(\phi)$.

Now we estimate $Q_{N,I}(\phi)$, which is more interesting as it uses the sharpest available bilinear estimate (3.8). We use duality and apply Lemma A.6. Taking $g \in V^2_{I,\Delta} L^2_x$ we need to show

$$\int_I \int \nabla [(W_N * |\phi|^2)\, \phi] \, g \, dx \, dt \lesssim N^{-\beta} (\log N)^7 \|\phi\|^3_{U^2_{I,\Delta} H^1_x} \|g\|_{V^2_{I,\Delta} L^2_x}$$

We distribute the derivative on the left to obtain two terms

$$\begin{aligned}&\int_I\int \nabla[(W_N*|\phi|^2)\,\phi]\;g\;dx\,dt\\&=2\int_I\int[W_N*\mathrm{Re}(\bar{\phi}\nabla\phi)]\,\phi\,g\,dx\,dt+\int_I\int[W_N*|\phi|^2]\,\nabla\phi\,g\,dx\,dt\\&=\mathrm{I}+\mathrm{II}\end{aligned}$$

For the second term, we estimate as

$$\mathrm{II}\lesssim\|W_N*|\phi|^2\|_{L_I^2L_x^3}\|\nabla\phi\|_{L_I^4L_x^3}\|g\|_{L_I^4L_x^3}$$

By Lemma A.5 with $s=1$ and $p=3$,

$$\begin{aligned}\mathrm{II}&\lesssim N^{-\beta}\|\nabla|\phi|^2\|_{L_I^2L_x^3}\|\nabla\phi\|_{L_I^4L_x^3}\|g\|_{L_I^4L_x^3}\\&\lesssim N^{-\beta}\|\nabla\phi\|_{L_I^2L_x^6}\|\phi\|_{L_I^\infty L_x^6}\|\nabla\phi\|_{L_I^4L_x^3}\|g\|_{L_I^4L_x^3}\\&\lesssim N^{-\beta}\|\phi\|^3_{U^2_{\Delta,I}H_x^1}\|g\|_{V^2_{I,\Delta}L_x^2}\\&\lesssim N^{-\beta}C_1^3\|g\|_{V^2_{I,\Delta}L_x^2}\end{aligned}$$

where we have applied (3.3) and also $V^2\hookrightarrow U^4$ embedding (see (3.1)) for the g term. For Term I, however, we use the dual structure and apply Lemma A.6. Applying Lemma A.6 slightly interpolated with the trivial estimate to insert the logarithmic terms

$$\mathrm{I}\lesssim N^{-\beta}(\log N)^7\left\|\frac{\langle\nabla\rangle^{1/2}}{(\log\langle\nabla\rangle)^3}[\bar{\phi}\;\nabla\phi]\right\|_{L_I^2L_x^2}\left\|\frac{\langle\nabla\rangle^{1/2}}{(\log\langle\nabla\rangle)^4}[\phi g]\right\|_{L_I^2L_x^2}$$

We can (nearly) rescue the $\frac{1}{2}$ derivative in each term using the bilinear Strichartz Lemma 3.3 (for each $L_I^2L_x^2$ term). Employing a Littlewood-Paley decomposition

$$\begin{aligned}\mathrm{I}\lesssim N^{-\beta}(\log N)^7\sum_{M_1,M_2,M_3,M_4}M_1\frac{\max(M_1,M_2)^{1/2}}{(\log\max(M_1,M_2))^3}\frac{\max(M_3,M_4)^{1/2}}{(\log\max(M_3,M_4))^4}\\\|P_{M_1}\phi\;\overline{P_{M_2}\phi}\|_{L_I^2L_x^2}\|P_{M_3}\phi\;P_{M_4}g\|_{L_I^2L_x^2}\end{aligned}$$

Applying the bilinear Strichartz estimate Lemma 3.3 to the first term, and (3.8) to the second term, which introduces the factor

$$\begin{aligned}&\min(M_1,M_2)\max(M_1,M_2)^{-1/2}\min(M_3,M_4)\\&\times\max(M_3,M_4)^{-1/2}\left(\log\frac{\max(M_3,M_4)}{\min(M_3,M_4)}+1\right)\end{aligned}$$

we obtain

$$\mathrm{I} \lesssim N^{-\beta}(\log N)^7 \sum_{M_1,M_2,M_3,M_4} \Bigg(\frac{M_1 \min(M_1, M_2)}{(\log \max(M_1, M_2))^3} \\ \times \frac{\min(M_3, M_4)}{(\log \max(M_3, M_4))^4} \left(\log \frac{\max(M_3, M_4)}{\min(M_3, M_4)} + 1 \right) \\ \times \quad \|P_{M_1}\phi\|_{U^2_{\Delta,I}L^2_x} \|P_{M_2}\phi\|_{U^2_{\Delta,I}L^2_x} \|P_{M_3}\phi\|_{U^2_{\Delta,I}L^2_x} \|P_{M_4}g\|_{V^2_{\Delta,I}L^2_x} \Bigg)$$

where the extra log factor comes from the need to get V^2 on the g term instead of U^2, as explained above (3.8). Distributing the derivatives onto each of the three ϕ factors,

$$\mathrm{I} \lesssim N^{-\beta}(\log N)^7 \sum_{M_1,M_2,M_3,M_4} \Bigg(\frac{\min(M_1, M_2)}{M_2(\log \max(M_1, M_2))^3} \\ \times \frac{\min(M_3, M_4)}{M_3(\log \max(M_3, M_4))^3} \quad \|P_{M_1}\phi\|_{U^2_{\Delta,I}H^1_x} \|P_{M_2}\phi\|_{U^2_{\Delta,I}H^1_x} \\ \times \|P_{M_3}\phi\|_{U^2_{\Delta,I}H^1_x} \|P_{M_4}g\|_{V^2_{\Delta,I}L^2_x} \Bigg)$$

Applying the estimates

$$\|P_{M_j}\phi\|_{U^2_{\Delta,I}H^1_x} \lesssim \|\phi\|_{U^2_{\Delta,I}H^1_x}, \qquad \|P_{M_4}g\|_{V^2_{\Delta,I}L^2_x} \lesssim \|g\|_{V^2_{\Delta,I}L^2_x}$$

we can carry out the sum to obtain $\mathrm{I} \lesssim N^{-\beta}(\log N)^7 C_1^3 \|g\|_{V^2_{I,\Delta}L^2_x}$. Collecting the estimates on I and II, we obtain

$$Q_{N,I}(\phi) \lesssim N^{-\beta}(\log N)^7 C_1^3 \tag{3.20}$$

□

Proposition 3.8 *Let $q > 1$. Suppose that on a time interval I the solutions to (1.1) and (1.11) satisfy*

$$\|\phi\|_{U^2_{I,\Delta}H^q_x} \le 2C_1, \qquad \|\phi_N\|_{U^2_{I,\Delta}H^q_x} \le 2C_1$$

for some constant C_1 and

$$\|\langle\nabla\rangle^q \phi\|_{L^2_I L^6_x} \le \delta, \qquad \|\langle\nabla\rangle^q \phi_N\|_{L^2_I L^6_x} \le \delta$$

Consider the difference

$$\tilde{\phi}(t) = \phi_N(t) - \phi(t)$$

with initial condition $\tilde{\phi}_0 = (\phi_N)_0 - \phi_0$ for the time interval I.

Provided $\delta > 0$ is chosen small, $\|\tilde{\phi}_0\|_{H_x^1}$ is sufficiently small, and N is sufficiently large (all of these thresholds are expressed in terms of C_1 only), then we have

$$\|\tilde{\phi}\|_{U^2_{I,\Delta} H^1_x} \lesssim \|\tilde{\phi}_0\|_{H^1_x} + N^{-q\beta}(\log N)^7 C_1^3 \tag{3.21}$$

Proof The proof follows that of Proposition 3.6, with the only modification needed in the treatment of the forcing term

$$Q_{N,I}(\phi) \stackrel{\text{def}}{=} \sup_{\substack{g \in V^2_{I,\Delta} L^2_x \\ \|g\|_{V^2_{I,\Delta} L^2_x} \le 1}} \int_I \int \langle \nabla \rangle [(W_N * |\phi|^2)\phi]\, g\, dx\, dt$$

After distributing the derivative on the left we obtain two terms

$$\begin{aligned}
&\int_I \int \nabla[(W_N * |\phi|^2)\, \phi]\; g\; dx\, dt \\
&= 2 \int_I \int [W_N * \mathrm{Re}(\bar{\phi} \nabla \phi)]\, \phi\, g\, dx\, dt + \int_I \int [W_N * |\phi|^2]\, \nabla \phi\, g\, dx\, dt \\
&= \mathrm{I} + \mathrm{II}
\end{aligned}$$

Term II is estimated as in the proof of Proposition 3.6, giving the bound

$$\mathrm{II} \lesssim N^{-\min(q,2)\beta} C_1^3 \|g\|_{V^2_{I,\Delta} L^2_x}$$

Term I is also estimated as in the proof of Proposition 3.6 . Moreover, we apply Lemma A.5 on the left product $W_N * (\phi\, \nabla \phi)$ with $s = q - \frac{1}{2}$, and apply lemma A.5 on the right product $W_N * (\phi g)$ with $s = \frac{1}{2}$ to obtain (dropping log factors for clarity)

$$\begin{aligned}
\mathrm{I} \lesssim (\log N)^7 N^{-q\beta} &\sum_{M_1, M_2, M_3, M_4} (\max(M_1, M_2)^q \min(M_1, M_2) \\
&\times \min(M_3, M_4) \|P_{M_1}\phi\|_{U^2_{\Delta,I} L^2_x} \|P_{M_2}\phi\|_{U^2_{\Delta,I} L^2_x} \|P_{M_3}\phi\|_{U^2_{\Delta,I} L^2_x} \|P_{M_4} g\|_{V^2_{\Delta,I} L^2_x})
\end{aligned}$$

Putting q derivatives onto each ϕ factor gives

$$\begin{aligned}
\mathrm{I} \lesssim (\log N)^7 N^{-q\beta} &\sum_{M_1, M_2, M_3, M_4} \min(M_1, M_2)^{-(q-1)} \min(M_3, M_4) M_3^{-q} \\
&\times \|P_{M_1}\phi\|_{U^2_{\Delta,I} H^q_x} \|P_{M_2}\phi\|_{U^2_{\Delta,I} H^q_x} \|P_{M_3}\phi\|_{U^2_{\Delta,I} H^q_x} \|P_{M_4} g\|_{V^2_{\Delta,I} L^2_x}
\end{aligned}$$

Now carry out the sum (recall that the introduction of $(\log N)^7$ provided $(\log M_i)$ factors that allow us to sum). □

We can now conclude the proof of Theorem 1.4. Recall there is a finite partition of $0 \le t < +\infty$

$$0 = t_0 < t_1 < \cdots < t_J = +\infty$$

such that for each $I = [t_{j-1}, t_j)$, the solutions ϕ and ϕ_M are small in the Strichartz norms, i.e. (3.9) holds. By Corollary 3.5, the U^2 norms of ϕ and ϕ_N are controlled, i.e. (3.10) holds. Thus, for each time interval $I = [t_{j-1}, t_j)$, the hypotheses of Proposition 3.6 are satisfied, and (3.12) holds. This implies, in particular, that

$$\|\tilde{\phi}(t)\|_{L^\infty_{[t_{j-1},t_j]}H^1_x} \lesssim \|\tilde{\phi}(t_{j-1})\|_{H^1_x} + CN^{-\beta}(\log N)^7$$

Therefore, the estimate on the jth interval feeds into the estimate for the $(j+1)$st interval, and since there are only a finite number of time intervals, we can reach all time.

For the H^q version, we apply a persistence of regularity argument, as in Bourgain [3], to deduce that the H^q norms of ϕ and ϕ_N are globally bounded, and that modifications to q regularity of Theorem 3.4 and Corollary 3.5 follow. Thus, on each time interval $I = [t_{j-1}, t_j)$, the hypotheses of Proposition 3.8 are satisfied, and (3.21) holds. This implies, in particular, that

$$\|\tilde{\phi}(t)\|_{L^\infty_{[t_{j-1},t_j]}H^1_x} \lesssim \|\tilde{\phi}(t_{j-1})\|_{H^1_x} + CN^{-q\beta}(\log N)^7$$

Therefore, the estimate on the jth interval feeds into the estimate for the $(j+1)$st interval, and since there are only a finite number of time intervals, we can reach all time.

Next we address the proof of the difference estimate (1.16). These estimates will follow from the lemma below since

$$\sum_{k=0}^{+\infty} Z^{-k}(2k)(3C_1)^{2k-1} < \infty$$

provided $Z > (3C_1)^2$.

Lemma 3.9 *Let*

$$G_{N,k} = \phi_N(x_1)\cdots\phi_N(x_k)\overline{\phi_N}(x_1')\cdots\overline{\phi_N}(x_k')$$
$$G_k = \phi(x_1)\cdots\phi(x_k)\overline{\phi}(x_1')\cdots\overline{\phi}(x_k')$$

and let $\tilde{\phi} = \phi_N - \phi$. *If*

$$\|\phi_N\|_{H^1} \le C_1 \quad \textit{and} \quad \|\phi\|_{H^1} \le C_1$$

then

$$\|G_{N,k} - G_k\|_{H^1_{\mathbf{x}_k,\mathbf{x}_k'}} \le 2k(3C_1)^{2k-1}\|\tilde{\phi}\|_{H^1}$$

Proof In the formula for $G_{N,k}$, replace each instance of ϕ_N by $\tilde{\phi} + \phi$, and expand to a sum of 2^{2k} terms, and note that passing to the difference $G_{N,k} - G_k$ removes one of these terms. Apply the $H^1_{\mathbf{x}_k,\mathbf{x}'_k}$ norm, bound via Minkowski's inequality by a sum with the norm on each of the individual terms. At this point, the $2^{2k}-1$ terms can be grouped into $2k$ terms:

$$\|G_{N,k}-G_k\|_{H^1_{\mathbf{x}_k,\mathbf{x}'_k}} \leq \sum_{\ell=1}^{2k}\binom{2k}{\ell}\|\tilde{\phi}\|^{\ell}_{H^1}\|\phi\|^{2k-\ell}_{H^1}$$

where we note that the sum starts at $\ell=1$ and not $\ell=0$. Applying the bound $\binom{2k}{\ell}\leq 2k\binom{2k-1}{\ell-1}$ and reindexing the sum with $j=\ell+1$,

$$\begin{aligned}
&\|G_{N,k}-G_k\|_{H^1_{\mathbf{x}_k,\mathbf{x}'_k}} \\
&\leq 2k\|\tilde{\phi}\|_{H^1}\sum_{j=0}^{2k-1}\binom{2k-1}{j}\|\tilde{\phi}\|^{j}_{H^1}\|\phi\|^{2k-1-j}_{H^1} \\
&= 2k\|\tilde{\phi}\|_{H^1}(\|\tilde{\phi}\|_{H^1}+\|\phi\|_{H^1})^{2k-1}
\end{aligned}$$

From this, the claimed estimate follows. □

3.1 Optimality via Space-time Resonance

We can in fact provide an example showing that $N^{-\beta q}$ is optimal for $\phi\in H^q$, $q\geq 1$. Consider the main forcing term in the equation for $\tilde{\phi}$

$$F(t)=\int_0^t e^{i(t-t')\Delta}[(W_N*|\phi|^2)\,\phi]\,dt' \tag{3.22}$$

where $\phi\in H^q$ is a scattering solution to NLS. For simplicity, let us replace ϕ by a linear solution, i.e. take

$$\phi(t)=e^{it\Delta}f$$

for some $f=f(x)$, which is a natural benchmark on which to assess $F(t)$ in (3.22) since the NLS solution ϕ scatters.

Lemma 3.10 *For $q\geq 1$, and for each $N\gg 1$, there exists a choice of f for which $\|f\|_{H^q}=1$ and $F(t)$ given by (3.22) with $\phi(t)=e^{it\Delta}f$ satisfies*

$$\|F(t)\|_{L^\infty_{t\in[0,1]}H^1_x}\gtrsim N^{-q\beta}$$

Proof The strategy is to concoct a choice for f in which the frequency support is sufficiently constrained so as to produce a resonant interaction.

Denote spatial coordinates by $x = (x_1, x_2, x_3)$ and frequency coordinates by $\xi = (\xi_1, \xi_2, \xi_3)$. Consider the following choice for f:

$$\hat{f}(\xi) = \Big(N^{\beta/2}\mathbf{1}_{(0,N^{-\beta})}(\xi_1) + N^{-\beta(q-\frac{1}{2})}\mathbf{1}_{(N^\beta,N^\beta+N^{-\beta})}(\xi_1)\Big)\mathbf{1}_{(0,1)}(\xi_2)\mathbf{1}_{(0,1)}(\xi_3)$$

so that $\|f\|_{H^q} \sim O(1)$. The reason for choosing intervals of width $N^{-\beta}$ is that $N^{2\beta} \le (N^\beta + \delta N^{-\beta})^2 \le N^{2\beta} + O(1)$ for $0 \le \delta \le 1$ – that is, we have $O(1)$ resolution of the square. With this choice,

$$\begin{aligned}
&(\hat{\phi}(t') * \hat{\bar{\phi}}(t'))(\eta) = \int \hat{\phi}(\eta-\rho,t')\hat{\bar{\phi}}(\eta,t')\,d\rho \\
&= \int_\rho e^{-it'(\eta-\rho)^2}e^{it'\rho^2}\hat{f}(\eta-\rho)\hat{f}(\rho)\,d\rho \\
&\sim e^{it'\cdot O(1)}\Big(\mathbf{1}_{(0,N^{-\beta})}(\eta_1) + N^{-q\beta}(e^{-it'N^{2\beta}} + e^{it'N^{2\beta}})\mathbf{1}_{(N^\beta,N^\beta+N^{-\beta})}(\eta_1) \\
&\quad + N^{-2q\beta}\mathbf{1}_{(2N^\beta,2N^\beta+N^{-\beta})}(\eta_1)\Big)\mathbf{1}_{(0,1)}(\eta_2)\mathbf{1}_{(0,1)}(\eta_3)
\end{aligned}$$

Applying $W_N *$ basically removes the first term $\mathbf{1}_{(0,N^{-\beta})}(\eta_1)$,

$$\begin{aligned}
&[W_N * |\phi(t')|^2]\hat{\ }(\eta) \\
&\quad\sim e^{it'\cdot O(1)}\Big(N^{-q\beta}(e^{-it'N^{2\beta}} + e^{it'N^{2\beta}})\mathbf{1}_{(N^\beta,N^\beta+N^{-\beta})}(\eta_1) \\
&\qquad + N^{-2q\beta}\mathbf{1}_{(2N^\beta,2N^\beta+N^{-\beta})}(\eta_1)\Big)\mathbf{1}_{(0,1)}(\eta_2)\mathbf{1}_{(0,1)}(\eta_3)
\end{aligned}$$

Now consider

$$[(W_N * |\phi(t')|^2)\phi(t')]\hat{\ }(\xi) = \int_\eta (W_N * |\phi(t')|^2)\hat{\ }(\eta)\widehat{\phi(t')}(\xi-\eta)\,d\eta \tag{3.23}$$

We have $\widehat{\phi(t')}(\xi-\eta) = e^{-it'(\xi-\eta)^2}\hat{f}(\xi-\eta)$. In the product inside the integrand in (3.23), we have either

$$N^\beta \le \eta_1 \le N^\beta + N^{-\beta}, \quad \text{or} \quad N^{2\beta} \le \eta_1 \le N^{2\beta} + N^{-\beta}$$

and

$$0 \le \xi_1 - \eta_1 \le N^{-\beta}, \quad \text{or} \quad N^\beta \le \xi_1 - \eta_1 \le N^\beta + N^{-\beta}$$

Thus, in the product, there are four terms resulting from all possible cross pairings.

The dominant term of interest for us (that produces the lower bound) will arise from the case when $N^\beta \le \eta_1 \le N^\beta + N^{-\beta}$ pairs with $0 \le \xi_1 - \eta_1 \le N^{-\beta}$, (so that the output frequency ξ satisfies $N^\beta \le \xi_1 \le N^\beta + N^{-\beta}$). The result is

$$\left([W_N * |\phi(t')|^2]\phi(t')\right)\widehat{}(\xi)$$
$$\sim e^{it'\cdot O(1)} N^{-\beta(q+\frac{1}{2})}(e^{-it'N^{2\beta}} + e^{it'N^{2\beta}})\mathbf{1}_{(N^\beta, N^\beta+N^{-\beta})}(\xi_1)\mathbf{1}_{(0,1)}(\xi_2)\mathbf{1}_{(0,1)}(\xi_3) \tag{3.24}$$

where we have dropped the terms that will have subordinate effect. The coefficient $N^{-\beta(q+\frac{1}{2})}$ comes from the product of three things: the $N^{-q\beta}$ coefficient in $[W_N * |\phi(t')|^2]\widehat{}(\eta)$, the $N^{\beta/2}$ coefficient in $\widehat{\phi(t')}(\xi-\eta)$, and the size of the η-integration support, which is $N^{-\beta}$. Finally we come to

$$\left[\int_0^t e^{i(t-t')\Delta}[W_N * |\phi(t')|^2]\phi(t')\,dt'\right]\widehat{}(\xi)$$
$$= e^{-it\xi^2}\int_0^t e^{it'\xi^2}\left([W_N|\phi(t'^2]\phi(t')\right)\widehat{}(\xi)\,dt'$$

Plugging in the term above, it is key to notice that in the phase product inside the integrand

$$e^{-it'\xi^2}(e^{-it'N^{2\beta}} + e^{it'N^{2\beta}}) = (e^{-2it'N^{2\beta}} + 1)$$

so there is a non-oscillatory (resonant) component. Thus, when the time integral is carried out, this term survives, and gives

$$\sim N^{-\beta(q+\frac{1}{2})}\mathbf{1}_{(N^\beta, N^\beta+N^{-\beta})}(\xi_1)\mathbf{1}_{(0,1)}(\xi_2)\mathbf{1}_{(0,1)}(\xi_3)$$

The H^1 norm of this term is $\sim N^{-q\beta}$. As other terms are subordinate, this term contributing $N^{-q\beta}$ to the norm becomes a lower bound. □

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Appendix A. Misc. Estimates

A.1 Collapsing Estimates and Strichartz Estimates

We use the original Klainerman-Machedon collapsing estimate as our iterating estimate in this paper.

Lemma A.1 *([13, 19, 54])* [14]*There is a C independent of V, j, k, and N such that, (for $f^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1})$ independent of t)*

$$\left\| S^{(1,k)} B_{N,j,k+1} U^{(k+1)}(t) f^{(k+1)} \right\|_{L_t^2 L^2_{\mathbf{x},\mathbf{x}'}} \le C \|V\|_{L^1} \left\| S^{(1,k+1)} f^{(k+1)} \right\|_{L^2_{\mathbf{x},\mathbf{x}'}}.$$

[14] For more estimates of this type, see [16, 18, 39, 40, 52].

To explore the time derivative gain by Duhamel type terms, we also need the $X_{s,b}$ version of Lemma A.1. As we are using $S^{(k)}$ to denote the space derivatives, we surpress the s notation in definition of the $X_{s,b}$ space and define the norm $X_b^{(k)}$ by

$$\|\alpha^{(k)}\|_{X_b^{(k)}} = \left(\int \langle \tau + |\boldsymbol{\xi}_k|^2 - |\boldsymbol{\xi}_k'|^2 \rangle^{2b} \left| \hat{\alpha}^{(k)}(\tau, \boldsymbol{\xi}_k, \boldsymbol{\xi}_k') \right|^2 d\tau \, d\boldsymbol{\xi}_k \, d\boldsymbol{\xi}_k' \right)^{1/2}$$

which is essentially a $X_{0,b}$ norm. We then have the Duhamel time-derivative gain property and the $X_{s,b}$ version of Lemma A.1.

Claim A.2 *([21]) Let $\frac{1}{2} < b < 1$ and $\theta(t)$ be a smooth cutoff. Then*

$$\left\| \theta(t) \int_0^t U^{(k)}(t-s)\beta^{(k)}(s)\, ds \right\|_{X_b^{(k)}} \lesssim \|\beta^{(k)}\|_{X_{b-1}^{(k)}} \tag{A.1}$$

Lemma A.3 *([21]) There is a C independent of $j, k,$ and N such that (for $\alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}_{k+1})$ dependent on t)*

$$\|S^{(1,k)} B_{N,j,k+1} \alpha^{(k+1)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2} \le C \|S^{(1,k+1)} \alpha^{(k+1)}\|_{X_{\frac{1}{2}+}^{(k+1)}}$$

In the above notation, the dual Strichartz estimates we need in this paper are the following:

Lemma A.4 *([21]) Let*

$$\beta^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') = N^{3\beta-1} V(N^\beta(x_i - x_j)) \gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k')$$

Then for $N \ge 1$, we have

$$\| \left|\nabla_{x_i}\right| \left|\nabla_{x_j}\right| \beta^{(k)} \|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim N^{\frac{5}{2}\beta - 1} \|\langle \nabla_{x_i}\rangle \langle \nabla_{x_j} \rangle \gamma^{(k)}\|_{L_t^2 L_{\mathbf{x}\mathbf{x}'}^2} \tag{A.2}$$

and

$$\|\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim N^{\frac{1}{2}\beta - 1} \|\langle \nabla_{x_i}\rangle \langle \nabla_{x_j} \rangle \gamma^{(k)}\|_{L_t^2 L_{\mathbf{x}\mathbf{x}'}^2}. \tag{A.3}$$

A.2. Convolution Estimates

Lemma A.5 *Let $W_N(x) = N^{3\beta} V(N^\beta x) - b_0 \delta(x)$, where $b_0 = \int V(x)\, dx$. For any $0 \le s \le 1$,*

$$\|W_N * f\|_{L_x^p} \lesssim N^{-\beta s} \|D^s f\|_{L_x^p}$$

for any $1 < p < \infty$. The implicit constant depends only on $\|\langle x \rangle V(x)\|_{L^1}$.

Proof The case $s = 0$ is just Young's inequality, since $\|V_N\|_{L^1} = \|V\|_{L^1} < \infty$, independent of N. We next establish the estimate for $s = 1$. Since $\hat{V}(0) = b_0$,

$$\widehat{W_N}(\xi) = \hat{V}(\xi N^{-\beta}) - b_0 = \int_{s=0}^{s=1} \frac{d}{ds}\hat{V}(s\xi N^{-\beta})\, ds = \int_{s=0}^{s=1} N^{-\beta}\xi \cdot \nabla\hat{V}(s\xi N^{-\beta})\, ds$$

and thus

$$\widehat{W_N}(\xi)\hat{f}(\xi) = N^{-\beta}\int_{s=0}^{s=1} \nabla\hat{V}(s\xi N^{-\beta}) \cdot \xi \hat{f}(\xi)\, ds$$

Let $Y(x) = xV(x)$ so that $\hat{Y}(\xi) = \nabla\hat{V}(\xi)$. It follows that

$$\begin{aligned}
&\int_{y\in\mathbb{R}^3} W_N(x-y) f(y)\, dy \\
&\quad = N^{-\beta}\int_{s=0}^{s=1}\int_{y\in\mathbb{R}^3} s^{-3}N^{3\beta}Y(s^{-1}N^{\beta}(x-y))\,\nabla f(y)\, dy\, ds
\end{aligned}$$

By Minkowski's inequality and Young's inequality,

$$\begin{aligned}
&\left\| \int_{y\in\mathbb{R}^3} W_N(x-y) f(y)\, dy \right\|_{L^p_x} \\
&\quad \lesssim N^{-\beta}\int_{s=0}^{1} \left\| \int_{y\in\mathbb{R}^3} s^{-3}N^{3\beta}Y(s^{-1}N^{\beta}(x-y))\,\nabla f(y)\, dy \right\|_{L^p_x} ds \\
&\quad \lesssim N^{-\beta}\|\nabla f\|_{L^p_x}
\end{aligned}$$

The cases $0 < s < 1$ follow by interpolation, as follows. Let P_M be the Littlewood-Paley projector for frequency $0 < M < \infty$. Then by the $s = 0$ and $s = 1$ cases,

$$\begin{aligned}
\|W_N * f\|_{L^p_x} &= \left\| W_N * \sum_M P_M f \right\|_{L^p_x} \le \sum_M \|W_N * P_M f\|_{L^p_x} \\
&\lesssim \sum_M \min(1, N^{-\beta}M)\|P_M f\|_{L^p_x} \lesssim \sum_M \min(1, N^{-\beta}M) M^{-s}\|D^s f\|_{L^p_x}
\end{aligned}$$

Divide the sum into the case $M \le N^{\beta}$, for which we use $\min(1, N^{-\beta}M) = N^{-\beta}M$, and the case $M \ge N^{\beta}$, for which we use $\min(1, N^{-\beta}M) = 1$.

$$\lesssim \left(\sum_{M\le N^{\beta}} N^{-\beta}M^{1-s} + \sum_{M\ge N^{\beta}} M^{-s} \right) \|D^s f\|_{L^p_x} \lesssim N^{-\beta s}\|D^s f\|_{L^p_x}$$

□

Lemma A.6 *Let $W_N(x) = N^{3\beta} V(N^\beta x) - b_0 \delta(x)$, where $b_0 = \int V(x)\, dx$.*

$$\int (W_N * f_1) f_2 \, dx \lesssim N^{-\beta} \| |\nabla|^{1/2} f_1 \|_{L^2_x} \| |\nabla|^{1/2} f_2 \|_{L^2_x}$$

Also, if f_j is replaced by $P_{M_j} f_j$, then the same estimate holds but in addition we must have $M_1 \sim M_2$ (or otherwise the left side is zero).

Proof By Plancherel

$$\int (W_N * f_1) f_2 \, dx = \int_\xi \hat{W}_N(\xi) f_1(\xi) f_2(\xi) \, d\xi \tag{A.4}$$

As in the proof of Lemma A.5,

$$\widehat{W_N}(\xi) = N^{-\beta} \hat{Q}_N(\xi) \cdot \xi \,, \qquad \hat{Q}_N(\xi) \stackrel{\text{def}}{=} \int_{s=0}^{s=1} \nabla \hat{V}(s \xi N^{-\beta}) \, ds$$

Since

$$\| \hat{Q}_N \|_{L^\infty_\xi} \leq \| \nabla \hat{V} \|_{L^\infty_\xi} = \| [x V(x)]^\frown \|_{L^\infty_\xi} \leq \| x V(x) \|_{L^1_x}$$

we can just complete the proof by Cauchy-Schwarz in (A.4) □

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