



Fractional elliptic equations in nondivergence form: definition, applications and Harnack inequality [☆]



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ABSTRACT

We define the fractional powers $L^s = (-a^{ij}(x)\partial_{ij})^s$, $0 < s < 1$, of nondivergence form elliptic operators $L = -a^{ij}(x)\partial_{ij}$ in bounded domains $\Omega \subset \mathbb{R}^n$, under minimal regularity assumptions on the coefficients $a^{ij}(x)$ and on the boundary $\partial\Omega$. We show that these fractional operators appear in several applications such as fractional Monge–Ampère equations, elasticity, and finance. The solution u to the nonlocal Poisson problem

$$\begin{cases} (-a^{ij}(x)\partial_{ij})^s u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is characterized by a local degenerate/singular extension problem. We develop the method of sliding paraboloids in the Monge–Ampère geometry and prove the interior Harnack inequality and Hölder estimates for solutions to the extension problem when the coefficients $a^{ij}(x)$ are bounded, measurable functions. This in turn implies the interior Harnack inequality and Hölder estimates for solutions u to the fractional problem.

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R É S U M É

On définit les puissances fractionnaires $L^s = (-a^{ij}(x)\partial_{ij})^s$, $0 < s < 1$, des opérateurs elliptiques sous forme non-divergence $L = -a^{ij}(x)\partial_{ij}$ dans des domaines bornés $\Omega \subset \mathbb{R}^n$, sous des hypothèses de régularité minimale sur les coefficients $a^{ij}(x)$ et à la frontière $\partial\Omega$. Nous montrons que ces opérateurs fractionnaires apparaissent dans plusieurs applications telles que équations fractionnaires de Monge–Ampère, élasticité et finance. La solution u au problème de Poisson non local

$$\begin{cases} (-a^{ij}(x)\partial_{ij})^s u = f & \text{dans } \Omega \\ u = 0 & \text{au } \partial\Omega \end{cases}$$

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se caractérise par un problème d'extension local dégénéré/singulier. Nous développons la méthode des paraboloides glissants dans la géométrie de Monge–Ampère et prouver l'inégalité intérieure de Harnack et estimations de Hölder pour les solutions à le problème d'extension lorsque les coefficients $a^{ij}(x)$ sont des fonctions mesurables bornées. Cela implique à son tour l'inégalité intérieure de Harnack et des estimations de Hölder pour les solutions u au problème fractionnaire.

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1. Introduction

In this paper, we inaugurate the analysis of equations driven by fractional powers of nondivergence form uniformly elliptic operators

$$L^s = (-a^{ij}(x)\partial_{ij})^s \quad \text{in } \Omega \quad \text{for } 0 < s < 1, \quad (1.1)$$

under minimal regularity assumptions on the coefficients $a^{ij}(x)$ and the boundary of the domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$. We show in Section 2 that fractional power operators as in (1.1) in such minimal regularity regime arise in applications to fractional Monge–Ampère equations, elasticity, and finance, among others.

The very first difficulty we need to overcome when considering (1.1) is that of giving a meaningful definition of the fractional power operator L^s when

$$L = -a^{ij}(x)\partial_{ij} \equiv - \sum_{i,j=1}^n a^{ij}(x)\partial_{ij} \quad x \in \Omega \quad (1.2)$$

is an elliptic operator in *nondivergence* form with *nonsmooth* coefficients. As in other well-known cases, this is not immediately obvious. For example, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ can be defined using the Fourier transform as $\widehat{(-\Delta)^s u} = |\xi|^{2s}\widehat{u}$. However, the nondivergence form operator (1.2) has nonsmooth coefficients in a bounded domain Ω , so the Fourier transform is not the most convenient tool. Indeed, (1.2) is not translation invariant and not defined in the whole space. If $-\Delta_D$ denotes the Laplacian in a bounded domain Ω subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$, then $(-\Delta_D)^s$ is naturally defined in a spectral way using the basis of eigenfunctions and the corresponding eigenvalues of $-\Delta_D$ in the Sobolev space $H_0^1(\Omega)$. In contrast, there is no natural Hilbert space structure for nondivergence form operators as in (1.2). The spectral method is also used to define fractional powers of *divergence form* elliptic operators $(-\partial_i(a^{ij}(x)\partial_j))^s$, see [6]. Nevertheless, our operator (1.2) has nonsmooth coefficients so it cannot be written in divergence form. We further remark that these definitions, though adequate from the operator theory point of view, do not immediately give explicit pointwise, nonlocal formulas.

Our idea to define (1.1) is to apply the *method of semigroups*. The main ingredient in this approach is the semigroup $\{e^{-tL}\}_{t \geq 0}$ generated by L . With this, we define

$$L^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tL} u(x) - u(x)) \frac{dt}{t^{1+s}} \quad (1.3)$$

for $0 < s < 1$, $x \in \Omega$, where Γ is the Gamma function. Using the semigroup, we can also give formulas for the solution u to the Poisson problem $L^s u = f$ as $u = L^{-s} f$ and for the solution U to local extension problems. Moreover, if L has a heat kernel, then one can derive explicit pointwise expressions for $L^s u(x)$, $L^{-s} f(x)$ and $U(x, z)$. These results are presented in Section 3. For details about the semigroup method applied to the fractional Laplacian in the whole space and to other different contexts, see [32] and the references therein.

We then consider the following fractional elliptic equation in nondivergence form:

$$\begin{cases} (-a^{ij}(x)\partial_{ij})^s u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain satisfying the uniform exterior cone condition. The coefficients $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ are symmetric $a^{ij}(x) = a^{ji}(x)$, $i, j = 1, \dots, n$, $a^{ij}(x) \in C(\Omega) \cap L^\infty(\Omega)$, and uniformly elliptic, meaning that there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda |\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega. \quad (1.5)$$

Under these conditions, the operator $L = -a^{ij}(x)\partial_{ij}$ generates a uniformly bounded C_0 -semigroup with exponential decay. Therefore, $L^s u$ is well-defined by means of (1.3). See Section 3 for these notions and the necessary notation.

The main regularity result for (1.4) in this paper is the following interior Harnack inequality and Hölder regularity estimate.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain that satisfies the uniform exterior cone condition, $a^{ij}(x) \in C^\alpha(\Omega) \cap L^\infty(\Omega)$, for some $0 < \alpha < 1$, are symmetric, satisfy (1.5), and $f \in C_0(\Omega)$. There exist positive constants $C_H = C_H(n, \lambda, \Lambda, s) > 1$, $\kappa = \kappa(n, s) < 1$, and $\hat{K} = \hat{K}(n, s) > 1$ such that for every ball $B_{\hat{K}R} = B_{\hat{K}R}(x_0)$ satisfying $B_{\hat{K}R} \subset \subset \Omega$ and every $u \in \text{Dom}(L)$, $u \geq 0$ in Ω , solution to*

$$(-a^{ij}(x)\partial_{ij})^s u = f \quad \text{in } B_{\hat{K}R}, \quad (1.6)$$

we have that

$$\sup_{B_{\kappa R}} u \leq C_H \left(\inf_{B_{\kappa R}} u + \|f\|_{L^\infty(B_{\hat{K}R})} R^{2s} \right). \quad (1.7)$$

Furthermore, there exist positive constants $\alpha_0 = \alpha_0(n, \lambda, \Lambda, s) < 1$ and $\hat{C} = \hat{C}(n, \lambda, \Lambda, s) > 0$ such that for any $u \in \text{Dom}(L)$ satisfying (1.6), we have that, for every $x \in B_{\hat{K}R}(x_0)$,

$$|u(x_0) - u(x)| \leq \frac{\hat{C}}{(\hat{K}R)^{\alpha_0}} |x_0 - x|^{\alpha_0} \left(\sup_{\Omega} |u| + \|f\|_{L^\infty(B_{\hat{K}R})} R^{2s} \right). \quad (1.8)$$

We mention that Grubb [15,16] and Seeley [30] studied fractional powers of nondivergence form elliptic operators with smooth coefficients in smooth domains from the operator theory and pseudo-differential operators points of view. Gradient estimates for fractional powers of constant coefficients, nondivergence form operators in $\mathbb{R}^n$ were considered in [2, Remark 1.10]. In particular, none of their results include the Harnack inequality and Hölder estimate in Theorem 1.1.

Our proof of Theorem 1.1 is based on the extension problem characterization of fractional power operators in Banach spaces given by the method of semigroups in [12] (see [33] for the case of Hilbert spaces). In our particular case, the extension result of [12] allows us to rewrite the nonlocal equation (1.4) in an equivalent way as a local PDE problem.

Theorem 1.2 (Particular case of [12]). *Assume that the bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the uniform exterior cone condition and that $a^{ij}(x) \in C(\Omega) \cap L^\infty(\Omega)$ are symmetric and satisfy (1.5). If $u \in \text{Dom}(L)$, then a solution $U \in C^\infty((0, \infty); \text{Dom}(L)) \cap C([0, \infty); C_0(\Omega))$ to the extension problem*

$$\begin{cases} a^{ij}(x)\partial_{ij}U + z^{2-\frac{1}{s}}\partial_{zz}U = 0 & \text{in } \Omega \times \{z > 0\} \\ U(x, 0) = u(x) & \text{on } \Omega \times \{z = 0\} \\ U = 0 & \text{on } \partial\Omega \times \{z \geq 0\} \end{cases} \quad (1.9)$$

is given by

$$U(x, z) = \frac{s^{2s}z}{\Gamma(s)} \int_0^\infty e^{-s^2 z^{1/s}/t} e^{-tL} u(x) \frac{dt}{t^{1+s}} \quad (1.10)$$

and satisfies

$$\|U(\cdot, z)\|_{L^\infty(\Omega)} \leq M \|u\|_{L^\infty(\Omega)} \quad \text{for some } M > 0.$$

Furthermore, $U_{z+} \in C([0, \infty); C_0(\Omega))$ and

$$-\partial_{z+}U(x, 0) = d_s L^s u(x) \in C_0(\Omega)$$

where $d_s = \frac{s^{2s}\Gamma(1-s)}{\Gamma(1+s)} > 0$ and

$$\partial_{z+}U(x, 0) = \lim_{z \rightarrow 0^+} \frac{U(x, z) - U(x, 0)}{z} \quad \text{for all } x \in \Omega.$$

If, in addition, $a^{ij}(x) \in C^\alpha(\Omega)$ for some $0 < \alpha < 1$, then the solution U in (1.10) is the unique classical solution $U \in C^2(\Omega \times (0, \infty)) \cap C(\overline{\Omega} \times [0, \infty))$ such that $\lim_{z \rightarrow \infty} \|U(\cdot, z)\|_{L^\infty(\Omega)} = 0$.

Hence, to prove Theorem 1.1, we will show interior Harnack inequalities and Hölder estimates for solutions U to the local, degenerate/singular elliptic equation in (1.9) subject to the Neumann boundary condition $-\partial_{z+}U(x, 0) = f(x)$ up to $\{z = 0\}$, and then take the trace at $\{z = 0\}$. Towards this end, we define the even reflection of U in the variable z by $\tilde{U}(x, z) = U(x, |z|)$, for $x \in \Omega$, $z \in \mathbb{R}$. For convenience, we continue to use the notation U instead of $\tilde{U}$ and notice that U , being symmetric across $\{z = 0\}$, satisfies the equation

$$a^{ij}(x)\partial_{ij}U + |z|^{2-\frac{1}{s}}\partial_{zz}U = 0 \quad \text{in } \Omega \times \{z \neq 0\}. \quad (1.11)$$

Furthermore, if $f(x) \neq 0$ then $\partial_z U$ is discontinuous across $(x, 0)$. Since $0 < s < 1$, the equation (1.11) either degenerates or blows up at $z = 0$, unless $s = 1/2$.

It turns out that we can recast (1.11) as an equation comparable to a linearized Monge–Ampère equation. Recall that the Monge–Ampère equation for a convex function ψ is given by $\det D^2\psi = G$. By taking the directional derivative ∂_e along a unit direction e to the equation and defining $v = \partial_e\psi$ and $g = \partial_e G$, we find that v satisfies the linearized Monge–Ampère equation

$$\text{tr}(A_\psi(x)D^2v) = g. \quad (1.12)$$

Here, $A_\psi(x) = \det(D^2\psi(x))(D^2\psi(x))^{-1}$ is the matrix of cofactors of $D^2\psi(x)$. Notice that (1.12) is a linear equation in nondivergence form that is elliptic as soon as $D^2\psi > 0$ and $G > 0$. However, it is not uniformly elliptic in general since the eigenvalues of $A_\psi(x)$ are not *a priori* controlled.

For our degenerate equation (1.11), we consider the strictly convex function $\Phi = \Phi(x, z) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ given by

$$\Phi(x, z) = \frac{1}{2} |x|^2 + \frac{s^2}{1-s} |z|^{\frac{1}{s}}, \quad 0 < s < 1.$$

Then Φ is in $C^1(\mathbb{R}^{n+1})$ but, when $s > 1/2$, is not in $C^2(\mathbb{R}^{n+1})$. Since the Hessian of Φ is

$$D^2\Phi(x, z) = \begin{pmatrix} I & 0 \\ 0 & |z|^{\frac{1}{s}-2} \end{pmatrix},$$

where I denotes the identity matrix of size $n \times n$, the linearized Monge–Ampère equation associated with Φ is

$$\mathrm{tr}((D^2\Phi)^{-1}D^2U) = \Delta_x U + |z|^{2-\frac{1}{s}} \partial_{zz} U = 0 \quad \text{for } z \neq 0. \quad (1.13)$$

As the coefficients $a^{ij}(x)$ are uniformly elliptic, see (1.5), we see that the coefficients in (1.11) are comparable to the coefficients in (1.13).

An important feature of the linearized Monge–Ampère equation is its intrinsic geometry that was first discovered by Caffarelli–Gutiérrez [5]. They proved a Harnack inequality for classical nonnegative solutions v to (1.12) when $\psi \in C^2$ and $g \equiv 0$, where the Euclidean balls and distance are replaced by Monge–Ampère sections and the Monge–Ampère quasi-distance, respectively. The Monge–Ampère sections associated to a convex, C^1 function ψ are the sublevel sets of $\psi - \ell$ where ℓ is any linear function, while the corresponding quasi-distance is given by $\delta_\psi(x_0, x) = \psi(x) - \psi(x_0) - \langle \nabla \psi(x_0), x - x_0 \rangle$.

We show that the geometry for our degenerate/singular equation (1.11) with Neumann boundary condition at $\{z = 0\}$ is given by the Monge–Ampère sections S_R associated to the strictly convex function Φ , that is, the sublevel sets of $\Phi - \ell$, and the Monge–Ampère quasi-distance δ_Φ in $\mathbb{R}^{n+1}$. See Section 4 for more details. We prove the following Harnack inequality and Hölder regularity estimate for the extension equation in such sections.

Theorem 1.3. *Let Ω be a bounded domain, $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ be bounded, measurable functions that satisfy (1.5) and let $f \in L^\infty(\Omega)$. There exist positive constants $C_H = C_H(n, \lambda, \Lambda, s) > 1$, $\kappa_0 = \kappa_0(n, s) < 1$, and $\hat{K}_0 = \hat{K}_0(n, s) > 1$ such that for every section $S_{\hat{K}_0 R} = S_{\hat{K}_0 R}(x_0, z_0) \subset \subset \Omega \times \mathbb{R}$ and every nonnegative solution $U \in C^2(S_{\hat{K}_0 R} \setminus \{z = 0\}) \cap C(S_{\hat{K}_0 R})$ such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(S_{\hat{K}_0 R} \cap \{z \geq 0\})$ to*

$$\begin{cases} a^{ij}(x) \partial_{ij} U + |z|^{2-\frac{1}{s}} \partial_{zz} U = 0 & \text{in } S_{\hat{K}_0 R} \cap \{z \neq 0\} \\ -\partial_{z+} U(x, 0) = f & \text{on } S_{\hat{K}_0 R} \cap \{z = 0\}, \end{cases} \quad (1.14)$$

we have that

$$\sup_{S_{\kappa_0 R}} U \leq C_H \left(\inf_{S_{\kappa_0 R}} U + \|f\|_{L^\infty(S_{\hat{K}_0 R} \cap \{z=0\})} R^s \right). \quad (1.15)$$

Consequently, there exist constants $0 < \alpha_1 = \alpha_1(n, \lambda, \Lambda, s) < 1/2$ and $\hat{C}_1 = \hat{C}_1(n, \lambda, \Lambda, s) > 1$ such that, for every solution $U \in C^2(S_{\hat{K}_0 R} \setminus \{z = 0\}) \cap C(S_{\hat{K}_0 R})$ to (1.14) such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(S_{\hat{K}_0 R} \cap \{z \geq 0\})$,

$$\begin{aligned} & |U(x_0, z_0) - U(x, z)| \\ & \leq \frac{\hat{C}_1}{(\hat{K}_0 R)^{\alpha_1}} [\delta_\Phi((x_0, z_0), (x, z))]^{\alpha_1} \left(\sup_{S_{\hat{K}_0 R}(x_0, z_0)} |U| + \|f\|_{L^\infty(S_{\hat{K}_0 R} \cap \{z=0\})} R^s \right) \end{aligned} \quad (1.16)$$

for every $(x, z) \in S_{\hat{K}_0 R}(x_0, z_0)$.

Note that if $S_{\hat{K}_0 R} \cap \{z = 0\} = \emptyset$, then $\|f\|_{L^\infty(S_{\hat{K}_0 R} \cap \{z=0\})} R^s$ does not appear in the right hand side of (1.15) and (1.16).

Regularity estimates, such as Harnack inequalities, for the linearized Monge–Ampère equation (1.12) have been studied by Caffarelli–Gutiérrez [5], Forzani–Maldonado [10], Le [20], Maldonado [21,24], Savin [28], among others. In each case, they either assume that $\det(D^2\psi)$ is bounded away from zero and infinity (that is, the Monge–Ampère measure $\mu_\psi(E) = |\nabla\psi(E)|$, $E \subset \mathbb{R}^n$, is comparable to the Lebesgue measure), or that ψ is sufficiently regular, e.g. $\psi \in C^2$. For our function Φ , we have that $D^2\Phi$ either degenerates or blows up near $\{z = 0\}$ when $s \neq 1/2$, and, moreover, $\Phi \notin C^2$ when $s > 1/2$. Therefore, (1.11) is not covered by such previous results. On the other hand, Maldonado proved Harnack inequality for degenerate elliptic equations associated with convex functions of the form $\psi(x) = |x|^p$, $p \geq 2$, see [22] and also [23]. However, not only are his techniques different than the ones presented here but also his work does not include the singular case in which $s > 1/2$. Moreover, when we write (1.14) for $s = 1/2$ in $\Omega \times \mathbb{R}$ as a single equation, we see that U satisfies (1.11) in $\Omega \times \mathbb{R}$ with a right hand side that is a singular measure with density $f(x)$ supported on $\{z = 0\}$.

We develop a method of sliding paraboloids inspired by the work of Savin for fully nonlinear uniformly elliptic equations [29]. For our setting, we work with a Neumann problem in a Monge–Ampère geometry that brings additional challenges because Φ is only C^1 and $D^2\Phi$ is degenerate/singular. For this, we define *paraboloids* $P : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ of opening $a > 0$ with vertex (x_v, z_v) by

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + c$$

where c is a constant. We lift these paraboloids from below until they touch the graph of U in a section S_R for the first time. We estimate the Monge–Ampère measure of the resulting set of contact points by the Monge–Ampère measure of the set of vertices. Observe that, since our equation (1.11) is degenerate/singular and $-\partial_{z+}U(x, 0) = f(x)$, we need to be able to control the contact points (x, z) for which $z = 0$ in terms of the size of f . Next, we show that, by increasing the opening of these paraboloids, they almost cover the section S_R in measure. This relies on explicit barriers whose construction is very delicate because of the Neumann boundary condition and the degeneracy/singularity of (1.11). Then, we build a refined geometric argument to obtain a localization estimate. Thus, using a covering argument, we can conclude the proof of Theorem 1.3 and deduce Theorem 1.1.

Our function Φ was also considered in [26] to study the fractional nonlocal linearized Monge–Ampère equation. They established Harnack inequality and Hölder estimates for solutions to (1.4) when the coefficients $a^{ij}(x)$ are given by the matrix of cofactors of $D^2\psi$, where ψ is a C^3 strictly convex function and Ω is a section of ψ . Observe that in [26] the weak Harnack inequality is proved using the divergence form structure of the equation. Whereas, in (1.4), we not only consider general elliptic coefficients $a^{ij}(x)$, but also the equation cannot be written in divergence form. Nevertheless, since the proof of the local boundedness of the solution to the extension problem in [26] uses purely nondivergence form techniques, one can easily check that solutions to our extension problem (1.9) satisfy the same local boundedness estimate as that of [26, Theorem 11.3].

We additionally mention that Le in [20] proved Harnack inequality for the linearized Monge–Ampère equation (1.12) when $\psi \in C^2$ and $0 < \lambda \leq \det(D^2\psi(x)) \leq \Lambda$, by using sliding paraboloids within the roadmap of the proof of Caffarelli–Gutiérrez [5]. Again, our methods (inspired by Savin [29]) and results are different and independent of [20] (in particular, Φ is not smooth when $s > 1/2$, $D^2\Phi$ is degenerate/singular, and we have the Neumann boundary condition $-\partial_{z+}U(x, 0) = f(x)$).

Theorem 1.3 holds for bounded domains Ω and bounded, measurable coefficients $a^{ij}(x)$. In Theorem 1.1 we additionally require that Ω satisfies the uniform exterior cone condition and that $a^{ij}(x)$ are Hölder continuous. There are several reasons for these technical assumptions. First, the uniform exterior cone condition and the hypothesis $a^{ij}(x) \in C(\Omega) \cap L^\infty(\Omega)$ give us the existence of an appropriate C_0 -semigroup

generated by L , so the fractional power operator L^s can be defined using (1.3). Furthermore, under these conditions, the extension problem characterization in Theorem 1.2 holds. Second, our proof of Theorem 1.3 is for *classical* solutions U to the extension problem and does not require any continuity assumptions on $a^{ij}(x)$ nor geometric conditions on Ω . Third, to apply Theorem 1.3, we need to ensure that the solution U given in Theorem 1.2 is classical, and for this we must require $a^{ij}(x) \in C^\alpha(\Omega)$. It is an open problem and will be the object of future work to define $(-a^{ij}(x)\partial_{ij})^s$ in bounded domains when the coefficients are only bounded, measurable and to establish a corresponding extension equation and Harnack inequality for viscosity solutions to (1.4).

The paper is organized as follows. First, in Section 2, we show several applications of fractional powers of nondivergence form operators (1.1). Then, in Section 3, we precisely define the fractional operator $(-a^{ij}(x)\partial_{ij})^s$ and prove the extension characterization. In Section 4, we provide the necessary Monge–Ampère background associated to our function Φ . We prove a sequence of reductions of Theorem 1.3 in Section 5. Section 6 contains preliminary results on the Monge–Ampère paraboloids P associated to Φ . Next, we establish several key results that will be used to prove the final reduction of Theorem 1.3. In Section 7, we estimate the Monge–Ampère measure of the set of contact points for sliding paraboloids of fixed opening by the measure of the set of vertices. The delicate construction of the barriers is done in Section 8. These are used in Section 9 to prove a localization estimate by means of a refined geometric argument. A Calderón–Zygmund-type covering lemma is proved in Section 10. Finally, in Section 11, we present the proof of the final reduction of Theorem 1.3 and the proof of Theorem 1.1.

2. Applications

In this section we present some applications where fractional powers of nondivergence form elliptic operators naturally arise.

2.1. Fractional Monge–Ampère equations

If $u = u(x)$ is a convex, C^2 function, then one can check that the Monge–Ampère operator acting on u at a point x can be written as

$$\begin{aligned} n \det(D^2 u(x))^{1/n} &= \inf \{ \Delta(u \circ B)(B^{-1}x) : B \in \mathcal{M} \} \\ &= \inf \{ a^{ij} \partial_{ij} u(x) : (a^{ij}) = B^2, B \in \mathcal{M} \}, \end{aligned}$$

where the infimum is taken over the class $\mathcal{M}$ of all positive definite, symmetric matrices B of size $n \times n$ such that $\det(B) = 1$. Motivated by these identities, Caffarelli–Charro defined in [4] the fractional Monge–Ampère operator by

$$\mathcal{D}_s u(x) = \inf \{ -(-\Delta)^s(u \circ B)(B^{-1}x) : B \in \mathcal{M} \}, \quad 1/2 < s < 1. \quad (2.1)$$

On the other hand, it was shown in [19] that the operator in (2.1) can also be written as

$$\mathcal{D}_s u(x) = \inf \{ -(-a^{ij} \partial_{ij})^s u(x) : (a^{ij}) = B^2, B \in \mathcal{M} \}, \quad (2.2)$$

where $(-a^{ij} \partial_{ij})^s$ is the fractional power of the constant coefficients operator $-a^{ij} \partial_{ij}$.

The fractional Monge–Ampère operator (2.1) is degenerate elliptic because the eigenvalues of the matrices $B \in \mathcal{M}$ are not *a priori* controlled from below or above. Nevertheless, it is proved in [4] that if u is Lipschitz, semiconcave, and $\mathcal{D}_s u \geq \eta_0 > 0$ in a bounded domain Ω , then $\mathcal{D}_s$ becomes uniformly elliptic in u , that is, there is a constant $\lambda > 0$ such that

$$\mathcal{D}_s u(x) = \mathcal{D}_s^\lambda u(x) := \inf \left\{ -(-\Delta)^s(u \circ B)(B^{-1}x) : B \in \mathcal{M}, B \geq \lambda I \right\}$$

for all $x \in \Omega$. Equivalently, in the description of (2.2),

$$\mathcal{D}_s^\lambda u(x) = \inf \left\{ -(-a^{ij} \partial_{ij})^s u(x) : (a^{ij}) = B^2, B \in \mathcal{M}, B \geq \lambda I \right\}.$$

It was observed in [34] that, for each $x \in \Omega$, the infimum above is attained at some matrix $a^{ij} = a^{ij}(x)$. Therefore, the fractional Monge–Ampère operator in the uniformly elliptic regime is in fact given by

$$\mathcal{D}_s^\lambda u(x) = -(-a^{ij}(x) \partial_{ij})^s u(x) \quad \text{for every } x \in \Omega.$$

In other words, $\mathcal{D}_s^\lambda u(x)$ is the fractional power of the nondivergence form uniformly elliptic operator $L = -a^{ij}(x) \partial_{ij}$, where $a^{ij}(x)$ are bounded, measurable coefficients.

2.2. Elasticity

Consider an anisotropic elastic membrane represented by the graph of a function $U(x, z)$, for $(x, z) \in \Omega \times [0, \infty)$. Suppose that we place a thin obstacle $\phi : \Omega \rightarrow \mathbb{R}$ on the hyperplane $\{z = 0\}$, such that $\phi \leq 0$ on $\partial\Omega$, which pushes U from below at $\{z = 0\}$. By fixing $U = 0$ on $\partial\Omega \times [0, \infty)$, this problem is modeled by the following thin obstacle problem:

$$\begin{cases} a^{ij}(x) \partial_{ij} U + \partial_{zz} U = 0 & \text{in } \Omega \times \{z > 0\} \\ U(x, z) = 0 & \text{on } \partial\Omega \times \{z \geq 0\} \\ U(x, 0) \geq \phi(x) & \text{on } \Omega \\ -\partial_{z+} U(x, 0) \geq 0 & \text{on } \Omega \\ -\partial_{z+} U(x, 0) = 0 & \text{on } \{U(x, 0) > \phi(x)\}. \end{cases} \quad (2.3)$$

The last two conditions are called the *Signorini complementary conditions*. They follow from the fact that ϕ is pushing U upwards, while U is actually free in the noncoincidence set $\{U(x, 0) > \phi(x)\}$. The coefficients $a^{ij}(x)$ encode the heterogeneity of the membrane. The thin obstacle problem (2.3) is equivalent to the problem of semipermeable cell membranes in biology (see [9]), where $a^{ij}(x)$ are a model for the cytoplasm inside the cell.

It follows from the extension problem characterization (see Theorem 1.2) with $s = 1/2$, that the trace $u(x) := U(x, 0)$ satisfies

$$-\partial_{z+} U(x, 0) = (-a^{ij}(x) \partial_{ij})^{1/2} u(x) \quad \text{in } \Omega.$$

Therefore, U solves the thin obstacle problem (2.3) if and only if its trace u is the solution to the following fractional obstacle problem

$$\begin{cases} (-a^{ij}(x) \partial_{ij})^{1/2} u \geq 0 & \text{in } \Omega \\ (-a^{ij}(x) \partial_{ij})^{1/2} u = 0 & \text{in } \Omega \cap \{u > \phi\} \\ u \geq \phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2.3. Finance

Consider a particle moving randomly in a heterogeneous domain Ω that is killed when it reaches the boundary $\partial\Omega$. This random behavior can be modeled by a diffusion process X_t whose infinitesimal generator is a nondivergence form elliptic operator $L = -a^{ij}(x)\partial_{ij}$ in Ω , subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$. In this situation, the coefficients $a^{ij}(x)$ serve as a measure of the anisotropy of the medium, or the preferred directions the particle chooses at every point x . A model for particles randomly jumping inside a heterogeneous medium that are killed as soon as they reach or try to cross the boundary can be given by subordinating the process X_t with an s -stable Lévy subordinator T_t , for $0 < s < 1$. The resulting subordinated process $Y_t = X_{T_t}$ is then generated by the fractional power operator $L^s = (-a^{ij}(x)\partial_{ij})^s$, $0 < s < 1$. See [18] for the case of smooth coefficients and domains, and [31] for the case when X_t is a Wiener process.

Next, let τ be the optimal stopping time that maximizes the function

$$u(x) = \sup_{\tau} \mathbb{E}[\phi(Y_{\tau}); \tau < +\infty],$$

where $\phi \in C_0(\Omega)$ (see (3.3)), $\mathbb{E}$ denotes the expected value, and the process Y is set to start at $x \in \Omega$. It turns out that u is the solution to the following obstacle problem:

$$\begin{cases} (-a^{ij}(x)\partial_{ij})^s u \geq 0 & \text{in } \Omega \\ (-a^{ij}(x)\partial_{ij})^s u = 0 & \text{in } \Omega \cap \{u > \phi\} \\ u \geq \phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

These free boundary problems appear in financial models (see [8]) where u is the value of a perpetual American option in which the asset prices are modeled by Y_t and ϕ is the payoff function.

3. Fractional powers of elliptic operators and extension problem

Here, we give the precise definition of the fractional power operator $L^s = (-a^{ij}(x)\partial_{ij})^s$ in (1.4) and present the extension problem characterization, i.e. Theorem 1.2. For this, we apply the method of semigroups of [12,33] (see also [32]) which we describe next.

3.1. Method of semigroups for fractional power operators

A family $\{T_t\}_{t \geq 0}$ of bounded, linear operators on a Banach space X is a *semigroup* on X if

$$T_0 = Id \quad \text{and} \quad T_{t_1} \circ T_{t_2} = T_{t_1+t_2} \quad \text{for every } t_1, t_2 \geq 0,$$

where Id denotes the identity operator. We say that a semigroup $\{T_t\}_{t \geq 0}$ is a C_0 -semigroup if $T_t u \rightarrow u$ as $t \rightarrow 0^+$ for all $u \in X$. A semigroup $\{T_t\}_{t \geq 0}$ is *uniformly bounded* if its operator norm is uniformly bounded in t , that is, there is a constant $M \geq 1$ such that $\|T_t\| \leq M$ for all $t \geq 0$. The *infinitesimal generator* A of a semigroup $\{T_t\}_{t \geq 0}$ is the closed linear operator defined as

$$-Au = \lim_{t \rightarrow 0^+} \frac{T_t u - u}{t} \quad (3.1)$$

in the domain $\text{Dom}(A) = \{u \in X : \text{the limit in (3.1) exists}\} \subset X$. In this case, we write $T_t = e^{-tA}$. Hence, if A is the infinitesimal generator of a C_0 -semigroup $\{e^{-tA}\}_{t \geq 0}$ on X , then the function $v = e^{-tA}u$, for $u \in \text{Dom}(A)$, satisfies the heat equation for A :

$$\begin{cases} \partial_t v = -Av & \text{for } t > 0 \\ v = u & \text{for } t = 0. \end{cases}$$

Conversely, a linear operator $(A, \text{Dom}(A))$ on X is said to *generate* a semigroup if there is a semigroup $\{T_t\}_{t \geq 0}$ for which A is its infinitesimal generator, that is, $T_t = e^{-tA}$. Given a uniformly bounded C_0 -semigroup $\{T_t = e^{-tA}\}_{t \geq 0}$ on X , the fractional power A^s of its infinitesimal generator is defined as

$$A^s u = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tA} u - u) \frac{dt}{t^{1+s}}, \quad \text{for all } u \in \text{Dom}(A) \subset X,$$

where $0 < s < 1$. If the semigroup $\{e^{-tA}\}_{t \geq 0}$ has *exponential decay*, that is, $\|e^{-tA}\| \leq M e^{-\varepsilon t}$, for some $\varepsilon > 0$, for all $t \geq 0$, then the negative power A^{-s} , $s > 0$, is given by

$$A^{-s} f = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tA} f \frac{dt}{t^{1-s}}, \quad \text{for all } f \in X.$$

Thus, under the exponential decay assumption on $\{e^{-tA}\}_{t \geq 0}$, given $f \in X$, the solution $u \in \text{Dom}(A^s)$ to the fractional problem $A^s u = f$ is $u = A^{-s} f$. Here $\text{Dom}(A^s)$ is defined as the range of A^{-s} . For all these details, see [27,35].

Fractional powers $A^{\pm s}$ of infinitesimal generators A of uniformly bounded C_0 -semigroups can be characterized by extension problems. For the case when X is a Hilbert space see [33], while for the case when X is a general Banach space see [12].

Theorem 3.1 (See [12, Theorems 1.1 and 2.1, Remark 2.2]). *Let $(A, \text{Dom}(A))$ be the infinitesimal generator of a uniformly bounded C_0 -semigroup $\{e^{-tA}\}_{t \geq 0}$ on a Banach space X . Let $0 < s < 1$. Define, for $y > 0$ and any $u \in X$,*

$$U(y) = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-tA} u \frac{dt}{t^{1+s}}. \quad (3.2)$$

Then $U \in C^\infty((0, \infty), \text{Dom}(A)) \cap C([0, \infty), X)$ is a solution to the extension problem

$$\begin{cases} -AU + \frac{1-2s}{y} \partial_y U + \partial_{yy} U = 0 & \text{for } y > 0 \\ \lim_{y \rightarrow 0^+} U(y) = u & \text{in } X. \end{cases}$$

Moreover, $\|U(y)\|_X \leq M \|u\|_X$, for all $y \geq 0$. Furthermore, if $u \in \text{Dom}(A)$ then

$$-\lim_{y \rightarrow 0^+} y^{1-2s} \partial_y U(y) = c_s A^s u = -2s \lim_{y \rightarrow 0^+} \frac{U(y) - u}{y^{2s}} \quad \text{in } X$$

where $c_s = \frac{\Gamma(1-s)}{4^{s-1/2} \Gamma(s)} > 0$. If, in addition, $\{e^{-tA}\}_{t \geq 0}$ has exponential decay and $u \in \text{Dom}(A)$ satisfies $A^s u = f$, for some $f \in X$, then the solution U in (3.2) can also be written as

$$U(y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-tA} f \frac{dt}{t^{1-s}}$$

and, in particular, $-\lim_{y \rightarrow 0^+} y^{1-2s} \partial_y U(y) = c_s f$ and $U(0) = u$.

3.2. Fractional powers of nondivergence form elliptic operators

To give the definition of (1.1), we need conditions on $a^{ij}(x)$ and Ω so that L as in (1.2) generates a uniformly bounded C_0 -semigroup with exponential decay in an appropriate Banach space X . For this, we assume that the bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the *uniform exterior cone condition*, namely, that there is a right circular cone C such that for all $x \in \partial\Omega$ there is a cone C_x with vertex x that is congruent to C such that $\overline{\Omega} \cap C_x = \{x\}$. We define the Banach space $X = C_0(\Omega)$ by

$$C_0(\Omega) = \{u \in C(\overline{\Omega}) : u \equiv 0 \text{ on } \partial\Omega\}, \quad (3.3)$$

endowed with the L^∞ -norm. Let L be the linear operator on $C_0(\Omega)$ given by

$$L = -a^{ij}(x)\partial_{ij}, \quad \text{Dom}(L) = \{u \in C_0(\Omega) \cap W_{\text{loc}}^{2,n}(\Omega) : Lu \in C_0(\Omega)\}, \quad (3.4)$$

where the coefficients $a^{ij}(x) \in C(\Omega) \cap L^\infty(\Omega)$ are symmetric and satisfy (1.5). Under these hypotheses, L generates a uniformly bounded C_0 -semigroup on $C_0(\Omega)$ with exponential decay.

Proposition 3.2 (See [1, Proposition 4.7]). Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain that satisfies the uniform exterior cone condition and that $a^{ij}(x) \in C(\Omega) \cap L^\infty(\Omega)$ are symmetric and satisfy (1.5). The operator L defined by (3.4) generates a uniformly bounded C_0 -semigroup, denoted by $\{e^{-tL}\}_{t \geq 0}$, on $C_0(\Omega)$, such that if $u \in C_0(\Omega)$ satisfies $u \geq 0$, then $e^{-tL}u \geq 0$, for all $t \geq 0$. Moreover, there are constants $M \geq 1$ and $\varepsilon > 0$ such that

$$\|e^{-tL}u\|_{C_0(\Omega)} \leq M e^{-\varepsilon t} \|u\|_{C_0(\Omega)}, \quad \text{for all } t \geq 0. \quad (3.5)$$

In other words, by Proposition 3.2 and the maximum principle for parabolic equations (see [11]), for any $u \in \text{Dom}(L)$, the function $v(x, t) = e^{-tL}u(x) \in C^1((0, \infty), \text{Dom}(L)) \cap C([0, \infty), C_0(\Omega))$ is the unique solution to the heat equation driven by L with initial data u :

$$\begin{cases} \partial_t v(x, t) = a^{ij}(x)\partial_{ij}v(x, t) & \text{in } \Omega \times \{t > 0\} \\ v(x, t) = 0 & \text{on } \partial\Omega \times \{t \geq 0\} \\ v(x, 0) = u(x) & \text{on } \Omega \times \{t = 0\}. \end{cases} \quad (3.6)$$

Now we can formalize the definition of the fractional power operator (1.1).

Definition 3.3. Assume that the bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the uniform exterior cone condition and that $a^{ij}(x) \in C(\Omega) \cap L^\infty(\Omega)$ are symmetric and satisfy (1.5). Consider the Banach space $C_0(\Omega)$ and the operator $L = -a^{ij}(x)\partial_{ij}$ given by (3.4). We define the fractional power operator $L^s = (-a^{ij}(x)\partial_{ij})^s : \text{Dom}(L) \rightarrow C_0(\Omega)$ by

$$(-a^{ij}(x)\partial_{ij})^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tL}u(x) - u(x)) \frac{dt}{t^{1+s}}, \quad 0 < s < 1, \quad x \in \Omega. \quad (3.7)$$

Remark 3.4 (*Pointwise formula*). It is known that, under certain conditions on L , the semigroup $\{e^{-tL}\}_{t \geq 0}$ has a heat kernel, namely, there is a function $H_t(x, y)$ such that

$$e^{-tL}u(x) = \int_{\Omega} H_t(x, y)u(y) dy \quad \text{for all } t > 0, x \in \Omega.$$

For example, the heat kernel exists and satisfies the Gaussian estimate

$$0 \leq H_t(x, y) \leq C \frac{e^{-c|x-y|^2/t}}{t^{n/2}} \quad \text{for all } t > 0, x, y \in \Omega, \quad (3.8)$$

for some constants $C, c > 0$, whenever the coefficients $a^{ij}(x)$ are Hölder continuous, see [11]. In this situation, it follows from (3.7) that for any smooth function $u \in \text{Dom}(L)$,

$$(-a^{ij}(x)\partial_{ij})^s u(x) = \int_{\Omega} (u(x) - u(y))K_s(x, y) dy + B_s(x)u(x) \quad \text{for all } x \in \Omega$$

where $0 \leq K_s(x, y) \leq C_{n,s}|x - y|^{-(n+2s)}$, for $x, y \in \Omega$, $x \neq y$, and $B_s(x) \in L^\infty(\Omega)$. Therefore, the fractional operator L^s is a nonlocal, integro-differential operator in Ω .

Remark 3.5 (*Negative fractional powers*). Let $f \in C_0(\Omega)$ and assume that $u \in \text{Dom}(L^s)$ is a solution to (1.4), that is, $(-a^{ij}(x)\partial_{ij})^s u = f$ in Ω . By Proposition 3.2, the semigroup $\{e^{-tL}\}_{t \geq 0}$ has exponential decay. Then u can be written as

$$u(x) = (-a^{ij}(x)\partial_{ij})^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{1-s}} \quad \text{for all } x \in \Omega. \quad (3.9)$$

If the coefficients $a^{ij}(x)$ are Hölder continuous then we can use the heat kernel from Remark 3.4 into (3.9) to write

$$u(x) = (-a^{ij}(x)\partial_{ij})^{-s} f(x) = \int_{\Omega} K_{-s}(x, y)f(y) dy \quad \text{for all } x \in \Omega$$

where, by the estimate (3.8), $0 \leq K_{-s}(x, y) \leq C_{n,-s}|x - y|^{-(n-2s)}$, for all $x, y \in \Omega$, $x \neq y$.

Proof of Theorem 1.2. We choose $X = C_0(\Omega)$ and $A = L$ as in (3.4) so that, by Proposition 3.2, L generates a uniformly bounded C_0 -semigroup on $C_0(\Omega)$ with exponential decay. Then, the solution $U(y)$ in (3.2) satisfies the properties stated in Theorem 3.1. With the change of variables $z = (y/(2s))^{2s}$, we obtain that $U(z) \equiv U(x, z)$ verifies the formulas and properties of Theorem 1.2. If the coefficients $a^{ij}(x)$ are also Hölder continuous then, by interior Schauder estimates (see [13, Theorem 9.19]), the solution U is classical. Moreover, by the weak maximum principle (see [13, Theorem 3.1]), there is at most one classical solution to (1.9) such that $\lim_{z \rightarrow \infty} \|U(\cdot, z)\|_{L^\infty(\Omega)} = 0$. Using (3.5) it is easy to check that the solution U given by (1.10) has such decay at infinity and hence is the unique solution. $\square$

4. Monge–Ampère setting

We present the necessary background for the Monge–Ampère geometry associated to equation (1.11) as well as the notation that will be used throughout the rest of this work. We reference the reader to [10, 17] for details about the Monge–Ampère geometry associated to general convex functions.

Given $0 < s < 1$, we define the functions $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(x) = \frac{1}{2} |x|^2 \quad \text{and} \quad h(z) = \frac{s^2}{1-s} |z|^{\frac{1}{s}}. \quad (4.1)$$

Notice that $\varphi \in C^\infty(\mathbb{R})$ and $h \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ are strictly convex. Set

$$\Phi(x, z) = \varphi(x) + h(z) \quad \text{for all } (x, z) \in \mathbb{R}^{n+1}. \quad (4.2)$$

We note that

$$h'(z) = \frac{s}{1-s} |z|^{\frac{1}{s}-2} z, \quad h''(z) = |z|^{\frac{1}{s}-2}, \quad D^2\Phi(x, z) = \begin{pmatrix} I & 0 \\ 0 & |z|^{\frac{1}{s}-2} \end{pmatrix}.$$

It is clear that $h'(-z) = -h'(z)$ and $h'(0) = 0$.

The *Monge-Ampère measure* associated to a strictly convex function $\psi \in C^1(\mathbb{R}^n)$ is the Borel measure given by

$$\mu_\psi(E) = |\nabla\psi(E)| \quad \text{for every Borel set } E \subset \mathbb{R}^n,$$

where $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. Since $\nabla\varphi(x) = x$, it is clear that $\mu_\varphi(E) = |E|$.

Lemma 4.1. *For a Borel set $I \subset \mathbb{R}$,*

$$\mu_h(I) = \int_I h''(z) dz.$$

Consequently, for a Borel set $E \subset \mathbb{R}^{n+1}$,

$$\mu_\Phi(E) = \int_E h''(z) dz dx.$$

Proof. Consider an interval $(a, b) \subset \mathbb{R}$ such that $0 \in (a, b)$. Note that h' is monotone increasing, injective, and $h'(z) = 0$ if and only if $z = 0$. Since h is C^2 and strictly convex in $\mathbb{R} \setminus \{z = 0\}$, we have that

$$\begin{aligned} \mu_h((a, b)) &= |h'((a, b))| = |h'((a, 0))| + |h'(0)| + |h'((0, b))| \\ &= \int_a^0 h''(z) dz + 0 + \int_0^b h''(z) dz = \int_a^b h''(z) dz. \end{aligned}$$

The result follows for any interval and hence for any Borel set $I \subset \mathbb{R}$. $\square$

The *Monge-Ampère (quasi)-distance* associated to a strictly convex function $\psi \in C^1(\mathbb{R}^n)$ is given by

$$\delta_\psi(x_0, x) = \psi(x) - \psi(x_0) - \langle \nabla\psi(x_0), x - x_0 \rangle.$$

By convexity, $\delta_\psi \geq 0$, and $\delta_\psi(x_0, x) = 0$ if and only if $x_0 = x$. We use the terminology quasi-distance when there exists a constant $K \geq 1$ such that

$$\delta_\psi(x_1, x_2) \leq K (\min\{\delta_\psi(x_1, x_3), \delta_\psi(x_3, x_1)\} + \min\{\delta_\psi(x_2, x_3), \delta_\psi(x_3, x_2)\})$$

for all $x_1, x_2, x_3 \in \mathbb{R}^n$. For our functions φ , h , and Φ in (4.1) and (4.2), we have

$$\begin{aligned}\delta_\varphi(x_0, x) &= \frac{1}{2}|x|^2 - \frac{1}{2}|x_0|^2 - \langle x_0, x - x_0 \rangle = \frac{1}{2}|x - x_0|^2 \\ \delta_h(z_0, z) &= h(z) - h(z_0) - h'(z_0)(z - z_0) \\ \delta_\Phi((x_0, z_0), (x, z)) &= \delta_\varphi(x_0, x) + \delta_h(z_0, z).\end{aligned}$$

We will later show that δ_h , δ_φ , and δ_Φ are indeed quasi-distances (see Corollary 4.7).

A *Monge–Ampère section* of radius $R > 0$, centered at $x_0 \in \mathbb{R}^n$ associated to a strictly convex function $\psi \in C^1(\mathbb{R}^n)$ is defined as

$$S_\psi(x_0, R) = \{x \in \mathbb{R}^n : \delta_\psi(x_0, x) < R\}.$$

The supporting hyperplane to ψ at x_0 is given by $\ell(x) = \psi(x_0) + \langle \nabla \psi(x_0), x - x_0 \rangle$. Then, $S_\psi(x_0, R) = \{x : \psi(x) - \ell(x) < R\}$, and we see that the Monge–Ampère sections for ψ are the sublevel sets of $\psi - \ell$. In the case of φ in (4.1), the sections correspond to Euclidean balls with the same center

$$S_\varphi(x_0, R) = \left\{x : \frac{1}{2}|x - x_0|^2 < R\right\} = B_{\sqrt{2R}}(x_0). \quad (4.3)$$

The sections for h in (4.1) with radius $R > 0$ correspond to intervals in $\mathbb{R}$.

We say that the Monge–Ampère measure μ_ψ is *doubling with respect to the center of mass* on the sections of ψ , written $\mu_\psi \in (DC)_\psi$, if there is a constant $C_d > 0$ such that

$$\mu_\psi(S_\psi(x, R)) \leq C_d \mu_\psi\left(\frac{1}{2}S_\psi(x, R)\right) \quad \text{for all sections } S_\psi(x, R). \quad (4.4)$$

Here, we use the notation $\alpha S_\psi(x, R) = \{\alpha(y - x^*) + x^* : y \in S_\psi(x, R)\}$, for $\alpha > 0$, where x^* is the center of mass of $S_\psi(x, R)$. On the other hand, we say that μ_ψ is *doubling with respect to the parameter* on the sections of ψ if there is a constant $C'_d > 0$ such that

$$\mu_\psi(S_\psi(x, R)) \leq C'_d \mu_\psi\left(S_\psi\left(x, \frac{R}{2}\right)\right) \quad \text{for all sections } S_\psi(x, R). \quad (4.5)$$

It can be seen that (4.4) implies (4.5), but the converse is not true in general, see [17].

Finally, we say that ψ satisfies the *engulfing property* if there is a constant $\theta \geq 1$ such that, for every section $S_\psi(x, R)$, if $x_1 \in S_\psi(x, R)$, then $S_\psi(x, R) \subset S_\psi(x_1, \theta R)$.

For the next result, see Theorem 5 in [10] and the comments following it and also Lemma 2.1 in [25].

Theorem 4.2. *Let $\psi \in C^1(\mathbb{R}^n)$ be a strictly convex function. The following are equivalent.*

1. $\mu_\psi \in (DC)_\psi$;
2. ψ satisfies the engulfing property;
3. there are constants $c, C > 0$ such that

$$cR^n \leq |S_\psi(x, R)| \mu_\psi(S_\psi(x, R)) \leq CR^n$$

for all sections $S_\psi(x, R)$;

4. δ_ψ is a quasi-distance.

All the statements are equivalent in the sense that the constants in each property only depend on each other. If $\mu_\psi \in (DC)_\psi$, then there exists a constant $K_d > 1$, depending only on the doubling constant C_d in (4.4) and on dimension, such that

$$\mu_\psi(S_\psi(x, r_2)) \leq K_d \left(\frac{r_2}{r_1}\right)^n \mu_\psi(S_\psi(x, r_1)) \quad \text{for all } x \in \mathbb{R}^n, \quad 0 < r_1 < r_2. \quad (4.6)$$

In order to show that our convex function Φ in (4.2) satisfies Theorem 4.2, we need to introduce the notion of Monge–Ampère cubes associated with Φ . Many of our proofs will rely on the fact that $\Phi(x, z) = \varphi(x) + h(z)$ has separated variables.

Definition 4.3. A Monge–Ampère cube of radius $R > 0$, centered at $(x, z) \in \mathbb{R}^{n+1}$, associated to Φ is given by

$$Q_R(x, z) = S_{\varphi_1}(x_1, R) \times \cdots \times S_{\varphi_n}(x_n, R) \times S_h(z, R)$$

where $x = (x_1, \dots, x_n)$ and $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\varphi_i(x_i) = \frac{1}{2} |x_i|^2$ for $i = 1, \dots, n$.

Notation 4.4. We will always use the following notation.

- $x = (x_1, \dots, x_n) \in \mathbb{R}^n, z \in \mathbb{R}$.
- $S_R(x) \subset \mathbb{R}^n$ is a section of radius $R > 0$ associated with φ centered at x .
- $S_R(z) \subset \mathbb{R}$ is a section of radius $R > 0$ associated with h , centered at z .
- $S_R(x, z) \subset \mathbb{R}^{n+1}$ is a section of radius $R > 0$ associated with Φ , centered at (x, z) .
- $Q_R(x) \subset \mathbb{R}^n$ is a cube of radius $R > 0$ associated with φ centered at x .
- $Q_R(z) \subset \mathbb{R}$ is a cube of radius $R > 0$ associated with h centered at z .
- $Q_R(x, z) \subset \mathbb{R}^{n+1}$ is a cube of radius $R > 0$ associated with Φ centered at (x, z) .

The relation between Monge–Ampère cubes and sections is given by the following result.

Lemma 4.5 (Lemma 6 in [10]). Fix $m \in \mathbb{N}$. For each $j = 1, \dots, m$, let $\psi_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}$ be strictly convex, differentiable functions. Set $n = \sum_{j=1}^m n_j$ and define

$$\psi(x) = \sum_{j=1}^m \psi_j(x_j), \quad x = (x_1, \dots, x_m) \in \mathbb{R}^n, x_j \in \mathbb{R}^{n_j}.$$

Then

$$S_\psi(x, R) \subset \prod_{j=1}^m S_{\psi_j}(x_j, R) \subset S_\psi(x, mR)$$

for all $x = (x_1, \dots, x_m) \in \mathbb{R}^n$ and $R > 0$.

In particular, if ψ_j satisfy the engulfing property with corresponding constants θ_j for all $j = 1, \dots, m$, then ϕ satisfies the engulfing property with $\theta = m \max_j \{\theta_j\}$. Conversely, if ψ satisfies the engulfing property for some $\theta > 1$, then ψ_j satisfies the engulfing property with constant θ for all $j = 1, \dots, m$.

As a consequence of Lemma 4.5,

$$S_R(x, z) \subset S_R(x) \times S_R(z) \subset S_{2R}(x, z)$$

and

$$S_R(x, z) \subset Q_R(x, z) \subset S_{(n+1)R}(x, z)$$

for all $(x, z) \in \mathbb{R}^{n+1}$ and $R > 0$.

As discussed in [26, Section 7.1], $h''(z) = |z|^{1/s-2}$ is a Muckenhoupt $A_\infty(\mathbb{R})$ weight. In particular, the following lemma holds. See [14, Section 9.3] for definitions and properties of the class $A_\infty(\mathbb{R})$.

Lemma 4.6. *Given $0 < \varepsilon < 1$, there is $0 < \varepsilon_0 < 1$, depending only on ε and $0 < s < 1$, such that for any section $S_R = S_R(z_0)$ and any measurable set $E \subset S_R$,*

$$\frac{|E|}{|S_R|} < \varepsilon_0 \quad \text{implies} \quad \frac{\mu_h(E)}{\mu_h(S_R)} < \varepsilon.$$

We can now establish Theorem 4.2 for $\psi = \Phi$.

Corollary 4.7. *We have $\mu_\varphi \in (DC)_\varphi$ and $\mu_h \in (DC)_h$, so that (1)–(4) in Theorem 4.2 hold and are equivalent for φ and h . Moreover, the following statements hold and are equivalent.*

1. $\mu_\Phi \in (DC)_\Phi$ with corresponding doubling constant $C_d = C_d(n, s) > 0$;
2. Φ satisfies the engulfing property with corresponding constant $\theta = \theta(n, s) > 0$;
3. there are positive constants $C = C(n, s)$, $c = c(n, s)$ such that

$$cR^{n+1} \leq |S_R(x, z)| \mu_\Phi(S_R(x, z)) \leq CR^{n+1}$$

for all sections $S_R(x, z)$;

4. δ_Φ is a quasi-distance with constant $K = K(n, s) \geq 1$.

Consequently, μ_φ , μ_h and μ_Φ satisfying the doubling estimate (4.6).

Proof. By (4.3), we can write

$$\mu_\varphi(S_R(x_0)) = |B_{\sqrt{2}R}(x_0)| \leq 2^n \left| \frac{1}{2} B_{\sqrt{2}R}(x_0) \right| = 2^n \mu_\varphi \left(\frac{1}{2} S_R(x_0) \right)$$

Hence $\varphi \in (DC)_\varphi$ with doubling constant $C_d^\varphi = C_d^\varphi(n)$. Since $h''(z)$ is a Muckenhoupt $A_\infty(\mathbb{R})$ weight for all $0 < s < 1$, we have that $\mu_h \in (DC)_h$ with doubling constant $C_d^h = C_d^h(s)$. It follows from Theorem 4.2 that μ_φ and μ_h satisfy the engulfing property and, by Lemma 4.5, so does μ_Φ . Hence, the conclusion follows from Theorem 4.2. $\square$

Remark 4.8. There is a constant q_s , depending only on s , so that $S_R(0) = B_{q_s R^s}(0)$ for any $R > 0$. Indeed, $z \in S_R(0)$ if and only if

$$\delta_h(0, z) = h(z) = \frac{s^2}{1-s} |z|^{1/s} < R$$

which is equivalent to

$$|z| < \left(\frac{1-s}{s^2} \right)^s R^s =: q_s R^s. \quad (4.7)$$

Notation 4.9. We will always use the following notation.

- θ is the engulfing constant associated with Φ .
- K is the quasi-triangle constant associated with Φ .
- K_d is the constant in (4.6).
- q_s is the constant in (4.7).

We end this section by presenting two lemmas that will be used later in the proofs.

Lemma 4.10. *Let $\psi \in C^1(\mathbb{R})$ be a strictly convex function. If $x_0 < x_1 < x_2$, then*

$$\delta_\psi(x_0, x_1) < \delta_\psi(x_0, x_2) \quad \text{and} \quad \delta_\psi(x_1, x_2) < \delta_\psi(x_0, x_2).$$

Consequently, for any $x_0, x_1 \in \mathbb{R}$ and $R > 0$, if $r > 0$ is such that $S_\psi(x_1, r) \subset S_\psi(x_0, R)$, then $r \leq R$. In particular, for any $(x_0, z_0), (x_1, z_1) \in \mathbb{R}^{n+1}$ and $R > 0$, if $r > 0$ is such that $Q_r(x_1, z_1) \subset Q_R(x_0, z_0)$, then $r \leq R$.

Proof. By the convexity of ψ ,

$$\psi'(x_0) < \frac{\psi(x_1) - \psi(x_2)}{x_1 - x_2},$$

so that

$$\begin{aligned} \delta_\psi(x_0, x_1) &= \psi(x_1) - \psi(x_0) - \psi'(x_0)(x_1 - x_0) \\ &< \psi(x_2) - \psi(x_0) - \psi'(x_0)(x_2 - x_0) = \delta_\psi(x_0, x_2). \end{aligned}$$

Next, define a function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Psi(x) := [\psi(x_1) + \psi'(x_1)(x - x_1)] - [\psi(x_0) + \psi'(x_0)(x - x_0)].$$

By the convexity of ψ , $\Psi'(x) = \psi'(x_1) - \psi'(x_0) > 0$, so Ψ is increasing. Since

$$\Psi(x_1) = \psi(x_1) - \psi(x_0) + \psi'(x_0)(x_1 - x_0) = \delta_\psi(x_0, x_1) > 0,$$

we know that

$$\begin{aligned} 0 < \Psi(x_2) &= [\psi(x_1) + \psi'(x_1)(x_2 - x_1)] - [\psi(x_0) + \psi'(x_0)(x_2 - x_0)] \\ &= [-\psi(x_2) + \psi(x_1) + \psi'(x_1)(x_2 - x_1)] - [-\psi(x_2) + \psi(x_0) + \psi'(x_0)(x_2 - x_0)]. \end{aligned}$$

Hence,

$$\begin{aligned} \delta_\psi(x_1, x_2) &= \psi(x_2) - \psi(x_1) - \psi'(x_1)(x_2 - x_1) \\ &< \psi(x_2) - \psi(x_0) - \psi'(x_0)(x_2 - x_0) = \delta_\psi(x_0, x_2). \end{aligned}$$

Lastly, fix $x_0, x_1 \in \mathbb{R}$ and $R > 0$. Suppose that $r > 0$ is such that $S_\psi(x_1, r) \subset S_\psi(x_0, R)$. Write

$$\begin{aligned} S_\psi(x_0, R) &= (x_0^L, x_0^R), \quad x_0^L < x_0 < x_0^R \\ S_\psi(x_1, r) &= (x_1^L, x_1^R), \quad x_1^L < x_1 < x_1^R. \end{aligned}$$

Without loss of generality, assume that $x_0 \leq x_1$ so that $x_0 \leq x_1 < x_1^R \leq x_0^R$. Then

$$r = \delta_\psi(x_1, x_1^R) \leq \delta_\psi(x_1, x_0^R) \leq \delta_\psi(x_0, x_0^R) = R. \quad \square$$

Lemma 4.11 (Theorem 3.3.10 in [17]). *There exist constants $C_0 > 0$, $p \geq 1$, depending only on n and s , such that for $0 < r_1 < r_2 \leq 1$, $t > 0$ and $(x_1, z_1) \in S_{r_1 t}(x_0, z_0)$, we have that*

$$S_{C_0(r_2-r_1)^p t}(x_1, z_1) \subset S_{r_2 t}(x_0, z_0).$$

5. Reductions of Theorem 1.3

In this section we show that, after a series of reductions, Theorem 1.3 will follow from Theorem 5.3. First, we show in Theorem 5.1 that it is enough to consider Monge–Ampère cubes, instead of Monge–Ampère sections, and to take a nonnegative right hand side f . The second reduction, Theorem 5.2, demonstrates how we only need to show that the supremum of U in a small cube is controlled by the value of U at the center of the cube. Finally, Theorem 5.3 is a normalized statement which says that if U is controlled at the center of the cube and $\|f\|_{L^\infty}$ is controlled with respect to the size of the section, then U is uniformly bounded in a smaller cube. Hence, the rest of the paper will be devoted to proving Theorem 5.3 and Theorem 1.1.

5.1. First reduction

We first show that it is enough to prove Theorem 1.3 in Monge–Ampère cubes and with $f \geq 0$.

Theorem 5.1. *Let Ω be a bounded domain, $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ be bounded, measurable functions that satisfy (1.5) and let $f \in L^\infty(\Omega)$ be nonnegative. There exist positive constants $C_H = C_H(n, \lambda, \Lambda, s) > 1$, $\kappa_1 = \kappa_1(n, s) < 1$, and $\hat{K}_1 = \hat{K}_1(n, s) > 1$ such that for every cube $Q_{\hat{K}_1 R} = Q_{\hat{K}_1 R}(x_0, z_0) \subset \Omega \times \mathbb{R}$ and every nonnegative solution $U \in C^2(Q_{\hat{K}_1 R} \setminus \{z = 0\}) \cap C(Q_{\hat{K}_1 R})$ such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(Q_{\hat{K}_1 R} \cap \{z \geq 0\})$ to*

$$\begin{cases} a^{ij}(x) \partial_{ij} U + |z|^{2-\frac{1}{s}} \partial_{zz} U = 0 & \text{in } Q_{\hat{K}_1 R} \cap \{z \neq 0\} \\ -\partial_{z+} U(x, 0) = f & \text{on } Q_{\hat{K}_1 R} \cap \{z = 0\}, \end{cases}$$

we have that

$$\sup_{Q_{\kappa_1 R}} U \leq C_H \left(\inf_{Q_{\kappa_1 R}} U + \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} R^s \right). \quad (5.1)$$

Proof of Theorem 1.3 from Theorem 5.1. Observe that

$$Q_{\hat{K}_1 R}(x_0, z_0) \subset Q_{\theta^2 \hat{K}_1 R}(x_0, z_0) \subset S_{(n+1)\theta^2 \hat{K}_1 R}(x_0, z_0)$$

and

$$S_{\kappa_1 R}(x_0, z_0) \subset Q_{\kappa_1 R}(x_0, z_0)$$

Let $\hat{K}_0 = (n+1)\theta^2 \hat{K}_1$ and $\kappa_0 = \kappa_1$.

Case 1: $Q_{\hat{K}_1 R}(x_0, z_0) \cap \{z = 0\} = \emptyset$. By Theorem 5.1 and the inclusion above, we get

$$\sup_{S_{\kappa_0 R}} U \leq \sup_{Q_{\kappa_1 R}} U \leq C_H \inf_{Q_{\kappa_1 R}} U \leq C_H \inf_{S_{\kappa_0 R}} U.$$

Case 2: $Q_{\hat{K}_1 R}(x_0, z_0) \cap \{z = 0\} \neq \emptyset$ and $z_0 = 0$. Define

$$V = U - \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} |z| + \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} |S_{\hat{K}_1 R}(0)|.$$

We claim that V is nonnegative in $Q_{\hat{K}_1 R}$. Indeed, let $(x, z) \in Q_{\hat{K}_1 R}$, so that $z \in S_{\hat{K}_1 R}(0) \subset \mathbb{R}$. By Remark 4.8, $S_{\hat{K}_1 R}(0) = B_{q_s \hat{K}_1^s R^s}(0)$. Thus, $|z| \leq |B_{q_s \hat{K}_1^s R^s}(0)| = |S_{\hat{K}_1 R}(0)|$. Consequently, for any $(x, z) \in Q_{\hat{K}_1 R}$, we have that

$$-\|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} |z| + \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} |S_{\hat{K}_1 R}(0)| \geq 0,$$

so that $V \geq U \geq 0$ in $Q_{\hat{K}_1 R}$.

Next, notice that V is symmetric across $\{z = 0\}$ and that $V \in C^2(Q_{\hat{K}_1 R} \setminus \{z = 0\}) \cap C(Q_{\hat{K}_1 R})$, $V_{z+} \in C(Q_{\hat{K}_1 R} \cap \{z \geq 0\})$. Moreover, for $(x, z) \in Q_{\hat{K}_1 R} \cap \{z \neq 0\}$, it is clear that

$$a^{ij}(x) \partial_{ij} V + |z|^{2-\frac{1}{s}} \partial_{zz} V = a^{ij}(x) \partial_{ij} U + |z|^{2-\frac{1}{s}} \partial_{zz} U = 0$$

and for $(x, 0) \in Q_{\hat{K}_1 R} \cap \{z = 0\}$,

$$-\partial_{z+} V(x, 0) = f(x) + \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} := g(x) \geq 0.$$

Therefore, V is a nonnegative solution to

$$\begin{cases} a^{ij}(x) \partial_{ij} V + |z|^{2-\frac{1}{s}} \partial_{zz} V = 0 & \text{in } Q_{\hat{K}_1 R} \cap \{z \neq 0\} \\ -\partial_{z+} V = g & \text{on } Q_{\hat{K}_1 R} \cap \{z = 0\}. \end{cases}$$

Therefore, by Theorem 5.1 applied to V and using that $|S_{\hat{K}_1 R}(0)| = |B_{q_s \hat{K}_1^s R^s}(0)| = C_{n,s} R^s$,

$$\begin{aligned} \sup_{S_{\kappa_0 R}} U &\leq \sup_{Q_{\kappa_1 R}} U \\ &\leq \sup_{Q_{\kappa_1 R}} V \\ &\leq C_H \left(\inf_{Q_{\kappa_1 R}} V + \|g\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} R^s \right) \\ &= C_H \left(\inf_{Q_{\kappa_1 R}} \left(U - \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} |z| + \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} |S_{\hat{K}_1 R}(0)| \right) \right. \\ &\quad \left. + \left\| f + \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} \right\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} R^s \right) \\ &\leq C_H \left(\inf_{Q_{\kappa_1 R}} U + (C_{n,s} + 2) \|f\|_{L^\infty(Q_{\hat{K}_1 R} \cap \{z=0\})} R^s \right) \\ &\leq C'_H \left(\inf_{S_{\kappa_0 R}} U + \|f\|_{L^\infty(S_{\hat{K}_0 R} \cap \{z=0\})} R^s \right). \end{aligned}$$

Case 3: $Q_{\hat{K}_1 R}(x_0, z_0) \cap \{z = 0\} \neq \emptyset$ and $z_0 \neq 0$. In this case, $0 \in S_{\hat{K}_1 R}(z_0)$. Then, by the engulfing property,

$$\begin{aligned} Q_{\hat{K}_1 R}(x_0, z_0) &= Q_{\hat{K}_1 R}(x_0) \times S_{\hat{K}_1 R}(z_0) \\ &\subset Q_{\theta \hat{K}_1 R}(x_0) \times S_{\theta \hat{K}_1 R}(0) = Q_{\theta \hat{K}_1 R}(x_0, 0). \end{aligned}$$

Again, applying the engulfing property,

$$\begin{aligned} Q_{\theta \hat{K}_1 R}(x_0, 0) &= Q_{\theta \hat{K}_1 R}(x_0) \times S_{\theta \hat{K}_1 R}(0) \\ &\subset Q_{\theta^2 \hat{K}_1 R}(x_0) \times S_{\theta^2 \hat{K}_1 R}(z_0) = Q_{\theta^2 \hat{K}_1 R}(x_0, z_0). \end{aligned}$$

Define

$$V = U - \|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z=0\})} |z| + \|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z=0\})} |S_{\theta \hat{K}_1 R}(0)|.$$

We claim that V is nonnegative in $Q_{\theta \hat{K}_1 R}(x_0, 0)$. Let $(x, z) \in Q_{\theta \hat{K}_1 R}(x_0, 0)$, so that, by Remark 4.8, $z \in S_{\theta \hat{K}_1 R}(0) = B_{q_s \theta^s \hat{K}_1^s R^s}(0) \subset \mathbb{R}$. In particular, $|z| \leq |B_{q_s \theta^s \hat{K}_1^s R^s}(0)| = |S_{\hat{K}_1 \theta R}(0)|$. Consequently, for any $(x, z) \in Q_{\hat{K}_1 R}(x_0, z_0)$, we have that

$$-\|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z=0\})} |z| + \|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z=0\})} |S_{\theta \hat{K}_1 R}(0)| \geq 0,$$

so that $V \geq U \geq 0$ in $Q_{\theta \hat{K}_1 R}(x_0, 0)$.

Next, notice that V is symmetric across $\{z = 0\}$ and that $V \in C^2(Q_{\theta \hat{K}_1 R}(x_0, 0) \setminus \{z = 0\}) \cap C(Q_{\theta \hat{K}_1 R}(x_0, 0))$, $V_{z+} \in C(Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z \geq 0\})$. Moreover, for $(x, z) \in Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z \neq 0\}$, it is clear that

$$a^{ij}(x) \partial_{ij} V + |z|^{2-\frac{1}{s}} \partial_{zz} V = a^{ij}(x) \partial_{ij} U + |z|^{2-\frac{1}{s}} \partial_{zz} U = 0$$

and for $(x, 0) \in Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z = 0\}$,

$$-\partial_{z+} V(x, 0) = f(x) + \|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z=0\})} := g(x) \geq 0.$$

Therefore, V is a nonnegative solution to

$$\begin{cases} a^{ij}(x) \partial_{ij} V + |z|^{2-\frac{1}{s}} \partial_{zz} V = 0 & \text{in } Q_{\hat{K}_1 R}(x_0, z_0) \cap \{z \neq 0\} \\ -\partial_{z+} V = g & \text{on } Q_{\hat{K}_1 R}(x_0, z_0) \cap \{z = 0\}. \end{cases}$$

Applying Theorem 5.1 to V and using that $|S_{\theta \hat{K}_1 R}(0)| = |B_{q_s \theta^s \hat{K}_1^s R^s}(0)| = C_{n,s} R^s$, we get

$$\begin{aligned} \sup_{S_{\kappa_0 R}(x_0, z_0)} U &\leq \sup_{Q_{\kappa_1 R}(x_0, z_0)} U \\ &\leq \sup_{Q_{\kappa_1 R}(x_0, z_0)} V \\ &\leq C_H \left(\inf_{Q_{\kappa_1 R}(x_0, z_0)} V + \|g\|_{L^\infty(Q_{\hat{K}_1 R}(x_0, z_0) \cap \{z=0\})} R^s \right) \\ &= C_H \left(\inf_{Q_{\kappa_1 R}(x_0, z_0)} \left(U - \|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0))} |z| + \|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0))} |S_{\theta \hat{K}_1 R}(0)| \right) \right. \\ &\quad \left. + \left\| f + \|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z=0\})} \right\|_{L^\infty(Q_{\hat{K}_1 R}(x_0, z_0) \cap \{z=0\})} R^s \right) \\ &\leq C_H \left(\inf_{Q_{\kappa_1 R}(x_0, z_0)} U + (C_{n,s} + 2) \|f\|_{L^\infty(Q_{\theta \hat{K}_1 R}(x_0, 0) \cap \{z=0\})} R^s \right) \\ &\leq C'_H \left(\inf_{S_{\kappa_0 R}(x_0, z_0)} U + \|f\|_{L^\infty(S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\})} R^s \right). \end{aligned}$$

Therefore, (1.15) holds in all cases.

It remains to prove the Hölder estimate (1.16). The proof follows by a standard argument (see, for example, [13, Sections 8.9 and 9.8]). We provide the details for completeness. Let $0 < r \leq R$ and define

$$M(r) = \sup_{S_r(x_0, z_0)} U \quad \text{and} \quad m(r) = \inf_{S_r(x_0, z_0)} U.$$

Apply (1.15) to $M(\hat{K}_0 r) - U \geq 0$ in $S_{\hat{K}_0 r}(x_0, z_0)$ to obtain

$$\sup_{S_{\kappa_0 r}(x_0, z_0)} (M(\hat{K}_0 r) - U) \leq C_H \left(\inf_{S_{\kappa_0 r}(x_0, z_0)} (M(\hat{K}_0 r) - U) + \|f\|_{L^\infty(S_{\hat{K}_0 r}(x_0, z_0) \cap \{z=0\})} r^s \right).$$

Therefore,

$$M(\hat{K}_0 r) - m(\kappa_0 r) \leq C_H \left(M(\hat{K}_0 r) - M(\kappa_0 r) + \|f\|_{L^\infty(S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\})} r^s \right). \quad (5.2)$$

Similarly, apply (1.15) to $U - m(\hat{K}_0 r) \geq 0$ in $S_{\hat{K}_0 r}(x_0, z_0)$ to obtain

$$\sup_{S_{\kappa_0 r}(x_0, z_0)} (U - m(\hat{K}_0 r)) \leq C_H \left(\inf_{S_{\kappa_0 r}(x_0, z_0)} (U - m(\hat{K}_0 r)) + \|f\|_{L^\infty(S_{\hat{K}_0 r}(x_0, z_0) \cap \{z=0\})} r^s \right),$$

so that

$$M(\kappa_0 r) - m(\hat{K}_0 r) \leq C_H \left(m(\kappa_0 r) - m(\hat{K}_0 r) + \|f\|_{L^\infty(S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\})} r^s \right). \quad (5.3)$$

Let $\omega(r) = M(r) - m(r)$. Adding (5.2) and (5.3) together, we get

$$\omega(\hat{K}_0 r) + \omega(\kappa_0 r) \leq C_H \left(\omega(\hat{K}_0 r) - \omega(\kappa_0 r) + 2 \|f\|_{L^\infty(S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\})} r^s \right).$$

After rearranging,

$$\omega(\kappa_0 r) \leq \gamma \omega(\hat{K}_0 r) + \sigma(r), \quad \gamma = \frac{C_H - 1}{C_H + 1} < 1$$

where

$$\sigma(r) = \begin{cases} \frac{2C_H}{C_H+1} \|f\|_{L^\infty(S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\})} r^s & \text{if } S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\} \neq \emptyset \\ 0 & \text{if } S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\} = \emptyset \end{cases}$$

is a non-decreasing function of r . Note that $\gamma = \gamma(n, \lambda, \Lambda, s)$. By [13, Lemma 8.23], for any $\mu \in (0, 1)$, there are constants $C = C(n, \lambda, \Lambda, s) > 0$ and $\alpha_1 = (1 - \mu) \log \gamma / \log(\kappa_0 / \hat{K}_0)$ such that

$$\omega(\hat{K}_0 r) \leq C \left(\left(\frac{r}{R} \right)^{\alpha_1} \omega(\hat{K}_0 R) + \sigma(r^\mu R^{1-\mu}) \right).$$

Choose $\mu = \mu(n, \lambda, \Lambda, s)$ so that $2\alpha_1 < \mu s$. Then,

$$\begin{aligned} \omega(\hat{K}_0 r) &\leq C \left(\frac{r}{R} \right)^{\alpha_1} \left(\omega(\hat{K}_0 R) + \sigma(r^\mu R^{1-\mu}) \left(\frac{R}{r} \right)^{\alpha_1} \right) \\ &\leq C \left(\frac{r}{R} \right)^{\alpha_1} \left(2 \sup_{S_{\hat{K}_0 R}(x_0, z_0)} |U| + \frac{2C_H}{C_H + 1} \|f\|_{L^\infty(S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\})} r^{\mu s} R^{(1-\mu)s} R^{\alpha_1} r^{-\alpha_1} \right) \end{aligned}$$

$$\leq \hat{C}_1 \left(\frac{\hat{K}_0 r}{\hat{K}_0 R} \right)^{\alpha_1} \left(\sup_{S_{\hat{K}_0 R}(x_0, z_0)} |U| + \|f\|_{L^\infty(S_{\hat{K}_0 R}(x_0, z_0) \cap \{z=0\})} R^s \right).$$

By taking $\hat{K}_0 r = \delta_\Phi((x_0, z_0), (x, z))$, the estimate in (1.16) follows. $\square$

5.2. Second reduction

Next, we show that Theorem 5.1 follows from the following result which says that the supremum of U in a small cube can be controlled by its value at the center.

Theorem 5.2. *Let Ω be a bounded domain, $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ be bounded, measurable functions that satisfy (1.5) and let $f \in L^\infty(\Omega)$ be nonnegative. There exist positive constants $C_H = C_H(n, \lambda, \Lambda, s) > 1$, $\kappa_2 = \kappa_2(n, s) < 1$, and $\hat{K}_3 = \hat{K}_3(n, s) > 1$ such that for every cube $Q_{\hat{K}_3 R} = Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z}) \subset \subset \Omega \times \mathbb{R}$ and every nonnegative solution $U \in C^2(Q_{\hat{K}_3 R} \setminus \{z = 0\}) \cap C(Q_{\hat{K}_3 R})$ such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(Q_{\hat{K}_3 R} \cap \{z \geq 0\})$ to*

$$\begin{cases} a^{ij}(x) \partial_{ij} U + |z|^{2-\frac{1}{s}} \partial_{zz} U = 0 & \text{in } Q_{\hat{K}_3 R} \cap \{z \neq 0\} \\ -\partial_{z+} U(x, 0) = f & \text{on } Q_{\hat{K}_3 R} \cap \{z = 0\}, \end{cases}$$

we have that

$$\sup_{Q_{\kappa_2 R}} U \leq C_H \left(U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} R^s \right).$$

Proof of Theorem 5.1 from Theorem 5.2. Let $\hat{K}_1 = \hat{K}_1(n, s)$ and $\kappa_1 = \kappa_1(n, s)$ be such that

$$1 < \theta \hat{K}_3 \leq \hat{K}_1 \quad \text{and} \quad \theta \kappa_1 \leq \kappa_2 < 1.$$

Let $(\tilde{x}, \tilde{z}) \in Q_{\kappa_1 R}(x_0, z_0)$. By the engulfing property,

$$Q_{\kappa_1 R}(x_0, z_0) \subset Q_{\theta \kappa_1 R}(\tilde{x}, \tilde{z}) \subset Q_{\kappa_2 R}(\tilde{x}, \tilde{z}).$$

Again applying the engulfing property, we have

$$Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z}) \subset Q_{\theta \hat{K}_3 R}(x_0, z_0) \subset Q_{\hat{K}_1 R}(x_0, z_0) \subset \subset \Omega \times \mathbb{R}.$$

By Theorem 5.2, we get

$$\begin{aligned} \sup_{Q_{\kappa_1 R}(x_0, z_0)} U &\leq \sup_{Q_{\kappa_2 R}(\tilde{x}, \tilde{z})} U \\ &\leq C_H \left(U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z}) \cap \{z=0\})} R^s \right) \\ &\leq C_H \left(U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_1 R}(x_0, z_0) \cap \{z=0\})} R^s \right). \end{aligned}$$

Taking the infimum over all $(\tilde{x}, \tilde{z}) \in Q_{\kappa_1 R}(x_0, z_0)$, the Harnack inequality (5.1) holds. $\square$

5.3. Third reduction

Here we will see that Theorem 5.2 follows from the next, and final, reduction which is a normalized statement.

Theorem 5.3. Fix $a > 0$. Let Ω be a bounded domain, $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ be bounded, measurable functions that satisfy (1.5) and let $f \in L^\infty(\Omega)$ be nonnegative. Let $\hat{K}_3$ be as in Theorem 5.2. There exist positive constants $C_H = C_H(n, \lambda, \Lambda, s) > 1$, $\kappa_2 = \kappa_2(n, s) < 1$, and $K_0 = K_0(n, s)$ such that for any cube $Q_{\hat{K}_3 R} = Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z}) \subset \subset \Omega \times \mathbb{R}$ and every nonnegative solution $U \in C^2(Q_{\hat{K}_3 R} \setminus \{z = 0\}) \cap C(Q_{\hat{K}_3 R})$ such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(Q_{\hat{K}_3 R} \cap \{z \geq 0\})$ to

$$\begin{cases} a^{ij}(x)\partial_{ij}U + |z|^{2-\frac{1}{s}}\partial_{zz}U = 0 & \text{in } Q_{\hat{K}_3 R} \cap \{z \neq 0\} \\ -\partial_{z+}U(x, 0) = f & \text{on } Q_{\hat{K}_3 R} \cap \{z = 0\}, \end{cases}$$

if

$$U(\tilde{x}, \tilde{z}) \leq \frac{aR}{2K_0}$$

and

$$\|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} \leq a\mu_h(S_{\hat{K}_3 R}(\tilde{z})),$$

then

$$U \leq C_H aR \quad \text{in } Q_{\kappa_2 R}. \quad (5.4)$$

Proof of Theorem 5.2 from Theorem 5.3. Let $\varepsilon > 0$. Define the nonnegative function W_ε in $Q_{\hat{K}_3 R}$ by

$$W_\varepsilon(x, z) = \frac{aR}{2K_0 U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} R/\mu_h(S_{\hat{K}_3 R}(\tilde{z})) + \varepsilon} U(x, z).$$

If $0 \notin S_{\hat{K}_3 R}(\tilde{z})$, then $\|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} R/\mu_h(S_{\hat{K}_3 R}(\tilde{z})) = 0$. Notice that W_ε is symmetric across $\{z = 0\}$ and $W_\varepsilon \in C^2(Q_{\hat{K}_3 R} \setminus \{z = 0\}) \cap C(Q_{\hat{K}_3 R})$, $(W_\varepsilon)_{z+} \in C(Q_{\hat{K}_3 R} \cap \{z \geq 0\})$. Moreover, in $Q_{\hat{K}_3 R} \cap \{z \neq 0\}$, we have

$$a^{ij}(x)\partial_{ij}W_\varepsilon + |z|^{2-\frac{1}{s}}\partial_{zz}W_\varepsilon = 0$$

and, in $Q_{\hat{K}_3 R} \cap \{z = 0\}$,

$$-\partial_{z+}W_\varepsilon(x, 0) = \frac{aR}{2K_0 U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} R/\mu_h(S_{\hat{K}_3 R}(\tilde{z})) + \varepsilon} f(x) =: g(x) \geq 0.$$

Therefore, W_ε is a nonnegative solution to

$$\begin{cases} a^{ij}(x)\partial_{ij}W_\varepsilon + |z|^{2-\frac{1}{s}}\partial_{zz}W_\varepsilon = 0 & \text{in } Q_{\hat{K}_3 R} \cap \{z \neq 0\} \\ -\partial_{z+}W_\varepsilon(x, 0) = g & \text{on } Q_{\hat{K}_3 R} \cap \{z = 0\}. \end{cases}$$

Clearly,

$$\|g\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} \leq a\mu_h(S_{\hat{K}_3 R}(\tilde{z}))$$

and

$$W_\varepsilon(\tilde{x}, \tilde{z}) \leq \frac{aR}{2K_0}.$$

By Theorem 5.3 applied to W_ε , we get

$$W_\varepsilon \leq C_H a R \quad \text{in } Q_{\kappa_2 R}$$

which implies

$$\sup_{Q_{\kappa_2 R}} U \leq C'_H \left(U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} \frac{R}{\mu_h(S_{\hat{K}_3 R}(\tilde{z}))} + \varepsilon \right).$$

If $0 \notin S_{\hat{K}_3 R}(\tilde{z})$, then

$$\sup_{Q_{\kappa_2 R}} U \leq C'_H (U(\tilde{x}, \tilde{z}) + \varepsilon).$$

If $0 \in S_{\hat{K}_3 R}(\tilde{z})$, then by the engulfing property and Remark 4.8,

$$S_{\hat{K}_3 R}(\tilde{z}) \subset S_{\theta \hat{K}_3 R}(0) = B_{q_s \theta^s \hat{K}_3^s R^s}(0).$$

Hence, $|S_{\hat{K}_3 R}(\tilde{z})| \leq C R^s$ for some $C = C(n, s) > 0$. With this and Corollary 4.7 part (3),

$$\begin{aligned} \sup_{Q_{\kappa_2 R}} U &\leq C'_H \left(U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} \frac{1}{\hat{K}_3} \frac{\hat{K}_3 R}{\mu_h(S_{\hat{K}_3 R}(\tilde{z}))} + \varepsilon \right) \\ &\leq C'_H \left(U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} \frac{1}{c\hat{K}_3} |S_{\hat{K}_3 R}(\tilde{z})| + \varepsilon \right) \\ &\leq C''_H \left(U(\tilde{x}, \tilde{z}) + \|f\|_{L^\infty(Q_{\hat{K}_3 R} \cap \{z=0\})} R^s + \varepsilon \right). \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, the conclusion follows. $\square$

6. Paraboloids associated to Φ

In this section, we define the Monge–Ampère paraboloids associated with Φ in (4.2) and study their basic properties and relations with respect to solutions to the extension problem.

Definition 6.1. Let $a > 0$. A *paraboloid* P of opening a in $\mathbb{R}^{n+1}$ is defined as

$$P(x, z) = -a\Phi(x, z) + \langle (y, w), (x, z) \rangle + b \quad (x, z) \in \mathbb{R}^{n+1} \quad (6.1)$$

for some $(y, w) \in \mathbb{R}^{n+1}$ and $b \in \mathbb{R}$.

Since $\Phi \in C^1(\mathbb{R}^{n+1})$ is strictly convex, the point where the maximum of P occurs, which we call the *vertex* (x_v, z_v) of P , is the unique solution to $\nabla P(x_v, z_v) = 0$.

We say that P *touches* a continuous function $U : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ *from below* at (x_0, z_0) in a convex set $S \subset \mathbb{R}^{n+1}$ if

$$P(x_0, z_0) = V(x_0, z_0) \quad \text{and} \quad P(x, z) \leq V(x, z) \quad \text{for all } (x, z) \in S.$$

Lemma 6.2. *A paraboloid P of opening $a > 0$ with vertex (x_v, z_v) given by (6.1) can be written as*

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + c \quad (6.2)$$

for some constant $c \in \mathbb{R}$. Moreover,

$$\nabla P(x, z) = -a(\nabla\Phi(x, z) - \nabla\Phi(x_v, z_v)) = -a(x - x_v, h'(z) - h'(z_v))$$

and

$$\partial_z P(x, 0) = ah'(z_v)$$

for all $(x, z) \in \mathbb{R}^{n+1}$. If P coincides with a continuous function $U : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ at a point (x_0, z_0) , i.e. $P(x_0, z_0) = U(x_0, z_0)$, then

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0).$$

Proof. Since $0 = \nabla P(x_v, z_v) = -a\nabla\Phi(x_v, z_v) + (y, w)$, we can write

$$P(x, z) = -a\Phi(x, z) + a\langle \nabla\Phi(x_v, z_v), (x, z) \rangle + b.$$

Consequently,

$$\nabla P(x, z) = -a(x - x_v, h'(z) - h'(z_v))$$

and

$$\partial_z P(x, 0) = -a(h'(z) - h'(z_v)) \big|_{(x, 0)} = ah'(z_v).$$

Moreover, we have

$$\begin{aligned} P(x, z) &= -a\Phi(x, z) + a\langle \nabla\Phi(x_v, z_v), (x, z) \rangle + b \\ &\quad + a\Phi(x_v, z_v) - a\Phi(x_v, z_v) - a\langle \nabla\Phi(x_v, z_v), (x_v, z_v) \rangle + a\langle \nabla\Phi(x_v, z_v), (x_v, z_v) \rangle \\ &= -a(\Phi(x, z) - \Phi(x_v, z_v) - \langle \nabla\Phi(x_v, z_v), (x, z) - (x_v, z_v) \rangle) \\ &\quad - a\Phi(x_v, z_v) + a\langle \nabla\Phi(x_v, z_v), (x_v, z_v) \rangle + b \\ &= -a\delta_\Phi((x_v, z_v), (x, z)) + c. \end{aligned}$$

If $P(x_0, z_0) = U(x_0, z_0)$, then $U(x_0, z_0) = -a\delta_\Phi((x_v, z_v), (x_0, z_0)) + c$ and, after solving for c , we conclude that $P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0)$. $\square$

For the remainder of the paper, we use the terminology *paraboloids* to reference those given by (6.1), or equivalently, (6.2).

Lemma 6.3. *Suppose that P is a paraboloid of opening $a > 0$ that touches a continuous function $U : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ from below at (x_0, z_0) in a convex set $S \subset \mathbb{R}^{n+1}$. For any $\tilde{a} \geq a$, there exists a paraboloid $\tilde{P}$ of opening $\tilde{a} > 0$ that touches U from below at (x_0, z_0) in S .*

Proof. Begin by writing

$$\begin{aligned}
 P(x, z) &= -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0) \\
 &= -a(\Phi(x, z) - \Phi(x_v, z_v) - \langle \nabla \Phi(x_v, z_v), (x, z) - (x_v, z_v) \rangle) \\
 &\quad + a(\Phi(x_0, z_0) - \Phi(x_v, z_v) - \langle \nabla \Phi(x_v, z_v), (x_0, z_0) - (x_v, z_v) \rangle) + U(x_0, z_0) \\
 &= -a\Phi(x, z) + a\langle \nabla \Phi(x_v, z_v), (x, z) \rangle + a\Phi(x_0, z_0) - a\langle \nabla \Phi(x_v, z_v), (x_0, z_0) \rangle + U(x_0, z_0) \\
 &= -a(\Phi(x, z) - \Phi(x_0, z_0) - \langle \nabla \Phi(x_0, z_0), (x, z) - (x_0, z_0) \rangle) \\
 &\quad + a\langle \nabla \Phi(x_v, z_v), (x, z) - (x_0, z_0) \rangle - a\langle \nabla \Phi(x_0, z_0), (x, z) - (x_0, z_0) \rangle + U(x_0, z_0) \\
 &= -a\delta_\Phi((x_0, z_0), (x, z)) + a\langle \nabla \Phi(x_v, z_v) - \nabla \Phi(x_0, z_0), (x, z) - (x_0, z_0) \rangle + U(x_0, z_0).
 \end{aligned}$$

Define $\tilde{P}$ by

$$\tilde{P}(x, z) = -\tilde{a}\delta_\Phi((x_0, z_0), (x, z)) + a\langle \nabla \Phi(x_v, z_v) - \nabla \Phi(x_0, z_0), (x, z) - (x_0, z_0) \rangle + U(x_0, z_0).$$

Note that $\tilde{P}$ is a paraboloid of opening $\tilde{a} > 0$ since it can be expressed as (6.1) in the following way

$$\begin{aligned}
 \tilde{P}(x, z) &= -\tilde{a}\Phi(x, z) + \tilde{a}\Phi(x_0, z_0) + \tilde{a}\langle \nabla \phi(x_0, z_0), (x, z) - (x_0, z_0) \rangle \\
 &\quad + a\langle \nabla \Phi(x_v, z_v) - \nabla \Phi(x_0, z_0), (x, z) - (x_0, z_0) \rangle + U(x_0, z_0) \\
 &= -\tilde{a}\Phi(x, z) + \langle \tilde{a}\nabla \Phi(x_0, z_0) + a\nabla \Phi(x_v, z_v) - a\nabla \Phi(x_0, z_0), (x, z) \rangle \\
 &\quad + \tilde{a}\Phi(x_0, z_0) - \langle \tilde{a}\nabla \phi(x_0, z_0) + a\nabla \Phi(x_v, z_v) - a\nabla \Phi(x_0, z_0), (x_0, z_0) \rangle + U(x_0, z_0).
 \end{aligned}$$

Since $\tilde{P}(x_0, z_0) = U(x_0, z_0)$ and

$$\begin{aligned}
 \tilde{P}(x, z) &\leq -a\delta_\Phi((x_0, z_0), (x, z)) + a\langle \nabla \Phi(x_v, z_v) - \nabla \Phi(x_0, z_0), (x, z) - (x_0, z_0) \rangle + U(x_0, z_0) \\
 &= P(x, z) \leq U(x, z),
 \end{aligned}$$

for every $(x, z) \in S$, we conclude that $\tilde{P}$ touches U from below at (x_0, z_0) in S . $\square$

The next two lemmas provide some observations regarding how the symmetry of U across $\{z = 0\}$ effects the geometry of the paraboloids that touch U from below.

Lemma 6.4. *Let $S \subset \mathbb{R}^{n+1}$ be an open, convex set that is symmetric across $\{z = 0\}$. Consider a continuous function $U : S \rightarrow \mathbb{R}$ which is symmetric across $\{z = 0\}$. Let P be a paraboloid of opening $a > 0$ with vertex (x_v, z_v) that touches U from below at (x_0, z_0) in S . If $z_0 > 0$, then $z_v \geq 0$, and if $z_0 < 0$, then $z_v \leq 0$. Moreover, the paraboloid $\tilde{P}(x, z) = P(x, -z)$ of opening $a > 0$ with vertex $(x_v, -z_v)$ that touches U from below at $(x_0, -z_0)$ in S .*

Proof. Assume that $z_0 > 0$. Write

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0)$$

and note that

$$\begin{aligned}
 P(x_0, -z_0) &= -a\delta_\Phi((x_v, z_v), (x_0, -z_0)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0) \\
 &= -a\delta_h(z_v, -z_0) + a\delta_h(z_v, z_0) + U(x_0, -z_0).
 \end{aligned}$$

Then

$$\begin{aligned} 0 &\leq U(x_0, -z_0) - P(x_0, -z_0) \\ &= a\delta_h(z_v, -z_0) - a\delta_h(z_v, z_0) \\ &= a(h(-z_0) - h(z_0)) + 2ah'(z_v)z_0 = 2ah'(z_v)z_0. \end{aligned}$$

Since $z_0 > 0$, it follows that $h'(z_v) \geq 0$. Hence, $z_v \geq 0$, as desired. The case for $z_0 < 0$ follows similarly.

Next, define $\tilde{P}$ by $\tilde{P}(x, z) = P(x, -z)$. Since $h'(-z_1) = -h'(z_1)$ and

$$\begin{aligned} \delta_h(z_1, -z_2) &= h(-z_2) - h(z_1) - h'(z_1)(-z_2 - z_1) \\ &= h(z_2) - h(-z_1) - h'(-z_1)(z_2 - (-z_1)) \\ &= \delta_h(-z_1, z_2) \end{aligned} \tag{6.3}$$

for all $z_1, z_2 \in \mathbb{R}$, we may write

$$\begin{aligned} \tilde{P}(x, z) &= P(x, -z) \\ &= -a\delta_\Phi((x_v, z_v), (x, -z)) + a\delta_\Phi((x_v, z_v), (x_0, z_0)) + U(x_0, z_0) \\ &= -a\delta_\varphi(x_v, x) + a\delta_\varphi(x_v, x_0) - a\delta_h(z_v, -z) + a\delta_h(z_v, z_0) + U(x_0, -z_0) \\ &= -a\delta_\varphi(x_v, x) + a\delta_\varphi(x_v, x_0) - a\delta_h(-z_v, z) + a\delta_h(-z_v, -z_0) + U(x_0, -z_0) \\ &= -a\delta_\Phi((x_v, -z_v), (x, z)) + a\delta_\Phi((x_v, -z_v), (x_0, -z_0)) + U(x_0, -z_0). \end{aligned}$$

Hence, $\tilde{P}$ is a paraboloid of opening $a > 0$ with vertex $(x_v, -z_v)$. Since

$$\tilde{P}(x, z) = P(x, -z) \leq U(x, -z) = U(x, z) \quad \text{for all } (x, z) \in S$$

and

$$\tilde{P}(x_0, -z_0) = P(x_0, z_0) = U(x_0, z_0) = U(x_0, -z_0),$$

we have that $\tilde{P}$ touches U from below at $(x_0, -z_0)$ in S . $\square$

Notation 6.5. Given $f : \Omega \rightarrow \mathbb{R}$, we define the functions $f^\pm$ by

$$f^-(x) = \min\{0, f(x)\} \leq 0 \quad \text{and} \quad f^+(x) = \max\{f(x), 0\} \geq 0.$$

Lemma 6.6. Let $f \in L^\infty(\Omega)$ and let $S \subset \subset \Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$ be an open, convex set such that $S \cap \{z = 0\} \neq \emptyset$. Suppose that a continuous function $U : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that $U_{z+} \in C([0, \infty); C(\Omega))$ is symmetric across $\{z = 0\}$ and satisfies

$$-\partial_{z+}U(x, 0) \geq f(x) \quad \text{on } S \cap \{z = 0\}.$$

If $f(x_0) > 0$, then U cannot be touched from below at $(x_0, 0)$ in S by any paraboloid. If $f(x_0) \leq 0$ and P is a paraboloid of opening $a > 0$ with vertex (x_v, z_v) that touches U from below in S at $(x_0, 0)$, then $|h'(z_v)| \leq |f^-(x_0)|/a$. Consequently, if $f(x_0) = 0$, then $z_v = 0$.

Proof. Suppose that P is a paraboloid of opening $a > 0$ that touches U from below at $(x_0, 0)$ in S . Write

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_0, 0)) + U(x_0, 0).$$

Let $\varepsilon > 0$ be small. Since $U - P$ attains a local minimum of 0 at $(x_0, 0)$, we know that

$$\frac{(U(x_0, \varepsilon) - P(x_0, \varepsilon)) - (U(x_0, 0) - P(x_0, 0))}{\varepsilon} = \frac{U(x_0, \varepsilon) - P(x_0, \varepsilon)}{\varepsilon} \geq 0.$$

Therefore, taking the limit as $\varepsilon \rightarrow 0^+$, we obtain

$$0 \leq \partial_{z+}U(x_0, 0) - \partial_z P(x_0, 0) \leq -f(x_0) - ah'(z_v). \quad (6.4)$$

We note that, by the symmetry of U across $\{z = 0\}$, we have that

$$\begin{aligned} \partial_{z-}U(x_0, 0) &= \lim_{h \rightarrow 0^-} \frac{U(x_0, h) - U(x_0, 0)}{h} \\ &= - \lim_{h \rightarrow 0^+} \frac{U(x_0, -h) - U(x_0, 0)}{-h} \\ &= - \lim_{h \rightarrow 0^+} \frac{U(x_0, h) - U(x_0, 0)}{h} \\ &= -\partial_{z+}U(x_0, 0) \geq f(x_0). \end{aligned}$$

For $\varepsilon > 0$ small, we have that

$$\frac{(U(x_0, -\varepsilon) - P(x_0, -\varepsilon)) - (U(x_0, 0) - P(x_0, 0))}{-\varepsilon} = \frac{-U(x_0, -\varepsilon) + P(x_0, -\varepsilon)}{\varepsilon} \leq 0.$$

Taking the limit as $\varepsilon \rightarrow 0^+$, we obtain

$$0 \geq \partial_{z-}U(x_0, 0) - \partial_z P(x_0, 0) \geq f(x_0) - ah'(z_v). \quad (6.5)$$

By combining (6.4) and (6.5),

$$f(x_0) \leq ah'(z_v) \leq -f(x_0).$$

If $f(x_0) > 0$, then the previous set of inequalities provides a contradiction, so P cannot touch U from below in S at $(x_0, 0)$. If $f(x_0) \leq 0$, then

$$-|f^-(x_0)| \leq ah'(z_v) \leq |f^-(x_0)|$$

as desired. If $f(x_0) = 0$, then $h'(z_v) = 0$ which implies that $z_v = 0$. $\square$

7. Estimate on the Monge–Ampère measure of the set of contact points

Our first key result is a measure estimate similar to the Alexandroff–Bakelman–Pucci estimate for fully nonlinear equations. We prove that if we lift paraboloids of fixed opening $a > 0$ with vertices in a closed, bounded set from below until they touch the graph of U for the first time, then, by using the equation and the Neumann boundary condition, the Monge–Ampère measure of the contact points is a universal proportion of the Monge–Ampère measure of the set of vertices.

Theorem 7.1. Assume that Ω is a bounded domain and that $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ are bounded, measurable functions that satisfy (1.5). Let $Q_R = Q_R(\tilde{x}, \tilde{z}) \subset \subset \Omega \times \mathbb{R}$ and $f \in L^\infty(Q_R \cap \{z = 0\})$. Suppose $U \in C^2(Q_R \setminus \{z = 0\}) \cap C(Q_R)$ such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(Q_R \cap \{z \geq 0\})$ is a supersolution to

$$\begin{cases} a^{ij}(x) \partial_{ij} U + |z|^{2-\frac{1}{s}} \partial_{zz} U \leq 0 & \text{in } Q_R \cap \{z \neq 0\} \\ -\partial_{z+} U \geq f & \text{on } Q_R \cap \{z = 0\}. \end{cases}$$

Let $B \subset \overline{Q_R}$ be a closed set and fix $a > 0$. For each $(x_v, z_v) \in B$, we slide paraboloids of opening $a > 0$ and vertex (x_v, z_v) from below until they touch the graph of U for the first time. Let A denote the set of contact points and assume that $A \subset Q_R$. Then A is compact and if

$$\mu_\Phi \left(B \cap \left\{ (x, z) : |h'(z)| \leq \frac{\|f^-\|_{L^\infty(Q_R \cap \{z=0\})}}{a} \right\} \right) \leq (1 - \varepsilon_0) \mu_\Phi(B),$$

for some $\varepsilon_0 > 0$, then there is a positive constant $c = c(n, \lambda, \Lambda) < 1$ such that

$$\mu_\Phi(A) \geq \varepsilon_0 c \mu_\Phi(B).$$

Proof. We first show that A is closed. Let $(x_k, z_k) \in A$ be such that $(x_k, z_k) \rightarrow (x_0, z_0)$. There exist corresponding polynomials P_k with vertices $(x_v^k, z_v^k) \in B$ such that P_k touches U from below at (x_k, z_k) in Q_R . Since $B \subset \overline{Q_R}$ is closed, B is compact. Thus, up to a subsequence, $(x_v^k, z_v^k) \rightarrow (x_v^0, z_v^0) \in B$. By the continuity of δ_Φ and U , as $k \rightarrow \infty$,

$$\begin{aligned} P_k(x, z) &= -a\delta_\Phi((x_v^k, z_v^k), (x, z)) + a\delta_\Phi((x_v^k, z_v^k), (x_k, z_k)) + U(x_k, z_k) \\ &\rightarrow -a\delta_\Phi((x_v^0, z_v^0), (x, z)) + a\delta_\Phi((x_v^0, z_v^0), (x_0, z_0)) + U(x_0, z_0) =: P(x, z). \end{aligned}$$

Since $P_k(x, z) \leq U(x, z)$, it must be that $P \leq U$ in Q_R . Moreover, $P(x_0, z_0) = U(x_0, z_0)$. Therefore, P is a paraboloid of opening $a > 0$ with vertex $(x_v^0, z_v^0) \in B$ that touches U from below at (x_0, z_0) . This shows that $(x_0, z_0) \in A$, so that A is closed and, moreover, compact.

Define the sets

$$\begin{aligned} B_0 &= B \cap \left\{ (x, z) : |h'(z)| \leq \frac{\|f^-\|_{L^\infty(Q_R \cap \{z=0\})}}{a} \right\} \\ B_1 &= B \setminus \left\{ (x, z) : |h'(z)| \leq \frac{\|f^-\|_{L^\infty(Q_R \cap \{z=0\})}}{a} \right\}, \end{aligned}$$

so that $B = B_0 \cup B_1$ and $B_0 \cap B_1 = \emptyset$. We lift paraboloids of opening $a > 0$ from below with vertices in B_0 and B_1 to form the contact sets A_0 and A_1 , respectively. Note that $A = A_0 \cup A_1$.

We will first show that $\mu_\Phi(B_1) \leq C\mu_\Phi(A_1)$ for some positive constant $C = C(n, \lambda, \Lambda)$.

Let $(x_0, z_0) \in A_1$. There exists a paraboloid P of opening $a > 0$ and vertex $(x_v, z_v) \in B_1$ that touches U from below at (x_0, z_0) . If $z_0 = 0$, then, by Lemma 6.6, it must be that $f(x_0) \leq 0$ and that

$$|h'(z_v)| \leq \frac{|f^-(x_0)|}{a} \leq \frac{\|f^-\|_{L^\infty(Q_R \cap \{z=0\})}}{a}$$

which contradicts that $(x_v, z_v) \in B_1$. Hence, $z_0 \neq 0$.

Since $U - P$ attains a local minimum at (x_0, z_0) ,

$$\nabla U(x_0, z_0) = \nabla P(x_0, z_0) = -a(x_0 - x_v, h'(z_0) - h'(z_v))$$

which implies

$$(x_v, h'(z_v)) = (x_0, h'(z_0)) + \frac{1}{a} \nabla U(x_0, z_0).$$

This is how the vertices $(x_v, z_v) \in B_1$ are uniquely determined by $(x_0, z_0) \in A_1$. Notice that this identity is equivalent to

$$\nabla \Phi(x_v, z_v) = \nabla \left(\Phi + \frac{1}{a} U \right) (x_0, z_0) \quad \text{for all } (x_v, z_v) \in B_1.$$

Consider the map $T : A_1 \rightarrow T(A_1) = \nabla \Phi(B_1)$ given by

$$T(x_0, z_0) = \nabla \left(\Phi + \frac{1}{a} U \right) (x_0, z_0).$$

For $\varepsilon > 0$, let $A_\varepsilon \subset A_1$ be given by

$$A_\varepsilon = A_1 \setminus \{(x, z) : |z| < \varepsilon\}.$$

Then, T is Lipschitz and injective on A_ε , so that, by the area formula for Lipschitz maps,

$$\begin{aligned} |T(A_\varepsilon)| &= \int_{T(A_\varepsilon)} dy dw = \int_{A_\varepsilon} |\det(\nabla T(x, z))| dz dx \\ &= \int_{A_\varepsilon} \left| \det \left(D^2 \left(\Phi + \frac{1}{a} U \right) (x, z) \right) \right| dz dx. \end{aligned}$$

We claim that there is a constant $C = C(n, \lambda, \Lambda) > 0$ such that for all $(x_0, z_0) \in A_1$

$$-aD^2\Phi(x_0, z_0) \leq D^2U(x_0, z_0) \leq CaD^2\Phi(x_0, z_0). \quad (7.1)$$

The first inequality is straightforward because P touches U from below at (x_0, z_0) . To prove the second inequality in (7.1), suppose, by way of contradiction, that

$$D^2U(x_0, z_0) > CaD^2\Phi(x_0, z_0) \quad \text{for all } C > 0. \quad (7.2)$$

Then

$$D^2U(x_0, z_0) > Ca \begin{pmatrix} e_k \otimes e_k & 0 \\ 0 & 0 \end{pmatrix} > Ca \begin{pmatrix} e_k \otimes e_k & 0 \\ 0 & 0 \end{pmatrix} - a \begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix}$$

where e_k , $k = 1, \dots, n$ are the standard basis vectors in $\mathbb{R}^n$. Since $\tilde{A} = \begin{pmatrix} A(x_0) & 0 \\ 0 & 0 \end{pmatrix} \geq 0$ and

$$D^2U(x_0, z_0) - Ca \begin{pmatrix} e_k \otimes e_k & 0 \\ 0 & 0 \end{pmatrix} + a \begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix} \geq 0,$$

we have that

$$\text{tr} \left(\tilde{A} D^2U(x_0, z_0) - Ca \tilde{A} \begin{pmatrix} e_k \otimes e_k & 0 \\ 0 & 0 \end{pmatrix} + a \tilde{A} \begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix} \right) \geq 0.$$

By ellipticity (see (1.5)),

$$a^{ij}(x_0)\partial_{ij}U(x_0, z_0) \geq (Ca)a^{kk}(x_0) - a \operatorname{tr}(A(x_0)) \geq Ca\lambda - an\Lambda. \quad (7.3)$$

Similarly, from (7.2),

$$D^2U(x_0, z_0) > Ca \begin{pmatrix} 0 & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix} > Ca \begin{pmatrix} 0 & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix} - a \begin{pmatrix} I & 0 \\ 0 & |z_0|^{\frac{1}{s}-2} \end{pmatrix}.$$

From the definition of positive definite matrices,

$$\partial_{zz}U(x_0, z_0) - Ca|z_0|^{\frac{1}{s}-2} + a|z_0|^{\frac{1}{s}-2} > 0.$$

Therefore,

$$|z_0|^{2-\frac{1}{s}} \partial_{zz}U(x_0, z_0) > Ca - a. \quad (7.4)$$

By (7.3) and (7.4), it follows that

$$\begin{aligned} 0 &\geq a^{ij}(x_0)\partial_{ij}U(x_0, z_0) + |z_0|^{2-\frac{1}{s}} \partial_{zz}U(x_0, z_0) \\ &> Ca\lambda - an\Lambda + Ca - a \\ &= [C(\lambda + 1) - (n\Lambda + 1)]a, \end{aligned}$$

which is a contradiction when $C = C(n, \lambda, \Lambda) > 0$ is sufficiently large. Thus, (7.1) holds.

From (7.1), we get

$$0 \leq D^2 \left(\Phi + \frac{1}{a}U \right) (x_0, z_0) \leq D^2\Phi(x_0, z_0) + CD^2\Phi(x_0, z_0) = (C+1)D^2\Phi(x_0, z_0)$$

for all $(x_0, z_0) \in A_1$. Hence,

$$\begin{aligned} |T(A_\varepsilon)| &= \int_{A_\varepsilon} \det \left(D^2 \left(\Phi + \frac{1}{a}U \right) (x, z) \right) dx dz \\ &\leq \int_{A_\varepsilon} \det ((C+1)D^2\Phi(x, z)) dx dz \\ &= (C+1)^{n+1} \mu_\Phi(A_\varepsilon) \\ &\leq (C+1)^{n+1} \mu_\Phi(A_1). \end{aligned}$$

As this holds for all $\varepsilon > 0$,

$$\mu_\Phi(B_1) = |\nabla\Phi(B_1)| = |T(A_1)| \leq (C+1)^{n+1} \mu_\Phi(A_1).$$

Thus,

$$\mu_\Phi(B) = \mu_\Phi(B_0) + \mu_\Phi(B_1) \leq (1 - \varepsilon_0)\mu_\Phi(B) + (C+1)^{n+1} \mu_\Phi(A_1)$$

from which it follows that

$$\mu_\Phi(A) \geq \mu_\Phi(A_1) \geq \frac{\varepsilon_0}{2(C+1)^{n+1}} \mu_\Phi(B) = c\varepsilon_0 \mu_\Phi(B). \quad \square$$

The following is a parallel result to that of Theorem 7.1 for subsolutions when paraboloids of opening $a < 0$ are lowered from above until they touch the graph of U for the first time. The proof is straightforward. We will apply this lemma in the proof of Theorem 5.3.

Theorem 7.2. Assume that Ω is a bounded domain and that $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ are bounded, measurable functions that satisfy (1.5). Let $Q_R = Q_R(\tilde{x}, \tilde{z}) \subset\subset \Omega \times \mathbb{R}$ and $f \in L^\infty(Q_R \cap \{z = 0\})$. Suppose $U \in C^2(Q_R \setminus \{z = 0\}) \cap C(Q_R)$ such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(Q_R \cap \{z \geq 0\})$ is a subsolution to

$$\begin{cases} a^{ij}(x)\partial_{ij}U + |z|^{2-\frac{1}{s}}\partial_{zz}U \geq 0 & \text{in } Q_R \cap \{z \neq 0\} \\ -\partial_{z+}U \leq f & \text{on } Q_R \cap \{z = 0\}. \end{cases}$$

Let $B \subset \overline{Q}_R$ be a closed set and fix $a < 0$. For each $(x_v, z_v) \in B$, we slide paraboloids of opening a and vertex (x_v, z_v) from above until they touch the graph of U for the first time. Let A denote the set of contact points and assume that $A \subset Q_R$. Then A is compact and if

$$\mu_\Phi \left(B \cap \left\{ (x, z) : |h'(z)| \leq \frac{\|f^+\|_{L^\infty(Q_R \cap \{z=0\})}}{|a|} \right\} \right) \leq (1 - \varepsilon_0)\mu_\Phi(B),$$

for some $\varepsilon_0 > 0$, then there is a positive constant $c = c(n, \lambda, \Lambda) < 1$ such that

$$\mu_\Phi(A) \geq \varepsilon_0 c \mu_\Phi(B).$$

Remark 7.3. By checking the proofs, it is easy to see that Theorems 7.1 and 7.2 are still valid when the cube Q_R is replaced by a section S_R .

8. Explicit barriers

This section contains the construction of the barriers that will be used in Section 9 to prove a localization estimate. This is a quite delicate task due to the degeneracy/singularity of the extension equation and the presence of the Neumann boundary condition.

The idea for the barrier is to use $\delta_\Phi((x_0, z_0), (x, z))^{-\alpha}$, for $\alpha > 0$ large, to construct subsolutions in a ring $S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)$. This depends heavily on whether s is smaller or larger than $1/2$. When $0 < s \leq 1/2$, the coefficient $|z|^{2-1/s}$ blows up at the origin. When $1/2 < s < 1$, the coefficient $|z|^{2-1/s}$ degenerates near $z = 0$. In the latter case, we need to use an auxiliary function that bypasses the points where $|z|^{2-1/s}$ is small. A similar auxiliary function will be used when $z_0 = 0$ to force the Neumann condition to be strictly positive. By the symmetry of the equation, it will be enough to consider the nonnegative side of the ring if $z_0 \geq 0$ and the nonpositive side if $z_0 \leq 0$.

The following is a preliminary result that will be used in the case when $0 < s \leq 1/2$.

Lemma 8.1. Let $0 < s \leq 1/2$ and $z_0 > 0$ be fixed. Define the function $Q : \mathbb{R} \rightarrow \mathbb{R}$ by

$$Q(z) = \frac{(h'(z) - h'(z_0))^2}{\delta_h(z_0, z)h''(z)}.$$

Then Q is a continuous function of $z > 0$, and $Q(z) \geq 1$ for all $z > 0$.

Proof. By L'Hôpital's rule, $\lim_{z \rightarrow z_0} Q(z) = 2$, so that $Q(z)$ is continuous for $z > 0$. Also, for $s = 1/2$ and all $z \neq z_0$, we have $Q(z) = 2$. Hence, let us assume for the remainder of the proof that $0 < s < 1/2$.

It is easy to see that $\lim_{z \rightarrow 0^+} Q(z) = \infty$ and that $\lim_{z \rightarrow \infty} Q(z) = \frac{1}{1-s} \geq 1$. Therefore, it is enough to prove that $Q(z)$ is decreasing for $z > 0$, $z \neq z_0$. To this end, we will show that $Q'(z) < 0$ for $z \neq z_0$. First, observe that

$$Q'(z) = \frac{(h'(z) - h'(z_0))h''(z)}{(\delta_h(z_0, z)h''(z))^2} \cdot I(z) \quad (8.1)$$

where

$$I(z) = 2\delta_h(z_0, z)h''(z) - (h'(z) - h'(z_0))^2 - \delta_h(z_0, z)(h'(z) - h'(z_0))\frac{h'''(z)}{h''(z)}.$$

We can write

$$I(z) = -\frac{s}{1-s}z_0^{\frac{2}{s}-2} + \frac{s}{1-s}z_0^{\frac{1}{s}}z^{\frac{1}{s}-2} - \frac{s(1-2s)}{1-s}z_0^{\frac{1}{s}-1}z^{\frac{1}{s}-1} + \frac{s(1-2s)}{1-s}z_0^{\frac{2}{s}-1}z^{-1}.$$

It follows that $I(z) > 0$ for all $z > 0$ if and only if

$$\psi(z) := -z_0^{\frac{2}{s}-1}z + z_0^{\frac{1}{s}+1}z^{\frac{1}{s}-1} - (1-2s)z_0^{\frac{1}{s}}z^{\frac{1}{s}} + (1-2s)z_0^{\frac{2}{s}} > 0,$$

for all $z > 0$. Note that $\psi(z_0) = 0$ and $\psi(0) = (1-2s)z_0^{\frac{2}{s}} > 0$. We claim that ψ is decreasing as function of $z > 0$. Indeed, $\psi'(z) < 0$ if and only if

$$-z_0^{\frac{2}{s}-1} + \left(\frac{1}{s} - 1\right)z_0^{\frac{1}{s}+1}z^{\frac{1}{s}-2} - \frac{(1-2s)}{s}z_0^{\frac{1}{s}}z^{\frac{1}{s}-1} < 0.$$

Multiplying both sides by $z_0^{-1/s}s/(1-s) > 0$ and rearranging, this is equivalent to

$$z_0z^{\frac{1}{s}-2} < \left(\frac{s}{1-s}\right)z_0^{\frac{1}{s}-1} + \left(\frac{1-2s}{1-s}\right)z^{\frac{1}{s}-1},$$

which is true by Young's inequality, and the claim follows. Thus, we conclude that

$$\psi(z) > 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad \psi(z) < 0 \quad \text{for } z > z_0.$$

This gives that

$$I(z) > 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad I(z) < 0 \quad \text{for } z > z_0.$$

Since, in addition,

$$h'(z) - h'(z_0) < 0 \quad \text{for } 0 < z < z_0 \quad \text{and} \quad h'(z) - h'(z_0) > 0 \quad \text{for } z > z_0,$$

we deduce from (8.1) that $Q'(z) < 0$ for all $z \neq z_0$. This completes the proof. $\square$

We now construct the barriers ϕ . For a set $S \subset \mathbb{R}^{n+1}$, we introduce the notation

$$S^+ = S \cap \{z \geq 0\} \quad \text{and} \quad S^- = S \cap \{z \leq 0\}.$$

To deal with the singularity at $z = 0$, we define ϕ in either the positive or negative half spaces. In particular, if $z_0 \geq 0$, then we consider the partial ring $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$. If $z_0 < 0$, then we consider the partial ring $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^-$. We will use the condensed notation

$$[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^\pm = \begin{cases} [S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ & \text{if } z_0 \geq 0 \\ [S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^- & \text{if } z_0 < 0. \end{cases}$$

Lemma 8.2. Fix $0 < \gamma < 1$ and consider a section $S_r(x_0, z_0) \subset \mathbb{R}^{n+1}$.

If $z_0 \geq 0$, then there exists a classical subsolution $\phi = \phi(x, z)$ to

$$\begin{cases} a^{ij}(x)\partial_{ij}\phi + |z|^{2-\frac{1}{s}}\partial_{zz}\phi > a(n\Lambda + 1) & \text{in } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z \neq 0\} \\ -\partial_{z+}\phi(x, 0) < 0 & \text{on } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}. \end{cases} \quad (8.2)$$

If $z_0 \leq 0$, then there exists a classical subsolution $\phi = \phi(x, z)$ to

$$\begin{cases} a^{ij}(x)\partial_{ij}\phi + |z|^{2-\frac{1}{s}}\partial_{zz}\phi > a(n\Lambda + 1) & \text{in } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^- \cap \{z \neq 0\} \\ -\partial_{z-}\phi(x, 0) > 0 & \text{on } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^- \cap \{z = 0\}. \end{cases} \quad (8.3)$$

In each case, $\phi > 0$ in $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^\pm$, $\phi \leq 0$ on $[\partial S_{2r}(x_0, z_0)]^\pm$, and there is a constant $C = C(n, \lambda, \Lambda, \gamma) > 0$ such that $\phi \leq Car$ on $[\partial S_{\gamma r}(x_0, z_0)]^\pm$.

Proof. The proof of (8.3) will follow from (8.2) at the end by symmetry. The construction of the subsolution in (8.2) will depend on whether $z_0 > 0$ or $z_0 = 0$ and on whether $0 < s \leq 1/2$ or $1/2 < s < 1$.

Case 1: $z_0 > 0$ and $0 < s \leq 1/2$.

We begin by considering the function $(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha}$ for a large constant $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s) > 0$ which will be fixed later on. Let $\mathcal{Q}(z)$ be the function defined in Lemma 8.1. For a point $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \setminus \{z = 0\}$, we use ellipticity and Lemma 8.1 to estimate

$$\begin{aligned} & a^{ij}(x)\partial_{ij}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} + |z|^{2-\frac{1}{s}}\partial_{zz}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} \\ &= \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\ & \quad \left[(\alpha + 1)(a^{ij}(x)(x - x_0)_i(x - x_0)_j + |z|^{2-\frac{1}{s}}(h'(z) - h'(z_0))^2) \right. \\ & \quad \left. - (\text{tr}(A(x)) + 1)\delta_\Phi((x_0, z_0), (x, z)) \right] \\ & \geq \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\ & \quad \left[(\alpha + 1)(\lambda|x - x_0|^2 + |z|^{2-\frac{1}{s}}(h'(z) - h'(z_0))^2) - (n\Lambda + 1)\delta_\Phi((x_0, z_0), (x, z)) \right] \\ &= \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\ & \quad \left[(\alpha + 1)\left(2\lambda\delta_\varphi(x_0, x) + \frac{(h'(z) - h'(z_0))^2}{h''(z)\delta_h(z_0, z)}\delta_h(z_0, z)\right) - (n\Lambda + 1)(\delta_\varphi(x_0, x) + \delta_h(z_0, z)) \right] \\ &= \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\ & \quad \left[(2\lambda(\alpha + 1) - (n\Lambda + 1))\delta_\varphi(x_0, x) + (\mathcal{Q}(z)(\alpha + 1) - (n\Lambda + 1))\delta_h(z_0, z) \right] \\ & \geq \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \\ & \quad \left[(2\lambda(\alpha + 1) - (n\Lambda + 1))\delta_\varphi(x_0, x) + ((\alpha + 1) - (n\Lambda + 1))\delta_h(z_0, z) \right]. \end{aligned}$$

Choose $\alpha = \alpha(\gamma, n, \lambda, \Lambda)$ large so that

$$2\lambda(\alpha + 1) - (n\Lambda + 1) > 4\gamma^{-1}(n\Lambda + 1) \quad \text{and} \quad (\alpha + 1) - (n\Lambda + 1) > 4\gamma^{-1}(n\Lambda + 1).$$

Since $\gamma r \leq \delta_\Phi((x_0, z_0), (x, z)) = \delta_\varphi(x_0, x) + \delta_h(z_0, z)$, it must be that $\delta_\varphi(x_0, x) \geq \gamma r/2$ or $\delta_h(z_0, z) \geq \gamma r/2$. If $\delta_\varphi(x_0, x) \geq \gamma r/2$, then

$$\begin{aligned} & a^{ij}(x)\partial_{ij}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} + |z|^{2-\frac{1}{s}}\partial_{zz}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} \\ & > \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \left[4\gamma^{-1}(n\Lambda + 1)\delta_\varphi(x_0, x) + 0 \right] \\ & \geq \alpha(n\Lambda + 1)(2r)^{-\alpha-1}. \end{aligned}$$

If $\delta_h(z_0, z) \geq \gamma r/2$, then

$$\begin{aligned} & a^{ij}(x)\partial_{ij}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} + |z|^{2-\frac{1}{s}}\partial_{zz}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} \\ & > \alpha(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-2} \left[0 + 4\gamma^{-1}(n\Lambda + 1)\delta_h(z_0, z) \right] \\ & \geq \alpha(n\Lambda + 1)(2r)^{-\alpha-1}. \end{aligned}$$

Combining the previous two estimates, we have that, for all $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \setminus \{z = 0\}$,

$$a^{ij}(x)\partial_{ij}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} + |z|^{2-\frac{1}{s}}\partial_{zz}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} > \alpha(n\Lambda + 1)(2r)^{-\alpha-1}.$$

Define ϕ in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$ by

$$\phi(x, z) = \alpha^{-1}a(2r)^{\alpha+1}[(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha} - r^{-\alpha}].$$

Then $a^{ij}(x)\partial_{ij}\phi(x, z) + |z|^{2-\frac{1}{s}}\phi(x, z) > a(n\Lambda + 1)$. If $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\} \neq \emptyset$, we need to check the Neumann condition. In this case, let $(x, 0) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}$ and observe that

$$\begin{aligned} \partial_{z+}\phi(x, 0) &= -a(2r)^{\alpha+1}(\delta_\Phi((x_0, z_0), (x, z)))^{-\alpha-1}(h'(z) - h'(z_0))\big|_{z=0} \\ &= a(2r)^{\alpha+1}(\delta_\Phi((x_0, z_0), (x, 0)))^{-\alpha-1}h'(z_0) \\ &\geq ah'(z_0) > 0 \end{aligned}$$

since $z_0 > 0$. Therefore, ϕ defined in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$ is a subsolution to (8.2). It is easy to check that $\phi \leq 0$ in $[S_{2r}(x_0, z_0) \setminus S_r(x_0, z_0)]^+$ and $\phi > 0$ in $[S_r(x_0, z_0)]^+$. Lastly, for $(x, z) \in [\partial S_{\gamma r}(x_0, z_0)]^\pm$, we have that $\phi(x, z) = \alpha^{-1}a2^{\alpha+1}(\gamma^{-1} - 1)r = Car$, where $C = C(\gamma, n, \lambda, \Lambda) > 0$.

Case 2: $z_0 \geq 0$ and $1/2 < s < 1$.

Here we need to bypass the points where $|z|^{2-\frac{1}{s}}$ is small with respect to the size of the section $S_{2r}(z_0) \subset \mathbb{R}$. Let $0 < \varepsilon < 1$ be a small constant, to be chosen. Let $0 < \varepsilon_0 < 1$ be as in Lemma 4.6 and define the set H_ε by

$$\begin{aligned} H_\varepsilon &= \left\{ z \in S_{2r}(z_0) : |z|^{2-\frac{1}{s}} \leq \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} \right\} \\ &= \left\{ z \in S_{2r}(z_0) : 1 \leq \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} h''(z) \right\}. \end{aligned}$$

We first show that the measure of H_ε is small with respect to the measure of the section $S_{2r}(z_0)$. Indeed, using Lemma 4.1, we estimate

$$\begin{aligned} |H_\varepsilon| &= \int_{H_\varepsilon} dz \leq \int_{H_\varepsilon} \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} h''(z) dz \\ &\leq \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} \int_{S_{2r}(z_0)} h''(z) dz = \varepsilon_0 |S_{2r}(z_0)|. \end{aligned}$$

By Lemma 4.6, $\mu_h(H_\varepsilon) \leq \varepsilon \mu_h(S_{2r}(z_0))$.

We will construct a function h_ε in $[S_{2r}(z_0)]^+$ that bypasses the points in H_ε . Let $\tilde{H}_\varepsilon$ be an open interval such that

$$H_\varepsilon \subset \tilde{H}_\varepsilon \subset S_{2r}(z_0), \quad \mu_h(\tilde{H}_\varepsilon \setminus H_\varepsilon) \leq \varepsilon \mu_h(S_{2r}(z_0)),$$

and let $\psi_\varepsilon = \psi_\varepsilon(z)$ be a smooth function satisfying

$$\psi_\varepsilon = 1 \text{ in } H_\varepsilon, \quad \psi_\varepsilon = \varepsilon \text{ in } S_{2r}(z_0) \setminus \tilde{H}_\varepsilon, \quad \varepsilon \leq \psi_\varepsilon \leq 1 \text{ in } S_{2r}(z_0).$$

We use the notation

$$[S_{2r}(z_0)]^+ = (z_L, z_R), \quad \text{where } 0 \leq z_L \leq z_0 < z_R.$$

Note that $z_L = 0$ if $0 \in S_{2r}(z_0)$.

In $[S_{2r}(z_0)]^+$, let $h_\varepsilon = h_\varepsilon(z)$ be the strictly convex solution to

$$\begin{cases} h_\varepsilon'' = 2(n\Lambda + 1)\psi_\varepsilon h'' & \text{in } [S_{2r}(z_0)]^+ \\ h_\varepsilon(z_R) = 0 \\ h_\varepsilon'(z_L) = \varepsilon \mu_h(S_{2r}(z_0)). \end{cases}$$

We remark that $h_\varepsilon \in C^\infty((z_L, z_R))$ and, since $h \in C^1(\mathbb{R})$, we have $h_\varepsilon \in C^1([\overline{S_{2r}(z_0)}]^+)$. Since h_ε is strictly convex in $[S_{2r}(z_0)]^+$ and $h_\varepsilon \in C^1([\overline{S_{2r}(z_0)}]^+)$, it follows that $h_\varepsilon' > 0$ in $[S_{2r}(z_0)]^+$. Moreover, since h_ε is strictly increasing, h_ε achieves its maximum at $z = z_R$, so that $h_\varepsilon \leq 0$ in $[\overline{S_{2r}(z_0)}]^+$.

To bound h_ε and h_ε' , we first estimate

$$\begin{aligned} \int_{S_{2r}(z_0)} \psi_\varepsilon d\mu_h &= \int_{H_\varepsilon} \psi_\varepsilon d\mu_h + \int_{\tilde{H}_\varepsilon \setminus H_\varepsilon} \psi_\varepsilon d\mu_h + \int_{S_{2r}(z_0) \setminus \tilde{H}_\varepsilon} \psi_\varepsilon d\mu_h \\ &\leq \int_{H_\varepsilon} d\mu_h + \int_{\tilde{H}_\varepsilon \setminus H_\varepsilon} d\mu_h + \int_{S_{2r}(z_0) \setminus \tilde{H}_\varepsilon} \varepsilon d\mu_h \\ &= \mu_h(H_\varepsilon) + \mu_h(\tilde{H}_\varepsilon \setminus H_\varepsilon) + \varepsilon \mu_h(S_{2r}(z_0) \setminus \tilde{H}_\varepsilon) \\ &\leq \varepsilon \mu_h(S_{2r}(z_0)) + \varepsilon \mu_h(S_{2r}(z_0)) + \varepsilon \mu_h(S_{2r}(z_0)) = 3\varepsilon \mu_h(S_{2r}(z_0)). \end{aligned}$$

Let $\delta > 0$. For $z \in [S_{2r}(z_0)]^+$, by the previous estimate,

$$|h_\varepsilon'(z)| = h_\varepsilon'(z) = \int_{z_L + \delta}^z h_\varepsilon''(w) dw + h_\varepsilon'(z_L + \delta)$$

$$\begin{aligned}
&= \int_{z_L+\delta}^z 2(n\Lambda+1)\psi_\varepsilon h''(w) dw + h'_\varepsilon(z_L+\delta) \\
&\leq 2(n\Lambda+1) \int_{S_{2r}(z_0)} \psi_\varepsilon d\mu_h + h'_\varepsilon(z_L+\delta) \\
&\leq 6(n\Lambda+1)\varepsilon\mu_h(S_{2r}(z_0)) + h'_\varepsilon(z_L+\delta) \\
&= C\varepsilon\mu_h(S_{2r}(z_0)) + h'_\varepsilon(z_L+\delta)
\end{aligned}$$

for a constant $C = C(n, \Lambda) > 0$. Taking the limit as $\delta \rightarrow 0$, we have

$$h'_\varepsilon(z) \leq C\varepsilon\mu_h(S_{2r}(z_0)) + h'_\varepsilon(z_L) = C\varepsilon\mu_h(S_{2r}(z_0)) + \varepsilon\mu_h(S_{2r}(z_0)) = C_1\varepsilon\mu_h(S_{2r}(z_0))$$

for a constant $C_1 = C_1(n, \Lambda)$.

Again, let $\delta > 0$. For $z \in [S_{2r}(z_0)]^+$, by Corollary 4.7 part (3),

$$\begin{aligned}
|h_\varepsilon(z)| = -h_\varepsilon(z) &= \int_z^{z_R-\delta} h'_\varepsilon(w) dw - h_\varepsilon(z_R-\delta) \\
&\leq C_1\varepsilon\mu_h(S_{2r}(z_0)) \int_z^{z_R-\delta} dw - h_\varepsilon(z_R-\delta) \\
&\leq C_1\varepsilon\mu_h(S_{2r}(z_0)) |S_{2r}(z_0)| - h_\varepsilon(z_R-\delta) \\
&\leq C_2\varepsilon r - h_\varepsilon(z_R-\delta)
\end{aligned}$$

for a constant $C_2 = C_2(n, \Lambda, s) > 0$. Taking the limit as $\delta \rightarrow 0$, we have

$$|h_\varepsilon(z)| \leq C_2\varepsilon r - h_\varepsilon(z_R) = C_2\varepsilon r.$$

Suppose that $\gamma r/2 \leq \delta_h(z_0, z) < 2r$. By the convexity of $\delta_h(z_0, z)$ in the variable z , we obtain

$$0 = \delta_h(z_0, z_0) \geq \delta_h(z_0, z) + \partial_z \delta_h(z_0, z) \cdot (z_0 - z).$$

By Corollary 4.7 part (3), this implies

$$|\partial_z \delta_h(z_0, z)| \geq \frac{\delta_h(z_0, z)}{|z - z_0|} \geq \frac{\gamma r/2}{|S_{2r}(z_0)|} \geq C_3\mu_h(S_{2r}(z_0))$$

for a constant $C_3 = C_3(\gamma, s)$. Choose $\varepsilon = \varepsilon(\gamma, n, \Lambda, s) > 0$ small so that $C_1\varepsilon < C_3$. Then,

$$|\partial_z \delta_h(z_0, z) - h'_\varepsilon(z)| \geq |\partial_z \delta_h(z_0, z)| - |h'_\varepsilon(z)| \geq (C_3 - C_1\varepsilon)\mu_h(S_{2r}(z_0)) > 0$$

and

$$(\partial_z \delta_h(z_0, z) - h'_\varepsilon(z))^2 \geq (C_3 - C_1\varepsilon)^2 [\mu_h(S_{2r}(z_0))]^2 = C_4 [\mu_h(S_{2r}(z_0))]^2 \quad (8.4)$$

for a constant $C_4 = C_4(\gamma, n, \Lambda, s) > 0$.

For a large constant $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s) > 0$, we define the function $\tilde{\phi}$ on $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$ by

$$\tilde{\phi}(x, z) = (\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))^{-\alpha}.$$

Let $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \setminus \{z = 0\}$. Since $h_\varepsilon \leq 0$, we first note that

$$\gamma r \leq \delta_\Phi((x_0, z_0), (x, z)) \leq \delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z) < 2r + C_2\varepsilon r = (2 + C_2\varepsilon)r. \quad (8.5)$$

The equation for $\tilde{\phi}$ in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \setminus \{z = 0\}$ is

$$\begin{aligned} a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} \\ = \alpha(\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))^{-\alpha-2} \\ \left((\alpha+1) \left[a^{ij}(x)(x-x_0)_i(x-x_0)_j + |z|^{2-\frac{1}{s}}(\partial_z(\delta_\Phi((x_0, z_0), (x, z)))) - h'_\varepsilon(z))^2 \right] \right. \\ \left. - (\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z)) \left[\text{tr}(A(x)) + 1 - 2(n\Lambda + 1)\psi_\varepsilon \right] \right). \end{aligned}$$

Using ellipticity and

$$\partial_z\delta_\Phi((x_0, z_0), (x, z)) = \partial_z(\delta_\varphi(x_0, x) + \delta_h(z_0, z)) = \partial_z\delta_h(z_0, z), \quad (8.6)$$

we estimate

$$\begin{aligned} a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} \\ \geq \alpha(\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))^{-\alpha-2} \\ \left((\alpha+1) \left[2\lambda\delta_\varphi(x_0, x) + |z|^{2-\frac{1}{s}}(\partial_z\delta_h(z_0, z) - h'_\varepsilon(z))^2 \right] \right. \\ \left. - (\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))(1 - 2\psi_\varepsilon)(n\Lambda + 1) \right). \end{aligned}$$

Suppose that $z \in H_\varepsilon$. Since $\psi_\varepsilon(z) = 1$, we can use (8.5) to estimate

$$\begin{aligned} a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} \\ \geq \alpha(\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))^{-\alpha-2} \left(0 + (\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))(n\Lambda + 1) \right) \\ \geq \alpha(n\Lambda + 1)(2 + C_2\varepsilon)^{-\alpha-1}r^{-\alpha-1}. \end{aligned} \quad (8.7)$$

Next, suppose that $z \notin H_\varepsilon$. Since $\psi_\varepsilon(z) > 0$ and $|z|^{2-\frac{1}{s}} > \varepsilon_0 |S_{2r}(z_0)| / \mu_h(S_{2r}(z_0))$, we estimate

$$\begin{aligned} a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} \\ \geq \alpha(\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))^{-\alpha-2} \\ \left((\alpha+1) \left[2\lambda\delta_\varphi(x_0, x) + \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} (\partial_z\delta_h(z_0, z) - h'_\varepsilon(z))^2 \right] \right. \\ \left. - (\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z))(n\Lambda + 1) \right). \end{aligned} \quad (8.8)$$

Since $\delta_\Phi((x_0, z_0), (x, z)) \geq \gamma r$, we have that $\delta_\varphi(x_0, x) \geq \gamma r/2$ or $\delta_h(z_0, z) \geq \gamma r/2$. Suppose first that $\delta_\varphi(x_0, x) \geq \gamma r/2$. Then

$$2\lambda\delta_\varphi(x_0, x) + \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} (\partial_z\delta_h(z_0, z) - h'_\varepsilon(z))^2 \geq 2\lambda\delta_\varphi(x_0, x) \geq \lambda\gamma r.$$

Choose $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s)$ large enough to guarantee that

$$(\alpha + 1)\lambda\gamma - (n\Lambda + 1)(2 + C_2\varepsilon) > (n\Lambda + 1)(2 + C_2\varepsilon).$$

Then, from (8.8) and (8.5),

$$\begin{aligned} a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} &\geq \alpha(\delta_{\Phi}((x_0, z_0), (x, z)) - h_{\varepsilon}(z))^{-\alpha-2} \left((\alpha + 1)\lambda\gamma r - (n\Lambda + 1)(2 + C_2\varepsilon)r \right) \\ &> \alpha(\delta_{\Phi}((x_0, z_0), (x, z)) - h_{\varepsilon}(z))^{-\alpha-2} (n\Lambda + 1)(2 + C_2\varepsilon)r \\ &\geq \alpha(n\Lambda + 1)(2 + C_2\varepsilon)^{-\alpha-1} r^{-\alpha-1}. \end{aligned} \quad (8.9)$$

Next, suppose that $\delta_h(z_0, z) \geq \gamma r/2$. Since $S_{2r}(x_0, z_0) \subset S_{2r}(x_0) \times S_{2r}(z_0)$, we know that $\gamma r/2 \leq \delta_h(z_0, z) < 2r$. By (8.4) and Corollary 4.7 part (3), we obtain

$$\begin{aligned} 2\lambda\delta_{\varphi}(x_0, x) + \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} (\partial_z \delta_h(z_0, z) - h'_{\varepsilon}(z))^2 \\ \geq \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} (\partial_z \delta_h(z_0, z) - h'_{\varepsilon}(z))^2 \\ \geq \varepsilon_0 \frac{|S_{2r}(z_0)|}{\mu_h(S_{2r}(z_0))} C_4 [\mu_h(S_{2r}(z_0))]^2 \\ \geq C_5 \varepsilon_0 r \end{aligned}$$

for some constant $C_5 = C_5(\gamma, n, \Lambda, s) > 0$. Let $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s) > 0$ be large so that

$$(\alpha + 1)C_5\varepsilon_0 - (n\Lambda + 1)(2 + C_2\varepsilon) > (n\Lambda + 1)(2 + C_2\varepsilon).$$

Then, from (8.8), we use (8.5) to obtain

$$\begin{aligned} a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} &\geq \alpha(\delta_{\Phi}((x_0, z_0), (x, z)) - h_{\varepsilon}(z))^{-\alpha-2} \left((\alpha + 1)C_5\varepsilon_0 r - (n\Lambda + 1)(2 + C_2\varepsilon)r \right) \\ &> \alpha(\delta_{\Phi}((x_0, z_0), (x, z)) - h_{\varepsilon}(z))^{-\alpha-2} (n\Lambda + 1)(2 + C_2\varepsilon)r \\ &\geq \alpha(n\Lambda + 1)(2 + C_2\varepsilon)^{-\alpha-1} r^{-\alpha-1}. \end{aligned} \quad (8.10)$$

From (8.7), (8.9), and (8.10), there is an $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s) > 0$ such that for all $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \setminus \{z = 0\}$, we have

$$a^{ij}(x)\partial_{ij}\tilde{\phi} + |z|^{2-\frac{1}{s}}\partial_{zz}\tilde{\phi} > \alpha(n\Lambda + 1)(2 + C_2\varepsilon)^{-\alpha-1} r^{-\alpha-1}.$$

We define the barrier ϕ on $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$ by

$$\phi(x, z) = a\alpha^{-1}(2 + C_2\varepsilon)^{\alpha+1} r^{\alpha+1} (\tilde{\phi}(x, z) - (1 + C_2\varepsilon)^{-\alpha} r^{-\alpha}).$$

For $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \setminus \{z = 0\}$, it then follows that $a^{ij}(x)\partial_{ij}\phi + |z|^{2-\frac{1}{s}}\partial_{zz}\phi > a(n\Lambda + 1)$. If $z_L = 0$, we need to check the Neumann condition. In this case, let $(x, 0) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}$. Using (8.5), we see that

$$\begin{aligned}
\partial_{z+}\phi(x, 0) &= a(2 + C_2\varepsilon)^{\alpha+1}r^{\alpha+1}(\delta_\Phi((x_0, z_0), (x, 0)) - h_\varepsilon(0))^{-\alpha-1}(h'(z_0) + \varepsilon\mu_h(S_{2r}(z_0))) \\
&> a(2 + C_2\varepsilon)^{\alpha+1}r^{\alpha+1}(2 + C_2\varepsilon)^{-\alpha-1}r^{-\alpha-1}(h'(z_0) + \varepsilon\mu_h(S_{2r}(z_0))) \\
&= a(h'(z_0) + \varepsilon\mu_h(S_{2r}(z_0))) > 0,
\end{aligned}$$

since $z_0 \geq 0$. Therefore, ϕ is a subsolution to (8.2). In $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$, we have

$$\gamma r \leq \delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z) < (1 + C_2\varepsilon)r,$$

so that $\phi > 0$ in $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$. Choose $\varepsilon > 0$ small so that $2 > 1 + C_2\varepsilon$. Then, $\phi \leq 0$ on $[\partial S_{2r}(x_0, z_0)]^+$. Indeed, for $(x, z) \in [\partial S_{2r}(x_0, z_0)]^+$, we have that

$$-h_\varepsilon(z) \geq 0 > (1 + C_2\varepsilon - 2)r$$

which implies

$$\delta_\Phi((x_0, z_0), (x, z)) - h_\varepsilon(z) = 2r - h_\varepsilon(z) > (1 + C_2\varepsilon)r.$$

Thus, $\phi(x, z) \leq 0$. Lastly, let $(x, z) \in [\partial S_{\gamma r}(x_0, z_0)]^+$ and observe that

$$\begin{aligned}
\phi(x, z) &= a\alpha^{-1}(2 + C_2\varepsilon)^{\alpha+1}r^{\alpha+1}((\gamma r - h_\varepsilon(z))^{-\alpha} - (1 + C_2\varepsilon)^{-\alpha}r^{-\alpha}) \\
&\leq a\alpha^{-1}(2 + C_2\varepsilon_2)^{\alpha+1}r^{\alpha+1}((\gamma r + 0)^{-\alpha} - 0) = Car
\end{aligned}$$

for $C = C(\gamma, n, \lambda, \Lambda, s) > 0$.

Case 3: $z_0 = 0$ and $0 < s \leq 1/2$.

For the barrier constructed in Case 1, the inequality for the Neumann condition was not strict for $z_0 = 0$. We will add a function g_ε to the quasi-distance function δ_Φ to adjust the barrier as we did in Case 2.

Let $(x, z) \in [S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+$. Since $(x, z) \in S_{2r}(x_0, 0) \subset S_{2r}(x_0) \times S_{2r}(0)$, we know that $z \in S_{2r}(0) = B_{q_s 2^s r^2}(0)$ by Remark 4.8. That is,

$$|z| < q_s 2^s r^s = \bar{C}_2 r^s. \quad (8.11)$$

Also, since $2 - \frac{1}{s} \leq 0$,

$$|z|^{2-\frac{1}{s}} \geq \bar{C}_2^{\frac{2s-1}{s}} r^{2s-1}. \quad (8.12)$$

Given $\varepsilon > 0$, define g_ε in $[S_{2r}(0)]^+$ by

$$g_\varepsilon(z) = \varepsilon r^{1-s} z - \varepsilon \bar{C}_2 r.$$

For all $z \in [S_{2r}(0)]^+$, we have that $g_\varepsilon \leq 0$ by (8.11). We also have that

$$|g_\varepsilon(z)| = \varepsilon \bar{C}_2 r - \varepsilon r^{1-s} z \leq \bar{C}_2 \varepsilon r, \quad \text{and} \quad g'_\varepsilon(z) = \varepsilon r^{1-s} > 0.$$

Let z be such that $\gamma r/2 \leq \delta_h(0, z) < 2r$. As in Case 2 above, since $z \in S_{2r}(0) = B_{\bar{C}_2 r^s}(0)$, we can use the convexity of $\delta_h(0, z)$ in the variable z to obtain

$$|\partial_z \delta_h(0, z)| \geq \frac{\delta_h(0, z)}{|z|} \geq \frac{\gamma r/2}{\bar{C}_2 r^s} = \bar{C}_3 r^{1-s}$$

for a constant $\bar{C}_3 = \bar{C}_3(\gamma, s)$. Choose $\varepsilon = \varepsilon(\gamma, s) > 0$ small so that $\varepsilon < \bar{C}_3$. Then,

$$|\partial_z \delta_h(0, z) - g'_\varepsilon(z)| \geq |\partial_z \delta_h(0, z)| - |g'_\varepsilon(z)| \geq (\bar{C}_3 - \varepsilon)r^{1-s} > 0$$

and

$$(\partial_z \delta_h(0, z) - g'_\varepsilon(z))^2 \geq (\bar{C}_3 - \varepsilon)^2 r^{2-2s} = \bar{C}_4 r^{2-2s} \quad (8.13)$$

for a constant $\bar{C}_4 = \bar{C}_4(\gamma, s) > 0$.

Let $(x, z) \in [S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+$. Since $-g_\varepsilon \geq 0$, we have that

$$\gamma r \leq \delta_\Phi((x_0, 0), (x, z)) \leq \delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z) < 2r + \varepsilon \bar{C}_2 r = (2 + \varepsilon \bar{C}_2)r. \quad (8.14)$$

We define a function $\bar{\phi}$ on $[S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+$ by

$$\bar{\phi}(x, z) = (\delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z))^{-\alpha}.$$

Let $(x, z) \in [S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+ \setminus \{z = 0\}$. Using ellipticity, (8.6), (8.12), and (8.14), we estimate the equation for $\bar{\phi}$ as follows:

$$\begin{aligned} & a^{ij}(x) \partial_{ij} \bar{\phi} + |z|^{2-\frac{1}{s}} \partial_{zz} \bar{\phi} \\ & \geq \alpha (\delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z))^{-\alpha-2} \\ & \quad \left((\alpha + 1) \left[2\lambda \delta_\varphi(x_0, x) + \bar{C}_2^{\frac{2s-1}{s}} r^{2s-1} (\partial_z \delta_h(0, z) - g'_\varepsilon(z))^2 \right] \right. \\ & \quad \left. - (\delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z))(n\Lambda + 1) \right) \\ & \geq \alpha (\delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z))^{-\alpha-2} \\ & \quad \left((\alpha + 1) \left[2\lambda \delta_\varphi(x_0, x) + \bar{C}_2^{\frac{2s-1}{s}} r^{2s-1} (\partial_z \delta_h(0, z) - g'_\varepsilon(z))^2 \right] \right. \\ & \quad \left. - (n\Lambda + 1)(2 + \varepsilon \bar{C}_2)r \right). \end{aligned} \quad (8.15)$$

Since $\delta_\Phi((x_0, 0), (x, z)) \geq \gamma r$, we know that $\delta_\varphi(x_0, x) \geq \gamma r/2$ or $\delta_h(0, z) \geq \gamma r/2$. Suppose first that $\delta_\varphi(x_0, x) \geq \gamma r/2$. Then

$$2\lambda \delta_\varphi(x_0, x) + \bar{C}_2^{\frac{2s-1}{s}} r^{2s-1} (\partial_z \delta_h(0, z) - g'_\varepsilon(z))^2 \geq 2\lambda \delta_\varphi(x_0, x) \geq \lambda \gamma r.$$

Choose $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s)$ large enough to guarantee that

$$(\alpha + 1)\lambda\gamma - (n\Lambda + 1)(2 + \bar{C}_2\varepsilon) > (n\Lambda + 1)(2 + \bar{C}_2\varepsilon).$$

Then, from (8.15) and using (8.14), we have that

$$\begin{aligned} & a^{ij}(x) \partial_{ij} \bar{\phi} + |z|^{2-\frac{1}{s}} \partial_{zz} \bar{\phi} \\ & \geq \alpha (\delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z))^{-\alpha-2} ((\alpha + 1)\lambda\gamma r - (n\Lambda + 1)(2 + \bar{C}_2\varepsilon)r) \\ & > \alpha (\delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z))^{-\alpha-2} (n\Lambda + 1)(2 + \bar{C}_2\varepsilon)r \\ & \geq \alpha (n\Lambda + 1)(2 + \bar{C}_2\varepsilon)^{-\alpha-1} r^{-\alpha-1}. \end{aligned} \quad (8.16)$$

Next, suppose that $\delta_h(0, z) \geq \gamma r/2$. We further know that $\gamma r/2 \leq \delta_h(0, z) < 2r$, so, by (8.13),

$$\begin{aligned} 2\lambda\delta_\varphi(x_0, x) + \bar{C}_2^{\frac{2s-1}{s}} r^{2s-1} (\partial_z \delta_h(0, z) - g_\varepsilon(z))^2 \\ \geq 0 + \bar{C}_2^{\frac{2s-1}{s}} r^{2s-1} \bar{C}_4 r^{2-2s} = \bar{C}_4 \bar{C}_2^{\frac{2s-1}{s}} r. \end{aligned}$$

Let $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s) > 0$ be large so that

$$(\alpha + 1) \bar{C}_4 \bar{C}_2^{\frac{2s-1}{s}} - (n\Lambda + 1)(2 + \bar{C}_2 \varepsilon) > (n\Lambda + 1)(2 + \bar{C}_2 \varepsilon).$$

Then, from (8.15) and using (8.14),

$$\begin{aligned} a^{ij}(x) \partial_{ij} \bar{\phi} + |z|^{2-\frac{1}{s}} \partial_{zz} \bar{\phi} \\ \geq \alpha(\delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z))^{-\alpha-2} \left((\alpha + 1) \bar{C}_4 \bar{C}_2^{\frac{2s-1}{s}} r - (n\Lambda + 1)(2r + \bar{C}_2 \varepsilon)r \right) \\ > \alpha(\delta_\Phi((x_0, 0), (x, z)) - g_\varepsilon(z))^{-\alpha-2} (n\Lambda + 1)(2 + \bar{C}_1 \varepsilon)r \\ \geq \alpha(n\Lambda + 1)(2 + \bar{C}_2 \varepsilon)^{-\alpha-1} r^{-\alpha-1}. \end{aligned} \quad (8.17)$$

From (8.16) and (8.17), there is an $\alpha = \alpha(\gamma, n, \lambda, \Lambda, s) > 0$ such that for all $(x, z) \in [S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+ \setminus \{z = 0\}$, we have

$$a^{ij}(x) \partial_{ij} \bar{\phi} + |z|^{2-\frac{1}{s}} \partial_{zz} \bar{\phi} > \alpha(n\Lambda + 1)(2 + \bar{C}_2 \varepsilon)^{-\alpha-1} r^{-\alpha-1}.$$

We define the barrier ϕ on $[S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+$ by

$$\phi(x, z) = a\alpha^{-1}(2 + \bar{C}_2 \varepsilon)^{\alpha+1} r^{\alpha+1} (\bar{\phi}(x, z) - (1 + \bar{C}_1 \varepsilon)^{-\alpha} r^{-\alpha}).$$

For $(x, z) \in [S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+ \setminus \{z = 0\}$, it follows that $a^{ij}(x) \partial_{ij} \phi + |z|^{2-\frac{1}{s}} \partial_{zz} \phi > a(n\Lambda + 1)$. If $(x, 0) \in [S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+ \cap \{z = 0\}$, by (8.14),

$$\begin{aligned} \partial_{z+} \phi(x, 0) &= a(2 + \bar{C}_2 \varepsilon)^{\alpha+1} r^{\alpha+1} (\delta_\Phi((x_0, 0), (x, 0)) - g_\varepsilon(0))^{-\alpha-1} \varepsilon r^{1-s} \\ &\geq a(2 + \bar{C}_2 \varepsilon)^{\alpha+1} r^{\alpha+1} (2 + \bar{C}_2 \varepsilon)^{-\alpha-1} r^{-\alpha-1} \varepsilon r^{1-s} \\ &= a\varepsilon r^{1-s} > 0. \end{aligned}$$

Therefore, ϕ defined in $[S_{2r}(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+$ is a subsolution to (8.2). One can also check that $\phi > 0$ in $[S_r(x_0, 0) \setminus S_{\gamma r}(x_0, 0)]^+$ and that $\phi \leq 0$ on $[\partial S_{2r}(x_0, 0)]^+$ when $\varepsilon = \varepsilon(\gamma, s)$ is small enough to guarantee that $2 > 1 + \bar{C}_2 \varepsilon$. Moreover, there is a constant $C = C(\gamma, n, \lambda, \Lambda, s) > 0$ such that $\phi(x, z) \leq Car$ on $[\partial S_{\gamma r}(x_0, 0)]^+$.

Case 4: $z_0 \leq 0$ and $0 < s < 1$.

By (6.3), if $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^-$, then $(x, -z) \in [S_{2r}(x_0, -z_0) \setminus S_{\gamma r}(x_0, -z_0)]^+$. Define ψ in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^-$ to be the even reflection across $\{z = 0\}$ of the solution ϕ to (8.2) in $[S_{2r}(x_0, -z_0) \setminus S_{\gamma r}(x_0, -z_0)]^+$:

$$\psi(x, z) = \phi(x, -z), \quad \text{for } (x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^-.$$

Since $D^2\psi(x, z) = D^2\phi(x, -z)$, we know, for $(x, z) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^- \setminus \{z = 0\}$, that

$$\begin{aligned} a^{ij}(x)\partial_{ij}\psi(x, z) + |z|^{2-\frac{1}{s}}\partial_{zz}\psi(x, z) &= a^{ij}(x)\partial_{ij}\phi(x, -z) + |z|^{2-\frac{1}{s}}\partial_{zz}\phi(x, -z) \\ &> a(n\Lambda + 1). \end{aligned}$$

For $(x, 0) \in [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^- \cap \{z = 0\}$, we have $-\partial_{z-}\psi(x, 0) = \partial_{z+}\phi(x, 0) > 0$. Therefore, ψ is a subsolution to (8.3). It is straightforward to check that $\psi > 0$ in $[S_r(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^-$ and that $\psi \leq 0$ on $[\partial S_{2r}(x_0, z_0)]^-$. Lastly, if $(x, z) \in [\partial S_{\gamma r}(x_0, z_0)]^-$, then $(x, -z) \in [\partial S_{\gamma r}(x_0, -z_0)]^+$. This gives the desired estimate $\psi(x, z) = \phi(x, -z) \leq Car$ for $(x, z) \in [\partial S_{\gamma r}(x_0, z_0)]^-$. $\square$

9. Localization lemma

In this section, we prove the main localization estimate, Lemma 9.4. We show that if a supersolution U can be touched from below with a paraboloid P of opening $a > 0$ in a cube Q_r , then the set in which U can be touched from below by paraboloids of increased opening $Ca > 0$, where $C = C(n, \lambda, \Lambda, s) > 0$, in a smaller cube $Q_{\eta r}$ makes up a universal proportion of Q_r . To prove this result, we first use the barrier ϕ constructed in Lemma 8.2 to control how U detaches from a touching paraboloid P , see Lemma 9.2.

Before stating the main lemma of this section, we need to introduce some notation. First, we define a constant $\hat{K}_2$ to be large enough so that for any $(x_0, z_0), (\tilde{x}, \tilde{z}) \in \mathbb{R}^{n+1}$ and $R > 0$, if $Q_r(x_0, z_0) \subset Q_R(\tilde{x}, \tilde{z})$, then $Q_{2(n+1)r}(x_0, z_0) \subset Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$. By Lemma 4.10, we know that if $Q_r(x_0, z_0) \subset Q_R(\tilde{x}, \tilde{z})$ then $r \leq R$. If $(x, z) \in Q_{2(n+1)r}(x_0, z_0)$ then, by the quasi-triangle inequality (see Notation 4.9),

$$\begin{aligned} \delta_\varphi(\tilde{x}, x) &\leq K(\delta_\varphi(\tilde{x}, x_0) + \delta_\varphi(x_0, x)) < K(R + 2(n+1)r) < K(1 + 2(n+1))R \\ \delta_h(\tilde{z}, z) &\leq K(\delta_h(\tilde{z}, z_0) + \delta_h(z_0, z)) < K(1 + 2(n+1))R. \end{aligned}$$

We then take $\hat{K}_2 = \hat{K}_2(n, s)$ as

$$\hat{K}_2 = (2n + 3)K. \quad (9.1)$$

Let $\hat{K}_3 = \hat{K}_3(n, s)$ be given by

$$\hat{K}_3 = \theta^2 \hat{K}_2. \quad (9.2)$$

If $Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) \cap \{z = 0\} \neq \emptyset$, then $0 \in S_{\hat{K}_2 R}(\tilde{z})$ and, by the engulfing property,

$$Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) = Q_{\hat{K}_2 R}(\tilde{x}) \times S_{\hat{K}_2 R}(\tilde{z}) \subset Q_{\theta \hat{K}_2 R}(\tilde{x}) \times S_{\theta \hat{K}_2 R}(0) = Q_{\theta \hat{K}_2 R}(\tilde{x}, 0)$$

and

$$Q_{\theta \hat{K}_2 R}(\tilde{x}, 0) = Q_{\theta \hat{K}_2 R}(\tilde{x}) \times S_{\theta \hat{K}_2 R}(0) \subset Q_{\theta^2 \hat{K}_2 R}(\tilde{x}) \times S_{\theta^2 \hat{K}_2 R}(\tilde{z}) = Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z}).$$

We define a vertex set $B_v \subset \overline{Q}_{\hat{K}_3 R}(\tilde{x}, \tilde{z})$ by

$$B_v = \begin{cases} \overline{Q}_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) & \text{if } \tilde{z} = 0 \text{ or if } \overline{Q}_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) \cap \{z = 0\} = \emptyset \\ \overline{Q}_{\theta \hat{K}_2 R}(\tilde{x}, 0) & \text{if } \tilde{z} \neq 0 \text{ and } \overline{Q}_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) \cap \{z = 0\} \neq \emptyset, \end{cases}$$

so that B_v is symmetric with respect to $\{z = 0\}$ if $\overline{Q}_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) \cap \{z = 0\} \neq \emptyset$.

Define the contact set $A_{a,R}$ for a continuous function U on $Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$ by

$$A_{a,R} := \left\{ (x, z) \in Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z}) : U(x, z) \leq aR \text{ and there is } (x_v, z_v) \in B_v \text{ such that} \right. \\ \left. U \text{ can be touched from below at } (x, z) \text{ in } Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z}) \right. \\ \left. \text{by a paraboloid of opening } a > 0 \text{ with vertex } (x_v, z_v) \right\}. \quad (9.3)$$

Lemma 9.1. *The contact set $A_{a,R}$ is closed in $Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$.*

Proof. Let $(x_k, z_k) \in A_{a,R}$ and $(x_0, z_0) \in Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$ be such that $(x_k, z_k) \rightarrow (x_0, z_0)$. Since $U(x_k, z_k) \leq aR$ and U is continuous, $U(x_0, z_0) \leq aR$. By the same argument as in the proof of Theorem 7.1 with $B = B_v$, we can touch U from below in $Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z})$ at (x_0, z_0) by a paraboloid P of opening $a > 0$ with vertex $(x_v^0, z_v^0) \in B_v$. Therefore, $(x_0, z_0) \in A_{a,R}$ which shows that $A_{a,R}$ is closed in $Q_{\hat{K}_2 R}(\tilde{x}, \tilde{z})$. $\square$

Lemma 9.2. *Fix $0 < \gamma < 1$. Assume that Ω is a bounded domain and that $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ are bounded, measurable functions that satisfy (1.5). For a cube $Q_R = Q_R(\tilde{x}, \tilde{z}) \subset \mathbb{R}^{n+1}$, consider a cube $Q_{\hat{K}_3 R} = Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z})$ where $\hat{K}_3$ is as in (9.2). Let $f \in L^\infty(Q_{\hat{K}_3 R} \cap \{z = 0\})$ be nonnegative. Suppose $U \in C^2(Q_{\hat{K}_3 R} \setminus \{z = 0\}) \cap C(Q_{\hat{K}_3 R})$ such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(Q_{\hat{K}_3 R} \cap \{z \geq 0\})$ is a supersolution to*

$$\begin{cases} a^{ij}(x) \partial_{ij} U + |z|^{2-\frac{1}{s}} \partial_{zz} U \leq 0 & \text{in } Q_{\hat{K}_3 R} \cap \{z \neq 0\} \\ -\partial_{z+} U \geq f & \text{on } Q_{\hat{K}_3 R} \cap \{z = 0\}. \end{cases}$$

Assume that $Q_r(x_0, z_0) \subset Q_R$ for some point (x_0, z_0) such that $z_0 \geq 0$. Suppose that U is touched from below at $(x_1, z_1) \in [S_r(x_0, z_0)]^+ \cap A_{a,R}$ in $Q_{\hat{K}_3 R}$ by a paraboloid P of opening $a > 0$ with vertex (x_v, z_v) such that $z_v \geq 0$. Then, there exists a constant $C = C(\gamma, n, \lambda, \Lambda) > 0$ and a point $(x_2, z_2) \in [\bar{S}_{\gamma r}(x_0, z_0)]^+$ such that

$$U(x_2, z_2) - P(x_2, z_2) \leq Car.$$

Proof. If $(x_1, z_1) \in [\bar{S}_{\gamma r}(x_0, z_0)]^+$,

$$U(x_1, z_1) - P(x_1, z_1) = 0 \leq Car$$

for all $C > 0$, so we can take $(x_2, z_2) = (x_1, z_1)$. Therefore, we assume for the remainder of the proof that $(x_1, z_1) \in [S_r(x_0, z_0) \setminus \bar{S}_{\gamma r}(x_0, z_0)]^+$.

Let $W = U - P$. For $(x, z) \in Q_{\hat{K}_3 R} \setminus \{z = 0\}$, we have that

$$a^{ij}(x) \partial_{ij} P(x, z) + |z|^{2-\frac{1}{s}} \partial_{zz} P(x, z) = -a(\text{tr}(A(x)) + 1) \geq -a(n\Lambda + 1)$$

which implies

$$a^{ij}(x) \partial_{ij} W(x, z) + |z|^{2-\frac{1}{s}} \partial_{zz} W(x, z) \leq a(n\Lambda + 1).$$

Since $z_v \geq 0$, we also have that

$$-\partial_{z+} W(x, 0) \geq f(x) + ah'(z_v) \geq 0.$$

Let ϕ be the subsolution to (8.2) in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$. By the choice of $\hat{K}_2$ in (9.1), we have that $Q_r(x_0, z_0) \subset Q_R$ implies

$$S_{2r}(x_0, z_0) \subset Q_{2r}(x_0, z_0) \subset Q_{\hat{K}_2 R} \subset Q_{\hat{K}_3 R}$$

Therefore, $W - \phi$ satisfies

$$\begin{cases} a^{ij}(x)\partial_{ij}(W - \phi) + |z|^{2-\frac{1}{s}}\partial_{zz}(W - \phi) < 0 & \text{in } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z \neq 0\} \\ -\partial_{z+}(W - \phi)(x, 0) > 0 & \text{on } [S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}. \end{cases} \quad (9.4)$$

Let $(x_2, z_2) \in [\overline{S_{2r}(x_0, z_0)} \setminus S_{\gamma r}(x_0, z_0)]^+$ be such that

$$W(x_2, z_2) - \phi(x_2, z_2) = \min_{[\overline{S_{2r}(x_0, z_0)} \setminus S_{\gamma r}(x_0, z_0)]^+} (W - \phi).$$

By the maximum principle (see [13, Theorem 3.1]), the minimum of $W - \phi$ occurs on the boundary $\partial[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+$. That is,

$$(x_2, z_2) \in [\partial S_{2r}(x_0, z_0)]^+ \cup [\partial S_{\gamma r}(x_0, z_0)]^+ \cup [(S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)) \cap \{z = 0\}].$$

We claim that $(x_2, z_2) \in [\partial S_{\gamma r}(x_0, z_0)]^+$.

First, we will show that $(x_2, z_2) \notin [\partial S_{2r}(x_0, z_0)]^+$. Since $(x_1, z_1) \in [S_r(x_0, z_0)]^+$, we know that $\phi(x_1, z_1) > 0$ which implies $W(x_1, z_1) - \phi(x_1, z_1) = 0 - \phi(x_1, z_1) < 0$. Moreover, since $\phi \leq 0$ on $[\partial S_{2r}(x_0, z_0)]^+$, we have that $W(x, z) - \phi(x, z) \geq 0$ on $[\partial S_{2r}(x_0, z_0)]^+$. Therefore, the minimum is strictly negative and cannot occur on $[\partial S_{2r}(x_0, z_0)]^+$.

If $[S_{2r}(x_0, z_0)]^+ \cap \{z = 0\} = \emptyset$, then our claim holds. Suppose that $[S_{2r}(x_0, z_0)]^+ \cap \{z = 0\} \neq \emptyset$. Assume, by way of contradiction, that the minimum occurs on $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^+ \cap \{z = 0\}$, i.e. $z_2 = 0$. Then $-\partial_{z+}(W - \phi)(x_2, 0) \leq 0$, which contradicts (9.4). Therefore, it must be that the minimum occurs at $(x_2, z_2) \in [\partial S_{\gamma r}(x_0, z_0)]^+ \subset [\overline{S_{\gamma r}(x_0, z_0)}]^+$.

It follows from Lemma 8.2 that $\phi(x_2, z_2) \leq Car$ for $C = C(n, \lambda, \Lambda, \gamma) > 0$. Since $W(x_2, z_2) - \phi(x_2, z_2) < 0$, this implies that

$$U(x_2, z_2) - P(x_2, z_2) = W(x_2, z_2) < \phi(x_2, z_2) \leq Car. \quad \square$$

Remark 9.3. An analogue of Lemma 9.2 with $z_0, z_1, z_v \leq 0$ can be similarly proved using the subsolution ϕ to (8.3) in $[S_{2r}(x_0, z_0) \setminus S_{\gamma r}(x_0, z_0)]^-$.

To state the main result of this section, we define positive constants $K_0 > 1$ and $\eta < 1$ by

$$K_0 = 2K^2 + 2K \quad \text{and} \quad \eta = \frac{1}{K^2(2K K_0 + 1)}. \quad (9.5)$$

Lemma 9.4. Fix $a > 0$. Assume that Ω is a bounded domain and that $a^{ij}(x) : \Omega \rightarrow \mathbb{R}$ are bounded, measurable functions that satisfy (1.5). For a cube $Q_R = Q_R(\tilde{x}, \tilde{z}) \subset \mathbb{R}^{n+1}$, consider $Q_{\hat{K}_3 R} = Q_{\hat{K}_3 R}(\tilde{x}, \tilde{z})$ where $\hat{K}_3$ is as in (9.2). Let $f \in L^\infty(Q_{\hat{K}_3 R} \cap \{z = 0\})$ be nonnegative. Suppose $U \in C^2(Q_{\hat{K}_3 R} \setminus \{z = 0\}) \cap C(Q_{\hat{K}_3 R})$ such that U is symmetric across $\{z = 0\}$ and $U_{z+} \in C(Q_{\hat{K}_3 R} \cap \{z \geq 0\})$ is a supersolution to

$$\begin{cases} a^{ij}(x)\partial_{ij}U + |z|^{2-\frac{1}{s}}\partial_{zz}U \leq 0 & \text{in } Q_{\hat{K}_3 R} \cap \{z \neq 0\} \\ -\partial_{z+}U \geq f & \text{on } Q_{\hat{K}_3 R} \cap \{z = 0\}. \end{cases}$$

Let $Q_r(x_0, z_0)$ be such that

$$\overline{Q}_r(x_0, z_0) \subset Q_R \quad \text{and} \quad \overline{Q}_r(x_0, z_0) \cap A_{a,R} \neq \emptyset.$$

There exist positive constants $C = C(n, \lambda, \Lambda, s) > 1$ and $c = c(n, \lambda, \Lambda, s) < 1$ such that

$$\mu_\Phi(A_{Ca,R} \cap Q_{\eta r}(x_0, z_0)) \geq c\mu_\Phi(Q_r(x_0, z_0)),$$

where $\eta = \eta(n, s) < 1$ is as in (9.5).

Remark 9.5. Once the existence of $C = C(n, \lambda, \Lambda, s) > 1$ has been established in Lemma 9.4, one can always take C larger. Indeed, if $C' > C$ then, by Lemma 6.3, we have that $A_{Ca,R} \subset A_{C'a,R}$.

Proof of Lemma 9.4. Without loss of generality, we can assume that $Q_r(x_0, z_0) \cap A_{a,R} \neq \emptyset$. Otherwise, we replace r by $r + \varepsilon$ and then take the limit as $\varepsilon \rightarrow 0^+$ at the end. Let $(x_1, z_1) \in Q_r(x_0, z_0) \cap A_{a,R}$.

Since $(x_1, z_1) \in A_{a,R}$, there is a paraboloid P of opening $a > 0$ with vertex $(x_v, z_v) \in B_v$ that touches U from below in $Q_{\hat{K}_3 R}$ at (x_1, z_1) . We write P as

$$P(x, z) = -a\delta_\Phi((x_v, z_v), (x, z)) + a\delta_\Phi((x_v, z_v), (x_1, z_1)) + U(x_1, z_1).$$

As for z_0 , it must be that either $z_0 \geq 0$ or $z_0 < 0$. We may assume that z_1 has the same sign as z_0 , meaning that $z_0, z_1 \geq 0$ or that $z_0, z_1 \leq 0$. Indeed, suppose that $z_0 \geq 0$ and $z_1 < 0$. If $\overline{Q}_{\hat{K}_2 R} \cap \{z = 0\} = \emptyset$, this is a contradiction. If $\overline{Q}_{\hat{K}_2 R} \cap \{z = 0\} \neq \emptyset$ then, by Lemma 6.4, $\tilde{P}(x, z) = P(x, -z)$ touches U from below in $Q_{\hat{K}_3 R}$ at $(x_1, -z_1)$ with vertex $(x_v, -z_v) \in B_v$. Since

$$\begin{aligned} \delta_h(z_0, -z_1) &= h(z_1) - h(z_0) + h'(z_0)z_1 + h'(z_0)z_0 \\ &< h(z_1) - h(z_0) - h'(z_0)z_1 + h'(z_0)z_0 \quad \text{since } z_0 \geq 0 \text{ and } z_1 < 0 < -z_1 \\ &= \delta_h(z_0, z_1) < r, \end{aligned}$$

it follows that $(x_1, -z_1) \in Q_r(x_0, z_0) \cap A_{a,R}$. We proceed with the proof of the lemma using $\tilde{P}$ and $-z_1 > 0$ in place of P and $z_1 < 0$. The argument for $z_0 \leq 0$ and $z_1 > 0$ follows similarly.

Hence, without loss of generality, let us assume that $z_0, z_1 \geq 0$. Then, $z_v \geq 0$. Indeed, if $z_1 > 0$, then by Lemma 6.4, we know that $z_v \geq 0$. If $z_1 = 0$, then, since $f \geq 0$, by Lemma 6.6, it must be that $f(x_1) = 0$ and, consequently, $z_v = 0$.

Let $\gamma = \eta/(2\theta^2)$. Note that $(x_1, z_1) \in Q_r(x_0, z_0) \subset S_{(n+1)r}(x_0, z_0)$. We apply Lemma 9.2 with $r_0 = (n+1)r$ and $\gamma_0 = \gamma/(n+1)$ to find a point

$$(x_2, z_2) \in [\overline{S}_{\gamma_0 r_0}(x_0, z_0)]^+ = [\overline{S}_{\gamma r}(x_0, z_0)]^+ \subset \overline{S}_{\gamma r}(x_0, z_0)$$

and a constant $C = C(n, \lambda, \Lambda, s) > 0$ such that

$$U(x_2, z_2) - P(x_2, z_2) \leq Car.$$

Let $\alpha = \eta/(2\theta^3) < 1$ and let $C' = C'(n, \lambda, \Lambda, s) > 1$ be a large constant, to be determined. Slide from below the family of paraboloids

$$\bar{P}(x, z) = P(x, z) - C'a\delta_\Phi((\bar{x}_v, \bar{z}_v), (x, z)) + d, \quad \text{for } (\bar{x}_v, \bar{z}_v) \subset S_{\alpha r}(x_2, z_2) \quad (9.6)$$

until they touch the graph of U in $Q_{\hat{K}_3 R}$ for the first time. It is clear that

$$\bar{P}(x, z) = -a\delta_\varphi(x_v, x) - C'a\delta_\varphi(\bar{x}_v, x) - a\delta_h(z_v, z) - C'a\delta_h(\bar{z}_v, z) + d'$$

for some constant d' . Let $\xi \in \mathbb{R}$ be such that

$$h'(\xi) = \frac{h'(z_v) + C'h'(\bar{z}_v)}{C' + 1}.$$

It follows that, for some constant b' ,

$$-a\delta_h(z_v, z) - C'a\delta_h(\bar{z}_v, z) = -(C' + 1)a\delta_h(\xi, z) + b'.$$

Since

$$\frac{\nabla\varphi(x_v) + C'\nabla\varphi(\bar{x}_v)}{C' + 1} = \frac{x_v + C'\bar{x}_v}{C' + 1} = \nabla\varphi\left(\frac{x_v + C'\bar{x}_v}{C' + 1}\right),$$

we similarly write, for some constant b'' ,

$$-a\delta_\varphi(x_v, x) - C'a\delta_\varphi(\bar{x}_v, x) = -(C' + 1)a\delta_\varphi\left(\frac{x_v + C'\bar{x}_v}{C' + 1}, x\right) + b''.$$

Therefore

$$\bar{P}(x, z) = -(C' + 1)a\delta_\Phi\left(\left(\frac{x_v + C'\bar{x}_v}{C' + 1}, \xi\right), (x, z)\right) + d'',$$

for some constant d'' . Hence, the opening of $\bar{P}$ is $(C' + 1)a > 0$ and its vertex is of the form

$$\left(\frac{x_v + C'\bar{x}_v}{C' + 1}, \xi\right) \quad \text{where } h'(\xi) = \frac{h'(z_v) + C'h'(\bar{z}_v)}{C' + 1}.$$

Let B be the set of these vertices and let A denote the set of corresponding touching points.

Since $\bar{P}(x_2, z_2) \leq U(x_2, z_2)$, we have that $P(x_2, z_2) - C'a\delta_\Phi((\bar{x}_v, \bar{z}_v), (x_2, z_2)) + d \leq U(x_2, z_2)$. By the engulfing property, $S_{\alpha r}(x_2, z_2) \subset S_{\alpha\theta r}(\bar{x}_v, \bar{z}_v)$, so that $\delta_\Phi((\bar{x}_v, \bar{z}_v), (x_2, z_2)) < \alpha\theta r$. Therefore,

$$d \leq U(x_2, z_2) - P(x_2, z_2) + C'a\delta((\bar{x}_v, \bar{z}_v), (x_2, z_2)) \leq Car + C'\alpha\theta ar.$$

Since $(x_2, z_2) \in S_{\alpha\theta r}(\bar{x}_v, \bar{z}_v) \subset S_{2\alpha\theta r}(\bar{x}_v, \bar{z}_v)$, we again use the engulfing property to see that $S_{2\alpha\theta r}(\bar{x}_v, \bar{z}_v) \subset S_{2\alpha\theta^2 r}(x_2, z_2)$. Suppose that $(x, z) \in Q_{\tilde{K}_3 R}$ is such that $\delta_\Phi((x_2, z_2), (x, z)) \geq 2\alpha\theta^2 r$. Then $\delta_\Phi((\bar{x}_v, \bar{z}_v), (x, z)) \geq 2\alpha\theta r$ and

$$\begin{aligned} \bar{P}(x, z) &\leq P(x, z) - C'a(2\alpha\theta r) + (Car + C'\alpha\theta ar) \\ &= P(x, z) + (C - C'\theta\alpha)ar < P(x, z) \leq U(x, z) \end{aligned}$$

when $C' = C'(n, \lambda, \Lambda, s) > 1$ is such that $C' > C/(\theta\alpha)$. Hence, the contact points for $\bar{P}$ are inside $S_{2\alpha\theta^2 r}(x_2, z_2)$. That is, $A \subset S_{2\alpha\theta^2 r}(x_2, z_2)$.

Recall that $(x_2, z_2) \in \bar{S}_{\gamma r}(x_0, z_0)$. Since $\gamma = \alpha\theta$, we use the engulfing property to obtain

$$\begin{aligned} \bar{S}_{\gamma r}(x_0, z_0) &= \bar{S}_{\alpha\theta r}(x_0, z_0) \subset \bar{S}_{\alpha\theta^2 r}(x_2, z_2) \\ &\subset S_{2\alpha\theta^2 r}(x_2, z_2) \\ &\subset S_{2\alpha\theta^3 r}(x_0, z_0) = S_{\eta r}(x_0, z_0) \subset Q_{\eta r}(x_0, z_0). \end{aligned}$$

Consequently, $A \subset S_{2\alpha\theta^2r}(x_2, z_2) \subset Q_{\eta r}(x_0, z_0)$.

We now estimate

$$\begin{aligned}\bar{P}(x, z) &\leq P(x, z) + d \\ &\leq a\delta_\Phi((x_v, z_v), (x_1, z_1)) + U(x_1, z_1) + d \\ &\leq a\delta_\Phi((x_v, z_v), (x_1, z_1)) + aR + (Car + C'\alpha\theta ar) \\ &\leq aK(\delta_\Phi((\tilde{x}, \tilde{z}), (x_v, z_v)) + \delta_\Phi((\tilde{x}, \tilde{z}), (x_1, z_1))) + aR + (CaR + C'\alpha\theta ar) \\ &\leq aK(\hat{K}_3R + R) + aR + (CaR + C'\alpha\theta ar) \\ &= \left((\hat{K}_3 + 1)K + 1 + C + C'\alpha\theta\right)aR.\end{aligned}$$

If $C' = C'(n, \lambda, \Lambda, s) > 1$ is sufficiently large, then

$$\bar{P}(x, z) \leq (C' + 1)aR$$

which shows that $A \subset A_{(C'+1)a, R}$.

Since $f \geq 0$, we trivially have that

$$\mu_\Phi\left(B \cap \left\{(x, z) : |h'(z)| \leq \frac{\|f^-\|_{L^\infty(Q_{\hat{K}_3R})}}{(C' + 1)a}\right\}\right) = \mu_\Phi(B \cap \{(x, z) : z = 0\}) = 0.$$

Therefore, by Theorem 7.1,

$$\mu_\Phi(A_{(C'+1)a, R} \cap Q_{\eta r}(x_0, z_0)) \geq \mu_\Phi(A \cap Q_{\eta r}(x_0, z_0)) = \mu_\Phi(A) \geq c\mu_\Phi(B). \quad (9.7)$$

We claim that

$$c\mu_\Phi(B) \geq c'\mu_\Phi(Q_r(x_0, z_0)) \quad (9.8)$$

for a positive constant $c' = c'(n, \lambda, \Lambda, s) < 1$.

For the proof of (9.8), we first show that

$$\mu_\Phi(B) \geq \left(\frac{C'}{C' + 1}\right)^{n+1} \mu_\Phi(S_{\frac{\alpha r}{2}}(x_2, z_2)). \quad (9.9)$$

Observe that the B can be expressed as

$$B = \left\{(x, z) : x = \frac{x_v + C'\bar{x}_v}{C' + 1}, h'(z) = \frac{h'(z_v) + C'h'(\bar{z}_v)}{C' + 1}, (\bar{x}_v, \bar{z}_v) \in S_{\alpha r}(x_2, z_2)\right\}.$$

Define the sets B_1 and B_2 by

$$\begin{aligned}B_1 &= \left\{x = \frac{x_v + C'\bar{x}_v}{C' + 1} : \bar{x}_v \in S_{\alpha r/2}(x_2)\right\} \\ B_2 &= \left\{z = (h')^{-1}\left(\frac{h'(z_v) + C'h'(\bar{z}_v)}{C' + 1}\right) : \bar{z}_v \in S_{\alpha r/2}(z_2)\right\}.\end{aligned}$$

Since $S_{\alpha r/2}(x_2, z_2) \subset S_{\alpha r/2}(x_2) \times S_{\alpha r/2}(z_2) \subset S_{\alpha r}(x_2, z_2)$, we know that $B_1 \times B_2 \subset B$ and

$$\mu_{\Phi}(B) \geq \mu_{\Phi}(B_1 \times B_2) = \mu_{\varphi}(B_1)\mu_h(B_2). \quad (9.10)$$

By a change of variables,

$$\mu_{\varphi}(B_1) = \int_{B_1} dx = \left(\frac{C'}{C'+1} \right)^n \int_{S_{\alpha r/2}(x_2)} d\bar{x}_v = \left(\frac{C'}{C'+1} \right)^n \mu_{\varphi}(S_{\alpha r/2}(x_2)).$$

Notice that the set Z_0 given by

$$Z_0 = \left\{ \bar{z}_v \in \mathbb{R} : h'(\bar{z}_v) = -\frac{1}{C'} h'(z_v) \right\}$$

is a singleton. Then, by using a change of variables,

$$\begin{aligned} \mu_h(B_2) &= \int_{B_2 \setminus \{z=0\}} h''(z) dz \\ &= \int_{S_{\alpha r/2}(z_2) \setminus Z_0} h'' \left((h')^{-1} \left(\frac{h'(z_v) + C' h'(\bar{z}_v)}{C'+1} \right) \right) \partial_z (h')^{-1} \Big|_{\frac{h'(z_v) + C' h'(\bar{z}_v)}{C'+1}} \left(\frac{C'}{C'+1} \right) h''(\bar{z}_v) d\bar{z}_v \\ &= \frac{C'}{C'+1} \int_{S_{\alpha r/2}(z_2) \setminus Z_0} h''(\bar{z}_v) d\bar{z}_v = \frac{C'}{C'+1} \mu_h(S_{\alpha r/2}(z_2)). \end{aligned}$$

Combining these estimates into (9.10), we obtain

$$\begin{aligned} \mu_{\Phi}(B) &\geq \left(\frac{C'}{C'+1} \right)^{n+1} \mu_{\varphi}(S_{\alpha r/2}(x_2)) \mu_h(S_{\alpha r/2}(z_2)) \\ &\geq \left(\frac{C'}{C'+1} \right)^{n+1} \mu_{\Phi}(S_{\alpha r/2}(x_2, z_2)) \end{aligned}$$

and (9.9) holds.

For (9.8), observe that, by the doubling estimate (4.6) for μ_{Φ} ,

$$\mu_{\Phi}(S_{\gamma \theta r}(x_2, z_2)) \leq K_d \left(\frac{2\theta\gamma}{\alpha} \right)^{n+1} \mu_{\Phi}(S_{\frac{\alpha r}{2}}(x_2, z_2))$$

and

$$\mu_{\Phi}(S_{(n+1)r}(x_0, z_0)) \leq K_d \left(\frac{n+1}{\gamma} \right)^{n+1} \mu_{\Phi}(S_{\gamma r}(x_0, z_0)).$$

Since $(x_2, z_2) \in \bar{S}_{\gamma r}(x_0, z_0)$, the engulfing property gives $\bar{S}_{\gamma r}(x_0, z_0) \subset \bar{S}_{\gamma \theta r}(x_2, z_2)$. Hence, by using (9.9) and the previous two estimates,

$$\begin{aligned} c\mu_{\Phi}(B) &\geq c \left(\frac{C'}{C'+1} \right)^{n+1} \mu_{\Phi}(S_{\frac{\alpha r}{2}}(x_2, z_2)) \\ &\geq c \left(\frac{C'}{C'+1} \right)^{n+1} \frac{1}{K_d} \left(\frac{\alpha}{2\theta\gamma} \right)^{n+1} \mu_{\Phi}(S_{\gamma \theta r}(x_2, z_2)) \end{aligned}$$

$$\begin{aligned}
&\geq c \left(\frac{C'}{C'+1} \right)^{n+1} \frac{1}{K_d} \left(\frac{\alpha}{2\theta\gamma} \right)^{n+1} \mu_\Phi(S_{\gamma r}(x_0, z_0)) \\
&\geq c \left(\frac{C'}{C'+1} \right)^{n+1} \frac{1}{K_d^2} \left(\frac{\alpha}{2\theta(n+1)} \right)^{n+1} \mu_\Phi(S_{(n+1)r}(x_0, z_0)) \\
&\geq c' \mu_\Phi(Q_r(x_0, z_0)).
\end{aligned}$$

This completes the proof of (9.8).

From (9.7) and (9.8), the lemma follows. $\square$

10. Covering lemma

Here, we establish the following covering lemma.

Lemma 10.1. *Let $K_0 = K_0(n, s) > 1$, $\eta = \eta(n, s) < 1$ be as in (9.5), and fix $0 < c < 1$. Consider a cube $Q_{R/K_0} = Q_{R/K_0}(\tilde{x}, \tilde{z})$. Suppose there is a countable family of closed sets $D_k \subset \mathbb{R}^{n+1}$ that satisfy the following properties:*

- 1) $D_0 \subset D_1 \subset \cdots \subset D_k \subset \cdots \subset \overline{Q_{R/K_0}}$, $D_0 \neq \emptyset$;
- 2) for any $(x, z) \in \mathbb{R}^{n+1}$, $\rho > 0$ such that

$$Q_\rho(x, z) \subset Q_R(\tilde{x}, \tilde{z}), \quad Q_{\eta\rho}(x, z) \subset Q_{R/K_0}(\tilde{x}, \tilde{z}), \quad \overline{Q}_\rho(x, z) \cap D_k \neq \emptyset,$$

we have

$$\mu_\Phi(Q_{\eta\rho}(x, z) \cap D_{k+1}) \geq c\mu_\Phi(Q_\rho(x, z)).$$

Then

$$\mu_\Phi(Q_{R/K_0} \setminus D_k) \leq (1 - c)^k \mu_\Phi(Q_{R/K_0}).$$

Remark 10.2. Observe that Lemma 10.1 is similar the Calderón–Zygmund lemma in [3]. In fact, the sets $Q_{R/K_0} \setminus D_{k+1}$ and $Q_{R/K_0} \setminus D_k$, the parameter $1 - c$, the Monge–Ampère cubes Q_r and $Q_{\eta\rho}$, and the Monge–Ampère measure μ_Φ can be seen as analogues of the sets A and B , the parameter δ , the dyadic cubes $\tilde{Q}$ and Q , and the Lebesgue measure of Lemma 4.2 in [3], respectively. See also [29, Lemma 2.3].

To prove Lemma 10.1, we need the following simple consequence of [7, Theorem 1.2] for Monge–Ampère cubes.

Lemma 10.3. *Let $E \subset \mathbb{R}^{n+1}$ be a bounded subset. For each $(x, z) \in E$, consider a cube $Q_{r_{(x,z)}}(x, z)$ with radius $r_{(x,z)} > 0$. Then there is a countable subfamily of such cubes $\{Q_{r_i}(x_i, z_i)\}_{i=1}^\infty$ such that*

$$E \subset \bigcup_{i=1}^\infty Q_{r_i}(x_i, z_i), \quad \text{with } Q_{r_i/K_0}(x_i, z_i) \text{ pairwise disjoint.}$$

Proof of Lemma 10.1. For any $(x_0, z_0) \in E := Q_{R/K_0}(\tilde{x}, \tilde{z}) \setminus D_k$ and let r be given by

$$r = r_{(x_0, z_0)} = \inf\{r_0 : Q_{r_0}(x_0, z_0) \cap D_k \neq \emptyset\}. \quad (10.1)$$

The family $\{Q_r(x_0, z_0)\}$ covers E . By Lemma 10.3, there is a countable collection of cubes $\{Q_{r_i}(x_i, z_i)\}_{i=1}^\infty$ such that $E = Q_{R/K_0} \setminus D_k \subset \bigcup_i Q_{r_i}(x_i, z_i)$, with $Q_{r_i/K_0}(x_i, z_i)$ pairwise disjoint. Then,

$$\mu_\Phi(Q_{R/K_0} \setminus D_k) \leq \mu_\Phi \left(\bigcup_i Q_{r_i}(x_i, z_i) \cap Q_{R/K_0} \right) \leq \sum_i \mu_\Phi(Q_{r_i}(x_i, z_i) \cap Q_{R/K_0}).$$

We claim that, for any $(x_0, z_0) \in E$ and r given by (10.1),

$$\mu_\Phi(Q_r(x_0, z_0) \cap Q_{R/K_0}) \leq \frac{1}{c} \mu_\Phi(Q_{r/K_0}(x_0, z_0) \cap D_{k+1}). \quad (10.2)$$

Suppose for now that (10.2) holds. Then

$$\begin{aligned} \mu_\Phi(Q_{R/K_0} \setminus D_k) &\leq \sum_i \mu_\Phi(Q_{r_i}(x_i, z_i) \cap Q_{R/K_0}) \\ &\leq \sum_i \frac{1}{c} \mu_\Phi(Q_{r_i/K_0}(x_i, z_i) \cap D_{k+1}) \\ &= \frac{1}{c} \mu_\Phi \left(\bigcup_i Q_{r_i/K_0}(x_i, z_i) \cap (D_{k+1} \setminus D_k) \right) \\ &\leq \frac{1}{c} \mu_\Phi(D_{k+1} \setminus D_k). \end{aligned}$$

In the second to last estimate, we used our choice of r in (10.1). Since

$$\begin{aligned} \mu_\Phi(Q_{R/K_0} \setminus D_{k+1}) &= \mu_\Phi(Q_{R/K_0} \setminus D_k) - \mu_\Phi(D_{k+1} \setminus D_k) \\ &\leq \mu_\Phi(Q_{R/K_0} \setminus D_k) - c \mu_\Phi(Q_{R/K_0} \setminus D_k) \\ &= (1 - c) \mu_\Phi(Q_{R/K_0} \setminus D_k), \end{aligned}$$

by iteration, we finally obtain $\mu_\Phi(Q_{R/K_0} \setminus D_k) \leq (1 - c)^k \mu_\Phi(Q_{R/K_0})$, and the lemma is proved.

It is left to prove (10.2). We will present the proof for $n = 1$ for which

$$Q_{R/K_0}(\tilde{x}, \tilde{z}) = S_{R/K_0}(\tilde{x}) \times S_{R/K_0}(\tilde{z}) \subset \mathbb{R}^2.$$

The more general case follows similarly and is left to the reader.

First, we estimate r . Given any point $(x, z) \in Q_{R/K_0}$ and $(x_0, z_0) \in Q_{R/K_0} \setminus D_k$, we have

$$\delta_\varphi(x_0, x) \leq K (\delta_\varphi(\tilde{x}, x_0) + \delta_\varphi(\tilde{x}, x)) < \frac{2KR}{K_0},$$

and, similarly, $\delta_h(z_0, z) \leq 2KR/K_0$. Therefore, $r < 2KR/K_0$ whenever r is given by (10.1).

Let $(x_0, z_0) \in Q_{R/K_0} \setminus D_k$ and r as in (10.1) be fixed.

Next, let $(x, z) \in Q_r(x_0, z_0)$. By the quasi-triangle inequality, the choice of K_0 in (9.5), and the estimate on r ,

$$\delta_\varphi(\tilde{x}, x) \leq K (\delta_\varphi(\tilde{x}, x_0) + \delta_\varphi(x_0, x)) < K \left(\frac{R}{K_0} + r \right) \leq R.$$

Similarly, one can show that $\delta_h(\tilde{z}, z) < R$. Therefore, we have that

$$S_r(x_0) \times S_r(z_0) = Q_r(x_0, z_0) \subset Q_R(\tilde{x}, \tilde{z}) = S_R(\tilde{x}) \times S_R(\tilde{z}). \quad (10.3)$$

We will break into cases based on how far $(\tilde{x}, \tilde{z})$ is from (x_0, z_0) .

Case 1. Suppose that $\tilde{x} \in S_{r/K_0}(x_0)$, $\tilde{z} \in S_{r/K_0}(z_0)$.

We will show that $Q_r(x_0, z_0)$ satisfies the hypothesis 2) in the statement with $\rho = r$:

$$Q_r(x_0, z_0) \subset Q_R(\tilde{x}, \tilde{z}), \quad Q_{\eta r}(x_0, z_0) \subset Q_{R/K_0}(\tilde{x}, \tilde{z}), \quad \overline{Q}_r(x_0, z_0) \cap D_k \neq \emptyset.$$

We have already established (10.3). By the definition of r , we know that $\overline{Q}_r(x_0, z_0) \cap D_k \neq \emptyset$. Thus, it is left to show that $Q_{\eta r}(x_0, z_0) \subset Q_{R/K_0}(\tilde{x}, \tilde{z})$. Let $(x, z) \in Q_{\eta r}(x_0, z_0)$. By the quasi-triangle inequality and by choice of K_0 and η in (9.5), since $x \in S_{\eta r}(x_0)$,

$$\delta_\varphi(\tilde{x}, x) \leq K(\delta_\varphi(x_0, \tilde{x}) + \delta_\varphi(x_0, x)) < K\left(\frac{r}{K_0} + \eta r\right) \leq \frac{R}{K_0}.$$

We can similarly show that

$$\delta_h(\tilde{z}, z) < \frac{R}{K_0} \quad \text{since } z \in S_{\eta r}(z_0). \quad (10.4)$$

Hence, $Q_{\eta r}(x_0, z_0) \subset Q_{R/K_0}(\tilde{x}, \tilde{z})$.

Therefore, since $\eta \leq 1/K_0$, by property 2), we obtain the desired estimate:

$$\begin{aligned} \mu_\Phi(Q_{r/K_0}(x_0, z_0) \cap D_{k+1}) &\geq \mu_\Phi(Q_{\eta r}(x_0, z_0) \cap D_{k+1}) \\ &\geq c\mu_\Phi(Q_r(x_0, z_0)) \\ &\geq c\mu_\Phi(Q_r(x_0, z_0) \cap Q_{R/K_0}(\tilde{x}, \tilde{z})). \end{aligned}$$

Case 2. Suppose that $\tilde{x} \notin S_{r/K_0}(x_0)$, $\tilde{z} \in S_{r/K_0}(z_0)$.

It must be that $x_0 < \tilde{x}$ or $\tilde{x} < x_0$. Without loss of generality, we assume that $x_0 < \tilde{x}$.

From (10.3) and (10.4), we deduce that

$$S_r(z_0) \subset S_R(\tilde{z}), \quad S_{\eta r}(z_0) \subset S_{R/K_0}(\tilde{z}).$$

We will find x_1 between x_0 and $\tilde{x}$ such that

$$S_{r/(2K^2K_0)}(x_1) \subset S_{r/K_0}(x_0) \cap S_{R/K_0}(\tilde{x}). \quad (10.5)$$

Let $x_1 > x_0$ be such that $\delta_\varphi(x_0, x_1) = r/(2KK_0)$. We first show that $S_{r/(2KK_0)}(x_1) \subset S_{r/K_0}(x_0)$. Indeed, for $x \in S_{r/(2KK_0)}(x_1)$, we have that

$$\delta_\varphi(x_0, x) \leq K(\delta_\varphi(x_0, x_1) + \delta_\varphi(x_1, x)) < \frac{r}{K_0}.$$

Since

$$\frac{r}{2KK_0} = \delta_\varphi(x_0, x_1) \leq K\delta_\varphi(x_1, x_0) \leq K^2\delta_\varphi(x_0, x_1) = K^2\frac{r}{2KK_0},$$

we know that

$$\frac{r}{2K^2K_0} \leq \delta_\varphi(x_1, x_0) \leq \frac{r}{2K_0}.$$

Thus, $x_0 \notin S_{r/(2K^2K_0)}(x_1)$. Since the sections $S_{r/(2K^2K_0)}(x_1)$ and $S_{r/K_0}(x_0)$ are one-dimensional intervals, we can write them as

$$\begin{aligned} S_{r/(2K^2K_0)}(x_1) &= (x_L, x_R) \quad \text{where } x_L < x_1 < x_R \\ S_{r/K_0}(x_0) &= (x_L^0, x_R^0) \quad \text{where } x_L^0 < x_0 < x_R^0. \end{aligned}$$

Since $\tilde{x} \notin S_{r/K_0}(x_0)$ and $x_0 < \tilde{x}$, we know that

$$x_L^0 < x_0 < x_R^0 < \tilde{x}.$$

Since $x_0 < x_1$ and $S_{r/(2K^2K_0)}(x_1) \subset S_{r/K_0}(x_0)$, we have that

$$x_0 < x_L < x_1 < x_R < x_R^0 < \tilde{x}.$$

Thus, for any $x \in S_{r/(2K^2K_0)}(x_1)$, we know that $x_0 < x < \tilde{x}$. By Lemma 4.10,

$$\delta_\varphi(\tilde{x}, x) < \delta_\varphi(\tilde{x}, x_0) < \frac{R}{K_0}$$

Hence, $S_{r/(2K^2K_0)}(x_1) \subset S_{R/K_0}(\tilde{x})$ and we proved (10.5).

Define

$$\rho = \left(K + \frac{1}{2K_0} \right) r.$$

Clearly $S_r(z_0) \subset S_\rho(z_0)$. Let $x \in S_r(x_0)$. Then,

$$\delta_\varphi(x_1, x) \leq K (\delta_\varphi(x_0, x_1) + \delta_\varphi(x_0, x)) \leq K \left(\frac{r}{2K_0} + r \right) = \rho.$$

Hence, $S_r(x_0) \subset S_\rho(x_1)$. Therefore,

$$Q_r(x_0, z_0) = S_r(x_0) \times S_r(z_0) \subset S_\rho(x_1) \times S_\rho(z_0) = Q_\rho(x_1, z_0). \quad (10.6)$$

Since $\overline{Q}_r(x_0, z_0) \cap D_k \neq \emptyset$, we know by (10.6) that $\overline{Q}_\rho(x_1, z_0) \cap D_k \neq \emptyset$. Next, in order to apply property 2) in the statement, we will show that $Q_\rho(x_1, z_0)$ satisfies the following:

$$Q_\rho(x_1, z_0) \subset Q_R(\tilde{x}, \tilde{z}), \quad Q_{\eta\rho}(x_1, z_0) \subset Q_{R/K_0}(\tilde{x}, \tilde{z}), \quad Q_{\eta\rho}(x_1, z_0) \subset Q_{r/K_0}(x_0, z_0). \quad (10.7)$$

First, let us check that $Q_\rho(x_1, z_0) \subset Q_R(\tilde{x}, \tilde{z})$. Take $(x, z) \in Q_\rho(x_1, z_0)$ and observe that

$$\delta_\varphi(\tilde{x}, x) \leq K (\delta_\varphi(\tilde{x}, x_1) + \delta_\varphi(x_1, x)) < K \left(\frac{R}{K_0} + \rho \right) \leq R.$$

We can similarly show that $\delta_h(\tilde{z}, z) < R$. Hence, $Q_\rho(x_1, z_0) \subset Q_R(\tilde{x}, \tilde{z})$. Next, by the choice of η in (9.5), we know that

$$\eta\rho = \frac{r}{2K^2K_0} \leq \frac{r}{K_0}. \quad (10.8)$$

Then, by (10.5),

$$\begin{aligned} Q_{\eta\rho}(x_1, z_0) &= S_{r/(2K^2K_0)}(x_1) \times S_{r/(2K^2K_0)}(z_0) \\ &\subset S_{r/K_0}(x_0) \times S_{r/K_0}(z_0) = Q_{r/K_0}(x_0, z_0). \end{aligned}$$

Lastly, since $\tilde{z} \in S_{r/K_0}(z_0)$, for $z \in S_{\eta\rho}(z_0)$, by (9.5),

$$\delta_h(\tilde{z}, z) \leq K(\delta_h(z_0, \tilde{z}) + \delta_h(z_0, z)) < K\left(\frac{r}{K_0} + \eta\rho\right) \leq \frac{R}{K_0}.$$

Therefore, $S_{\eta\rho}(z_0) \subset S_{R/K_0}(\tilde{z})$. With this, (10.8), and (10.5), we obtain

$$\begin{aligned} Q_{\eta\rho}(x_1, z_0) &= S_{r/(2K^2K_0)}(x_1) \times S_{\eta\rho}(z_0) \\ &\subset S_{R/K_0}(\tilde{x}) \times S_{R/K_0}(\tilde{z}) = Q_{R/K_0}(\tilde{x}, \tilde{z}). \end{aligned}$$

We have shown that $Q_\rho(x_1, z_0)$ satisfies the hypotheses of property 2). Therefore, by using (10.7), the conclusion of 2), and (10.6), we obtain the desired estimate:

$$\begin{aligned} \mu_\Phi(Q_{r/K_0}(x_0, z_0) \cap D_{k+1}) &\geq \mu_\Phi(Q_{\eta\rho}(x_1, z_0) \cap D_{k+1}) \\ &\geq c\mu_\Phi(Q_\rho(x_1, z_0)) \\ &\geq c\mu_\Phi(Q_r(x_0, z_0)). \end{aligned}$$

Case 3. Suppose that $\tilde{x} \in S_{r/K_0}(x_0)$, $\tilde{z} \notin S_{r/K_0}(z_0)$.

This follows exactly as in Case 2 by switching the roles of $\tilde{x}$ and $\tilde{z}$ and using δ_h in place of δ_φ .

Case 4. Suppose that $\tilde{x} \notin S_{r/K_0}(x_0)$, $\tilde{z} \notin S_{r/K_0}(z_0)$.

This follows by combining the arguments in Case 2 and Case 3. $\square$

11. Proof of Theorem 5.3 and Theorem 1.1

11.1. Proof of Theorem 5.3

We begin by sliding a paraboloid P of opening $a > 0$ with vertex $(\tilde{x}, \tilde{z})$ from below until it touches the graph of U for the first time in $Q_{\hat{K}_3R}$, say at $(x_0, z_0) \in Q_{\hat{K}_3R}$. Then

$$P(x, z) = -a\delta_\Phi((\tilde{x}, \tilde{z}), (x, z)) + a\delta_\Phi((\tilde{x}, \tilde{z}), (x_0, z_0)) + U(x_0, z_0).$$

If $\delta_\Phi((\tilde{x}, \tilde{z}), (x_0, z_0)) > R/K_0$, then

$$\frac{aR}{2K_0} \geq U(\tilde{x}, \tilde{z}) \geq P(\tilde{x}, \tilde{z}) = a\delta_\Phi((\tilde{x}, \tilde{z}), (x_0, z_0)) + U(x_0, z_0) > \frac{aR}{K_0}.$$

Hence, $(x_0, z_0) \in \overline{S}_{R/K_0} = \overline{S}_{R/K_0}(\tilde{x}, \tilde{z}) \subset \overline{Q}_{R/K_0}$ and

$$U(x_0, z_0) = P(x_0, z_0) \leq P(\tilde{x}, \tilde{z}) < aR.$$

Thus, if $A_{a,R}$ is defined as in (9.3),

$$A_{a,R} \cap \overline{Q}_{R/K_0} \neq \emptyset.$$

In order to apply Lemma 10.1, we define the closed sets $D_k \subset \overline{Q}_{R/K_0}$ by

$$D_k := A_{aC^k, R} \cap \overline{Q}_{R/K_0}, \quad k \geq 0$$

where $C = C(n, \lambda, \Lambda, s) > 1$ is the constant from Lemma 9.4. If necessary, we can enlarge C to guarantee that

$$C - 2K \geq 2 \quad \text{and} \quad C - 2K - \frac{2K}{\theta} > 0, \quad (11.1)$$

see Remark 9.5. As a consequence of Lemma 6.3, we have

$$\emptyset \neq D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_k \subset \cdots \subset \overline{Q}_{R/K_0}.$$

Thus, hypothesis 1) of Lemma 10.1 is satisfied. To check that property 2) in Lemma 10.1 holds, let $(x, z) \in \mathbb{R}^{n+1}$, $\rho > 0$ be such that

$$Q_\rho(x, z) \subset Q_R(\tilde{x}, \tilde{z}), \quad Q_{\eta\rho}(x, z) \subset Q_{R/K_0}(\tilde{x}, \tilde{z}), \quad \overline{Q}_\rho(x, z) \cap D_k \neq \emptyset.$$

By Lemma 9.4, there is a positive constant $c = c(n, \lambda, \Lambda, s) < 1$ such that

$$\mu_\Phi(D_{k+1} \cap Q_{\eta\rho}(x, z)) = \mu_\Phi(A_{aC^{k+1}, R} \cap Q_{\eta\rho}(x, z)) \geq c\mu_\Phi(Q_{\eta\rho}(x, z)).$$

Hence, property 2) is satisfied. It follows from Lemma 10.1 that

$$\mu_\Phi(Q_{R/K_0} \setminus D_k) \leq (1 - c)^k \mu_\Phi(Q_{R/K_0}). \quad (11.2)$$

Also, from the definition of $A_{aC^k, R}$,

$$U(x, z) \leq aRC^k \quad \text{for } (x, z) \in D_k. \quad (11.3)$$

For $k \geq 0$, let $\rho_k = \rho_k(n, \lambda, \Lambda, s) < 1$ be a sequence of positive constants, to be determined, such that $\rho_k \searrow 0$ as $k \rightarrow \infty$. For convenience in the notation, let

$$\beta = \frac{1}{3K_0}.$$

Let $k_0 = k_0(n, \lambda, \Lambda, s) > 0$ be a large constant, to be determined.

Claim. Suppose that, for some $k \geq k_0$, there exists a point $(x_k, z_k) \in Q_{\beta R/(n+1)} \subset S_{\beta R} = S_{\beta R}(\tilde{x}, \tilde{z})$ such that

$$U(x_k, z_k) \geq aRC^{k+1}.$$

Then there is a point $(x_{k+1}, z_{k+1}) \in \partial S_{\rho_k R}(x_k, z_k)$ such that

$$U(x_{k+1}, z_{k+1}) \geq aRC^{k+2}.$$

Proof of claim. Suppose, by way of contradiction, that $U < aRC^{k+2}$ on $\partial S_{\rho_k R}(x_k, z_k)$. In the section

$$S_k = \overline{S}_{\rho_k R}(x_k, z_k)$$

we lower paraboloids of the form

$$P(x, z) = \frac{2aKC^{k+2}}{\rho_k} \delta_\Phi((x_v, z_v), (x, z)) + c_v, \quad (x_v, z_v) \in S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k) \quad (11.4)$$

from above until they touch the graph of U for the first time in S_k . Let A denote the set of contact points. Fix a point $(x_0, z_0) \in A$ and a corresponding paraboloid P as in (11.4) that touches U from above in S_k at (x_0, z_0) .

If necessary, slide P further until it intersects U at (x_k, z_k) and let us denote this paraboloid by $\tilde{P}$. By Lemma 6.2, we can write

$$\tilde{P}(x, z) = \frac{2aKC^{k+2}}{\rho_k} \delta_\Phi((x_v, z_v), (x, z)) - \frac{2aKC^{k+2}}{\rho_k} \delta_\Phi((x_v, z_v), (x_k, z_k)) + U(x_k, z_k).$$

Since $(x_v, z_v) \in S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k)$, by the engulfing property, $S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k) \subset S_{\frac{\rho_k R}{C^2}}(x_v, z_v)$. In particular, $\delta_\Phi((x_v, z_v), (x_k, z_k)) \leq \frac{\rho_k R}{C^2}$. Therefore, for $(x, z) \in S_k$,

$$\begin{aligned} \tilde{P}(x, z) &\geq \frac{2aKC^{k+2}}{\rho_k} \delta_\Phi((x_v, z_v), (x, z)) - \frac{2aKC^{k+2}}{\rho_k} \frac{\rho_k R}{C^2} + aRC^{k+1} \\ &\geq \frac{2aKC^{k+2}}{\rho_k} \delta_\Phi((x_v, z_v), (x, z)) + 2aRC^k, \end{aligned} \quad (11.5)$$

where we used (11.1). Therefore,

$$U(x_0, z_0) = P(x_0, z_0) \geq \tilde{P}(x_0, z_0) \geq 2aRC^k$$

which shows that

$$A \subset \{(x_0, z_0) \in \bar{S}_{\rho_k R}(x_k, z_k) : U(x_0, z_0) \geq 2aRC^k\}.$$

We will next prove that $(x_0, z_0) \in S_{\rho_k R}(x_k, z_k)$; that is, the contact points in A are interior points of the section $S_{\rho_k R}(x_k, z_k)$. Assume, by way of contradiction, that $\delta_\Phi((x_k, z_k), (x_0, z_0)) = \rho_k R$. By the quasi-triangle inequality,

$$\begin{aligned} \rho_k R &\leq K (\delta_\Phi((x_k, z_k), (x_v, z_v)) + \delta_\Phi((x_v, z_v), (x_0, z_0))) \\ &< K \left(\frac{\rho_k R}{\theta C^2} + \delta_\Phi((x_v, z_v), (x_0, z_0)) \right), \end{aligned}$$

so that

$$\delta_\Phi((x_v, z_v), (x_0, z_0)) > \rho_k R \left(\frac{1}{K} - \frac{1}{\theta C^2} \right).$$

Since $(x_0, z_0) \in S_k$, from (11.5) and (11.1), we get

$$\begin{aligned} U(x_0, z_0) &= P(x_0, z_0) \geq \tilde{P}(x_0, z_0) \\ &\geq \frac{2aKC^{k+2}}{\rho_k} \delta_\Phi((x_v, z_v), (x_0, z_0)) + aRC^k (C - 2K) \\ &> \frac{2aKC^{k+2}}{\rho_k} \rho_k R \left(\frac{1}{K} - \frac{1}{\theta C^2} \right) + aRC^k (C - 2K) > 2aRC^{k+2}, \end{aligned}$$

which contradicts our assumption that $U < aRC^{k+2}$ on $\partial S_{\rho_k R}(x_k, z_k)$. Therefore, it must be that $(x_0, z_0) \in S_{\rho_k R}(x_k, z_k)$. Consequently,

$$A \subset \{(x_0, z_0) \in S_{\rho_k R}(x_k, z_k) : U(x_0, z_0) \geq 2aRC^k\}. \quad (11.6)$$

Next, we want to apply Theorem 7.2 with Remark 7.3 in S_k with $\varepsilon_0 = 1/2$. For this, we need to choose $k_0 = k_0(n, \lambda, \Lambda, s)$ sufficiently large to guarantee that

$$\mu_\Phi \left(S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k) \cap \left\{ (x, z) : |h'(z)| \leq \frac{\|f^+\|_{L^\infty(S_k \cap \{z=0\})}}{(2aKC^{k+2}/\rho_k)} \right\} \right) \leq \frac{1}{2} \mu_\Phi(S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k)) \quad (11.7)$$

for all $k \geq k_0$. Indeed, observe that

$$\begin{aligned} & \mu_\Phi \left(S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k) \cap \left\{ (x, z) : |h'(z)| \leq \frac{\|f^+\|_{L^\infty(S_k \cap \{z=0\})}}{(2aKC^{k+2}/\rho_k)} \right\} \right) \\ & \leq \mu_\varphi \left(S_{\frac{\rho_k R}{\theta C^2}}(x_k) \right) \mu_h \left(\left\{ z \in \mathbb{R} : |h'(z)| \leq \frac{\|f\|_{L^\infty(S_k \cap \{z=0\})}}{(2aKC^{k+2}/\rho_k)} \right\} \right). \end{aligned}$$

Notice that

$$\begin{aligned} \mu_h \left(\left\{ z \in \mathbb{R} : |h'(z)| \leq \frac{\|f\|_{L^\infty(S_k \cap \{z=0\})}}{(2aKC^{k+2}/\rho_k)} \right\} \right) &= 2 \frac{\|f\|_{L^\infty(S_k \cap \{z=0\})}}{(2aKC^{k+2}/\rho_k)} \\ &= \frac{\rho_k}{KC^{k+2}} \frac{\|f\|_{L^\infty(S_k \cap \{z=0\})}}{a} \\ &\leq \frac{\rho_k}{KC^{k+2}} \mu_h(S_R(\tilde{z})). \end{aligned}$$

Since $z_k \in S_{\beta R/(n+1)}(\tilde{z})$, by the engulfing property, we have $S_{\beta R/(n+1)}(\tilde{z}) \subset S_{\theta \beta R/(n+1)}(z_k)$. With this and the doubling property (4.6) for μ_h ,

$$\begin{aligned} \mu_h(S_R(\tilde{z})) &\leq K_d \left(\frac{R}{\beta R/(n+1)} \right)^1 \mu_h(S_{\frac{\beta R}{n+1}}(\tilde{z})) \\ &= K_d \left(\frac{n+1}{\beta} \right) \mu_h(S_{\frac{\beta R}{n+1}}(\tilde{z})) \\ &\leq K_d \left(\frac{n+1}{\beta} \right) \mu_h(S_{\frac{\theta \beta R}{n+1}}(z_k)) \\ &\leq K_d \left(\frac{n+1}{\beta} \right) K_d \left(\frac{\theta \beta R/(n+1)}{\rho_k R/(\theta C^2)} \right)^1 \mu_h(S_{\frac{\rho_k R}{\theta C^2}}(z_k)) \\ &= \frac{K_d^2 \theta^2 C^2}{\rho_k} \mu_h(S_{\frac{\rho_k R}{\theta C^2}}(z_k)). \end{aligned}$$

Hence

$$\begin{aligned} \mu_h \left(\left\{ z \in \mathbb{R} : |h'(z)| \leq \frac{\|f\|_{L^\infty(S_k \cap \{z=0\})}}{(2aKC^{k+2}/\rho_k)} \right\} \right) &\leq \frac{\rho_k}{KC^{k+2}} \frac{K_d^2 \theta^2 C^2}{\rho_k} \mu_h(S_{\frac{\rho_k R}{\theta C^2}}(z_k)) \\ &= \frac{K_d^2 \theta^2}{KC^k} \mu_h(S_{\frac{\rho_k R}{\theta C^2}}(z_k)). \end{aligned}$$

This and the doubling property (4.6) for μ_Φ give

$$\begin{aligned} \mu_\Phi & \left(S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k) \cap \left\{ (x, z) : |h'(z)| \leq \frac{\|f^+\|_{L^\infty(S_k \cap \{z=0\})}}{(2aKC^{k+2}/\rho_k)} \right\} \right) \\ & \leq \frac{K_d^2 \theta^2}{KC^k} \mu_\varphi \left(S_{\frac{\rho_k R}{\theta C^2}}(x_k) \right) \mu_h \left(S_{\frac{\rho_k R}{\theta C^2}}(z_k) \right) \\ & \leq \frac{K_d^2 \theta^2}{KC^k} \mu_\Phi \left(S_{\frac{2\rho_k R}{\theta C^2}}(x_k, z_k) \right) \\ & \leq \frac{K_d^2 \theta^2}{KC^k} K_d 2^{n+1} \mu_\Phi \left(S_{\frac{\rho_k R}{\theta C^2}}(x_k, z_k) \right). \end{aligned}$$

Therefore, (11.7) holds if we choose $k_0 = k_0(n, \lambda, \Lambda, s)$ large enough so that

$$\frac{K_d^3 \theta^2 2^{n+1}}{KC^k} \leq \frac{1}{2} \quad \text{for all } k \geq k_0.$$

Hence, by Theorem 7.2 with Remark 7.3 for $\varepsilon_0 = 1/2$, it follows that

$$\mu_\Phi(A) \geq \frac{c}{2} \mu_\Phi(S_{\frac{\rho_k R}{C^2 \theta}}(x_k, z_k)). \quad (11.8)$$

Next, we will choose ρ_k in order to estimate $\mu_\Phi(S_{\frac{\rho_k R}{C^2 \theta}}(x_k, z_k))$ in (11.8) from below by $\mu_\Phi(Q_{R/K_0}(\tilde{x}, \tilde{z}))$ and get

$$\mu_\Phi(A) \geq 2(1-c)^k \mu_\Phi(Q_{R/K_0}(\tilde{x}, \tilde{z})). \quad (11.9)$$

In fact, since $\beta < 1/K_0$, we have that $(x_k, z_k) \in Q_{\beta R/(n+1)}(\tilde{x}, \tilde{z}) \subset S_{\beta R}(\tilde{x}, \tilde{z}) \subset S_{R/K_0}(\tilde{x}, \tilde{z})$, so that, by the engulfing property,

$$S_{R/K_0}(\tilde{x}, \tilde{z}) \subset S_{\theta R/K_0}(x_k, z_k).$$

As a consequence of the doubling property (4.6) for μ_Φ ,

$$\mu_\Phi(S_{\theta R/K_0}(x_k, z_k)) \leq K_d \left(\frac{C^2 \theta^2}{\rho_k K_0} \right)^{n+1} \mu_\Phi(S_{\frac{\rho_k R}{C^2 \theta}}(x_k, z_k))$$

and

$$\mu_\Phi(S_{R(n+1)/K_0}(\tilde{x}, \tilde{z})) \leq K_d (n+1)^{n+1} \mu_\Phi(S_{R/K_0}(\tilde{x}, \tilde{z})).$$

Combining these estimates, we obtain

$$\begin{aligned} \mu_\Phi(S_{\frac{\rho_k R}{C^2 \theta}}(x_k, z_k)) & \geq K_d^{-1} \left(\frac{\rho_k K_0}{C^2 \theta^2} \right)^{n+1} \mu_\Phi(S_{\theta R/K_0}(x_k, z_k)) \\ & \geq K_d^{-1} \left(\frac{\rho_k K_0}{C^2 \theta^2} \right)^{n+1} \mu_\Phi(S_{R/K_0}(\tilde{x}, \tilde{z})) \\ & \geq K_d^{-1} \left(\frac{\rho_k K_0}{C^2 \theta^2} \right)^{n+1} K_d^{-1} (n+1)^{-(n+1)} \mu_\Phi(S_{R(n+1)/K_0}(\tilde{x}, \tilde{z})) \\ & \geq K_d^{-1} \left(\frac{\rho_k K_0}{C^2 \theta^2} \right)^{n+1} K_d^{-1} (n+1)^{-(n+1)} \mu_\Phi(Q_{R/K_0}(\tilde{x}, \tilde{z})). \end{aligned}$$

If we take

$$\rho_k = c_0(1 - c)^{k/(n+1)}, \quad c_0 = \frac{C^2\theta^2(n+1)}{K_0} \left(\frac{4K_d^2}{c} \right)^{1/(n+1)}$$

we arrive at (11.9).

We next show, by enlarging k_0 if necessary, that

$$S((x_k, z_k), \rho_k R) = S_{\rho_k R}(x_k, z_k) \subset\subset Q_{R/K_0}(\tilde{x}, \tilde{z})$$

so that

$$A = A \cap S_{\rho_k R}(x_k, z_k) = A \cap Q_{R/K_0}(\tilde{x}, \tilde{z}). \quad (11.10)$$

Let $C_0 > 0$ and $p > 1$ be the constants in Lemma 4.11. Since $(x_k, z_k) \in S_{\beta R}(\tilde{x}, \tilde{z})$, we know by Lemma 4.11 with $r_1 = \beta$, $r_2 = \beta + (\rho_k/C_0)^{1/p}$, and $t = R$, that

$$S((x_k, z_k), \rho_k R) \subset S\left((\tilde{x}, \tilde{z}), \left(\beta + \left(\frac{\rho_k}{C_0}\right)^{1/p}\right) R\right).$$

If necessary, make $k_0 = k_0(n, \lambda, \Lambda, s)$ larger to guarantee that

$$\sum_{j=k_0}^{\infty} \left(\frac{\rho_j}{C_0}\right)^{1/p} < \frac{1}{2K_0} - \beta. \quad (11.11)$$

In particular,

$$\beta + \left(\frac{\rho_k}{C_0}\right)^{1/p} \leq \frac{1}{2K_0} \quad \text{for all } k \geq k_0.$$

Therefore, $S_{\rho_k R}(x_k, z_k) \subset S_{R/(2K_0)}(\tilde{x}, \tilde{z}) \subset\subset S_{R/K_0}(\tilde{x}, \tilde{z}) \subset Q_{R/K_0}(\tilde{x}, \tilde{z})$, which shows (11.10).

By the definition of D_k ,

$$\{(x, z) : U(x, z) > aRC^k\} \cap \overline{Q}_{R/K_0} \subset \overline{Q}_{R/K_0} \setminus D_k.$$

With this, (11.2), (11.9), (11.10), and (11.6), we estimate

$$\begin{aligned} \mu_{\Phi}(\{U > aRC^k\} \cap Q_{R/K_0}) &\leq \mu_{\Phi}(Q_{R/K_0} \setminus D_k) \\ &\leq (1 - c)^k \mu_{\Phi}(Q_{R/K_0}) \\ &\leq \frac{1}{2} \mu_{\Phi}(A) \\ &= \frac{1}{2} \mu_{\Phi}(A \cap Q_{R/K_0}) \\ &\leq \frac{1}{2} \mu_{\Phi}(\{U \geq 2aRC^k\} \cap Q_{R/K_0}) \\ &\leq \frac{1}{2} \mu_{\Phi}(\{U > aRC^k\} \cap Q_{R/K_0}), \end{aligned}$$

which is a contradiction. This completes the proof of the claim. $\square$

We now use the claim to prove (5.4) with $\kappa_2 = \beta/(n+1)$ and $C_H = C^{k_0+1}$. Suppose, by way of contradiction, that there is a point $(x_{k_0}, z_{k_0}) \in Q_{\beta R/(n+1)}$ such that

$$\sup_{Q_{\beta R/(n+1)}} U \geq U(x_{k_0}, z_{k_0}) > aRC^{k_0+1}.$$

By the claim, there is a point $(x_{k_0+1}, z_{k_0+1}) \in \partial S_{\rho_{k_0} R}(x_{k_0}, z_{k_0})$ such that

$$U(x_{k_0+1}, z_{k_0+1}) > aRC^{k_0+2}.$$

Repeating this process, we can find a sequence $(x_{k+1}, z_{k+1}) \in \partial S_{\rho_k R}(x_k, z_k)$ such that

$$U(x_{k+1}, z_{k+1}) > aRC^{k+2} \quad \text{for } k > k_0.$$

For all $k \geq k_0$, by Lemma 4.11 with

$$r_1 = \beta + \sum_{j=k_0}^k \left(\frac{\rho_j}{C_0} \right)^{1/p}, \quad r_2 = \beta + \sum_{j=k_0}^{k+1} \left(\frac{\rho_j}{C_0} \right)^{1/p}, \quad t = R,$$

and by (11.11), we obtain

$$\begin{aligned} S((x_{k+1}, z_{k+1}), \rho_{k+1} R) &\subset S \left((\tilde{x}, \tilde{z}), \left(\beta + \sum_{j=k_0}^{k+1} \left(\frac{\rho_j}{C_0} \right)^{1/p} \right) R \right) \\ &\subset S \left((\tilde{x}, \tilde{z}), \frac{R}{2K_0} \right) \subset Q \left((\tilde{x}, \tilde{z}), \frac{R}{2K_0} \right). \end{aligned}$$

Therefore, $(x_{k+1}, z_{k+1}) \in Q_{R/(2K_0)}$ for all $k \geq k_0$. In particular, U is unbounded in $\overline{Q}_{R/(2K_0)}$. This is a contradiction and completes the proof. $\square$

11.2. Proof of Theorem 1.1

Let $\kappa = \kappa(n, s) < 1$ and $\hat{K} = \hat{K}(n, s) > 1$ be such that

$$\kappa = \sqrt{\kappa_0} \quad \text{and} \quad \sqrt{2\hat{K}_0} = \hat{K}$$

where κ_0 and $\hat{K}_0$ are the constants from Theorem 1.3. We recall from (4.3) that

$$B_r(x_0) = S_{r^2/2}(x_0) \quad \text{for any } r > 0.$$

By taking $r = \sqrt{\kappa_0} R$,

$$B_{\kappa R}(x_0) \times \{z = 0\} \subset S_{\kappa_0 R^2/2}(x_0) \times S_{\kappa_0 R^2/2}(0) \subset S_{\kappa_0 R^2}(x_0, 0). \quad (11.12)$$

By taking $r = \sqrt{2\hat{K}_0} R$,

$$\begin{aligned} S_{\hat{K}_0 R^2}(x_0, 0) &\subset B_{\sqrt{2\hat{K}_0} R}(x_0) \times S_{\hat{K}_0 R^2}(0) \\ &= B_{\hat{K} R}(x_0) \times S_{\hat{K}_0 R^2}(0) \subset \subset \Omega \times \mathbb{R}. \end{aligned}$$

We also note that

$$S_{\hat{K}_0 R^2}(x_0, 0) \cap \{z = 0\} = B_{\hat{K}R}(x_0) \times \{z = 0\}. \quad (11.13)$$

Let U be as in Theorem 1.2. By Proposition 3.2 and (1.10), it follows that $U \geq 0$. Let $\tilde{U}$ be the even reflection of U so that $\tilde{U}$ is symmetric across $\{z = 0\}$. Notice that $\tilde{U} \in C^2(S_{\hat{K}_0 R^2}(x_0, 0) \setminus \{z = 0\}) \cap C(S_{\hat{K}_0 R^2}(x_0, 0))$, $\tilde{U}_{z+} \in C(S_{\hat{K}_0 R^2}(x_0, 0) \cap \{z \geq 0\})$ and that $\tilde{U}$ is a nonnegative solution to

$$\begin{cases} a^{ij}(x) \partial_{ij} \tilde{U} + z^{2-\frac{1}{s}} \partial_{zz} \tilde{U} = 0 & \text{in } S_{\hat{K}_0 R^2}(x_0, 0) \cap \{z \neq 0\} \\ -\partial_{z+} \tilde{U}(x, 0) = f(x) & \text{on } S_{\hat{K}_0 R^2}(x_0, 0) \cap \{z = 0\}. \end{cases}$$

Since $U(x, 0) = u(x)$, by (11.12), Theorem 1.3, and (11.13), we have that

$$\begin{aligned} \sup_{B_{\kappa R}(x_0)} u &\leq \sup_{S_{\kappa_0 R^2}(x_0, 0)} \tilde{U} \\ &\leq C_H \left(\inf_{S_{\kappa_0 R^2}(x_0, 0)} \tilde{U} + \|f\|_{L^\infty(S_{\hat{K}_0 R^2}(x_0, 0) \cap \{z=0\})} R^{2s} \right) \\ &\leq C_H \left(\inf_{B_{\kappa R}(x_0)} u + \|f\|_{L^\infty(B_{\hat{K}R}(x_0))} R^{2s} \right), \end{aligned}$$

which proves (1.7). Since u is bounded, the Hölder estimate (1.8) immediately follows for $R \leq |x - x_0| < \hat{K}R$. Assume that $|x - x_0| < R$. Note that

$$B_R(x_0) \times \{z = 0\} \subset S_{R^2}(x_0, 0) \subset S_{\hat{K}_0 R^2}(x_0, 0).$$

By this, (1.16), and (11.13), we have, for any $x \in B_R(x_0)$, that

$$\begin{aligned} |u(x_0) - u(x)| &= |\tilde{U}(x_0, 0) - \tilde{U}(x, 0)| \\ &\leq \frac{\hat{C}_1}{(\hat{K}_0 R^2)^{\alpha_1}} \delta_\Phi((x_0, 0), (x, 0))^{\alpha_1} \left(\sup_{S_{\hat{K}_0 R^2}(x_0, 0)} |\tilde{U}| + \|f\|_{L^\infty(S_{\hat{K}_0 R^2}(x_0, 0) \cap \{z=0\})} R^{2s} \right) \\ &\leq \frac{\hat{C}'_1}{(\hat{K}R)^{2\alpha_1}} |x_0 - x|^{2\alpha_1} \left(\sup_{B_{\hat{K}R}(x_0) \times S_{\hat{K}_0 R^2}(0)} |\tilde{U}| + \|f\|_{L^\infty(B_{\hat{K}R}(x_0))} R^{2s} \right). \end{aligned}$$

For each fixed $z \geq 0$, by (3.5),

$$\|U(\cdot, z)\|_{L^\infty(B_{\hat{K}R}(x_0))} \leq \frac{(2s)z}{4^s \Gamma(s)} \int_0^\infty e^{-\frac{s^2}{t} z^{\frac{1}{s}}} \|e^{-tL} u\|_{L^\infty(B_{\hat{K}R}(x_0))} \frac{dt}{t^{1+s}} \leq M \|u\|_{L^\infty(\Omega)}.$$

Letting $\hat{C} = M\hat{C}'_1$, and $\alpha_0 = 2\alpha_1 < 1$, we conclude (1.8). $\square$

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References

- [1] W. Arendt, R.M. Schätzle, Semigroups generated by elliptic operators in non-divergence form on $C_0(\Omega)$, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* 13 (2014) 1–18.
- [2] X. Cabré, S. Dipirro, E. Valdinoci, The Bernstein technique for integro-differential equations, arXiv:2010.00376, 2021, 57 pp.
- [3] L. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, Amer. Math. Soc. Colloquium Publications, vol. 43, American Mathematical Society, 1995.
- [4] L. Caffarelli, F. Charro, On a fractional Monge-Ampère operator, *Ann. PDE* 1 (2015) 4.
- [5] L.A. Caffarelli, C.E. Gutiérrez, Properties of the solutions of the linearized Monge-Ampère equation, *Am. J. Math.* 119 (1997) 423–465.
- [6] L.A. Caffarelli, P.R. Stinga, Fractional elliptic equations, Caccioppoli estimates and regularity, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 33 (2016) 767–807.
- [7] R.R. Coifman, G. Weiss, *Analyse Harmonique Non-commutative Sur Certains Espaces Homogènes*, Lecture Notes in Mathematics, vol. 242, Springer-Verlag, Berlin, Heidelberg, 1971.
- [8] R. Cont, P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, 2003.
- [9] G. Duvaut, J.L. Lions, *Inequalities in Mechanics and Physics*, Springer Verlag, Berlin, 1976.
- [10] L. Forzani, D. Maldonado, A mean-value inequality for nonnegative solutions to the linearized Monge-Ampère equation, *Potential Anal.* 30 (2009) 251–270.
- [11] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.
- [12] J.E. Galé, P.J. Miana, P.R. Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.* 13 (2013) 343–386.
- [13] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, Heidelberg, 2001.
- [14] L. Grafakos, *Modern Fourier Analysis*, second edition, Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009.
- [15] G. Grubb, Fractional Laplacians on domains, a development of Hörmander's theory of μ -transmission pseudodifferential operators, *Adv. Math.* 268 (2015) 478–528.
- [16] G. Grubb, Regularity of spectral fractional Dirichlet and Neumann problems, *Math. Nachr.* 289 (2016) 831–844.
- [17] C.E. Gutiérrez, *The Monge-Ampère Equation*, Progress in Nonlinear Differential Equations and Their Applications, vol. 44, Birkhäuser, Basel, 2001.
- [18] N.I. Jacob, R.L. Schilling, Some Dirichlet spaces obtained by subordinate reflected diffusions, *Rev. Mat. Iberoam.* 15 (1999) 59–91.
- [19] Y. Jhaveri, P.R. Stinga, The obstacle problem for a fractional Monge-Ampère equation, *Commun. Partial Differ. Equ.* 45 (2020) 457–482.
- [20] N.Q. Le, On the Harnack inequality for degenerate and singular elliptic equations with unbounded lower order terms via sliding parabolooids, *Commun. Contemp. Math.* 20 (2018), 38 pp.
- [21] D. Maldonado, Harnack's inequality for solutions to the linearized Monge-Ampère operator with lower-order terms, *J. Differ. Equ.* 256 (2014) 1987–2022.
- [22] D. Maldonado, On certain degenerate and singular elliptic PDEs I: nondivergence form operators with unbounded drifts and applications to subelliptic equations, *J. Differ. Equ.* 264 (2018) 624–678.
- [23] D. Maldonado, On certain degenerate and singular elliptic PDEs III: nondivergence form operators and RH_∞ -weights, *J. Differ. Equ.* 280 (2021) 805–840.
- [24] D. Maldonado, The Monge-Ampère quasi-metric structure admits a Sobolev inequality, *Math. Res. Lett.* 20 (2013) 527–536.
- [25] D. Maldonado, $W_\varphi^{1,p}$ -estimates for Green's functions on the linearized Monge-Ampère operator, *Manuscr. Math.* 152 (2017) 539–554.
- [26] D. Maldonado, P.R. Stinga, Harnack inequality for the fractional nonlocal linearized Monge-Ampère equation, *Calc. Var. Partial Differ. Equ.* 56 (2017) 56–103.
- [27] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
- [28] O. Savin, A Liouville theorem for solutions to the linearized Monge-Ampère equation, *Discrete Contin. Dyn. Syst.* 28 (2010) 865–873.
- [29] O. Savin, Small perturbation solutions for elliptic equations, *Commun. Partial Differ. Equ.* 32 (2007) 557–578.
- [30] R.T. Seeley, Norms and domains of the complex powers A_B^z , *Am. J. Math.* 93 (1971) 299–309.
- [31] R. Song, Z. Vondráček, Potential theory of subordinate killed Brownian motion in a domain, *Probab. Theory Relat. Fields* 125 (2003) 578–592.
- [32] P.R. Stinga, User's guide to the fractional Laplacian and the method of semigroups, in: *Handbook of Fractional Calculus with Applications*, vol. 2, De Gruyter, Berlin, 2019, pp. 235–265.
- [33] P.R. Stinga, J.L. Torrea, Extension problem and Harnack's inequality for some fractional operators, *Commun. Partial Differ. Equ.* 35 (2010) 2092–2122.
- [34] M. Vaughan, Analysis of nonlocal equations: One-sided weighted fractional Sobolev spaces and Harnack inequality for fractional nondivergence form elliptic equations, Thesis (Ph.D.)—Iowa State University, 2020, 148 pp. ProQuest LLC.
- [35] K. Yosida, *Functional Analysis*, reprint of the sixth (1980) edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.