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From the Peierls–Nabarro model to the equation of motion of the dislocation continuum

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ABSTRACT

We consider a semi-linear integro-differential equation in dimension one associated to the half Laplacian whose solution represents the atom dislocation in a crystal. The equation comprises the evolutive version of the classical Peierls–Nabarro model. We show that for a large number of dislocations, the solution, properly rescaled, converges to the solution of a well known equation called by Head (1972) "the equation of motion of the dislocation continuum". The limit equation is a model for the macroscopic crystal plasticity with density of dislocations. In particular, we recover the so called Orowan's law which states that dislocations move at a velocity proportional to the effective stress.

1. Introduction

In this paper we are interested in studying the behavior as $\varepsilon \to 0$ of the solution u^ε of the following integro-differential equation:

$$\begin{cases} \delta\partial_t u^\varepsilon = \mathcal{I}_1[u^\varepsilon] - \dfrac{1}{\delta} W'\left(\dfrac{u^\varepsilon}{\varepsilon}\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ u^\varepsilon(0,\cdot) = u_0(\cdot) & \text{on } \mathbb{R} \end{cases} \tag{1.1}$$

where ε, $\delta > 0$ are small scale parameters and $\delta = \delta_\varepsilon \to 0$ as $\varepsilon \to 0$, W is a periodic potential and we denote by $\mathcal{I}_1$ the so-called fractional Laplacian of order 1, $-(-\Delta)^{\frac{1}{2}}$, defined on the Schwartz class $\mathcal{S}(\mathbb{R})$ by

$$\widehat{(-\Delta)^{\frac{1}{2}} v}\,(\xi) = |\xi|\,\widehat{v}(\xi), \tag{1.2}$$

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where $\widehat{v}$ is the Fourier transform of v. It is well known, see e.g. [46], that $\mathcal{I}_1$ may be also represented as

$$\mathcal{I}_1[v](x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{v(y)-v(x)}{(y-x)^2} dy,$$

where PV stands for principal value. See also [45] or [13] for a basic introduction to the fractional Laplace operator.

We assume that W is a multi-well potential with nondegenerate minima at integer points. More precisely, we suppose that

$$\begin{cases} W \in C^{2,\beta}(\mathbb{R}) & \text{for some } 0<\beta<1 \\ W(u+1) = W(u) & \text{for any } u \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R} \setminus \mathbb{Z} \\ W''(0) > 0. \end{cases} \tag{1.3}$$

On the function u_0 we assume

$$\begin{cases} u_0 \in C^{1,1}(\mathbb{R}) \\ u_0 \text{ non-decreasing.} \end{cases} \tag{1.4}$$

Eq. (1.1) is a rescaled version of the so called Peierls–Nabarro model, which is a phase field model describing dislocations. Dislocations are line defects in crystals. Their typical length is of the order of 10^{-6} m and their thickness of order of 10^{-9} m. When the material is submitted to shear stress, these lines can move in the crystallographic planes (slip planes) and their dynamics is one of the main explanation of the plastic behavior of metals. We refer the reader to the book [24] for a tour in the theory of dislocations. Dislocations can be described at several scales by different models:

- atomic scale (Frenkel–Kontorova model),
- microscopic scale (Peierls–Nabarro model),
- mesoscopic scale (Discrete dislocation dynamics),
- macroscopic scale (Elasto-visco-plasticity with density of dislocations).

Our goal in this paper is to understand the large scale limit of the Peierls–Nabarro model for a *large* number of *parallel straight edge dislocation lines in the same slip plane with the same Burgers' vector, moving with self-interactions*. The number of dislocations is of order $1/\varepsilon$, while the distance between neighboring dislocations is (at microscopic scale) of order $1/\delta$. Rescaling the Peierls–Nabarro model leads to Eq. (1.1). The model is explained in further details in Section 1.1.

We show that at macroscopic scale the density of dislocations is governed by the following evolution law:

$$\begin{cases} \partial_t u = c_0 \partial_x u \, \mathcal{I}_1[u] & \text{in } (0,+\infty)\times\mathbb{R} \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R} \end{cases} \tag{1.5}$$

where $c_0 > 0$ is defined in the forthcoming (1.15). Under assumption (1.4), there exists a unique non-decreasing in x viscosity solution $\overline{u}$ of (1.5) (see Section 3). Our main result is the following:

Theorem 1.1. *Assume (1.3) and (1.4). Let u^ε be the viscosity solution of (1.1). Then, u^ε converges locally uniformly in $(0,+\infty)\times\mathbb{R}$ to the viscosity solution $\overline{u}$ of (1.5), as $\varepsilon \to 0$.*

Remark 1.2. We do not assume any assumption about how δ goes to 0 when $\varepsilon \to 0$.

The limit equation (1.5) represents the plastic flow rule for the macroscopic crystal plasticity with density of dislocations. The theorem says that in this regime, the plastic strain velocity $\partial_t u$ in (1.5) is proportional to the dislocation density u_x times the effective stress $\mathcal{I}_1[u]$. This physical law is known as Orowan's equation, see e.g. [44] p. 3739. Equation

$$\partial_t u = c_0 \partial_x u \mathcal{I}_1[u] \tag{1.6}$$

is an integrated form of a model studied by Head [23] for the self-dynamics of a dislocation density represented by u_x. Indeed, denoting $f = u_x$, differentiating (1.6), we see that, at least formally, f solves

$$\partial_t f = c_0 \partial_x (f \mathcal{H}[f]) \tag{1.7}$$

where $\mathcal{H}$ is Hilbert transform defined in Fourier variables by

$$\widehat{\mathcal{H}[v]}\ (\xi) = i\, \mathrm{sgn}(\xi)\ \widehat{v}(\xi),$$

for $v \in \mathcal{S}(\mathbb{R})$. The Hilbert transform has the following representation formula, see e.g. [46],

$$\mathcal{H}[v](x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{v(y)}{y-x} dy$$

and if $u \in C^{1,\alpha}(\mathbb{R})$ and $u_x \in L^p(\mathbb{R})$ with $1 < p < +\infty$, then

$$\mathcal{I}_1[u] = \mathcal{H}[u_x]. \tag{1.8}$$

Identity (1.8) can be easily proven by performing an integration by parts or using Fourier variables. The conservation of mass satisfied by the positive integrable solutions of (1.7) reflects the fact that if $f = u_x$ is the density of dislocations, no dislocations are created or annihilated.

Eq. (1.7) was also proposed by Constantin et al. [10] as a simplified one dimensional version of the 2-D quasi-geostrophic model. In [9], Castro and Còrdoba show that given an initial datum $f_0(x)$ which belongs to $C^\alpha(\mathbb{R}) \cap L^2(\mathbb{R})$ and is strictly positive, then there exists a smooth (analytic in x) global (for all times) solution of (1.7) that at time 0 is equal to $f_0(x)$. If $f_0(x)$ is non-negative and 0 at some point, the authors show the existence of a local solution that blows up in finite time. On the other hand, Carrillo, Ferreira and Precioso [8] apply transportation methods and show that the solution can be obtained as a gradient flow in the space of probability measures with bounded second moment. Finally, we mention that Eq. (1.7) is a particular case of the fractional porous medium equation

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u)$$

recently studied in [5–7]. Indeed, it corresponds to the case $s = 1/2$ and $m = 2$ in dimension 1. Self-similar solutions and decay estimates for Eq. (1.6) have been studied in [2].

From a mathematical point of view, as δ and ε go to 0 simultaneously, (1.1) is both a homogenization problem (even though there is no a cell problem and the limit equation is explicit) and a non-local Allen–Cahn type equation. As for an Allen–Cahn type problem, the solution gets closer and closer to the stable minima of the potential, that for the rescaled potential $W(\cdot/\varepsilon)$, by (1.3), are the points of the set $\varepsilon\mathbb{Z}$, and converges to a continuous function, the solution of (1.5), when ε goes to 0. To prove Theorem 1.1, the idea is to approximate the dislocation particles with points $x_i(t)$ where the limit function u attains the value εi at time t. We then provide a discrete approximation formula for the operator $\mathcal{I}_1$ with uniform error estimates over $\mathbb{R}$, which holds true for any $C^{1,1}$ function, and we use it to show that

$$\dot{x}_i = -\frac{\partial u_t(t, x_i(t))}{\partial_x u(t, x_i(t))} \simeq -c_0 \mathcal{I}_1[u(t,\cdot)](x_i(t)).$$

The strategy and the heuristic of the proofs are explained in Section 2.

1.1. The 1-D Peierls–Nabarro

The Peierls–Nabarro model [35,36,42] is a phase field model for dislocation dynamics incorporating atomic features into continuum framework. In a phase field approach, the dislocations are represented by transition of a continuous field. We briefly review the model in the case of an *edge straight* dislocation in a crystal with simple cubic lattice. In a Cartesian system of coordinates $x_1x_2x_3$, we assume that the straight dislocation is located in the slip plane x_1x_3 (where the dislocation can move) and perpendicular to the axis x_1. In the case of an edge dislocation the Burgers' vector (i.e. a fixed vector associated to the dislocation) is perpendicular to the dislocation line, thus in the direction of the axis x_1. We write this Burgers' vector as be_1 for a real b. After a section of the three-dimensional crystal with the plane x_1x_2, the dislocation line can be identified with a point on the x_1 axis. The disregistry of the upper half crystal $\{x_2 > 0\}$ relative to the lower half $\{x_2 < 0\}$ in the direction of the Burgers' vector is $\phi(x_1)$, where ϕ is a phase parameter between 0 and b. Then the dislocation point can be for instance localized by the level set $\phi = b/2$. In the Peierls–Nabarro model, the total energy is given by

$$\mathcal{E} = \mathcal{E}^{el} + \mathcal{E}^{mis}. \tag{1.9}$$

In (1.9), $\mathcal{E}^{el}$ is the elastic energy induced by the dislocation. In the isotropic case and for a straight dislocation line it takes the form

$$\mathcal{E}^{el} = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^+} |\nabla U|^2\, dx_1\, dx_2,$$

where $U : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ represents the displacement which is such that $U(x_1,0) = \phi(x_1)$. $\mathcal{E}^{mis}$ is the so called misfit energy due to the nonlinear atomic interaction across the slip plane,

$$\mathcal{E}^{mis}(\phi) = \int_{\mathbb{R}} W(U(x_1,0))\, dx_1 = \int_{\mathbb{R}} W(\phi(x_1))\, dx_1, \tag{1.10}$$

where $W(\phi)$ is the interplanar potential. In a general model, one can consider a potential W satisfying assumptions (1.3). The periodicity of W reflects the periodicity of the crystal, while the minimum property is consistent with the fact that the perfect crystal is assumed to minimize the energy. The equilibrium configuration of the edge dislocation is obtained by minimizing the total energy with respect to U, under the constraint that far from the dislocation core, the function ϕ tends to 0 in one half line and to b in the other half line. The corresponding Euler–Lagrange equation can be written in terms of the phase transition ϕ as

$$\mathcal{I}_1[\phi] = W'(\phi).$$

Assume for simplicity $b = 1$, if we fix the value of ϕ at the origin to be $1/2$, then for $x = x_1$ the 1-D phase transition is solution to:

$$\begin{cases} \mathcal{I}_1[\phi] = W'(\phi) & \text{in } \mathbb{R} \\ \phi' > 0 & \text{in } \mathbb{R} \\ \lim_{x\to-\infty}\phi(x) = 0, \quad \lim_{x\to+\infty}\phi(x) = 1, \quad \phi(0) = \dfrac{1}{2}. \end{cases} \tag{1.11}$$

Existence of a unique solution of (1.11) has been proven in [4]. In the classical Peierls–Nabarro model the potential is given by $W(u) = \frac{b^2}{4\pi^2 d}\left(1-\cos\left(\frac{2\pi u}{b}\right)\right)$, where d is the lattice spacing perpendicular to the slip plane, and the 1-D phase transition, found by Nabarro [35], is explicit: $\phi(x) = \frac{b}{2} + \frac{b}{\pi}\arctan\left(\frac{2x}{d}\right)$.

In the face cubic structured (FCC) observed in many metals and alloys, dislocations move at low temperature on the slip plane. The dynamics for a collection of straight dislocations lines with the same Burgers' vector and all contained in a single slip plane, moving with self-interactions (no exterior forces) is then described by the evolutive version of the Peierls–Nabarro model (see for instance [34] and [12]):

$$\partial_t u = \mathcal{I}_1[u(t,\cdot)] - W'(u) \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}. \tag{1.12}$$

In this paper we consider Eq. (1.12) when the number of dislocations is of order $1/\varepsilon$ and neighboring dislocations are at distance at microscopic scale of order $1/\delta$. This can be represented by the following initial condition

$$u(0,x) = \sum_{i=1}^{N_\varepsilon} \phi\left(x - \frac{y_i}{\delta}\right),$$

where ϕ is the solution of (1.11), $N_\varepsilon \sim 1/\varepsilon$ and

$$0 \leqslant y_{i+1} - y_i \sim 1.$$

We want to identify at *large (macroscopic) scale* the evolution model for the dynamics of a density of dislocations. We consider the following rescaling

$$u^\varepsilon(t,x) = \varepsilon u\left(\frac{t}{\varepsilon\delta^2}, \frac{x}{\varepsilon\delta}\right),$$

then we see that u^ε is solution of (1.1) with initial datum

$$u^\varepsilon(0,x) = \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - \varepsilon y_i}{\varepsilon\delta}\right). \tag{1.13}$$

Here ε describes the ratio between the microscopic scale and the macroscopic scale. After the rescaling we see that the distance between neighboring dislocations is of order $\varepsilon \sim 1/N_\varepsilon$. Every dislocation point is described by a phase transition $\varepsilon\phi\left(\frac{x-\varepsilon y_i}{\varepsilon\delta}\right)$ whose derivative is of order $1/\delta$.

More in general, we consider an initial datum u_0 satisfying (1.4). One can actually prove (see Proposition 4.12) that any function satisfying (1.4), normalized such that the infimum is 0, can be approximated by a function of the form (1.13). The monotonicity of u_0 reflects the fact that the dislocations have all the same orientation so that no annihilations occur.

1.2. The discrete dislocation dynamics ($\delta = 0$)

When $\varepsilon = 1$, (1.1) is a non-local Allen–Cahn equation. In [22], González and Monneau, show that the solution converges as $\delta \to 0$ to the stable minima of the potential W, that is integers. More precisely, if the initial datum is well prepared, the solution converges to a sum of Heaviside functions of the form $\sum_{i=1}^N H(x - y_i(t))$, where the interface points $y_i(t)$, $i = 1, \dots, N$ evolve in time driven by the following system of ODE's:

$$\begin{cases} \dot{y}_i = \dfrac{c_0}{\pi} \displaystyle\sum_{j \neq i} \dfrac{1}{y_i - y_j} & \text{in } (0, +\infty) \\ y_i(0) = y_i^0. \end{cases} \tag{1.14}$$

Here the points y_i^0, $i = 1, \dots, N$, are given in the initial condition and

$$c_0 = \left(\int_{\mathbb{R}} (\phi')^2\right)^{-1}, \tag{1.15}$$

with ϕ the solution of (1.11). System (1.14) corresponds to the classical discrete dislocation dynamics in the particular case of parallel straight edge dislocation lines in the same slip plane with the same Burgers' vector and describe the dynamics of dislocation particles at mesoscopic scale.

1.3. Brief review of the literature

When $\delta = 1$, (1.1) is a homogenization problem and the convergence of the solution when $\varepsilon \to 0$ have been studied by Monneau and the first author in [32] in any dimension. In this case it is proven that u^ε converges to the solution of an homogenized equation of type $\partial_t u = \overline{H}(\nabla u, \mathcal{I}_1[u])$, where the effective Hamiltonian $\overline{H}$ is implicitly defined through a cell problem. In [31] it is proven that in dimension 1 $\overline{H}(\delta p, \delta L) \simeq c_0\delta^2|p|L$ as $\delta \to 0$. See also [39] for fractional operators of any order $s \in (0, 2)$. The proofs of [31,32] cannot be adapted here as the errors obtained blow up when ε and δ converge to 0 simultaneously. For more results about homogenization of local and nonlocal first order operators with $u^\varepsilon/\varepsilon$ dependence we refer to [1,25,26]. Collisions of dislocation particles and/or long time behavior for the solution of (1.1) with $\varepsilon = 1$ have been studied in [11,38,40,41]. In [20] and [21], Garroni and Muller study a variational model for dislocations that is the variational formulation of the stationary Peierls–Nabarro equation in dimension 2, and they derive a line tension model.

The passage from discrete models of type (1.14) ($\delta = 0$) to continuum models has been studied in several papers. In [17], Forcadel, Imbert and Monneau prove that the function $\sum_{i=1}^{N_\varepsilon} H(x - y_i(t))$, where y_i, $i = 1, \dots, N_\varepsilon$ solve (1.14), properly rescaled, converges to the continuous viscosity solution of an homogenized equation, which is (1.6) when the forcing term is 0. In [28], van Meurs and Morandotti present a discrete-to-continuum limit passage for a system of dislocation particles with a regularized potential, which includes annihilation. Convergence of evolving interacting particle systems in dimension 2 has been studied in [19]. For further related results we refer the reader to [18,29,30,33,43] and references therein.

1.4. Organization of the paper

The paper is organized as follows. In Section 2 we present the strategy and the heuristic of the proof of Theorem 1.1. In Section 3 we recall some general auxiliary results that will be used in the rest of the paper. In Section 4 we prove a discrete approximation formula for the operator $\mathcal{I}_1$. Section 5 is devoted to the proof of our main result, Theorem 1.1. The main comparison result used in the proof of the theorem is shown in Section 6. Finally the proofs of some auxiliary lemmas are given in Section 7.

1.5. Notations

We denote by $B_r(x)$ the ball of radius r centered at x. The cylinder $(t-\tau, t+\tau) \times B_r(x)$ is denoted by $Q_{\tau,r}(t,x)$. $\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the floor and the ceil integer parts of a real number x.

For $r > 0$, we denote

$$\mathcal{I}_1^{1,r}[v](x) = \frac{1}{\pi} PV \int_{|y-x|\leqslant r} \frac{v(y) - v(x)}{(y-x)^2} dy, \tag{1.16}$$

and

$$\mathcal{I}_1^{2,r}[v](x) = \frac{1}{\pi} \int_{|y-x|>r} \frac{v(y) - v(x)}{(y-x)^2} dy. \tag{1.17}$$

Then we can write

$$\mathcal{I}_1[v](x) = \mathcal{I}_1^{1,r}[v](x) + \mathcal{I}_1^{2,r}[v](x).$$

We denote by $USC_b((0,+\infty)\times\mathbb{R})$ (resp., $LSC_b((0,+\infty)\times\mathbb{R})$) the set of upper (resp., lower) semicontinuous functions on $(0,+\infty)\times\mathbb{R}$ which are bounded on $(0,T)\times\mathbb{R}$ for any $T > 0$ and we set $C_b((0,+\infty)\times\mathbb{R}) := USC_b((0,+\infty)\times\mathbb{R}) \cap LSC_b((0,+\infty)\times\mathbb{R})$. We denote by $C_b^2((0,+\infty)\times\mathbb{R})$ the subset of functions of $C_b((0,+\infty)\times\mathbb{R})$ with continuous second derivatives. Finally, $C^{1,1}(\mathbb{R})$ is the set of functions with bounded $C^{1,1}$ norm over $\mathbb{R}$.

Given a sequence $\{u^\varepsilon\}$ we denote

$$\limsup_{\varepsilon\to 0}{}^* u^\varepsilon(t,x) = \sup\Big\{\limsup_{\varepsilon\to 0} u^\varepsilon(x_\varepsilon) \mid x_\varepsilon \to x\Big\},$$

and

$$\liminf_{\varepsilon\to 0}{}_* u^\varepsilon(t,x) = \inf\Big\{\liminf_{\varepsilon\to 0} u^\varepsilon(x_\varepsilon) \mid x_\varepsilon \to x\Big\}.$$

Given a quantity $E = E(x)$, we write $E = O(A)$ is there exists a constant $C > 0$ such that, for all x,

$$|E| \leqslant CA.$$

We write $E = o_\varepsilon(1)$ if

$$\lim_{\varepsilon\to 0} E = 0,$$

uniformly in x.

2. Strategy and heuristic of the proofs

In this section we explain the steps that we will follow to prove Theorem 1.1 and the heuristic of the main proofs.

2.1. Approximation of $\mathcal{I}_1$

The first result is a discrete approximation formula for the fractional Laplace $\mathcal{I}_1$ of non-decreasing $C^{1,1}$ functions (Propositions 4.4 and 4.7, see also Remark 4.9). Let $v \in C^{1,1}(\mathbb{R})$. Assume for simplicity that v is strictly increasing. Let $\varepsilon > 0$ be a small parameter. Let us define the points x_i as follows,

$$v(x_i) = \varepsilon i, \quad i = M_\varepsilon, \dots, N_\varepsilon \tag{2.1}$$

where $M_\varepsilon := \left\lceil \frac{\inf_{\mathbb{R}} v + \varepsilon}{\varepsilon} \right\rceil$ and $N_\varepsilon = \left\lfloor \frac{\sup_{\mathbb{R}} v - \varepsilon}{\varepsilon} \right\rfloor$. By the monotonicity of v the points x_i are ordered,

$$x_i < x_{i+1} \quad \text{for all } i.$$

Then, we show that

$$\mathcal{I}_1[v](x_{i_0}) \simeq \frac{1}{\pi} \sum_{i \neq i_0} \frac{\varepsilon}{x_i - x_{i_0}}, \tag{2.2}$$

where the error goes to 0 when $\varepsilon \to 0$. To show (2.2), we consider a small radius $r = r_\varepsilon$ such that $r \to 0$ as $\varepsilon \to 0$ and we split

$$\sum_{i\neq i_0} \frac{\varepsilon}{x_i - x_{i_0}} = \sum_{\substack{i\neq i_0 \\ |x_i - x_{i_0}| \leqslant r}} \frac{\varepsilon}{x_i - x_{i_0}} + \sum_{|x_i - x_{i_0}| > r} \frac{\varepsilon}{x_i - x_{i_0}}.$$

Then, we have

$$\begin{aligned}
\frac{1}{\pi} \sum_{|x_i - x_{i_0}| > r} \frac{\varepsilon}{x_i - x_{i_0}} &= \frac{1}{\pi} \sum_{|x_i - x_{i_0}| > r} \frac{v(x_{i+1}) - v(x_i)}{x_i - x_{i_0}} \\
&\simeq \frac{1}{\pi} \sum_{|x_i - x_{i_0}| > r} \frac{v_x(x_i)(x_{i+1} - x_i)}{x_i - x_{i_0}} \\
&\simeq \frac{1}{\pi} \int_{|x - x_{i_0}| > r} \frac{v_x(x)}{x - x_{i_0}} dx \\
&= \frac{1}{\pi} \int_{|x - x_{i_0}| > r} \frac{v(x) - v(x_{i_0})}{(x - x_{i_0})^2} dx - \frac{1}{\pi} \frac{v(x_{i_0} + r) + v(x_{i_0} - r) - 2v(x_{i_0})}{r} \\
&\simeq \mathcal{I}_1[v](x_{i_0}),
\end{aligned}$$

where we have performed an integration by parts in the fourth equality. We can control the error produced in the approximation by choosing r not too small (r such that $\varepsilon/r \to 0$ as $\varepsilon \to 0$).

On the other hand, for $i \neq i_0$,

$$\varepsilon(i - i_0) = v(x_i) - v(x_{i_0}) \simeq v_x(x_{i_0})(x_i - x_{i_0})$$

from which (Lemma 4.6).

$$\begin{aligned}\sum_{\substack{i\neq i_0\\ |x_i - x_{i_0}|\leqslant r}} \frac{\varepsilon}{x_i - x_{i_0}} &\simeq v_x(x_{i_0}) \sum_{\substack{i\neq i_0\\ |i-i_0|\leqslant v_x(x_{i_0})\frac{r}{\varepsilon}}} \frac{1}{(i-i_0)} \\ &\simeq v_x(x_{i_0}) \left(\sum_{i\leqslant i_0 - 1} \frac{1}{(i-i_0)} + \sum_{i\geqslant i_0+1} \frac{1}{(i-i_0)} \right) \\ &= v_x(x_{i_0}) \left(-\sum_{k\geqslant 1} \frac{1}{k} + \sum_{k\geqslant 1} \frac{1}{k} \right) \\ &= 0.\end{aligned}$$

We can control the error produced by choosing r sufficiently small ($r \leqslant \varepsilon^{\frac{1}{2}}$). Combining the two estimates, we obtain (2.2).

We actually show that for any x,

$$\mathcal{I}_1[v](x) \simeq \frac{1}{\pi} \sum_{|x_i - x| > r} \frac{\varepsilon}{x_i - x}$$

where the error is uniform over $\mathbb{R}$, that is do not depend on the point x, while the sum

$$\sum_{\substack{i\neq i_0\\ |x_i - x|\leqslant r}} \frac{\varepsilon}{x_i - x}$$

may not be zero but depends on the distance of x from the closest x_i.

All our estimates hold true for any non-decreasing (non necessarily strictly increasing) $C^{1,1}$ function.

2.2. Approximation of v

Let ϕ be the transition layer defined by (1.11). It is known (see Lemma 3.1) that if $H(x)$ is the Heaviside function, then ϕ exhibits the following behavior at infinity: for $|x| \gg 1$,

$$\phi(x) \simeq H(x) - \frac{1}{\alpha \pi x}, \tag{2.3}$$

where $\alpha = W''(0)$. Using estimates (2.3) and (2.2), we show (Proposition 4.12) that if $v \in C^{1,1}(\mathbb{R})$ is non-decreasing and x_i are defined by (2.1), then

$$v(x) \simeq \sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon \phi \left(\frac{x - x_i}{\varepsilon \delta} \right) + \varepsilon M_\varepsilon. \tag{2.4}$$

Notice that $\varepsilon M_\varepsilon \simeq \inf_{\mathbb{R}} v$. Indeed, assume for simplicity that $x = x_{i_0}$ for some $M_\varepsilon \leqslant i_0 \leqslant N_\varepsilon$. Then, for ε and δ small: $(x_{i_0} - x_i)/(\delta\varepsilon) \gg 1$ if $i \leqslant i_0 - 1$, $(x_{i_0} - x_i)/(\delta\varepsilon) \ll -1$ if $i \geqslant i_0 + 1$. Then, by (2.3) and (2.2),

$$\begin{aligned}\sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_{i_0} - x_i}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon &= \sum_{i=M_\varepsilon}^{i_0 - 1} \varepsilon\phi\left(\frac{x_{i_0} - x_i}{\varepsilon\delta}\right) + \varepsilon\phi(0) + \sum_{i=i_0+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_{i_0} - x_i}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon \\ &\simeq \sum_{i=M_\varepsilon}^{i_0-1} \varepsilon \left(1 + \frac{\varepsilon\delta}{\alpha\pi(x_i - x_{i_0})}\right) + \frac{\varepsilon\delta}{\alpha\pi} \sum_{i=i_0+1}^{N_\varepsilon} \frac{\varepsilon}{x_i - x_{i_0}} + \varepsilon M_\varepsilon\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varepsilon\delta}{\alpha\pi}\sum_{i\neq i_0}\frac{\varepsilon}{x_i - x_{i_0}} + \varepsilon i_0 \\
&\simeq \frac{\varepsilon\delta}{\alpha}\mathcal{I}_1[v](x_{i_0}) + \varepsilon i_0 \\
&\simeq \varepsilon i_0 \\
&= v(x_{i_0}).
\end{aligned}
$$

We prove that estimate (2.4) holds true for any non-decreasing $C^{1,1}$ function v and that the error is independent of the point x.

2.3. Heuristic of the proof of Theorem 1.1

As for a homogenization problem we fix $(t_0, x_0) \in (0, +\infty)\times\mathbb{R}$ and find an ansatz for u^ε in a small box Q_R of size R centered at the point. Let u be the limit solution (that here we suppose to exist and be smooth). For R small, all the derivatives of u can be considered constant in Q_R:

$$
\partial_t u(t,x) \simeq \partial_t u(t_0,x_0), \quad \partial_x u(t,x) \simeq \partial_x u(t_0,x_0)
$$

and

$$
\mathcal{I}_1[u(t,\cdot)](x) \simeq \mathcal{I}_1[u(t_0,\cdot)](x_0) =: L_0.
$$

By the comparison principle u^ε and thus u is non-decreasing in x. Assume that

$$
\partial_x u(t_0,x_0) > 0.
$$

In particular u is strictly increasing in x in Q_R. For t close to t_0, we define the points $x_i(t)$ such that

$$
u(t, x_i(t)) = \varepsilon i. \tag{2.5}
$$

Since u is strictly increasing in x in Q_R, if $(t, x_i(t))$, $(t, x_{i+1}(t)) \in Q_R$ then $0 < x_{i+1}-x_i \simeq \varepsilon$ (see Lemma 4.1). For i such that $(t, x_i(t)) \in Q_R$, by differentiating (2.5) we get

$$
\partial_t u(t, x_i(t)) + \partial_x u(t, x_i(t))\dot{x}_i(t) = 0,
$$

from which

$$
\dot{x}_i(t) = -\frac{\partial_t u(t,x_i(t))}{\partial_x u(t,x_i(t))} \simeq -\frac{\partial_t u(t_0,x_0)}{\partial_x u(t_0,x_0)}. \tag{2.6}
$$

Next we consider as ansatz for u^ε the approximation of u given by (2.4) plus a small correction:

$$
\Phi^\varepsilon(t,x) := \sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\left(\phi\left(\frac{x-x_i(t)}{\varepsilon\delta}\right) + \delta\psi\left(\frac{x-x_i(t)}{\varepsilon\delta}\right)\right) + \varepsilon M_\varepsilon.
$$

The function ψ is defined in the forthcoming equation (3.3) with $L = L_0$. For a detailed heuristic motivation of this correction, see Section 3.1 of [22]. By (2.4), $\Phi^\varepsilon(t,x) \to u(t,x)$ as $\varepsilon \to 0$. Fix $(t,x) \in Q_R$ and let $x_{i_0}(t)$ be the closest point among the $x_i(t)$'s to x and $z_i = (x - x_i(t))/(\varepsilon\delta)$. Plugging into (1.1), we get (see proof of (5.21) in Section 5)

$$
\begin{aligned}
0 &= \delta\partial_t\Phi^\varepsilon(t,x) - \mathcal{I}_1[\Phi^\varepsilon(t,\cdot)](x) + \frac{1}{\delta}W'\left(\frac{\Phi_\varepsilon(t,x)}{\varepsilon}\right) \\
&\simeq -\phi'(z_{i_0})(\dot{x}_{i_0}(t) + c_0 L_0) + (W''(\phi(z_{i_0})) - W''(0))\left(\frac{1}{\delta}\sum_{i\neq i_0}\tilde{\phi}(z_i) - \frac{L_0}{\alpha}\right)
\end{aligned}
$$

where $\tilde{\phi}(z) = \phi(z) - H(z)$. Suppose for simplicity that $x = x_{i_0}(t)$, then by (2.3) and (2.2)

$$\frac{1}{\delta}\sum_{i\neq i_0}\tilde{\phi}(z_i) - \frac{L_0}{\alpha} \simeq \frac{1}{\alpha\pi}\sum_{i\neq i_0}\frac{\varepsilon}{x_i - x_{i_0}} - \frac{L_0}{\alpha} \simeq 0.$$

Since $\phi' > 0$, we must have

$$\dot{x}_{i_0}(t) \simeq -c_0 L_0$$

that is, by (2.6),

$$\partial_t u(t_0, x_0) \simeq c_0 \partial_x u(t_0, x_0)\mathcal{I}_1[u(t_0, \cdot)](x_0).$$

Notice that if we define

$$y_i(\tau) := \frac{x_i(\varepsilon\tau)}{\varepsilon}$$

then the y_i's solve

$$\dot{y}_i(\tau) = \dot{x}_i(\varepsilon\tau) \simeq -c_0 L_0 \simeq \frac{c_0}{\pi}\sum_{j\neq i}\frac{\varepsilon}{x_i - x_j} = \frac{c_0}{\pi}\sum_{j\neq i}\frac{1}{y_i - y_j},$$

which is the discrete dislocations dynamics given in (1.14).

2.4. Viscosity sub and supersolutions

By using the comparison principle we show that the functions u^ε are bounded uniformly in ε (see Section 5). In particular, $u^+ := \limsup_{\varepsilon\to 0}{}^* u^\varepsilon$ and $u^- := \liminf_{\varepsilon\to 0*} u^\varepsilon$ are everywhere finite. To formally prove the convergence result following the idea of Section 2.3, we show that u^+ and u^- are respectively viscosity sub and supersolution of (1.5). As in the perturbed test function method by Evans [16] in homogenization problems, we will proceed by contradiction.

2.5. Comparison with the solution of (1.5)

We prove that u^+ and u^- are respectively viscosity sub and supersolution of (1.5), when testing with functions whose derivative in x is different than 0. This is not enough to conclude that by the comparison principle $u^+ \leqslant u^-$. Thus, we consider the approximation $F_\varepsilon(x)$ of the initial datum u_0 provided by (2.4). Since $\phi' \in L^p(\mathbb{R})$ for all $p \in [1, \infty]$ and $\phi' > 0$ (see Lemma 3.1), for fixed ε, $\delta > 0$, the derivative of $F_\varepsilon(x)$ belongs to $L^p(\mathbb{R})$ for all $p \in [1, \infty]$ and is strictly positive. By the results of [9] about Eq. (1.7) (see Theorem 3.9 in Section 3), we can construct a solution $w^\varepsilon(t, x)$ of (1.5) such that w^ε is smooth, $\partial_x w^\varepsilon > 0$, $w^\varepsilon(0, x) \simeq u_0(x)$ and $w^\varepsilon \simeq \overline{u}$, with $\overline{u}$ the viscosity solution of (1.5). We then show that

$$\lim_{|x|\to+\infty} u^+(t, x) - w^\varepsilon(t, x) \simeq 0, \tag{2.7}$$

moreover,

$$u^+(0, x) - w^\varepsilon(0, x) \simeq 0. \tag{2.8}$$

We finally prove that $u^+(t, x) - w^\varepsilon(t, x) \leqslant o_1(\varepsilon)$. Indeed, if not, by (2.7) and (2.8), $u^+ - w^\varepsilon$ must attain a global positive maximum at some point in $(0, +\infty) \times \mathbb{R}$. Then, using w^ε (whose derivative in x is strictly positive) as test function for u^+ we get a contradiction. Passing to the limit as $\varepsilon \to 0$, this shows that $u^+ \leqslant \overline{u}$. Similarly, one can prove that $u^- \geqslant \overline{u}$. Since the reverse inequality $u^- \leqslant u^+$ always holds true, we conclude that $u^- = u^+ = \overline{u}$.

As a byproduct of our proof we show that the viscosity solution $\overline{u}$ of (1.5) satisfies, for all $t \geqslant 0$,

$$\lim_{x\to-\infty}\overline{u}(t, x) = \inf_{\mathbb{R}} u_0 \quad \text{and} \quad \lim_{x\to+\infty}\overline{u}(t, x) = \sup_{\mathbb{R}} u_0,$$

which is equivalent to say that the mass of the non-negative function $\partial_x\overline{u}(t, x)$ is conserved: for all $t \geqslant 0$,

$$\|\partial_x\overline{u}(t, \cdot)\|_{L^1(\mathbb{R})} = \|\partial_x u_0\|_{L^1(\mathbb{R})}.$$

3. Preliminary results

In this section we recall some general auxiliary results that will be used in the rest of the paper.

3.1. Short and long range interaction

We start by recalling a basic fact about the operator $\mathcal{I}_1$. Given $v \in C^{1,1}(\mathbb{R})$ and $r > 0$ we can split $\mathcal{I}_1[v]$ into the short and long range interaction as follows,

$$\mathcal{I}_1[v](x) = \mathcal{I}_1^{1,r}[v](x) + \mathcal{I}_1^{2,r}[v](x),$$

where $\mathcal{I}_1^{1,r}[v](x)$, $\mathcal{I}_1^{2,r}[v](x)$ are defined respectively by (1.16) and (1.17). The short range interaction can be rewritten as

$$\mathcal{I}_1^{1,r}[v](x) = \frac{1}{2\pi}\int_{|y|<r} \frac{v(x+y)+v(x-y)-2v(x)}{y^2}dy,$$

Therefore,

$$|\mathcal{I}_1^{1,r}[v](x)| \leqslant \frac{r}{\pi}\|v\|_{C^{1,1}(\mathbb{R})}.$$

The long range interaction can be bounded as follows

$$|\mathcal{I}_1^{2,r}[v](x)| \leqslant \frac{4}{r\pi}\|v\|_\infty.$$

3.2. The functions ϕ and ψ

In what follows we denote by $H(x)$ the Heaviside function. Let $\alpha := W''(0) > 0$.

Lemma 3.1. *Assume that (1.3) holds, then there exists a unique solution ϕ of (1.11). Furthermore $\phi \in C^{2,\beta}(\mathbb{R})$ and there exist constants $K_0, K_1 > 0$ such that*

$$\left|\phi(x) - H(x) + \frac{1}{\alpha\pi x}\right| \leqslant \frac{K_1}{x^2}, \quad \text{for } |x| \geqslant 1, \tag{3.1}$$

and for any $x \in \mathbb{R}$

$$0 < \frac{K_0}{1+x^2} \leqslant \phi'(x) \leqslant \frac{K_1}{1+x^2}. \tag{3.2}$$

Proof. The existence of a unique solution of (1.11) and estimate (3.2) are proven in [4]. Estimate (3.1) is proven in [22]. □

Let c_0 be defined as in (1.15). Let us introduce the function ψ to be the solution of

$$\begin{cases} \mathcal{I}_1[\psi] = W''(\phi)\psi + \frac{L}{\alpha}(W''(\phi) - W''(0)) + c_0 L\phi' & \text{in} \quad \mathbb{R} \\ \lim_{x\to\pm\infty}\psi(x) = 0. \end{cases} \tag{3.3}$$

For later purposes, we recall the following decay estimate on the solution of (3.3):

Lemma 3.2. *Assume that (1.3) holds, then there exists a unique solution ψ to (3.3). Furthermore $\psi \in C^{1,\beta}(\mathbb{R})$ and for any $L \in \mathbb{R}$ there exist constants K_2 and K_3, with $K_3 > 0$, depending on L such that*

$$\left|\psi(x) - \frac{K_2}{x}\right| \leqslant \frac{K_3}{x^2}, \quad \text{for } |x| \geqslant 1, \tag{3.4}$$

and for any $x \in \mathbb{R}$

$$-\frac{K_3}{1+x^2} \leqslant \psi'(x) \leqslant \frac{K_3}{1+x^2}. \tag{3.5}$$

Proof. The existence of a unique solution of (3.3) is proven in [22]. Estimates (3.4) and (3.5) are shown in [31]. □

The results of Lemmas 3.1 and 3.2 have been generalized in [3,14,15,37,39] to the case when the fractional operator is $-(-\Delta)^s$ for any $s \in (0,1)$.

3.3. Definition of viscosity solution

We first recall the definition of viscosity solution for a general first order non-local equation

$$\partial_t u = F(t,x,u,\partial_x u,\mathcal{I}_1[u]) \quad \text{in} \quad (0,+\infty)\times\Omega \tag{3.6}$$

where Ω is an open subset of $\mathbb{R}$ and $F(t,x,u,p,L)$ is continuous and non-decreasing in L.

Definition 3.1. A function $u \in USC_b((0,+\infty)\times\mathbb{R})$ (resp., $u \in LSC_b((0,+\infty)\times\mathbb{R})$) is a viscosity subsolution (resp., supersolution) of (3.6) if for any $(t_0,x_0)\in(0,+\infty)\times\Omega$, and any test function $\varphi \in C_b^2((0,+\infty)\times\mathbb{R})$ such that $u-\varphi$ attains a global maximum (resp., minimum) at the point (t_0,x_0), then

$$\begin{aligned}&\partial_t\varphi(t_0,x_0) - F(t_0,x_0,u(t_0,x_0),\partial_x\varphi(t_0,x_0),\mathcal{I}_1[\varphi(t_0,\cdot)](x_0)) \leqslant 0\\ &(\text{resp.,} \ \geqslant 0).\end{aligned}$$

A function $u \in C_b((0,+\infty)\times\mathbb{R})$ is a viscosity solution of (3.7) if it is a viscosity sub and supersolution of (3.6).

Remark 3.3. It is classical that the maximum (resp., the minimum) in Definition 3.1 can be assumed to be strict and that

$$\varphi(t_0,x_0) = u(t_0,x_0).$$

This will be used later.

Next, let us consider the initial value problem

$$\begin{cases}\partial_t u = F(t,x,u,\partial_x u,\mathcal{I}_1[u]) & \text{in} \quad (0,+\infty)\times\mathbb{R}\\ u(0,x)=u_0(x) & \text{on} \quad \mathbb{R},\end{cases} \tag{3.7}$$

where u_0 is a continuous function.

Definition 3.2. A function $u \in USC_b((0,+\infty)\times\mathbb{R})$ (resp., $u \in LSC_b((0,+\infty)\times\mathbb{R})$) is a viscosity subsolution (resp., supersolution) of the initial value problem (3.7) if $u(0,x) \leqslant (u_0)(x)$ (resp., $u(0,x) \geqslant (u_0)(x)$) and u is viscosity subsolution (resp., supersolution) of the equation

$$\partial_t u = F(t,x,u,\partial_x u,\mathcal{I}_1[u]) \quad \text{in} \quad (0,+\infty)\times\mathbb{R}.$$

A function $u \in C_b((0,+\infty)\times\mathbb{R})$ is a viscosity solution of (3.7) if it is a viscosity sub and supersolution of (3.7).

It is a classical result that smooth solutions are also viscosity solutions.

Proposition 3.4. *If $u \in C^1((0,+\infty); C^{1,\beta}_{loc}(\Omega)\cap L^\infty(\mathbb{R}))$ for some $0<\beta\leqslant 1$, and u satisfies pointwise*

$$\partial_t u - F(t,x,u,\partial_x u,\mathcal{I}_1[u]) \leqslant 0 \ (\textit{resp.} \geqslant 0) \quad \textit{in} \quad (0,+\infty)\times\Omega,$$

then u is a viscosity subsolution (resp., supersolution) of (3.6).

3.4. Comparison principle and existence results

In this subsection, we successively give comparison principles and existence results for (1.1) and (1.5). The following comparison theorem is shown in [27] for more general parabolic integro-PDEs.

Proposition 3.5 (*Comparison Principle for* (1.1)). *Consider $u \in USC_b((0,+\infty)\times\mathbb{R})$ subsolution and $v \in LSC_b((0,+\infty)\times\mathbb{R})$ supersolution of* (1.1)*, then $u \leqslant v$ on $(0,+\infty)\times\mathbb{R}$.*

Following [27] it can also be proven the comparison principle for (1.1) in bounded domains. Since we deal with a non-local equation, we need to compare the sub and the supersolution everywhere outside the domain.

Proposition 3.6 (*Comparison Principle on Bounded Domains for* (1.1)). *Let Ω be a bounded domain of $(0,+\infty)\times\mathbb{R}$ and let $u \in USC_b((0,+\infty)\times\mathbb{R})$ and $v \in LSC_b((0,+\infty)\times\mathbb{R})$ be respectively a sub and a supersolution of*

$$\delta\partial_t u = \mathcal{I}_1[u(t,\cdot)] - \frac{1}{\delta}W'\left(\frac{u}{\varepsilon}\right) \quad \text{in } \Omega.$$

If $u \leqslant v$ outside Ω, then $u \leqslant v$ in Ω.

Proposition 3.7 (*Existence for* (1.1)). *For $\varepsilon, \delta > 0$ there exists $u^\varepsilon \in C_b([0,+\infty)\times\mathbb{R})$ (unique) viscosity solution of* (1.1)*. Moreover, u^ε is non-decreasing in x.*

Proof. We can construct a solution by Perron's method if we construct sub and supersolutions of (1.1) which are equal to $u_0(x)$ at $t=0$. Since $u_0 \in C^{1,1}(\mathbb{R})$, the two functions $u^\pm(t,x) := u_0(x) \pm \frac{C}{\delta^2}t$ are respectively a super and a subsolution of (1.1), if

$$C \geqslant \frac{4\delta}{\pi}\|u_0\|_{C^{1,1}(\mathbb{R})} + \|W'\|_\infty.$$

Moreover $u^+(0,x) = u^-(0,x) = u_0(x)$. Since u_0 is non-decreasing, the comparison principle implies that u^ε is non-decreasing in x. □

We next recall the comparison and the existence results for (1.5), see e.g. [26], Proposition 3.

Proposition 3.8. *If $u \in USC_b([0,+\infty)\times\mathbb{R})$ and $v \in LSC_b([0,+\infty)\times\mathbb{R})$ are respectively a sub and a supersolution of*

$$\begin{cases} \partial_t u = c_0|\partial_x u|\mathcal{I}_1[u] & \text{in } (0,+\infty)\times\mathbb{R} \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R}, \end{cases} \tag{3.8}$$

then $u \leqslant v$ on $(0,+\infty)\times\mathbb{R}$. Moreover, under assumption (1.4)*, there exists a (unique) viscosity solution of* (3.8) *which is non-decreasing in x and thus is viscosity solution of* (1.5)*.*

3.5. Existence of global solutions of Eq. (1.7)

Theorem 3.9 (*[9], Theorem 2.1*). *Let $f_0 \in L^2(\mathbb{R}) \cap C^\beta(\mathbb{R})$, for some $0 < \beta \leqslant 1$ and $f_0 > 0$ in $\mathbb{R}$ (vanishing at infinity). Then, there exists a global solution v of Eq.* (1.7) *in $C^1((0;+\infty);analytic)$ with $f(0,x) = f_0(x)$. Moreover, f is vanishing at infinity and $H(f(t,\cdot)) \in L^\infty(\mathbb{R})$ for all $t \geqslant 0$. If $f_0 \in L^2(\mathbb{R}) \cap C^{1,\beta}(\mathbb{R})$, the solution is unique.*

4. A discrete approximation of the operator $\mathcal{I}_1$

Let $v \in C^{0,1}(\mathbb{R})$ be non-decreasing and non-constant. For $0 < \varepsilon < 1$, define the points x_i as follows

$$x_i := \inf\{x \in \mathbb{R} \mid v(x) = \varepsilon i\} \qquad i = M_\varepsilon, \dots, N_\varepsilon, \tag{4.1}$$

where

$$M_\varepsilon := \left\lceil \frac{\inf_{\mathbb{R}} v + \varepsilon}{\varepsilon} \right\rceil \quad \text{and} \quad N_\varepsilon := \left\lfloor \frac{\sup_{\mathbb{R}} v - \varepsilon}{\varepsilon} \right\rfloor. \tag{4.2}$$

Since v is continuous,

$$v(x_i) = \varepsilon i,$$

and since v is non-decreasing,

$$x_i < x_{i+1} \quad \text{for all } i = M_\varepsilon, \dots, N_\varepsilon - 1.$$

Notice that if v is strictly increasing then

$$x_i = v^{-1}(\varepsilon i).$$

In what follows given $\overline{x} \in \mathbb{R}$, we denote by x_{i_0} the closest point among the x_i's to $\overline{x}$.

Lemma 4.1. *Let $v \in C^{0,1}(\mathbb{R})$ be non-decreasing and non-constant with $\|v_x\|_\infty \leqslant L$, and let x_i be defined as in (4.1). Then,*

$$x_{i+1} - x_i \geqslant \varepsilon L^{-1} \qquad \text{for all } i = M_\varepsilon, \dots, N_\varepsilon - 1. \tag{4.3}$$

Moreover, there exists $c > 0$ independent of v such that for any $\overline{x} \in \mathbb{R}$

$$\sum_{\substack{i=M_\varepsilon \\ i \neq i_0}}^{N_\varepsilon} \frac{\varepsilon^2}{(x_i - \overline{x})^2} \leqslant cL^2. \tag{4.4}$$

If in addition $v_x \geqslant a > 0$ on an interval I, then for all x_{i+1}, $x_i \in I$, we have

$$x_{i+1} - x_i \leqslant \varepsilon a^{-1}. \tag{4.5}$$

Proof. We have

$$\varepsilon = v(x_{i+1}) - v(x_i) \leqslant L(x_{i+1} - x_i),$$

from which (4.3) follows.

Next, by (4.3), if x_{i_0} is the closest point to $\overline{x}$, then

$$|x_i - \overline{x}| \geqslant \frac{|i - i_0|\varepsilon}{2L} \quad \text{for all } i.$$

Therefore,

$$\sum_{\substack{i=M_\varepsilon \\ i \neq i_0}}^{N_\varepsilon} \frac{\varepsilon^2}{(x_i - \overline{x})^2} \leqslant 4L^2 \sum_{\substack{i=M_\varepsilon \\ i \neq i_0}}^{N_\varepsilon} \frac{1}{(i - i_0)^2} \leqslant 8L^2 \sum_{i=1}^{\infty} \frac{1}{i^2} = cL^2,$$

which proves (4.4).

Finally, if $v_x \geqslant a$, then

$$\varepsilon = v(x_{i+1}) - v(x_i) \geqslant a(x_{i+1} - x_i)$$

from which (4.5) follows. □

Lemma 4.2 (*Short Range Interaction*)**.** *Let $v \in C^{1,1}(\mathbb{R})$ be non-decreasing and non-constant and x_i defined as in (4.1). Let $r = r_\varepsilon$ be such that $r \to 0$ and $\varepsilon/r \to 0$ as $\varepsilon \to 0$. Let $\rho \geqslant r$ and $\overline{x} \in (x_{M_\varepsilon} + \rho, x_{N_\varepsilon} - \rho)$, then*

$$\frac{1}{\pi} \sum_{\substack{i \neq i_0 \\ r \leqslant |x_i - \overline{x}| \leqslant \rho}} \frac{\varepsilon}{x_i - \overline{x}} = \mathcal{I}_1^{1,\rho}[v](\overline{x}) + \frac{1}{\pi} \frac{v(\overline{x}+\rho) + v(\overline{x}-\rho) - 2v(\overline{x})}{\rho} + o_\varepsilon(1). \tag{4.6}$$

Proof. Since $v \in C^{1,1}(\mathbb{R})$ and $r = o_\varepsilon(1)$, there exists $C > 0$ such that

$$|\mathcal{I}_1^{1,r}[v](\overline{x})| \leqslant Cr = o_\varepsilon(1).$$

Therefore, we have

$$\mathcal{I}_1^{1,\rho}[v](\overline{x}) = \frac{1}{\pi} \int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x) - v(\overline{x})}{(x-\overline{x})^2} dx + \frac{1}{\pi} \int_{\overline{x}+r}^{\overline{x}+\rho} \frac{v(x) - v(\overline{x})}{(x-\overline{x})^2} dx + o_\varepsilon(1). \tag{4.7}$$

Let us estimate from above and below the first and second term in the right-hand side of (4.7). We split

$$\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x) - v(\overline{x})}{(x-\overline{x})^2} dx = \int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2} dx - \int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(\overline{x})}{(x-\overline{x})^2} dx.$$

Notice that we can integrate the second term as follows,

$$\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(\overline{x})}{(x-\overline{x})^2} dx = v(\overline{x}) \int_{\overline{x}-\rho}^{\overline{x}-r} \frac{1}{(x-\overline{x})^2} dx = \frac{v(\overline{x})}{r} - \frac{v(\overline{x})}{\rho}. \tag{4.8}$$

Next, we denote by M_ρ and M_r respectively the lowest and the biggest integer i such that $x_i \in [\overline{x}-\rho, \overline{x}-r]$, that is

$$x_{M_\rho - 1} < \overline{x} - \rho \leqslant x_{M_\rho} \leqslant x_{M_r} \leqslant \overline{x} - r < x_{M_r+1}.$$

Then, we split

$$\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2} dx = \int_{\overline{x}-\rho}^{x_{M_\rho}} \frac{v(x)}{(x-\overline{x})^2} dx + \sum_{i=M_\rho}^{M_r-1} \int_{x_i}^{x_{i+1}} \frac{v(x)}{(x-\overline{x})^2} dx + \int_{x_{M_r}}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2} dx. \tag{4.9}$$

By using the monotonicity of v, we obtain

$$\begin{aligned} \int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2} dx &\leqslant \int_{\overline{x}-\rho}^{x_{M_\rho}} \frac{v(x_{M_\rho})}{(x-\overline{x})^2} dx + \sum_{i=M_\rho}^{M_r-1} \int_{x_i}^{x_{i+1}} \frac{v(x_{i+1})}{(x-\overline{x})^2} dx + \int_{x_{M_r}}^{\overline{x}-r} \frac{v(\overline{x}-r)}{(x-\overline{x})^2} dx \\ &= -\frac{v(x_{M_\rho})}{\rho} - \frac{v(x_{M_\rho})}{x_{M_\rho} - \overline{x}} + \sum_{i=M_\rho}^{M_r-1} \left(\frac{v(x_{i+1})}{x_i - \overline{x}} - \frac{v(x_{i+1})}{x_{i+1} - \overline{x}} \right) \\ &\quad + \frac{v(\overline{x}-r)}{x_{M_r} - \overline{x}} + \frac{v(\overline{x}-r)}{r}. \end{aligned} \tag{4.10}$$

Recalling that $v(x_i) = \varepsilon i$, we compute

$$\begin{aligned} \sum_{i=M_\rho}^{M_r-1} \left(\frac{v(x_{i+1})}{x_i - \overline{x}} - \frac{v(x_{i+1})}{x_{i+1} - \overline{x}} \right) &= \sum_{i=M_\rho}^{M_r-1} \left(\frac{\varepsilon(i+1)}{x_i - \overline{x}} - \frac{\varepsilon(i+1)}{x_{i+1} - \overline{x}} \right) \\ &= \sum_{i=M_\rho}^{M_r-1} \frac{\varepsilon(i+1)}{x_i - \overline{x}} - \sum_{i=M_\rho+1}^{M_r} \frac{\varepsilon i}{x_i - \overline{x}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=M_\rho+1}^{M_r-1} \frac{\varepsilon}{x_i-\overline{x}} + \frac{\varepsilon(M_\rho+1)}{x_{M_\rho}-\overline{x}} - \frac{\varepsilon M_r}{x_{M_r}-\overline{x}}\\
&= \sum_{i=M_\rho}^{M_r} \frac{\varepsilon}{x_i-\overline{x}} + \frac{\varepsilon M_\rho}{x_{M_\rho}-\overline{x}} - \frac{\varepsilon(M_r+1)}{x_{M_r}-\overline{x}}\\
&= \sum_{i=M_\rho}^{M_r} \frac{\varepsilon}{x_i-\overline{x}} + \frac{v(x_{M_\rho})}{x_{M_\rho}-\overline{x}} - \frac{v(x_{M_r})}{x_{M_r}-\overline{x}} - \frac{\varepsilon}{x_{M_r}-\overline{x}}\\
&\leqslant \sum_{i=M_\rho}^{M_r} \frac{\varepsilon}{x_i-\overline{x}} + \frac{v(x_{M_\rho})}{x_{M_\rho}-\overline{x}} - \frac{v(x_{M_r})}{x_{M_r}-\overline{x}} + \frac{\varepsilon}{r}.
\end{aligned}
$$

Plugging into (4.10), we obtain

$$
\begin{aligned}
\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2}dx &\leqslant \sum_{i=M_\rho}^{M_r} \frac{\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x}-r)-v(x_{M_r})}{x_{M_r}-\overline{x}} - \frac{v(x_{M_\rho})}{\rho} + \frac{v(\overline{x}-r)}{r} + \frac{\varepsilon}{r}\\
&\leqslant \sum_{i=M_\rho}^{M_r} \frac{\varepsilon}{x_i-\overline{x}} - \frac{v(x_{M_\rho})}{\rho} + \frac{v(\overline{x}-r)}{r} + \frac{\varepsilon}{r},
\end{aligned}
$$

where in the last inequality we have used that $v(\overline{x}-r) \geqslant v(x_{M_r})$ and $x_{M_r} < \overline{x}$. Combining with (4.8) and using that $v(x_{M_\rho}) \geqslant v(\overline{x}-\rho)$, we obtain

$$
\begin{aligned}
\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx &\leqslant \sum_{i=M_\rho}^{M_r} \frac{\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x}-r)-v(\overline{x})}{r} - \frac{v(x_{M_\rho})-v(\overline{x})}{\rho} + \frac{\varepsilon}{r}\\
&\leqslant \sum_{i=M_\rho}^{M_r} \frac{\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x}-r)-v(\overline{x})}{r} - \frac{v(\overline{x}-\rho)-v(\overline{x})}{\rho} + \frac{\varepsilon}{r}.
\end{aligned}
\tag{4.11}
$$

Next, we will get a similar estimate for the second term in the right-hand side of (4.7). As before, we split

$$
\begin{aligned}
\int_{\overline{x}+r}^{\overline{x}+\rho} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx &= \int_{\overline{x}+r}^{\overline{x}+\rho} \frac{v(x)}{(x-\overline{x})^2}dx - \int_{\overline{x}+r}^{\overline{x}+\rho} \frac{v(\overline{x})}{(x-\overline{x})^2}dx\\
&= \int_{\overline{x}+r}^{\overline{x}+\rho} \frac{v(x)}{(x-\overline{x})^2}dx - \frac{v(\overline{x})}{r} + \frac{v(\overline{x})}{\rho}.
\end{aligned}
\tag{4.12}
$$

Let N_r and N_ρ be respectively the lowest and the biggest index i such that $x_i \in [\overline{x}+r, \overline{x}+\rho]$, that is

$$
x_{N_r-1} < \overline{x}+r \leqslant x_{N_r} \leqslant x_{N_\rho} \leqslant \overline{x}+\rho < x_{N_\rho+1}.
$$

By the monotonicity of v,

$$
0 \leqslant v(\overline{x}+\rho) - v(x_{N_\rho}) \leqslant v(x_{N_\rho+1}) - v(x_{N_\rho}) = \varepsilon \tag{4.13}
$$

and

$$
0 \leqslant v(x_{N_r}) - v(\overline{x}+r) \leqslant v(x_{N_r}) - v(x_{N_r-1}) = \varepsilon. \tag{4.14}
$$

By using again the monotonicity of v, we get

$$\begin{aligned}\int_{\overline{x}+r}^{\overline{x}+\rho}\frac{v(x)}{(x-\overline{x})^2}dx&=\int_{\overline{x}+r}^{x_{N_r}}\frac{v(x)}{(x-\overline{x})^2}dx+\sum_{i=N_r}^{N_\rho-1}\int_{x_i}^{x_{i+1}}\frac{v(x)}{(x-\overline{x})^2}dx+\int_{x_{N_\rho}}^{\overline{x}+\rho}\frac{v(x)}{(x-\overline{x})^2}dx\\&\leqslant\int_{\overline{x}+r}^{x_{N_r}}\frac{v(x_{N_r})}{(x-\overline{x})^2}dx+\sum_{i=N_r}^{N_\rho-1}\int_{x_i}^{x_{i+1}}\frac{v(x_{i+1})}{(x-\overline{x})^2}dx+\int_{x_{N_\rho}}^{\overline{x}+\rho}\frac{v(\overline{x}+\rho)}{(x-\overline{x})^2}dx\\&=\frac{v(x_{N_r})}{r}-\frac{v(x_{N_r})}{x_{N_r}-\overline{x}}+\sum_{i=N_r}^{N_\rho-1}\left(\frac{v(x_{i+1})}{x_i-\overline{x}}-\frac{v(x_{i+1})}{x_{i+1}-\overline{x}}\right)\\&+\frac{v(\overline{x}+\rho)}{x_{N_\rho}-\overline{x}}-\frac{v(\overline{x}+\rho)}{\rho}.\end{aligned}\tag{4.15}$$

As before, we compute

$$\begin{aligned}\sum_{i=N_r}^{N_\rho-1}\left(\frac{v(x_{i+1})}{x_i-\overline{x}}-\frac{v(x_{i+1})}{x_{i+1}-\overline{x}}\right)&=\sum_{i=N_r}^{N_\rho-1}\left(\frac{\varepsilon(i+1)}{x_i-\overline{x}}-\frac{\varepsilon(i+1)}{x_{i+1}-\overline{x}}\right)\\&=\sum_{i=N_r}^{N_\rho-1}\frac{\varepsilon(i+1)}{x_i-\overline{x}}-\sum_{i=N_r+1}^{N_\rho}\frac{\varepsilon i}{x_i-\overline{x}}\\&=\sum_{i=N_r+1}^{N_\rho-1}\frac{\varepsilon}{x_i-\overline{x}}+\frac{\varepsilon(N_r+1)}{x_{N_r}-\overline{x}}-\frac{\varepsilon N_\rho}{x_{N_\rho}-\overline{x}}\\&=\sum_{i=N_r}^{N_\rho-1}\frac{\varepsilon}{x_i-\overline{x}}+\frac{\varepsilon N_r}{x_{N_r}-\overline{x}}-\frac{\varepsilon N_\rho}{x_{N_\rho}-\overline{x}}\\&=\sum_{i=N_r}^{N_\rho-1}\frac{\varepsilon}{x_i-\overline{x}}+\frac{v(x_{N_r})}{x_{N_r}-\overline{x}}-\frac{v(x_{N_\rho})}{x_{N_\rho}-\overline{x}}.\end{aligned}$$

Plugging into (4.15) and using (4.13) and (4.14), we obtain

$$\begin{aligned}\int_{\overline{x}+r}^{\overline{x}+\rho}\frac{v(x)}{(x-\overline{x})^2}dx&\leqslant\sum_{i=N_r}^{N_\rho-1}\frac{\varepsilon}{x_i-\overline{x}}+\frac{v(\overline{x}+\rho)-v(x_{N_\rho})}{x_{N_\rho}-\overline{x}}+\frac{v(x_{N_r})}{r}-\frac{v(\overline{x}+\rho)}{\rho}\\&\leqslant\sum_{i=N_r}^{N_\rho-1}\frac{\varepsilon}{x_i-\overline{x}}+\frac{\varepsilon}{x_{N_\rho}-\overline{x}}+\frac{v(x_{N_r})}{r}-\frac{v(\overline{x}+\rho)}{\rho}\\&=\sum_{i=N_r}^{N_\rho}\frac{\varepsilon}{x_i-\overline{x}}+\frac{v(x_{N_r})}{r}-\frac{v(\overline{x}+\rho)}{\rho}\\&\leqslant\sum_{i=N_r}^{N_\rho}\frac{\varepsilon}{x_i-\overline{x}}+\frac{v(\overline{x}+r)}{r}-\frac{v(\overline{x}+\rho)}{\rho}+\frac{\varepsilon}{r}.\end{aligned}$$

Inserting into (4.12), we get

$$\int_{\overline{x}+r}^{\overline{x}+\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\leqslant\sum_{i=N_r}^{N_\rho}\frac{\varepsilon}{x_i-\overline{x}}+\frac{v(\overline{x}+r)-v(\overline{x})}{r}-\frac{v(\overline{x}+\rho)-v(\overline{x})}{\rho}+\frac{\varepsilon}{r}.\tag{4.16}$$

Combining (4.11) and (4.16), we obtain the upper bound

$$\begin{aligned}\int_{r\leqslant|x-\overline{x}|\leqslant\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \leqslant \sum_{r\leqslant|x_i-\overline{x}|\leqslant\rho}\frac{\varepsilon}{x_i-\overline{x}}+\frac{v(\overline{x}+r)+v(\overline{x}-r)-2v(\overline{x})}{r}\\ -\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho}+\frac{2\varepsilon}{r}.\end{aligned} \tag{4.17}$$

Similarly, one can get the following lower bound estimate

$$\begin{aligned}\int_{r\leqslant|x-\overline{x}|\leqslant\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \geqslant \sum_{r\leqslant|x_i-\overline{x}|\leqslant\rho}\frac{\varepsilon}{x_i-\overline{x}}+\frac{v(\overline{x}+r)+v(\overline{x}-r)-2v(\overline{x})}{r}\\ -\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho}-\frac{2\varepsilon}{r}.\end{aligned} \tag{4.18}$$

Since $v\in C^{1,1}(\mathbb{R})$, there exists a constant $C>0$ such that

$$\left|\frac{v(\overline{x}+r)+v(\overline{x}-r)-2v(\overline{x})}{r}\right|\leqslant Cr=o_\varepsilon(1).$$

Therefore, combining (4.17) and (4.18), then dividing both sides by π and using that $\varepsilon/r=o_\varepsilon(1)$, we finally obtain

$$\frac{1}{\pi}\sum_{r\leqslant|x_i-\overline{x}|\leqslant\rho}\frac{\varepsilon}{x_i-\overline{x}}=\frac{1}{\pi}\int_{r\leqslant|x-\overline{x}|\leqslant\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\frac{1}{\pi}\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho}+o_\varepsilon(1),$$

which together with (4.7) gives (4.6). □

Lemma 4.3 (*Long Range Interaction*)**.** *Under the assumptions of Lemma 4.2 and for r as in the lemma, for any $\rho\geqslant r$ and $\overline{x}\in(x_{M_\varepsilon}+\rho,x_{N_\varepsilon}-\rho)$,*

$$\frac{1}{\pi}\sum_{|x_i-\overline{x}|>\rho}\frac{\varepsilon}{x_i-\overline{x}}=\mathcal{I}_1^{2,\rho}[v](\overline{x})-\frac{1}{\pi}\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho}+o_\varepsilon(1). \tag{4.19}$$

Proof. We decompose $\mathcal{I}_1^{2,\rho}[v](\overline{x})$ as follows

$$\mathcal{I}_1^{2,\rho}[v](\overline{x})=\int_{-\infty}^{x_{M_\varepsilon}}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{x_{M_\varepsilon}}^{\overline{x}-\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{\overline{x}+\rho}^{x_{N_\varepsilon}}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{x_{N_\varepsilon}}^{+\infty}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx. \tag{4.20}$$

By the monotonicity of v, we get

$$\int_{-\infty}^{x_{M_\varepsilon}}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\leqslant\int_{-\infty}^{x_{M_\varepsilon}}\frac{v(x_{M_\varepsilon})-v(\overline{x})}{(x-\overline{x})^2}dx=\frac{v(x_{M_\varepsilon})-v(\overline{x})}{\overline{x}-x_{M_\varepsilon}}, \tag{4.21}$$

and

$$\int_{x_{N_\varepsilon}}^{+\infty}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\leqslant\int_{x_{N_\varepsilon}}^{+\infty}\frac{\sup_{\mathbb{R}}v-v(\overline{x})}{(x-\overline{x})^2}dx=\frac{\sup_{\mathbb{R}}v-v(\overline{x})}{x_{N_\varepsilon}-\overline{x}}. \tag{4.22}$$

One can similarly obtain a lower bound as follows

$$\int_{-\infty}^{x_{M_\varepsilon}}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\geqslant\int_{-\infty}^{x_{M_\varepsilon}}\frac{\inf_{\mathbb{R}}v-v(\overline{x})}{(x-\overline{x})^2}dx=\frac{\inf_{\mathbb{R}}v-v(\overline{x})}{\overline{x}-x_{M_\varepsilon}}, \tag{4.23}$$

and

$$\int_{x_{N_\varepsilon}}^{+\infty}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\geqslant\int_{x_{N_\varepsilon}}^{+\infty}\frac{v(x_{N_\varepsilon})-v(\overline{x})}{(x-\overline{x})^2}dx=\frac{v(x_{N_\varepsilon})-v(\overline{x})}{x_{N_\varepsilon}-\overline{x}}. \tag{4.24}$$

To get the estimates for the middle two terms in the right-hand side of (4.20), we will proceed as in the proof of Lemma 4.2. By respectively replacing $\overline{x}-\rho$, $\overline{x}-r$ with x_{M_ε} and $\overline{x}-\rho$ in (4.11) and $\overline{x}+r$, $\overline{x}+\rho$ with $\overline{x}+\rho$ and x_{N_ε} in (4.16) we obtain

$$\begin{aligned}\int_{x_{M_\varepsilon}}^{\overline{x}-\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{\overline{x}+\rho}^{x_{N_\varepsilon}}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\leqslant\sum_{|x_i-\overline{x}|\geqslant\rho}\frac{\varepsilon}{x_i-\overline{x}}+\frac{2\varepsilon}{\rho}\\+\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho}-\frac{v(x_{M_\varepsilon})-v(\overline{x})}{\overline{x}-x_{M_\varepsilon}}-\frac{v(x_{N_\varepsilon})-v(\overline{x})}{x_{N_\varepsilon}-\overline{x}}.\end{aligned}\tag{4.25}$$

Similarly,

$$\begin{aligned}\int_{x_{M_\varepsilon}}^{\overline{x}-\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{\overline{x}+\rho}^{x_{N_\varepsilon}}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\geqslant\sum_{|x_i-\overline{x}|\geqslant\rho}\frac{\varepsilon}{x_i-\overline{x}}-\frac{2\varepsilon}{\rho}\\+\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho}-\frac{v(x_{M_\varepsilon})-v(\overline{x})}{\overline{x}-x_{M_\varepsilon}}-\frac{v(x_{N_\varepsilon})-v(\overline{x})}{x_{N_\varepsilon}-\overline{x}}.\end{aligned}\tag{4.26}$$

Combining (4.21), (4.22) and (4.25), we get

$$\begin{aligned}\int_{-\infty}^{\overline{x}-\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{\overline{x}+\rho}^{+\infty}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\leqslant\sum_{|x_i-\overline{x}|\geqslant\rho}\frac{\varepsilon}{x_i-\overline{x}}\\+\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho}+\frac{\sup_{\mathbb{R}}v-\varepsilon N_\varepsilon}{x_{N_\varepsilon}-\overline{x}}+\frac{2\varepsilon}{\rho}.\end{aligned}\tag{4.27}$$

Combining (4.23), (4.24) and (4.26), we get

$$\begin{aligned}\int_{-\infty}^{\overline{x}-\rho}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{\overline{x}+\rho}^{+\infty}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\geqslant\sum_{|x_i-\overline{x}|\geqslant\rho}\frac{\varepsilon}{x_i-\overline{x}}\\+\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho}-\frac{\varepsilon M_\varepsilon-\inf_{\mathbb{R}}v}{\overline{x}-x_{M_\varepsilon}}-\frac{2\varepsilon}{\rho}.\end{aligned}\tag{4.28}$$

Recalling the definition (4.2) of N_ε and M_ε, we see that $0\leqslant\sup_{\mathbb{R}}v-\varepsilon N_\varepsilon\leqslant 2\varepsilon$ and $0\leqslant\varepsilon M_\varepsilon-\inf_{\mathbb{R}}v\leqslant 2\varepsilon$. Since in addition $x_{N_\varepsilon}-\overline{x}>\rho$, $\overline{x}-x_{M_\varepsilon}>\rho$, $\rho\geqslant r$ and $\varepsilon/r=o_\varepsilon(1)$, from (4.27) and (4.28) we finally get (4.19). □

The following proposition is an immediate consequence of Lemmas 4.2 and 4.3.

Proposition 4.4. *Let $v\in C^{1,1}(\mathbb{R})$ be non-decreasing and non-constant and x_i defined as in (4.1). Let $r=r_\varepsilon$ be such that $r\to 0$ and $\varepsilon/r\to 0$ as $\varepsilon\to 0$. Then, for any $\overline{x}\in(x_{M_\varepsilon}+r,x_{N_\varepsilon}-r)$,*

$$\frac{1}{\pi}\sum_{|x_i-\overline{x}|\geqslant r}\frac{\varepsilon}{x_i-\overline{x}}=\mathcal{I}_1[v](\overline{x})+o_\varepsilon(1).$$

Remark 4.5. Notice that in Lemmas 4.2, 4.3 and Proposition 4.4, the error $o_\varepsilon(1)$ satisfies

$$o_\varepsilon(1)=O(r)+O\left(\frac{\varepsilon}{r}\right).\tag{4.29}$$

Lemma 4.6. *Under the assumptions of Lemma 4.2, let $\overline{x}=x_{i_0}+\varepsilon\gamma$. Then, there exists $r=r_\varepsilon$ satisfying $\varepsilon^{\frac{5}{8}}\leqslant r\leqslant c\varepsilon^{\frac{1}{2}}$, with c depending on the $C^{1,1}$ norm of v, such that*

$$\frac{1}{\pi}\sum_{\substack{i\neq i_0\\|x_i-\overline{x}|<r}}\frac{\varepsilon}{x_i-\overline{x}}=O(\varepsilon^{\frac{1}{8}})+O(\gamma).\tag{4.30}$$

Proof. In what follows we denote by c and C different constants independent of ε and $\overline{x}$. Let $K>0$ be such that $\|v_{xx}\|_{L^\infty(\mathbb{R})}\leqslant K$. We divide the proof into three cases.

Case 1: $v_x(x_{i_0})\leqslant 12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$.

By making a Taylor expansion, we get

$$\begin{aligned}\varepsilon=v(x_{i_0+1})-v(x_{i_0})&\leqslant v_x(x_{i_0})(x_{i_0+1}-x_{i_0})+\frac{K}{2}(x_{i_0+1}-x_{i_0})^2\\&\leqslant\frac{v_x(x_{i_0})^2}{2(12)^2K}+\left(\frac{12^2K}{2}+\frac{K}{2}\right)(x_{i_0+1}-x_{i_0})^2\\&\leqslant\frac{\varepsilon}{2}+\frac{12^2+1}{2}K(x_{i_0+1}-x_{i_0})^2,\end{aligned}$$

from which

$$x_{i_0+1}-x_{i_0}\geqslant c\varepsilon^{\frac{1}{2}}.$$

Similarly, one can prove that

$$x_{i_0}-x_{i_0-1}\geqslant c\varepsilon^{\frac{1}{2}}.$$

Since x_{i_0} is the closest point to $\overline{x}$, we must have that $\overline{x}-x_{i_0-1}\geqslant c\varepsilon^{\frac{1}{2}}/2$ and $x_{i_0+1}-\overline{x}\geqslant c\varepsilon^{\frac{1}{2}}/2$. Therefore, if we choose $r=r_\varepsilon=c\varepsilon^{\frac{1}{2}}/4$, there is no index $i\neq i_0$ for which $|\overline{x}-x_i|\leqslant r$ and thus (4.30) is trivially true.

Next, we show that

$$\frac{1}{\pi}\sum_{\substack{i\neq i_0\\|x_i-\overline{x}|<r}}\frac{\varepsilon}{x_i-x_{i_0}}=O(\varepsilon^{\frac{1}{8}}).\tag{4.31}$$

We consider two more cases.

Case 2: $12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\leqslant v_x(x_{i_0})\leqslant\varepsilon^{\frac{1}{2}-\tau}$, *for some* $\tau\in(0,1/4)$.

If $|\overline{x}-x_{i_0}|\geqslant\varepsilon^{\frac{1}{2}}/(4K^{\frac{1}{2}})$, then we choose $r=\varepsilon^{\frac{1}{2}}/(8K^{\frac{1}{2}})$ and as in Case 1, there is no index $i\neq i_0$ for which $|\overline{x}-x_i|\leqslant r$. Thus (4.30) holds true.

Now, assume $|\overline{x}-x_{i_0}|\leqslant\varepsilon^{\frac{1}{2}}/(4K^{\frac{1}{2}})$ and define

$$r:=\frac{\varepsilon^{\frac{1}{2}}}{2K^{\frac{1}{2}}}\geqslant 2|\overline{x}-x_{i_0}|.\tag{4.32}$$

Let M_r and N_r be respectively the smallest and the larger index i such that $x_i\in(\overline{x}-r,\overline{x}+r)$, that is

$$\begin{aligned}&x_{M_r-1}\leqslant\overline{x}-r<x_{M_r}\\&x_{N_r}<\overline{x}+r\leqslant x_{N_r+1}.\end{aligned}\tag{4.33}$$

By the monotonicity of v and (4.33),

$$\begin{aligned}&-\varepsilon=v(x_{i_0})-v(x_{i_0+1})\leqslant v(x_{i_0})-v(\overline{x})\leqslant v(x_{i_0})-v(x_{i_0-1})=\varepsilon,\\&-\varepsilon=v(x_{N_r})-v(x_{N_r+1})\leqslant v(x_{N_r})-v(\overline{x}+r)\leqslant 0\\&0\leqslant v(x_{M_r})-v(\overline{x}-r)\leqslant v(x_{M_r})-v(x_{M_r-1})=\varepsilon.\end{aligned}\tag{4.34}$$

By making a Taylor expansion, we get, for $i=M_r,\dots,N_r$

$$\varepsilon(i-i_0)=v(x_i)-v(x_{i_0})=v_x(x_{i_0})(x_i-x_{i_0})+O(r^2),$$

where $|O(r^2)|\leqslant K(2r)^2/2=\varepsilon/2$, from which

$$x_i-x_{i_0}=\frac{\varepsilon(i-i_0)+O(r^2)}{v_x(x_{i_0})}.\tag{4.35}$$

Therefore, we can write

$$\begin{aligned}\sum_{\substack{i\neq i_0\\|x_i-\overline{x}|<r}}\frac{\varepsilon}{x_i-x_{i_0}}&=\sum_{\substack{i=M_r\\i\neq i_0}}^{N_r}\frac{v_x(x_{i_0})\varepsilon}{\varepsilon(i-i_0)+O(r^2)}\\&=\sum_{i=M_r}^{i_0-1}\frac{v_x(x_{i_0})\varepsilon}{\varepsilon(i-i_0)+O(r^2)}+\sum_{i=i_0+1}^{N_r}\frac{v_x(x_{i_0})\varepsilon}{\varepsilon(i-i_0)+O(r^2)}.\end{aligned}\tag{4.36}$$

Now, suppose without loss of generality that $N_r-i_0\leqslant i_0-M_r$. Then,

$$\begin{aligned}&\sum_{i=M_r}^{i_0-1}\frac{\varepsilon}{\varepsilon(i-i_0)+O(r^2)}+\sum_{i=i_0+1}^{N_r}\frac{\varepsilon}{\varepsilon(i-i_0)+O(r^2)}\\&=\sum_{k=1}^{i_0-M_r}\frac{\varepsilon}{-\varepsilon k+O(r^2)}+\sum_{k=1}^{N_r-i_0}\frac{\varepsilon}{\varepsilon k+O(r^2)}\\&=\sum_{k=1}^{N_r-i_0}\varepsilon\left(\frac{1}{-\varepsilon k+O(r^2)}+\frac{1}{\varepsilon k+O(r^2)}\right)+\sum_{k=N_r-i_0+1}^{i_0-M_r}\frac{\varepsilon}{-\varepsilon k+O(r^2)}.\end{aligned}\tag{4.37}$$

We can bound the first term of the right hand-side of the last equality as follows

$$\begin{aligned}\left|\sum_{k=1}^{N_r-i_0}\varepsilon\left(\frac{1}{-\varepsilon k+O(r^2)}+\frac{1}{\varepsilon k+O(r^2)}\right)\right|&=\frac{2|O(r^2)|}{\varepsilon}\left|\sum_{k=1}^{N_r-i_0}\frac{1}{(-k+\frac{O(r^2)}{\varepsilon})(k+\frac{O(r^2)}{\varepsilon})}\right|\\&\leqslant\sum_{k=1}^{\infty}\frac{1}{k^2-\frac{1}{4}}\\&=C,\end{aligned}\tag{4.38}$$

where we used that $|O(r^2)|/\varepsilon\leqslant 1/2$. Therefore,

$$v_x(x_{i_0})\left|\sum_{k=1}^{N_r-i_0}\varepsilon\left(\frac{1}{-\varepsilon k+O(r^2)}+\frac{1}{\varepsilon k+O(r^2)}\right)\right|\leqslant Cv_x(x_{i_0})\leqslant C\varepsilon^{\frac{1}{2}-\tau}.\tag{4.39}$$

Next, by using that $\sum_{k=n}^{m}1/k\leqslant(m-n+1)/n$, we get

$$\begin{aligned}\left|\sum_{k=N_r-i_0+1}^{i_0-M_r}\frac{\varepsilon}{-\varepsilon k+O(r^2)}\right|&\leqslant\sum_{k=N_r-i_0+1}^{i_0-M_r}\frac{\varepsilon}{\varepsilon k-|O(r^2)|}\\&\leqslant\frac{-(\varepsilon N_r+\varepsilon M_r-2\varepsilon i_0)}{\varepsilon(N_r+1)-\varepsilon i_0-|O(r^2)|}\\&=\frac{-(v(x_{N_r})+v(x_{M_r})-2v(x_{i_0}))}{v(x_{N_r})-v(x_{i_0})+\varepsilon-|O(r^2)|}\\&\leqslant\frac{-(v(x_{N_r})+v(x_{M_r})-2v(x_{i_0}))}{v(x_{N_r})-v(x_{i_0})}.\end{aligned}\tag{4.40}$$

By (4.34) and the regularity of v,

$$0\leqslant-(v(x_{N_r})+v(x_{M_r})-2v(x_{i_0}))\leqslant-(v(\overline{x}+r)+v(\overline{x}-r)-2v(\overline{x}))+3\varepsilon\leqslant Kr^2+3\varepsilon\leqslant C\varepsilon.\tag{4.41}$$

Now, by using that $v_x(x_{i_0}) \geqslant 12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ and that $|\overline{x} - x_{i_0}| \leqslant r/2$, and by (4.34), we get

$$\begin{aligned} v(x_{N_r}) - v(x_{i_0}) &\geqslant v(\overline{x} + r) - v(x_{i_0}) - \varepsilon \\ &\geqslant v_x(x_{i_0})(r - |\overline{x} - x_{i_0}|) - \frac{K}{2}(2r)^2 - \varepsilon \\ &\geqslant v_x(x_{i_0})\frac{r}{2} - \frac{3}{2}\varepsilon \\ &= v_x(x_{i_0})\frac{r}{2} - 12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\frac{r}{4} \\ &\geqslant v_x(x_{i_0})\frac{r}{4} \\ &= v_x(x_{i_0})\frac{\varepsilon^{\frac{1}{2}}}{8K^{\frac{1}{2}}}. \end{aligned} \tag{4.42}$$

From (4.40), (4.41) and (4.42), we infer that

$$\left| \sum_{i=i_0-M_r+1}^{N_r-i_0} \frac{\varepsilon v_x(x_{i_0})}{-\varepsilon k + O(r^2)} \right| \leqslant \frac{v_x(x_{i_0}) C\varepsilon 8K^{\frac{1}{2}}}{v_x(x_{i_0})\varepsilon^{\frac{1}{2}}} \leqslant C\varepsilon^{\frac{1}{2}}. \tag{4.43}$$

Finally, (4.36), (4.37), (4.39) and (4.43) imply

$$\left| \frac{1}{\pi} \sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| < r}} \frac{\varepsilon}{x_i - x_{i_0}} \right| \leqslant C\varepsilon^{\frac{1}{2}-\tau} \leqslant C\varepsilon^{\frac{1}{4}},$$

which gives (4.31).

Case 3: $v_x(x_{i_0}) \geqslant \varepsilon^{\frac{1}{2}-\tau}$, for some $\tau \in (0, 1/4)$.

As in Case 2, we can assume that $|\overline{x} - x_{i_0}| \leqslant \varepsilon^{\frac{1+\tau}{2}}$. Then, we define

$$r := 2\varepsilon^{\frac{1+\tau}{2}} \geqslant 2|\overline{x} - x_{i_0}|. \tag{4.44}$$

Notice that $r \geqslant \varepsilon^{\frac{5}{8}}$. Assume, without loss of generality, that $N_r - i_0 \leqslant i_0 - M_r$. Then as before, we write

$$\sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| < r}} \frac{\varepsilon}{x_i - x_{i_0}} = \sum_{k=1}^{N_r - i_0} \varepsilon v_x(x_{i_0}) \left(\frac{1}{-\varepsilon k + O(r^2)} + \frac{1}{\varepsilon k + O(r^2)} \right) + \sum_{k=N_r-i_0+1}^{i_0 - M_r} \frac{\varepsilon v_x(x_{i_0})}{-\varepsilon k + O(r^2)}. \tag{4.45}$$

By (4.38) and the definition (4.44) of r,

$$\left| \sum_{k=1}^{N_r - i_0} \varepsilon v_x(x_{i_0}) \left(\frac{1}{-\varepsilon k + O(r^2)} + \frac{1}{\varepsilon k + O(r^2)} \right) \right| \leqslant C v_x(x_{i_0}) \frac{|O(r^2)|}{\varepsilon} \leqslant C\varepsilon^{\tau}. \tag{4.46}$$

By (4.40), (4.41) and (4.42), and by using that $v_x(x_{i_0}) \geqslant C\varepsilon^{\frac{1}{2}-\tau}$ and (4.44), we get

$$\left| \sum_{k=N_r-i_0+1}^{i_0-M_r} \frac{\varepsilon}{-\varepsilon k + O(r^2)} \right| \leqslant \frac{C\varepsilon}{v_x(x_{i_0})\frac{r}{2} - \frac{3}{2}\varepsilon} \leqslant C\varepsilon^{\frac{\tau}{2}}, \tag{4.47}$$

for ε small enough (independently of $\overline{x}$). Estimates (4.45), (4.46) and (4.47) imply

$$\left| \sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| < r}} \frac{\varepsilon}{x_i - x_{i_0}} \right| \leqslant C\varepsilon^{\frac{\tau}{2}} \leqslant C\varepsilon^{\frac{1}{8}},$$

which gives (4.31)

Finally, to prove (4.30), we estimate

$$\left| \sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| < r}} \frac{\varepsilon}{x_i - \overline{x}} - \sum_{\substack{i \neq i_0 \\ |x_i - x_{i_0}| < r}} \frac{\varepsilon}{x_i - x_{i_0}} \right| = \left| \sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| < r}} \frac{\varepsilon^2 \gamma}{(x_i - \overline{x})(x_i - x_{i_0})} \right|. \tag{4.48}$$

Assume, without loss of generality that $\overline{x} = x_{i_0} + \varepsilon\gamma$, with $\gamma \geqslant 0$, that is, $\overline{x} \in [x_{i_0}, x_{i_0+1})$. Then,

$$|x_i - \overline{x}| \geqslant \begin{cases} |x_i - x_{i_0}| & \text{if } i \leqslant i_0 - 1 \\ \frac{x_{i_0+1} - x_{i_0}}{2} & \text{if } i = i_0 + 1 \\ x_i - x_{i_0+1} & \text{if } i \geqslant i_0 + 2. \end{cases}$$

Moreover, by (4.3), $x_{i_0+1} - x_{i_0} \geqslant \varepsilon L^{-1}$. Therefore,

$$\left| \sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| < r}} \frac{\varepsilon^2 \gamma}{(x_i - \overline{x})(x_i - x_{i_0})} \right| \leqslant \sum_{i \leqslant i_0 - 1} \frac{\varepsilon^2 \gamma}{(x_i - x_{i_0})^2} + 2L^2\gamma + \sum_{i \geqslant i_0 + 2} \frac{\varepsilon^2 \gamma}{(x_i - x_{i_0+1})^2} \leqslant C\gamma, \tag{4.49}$$

where in the last inequality we used (4.4). By (4.48) and (4.49) we get

$$\left| \sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| < r}} \frac{\varepsilon}{x_i - \overline{x}} - \sum_{\substack{i \neq i_0 \\ |x_i - x_{i_0}| < r}} \frac{\varepsilon}{x_i - x_{i_0}} \right| \leqslant C\gamma,$$

which together with (4.31) gives (4.30). □

The following proposition is an immediate consequence of Lemma 4.2, Proposition 4.4 and Lemma 4.6.

Proposition 4.7. *Let $v \in C^{1,1}(\mathbb{R})$ be non-decreasing and non-constant and x_i defined as in (4.1). Then, there exists $c > 0$ depending on the $C^{1,1}$ norm of v such that if $\rho \geqslant c\varepsilon^{\frac{1}{2}}$, and $\overline{x} \in (x_{M_\varepsilon} + \rho, x_{N_\varepsilon} - \rho)$, $\overline{x} = x_{i_0} + \varepsilon\gamma$, then*

$$\frac{1}{\pi} \sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| \leqslant \rho}} \frac{\varepsilon}{x_i - \overline{x}} = \mathcal{I}_1^{1,\rho}[v](\overline{x}) + O(\gamma) + \frac{1}{\pi} \frac{v(\overline{x} + \rho) + v(\overline{x} - \rho) - 2v(\overline{x})}{\rho} + o_\varepsilon(1), \tag{4.50}$$

and

$$\frac{1}{\pi} \sum_{i \neq i_0} \frac{\varepsilon}{x_i - \overline{x}} = \mathcal{I}_1[v](\overline{x}) + o_\varepsilon(1) + O(\gamma). \tag{4.51}$$

Proof. Fix $\overline{x}$ and let r and c be given by Lemma 4.6. Then $\varepsilon^{\frac{5}{8}} \leqslant r \leqslant c\varepsilon^{\frac{1}{2}} \leqslant \rho$. By Lemma 4.2 and recalling (4.29),

$$\frac{1}{\pi} \sum_{\substack{i \neq i_0 \\ r \leqslant |x_i - \overline{x}| \leqslant \rho}} \frac{\varepsilon}{x_i - \overline{x}} = \mathcal{I}_1^{1,\rho}[v](\overline{x}) + \frac{1}{\pi} \frac{v(\overline{x} + \rho) + v(\overline{x} - \rho) - 2v(\overline{x})}{\rho} + O\left(\varepsilon^{\frac{3}{8}}\right).$$

Combining this estimate with (4.30) yields (4.50).

Similarly, by Proposition 4.4 and Lemma 4.6, we get (4.51). □

Remark 4.8. If $\varepsilon|\gamma| = |\overline{x} - x_{i_0}| > c\varepsilon^{\frac{1}{2}} \geqslant r$, then $|\overline{x} - x_i| > r$ for all i and

$$\frac{1}{\pi}\sum_{\substack{i\neq i_0\\ |x_i-\overline{x}|\leqslant\rho}} \frac{\varepsilon}{x_i-\overline{x}} = \frac{1}{\pi}\sum_{r<|x_i-\overline{x}|\leqslant\rho} \frac{\varepsilon}{x_i-\overline{x}}$$
$$= \mathcal{I}_1^{1,\rho}[v](\overline{x}) + \frac{1}{\pi}\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho} + o_\varepsilon(1).$$

Remark 4.9. If $\overline{x} = x_{i_0}$, then $\gamma = 0$ and

$$\frac{1}{\pi}\sum_{i\neq i_0}\frac{\varepsilon}{x_i - x_{i_0}} = \mathcal{I}_1[v](x_{i_0}) + o_\varepsilon(1). \tag{4.52}$$

Lemma 4.10. *Let $v \in C^{1,1}(\mathbb{R})$ be non-decreasing and non-constant and x_i be defined as in (4.1). Let ϕ be defined by (1.11). Let $M_\varepsilon \leqslant M < N \leqslant N_\varepsilon$ and $R \geqslant c\varepsilon^{\frac{1}{2}}$, with $c > 0$ given by Proposition 4.7. Then, for all $x \in (x_M + R, x_N - R)$*

$$\left|\sum_{i=M}^{N}\varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) + \varepsilon M - v(x)\right| \leqslant o_\varepsilon(1)\left(1+\frac{\delta}{R}\right),$$

with $o_\varepsilon(1)$ independent of R and x.

Proof. Fix $x \in (x_M + R, x_N - R)$, and let x_{i_0} be the closest point among the x_i's to x. Then, $x_{i_0-1} < x < x_{i_0+1}$ and by the monotonicity of v,

$$\varepsilon(i_0 - 1) = v(x_{i_0-1}) \leqslant v(x) \leqslant v(x_{i_0+1}) = \varepsilon(i_0 + 1). \tag{4.53}$$

By using (4.53), estimate (3.1) and that $\phi \leqslant 1$, we get

$$\begin{aligned}
&\sum_{i=M}^{N}\varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) + \varepsilon M - v(x)\\
&= \sum_{i=M}^{i_0-1}\varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) + \varepsilon\phi\left(\frac{x-x_{i_0}}{\varepsilon\delta}\right) + \sum_{i=i_0+1}^{N}\varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) + \varepsilon M - v(x)\\
&\leqslant \sum_{i=M}^{i_0-1}\varepsilon\left(1 + \frac{\varepsilon\delta}{\alpha\pi(x_i-x)} + \frac{K_1\varepsilon^2\delta^2}{(x_i-x)^2}\right) + \varepsilon\\
&+ \sum_{i=i_0+1}^{N}\varepsilon\left(\frac{\varepsilon\delta}{\alpha\pi(x_i-x)} + \frac{K_1\varepsilon^2\delta^2}{(x_i-x)^2}\right) + \varepsilon M - \varepsilon(i_0-1)\\
&= \varepsilon\delta\sum_{\substack{i=M\\ i\neq i_0}}^{N}\frac{\varepsilon}{\alpha\pi(x_i-x)} + \varepsilon\delta^2K_1\sum_{\substack{i=M\\ i\neq i_0}}^{N}\frac{\varepsilon^2}{(x_i-x)^2} + 2\varepsilon\\
&= \varepsilon\delta\sum_{\substack{i\neq i_0\\ |x_i-x|\leqslant R}}\frac{\varepsilon}{\alpha\pi(x_i-x)} + \varepsilon\delta\sum_{\substack{i=M\\ |x_i-x|>R}}^{N}\frac{\varepsilon}{\alpha\pi(x_i-x)} + \varepsilon\delta^2K_1\sum_{\substack{i=M\\ i\neq i_0}}^{N}\frac{\varepsilon^2}{(x_i-x)^2} + 2\varepsilon.
\end{aligned}$$

We can bound the second term above as follows

$$\varepsilon\delta\left|\sum_{\substack{i=M\\ |x_i-x|>R}}^{N}\frac{\varepsilon}{\alpha\pi(x_i-x)}\right| \leqslant \varepsilon\delta\sum_{\substack{i=M\\ |x_i-x|>R}}^{N}\frac{\varepsilon}{\alpha\pi|x_i-x|} \leqslant \frac{\varepsilon\delta(\varepsilon N - \varepsilon M + \varepsilon)}{\alpha\pi R}$$
$$= \frac{\varepsilon\delta(v(x_N) - v(x_M) + \varepsilon)}{\alpha\pi R} \leqslant \varepsilon(2\|v\|_\infty + \varepsilon)\frac{\delta}{\alpha\pi R}.$$

Therefore, by Proposition 4.7, Remark 4.8 and (4.4), we get

$$\begin{aligned}\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right)+\varepsilon M-v(x) &\leqslant \frac{\varepsilon\delta}{\alpha}\left(\mathcal{I}_1^{1,R}[v](\overline{x})+O(\varepsilon^{-\frac{1}{2}})+C\right)\\ &\quad+\varepsilon(2\|v\|_\infty+\varepsilon)\frac{\delta}{\alpha\pi R}+C\varepsilon\delta^2+2\varepsilon\\ &\leqslant o_\varepsilon(1)\left(1+\frac{\delta}{R}\right).\end{aligned}$$

Similarly, one can prove that

$$\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right)+\varepsilon M-v(x) \geqslant o_\varepsilon(1)\left(1+\frac{\delta}{R}\right)$$

and this concludes the proof of the lemma. □

Lemma 4.11. *Under the assumptions of Lemma 4.10, there exists $C>0$ independent of ε and R such that for all $x>x_N+R$,*

$$\left|\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right)+\varepsilon M-v(x_N)\right| \leqslant C\varepsilon\left(1+\frac{\delta}{R}\right), \tag{4.54}$$

and for all $x<x_M-R$,

$$\left|\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right)\right| \leqslant C\varepsilon\left(1+\frac{\delta}{R}\right). \tag{4.55}$$

Proof. Let $x>x_N+R$, then $x-x_i>R$ for all $i=M,\dots,N$ and by using that $\phi\leqslant 1$, we get

$$\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right)+\varepsilon M \leqslant (N+1)\varepsilon = v(x_N)+\varepsilon.$$

On the other hand, by (3.1) and (4.4),

$$\begin{aligned}\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right)+\varepsilon M &\geqslant \sum_{i=M}^{N} \varepsilon\left(1+\frac{\varepsilon\delta}{\alpha\pi(x_i-x)}-\frac{K_1\delta^2\varepsilon^2}{(x_i-x)^2}\right)+\varepsilon M\\ &\geqslant (N+1)\varepsilon-\frac{\varepsilon}{\alpha\pi}(\varepsilon N-\varepsilon M+\varepsilon)\frac{\delta}{R}-C\varepsilon\delta^2\\ &= v(x_N)+\varepsilon-\frac{\varepsilon}{\alpha\pi}(v(x_N)-v(x_M)+\varepsilon)\frac{\delta}{R}-C\varepsilon\delta^2\\ &\geqslant v(x_N)-C\varepsilon\left(1+\frac{\delta}{R}\right).\end{aligned}$$

This proves (4.54).

Now, let $x<x_M-R$, then $x-x_i<-R$ for all $i=M,\dots,N$ and by (3.1) and (4.4),

$$\begin{aligned}\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) &\leqslant \sum_{i=M}^{N} \varepsilon\left(\frac{\varepsilon\delta}{\alpha\pi(x_i-x)}+\frac{K_1\delta^2\varepsilon^2}{(x-x_i)^2}\right)\\ &\leqslant \frac{\varepsilon}{\alpha\pi}(\varepsilon N-\varepsilon M+\varepsilon)\frac{\delta}{R}+C\varepsilon\delta^2\\ &= \frac{\varepsilon}{\alpha\pi}(v(x_N)-v(x_M)+\varepsilon)\frac{\delta}{R}+C\varepsilon\delta^2\\ &\leqslant C\varepsilon\left(1+\frac{\delta}{R}\right).\end{aligned}$$

On the other hand

$$\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) \geqslant 0.$$

This concludes the proof of (4.55) and of the lemma. □

Proposition 4.12. *Let $v \in C^{1,1}(\mathbb{R})$ be non-decreasing and non-constant and x_i be defined as in (4.1). Let ϕ be defined by (1.11). Then, for all $x \in \mathbb{R}$,*

$$\left|\sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - v(x)\right| \leqslant o_\varepsilon(1), \tag{4.56}$$

where $o_\varepsilon(1)$ is independent of x.

Proof. Let $R = R_\varepsilon := \max\{\delta, c\varepsilon^{\frac{1}{2}}\}$, with c given in Proposition 4.7. If $x \in (x_{M_\varepsilon} + R, x_{N_\varepsilon} - R)$, then (4.56) follows from Lemma 4.10.

Next, let us assume $x > x_{N_\varepsilon} + R$. Then, by (4.54)

$$\left|\sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - v(x)\right| \leqslant |v(x_{N_\varepsilon}) - v(x)| + C\varepsilon.$$

Now, by the monotonicity of v,

$$v(x_{N_\varepsilon}) - v(x) \leqslant 0.$$

On the other hand, by the definition (4.2) of N_ε, we have

$$v(x) - v(x_{N_\varepsilon}) = v(x) - \varepsilon N_\varepsilon \leqslant \sup_{\mathbb{R}} v - \varepsilon N_\varepsilon \leqslant 2\varepsilon.$$

This proves (4.56) when $x > x_{N_\varepsilon} + R$. By using (4.55), one can similarly prove (4.56) when $x < x_{M_\varepsilon} - R$.

Now, assume $x_{N_\varepsilon} - R \leqslant x \leqslant x_{N_\varepsilon} + R$. Then by (4.56) applied at $x_{N_\varepsilon} - 2R$ and $x_{N_\varepsilon} + 2R$, the monotonicity of ϕ and the regularity of v, we get

$$\begin{aligned}
&\sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - v(x) \\
&\leqslant \sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_{N_\varepsilon}+2R-x_i}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - v(x_{N_\varepsilon}+2R) + O(R) \\
&\leqslant o_\varepsilon(1),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - v(x) \\
&\geqslant \sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_{N_\varepsilon}-2R-x_i}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - v(x_{N_\varepsilon}-2R) + O(R) \\
&\geqslant o_\varepsilon(1).
\end{aligned}$$

This proves (4.56) when $x_{N_\varepsilon} - R \leqslant x \leqslant x_{N_\varepsilon} + R$. Similarly, one can prove (4.56) when $x_{M_\varepsilon} - R \leqslant x \leqslant x_{M_\varepsilon} + R$ and the proof of the proposition is completed. □

We conclude this section with the following lemma that will be used later on.

Lemma 4.13. *Let $v \in C^{1,1}(\mathbb{R})$ be non-decreasing and non-constant and x_i be defined as in (4.1). Then, there exists $C > 0$ such that for all $x \in \mathbb{R}$,*

$$\left|\sum_{i\neq i_0} \frac{\varepsilon}{x_i - x}\right| \leqslant C. \tag{4.57}$$

Proof. Let us fix $\overline{x} \in \mathbb{R}$. In what follows we denote by C several positive constants independent of ε and $\overline{x}$. Let x_{i_0} be the closest point to $\overline{x}$ among the x_i's. Then, by (4.3), $|x_i - \overline{x}| \geqslant \varepsilon/(2L)$ for $i \neq i_0$. Since $v \in C^{1,1}(\mathbb{R})$, there exists $C > 0$ such that $|\mathcal{I}_1[v](\overline{x})| \leqslant C$. Moreover,

$$\begin{aligned}\mathcal{I}_1[v](\overline{x}) &= \frac{1}{\pi}\int_{-\infty}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \frac{1}{\pi}PV\int_{\overline{x}-\frac{\varepsilon}{2L}}^{\overline{x}+\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \frac{1}{\pi}\int_{\overline{x}+\frac{\varepsilon}{2L}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \frac{1}{\pi}\int_{\overline{x}+\frac{\varepsilon}{2L}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + O(\varepsilon),\end{aligned} \tag{4.58}$$

where $|O(\varepsilon)| \leqslant C\varepsilon$. If $\overline{x} \in (x_{M_\varepsilon} + \varepsilon/(2L), x_{N_\varepsilon} - \varepsilon/(2L))$, then we write

$$\int_{-\infty}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx = \int_{-\infty}^{x_{M_\varepsilon}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \sum_{i=M_\varepsilon}^{i_0-2}\int_{x_i}^{x_{i+1}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \int_{x_{i_0-1}}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx,$$

where we define $x_{i_0-1} = x_{M_\varepsilon}$ if $i_0 = M_\varepsilon$. By the monotonicity of v,

$$0 \geqslant \int_{-\infty}^{x_{M_\varepsilon}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \geqslant -\frac{\inf_{\mathbb{R}} v - v(\overline{x})}{x_{M_\varepsilon} - \overline{x}}.$$

As in the proof of Lemma 4.2,

$$\begin{aligned}\sum_{i=M_\varepsilon}^{i_0-2}\int_{x_i}^{x_{i+1}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx &\leqslant \sum_{i=M_\varepsilon}^{i_0-1}\frac{\varepsilon}{x_i-\overline{x}} + \frac{\varepsilon M_\varepsilon - v(\overline{x})}{x_{M_\varepsilon}-\overline{x}} - \frac{\varepsilon(i_0-1) - v(\overline{x})}{x_{i_0-1}-\overline{x}} + \frac{\varepsilon}{\overline{x}-x_{i_0-1}} \\ &= \sum_{i=M_\varepsilon}^{i_0-1}\frac{\varepsilon}{x_i-\overline{x}} + \frac{v(x_{M_\varepsilon}) - v(\overline{x})}{x_{M_\varepsilon}-\overline{x}} - \frac{v(x_{i_0-1}) - v(\overline{x})}{x_{i_0-1}-\overline{x}} + \frac{\varepsilon}{\overline{x}-x_{i_0-1}},\end{aligned}$$

and

$$\sum_{i=M_\varepsilon}^{i_0-2}\int_{x_i}^{x_{i+1}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \geqslant \sum_{i=M_\varepsilon}^{i_0-1}\frac{\varepsilon}{x_i-\overline{x}} + \frac{v(x_{M_\varepsilon}) - v(\overline{x})}{x_{M_\varepsilon}-\overline{x}} - \frac{v(x_{i_0-1}) - v(\overline{x})}{x_{i_0-1}-\overline{x}} + \frac{\varepsilon}{\overline{x}-x_{M_\varepsilon}}.$$

Therefore, by the Lipschitz regularity of v and using that $\overline{x} - x_{i_0-1} \geqslant \varepsilon/(2L)$, we get

$$\begin{aligned}&\int_{-\infty}^{x_{M_\varepsilon}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \sum_{i=M_\varepsilon}^{i_0-2}\int_{x_i}^{x_{i+1}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \\ &\leqslant \sum_{i=M_\varepsilon}^{i_0-1}\frac{\varepsilon}{x_i-\overline{x}} + \frac{v(x_{M_\varepsilon}) - v(\overline{x})}{x_{M_\varepsilon}-\overline{x}} - \frac{v(x_{i_0-1}) - v(\overline{x})}{x_{i_0-1}-\overline{x}} + \frac{\varepsilon}{\overline{x}-x_{i_0-1}} \\ &\leqslant \sum_{i=M_\varepsilon}^{i_0-1}\frac{\varepsilon}{x_i-\overline{x}} + 4L,\end{aligned}$$

and

$$\begin{aligned}&\int_{-\infty}^{x_{M_\varepsilon}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\sum_{i=M_\varepsilon}^{i_0-2}\int_{x_i}^{x_{i+1}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\\ &\geqslant \sum_{i=M_\varepsilon}^{i_0-1}\frac{\varepsilon}{x_i-\overline{x}}+\frac{v(x_{M_\varepsilon})-\inf_{\mathbb{R}} v}{x_{M_\varepsilon}-\overline{x}}-\frac{v(x_{i_0-1})-v(\overline{x})}{x_{i_0-1}-\overline{x}}\\ &\geqslant \sum_{i=M_\varepsilon}^{i_0-1}\frac{\varepsilon}{x_i-\overline{x}}-5L,\end{aligned}$$

where in the last inequality we used that $v(x_{M_\varepsilon})-\inf_{\mathbb{R}} v\leqslant 2\varepsilon$ and $x_{M_\varepsilon}-\overline{x}\leqslant -\varepsilon/(2L)$.

Next, using that $v(\overline{x})-v(x_{i_0-1})\leqslant v(x_{i_0+1})-v(x_{i_0-1})=2\varepsilon$, the monotonicity of v and that $\overline{x}-x_{i_0-1}\geqslant \varepsilon/(2L)$, we have

$$0\geqslant \int_{x_{i_0-1}}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\geqslant (v(x_{i_0-1})-v(\overline{x}))\left(\frac{2L}{\varepsilon}-\frac{1}{\overline{x}-x_{i_0-1}}\right)\geqslant -C.$$

We conclude that

$$\int_{-\infty}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx-C\leqslant \sum_{i=M_\varepsilon}^{i_0-1}\frac{\varepsilon}{x_i-\overline{x}}\leqslant \int_{-\infty}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+C. \tag{4.59}$$

Similarly,

$$\int_{\overline{x}+\frac{\varepsilon}{2L}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx-C\leqslant \sum_{i=i_0+1}^{N_\varepsilon}\frac{\varepsilon}{x_i-\overline{x}}\leqslant \int_{\overline{x}+\frac{\varepsilon}{2L}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+C. \tag{4.60}$$

From (4.58), (4.59) and (4.60),

$$\sum_{i\neq i_0}\frac{\varepsilon}{x_i-x}\leqslant \int_{-\infty}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{\overline{x}+\frac{\varepsilon}{2L}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+C\leqslant \mathcal{I}_1[v](\overline{x})+C\leqslant C,$$

and

$$\sum_{i\neq i_0}\frac{\varepsilon}{x_i-x}\geqslant \int_{-\infty}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{\overline{x}+\frac{\varepsilon}{2L}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx-C\geqslant \mathcal{I}_1[v](\overline{x})-C\geqslant -C,$$

which gives (4.57).

If $x\leqslant x_{M_\varepsilon}+\varepsilon/(2L)$, then $x_{i_0}=x_{M_\varepsilon}$ and we write

$$\int_{\overline{x}+\frac{\varepsilon}{2L}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx=\int_{\overline{x}+\frac{\varepsilon}{2L}}^{x_{M_\varepsilon+1}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\sum_{i=M_\varepsilon+1}^{N_\varepsilon-1}\int_{x_i}^{x_{i+1}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{x_{N_\varepsilon}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx.$$

If $x\geqslant x_{N_\varepsilon}-\varepsilon/(2L)$, then $x_{i_0}=x_{N_\varepsilon}$ and we write

$$\int_{-\infty}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx=\int_{-\infty}^{x_{M_\varepsilon}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\sum_{i=M_\varepsilon}^{N_\varepsilon-2}\int_{x_i}^{x_{i+1}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx+\int_{x_{N_\varepsilon-1}}^{\overline{x}-\frac{\varepsilon}{2L}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx.$$

Similar computations as before show (4.57). This concludes the proof of the lemma. □

5. Proof of Theorem 1.1

We first show that the functions u^ε are bounded uniformly in ε. Since $W'(z) = 0$ for any $z \in \mathbb{Z}$, integers are stationary solutions to (1.1). Let $k_1, k_2 \in \mathbb{Z}$ be such that $k_1 \leqslant \inf_{\mathbb{R}} u_0 \leqslant \sup_{\mathbb{R}} u_0 \leqslant k_2$. Then by the comparison principle we have that for any $\varepsilon > 0$

$$k_1 \leqslant u^\varepsilon(t,x) \leqslant k_2 \quad \text{for all } (t,x) \in (0,+\infty) \times \mathbb{R}.$$

In particular, $u^+ := \limsup^*_{\varepsilon\to 0} u^\varepsilon$ is everywhere finite. We will prove that u^+ is a viscosity subsolution of (1.5) when testing with test functions whose derivative in x at the maximum point is different than 0. Similarly, we can prove that $u^- := \liminf_{*\varepsilon\to 0} u^\varepsilon$ is a supersolution of (1.5) when testing with functions whose derivative in x at the minimum point is different than 0. We will show that this is enough to conclude that the following comparison principle holds true: if $\overline{u}$ is the viscosity solution of (1.5), then

$$u^+ \leqslant \overline{u} \leqslant u^-. \tag{5.1}$$

Since the reverse inequality $u^- \leqslant u^+$ always holds true, we conclude that the two functions coincide with $\overline{u}$ and that $u^\varepsilon \to \overline{u}$ as $\varepsilon \to 0$, uniformly on compact sets. We will prove (5.1) in Section 6.

Let $\eta \in C^2_b((0,+\infty)\times\mathbb{R})$ be such that

$$u^+(t,x) - \eta(t,x) < u^+(t_0,x_0) - \eta(t_0,x_0) = 0 \quad \text{for all } (t,x) \neq (t_0,x_0), \tag{5.2}$$

and assume $\partial_x\eta(t_0,x_0) \neq 0$. By the comparison principle, u^ε is non-decreasing in x, and thus also u^+ is non-decreasing in x. The monotonicity of u^+ and (5.2) imply that $\partial_x\eta(t_0,x_0) \geqslant 0$. Therefore, we have

$$\partial_x\eta(t_0,x_0) > 0. \tag{5.3}$$

The goal is to show that

$$\partial_t\eta(t_0,x_0) \leqslant c_0\partial_t\eta_x(t_0,x_0)\,\mathcal{I}_1[\eta(t_0,\cdot)](x_0). \tag{5.4}$$

Assume by contradiction that

$$\partial_t\eta(t_0,x_0) > c_0\partial_t\eta_x(t_0,x_0)\,\mathcal{I}_1[\eta(t_0,\cdot)](x_0). \tag{5.5}$$

Denote

$$L_0 := \mathcal{I}_1[\eta(t_0,\cdot)](x_0).$$

By (5.3) and (5.5), there exists $0 < \rho < 1$ and $L_1 > 0$ such that

$$\partial_x\eta(t,x) \geqslant \frac{\partial_x\eta(t_0,x_0)}{2} > 0 \quad \text{for all } (t,x) \in Q_{2\rho,2\rho}(t_0,x_0), \tag{5.6}$$

and

$$\partial_t\eta(t,x) \geqslant c_0\partial_x\eta(t,x)(L_0+L_1) \quad \text{for all } (t,x) \in Q_{2\rho,2\rho}(t_0,x_0). \tag{5.7}$$

By (5.6), η is increasing in x over $Q_{2\rho,2\rho}(t_0,x_0)$. Without loss of generality, we can assume $\eta(t,\cdot)$ to be non-decreasing over $\mathbb{R}$, for $|t-t_0| < 2\rho$. Indeed, if not, since $\eta > u^+$ outside $Q_{2\rho,2\rho}(t_0,x_0)$ and $u^+(t,\cdot)$ is non-decreasing over $\mathbb{R}$, we can replace η with $\tilde\eta$ such that $\eta = \tilde\eta$ in $Q_{2\rho,2\rho}(t_0,x_0)$, $\tilde\eta(t,\cdot)$ is non-decreasing over $\mathbb{R}$ for $|t-t_0| < 2\rho$, $\tilde\eta \in C^2_b((t_0-2\rho,t_0+2\rho)\times\mathbb{R})$, $u^+ \leqslant \tilde\eta \leqslant \eta$ in $(t_0-2\rho,t_0+2\rho)\times(-K,K)$. If we prove (5.4) for $\tilde\eta$, then, since $\partial_t\tilde\eta(t_0,x_0) = \partial_t\eta(t_0,x_0)$, $\partial_x\tilde\eta(t_0,x_0) = \partial_x\eta(t_0,x_0)$ and $\mathcal{I}_1^{1,K}[\tilde\eta(t_0,\cdot)](x_0) \leqslant \mathcal{I}_1^{1,K}[\eta(t_0,\cdot)](x_0)$, by letting K go to $+\infty$, (5.4) holds true for η. Therefore in what follows we assume η non-decreasing with respect to x over $\mathbb{R}$ for $|t-t_0| < 2\rho$.

We then define the points

$$x^0_{M_\varepsilon} < \cdots < x^0_i < x^0_{i+1} < \cdots < x^0_{N_\varepsilon}$$

such that

$$x^0_i := \inf\{x \mid \eta(t_0, x) = \varepsilon i\} \qquad i = M_\varepsilon, \dots, N_\varepsilon,$$

where

$$M_\varepsilon := \left\lceil \frac{\inf_{\mathbb{R}} \eta(t_0, \cdot) + \varepsilon}{\varepsilon} \right\rceil \quad \text{and} \quad N_\varepsilon := \left\lfloor \frac{\sup_{\mathbb{R}} \eta(t_0, \cdot) - \varepsilon}{\varepsilon} \right\rfloor .$$

Next, for $0 < R \ll \rho$ to be determined, let M_ρ be the biggest integer such that $x^0_{M_\rho}$ is smaller than $x_0 - (\rho + R)$ and N_ρ is the lowest integer such that $x^0_{N_\rho}$ is bigger than $x_0 + (\rho + R)$, that is

$$x^0_{M_\rho} < x_0 - (\rho + R) \leqslant x^0_{M_\rho + 1} \tag{5.8}$$

and

$$x^0_{N_\rho - 1} \leqslant x_0 + (\rho + R) < x^0_{N_\rho}. \tag{5.9}$$

Then, we define the points $x_i(t)$ as follows

$$x_i(t) := \inf\{x \mid \eta(t, x) = \varepsilon i\} \qquad \text{for } i = M_\rho, \dots, N_\rho. \tag{5.10}$$

By definition,

$$\eta(t, x_i(t)) = \varepsilon i, \tag{5.11}$$

moreover,

$$x_i(t_0) = x^0_i. \tag{5.12}$$

Lemma 5.1. *Let $B_0 := \partial_x \eta(t_0, x_0)/(2\|\partial_t \eta\|_\infty)$ and $x_i(t)$ be defined by (5.10), $i = M_\rho, \dots, N_\rho$. Then, there exists $\varepsilon_0 = \varepsilon_0(\rho)$ such that for $\varepsilon < \varepsilon_0$ and $R < \rho/3$, $x_i \in C^1(t_0 - B_0 R, t_0 + B_0 R)$ and for $|t - t_0| < B_0 R$,*

$$|\dot{x}_i(t)| \leqslant B_0^{-1}, \tag{5.13}$$

$$x_0 + \rho < x_{N_\rho}(t) < x_0 + \rho + 3R, \tag{5.14}$$

$$x_0 - (\rho + 3R) < x_{M_\rho}(t) < x_0 - \rho. \tag{5.15}$$

In particular $(t, x_i(t)) \in Q_{2\rho,2\rho}(t_0, x_0)$.

We postpone the proof of Lemma 5.1 to Section 7.

Now, since by the lemma the $x_i(t)$'s are of class C^1 and $(t, x_i(t)) \in Q_{2\rho,2\rho}(t_0, x_0)$, we can differentiate in t Eq. (5.11)

$$\partial_t \eta(t, x_i(t)) + \partial_x \eta(t, x_i(t)) \dot{x}_i(t) = 0$$

and use (5.7) to get, for $|t - t_0| < B_0 R$,

$$-\dot{x}_i(t) \geqslant c_0(L_0 + L_1), \quad i = M_\rho, \dots, N_\rho. \tag{5.16}$$

Next, we are going to construct a supersolution of (1.1) in $Q_{B_0 R, R}(t_0, x_0)$ for $R \ll \rho < 1$.

Since the maximum of $u^+ - \eta$ is strict, there exists $\gamma_R > 0$ such that

$$u^+ - \eta \leqslant -2\gamma_R < 0 \quad \text{in } Q_{2\rho,2\rho}(t_0, x_0) \setminus Q_{B_0 R, R}(t_0, x_0). \tag{5.17}$$

Then, we define

$$\Phi^\varepsilon(t,x) := \begin{cases} h^\varepsilon(t,x) + \varepsilon M_\varepsilon + \frac{\varepsilon\delta L_1}{\alpha} - \varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor & \text{for } (t,x)\in Q_{B_0R,\frac{\rho}{2}}(t_0,x_0) \\ u^\varepsilon(t,x) & \text{outside} \end{cases} \tag{5.18}$$

where

$$\begin{aligned} h^\varepsilon(t,x) = &\sum_{i=M_\rho}^{N_\rho} \varepsilon\left(\phi\left(\frac{x-x_i(t)}{\varepsilon\delta}\right) + \delta\psi\left(\frac{x-x_i(t)}{\varepsilon\delta}\right)\right) \\ &+ \sum_{i=M_\varepsilon}^{M_\rho-1} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}\right) + \sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}\right), \end{aligned} \tag{5.19}$$

with ϕ solution of (1.11) and ψ solution of (3.3) with $L = L_0 + L_1$.

Remark 5.2. We choose $x_i(t) = x_i^0$ to be constant in time for $i = M_\varepsilon,\dots,M_\rho-1$ and $i = N_\rho+1,\dots,N_\varepsilon$, because we cannot bound the derivative $\dot{x}_i(t)$ for all $i = M_\varepsilon,\dots,N_\varepsilon$. This will produce an error $O(R)$ when comparing Φ^ε with η when $|t-t_0| < B_0R$ and $|x-x_0| \geqslant \rho - R$, see Lemma 5.6.

Lemma 5.3. *There exists $0 < R \ll \rho$ and $\varepsilon_0 = \varepsilon_0(R,\rho) > 0$ such that for any $\varepsilon < \varepsilon_0$, the function Φ^ε defined by (5.18) satisfies*

$$\Phi^\varepsilon \geqslant u^\varepsilon \quad \textit{outside } Q_{B_0R,R}(t_0,x_0), \tag{5.20}$$

$$\delta\partial_t\Phi^\varepsilon \geqslant \mathcal{I}_1[\Phi^\varepsilon] - \frac{1}{\delta}W'\left(\frac{\Phi_\varepsilon}{\varepsilon}\right) \quad \textit{in } Q_{B_0R,R}(t_0,x_0), \tag{5.21}$$

and

$$\Phi^\varepsilon \leqslant \eta + o_\varepsilon(1) - \varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor \quad \textit{in } Q_{B_0R,R}(t_0,x_0). \tag{5.22}$$

We are now in position to conclude the proof of Theorem 1.1.

By (5.20) and (5.21) and the comparison principle, Proposition 3.6, we have

$$u^\varepsilon(t,x) \leqslant \Phi^\varepsilon(t,x) \quad \text{for all } (t,x)\in Q_{B_0R,R}(t_0,x_0).$$

Passing to the upper limit as $\varepsilon\to 0$ and using (5.22) and that $u^+(t_0,x_0) = \eta(t_0,x_0)$, we obtain

$$0 \leqslant -\gamma_R,$$

a contradiction. This concludes the proof of Theorem 1.1.

5.1. Proof of Lemma 5.3

We divide the proof of Lemma 5.3 in several steps. We start with the following lemma.

Lemma 5.4. *There exists $\varepsilon_0 = \varepsilon_0(R,\rho) > 0$ such that for any $\varepsilon < \varepsilon_0$ and for any $(t,x)\in Q_{B_0R,\rho-R}(t_0,x_0)$, we have*

$$|h^\varepsilon(t,x) + \varepsilon M_\varepsilon - \eta(t,x)| \leqslant o_\varepsilon(1).$$

We postpone the proof of Lemma 5.4 to Section 7.

Proof of (5.20). Outside $Q_{B_0R,\frac{\rho}{2}}(t_0,x_0)$, by definition (5.18) of Φ^ε, $\Phi^\varepsilon(t,x) = u^\varepsilon(t,x)$.

Next, by Lemma 5.4 and (5.17), for $(t,x)\in Q_{B_0R,\frac{\rho}{2}}(t_0,x_0)\setminus Q_{B_0R,R}(t_0,x_0)$,

$$\begin{aligned}\Phi^\varepsilon(t,x)&=h^\varepsilon(t,x)+\varepsilon M_\varepsilon+\frac{\varepsilon\delta L_1}{\alpha}-\varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor\\&\geqslant\eta(t,x)+o_\varepsilon(1)-\varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor\\&\geqslant u^+(t,x)+o_\varepsilon(1)+2\gamma_R-\varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor\\&\geqslant u^\varepsilon(t,x)\end{aligned}$$

for ε small enough, where in the last inequality we have used that $u^+(t,x)\geqslant u^\varepsilon(t,x)+o_\varepsilon(1)$ and $2\gamma_R-\varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor\to\gamma_R>0$ as $\varepsilon\to 0$. This concludes the proof of (5.20).

Proof of (5.22). By Lemma 5.4, for $(t,x)\in Q_{B_0R,R}(t_0,x_0)$

$$\Phi^\varepsilon(t,x)=h^\varepsilon(t,x)+\varepsilon M_\varepsilon+\frac{\varepsilon\delta L_1}{\alpha}-\varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor\leqslant\eta(t,x)+o_\varepsilon(1)-\varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor,$$

which gives (5.22).

Next, we need some preliminary results in order to prove (5.21).

Lemma 5.5. *There exists $C>0$ independent of ε and ρ such that, for any $x\in\mathbb{R}$,*

$$\left|\sum_{i=M_\rho}^{N_\rho}\varepsilon\delta\psi\left(\frac{x-x_i(t)}{\varepsilon\delta}\right)\right|\leqslant C\delta.$$

Proof. We have,

$$\begin{aligned}\left|\sum_{i=M_\rho}^{N_\rho}\varepsilon\delta\psi\left(\frac{x-x_i(t)}{\varepsilon\delta}\right)\right|&\leqslant\delta\|\psi\|_\infty\varepsilon(N_\rho-M_\rho+1)\\&=\delta\|\psi\|_\infty(\eta(t,x_{N_\rho}(t))-\eta(t,x_{M_\rho}(t))+\varepsilon)\\&\leqslant C\delta.\quad\square\end{aligned}$$

Lemma 5.6. *There exists $\varepsilon_0=\varepsilon_0(R,\rho)>0$ such that for any $\varepsilon<\varepsilon_0$, if $|t-t_0|<B_0R$, and $|x-x_0|\geqslant\rho-R$, then*

$$|h^\varepsilon(t,x)+\varepsilon M_\varepsilon-\eta(t,x)|\leqslant o_\varepsilon(1)+O(R).$$

We postpone the proof of Lemma 5.6 to Section 7.

Corollary 5.7. *There exists $\varepsilon_0=\varepsilon_0(R,\rho)>0$ such that for any $\varepsilon<\varepsilon_0$, $R<\rho/4$, and any $(t,x)\in Q_{B_0R,R}(t_0,x_0)$, we have*

$$\mathcal{I}_1[\Phi^\varepsilon(t,\cdot)](x)\leqslant\mathcal{I}_1[h^\varepsilon(t,\cdot)](x)+o_\varepsilon(1)+\frac{o_R(1)}{\rho}.\tag{5.23}$$

Proof. We have

$$\mathcal{I}_1[\Phi^\varepsilon(t,\cdot)](x)=\mathcal{I}_1^{1,\frac{\rho}{4}}[\Phi^\varepsilon(t,\cdot)](x)+\frac{1}{\pi}\int_{\frac{\rho}{4}<|y-x|<\rho}\frac{\Phi^\varepsilon(t,y)-\Phi^\varepsilon(t,x)}{(y-x)^2}dy+\mathcal{I}_1^{2,\rho}[\Phi^\varepsilon(t,\cdot)](x).$$

If $(t,x) \in Q_{B_0 R,R}(t_0,x_0)$ and $|y-x| < \rho/4$ then for $R < \rho/4$, $|y-x_0| < \rho/2$, that is $(t,y) \in Q_{B_0 R,\frac{\rho}{2}}(t_0,x_0)$. Therefore, by the definition (5.18) of Φ^ε,

$$\mathcal{I}_1^{1,\frac{\rho}{4}}[\Phi^\varepsilon(t,\cdot)](x) = \mathcal{I}_1^{1,\frac{\rho}{4}}[h^\varepsilon(t,\cdot)](x). \tag{5.24}$$

If $(t,x) \in Q_{B_0 R,R}(t_0,x_0)$ and $|y-x| > \rho$ then $|y-x_0| > \rho/2$, therefore $\Phi^\varepsilon(t,y) = u^\varepsilon(t,y)$. Then, by Lemmas 5.4, 5.6 and using that $u^\varepsilon \leqslant u^+ + o_\varepsilon(1) \leqslant \eta + o_\varepsilon(1)$, we get

$$\begin{aligned}
\mathcal{I}_1^{2,\rho}[\Phi^\varepsilon(t,\cdot)](x) &= \frac{1}{\pi}\int_{|y-x|>\rho} \frac{\Phi^\varepsilon(t,y)-\Phi^\varepsilon(t,x)}{(y-x)^2}dy \\
&= \frac{1}{\pi}\int_{|y-x|>\rho} \frac{u^\varepsilon(t,y)-(h^\varepsilon(t,x)+\varepsilon M_\varepsilon + O(\varepsilon)+O(\gamma_R))}{(y-x)^2}dy \\
&\leqslant \frac{1}{\pi}\int_{|y-x|>\rho} \frac{\eta(t,y)-(h^\varepsilon(t,x)+\varepsilon M_\varepsilon)}{(y-x)^2}dy + \frac{o_\varepsilon(1)+O(\gamma_R)}{\rho} \\
&\leqslant \frac{1}{\pi}\int_{|y-x|>\rho} \frac{h^\varepsilon(t,y)-h^\varepsilon(t,x)}{(y-x)^2}dy + \frac{o_\varepsilon(1)+O(\gamma_R)+O(R)}{\rho}.
\end{aligned}$$

Therefore,

$$\mathcal{I}_1^{2,\rho}[\Phi^\varepsilon(t,\cdot)](x) \leqslant \mathcal{I}_1^{2,\rho}[h^\varepsilon(t,\cdot)](x) + \frac{o_\varepsilon(1)+o_R(1)}{\rho}. \tag{5.25}$$

Finally, if $\rho/4 < |y-x| < \rho$ then either $\Phi^\varepsilon(t,y) = u^\varepsilon(t,y)$ and by Lemmas 5.4 and 5.6, $\Phi^\varepsilon(t,y) \leqslant h^\varepsilon(t,y) + \varepsilon M_\varepsilon + o_\varepsilon(1) + O(R)$ or $\Phi^\varepsilon(t,y) = h^\varepsilon(t,y) + \varepsilon M_\varepsilon + o_\varepsilon(1) + o_R(1)$. In both cases,

$$\int_{\frac{\rho}{4}<|y-x|<\rho} \frac{\Phi^\varepsilon(t,y)-\Phi^\varepsilon(t,x)}{(y-x)^2}dy \leqslant \int_{\frac{\rho}{4}<|y-x|<\rho} \frac{h^\varepsilon(t,y)-h^\varepsilon(t,x)}{(y-x)^2}dy + \frac{o_\varepsilon(1)+o_R(1)}{\rho}. \tag{5.26}$$

From (5.24), (5.25) and (5.26), inequality (5.23) follows. □

Now, we are ready to prove (5.21).

Proof of (5.21). Denote

$$\Lambda := \delta\partial_t\Phi^\varepsilon - \mathcal{I}_1[\Phi^\varepsilon] + \frac{1}{\delta}W'\left(\frac{\Phi_\varepsilon}{\varepsilon}\right).$$

We want to show that $\Lambda(t,x) \geqslant 0$ for all $(t,x) \in Q_{B_0 R,R}(t_0,x_0)$. Fix $(\bar{t},\bar{x}) \in Q_{B_0 R,R}(t_0,x_0)$. By Corollary 5.7,

$$\begin{aligned}
\mathcal{I}_1[\Phi^\varepsilon(\bar{t},\cdot)](\bar{x}) &\leqslant \mathcal{I}_1[h^\varepsilon(\bar{t},\cdot)](\bar{x}) + o_\varepsilon(1) + \frac{o_R(1)}{\rho} \\
&= \sum_{i=M_\rho}^{N_\rho} \frac{1}{\delta}\mathcal{I}_1[\phi](z_i) + \sum_{i=M_\varepsilon}^{M_\rho-1} \frac{1}{\delta}\mathcal{I}_1[\phi](z_i^0) + \sum_{i=N_\rho+1}^{N_\varepsilon} \frac{1}{\delta}\mathcal{I}_1[\phi](z_i^0) \\
&\quad + \sum_{i=M_\rho}^{N_\rho} \mathcal{I}_1[\psi](z_i) + o_\varepsilon(1) + \frac{o_R(1)}{\rho},
\end{aligned} \tag{5.27}$$

where we denote $z_i^0 = (\bar{x}-x_i^0)/(\varepsilon\delta)$ and $z_i = (\bar{x}-x_i(\bar{t}))/(\varepsilon\delta)$. Let i_0 be such that $x_{i_0}(\bar{t})$ is the closest point to $\bar{x}$. Since $(\bar{t},\bar{x}) \in Q_{B_0 R,R}(t_0,x_0)$, by Lemma 5.1 we have $M_\rho < i_0 < N_\rho$. If $\bar{x} = x_{i_0} + \varepsilon\gamma$, then (4.5) and

(5.6) imply that $|\gamma| \leqslant 2/\partial_x\eta(t_0,x_0)$. Note that $z_{i_0} = \gamma/\delta$. By (5.27), we have

$$\begin{aligned}\Lambda(\bar t,\bar x) &= \delta\partial_t \Phi^\varepsilon(\bar t,\bar x) - \mathcal{I}_1[\Phi^\varepsilon(\bar t,\cdot)](\bar x) + \frac{1}{\delta}W'\left(\frac{\Phi_\varepsilon(\bar t,\bar x)}{\varepsilon}\right)\\
&\geqslant \sum_{i=M_\rho}^{N_\rho}\left[-\dot x_i(\bar t)\phi'(z_i) - \delta\dot x_i(\bar t)\psi'(z_i)\right] + o_\varepsilon(1) + \frac{o_R(1)}{\rho}\\
&\quad - \sum_{i=M_\rho}^{N_\rho}\frac{1}{\delta}\mathcal{I}_1[\phi](z_i) - \sum_{i=M_\varepsilon}^{M_\rho-1}\frac{1}{\delta}\mathcal{I}_1[\phi](z_i^0) - \sum_{i=N_\rho+1}^{N_\varepsilon}\frac{1}{\delta}\mathcal{I}_1[\phi](z_i^0) - \sum_{i=M_\rho}^{N_\rho}\mathcal{I}_1[\psi](z_i)\\
&\quad + \frac{1}{\delta}W'\left(\sum_{i=M_\rho}^{N_\rho}[\phi(z_i)+\delta\psi(z_i)] + \sum_{i=M_\varepsilon}^{M_\rho-1}\phi(z_i^0) + \sum_{i=N_\rho+1}^{N_\varepsilon}\phi(z_i^0) + \frac{\delta L_1}{\alpha}\right),\end{aligned}$$

where we have used the periodicity of W' in the last term. Let us denote

$$E_0 := o_\varepsilon(1) + \frac{o_R(1)}{\rho}, \tag{5.28}$$

and

$$\tilde\phi(z) := \phi(z) - H(z),$$

where H is the Heaviside function. Then, by (5.16), (1.11), the periodicity of W' and making a Taylor expansion of W' around $\phi(z_{i_0})$, we obtain

$$\begin{aligned}\Lambda(\bar t,\bar x) &\geqslant c_0(L_0+L_1)\phi'(z_{i_0})\\
&\quad + \frac{1}{\delta}\left(-W'(\phi(z_{i_0})) - \sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}W'(\tilde\phi(z_i)) - \sum_{i=M_\varepsilon}^{M_\rho-1}W'(\tilde\phi(z_i^0)) - \sum_{i=N_\rho+1}^{N_\varepsilon}W'(\tilde\phi(z_i^0))\right)\\
&\quad - \sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\mathcal{I}_1[\psi](z_i) - \mathcal{I}_1[\psi](z_{i_0}) + \frac{1}{\delta}W'(\phi(z_{i_0}))\\
&\quad + \frac{1}{\delta}W''(\tilde\phi(z_{i_0}))\left(\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}[\tilde\phi(z_i)+\delta\psi(z_i)] + \delta\psi(z_{i_0}) + \sum_{i=M_\varepsilon}^{M_\rho-1}\tilde\phi(z_i^0) + \sum_{i=N_\rho+1}^{N_\varepsilon}\tilde\phi(z_i^0) + \frac{\delta L_1}{\alpha}\right)\\
&\quad + E_0 + E_1 + E_2,\end{aligned}$$

where we define E_1 as follows

$$E_1 := -\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\dot x_i(\bar t)\phi'(z_i) - \delta\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\dot x_i(\bar t)\psi'(z_i) - \delta\dot x_{i_0}(\bar t)\psi'(z_{i_0}),$$

and E_2 as the error from the Taylor expansion,

$$E_2 := \frac{1}{\delta}O\left(\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}[\tilde\phi(z_i)+\delta\psi(z_i)] + \delta\psi(z_{i_0}) + \sum_{i=M_\varepsilon}^{M_\rho-1}\tilde\phi(z_i^0) + \sum_{i=N_\rho+1}^{N_\varepsilon}\tilde\phi(z_i^0) + \frac{\delta L_1}{\alpha}\right)^2.$$

Making a Taylor expansion of W' around 0, using that $W'(0) = 0$ and rearranging the terms, we obtain

$$\begin{aligned}\Lambda(\bar{t},\bar{x}) \geqslant\ & c_0(L_0+L_1)\phi'(z_{i_0}) - \mathcal{I}_1[\psi](z_{i_0}) + W''(\phi(z_{i_0}))\psi(z_{i_0}) \\ & + \frac{1}{\delta}\left(-W''(0)\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\tilde{\phi}(z_i) - W''(0)\sum_{i=M_\varepsilon}^{M_\rho-1}\tilde{\phi}(z_i^0) - W''(0)\sum_{i=N_\rho+1}^{N_\rho}\tilde{\phi}(z_i^0)\right) \\ & - \sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\mathcal{I}_1[\psi](z_i) \\ & + \frac{1}{\delta}W''(\tilde{\phi}(z_{i_0}))\left(\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}[\tilde{\phi}(z_i)+\delta\psi(z_i)] + \sum_{i=M_\varepsilon}^{M_\rho-1}\tilde{\phi}(z_i^0) + \sum_{i=N_\rho+1}^{N_\varepsilon}\tilde{\phi}(z_i^0) + \frac{\delta L_1}{\alpha}\right) \\ & + E_0+E_1+E_2+E_3,\end{aligned}$$

where E_3 is defined by

$$E_3 := \frac{1}{\delta}\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho} O(\tilde{\phi}(z_i))^2 + \frac{1}{\delta}\sum_{i=M_\varepsilon}^{M_\rho-1} O(\tilde{\phi}(z_i^0))^2 + \frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon} O(\tilde{\phi}(z_i^0))^2.$$

Since ψ solves (3.3) with $L = L_0 + L_1$, we have that

$$c_0(L_0+L_1)\phi'(z_{i_0}) - \mathcal{I}_1[\psi](z_{i_0}) + W''(\phi(z_{i_0}))\psi(z_{i_0}) = -\frac{L_0+L_1}{\alpha}(W''(\phi(z_{i_0})) - W''(0)).$$

Therefore,

$$\begin{aligned}\Lambda(\bar{t},\bar{x}) \geqslant & -\frac{L_0+L_1}{\alpha}(W''(\phi(z_{i_0})) - W''(0)) \\ & + (W''(\phi(z_{i_0})) - W''(0))\left(\frac{1}{\delta}\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\tilde{\phi}(z_i) + \frac{1}{\delta}\sum_{i=M_\varepsilon}^{M_\rho-1}\tilde{\phi}(z_i^0) + \frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon}\tilde{\phi}(z_i^0)\right) \\ & + W''(\tilde{\phi}(z_{i_0}))\frac{L_1}{\alpha} + W''(\tilde{\phi}(z_{i_0}))\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\psi(z_i) - \sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\mathcal{I}_1[\psi](z_i) \\ & + E_0+E_1+E_2+E_3.\end{aligned}$$

Rearranging the terms and recalling that $\alpha = W''(0)$, we finally get

$$\begin{aligned}\Lambda(\bar{t},\bar{x}) \geqslant\ & (W''(\phi(z_{i_0})) - W''(0))\left(\frac{1}{\delta}\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\tilde{\phi}(z_i) + \frac{1}{\delta}\sum_{i=M_\varepsilon}^{M_\rho-1}\tilde{\phi}(z_i^0) + \frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon}\tilde{\phi}(z_i^0) - \frac{L_0}{\alpha}\right) \\ & + L_1 + E_0+E_1+E_2+E_3+E_4,\end{aligned} \tag{5.29}$$

where E_4 is given by

$$E_4 := W''(\tilde{\phi}(z_{i_0}))\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\psi(z_i) - \sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\mathcal{I}_1[\psi](z_i). \tag{5.30}$$

Next, for fixed $L_1 > 0$, we are going to show that all the other terms on the right-hand side of (5.29) are small. Recall that

$$L_0 = \mathcal{I}_1[\eta(t_0,\cdot)](x_0) = \mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0) + \mathcal{I}_1^{2,\rho}[\eta(t_0,\cdot)](x_0).$$

Lemma 5.8. *We have,*

$$(W''(\phi(z_{i_0})) - W''(0)) \left(\frac{1}{\delta} \sum_{\substack{i=M_\rho \\ i\neq i_0}}^{N_\rho} \tilde{\phi}(z_i) - \frac{1}{\alpha}\mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0) \right) = o_\varepsilon(1) + o_R(1) + o_\rho(1) + O\left(\frac{R}{\rho}\right), \tag{5.31}$$

and

$$\frac{1}{\delta}\sum_{i=M_\varepsilon}^{M_\rho-1} \tilde{\phi}(z_i^0) + \frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon} \tilde{\phi}(z_i^0) - \frac{1}{\alpha}\mathcal{I}_1^{2,\rho}[\eta(t_0,\cdot)](x_0) = o_\varepsilon(1) + o_\rho(1) + O\left(\frac{R}{\rho}\right). \tag{5.32}$$

Proof. Let us prove (5.31). By (4.3), for $i \neq i_0$, and ε (thus δ) small enough

$$|z_i| = \left| \frac{\overline{x} - x_i(\overline{t})}{\varepsilon\delta} \right| \geqslant \frac{L^{-1}}{2\delta} \geqslant 1.$$

Then, by (3.1), for $i \neq i_0$,

$$\left| \tilde{\phi}(z_i) + \frac{\varepsilon\delta}{\alpha\pi(\overline{x} - x_i(\overline{t}))} \right| \leqslant \frac{K_1\varepsilon^2\delta^2}{(\overline{x} - x_i(\overline{t}))^2},$$

which implies that

$$\Gamma_1 - \Gamma_2 \leqslant \sum_{\substack{i=M_\rho \\ i\neq i_0}}^{N_\rho} \frac{\tilde{\phi}(z_i)}{\delta} - \frac{1}{\alpha}\mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0) \leqslant \Gamma_1 + \Gamma_2,$$

where Γ_1 and Γ_2 are respectively defined by

$$\Gamma_1 := \frac{1}{\alpha}\left(\frac{1}{\pi} \sum_{\substack{i=M_\rho \\ i\neq i_0}}^{N_\rho} \frac{\varepsilon}{x_i(\overline{t}) - \overline{x}} - \mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0) \right) \quad \text{and} \quad \Gamma_2 := K_1 \sum_{\substack{i=M_\rho \\ i\neq i_0}}^{N_\rho} \frac{\varepsilon^2\delta}{(x_i - \overline{x}(\overline{t}))^2}.$$

Since $(\overline{t},\overline{x}) \in Q_{B_0R,R}(t_0,x_0)$, by Lemma 5.1 we have that $x_{N_\rho}(\overline{t}) - \overline{x} > x_0 + \rho - \overline{x} > \rho - R$ and $\overline{x} - x_{M_\rho}(\overline{t}) > \overline{x} - x_0 + \rho > \rho - R$. Then,

$$\begin{aligned}
\sum_{\substack{i=M_\rho \\ i\neq i_0}}^{N_\rho} \frac{\varepsilon}{x_i(\overline{t}) - \overline{x}} &= \sum_{\substack{i=M_\rho \\ i\neq i_0 \\ |x_i(\overline{t})-\overline{x}|\leqslant\rho-R}}^{N_\rho} \frac{\varepsilon}{x_i(\overline{t}) - \overline{x}} + \sum_{\substack{i=M_\rho \\ |x_i(\overline{t})-\overline{x}|>\rho-R}}^{N_\rho} \frac{\varepsilon}{x_i(\overline{t}) - \overline{x}} \\
&= \sum_{\substack{i\neq i_0 \\ |x_i(\overline{t})-\overline{x}|\leqslant\rho-R}} \frac{\varepsilon}{x_i(\overline{t}) - \overline{x}} + \sum_{\substack{i=M_\rho \\ |x_i(\overline{t})-\overline{x}|>\rho-R}}^{N_\rho} \frac{\varepsilon}{x_i(\overline{t}) - \overline{x}}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\mathcal{I}_1^{1,\rho}[\eta(\overline{t},\cdot)](\overline{x}) - \mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0) = o_R(1), \\
&\mathcal{I}_1^{1,\rho}[\eta(\overline{t},\cdot)](\overline{x}) - \mathcal{I}_1^{1,\rho-R}[\eta(\overline{t},\cdot)](\overline{x}) = o_R(1).
\end{aligned} \tag{5.33}$$

By (5.33) and Proposition 4.7, we have

$$\sum_{\substack{i\neq i_0\\|x_i(\bar t)-\bar x|\leqslant\rho-R}}\frac{\varepsilon}{x_i(\bar t)-\bar x}=\mathcal{I}_1^{1,\rho-R}[\eta(\bar t,\cdot)](\bar x)+o_\varepsilon(1)+o_\rho(1)+O(\gamma)$$
$$=\mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0)+o_R(1)+o_\varepsilon(1)+o_\rho(1)+O(\gamma).$$

Next, let n be the number of points $x_i(\bar t)$, $i=M_\rho,\dots,N_\rho$, such that $|x_i(\bar t)-\bar x|>\rho-R$. Since $|\bar x-x_0|<R$ and by Lemma 5.1 $x_i(\bar t)\in(x_0-(\rho+3R),x_0+\rho+3R)$, such points must belong to the set $\{\rho-2R<|x-x_0|<\rho+3R\}$ whose length is $10R$. Therefore, by (4.3), $n\leqslant CR/\varepsilon$. Then,

$$\left|\sum_{\substack{i=M_\rho\\|x_i(\bar t)-\bar x|>\rho-R}}^{N_\rho}\frac{\varepsilon}{x_i(\bar t)-\bar x}\right|\leqslant\sum_{\substack{i=M_\rho\\|x_i(\bar t)-\bar x|>\rho-R}}^{N_\rho}\frac{\varepsilon}{\rho-R}=\frac{\varepsilon}{\rho-R}n\leqslant\frac{\varepsilon}{\rho-R}\cdot\frac{CR}{\varepsilon}=O\left(\frac{R}{\rho}\right).$$

We conclude that

$$\Gamma_1=o_\varepsilon(1)+o_R(1)+o_\rho(1)+O(\gamma)+O\left(\frac{R}{\rho}\right).$$

Since in addition, by (4.4), $\Gamma_2=O(\delta)$, we have proven that

$$\sum_{\substack{i=M_\rho\\i\neq i_0}}^{N_\rho}\frac{\tilde\phi(z_i)}{\delta}-\frac{1}{\alpha}\mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0)=o_\varepsilon(1)+o_R(1)+o_\rho(1)+O(\gamma)+O\left(\frac{R}{\rho}\right). \tag{5.34}$$

Notice that $O(\gamma)$ is not necessarily small. Next, we consider two cases.

Case 1: $|\gamma|<\delta$. Then, $O(\gamma)=o_\varepsilon(1)$ and

$$|W''(\tilde\phi(z_{i_0}))-W''(0)|\left(\sum_{\substack{i=M_\rho\\i\neq i_0}}^{N_\rho}\frac{\tilde\phi(z_i)}{\delta}-\frac{1}{\alpha}\mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0)\right)$$
$$\leqslant 2\|W''\|_\infty\left(o_\varepsilon(1)+o_R(1)+o_\rho(1)+O\left(\frac{R}{\rho}\right)\right),$$

and (5.31) is proven.

Case 2: $|\gamma|\geqslant\delta$. By (3.1), and using the fact that $z_{i_0}=\gamma/\delta$, we have

$$\left|\tilde\phi(z_{i_0})+\frac{\delta}{\alpha\pi\gamma}\right|\leqslant K_1\frac{\delta^2}{\gamma^2},$$

which implies that

$$|W''(\tilde\phi(z_{i_0}))-W''(0)|\leqslant|W'''(0)|\,|\tilde\phi(z_{i_0})|+O(\tilde\phi(z_{i_0}))^2\leqslant C\left(\frac{\delta}{|\gamma|}+\frac{\delta^2}{\gamma^2}\right)\leqslant C\frac{\delta}{|\gamma|}.$$

Hence, it follows that

$$|W''(\tilde\phi(z_{i_0}))-W''(0)|\left(\sum_{\substack{i=M_\rho\\i\neq i_0}}^{N_\rho}\frac{\tilde\phi(z_i)}{\delta}-\frac{1}{\alpha}\mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0)\right)$$
$$\leqslant C\frac{\delta}{|\gamma|}\left(o_\varepsilon(1)+o_R(1)+o_\rho(1)+O(\gamma)+O\left(\frac{R}{\rho}\right)\right)$$
$$\leqslant o_\varepsilon(1)+o_R(1)+o_\rho(1)+O\left(\frac{R}{\rho}\right).$$

This completes the proof of (5.31).

Let us now turn to the proof of (5.32). As before, by (3.1), for $i = M_\varepsilon, \dots, M_\rho - 1$ and $i = N_\rho + 1, \dots, N_\varepsilon$,

$$\left|\tilde{\phi}(z_i^0) + \frac{\varepsilon\delta}{\alpha\pi(\overline{x} - x_i^0)}\right| \leqslant \frac{K_1\varepsilon^2\delta^2}{(\overline{x} - x_i^0)^2}.$$

Hence, we obtain

$$\begin{aligned}\frac{1}{\delta}\sum_{i=M_\varepsilon}^{M_\rho-1}\tilde{\phi}(z_i^0) + \frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon}\tilde{\phi}(z_i^0) \leqslant \frac{1}{\alpha\pi}\sum_{i=M_\varepsilon}^{M_\rho-1}\frac{\varepsilon}{x_i^0 - \overline{x}} + K_1\sum_{i=M_\varepsilon}^{M_\rho-1}\frac{\varepsilon^2\delta}{(x_i^0 - \overline{x})^2} \\ + \frac{1}{\alpha\pi}\sum_{i=N_\rho+1}^{N_\varepsilon}\frac{\varepsilon}{x_i^0 - \overline{x}} + K_1\sum_{i=N_\rho+1}^{N_\varepsilon}\frac{\varepsilon^2\delta}{(x_i^0 - \overline{x})^2},\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}\frac{1}{\delta}\sum_{i=M_\varepsilon}^{M_\rho-1}\tilde{\phi}(z_i^0) + \frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon}\tilde{\phi}(z_i^0) \geqslant \frac{1}{\alpha\pi}\sum_{i=M_\varepsilon}^{M_\rho-1}\frac{\varepsilon}{x_i^0 - \overline{x}} - K_1\sum_{i=M_\varepsilon}^{M_\rho-1}\frac{\varepsilon^2\delta}{(x_i^0 - \overline{x})^2} \\ + \frac{1}{\alpha\pi}\sum_{i=N_\rho+1}^{N_\varepsilon}\frac{\varepsilon}{x_i^0 - \overline{x}} - K_1\sum_{i=N_\rho+1}^{N_\varepsilon}\frac{\varepsilon^2\delta}{(x_i^0 - \overline{x})^2}.\end{aligned} \tag{5.36}$$

By (4.4), we have

$$K_1\sum_{i=M_\varepsilon}^{M_\rho-1}\frac{\varepsilon^2\delta}{(x_i^0 - \overline{x})^2} + K_1\sum_{i=N_\rho+1}^{N_\varepsilon}\frac{\varepsilon^2\delta}{(x_i^0 - \overline{x})^2} = O(\delta). \tag{5.37}$$

Moreover, since $|\overline{x} - x_0| < R$ and $|x_i^0 - x_0| > \rho + R$, for $i = M_\varepsilon, \dots, M_\rho - 1$ and $i = N_\rho + 1, \dots, N_\varepsilon$, it follows that $|x_i^0 - \overline{x}| > \rho$, thus,

$$\frac{1}{\pi}\sum_{i=M_\varepsilon}^{M_\rho-1}\frac{\varepsilon}{x_i^0 - \overline{x}} + \frac{1}{\pi}\sum_{i=N_\rho+1}^{N_\varepsilon}\frac{\varepsilon}{x_i^0 - \overline{x}} = \frac{1}{\pi}\sum_{\substack{i=M_\varepsilon \\ |x_i^0 - \overline{x}| > \rho}}^{N_\varepsilon}\frac{\varepsilon}{x_i^0 - \overline{x}} - \frac{1}{\pi}\sum_{\substack{i=M_\rho \\ |x_i^0 - \overline{x}| \geqslant \rho}}^{N_\rho}\frac{\varepsilon}{x_i^0 - \overline{x}}.$$

By Lemma 4.3,

$$\frac{1}{\pi}\sum_{\substack{i=M_\varepsilon \\ |x_i^0 - \overline{x}| > \rho}}^{N_\varepsilon}\frac{\varepsilon}{x_i^0 - \overline{x}} = \mathcal{I}_1^{2,\rho}[\eta(t_0, \cdot)](x_0) + o_\varepsilon(1) + o_\rho(1),$$

and as before,

$$\left|\frac{1}{\pi}\sum_{\substack{i=M_\rho \\ |x_i^0 - \overline{x}| \geqslant \rho}}^{N_\rho}\frac{\varepsilon}{x_i^0 - \overline{x}}\right| \leqslant C\frac{R}{\rho}.$$

Therefore,

$$\frac{1}{\pi}\sum_{i=M_\varepsilon}^{M_\rho-1}\frac{\varepsilon}{x_i^0 - \overline{x}} + \frac{1}{\pi}\sum_{i=N_\rho+1}^{N_\varepsilon}\frac{\varepsilon}{x_i^0 - \overline{x}} = \mathcal{I}_1^{2,\rho}[\eta(t_0, \cdot)](x_0) + o_\varepsilon(1) + o_\rho(1) + O\left(\frac{R}{\rho}\right). \tag{5.38}$$

Combining (5.35), (5.36), (5.37) and (5.38), yields (5.32). This concludes the proof of the lemma. □

Next, we have a control over the remaining errors.

Lemma 5.9. *For $i \geqslant 1$, the error E_i satisfies*

$$E_i = O(\delta).$$

We postpone the proof of Lemma 5.9 to Section 7.

Let us finally complete the proof of (5.21). By (5.29), Lemmas 5.8, 5.9 and recalling the definition (5.28) of E_0, we obtain

$$\Lambda(\bar{t}, \bar{x}) \geqslant L_1 + o_\varepsilon(1) + o_R(1) + o_\rho(1) + \frac{o_R(1)}{\rho}.$$

We choose $R \ll \rho \ll 1$ and ε_0 so small that for any $\varepsilon < \varepsilon_0$,

$$\left| o_\varepsilon(1) + o_R(1) + o_\rho(1) + \frac{o_R(1)}{\rho} \right| < \frac{L_1}{2}.$$

Then,

$$\Lambda(\bar{t}, \bar{x}) > \frac{L_1}{2} > 0.$$

This completes the proof of (5.21).

6. Comparison between u^+ and u^-: proof of (5.1)

Let us consider the approximation of the initial datum $u_0 \in C^{1,1}(\mathbb{R})$, given by Proposition 4.12:

$$\sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_{0,i}}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon, \tag{6.1}$$

where

$$\begin{gathered} x_{0,i} := \inf\{x \in \mathbb{R} \mid u_0(x) = \varepsilon i\} \qquad i = M_\varepsilon, \dots, N_\varepsilon, \\ M_\varepsilon := \left\lceil \frac{\inf_{\mathbb{R}} u_0 + \varepsilon}{\varepsilon} \right\rceil \quad \text{and} \quad N_\varepsilon := \left\lfloor \frac{\sup_{\mathbb{R}} u_0 - \varepsilon}{\varepsilon} \right\rfloor. \end{gathered} \tag{6.2}$$

Then, for all $x \in \mathbb{R}$,

$$\left| \sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_{0,i}}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - u_0(x) \right| \leqslant o_\varepsilon(1). \tag{6.3}$$

Let us first show the following asymptotic behavior of u^+ and u^-.

Lemma 6.1. *For all $t > 0$,*

$$\lim_{x \to -\infty} u^-(t, x) = \lim_{x \to -\infty} u^+(t, x) = \inf_{\mathbb{R}} u_0, \tag{6.4}$$

and

$$\lim_{x \to +\infty} u^-(t, x) = \lim_{x \to +\infty} u^+(t, x) = \sup_{\mathbb{R}} u_0. \tag{6.5}$$

Moreover, for all $x \in \mathbb{R}$,

$$u^+(0, x) = u^-(0, x) = u_0(x). \tag{6.6}$$

Proof. To prove the asymptotic behavior at infinity of u^+ and u^-, we will construct sub and supersolutions of (1.1). Let $x_i(t)$, $i = M_\varepsilon, \dots, N_\varepsilon$ be the solutions of

$$\begin{cases} \dot{x}_i(t) = -c_0 L, & t > 0 \\ x_i(0) = x_{0,i}, \end{cases}$$

with $L > 0$ to be chosen and $x_{0,i}$, M_ε, N_ε defined by (6.2), that is $x_i(t) = x_{0,i} - c_0 L t$. Consider the function

$$h_0^\varepsilon(t,x) := \sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon \left(\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}\right) + \delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}\right)\right) + \frac{\varepsilon\delta L}{\alpha} + \varepsilon M_\varepsilon + \varepsilon\left\lceil \frac{o_\varepsilon(1)}{\varepsilon}\right\rceil,$$

where ϕ and ψ are respectively solution of (1.11) and (3.3). By the fact that

$$\sum_{i=M_\varepsilon}^{N_\varepsilon} \left| \varepsilon\delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}\right)\right| \leqslant \varepsilon(N_\varepsilon - M_\varepsilon + 1)\delta\|\psi\|_\infty \leqslant (\sup_{\mathbb{R}} u_0 - \inf_{\mathbb{R}} u_0 + \varepsilon)\delta\|\psi\|_\infty, \tag{6.7}$$

and (6.3), we can choose $o_\varepsilon(1)$ such that

$$u_0(x) \leqslant h_0^\varepsilon(0,x). \tag{6.8}$$

We are going to show that for $L > 0$ large enough, h^ε is supersolution of (1.1). Fix $(\bar{t},\bar{x}) \in (0,+\infty)\times\mathbb{R}$. Let $x_{i_0}(\bar{t})$ be the closest point to $\bar{x}$ and let us denote $z_i := (\bar{x} - x_i(\bar{t}))/(\varepsilon\delta)$. As in the proof of Lemma 5.3, we compute

$$\begin{aligned}
\Lambda(\bar{t},\bar{x}) &:= \delta\partial_t h_0^\varepsilon(\bar{t},\bar{x}) - \mathcal{I}_1[h_0^\varepsilon(\bar{t},\cdot)](\bar{x}) + \frac{1}{\delta}W'\left(\frac{h_0^\varepsilon(\bar{t},\bar{x})}{\varepsilon}\right)\\
&= \sum_{i=M_\varepsilon}^{N_\varepsilon} [c_0 L\phi'(z_i) + \delta c_0 L\psi'(z_i)] - \sum_{i=M_\varepsilon}^{N_\varepsilon} \frac{1}{\delta}\mathcal{I}_1[\phi](z_i) - \sum_{i=M_\varepsilon}^{N_\varepsilon} \mathcal{I}_1[\psi](z_i)\\
&\quad + \frac{1}{\delta}W'\left(\sum_{i=M_\varepsilon}^{N_\varepsilon}[\phi(z_i) + \delta\psi(z_i)] + \frac{\delta L}{\alpha}\right).
\end{aligned}$$

By (1.11) and making a Taylor expansion of W' around $\phi(z_{i_0})$, we get

$$\begin{aligned}
\Lambda(\bar{t},\bar{x}) &= c_0 L\phi'(z_{i_0}) - \mathcal{I}_1[\psi](z_{i_0}) + W''(\tilde{\phi}(z_{i_0}))\psi(z_{i_0})\\
&\quad - \frac{1}{\delta}W'(\phi(z_{i_0})) - \frac{1}{\delta}\sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon} W'(\tilde{\phi}(z_i)) - \sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon} \mathcal{I}_1[\psi](z_i)\\
&\quad + \frac{1}{\delta}W'(\phi(z_{i_0})) + \frac{1}{\delta}W''(\phi(z_{i_0}))\left(\sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon}[\tilde{\phi}(z_i) + \delta\psi(z_i)] + \frac{\delta L}{\alpha}\right)\\
&\quad + \frac{1}{\delta}O\left(\sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon}[\tilde{\phi}(z_i) + \delta\psi(z_i)] + \frac{\delta L}{\alpha}\right)^2 + c_0 L\sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon}[\phi'(z_i) + \delta\psi'(z_i)] + \delta\psi'(z_{i_0}),
\end{aligned}$$

where $\tilde{\phi}(z) = \phi(z) - H(z)$ with H the Heaviside function. By (3.3) and making a Taylor expansion of W' around 0, we obtain

$$\begin{aligned}
\Lambda(\bar{t},\bar{x}) &= -\frac{L}{\alpha}(W''(\phi(z_{i_0})) - W''(0))\\
&\quad + (W''(\phi(z_{i_0})) - W''(0))\sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon}\frac{\tilde{\phi}(z_i)}{\delta} + W'(\phi(z_{i_0}))\frac{L}{\alpha}\\
&\quad + \frac{1}{\delta}O\left(\sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon}[\tilde{\phi}(z_i) + \delta\psi(z_i)] + \frac{\delta L}{\alpha}\right)^2 + \frac{1}{\delta}\sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon} O(\tilde{\phi}(z_i))^2
\end{aligned}$$

$$+ W''(\phi(z_{i_0})) \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \psi(z_i) - \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \mathcal{I}_1[\psi](z_i)$$

$$+ c_0 L \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} [\phi'(z_i) + \delta\psi'(z_i)] + \delta\psi'(z_{i_0}).$$

Thus, recalling that $\alpha = W''(0)$,

$$\begin{aligned}
\Lambda(\bar{t}, \bar{x}) &= (W''(\phi(z_{i_0})) - W''(0)) \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \frac{\tilde{\phi}(z_i)}{\delta} + L \\
&+ \frac{1}{\delta} O\left(\sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} [\tilde{\phi}(z_i) + \delta\psi(z_i)] + \frac{\delta L}{\alpha} \right)^2 + \frac{1}{\delta} \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} O(\tilde{\phi}(z_i))^2 \\
&+ W''(\phi(z_{i_0})) \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \psi(z_i) - \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \mathcal{I}_1[\psi](z_i) \\
&+ c_0 L \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} [\phi'(z_i) + \delta\psi'(z_i)] + \delta\psi'(z_{i_0}).
\end{aligned}$$

Notice that if $x_{i_0}(\bar{t})$ is the closest point to $\bar{x}$, then x_{0,i_0} is the closest point to $\bar{x} + c_0 L\bar{t}$ and $\bar{x} - x_i(\bar{t}) = (\bar{x} + c_0 L\bar{t}) - x_{0,i}$. Then, by (3.1), Lemma 4.13 applied to $u_0 \in C^{1,1}(\mathbb{R})$, and (4.4),

$$\left| \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \frac{\tilde{\phi}(z_i)}{\delta} \right| \leqslant \frac{1}{\alpha\pi} \left| \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \frac{\varepsilon}{x_{0,i} - (\bar{x} + c_0 L\bar{t})} \right| + K_1 \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \frac{\varepsilon^2}{(x_{0,i} - (\bar{x} + c_0 L\bar{t}))^2} \leqslant C.$$

Moreover, as in the proof of Lemma 5.3,

$$\begin{aligned}
&\frac{1}{\delta} O\left(\sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} [\tilde{\phi}(z_i) + \delta\psi(z_i)] + \frac{\delta L}{\alpha} \right)^2 + \frac{1}{\delta} \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} O(\tilde{\phi}(z_i))^2 \\
&W''(\phi(z_{i_0})) \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \psi(z_i) - \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} \mathcal{I}_1[\psi](z_i) \\
&+ c_0 L \sum_{\substack{i=M_\varepsilon \\ i\neq i_0}}^{N_\varepsilon} [\phi'(z_i) + \delta\psi'(z_i)] + \delta\psi'(z_{i_0}) \\
&= O(\delta).
\end{aligned}$$

We conclude that

$$\Lambda(\bar{t}, \bar{x}) \geqslant -C + L \geqslant 0,$$

choosing $L > 0$ large enough (but independent of ε and $(\bar{t}, \bar{x})$). Since in addition (6.8) holds true, by the comparison principle, for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$u^\varepsilon(t, x) \leqslant h_0^\varepsilon(t, x). \tag{6.9}$$

We will show that the previous inequality implies that for any $\tau > 0$ there exists $\tilde{K} = \tilde{K}(\tau, T)$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$ with $x < \tilde{K}$,

$$u^\varepsilon(t, x) \leqslant \inf_{\mathbb{R}} u_0 + \tau + o_\varepsilon(1). \tag{6.10}$$

Fix $\tau > 0$. Since $\lim_{x\to-\infty} u_0(x) = \inf_{\mathbb{R}} u_0$, there exists $K \in \mathbb{R}$ such that for all $x < K$,

$$u_0(x) \leqslant \inf_{\mathbb{R}} u_0 + \tau.$$

Given $T > 0$, let $\tilde{K} := K - c_0 LT$. Then, by (6.9), (6.3) and (6.7), for all $(t, x) \in [0, T] \times \mathbb{R}$ such that $x < \tilde{K}$,

$$\begin{aligned} u^\varepsilon(t, x) &\leqslant h_0^\varepsilon(t, x) \\ &= \sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x + c_0 Lt - x_{i,0}}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\ &\leqslant u_0(x + c_0 Lt) + o_\varepsilon(1) \\ &\leqslant \inf_{\mathbb{R}} u_0 + \tau + o_\varepsilon(1), \end{aligned}$$

which proves (6.10). On the other hand, by the comparison principle, $u^\varepsilon \geqslant \varepsilon\lfloor \inf_{\mathbb{R}} u_0/\varepsilon \rfloor$. Thus, (6.4) follows. Similarly one can prove that the limits (6.5) hold true.

Finally, to prove (6.6), take a sequence $(t_\varepsilon, x_\varepsilon) \to (0, x)$ as $\varepsilon \to 0$. Then by (6.9), (6.3) and (6.7),

$$u^\varepsilon(t_\varepsilon, x_\varepsilon) \leqslant u_0(x_\varepsilon + c_0 Lt_\varepsilon) + o_1(\varepsilon)$$

which implies that $u^+(0, x) \leqslant u_0(x)$. On the other hand, $u^+(0, x) \geqslant \limsup_{\varepsilon\to 0} u^\varepsilon(0, x) = u_0(x)$. We infer that $u^+(0, x) = u_0(x)$. Similarly, $u^-(0, x) = u_0(x)$. This proves (6.6) and concludes the proof of the lemma. □

Now, let f^ε be the smooth and positive global solution of Eq. (1.7) with initial datum

$$f_0^\varepsilon(x) = \frac{1}{\delta}\sum_{i=M_\varepsilon}^{N_\varepsilon} \phi'\left(\frac{x - x_{0,i}}{\varepsilon\delta}\right) > 0$$

provided by Theorem 3.9. Notice that by (3.2),

$$f_0^\varepsilon \in L^p(\mathbb{R}) \quad \text{for all } p \in [1, \infty].$$

Integrating equation (1.7) from a to b yields

$$\partial_t \int_a^b f^\varepsilon(t, y)\, dy = c_0 f^\varepsilon(t, b)\, \mathcal{H}[f^\varepsilon(t, \cdot)](b) - c_0 f^\varepsilon(t, a)\, \mathcal{H}[f^\varepsilon(t, \cdot)](a). \tag{6.11}$$

Sending $a \to -\infty$ and $b \to +\infty$ and using that $f^\varepsilon > 0$ is vanishing at infinity and $\mathcal{H}[f^\varepsilon(t, \cdot)] \in L^\infty(\mathbb{R})$, we see that $f^\varepsilon(t, \cdot) \in L^1(\mathbb{R})$ for all $t \geqslant 0$ and

$$\|f^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} = \|f_0^\varepsilon\|_{L^1(\mathbb{R})}.$$

Following [2], one can actually show that for all $p \in [1, \infty)$,

$$\|f^\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \leqslant \|f_0^\varepsilon\|_{L^p(\mathbb{R})}, \quad \|f^\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \leqslant C_p \|f_0^\varepsilon\|_{L^1(\mathbb{R})}^{\frac{p+1}{2p}} t^{-\frac{p-1}{2p}}.$$

By taking $b = x$ and $a = -\infty$ in (6.11), we see that the function

$$F^\varepsilon(t, x) = \int_{-\infty}^x f^\varepsilon(t, y)\, dy,$$

is a solution of (1.5) with initial datum

$$\sum_{i=M_\varepsilon}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_{0,i}}{\varepsilon\delta}\right).$$

Note that for all $t>0$,

$$\lim_{x\to-\infty} F^\varepsilon(t,x)=0, \tag{6.12}$$

and by using that $\lim_{x\to+\infty}\phi(x)=1$ and $\lim_{x\to-\infty}\phi(x)=0$,

$$\lim_{x\to+\infty} F^\varepsilon(t,x)=\|f^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})}=\|f_0^\varepsilon\|_{L^1(\mathbb{R})}=\sum_{i=M_\varepsilon}^{N_\varepsilon}\varepsilon=\varepsilon(N_\varepsilon-M_\varepsilon+1). \tag{6.13}$$

Finally, $w^\varepsilon(t,x)=F^\varepsilon(t,x)+\varepsilon M_\varepsilon$ is the unique (and smooth) viscosity solution of (1.5) with initial datum (6.1). Moreover $\partial_x w^\varepsilon(t,x)=f^\varepsilon(t,x)>0$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$. By (6.12) and (6.13), we see that

$$\lim_{x\to-\infty} w^\varepsilon(t,x)=\varepsilon M_\varepsilon \quad\text{and}\quad \lim_{x\to+\infty} w^\varepsilon(t,x)=\varepsilon(N_\varepsilon+1).$$

In particular, by Lemma 6.1 and the fact that $0\leqslant\varepsilon M_\varepsilon-\inf_{\mathbb{R}}u_0\leqslant 2\varepsilon$ and $0\leqslant\sup_{\mathbb{R}}u_0-\varepsilon N_\varepsilon\leqslant 2\varepsilon$, we have that

$$\lim_{x\to-\infty}(u^+(t,x)-w^\varepsilon(t,x))\leqslant 0 \quad\text{and}\quad \lim_{x\to+\infty}(u^+(t,x)-w^\varepsilon(t,x))\leqslant\varepsilon. \tag{6.14}$$

Moreover, by (6.3) and (6.6),

$$u^+(0,x)-w^\varepsilon(0,x)=u_0(x)-\sum_{i=M_\varepsilon}^{N_\varepsilon}\varepsilon\phi\left(\frac{x-x_{0,i}}{\varepsilon\delta}\right)-\varepsilon M_\varepsilon\leqslant o_1(\varepsilon). \tag{6.15}$$

We next show that

$$u^+(t,x)-w^\varepsilon(t,x)\leqslant o_\varepsilon(1)\quad\text{for all }(t,x)\in(0,+\infty)\times\mathbb{R}, \tag{6.16}$$

for $o_\varepsilon(1)\geqslant\varepsilon$ for which such that (6.15) holds true.

Suppose by contradiction that for some $T>0$,

$$\sup_{(t,x)\in(0,T)\times\mathbb{R}} u^+(t,x)-w^\varepsilon(t,x)>o_\varepsilon(1). \tag{6.17}$$

Then, for $\vartheta>0$ small enough the supremum of the function

$$u^+(t,x)-w^\varepsilon(t,x)-\frac{\vartheta}{T-t}-o_\varepsilon(1)$$

is positive and by (6.14) and (6.15) attended at some point $(\bar{t},\bar{x})\in(0,T)\times\mathbb{R}$. Then, $\eta(t,x)=w^\varepsilon(t,x)+\frac{\vartheta}{T-t}+o_\varepsilon(1)$ is a test function for u^+ as subsolution with $\partial_x\eta(\bar{t},\bar{x})=\partial_x w^\varepsilon(\bar{t},\bar{x})>0$, and by (5.4),

$$\partial_t w^\varepsilon(\bar{t},\bar{x})<\frac{\vartheta}{(T-\bar{t})^2}+\partial_t w^\varepsilon(\bar{t},\bar{x})\leqslant c_0\partial_x w^\varepsilon(\bar{t},\bar{x})\mathcal{I}_1[w^\varepsilon(\bar{t},\cdot)](\bar{x}).$$

On the other hand, since w^ε is a smooth solution of (1.5) we have

$$\partial_t w^\varepsilon(\bar{t},\bar{x})=c_0\partial_x w^\varepsilon(\bar{t},\bar{x})\mathcal{I}_1[w^\varepsilon(\bar{t},\cdot)](\bar{x}).$$

We have reached a contradiction. This proves (6.16). Moreover, by (6.3) and the comparison principle, $|w^\varepsilon-\overline{u}|\leqslant o_\varepsilon(1)$. Therefore, passing to the limit as $\varepsilon\to 0$, we finally obtain $u^+\leqslant\overline{u}$. Similarly we can prove that $\overline{u}\leqslant u^-$. This completes the proof of (5.1).

Remark 6.2. Notice that the viscosity solution $\overline{u}=u^+=u^-$ of (1.5) satisfies

$$\lim_{x\to-\infty}\overline{u}(t,x)=\inf_{\mathbb{R}}u_0 \quad\text{and}\quad \lim_{x\to+\infty}\overline{u}(t,x)=\sup_{\mathbb{R}}u_0.$$

7. Proofs of Lemmas 5.1, 5.4, 5.6 and 5.9

7.1. Proof of Lemma 5.1

Recall that by (5.9)

$$x^0_{N_\rho} - x_0 > \rho + R, \tag{7.1}$$

and by (5.9), (5.6) and (4.5),

$$x^0_{N_\rho} - x_0 = (x^0_{N_\rho} - x^0_{N_\rho - 1}) + (x^0_{N_\rho - 1} - x_0) \leqslant C\varepsilon + \rho + R \leqslant \rho + 2R < 2\rho, \tag{7.2}$$

for ε small enough. Similarly,

$$-(\rho + 2R) < x^0_{M_\rho} - x_0 < -(\rho + R).$$

In particular, for all $i = M_\rho, \dots, N_\rho$, $(t_0, x^0_i) \in Q_{2\rho,2\rho}(t_0, x_0)$. Then by the regularity of η and (5.6), the ODE

$$\dot{y}_i(t) = -\frac{\partial_t \eta(t, y_i(t))}{\partial_x \eta(t, y_i(t))} \tag{7.3}$$

has a unique local solution $y_i(t)$ such that $y_i(t_0) = x^0_i$ which is of class C^1 as long as $(t, y_i(t)) \in Q_{2\rho,2\rho}(t_0, x_0)$. Since in addition $\eta(t, y_i(t)) = \varepsilon i$ and η is strictly increasing in $Q_{2\rho,2\rho}(t_0, x_0)$, we must have $y_i = x_i$. Moreover, as long as $(t, x_i(t)) \in Q_{2\rho,2\rho}(t_0, x_0)$, by (5.6),

$$|\dot{x}_i(t)| \leqslant \frac{2\|\partial_t \eta\|_\infty}{\partial_x \eta(t_0, x_0)} = B_0^{-1}. \tag{7.4}$$

Next, let $-\infty \leqslant t^* \leqslant +\infty$ be the first time such that

$$|x_{N_\rho}(t^*) - x^0_{N_\rho}| = R,$$

and

$$\tau := \min\{2\rho, |t^* - t_0|\}.$$

Then, for t such that $|t - t_0| < \tau$,

$$|x_{N_\rho}(t) - x^0_{N_\rho}| < R \tag{7.5}$$

and by (7.2),

$$x_{N_\rho}(t) - x_0 \leqslant |x_{N_\rho}(t) - x^0_{N_\rho}| + x^0_{N_\rho} - x_0 \leqslant \rho + 3R, \tag{7.6}$$

In particular, $(t, x_{N_\rho}(t)) \in Q_{2\rho,2\rho}(t_0, x_0)$ and (7.4) holds true. Therefore, if $|t^* - t_0| < 2\rho$,

$$R = |x_{N_\rho}(t^*) - x_{N_\rho}(t_0)| = \left| \int_{t_0}^{t^*} \dot{x}_{N_\rho}(t) dt \right| \leqslant \frac{|t^* - t_0|}{B_0},$$

which implies that $|t^* - t_0| \geqslant B_0 R$. Hence, for t such that $|t - t_0| < B_0 R$, (7.6) holds true which proves the upper bound in (5.14). For the lower bound, for t such that $|t - t_0| < B_0 R$, by (7.5) and (7.1), we have

$$x_{N_\rho}(t) - x_0 \geqslant x^0_{N_\rho} - x_0 - |x_{N_\rho}(t) - x^0_{N_\rho}| > \rho + R - R = \rho.$$

This completes the proof of (5.14). Similarly, one can prove (5.15). By the monotonicity of η, for $i = M_\rho, \dots, N_\rho$, $x_{M_\rho}(t) < x_i(t) < x_{N_\rho}(t)$ and by (5.14) and (5.15), for $|t - t_0| < B_0 R$, $|x_i(t) - x_0| \leqslant \rho + 3R \leqslant 2\rho$. Therefore, $x_i \in C^1(t_0 - B_0 R, t_0 + B_0 R)$ and (7.4) holds true. This proves (5.13) and concludes the proof of the lemma.

7.2. *Proof of Lemma 5.4*

We divide the proof of the lemma into three claims.

Claim 1. $\left|\sum_{i=M_\rho}^{N_\rho} \varepsilon\phi\left(\frac{x-x_i(t)}{\varepsilon\delta}\right) + \varepsilon M_\rho - \eta(t,x)\right| \leqslant o_\varepsilon(1)\left(1+\frac{\delta}{R}\right)$.

Proof of Claim 1. By Lemma 5.1, if $(t,x) \in Q_{B_0R,\rho-R}(t_0,x_0)$, then $x \in (x_{M_\rho}(t)+R, x_{N_\rho}(t)-R)$. Therefore, Claim 1 immediately follows from Lemma 4.10.

Claim 2. $\left|\sum_{i=M_\varepsilon}^{M_\rho-1} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - \varepsilon M_\rho\right| \leqslant C\varepsilon\left(1+\frac{\delta}{R}\right)$.

Proof of Claim 2. By (5.8), if $(t,x) \in Q_{B_0R,\rho-R}(t_0,x_0)$, then $x > x^0_{M_\rho-1}+R$. Claim 2 then follows from (4.54) and the fact that $\varepsilon M_\rho = \eta(t_0, x^0_{M_\rho-1}) + \varepsilon$.

Claim 3. $0 \leqslant \sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}\right) \leqslant C\varepsilon\left(1+\frac{\delta}{R}\right)$

Proof of Claim 3. By (5.9), if $(t,x) \in Q_{B_0R,\rho-R}(t_0,x_0)$, then $x < x^0_{N_\rho+1} - R$. Claim 3 then immediately follows from (4.55).

Finally, the lemma is a consequence of Claims 1–3 and Lemma 5.5, by choosing ε so small that $\delta/R \leqslant 1$.

7.3. *Proof of Lemma 5.6*

We first consider the case

$$|x-x_0| > \rho + 4R.$$

Let us assume $x > x_0+\rho+4R$. Similarly one can prove the lemma for $x < x_0 - (\rho+4R)$.
We divide the proof into three claims.

Claim 1. $\left|\sum_{i=M_\rho}^{N_\rho} \varepsilon\phi\left(\frac{x-x_i(t)}{\varepsilon\delta}\right) + \varepsilon M_\rho - \varepsilon N_\rho\right| \leqslant C\varepsilon\left(1+\frac{\delta}{R}\right)$.

Proof of Claim 1. By Lemma 5.1, if $|t-t_0| < B_0R$ and $x > x_0+\rho+4R$, then $x > x_{N_\rho}(t)+R$. Therefore, Claim 1 immediately follows from (4.54) and the fact that $\varepsilon N_\rho = \eta(t, x_{N_\rho}(t))$.

Claim 2. $\left|\sum_{i=M_\varepsilon}^{M_\rho-1} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}\right) + \varepsilon M_\varepsilon - \varepsilon M_\rho\right| \leqslant C\varepsilon\left(1+\frac{\delta}{R}\right)$.

Proof of Claim 2. By (5.8), if $x > x_0+\rho+4R$, then $x > x^0_{M_\rho}+R$. Claim 2 then follows from (4.54) and the fact that $\varepsilon M_\rho = \eta(t_0, x^0_{M_\rho-1}) + \varepsilon$.

Claim 3. $\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}\right) + \varepsilon N_\varepsilon - \eta(t,x)\right| \leqslant o_\varepsilon(1)\left(1+\frac{\delta}{R}\right) + O(R)$.

Proof of Claim 3. By (5.9), if $x > x_0 + \rho + 4R$ and in addition $x < x^0_{N_\varepsilon} - R$, then $x \in (x^0_{N_\rho} + R, x^0_{N_\varepsilon} - R)$. Therefore by Lemma 4.10 and the fact that $|t - t_0| < B_0 R$,

$$\begin{aligned}\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) + \varepsilon N_\varepsilon - \eta(t,x)\right| &= \left|\sum_{i=N_\rho}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) + \varepsilon N_\varepsilon - \eta(t,x)\right| + o_\varepsilon(1)\\ &\leqslant \left|\sum_{i=N_\rho}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) + \varepsilon N_\varepsilon - \eta(t_0,x)\right|\\ &+ |\eta(t_0,x) - \eta(t,x)| + o_\varepsilon(1)\\ &\leqslant o_\varepsilon(1)\left(1 + \frac{\delta}{R}\right) + O(R),\end{aligned}$$

and the claim is proven for $x_0 + \rho + 4R < x < x^0_{N_\varepsilon} - R$.

Next, if $x > x^0_{N_\varepsilon} + R$, then by (4.55),

$$\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right)\right| \leqslant C\varepsilon\left(1 + \frac{\delta}{R}\right). \tag{7.7}$$

Moreover, since $\varepsilon N_\varepsilon \to \sup_{\mathbb{R}} \eta(t_0,\cdot)$ as $\varepsilon \to 0$, $|t - t_0| < B_0 R$ and η is non-decreasing,

$$\begin{aligned}\eta(t,x) = \eta(t_0,x) + O(R) \leqslant \sup_{\mathbb{R}} \eta(t_0,\cdot) + O(R) = \varepsilon N_\varepsilon + o_\varepsilon(1) + O(R),\\ \eta(t,x) \geqslant \eta(t, x^0_{N_\varepsilon} + R) = \eta(t_0, x^0_{N_\varepsilon}) + O(R) = \varepsilon N_\varepsilon + O(R).\end{aligned} \tag{7.8}$$

Estimates (7.7) and (7.8) imply Claim 3 for $x > x^0_{N_\varepsilon} + R$.

Finally, let us assume $x^0_{N_\varepsilon} - R \leqslant x \leqslant x^0_{N_\varepsilon} + R$. Then, by using the monotonicity of ϕ and that the claim holds true for $x = x^0_{N_\varepsilon} - 2R$ and $x = x^0_{N_\varepsilon} + 2R$, we get

$$\begin{aligned}&\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) + \varepsilon N_\varepsilon - \eta(t,x)\\ &\leqslant \sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x^0_{N_\varepsilon} + 2R - x_i^0}{\varepsilon\delta}\right) + \varepsilon N_\varepsilon - \eta(t, x^0_{N_\varepsilon} + 2R) + O(R)\\ &\leqslant o_\varepsilon(1)\left(1 + \frac{\delta}{R}\right) + O(R),\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) + \varepsilon N_\varepsilon - \eta(t,x)\\ &\geqslant \sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x^0_{N_\varepsilon} - 2R - x_i^0}{\varepsilon\delta}\right) + \varepsilon N_\varepsilon - \eta(t, x^0_{N_\varepsilon} - 2R) + O(R)\\ &\geqslant o_\varepsilon(1)\left(1 + \frac{\delta}{R}\right) + O(R).\end{aligned}$$

This concludes the proof of Claim 3.

The lemma for $|x - x_0| > \rho + 4R$ is then a consequence of Claims 1–3 and Lemma 5.5, by choosing ε so small that $\delta/R \leqslant 1$.

Next, let us consider the case

$$\rho - R \leqslant |x - x_0| \leqslant \rho + 4R.$$

Assume without loss of generality that $\rho - R \leqslant x - x_0 \leqslant \rho + 4R$. Then, by using Lemma 5.4 at the point $x_0 + \rho - 2R$, Lemma 5.5 and the monotonicity of ϕ, we get

$$\begin{aligned} h^\varepsilon(t,x) + \varepsilon M_\varepsilon &\geqslant \sum_{i=M_\rho}^{N_\rho} \varepsilon\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}\right) + \sum_{i=M_\varepsilon}^{M_\rho - 1} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) + \sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) - C\delta \\ &\geqslant \sum_{i=M_\rho}^{N_\rho} \varepsilon\phi\left(\frac{x_0 + \rho - 2R - x_i(t)}{\varepsilon\delta}\right) + \sum_{i=M_\varepsilon}^{M_\rho - 1} \varepsilon\phi\left(\frac{x_0 + \rho - 2R - x_i^0}{\varepsilon\delta}\right) \\ &+ \sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_0 + \rho - 2R - x_i^0}{\varepsilon\delta}\right) - C\delta \\ &\geqslant \eta(t, x_0 + \rho - 2R) + o_\varepsilon(1) \\ &\geqslant \eta(t,x) + o_\varepsilon(1) + O(R). \end{aligned}$$

Moreover, by using that the lemma holds true at the point $x_0 + \rho + 5R$, Lemma 5.5 and the monotonicity of ϕ, we get

$$\begin{aligned} h^\varepsilon(t,x) + \varepsilon M_\varepsilon &\leqslant \sum_{i=M_\rho}^{N_\rho} \varepsilon\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}\right) + \sum_{i=M_\varepsilon}^{M_\rho - 1} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) + \sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}\right) + C\delta \\ &\leqslant \sum_{i=M_\rho}^{N_\rho} \varepsilon\phi\left(\frac{x_0 + \rho + 5R - x_i(t)}{\varepsilon\delta}\right) + \sum_{i=M_\varepsilon}^{M_\rho - 1} \varepsilon\phi\left(\frac{x_0 + \rho + 5R - x_i^0}{\varepsilon\delta}\right) \\ &+ \sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_0 + \rho + 5R - x_i^0}{\varepsilon\delta}\right) + C\delta \\ &\leqslant \eta(t, x_0 + \rho + 5R) + o_\varepsilon(1) + O(R) \\ &\leqslant \eta(t,x) + o_\varepsilon(1) + O(R). \end{aligned}$$

This concludes the proof of the lemma.

7.4. Proof of Lemma 5.9

By (5.13), (3.2), (3.5) and (4.4), we have

$$\begin{aligned} |E_1| &\leqslant B_0^{-1}\left(\sum_{\substack{i=M_\rho \\ i\neq i_0}}^{N_\rho} \phi'(z_i) + \delta \sum_{\substack{i=M_\rho \\ i\neq i_0}}^{N_\rho} |\psi'(z_i)| + \delta|\psi'(z_{i_0})|\right) \\ &\leqslant B_0^{-1}\left((K_1 + \delta K_3)\delta^2 \sum_{\substack{i=M_\rho \\ i\neq i_0}}^{N_\rho} \frac{\varepsilon^2}{(x_i - \overline{x})^2} + \delta\|\psi'\|_\infty\right) \\ &\leqslant C\delta, \end{aligned}$$

which gives $E_1 = O(\delta)$.

Now, for E_2, using (3.1), (3.4), and (4.4) we get

$$
\begin{aligned}
|E_2| &\leqslant \frac{C}{\delta}\left(\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\tilde{\phi}(z_i)^2+\delta^2\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\psi(z_i)^2+\delta^2\psi(z_{i_0})^2+\sum_{i=M_\varepsilon}^{M_\rho-1}\tilde{\phi}(z_i^0)^2+\sum_{i=N_\rho+1}^{N_\varepsilon}\tilde{\phi}(z_i^0)^2+\frac{\delta^2L_1^2}{\alpha^2}\right)\\
&\leqslant \frac{C}{\delta}\left(\sum_{\substack{i=M_\varepsilon\\ i\neq i_0}}^{N_\varepsilon}\frac{\varepsilon^2\delta^2}{(x_i-\overline{x})^2}+\frac{\delta^2L_1^2}{\alpha^2}+\delta^2\|\psi\|_\infty^2\right)\\
&\leqslant C\delta,
\end{aligned}
$$

that is, $E_2=O(\delta)$.

Similarly, (3.1) and (4.4) imply that $E_3=O(\delta)$.

Finally, consider E_4 defined by (5.30). From (3.4), (4.4), Proposition 4.7 and the fact that $|\gamma|\leqslant 2/\partial_x\eta(t_0,x_0)$,

$$
\left|W''(\tilde{\phi}(z_{i_0}))\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\psi(z_i)\right|\leqslant C\delta\left|\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\frac{\varepsilon}{x_i-\overline{x}}\right|+C\delta^2\left|\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\frac{\varepsilon^2}{(x_i-\overline{x})^2}\right|\leqslant C\delta.
$$

Now, using (3.3) and a Taylor expansion, we get

$$
\begin{aligned}
\mathcal{I}_1[\psi](z_i) &= W''(\tilde{\phi}(z_i))\psi(z_i)+\frac{L_0+L_1}{\alpha}(W''(\tilde{\phi}(z_i))-W''(0))+c_0(L_0+L_1)\phi'(z_i)\\
&= W''(0)\psi(z_i)+\frac{L_0+L_1}{\alpha}W'''(0)\tilde{\phi}(z_i)+O(\tilde{\phi}(z_i))\psi(z_i)+O(\tilde{\phi})^2\\
&+\ c_0(L_0+L_1)\phi'(z_i).
\end{aligned}
$$

Hence, again from (3.1), (3.2), (3.4), (4.4) and Proposition 4.7, we obtain

$$
\sum_{\substack{i=M_\rho\\ i\neq i_0}}^{N_\rho}\mathcal{I}_1[\psi](z_i)=O(\delta).
$$

We infer that $E_4=O(\delta)$. This completes the proof of the lemma.

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