



Robust k -center with two types of radii

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Abstract

In the non-uniform k -center problem, the objective is to cover points in a metric space with specified number of balls of different radii. Chakrabarty, Goyal, and Krishnaswamy [ICALP 2016, Trans. on Algs. 2020] (CGK, henceforth) give a constant factor approximation when there are two types of radii. In this paper, we give a constant factor approximation for the two radii case in the presence of outliers. To achieve this, we need to bypass the technical barrier of bad integrality gaps in the CGK approach. We do so using “the ellipsoid method inside the ellipsoid method”: use an outer layer of the ellipsoid method to reduce to stylized instances and use an inner layer of the ellipsoid method to solve these specialized instances. This idea is of independent interest and could be applicable to other problems.

Keywords Approximation · Clustering · Outliers · Round-or-cut.

Mathematics Subject Classification 68, Computer Science

1 Introduction

In the non-uniform k -center (NUkC) problem, one is given a metric space (X, d) and balls of different radii $r_1 > \dots > r_t$, with k_i balls of radius type r_i . The objective is to find a placement $C \subseteq X$ of centers of these $\sum_i k_i$ balls, such that they cover X with as little dilation as possible. More precisely, for every point $x \in X$ there must exist a center $c \in C$ of some radius type r_i such that $d(x, c) \leq \alpha \cdot r_i$ and the objective is to find C with α as small as possible.

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Chakrabarty et al. [10] introduced this problem as a generalization to the vanilla k -center problem [16–18] which one obtains with only one type of radius. One motivation arises from source location and vehicle routing: imagine you have a fleet of t -types of vehicles of different speeds and your objective is to find depot locations so that any client point can be served as fast as possible. This can be modeled as an $\text{NU}k\text{C}$ problem. The second motivation arises in clustering data. The k -center objective forces one towards clustering with equal sized balls, while the $\text{NU}k\text{C}$ objective gives a more nuanced way to model the problem. Indeed, $\text{NU}k\text{C}$ generalizes the *robust* k -center problem [13] which allows the algorithm to throw away z points as outliers. This is precisely the $\text{NU}k\text{C}$ problem with two types of radii, $r_1 = 1$, $k_1 = k$, $r_2 = 0$, and $k_2 = z$.

Chakrabarty et al. [10] give a 2-approximation for the special case of robust k -center which is the best possible [16, 17]. Furthermore, they give a $(1 + \sqrt{5})$ -factor approximation algorithm for the $\text{NU}k\text{C}$ problem with two types of radii (henceforth, the 2- $\text{NU}k\text{C}$ problem). [10] also prove that when t , the number of types of radii, is part of the input, there is no constant factor approximation algorithm unless $\text{P}=\text{NP}$. They explicitly leave open the case when the number of different radii types is a constant, conjecturing that constant-factor approximations should be possible. We take the first step towards this by looking at the *robust* 2- $\text{NU}k\text{C}$ problem. That is, the $\text{NU}k\text{C}$ problem with two kinds of radii when we can throw away z outliers. This is the case of 3-radii with $r_3 = 0$.

Theorem 1 *There is a 10-approximation for the Robust 2- $\text{NU}k\text{C}$ problem.*

Although the above theorem seems a modest step towards the CGK conjecture, it is in fact a non-trivial one which bypasses multiple technical barriers in the [10] approach. To do so, our algorithm applies a two-layered round-or-cut framework, and it is foreseeable that this idea will form a key ingredient for the constantly many radii case as well. In the rest of this section, we briefly describe the [10] approach, the technical bottlenecks one faces to move beyond 2 types of radii, and our approach to bypass them. A more detailed description appears in Sect. 2.

One key observation of [10] connects $\text{NU}k\text{C}$ with the *firefighter problem on trees* [1, 12, 15]. In the latter problem, one is given a tree where there is a fire at the root. The objective is to figure out if a specified number of firefighters can be placed in each layer of the tree, so that the leaves can be saved. To be precise, the objective is to select k_i nodes from layer i of the tree so that every leaf-to-root path contains at least one of these selected nodes.

Chakrabarty et al. [10] use the *integrality* of a natural LP relaxation for the firefighter problem on height-2 trees to obtain their constant factor approximation for 2- $\text{NU}k\text{C}$. In particular, they show how to convert a fractional solution of the standard LP relaxation of the 2- $\text{NU}k\text{C}$ problem to a feasible fractional solution for the firefighter LP. Since the latter LP is integral¹ for height-2 trees, they obtain an integral firefighting solution from which they construct an $O(1)$ -approximate solution for the 2- $\text{NU}k\text{C}$ problem. Unfortunately, this idea breaks down in the presence of outliers as the firefighter LP

¹ In the sense that, although it may not be integral at every basic feasible solution, if it is feasible for a given dilation of the per-level budget, then it is integrally feasible.

on height-2 trees *when certain leaves can be burnt* (outlier leaves, so to speak) is not integral anymore. In fact, the standard LP-relaxation for Robust 2-NUkC has unbounded integrality gap. This is the first bottleneck in the CGK approach.

Although the LP relaxation for the firefighter problem on height-2 trees is not integral when some leaves can be burnt, the problem itself (in fact for any constant height) is solvable in polynomial time using dynamic programming (DP) [1]. Using the DP, one can then obtain (see, for instance, [21]) a *polynomial sized integral* LP formulation for the firefighting problem.

This suggests the following enhancement of the CGK approach using the ellipsoid method. Given a fractional solution \mathbf{x} to Robust 2-NUkC, use the CGK approach to obtain a fractional solution \mathbf{y} to the firefighting problem. If \mathbf{y} is feasible for the integral LP formulation, then we get an integral solution to the firefighting problem which in turn gives an $O(1)$ -approximation for the Robust 2-NUkC instance via the CGK approach. Otherwise, we would get a *separating hyperplane* for \mathbf{y} and the poly-sized integral formulation for firefighting. If we could only use this to separate the fractional solution \mathbf{x} from the integer hull of the Robust 2-NUkC problem, then we could use the ellipsoid method to approximate Robust 2-NUkC. This is the so-called “round-or-cut” technique in approximation algorithms.

Unfortunately, this method also fails and indicates a much more serious bottleneck in the CGK approach. Specifically, there is an instance of Robust 2-NUkC and an \mathbf{x} in the integer hull of its solutions, such that the firefighting instance output by the CGK has *no* integral solution! Thus, one needs to enhance the CGK approach in order to obtain $O(1)$ -approximations even for the Robust 2-NUkC problem. The main contribution of this paper is to provide such an approach. We show that if the firefighting instance does not have an integral solution, then we can tease out many stylized Robust 2-NUkC instances on which the round-or-cut method provably succeeds, and an $O(1)$ -approximation to any one of them gives an $O(1)$ -approximation to the original Robust 2-NUkC instance.

Our Approach Any solution \mathbf{x} in the integer hull of NUkC solutions gives an indication of where different radii centers are opened. As it turns out, the key factor towards obtaining algorithms for the Robust 2-NUkC problem is observing where the large radii (that is, radius r_1) balls are opened. Our first step is showing that if the fractional solution \mathbf{x} tends to open the r_1 -centers only on “well-separated” locations then in fact, a round-or-cut algorithm from a previous paper [11] works. More precisely, if the Robust 2-NUkC instance is for some reason forced to open its r_1 centers on points which are at least cr_1 apart from each other for some constant $c > 4$, then the CGK approach plus round-or-cut leads to an $O(1)$ -approximation for the Robust 2-NUkC problem. We stress that this is far from trivial and the natural LP relaxations have bad gaps even in this case, hence, we rely on [11] to handle these well-separated instances.

But how and why would such well-separated instances arise? This is where we use ideas from recent papers on fair colorful clustering [3, 7, 20]. If \mathbf{x} suggested that the r_1 -radii centers are not well-separated, then one does not need that many balls if one allows dilation. In particular, if p and q are two r_1 -centers of a feasible integral solution, and $d(p, q) \leq cr_1$, then just opening one ball at either p or q with radius $(c + 1)r_1$ would cover every point that they each cover with radius r_1 -balls. Thus, in this case,

the approximation algorithm gets a “saving” in the budget of how many balls it can open. We exploit this savings in the budget by utilizing yet another observation from Adjiashvili, Baggio, and Zenklusen [1] on the natural LP relaxation for the firefighter problem on trees. This asserts that although the natural LP relaxation for constant height trees is not integral, one can get integral solutions by violating the constraints *additively* by a constant. The aforementioned savings allow us to get a solution without violating the budget constraints.

In summary, given an instance of the Robust 2-NUkC problem, we run an outer round-or-cut framework and use it to check whether an instance is well-separated or not. If not, we straightaway get an approximate solution via the CGK approach and the ABZ observation. Otherwise, we use enumeration (similar to [3]) to obtain $O(n)$ many different *well-separated* instances and for each, run an inner round-or-cut framework. If any of these well-separated instances are feasible, we get an approximate solution for the initial Robust 2-NUkC instance. Otherwise, we can assert a separating hyperplane for the outer round-or-cut framework.

Recent Work Shortly after this paper appeared in Integer Programming and Combinatorial Optimization [IPCO] 2021, Jia et al. [19] proved that an α -approximation for *robust* NUkC with t radii, translates to a $(2\alpha + 2)$ -approximation for NUkC with $t + 1$ radii. This was also brought to our attention independently by an anonymous reviewer.

Related Work NUkC was introduced in [10] as a generalization to the k -center problem [16–18] and the robust k -center problem [13]. In particular CGK reduce NUkC to the firefighter problem on trees which has constant approximations [1, 12, 15] and recently, a quasi-PTAS [24]. NUkC has also been studied in the *perturbation resilient* [4, 5, 14] settings. An instance is ρ -perturbation resilient if the optimal clustering does not change even when the metric is perturbed up to factor ρ . Bandapadhyay [6] gives an exact polynomial time algorithm for 2-perturbation resilient instances with constant number of radii.

As mentioned above, part of our approach is inspired by ideas from fair colorful k -center clustering [3, 7, 20] problems studied recently. In this problem, the points are divided into t color classes and we are asked to cover m_i , $i \in \{1, \dots, t\}$ many points from each color by opening k -centers. The idea of moving to well-separated instances are present in these papers. We should mention, however, that the problems are different, and their results do not imply ours.

The round-or-cut framework is a powerful approximation algorithm technique first used in a paper by Carr et al. [8] for the minimum knapsack problem, and since then has found use in other areas such as network design [9] and clustering [2, 3, 11, 22, 23]. Our multi-layered round-or-cut approach may find uses in other optimization problems as well.

2 Detailed description of our approach

In this section, we provide the necessary technical preliminaries required for proving Theorem 1 and give a more detailed description of the CGK bottleneck and our

approach. We start with notations. Let (X, d) be a metric space on a set of points X with distance function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ satisfying the triangle inequality. For any $u \in X$ we let $B(u, r)$ denote the set of points in a ball of radius r around u , that is, $B(u, r) = \{v \in X : d(u, v) \leq r\}$. For any set $U \subseteq X$ and function $f : U \rightarrow \mathbb{R}$, we use the shorthand notation $f(U) := \sum_{u \in U} f(u)$. For a set $U \subseteq X$ and any $v \in X$ we use $d(v, U)$ to denote $\min_{u \in U} d(v, u)$.

The 2-radii NUKC problem and the robust version are formally defined as follows.

Definition 1 [2-NUkC and Robust 2-NUkC] The input to 2-NUkC is a metric space (X, d) along with two radii $r_1 > r_2 \geq 0$ with respective budgets $k_1, k_2 \in \mathbb{N}$. The objective of 2-NUkC is to find the minimum $\rho \geq 1$ for which there exists subsets $S_1, S_2 \subseteq X$ such that (a) $|S_i| \leq k_i$ for $i \in \{1, 2\}$, and (b) $\bigcup_i \bigcup_{u \in S_i} B(u, \rho r_i) = X$. The input to Robust 2-NUkC contains an extra parameter $m \in \mathbb{N}$, and the objective is the same, except that condition (b) is changed to $|\bigcup_i \bigcup_{u \in S_i} B(u, \rho r_i)| \geq m$.

An instance \mathcal{I} of Robust 2-NUkC is denoted as $((X, d), (r_1, r_2), (k_1, k_2), m)$. As is standard, we will focus on the *approximate feasibility* version of the problem. An algorithm for this problem takes input an instance \mathcal{I} of Robust 2-NUkC, and either asserts that \mathcal{I} is *infeasible*, that is, there is no solution with $\rho = 1$, or provides a solution with $\rho \leq \alpha$. Using binary search, such an algorithm implies an α -approximation for Robust 2-NUkC.

Linear Programming Relaxations The following is the natural LP relaxation for the feasibility version of Robust 2-NUkC. For every point $v \in X$, variable $\text{cov}_i(v)$ denotes its *coverage* by balls of radius r_i . Variable $x_{i,u}$ denotes the extent to which a ball of radius r_i is open at point u . If instance \mathcal{I} is feasible, then the following polynomial sized system of inequalities has a feasible solution.

$$\begin{aligned} \{(\text{cov}_i(v) : v \in X, i \in \{1, 2\}) : & \sum_{v \in X} \text{cov}(v) \geq m \quad (\text{Robust2-NUkCLP}) \\ & \sum_{u \in X} x_{i,u} \leq k_i \quad \forall i \in \{1, 2\} \\ \text{cov}_1(v) \leq \sum_{u \in B(v, r_1)} x_{1,u}, \quad \text{cov}_2(v) \leq \sum_{u \in B(v, r_2)} x_{2,u} & \quad \forall v \in X \\ \text{cov}(v) = \text{cov}_1(v) + \text{cov}_2(v) \leq 1 & \quad \forall v \in X \\ x_{i,u} \geq 0 & \quad \forall i \in \{1, 2\}, \forall u \in X \end{aligned}$$

For our algorithm, we will work with the following integer hull of all possible fractional coverages. Fix a Robust 2-NUkC instance $\mathcal{I} = ((X, d), (r_1, r_2), (k_1, k_2), m)$ and let \mathcal{F} be the set of all tuples of subsets (S_1, S_2) with $|S_i| \leq k_i$. For $v \in X$ and $i \in \{1, 2\}$, we say \mathcal{F} covers v with radius r_i if $d(v, S_i) \leq r_i$. Let $\mathcal{F}_i(v) \subseteq \mathcal{F}$ be the subset of solutions that cover v with radius r_i . Moreover, we would like $\mathcal{F}_1(v)$ and $\mathcal{F}_2(v)$ to be disjoint, so if $S \in \mathcal{F}_1(v)$, we do not include it in $\mathcal{F}_2(v)$. The following is the integer hull of the coverages, where the coverage variables are derived from a convex combination of the integral solutions using variables z_S for $S \in \mathcal{F}$. If \mathcal{I} is feasible,

there must exist a solution in $P_{\text{cov}}^{\text{I}}$.

$$\begin{aligned} \{(\text{cov}_i(v) : v \in X, i \in \{1, 2\}) : \sum_{v \in X} (\text{cov}_1(v) + \text{cov}_2(v)) &\geq m \quad (P_{\text{cov}}^{\text{I}}) \\ \forall v \in X, i \in \{1, 2\} \quad \text{cov}_i(v) - \sum_{S \in \mathcal{F}_i(v)} z_S &= 0 \quad (P_{\text{cov}}^{\text{I}}.1) \end{aligned}$$

$$\sum_{S \in \mathcal{F}} z_S = 1 \quad (P_{\text{cov}}^{\text{I}}.2)$$

$$\forall S \in \mathcal{F} \quad z_S \geq 0 \quad (P_{\text{cov}}^{\text{I}}.3)$$

Fact 1 $P_{\text{cov}}^{\text{I}}$ lies inside Robust 2-NUkC LP.

Firefighting on Trees As described in Sect. 1, the CGK approach [10] is via the firefighter problem on trees. Since we only focus on Robust 2-NUkC, the relevant problem is the *weighted* 2-level fire fighter problem. The input includes a set of height-2 trees (stars) with root nodes L_1 and leaf nodes L_2 . Each leaf $v \in L_2$ has a parent $p(v) \in L_1$ and an integer weight $w(v) \in \mathbb{N}$. We use $\text{Leaf}(u)$ to denote the leaves connected to a $u \in L_1$ (that is, $\{v \in L_2 : p(v) = u\}$). Observe that $\{\text{Leaf}(u) : u \in L_1\}$ partitions L_2 . So we could represent the edges of the trees by this Leaf partition. Hence the structure is identified as $(L_1, L_2, \text{Leaf}, w)$.

Definition 2 (*2-Level Fire Fighter (2-FF) Problem*) Given height-2 trees $(L_1, L_2, \text{Leaf}, w)$ along with budgets $k_1, k_2 \in \mathbb{N}$, a feasible solution is a pair $T = (T_1, T_2)$, $T_i \subseteq L_i$, such that $|T_i| \leq k_i$ for $i \in \{1, 2\}$. Let $\mathcal{C}(T) := \{v \in L_2 : v \in T_2 \vee p(v) \in T_1\}$ be the set of leaves covered by T . The objective is to maximize $w(\mathcal{C}(T))$. Hence a 2-FF instance is represented by $((L_1, L_2, \text{Leaf}, w), k_1, k_2)$.

The standard LP relaxation for this problem is quite similar to the Robust 2-NUkC LP. For each vertex $u \in L_1 \cup L_2$ there is a variable $0 \leq y_u \leq 1$ that shows the extent to which u is included in the solution. For a leaf v , $Y(v)$ is the fractional amount by which v is covered in the solution, or saved, through both itself and its parent.

$$\begin{aligned} \max \sum_{v \in L_2} w(v)Y(v) : \quad & \sum_{u \in L_i} y_u \leq k_i, \quad \forall i \in \{1, 2\}; \quad (2\text{-FF LP}) \\ & Y(v) := y_{p(v)} + y_v \leq 1, \quad \forall v \in L_2; \quad y_u \geq 0, \quad \forall u \in L_1 \cup L_2 \end{aligned}$$

Remark 1 The following figure shows an example where the above LP relaxation has an integrality gap. However, 2-FF can be solved via dynamic programming in $O(n^3)$ time and has similar sized integral LP relaxations (Fig. 1).

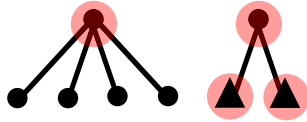


Fig. 1 A 2-FF instance with budgets $k_1 = k_2 = 1$. Multiplicity w is 1 for the circle leaves and 3 for the triangles. The highlighted nodes have $y = 1/2$ and the rest of the nodes have $y = 0$. The objective value for this y is $4 \times 1/2 + 6 = 8$ but no integral solution can get an objective value of more than 7

2.1 CGK's approach and its shortcomings

Given fractional coverages $(\text{cov}_1(v), \text{cov}_2(v) : v \in X)$, the CGK algorithm [10] runs the classic clustering subroutine by Hochbaum and Shmoys [17] in a greedy fashion. In English, the Hochbaum-Shmoys (HS) routine partitions a metric space such that the representatives of each part are well-separated with respect to an input parameter. The CGK algorithm obtains a 2-FF instance by applying the HS routine twice. Once on the whole metric space in decreasing order of $\text{cov}(v) = \text{cov}_1(v) + \text{cov}_2(v)$, and the set of representatives forms the leaf layer L_2 with weights being the size of the parts. The next time on L_2 itself in decreasing order of cov_1 and the representatives form the parent layer L_1 . These subroutines and the subsequent facts form a part of our algorithm and analysis.

Algorithm 1 HS

Input: Metric (U, d) , parameter $r \geq 0$, and assignment $\{\text{cov}(v) \in \mathbb{R}_{\geq 0} : v \in U\}$
 1: $R \leftarrow \emptyset$ ▷ The set of representatives
 2: **while** $U \neq \emptyset$ **do**
 3: $u \leftarrow \arg \max_{v \in U} \text{cov}(v)$ ▷ The first client in U in non-increasing cov order
 4: $R \leftarrow R \cup u$
 5: $\text{Child}(u) \leftarrow \{v \in U : d(u, v) \leq r\}$ ▷ Points in U at distance r from u (including u itself)
 6: $U \leftarrow U \setminus \text{Child}(u)$
 7: **end while**
Output: $R, \{\text{Child}(u) : u \in R\}$

Algorithm 2 CGK

Input: Robust 2-NUkC instance $((X, d), (r_1, r_2), (k_1, k_2), m)$, dilation factors $\alpha_1, \alpha_2 > 0$, and assignments $\text{cov}_1(v), \text{cov}_2(v) \in \mathbb{R}_{\geq 0}$ for all $v \in X$
 1: $(L_2, \{\text{Child}_2(v), v \in L_2\}) \leftarrow \text{HS}((X, d), \alpha_2 r_2, \text{cov} = \text{cov}_1 + \text{cov}_2)$
 2: $(L_1, \{\text{Child}_1(v), v \in L_2\}) \leftarrow \text{HS}((L_2, d), \alpha_1 r_1, \text{cov}_1)$
 3: $w(v) \leftarrow |\text{Child}_2(v)|$ for all $v \in L_2$
 4: $\text{Leaf}(u) \leftarrow \text{Child}_1(u)$ for all $u \in L_1$
Output: 2-FF instance $((L_1, L_2, \text{Leaf}, w), (k_1, k_2))$

Definition 3 [Valuable 2-FF instances] We call an instance T returned by the CGK algorithm valuable if it has an *integral* solution of total weight at least m .

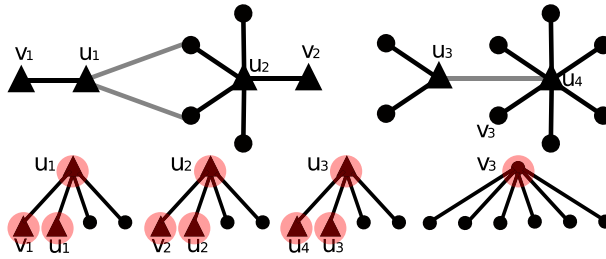


Fig. 2 At the top, there is a feasible Robust 2-NUkC instance with $k_1 = 2, k_2 = 3$, and $m = 24$. There are 6 triangles representing 3 collocated points each, along with 12 circles, each representing one point. The black edges are distance $r_1 > \alpha_2 r_2$ and the grey edges are distance $\alpha_1 r_1$. There are two integral solutions S and S' each covering exactly 24 points. $S_1 = \{u_1, u_4\}$, $S_2 = \{u_2, v_2, u_3\}$, $S'_1 = \{u_2, u_3\}$, and $S'_2 = \{u_1, v_1, u_4\}$. Having $z_S = z_{S'} = 1/2$ in $\mathcal{P}_{\text{cov}}^I$, gives cov_1 of $1/2$ for all the points and cov_2 of $1/2$ for the triangles. The output of Algorithm 2 is the 2-FF instance at the bottom. According to Proposition 1 the highlighted nodes have $y = 1/2$ and the rest of the nodes have $y = 0$ with objective value $12 \times 1/2 + 18 = 24$ but no integral solution can get an objective value of more than 23

Using a dynamic programming algorithm (see [1]), one can check whether a 2-FF instance is valuable in polynomial time.

Fact 2 The following are true regarding the output of HS: (a) $\forall u \in R, \forall v \in \text{Child}(u) : d(u, v) \leq r$, (b) $\forall u, v \in R : d(u, v) > r$, (c) The set $\{\text{Child}(u) : u \in R\}$ partitions U , and (d) $\forall u \in R, \forall v \in \text{Child}(u) : \text{cov}(u) \geq \text{cov}(v)$.

Lemma 1 [rewording of Lemma 3.4. in [10]] Let \mathcal{I} be a Robust 2-NUkC instance. If for any fractional coverages $(\text{cov}_1(v), \text{cov}_2(v))$ the instance 2-FF created by Algorithm 2 is valuable, then one obtains an $(\alpha_1 + \alpha_2)$ -approximation for \mathcal{I} .

This is simply by opening r_1 and r_2 centers on points corresponding to L_1 and L_2 firefighters respectively, and since the 2-FF instance is valuable, dilating the radii by $(\alpha_1 + \alpha_2)$ covers at least m points. See [10] for more detail.

Lemma 1 suggests that if we can find fractional coverages so that the corresponding 2-FF instance \mathcal{I} is valuable, then we are done. Unfortunately, the example illustrated in Fig. 2 shows that for any (α_1, α_2) there exists Robust 2-NUkC instances and fractional coverages $(\text{cov}_1(v), \text{cov}_2(v)) \in \mathcal{P}_{\text{cov}}^I$ in the integer hull, for which the CGK algorithm returns 2-FF instances that are not valuable.

2.2 Our idea

Although the 2-FF instance obtained by Algorithm 2 from fractional coverages $(\text{cov}_1(v), \text{cov}_2(v) : v \in X)$ may not be valuable, [10] proved that if these coverages come from Robust 2-NUkC LP, then there is always a fractional solution to 2-FF for this instance which has value at least m .

Proposition 1 (rewording of Lemma 3.1. in [10]) Let $(\text{cov}_1(v), \text{cov}_2(v) : v \in X)$ be any feasible solution to Robust 2-NUkC LP. As long as $\alpha_1, \alpha_2 \geq 2$, the following is a fractional solution of 2-FF with value at least m for the 2-FF instance output by Algorithm 2.

$$y_v = \begin{cases} \text{cov}_1(v) & v \in L_1 \\ \min\{\text{cov}_2(v), 1 - \text{cov}_1(p(v))\} & v \in L_2. \end{cases}$$

Therefore, the problematic instances are precisely 2-FF instances that are integrality gap examples for 2-FF. Our first observation stems from what Adjashvili, Baggio, and Zenklusen [1] call “the narrow integrality gap of the firefighter LP”.

Lemma 2 (From Lemma 6 of [1]) *Any basic feasible solution $\{y_v : i \in \{1, 2\}, v \in L_i\}$ of the 2-FF polytope has at most 2 loose variables. A variable y_v is loose if $0 < y_v < 1$ and $y_{p(v)} = 0$ in case $v \in L_2$.*

In particular, if $y(L_1) \leq k_1 - 2$, then the above lemma along with Proposition 1 implies there exists an *integral* solution with value $\geq m$. That is, the 2-FF instance is valuable (see Proposition 5). Conversely, the fact that the instance is *not* valuable asserts that $y(L_1) > k_1 - 2$ which in turn implies $\text{cov}_1(L_1) > k_1 - 2$. In English, the fractional coverage puts a lot of weight on the points in L_1 .

This is where we exploit the ideas in [3, 7, 20]. By choosing $\alpha_1 > 2$ to be large enough in Proposition 1, we can ensure that points in L_1 are “well-separated”. More precisely, we can ensure for any two $u, v \in L_1$ we have $d(u, v) > \alpha_1 r_1$ (from Fact 2). The well-separated condition implies that the same center cannot be fractionally covering two different points in L_1 . Therefore, if $\text{cov}_1(L_1) > k_1 - 2$ and $(\text{cov}_1, \text{cov}_2) \in \mathbb{P}_{\text{cov}}^{\mathbb{I}}$ is in the integer hull, then there must exist an integer solution which opens *at most* 1 radius r_1 center that does *not* cover points in L_1 (see Proposition 6). For the time being assume in fact no such center exists and $\text{cov}_1(L_1) = k_1$. Indeed, the integrality gap example in Fig. 2 satisfies this equality.

Our last piece of the puzzle is that if the cov_i ’s are concentrated on separated points, then indeed we can apply the round-or-cut framework to obtain an approximation algorithm. To this end, we make the following definition, and assert the following theorem.

Definition 4 (*Well-Separated Robust*) The input is the same as Robust 2-NUkC, along with $Y \subseteq X$ where $d(u, v) > 4r_1$ for all pairs $u, v \in Y$. The objective is to find S_1 and S_2 as in Definition 1 with the constraint that $S_1 \subseteq Y$.

Theorem 2 *Given a Well-Separated Robust 2-NUkC instance there is a polynomial time algorithm using the ellipsoid method that either gives a 4-approximate solution, or proves that the instance is infeasible.*

We remark the natural Robust 2-NUkC LP relaxation still has a bad integrality gap, and we need the round-or-cut approach. Formally, given fractional coverages $(\text{cov}_1, \text{cov}_2)$ we run Algorithm 2 (with $\alpha_1 = \alpha_2 = 2$) to get a 2-FF instance. If the instance is valuable, we are done by Lemma 1. Otherwise, we prove that $(\text{cov}_1, \text{cov}_2) \notin \mathbb{P}_{\text{cov}}^{\mathbb{I}}$ by exhibiting a separating hyperplane. This crucially uses the well-separated-ness of the instance and indeed, the bad example shown in Fig. 2 is *not* well-separated. This implies Theorem 2 using the ellipsoid method.

In summary, to prove Theorem 1, we start with $(\text{cov}_1, \text{cov}_2)$ purported to be in $\mathbb{P}_{\text{cov}}^{\mathbb{I}}$. Our goal is to either get a constant approximation, or separate $(\text{cov}_1, \text{cov}_2)$ from $\mathbb{P}_{\text{cov}}^{\mathbb{I}}$.

such that in the t -layered firefighter instance returned, *all* layers have “slack”, that is $\text{cov}_i(L_i) \leq k_i - t$, then one can repeatedly use Lemma 2 to show that the tree instance is indeed valuable.

The issue we do not know how to circumvent is when some layers have slack and some layers do not. In particular, even with 3 kinds of radii, we do not know how to handle the case when the first layer L_1 is well-separated and $\text{cov}_1(L_1) = k_1$, but the second layer has slack $\text{cov}_2(L_2) \leq k_2 - 3$. Lemma 2 does not help since all the loose vertices may be in L_1 , but they cannot all be picked without violating the budget. At the same time, we do not know how to separate such cov’s, or whether such a situation arises when cov’s are in the integer hull. We believe one needs more ideas to resolve the CGK conjecture.

3 Approximating well-separated robust 2-NUkC

In this section we prove Theorem 2 stated in Sect. 2.2. As mentioned there, the main idea is to run the round-or-cut method, and in particular use ideas from a previous paper [11] of ours. The main technical lemma is the following.

Lemma 3 *Given Well-Separated Robust 2-NUkC instance \mathbb{I} and fractional coverages $(\hat{\text{cov}}_1(v), \hat{\text{cov}}_2(v))$, if the output of the CGK Algorithm 2 is not valuable, there is a hyperplane separating $(\hat{\text{cov}}_1(v), \hat{\text{cov}}_2(v))$ from $\mathbb{P}_{\text{cov}}^{\mathbb{I}}$. Furthermore, the coefficients of this hyperplane are bounded in value by $|X|$.*

Remark 2 We need to be careful in one place. Recall that HS is used in the CGK Algorithm 2. We need to assert in HS, that points u with $d(u, Y) \leq r_1$ are prioritized over points v with $d(v, Y) > r_1$ to be taken in L_1 . This is w.l.o.g. since $\text{cov}_1(v) = 0$ if $d(v, Y) > r_1$ by definition of Well-Separated Robust 2-NUkC.

This remark is a simple technicality that only appears in the proof of Proposition 3. Using the ellipsoid method, the above lemma implies Theorem 2.

Proof (Proof of Theorem 2) The goal is to either prove $\mathbb{P}_{\text{cov}}^{\mathbb{I}}$ is empty, or give a 4-approximate solution. To do so, we run the ellipsoid algorithm. Each time the ellipsoid algorithm provides a purported fractional point $(\hat{\text{cov}}_1(v), \hat{\text{cov}}_2(v)) \in \mathbb{P}_{\text{cov}}^{\mathbb{I}}$ and asks for a separating hyperplane. Given such a solution, we first check if $\hat{\text{cov}}_1(v) = 0$ for all v with $d(v, Y) > r_1$. By the well-separatedness property of \mathbb{I} , this must be a valid equality and we can force the ellipsoid method to run over these equalities. Then we run CGK Algorithm 2 with this $(\hat{\text{cov}}_1, \hat{\text{cov}}_2)$ and $\alpha_1 = \alpha_2 = 2$. If the resulting 2-FF instance is valuable, we get a 4-approximate solution by Lemma 1. If not, Lemma 3 provides a separating hyperplane to feed to ellipsoid. Since our hyperplanes can be described in polynomial size, ellipsoid terminates in polynomial time, either giving us some $\hat{\text{cov}}$ leading to a 4-approximation along the way, or prompts that $\mathbb{P}_{\text{cov}}^{\mathbb{I}}$ is empty thereby proving \mathbb{I} is infeasible. \square

The rest of this section is dedicated to proving Lemma 3. Fix a well-separated Robust 2-NUkC instance \mathbb{I} . Recall that $Y \subseteq X$ is a subset of points, and the radius r_1 centers are only allowed to be opened at Y . Let \mathbb{T} be the 2-FF instance output by Algorithm

2 on \mathbb{I} and cov with $\alpha_1 = \alpha_2 = 2$. Recall, $\mathbb{T} = ((L_1, L_2, \text{Leaf}, w), k_1, k_2)$. The key part of the proof is the following valid inequality in case \mathbb{T} is not valuable.

Lemma 4 *If \mathbb{T} is not valuable, then $\sum_{v \in L_2} w(v) \text{cov}(v) \leq m - 1$ for any $\text{cov}(v) \in \mathbb{P}_{\text{cov}}^{\mathbb{I}}$.*

Before we prove Lemma 4, let us show how it proves Lemma 3. Given $(\hat{\text{cov}}_1, \hat{\text{cov}}_2)$ we first check³ that $\sum_{u \in X} \hat{\text{cov}}(u) \geq m$, or otherwise that would be the hyperplane separating it from $\mathbb{P}_{\text{cov}}^{\mathbb{I}}$. Now recall that in Algorithm 2, for $v \in L_2$, $w(v) = |\text{Child}_2(v)|$ which is the number of points assigned to v by HS (see Line 1 of Algorithm 1 of Algorithm 2). By definition of w and then parts d) and c) of Fact 2,

$$\sum_{v \in L_2} w(v) \hat{\text{cov}}(v) = \sum_{v \in L_2} \sum_{u \in \text{Child}(v)} \hat{\text{cov}}(v) \geq \sum_{v \in L_2} \sum_{u \in \text{Child}(v)} \hat{\text{cov}}(u) = \sum_{u \in X} \hat{\text{cov}}(u) \geq m.$$

That is, $(\hat{\text{cov}}_1, \hat{\text{cov}}_2)$ violates the valid inequality asserted in Lemma 4, and this would complete the proof of Lemma 3. All that remains is to prove the valid inequality lemma above.

Proof (of Lemma 4) Fix a solution $\text{cov} \in \mathbb{P}_{\text{cov}}^{\mathbb{I}}$ and note that this is a convex combination of coverages induced by integral feasible solutions in \mathcal{F} . The main idea of the proof is to use the solutions in \mathcal{F} to construct solutions to the tree instance \mathbb{T} . Since \mathbb{T} is not valuable, each of these solutions will have “small” value, and then we use this to prove the lemma. To this end, fix $S = (S_1, S_2) \in \mathcal{F}$ where $|S_i| \leq k_i$ for $i \in \{1, 2\}$. The corresponding solution $T = (T_1, T_2)$ for \mathbb{T} is defined as follows: For $i \in \{1, 2\}$ and any $u \in L_i$, u is in T_i iff $S_i \in \mathcal{F}_i(u)$. That is, $d(u, S_i) \leq r_i$.

Proposition 2 *T satisfies the budget constraints $|T_i| \leq k_i$ for $i \in \{1, 2\}$.*

Proof For $i \in \{1, 2\}$ and two different $u, v \in T_i$ by Fact 2 and our choice of $\alpha_i = 2$, $d(u, v) > 2r_i$. By the triangle inequality, a facility in S_i cannot cover both u and v meaning $|T_i| \leq |S_i| \leq k_i$. \square

The next claim is the only place where we need the well-separated-ness of \mathbb{I} . Basically, we will argue that the leaves saved by T_1 capture all the points covered by S_1 .

Proposition 3 *If $u \in L_1$ but $u \notin T_1$ then no $v \in \text{Leaf}(u)$ can be covered by a ball of radius r_1 in S_1 .*

Proof We will prove the contrapositive by showing that if $u = p(v)$ and v is covered through $f \in S_1$, then u as well must be covered by the same f and therefore, $u \in T_1$. Recall that Y is a set of well-separated points where r_1 centers are allowed. Consider the following two cases: either $d(u, Y) > r_1$ in which case, by our assumption on HS in Remark 2, v is prioritized over u to be chosen in L_1 so this cannot happen. Thus, we must have $d(u, Y) \leq r_1$ which means there is $f_u \in Y$ with $d(u, f_u) \leq r_1$. This f_u has to be equal to f otherwise, by definition of Y we must have $d(f, f_u) > 4r_1$ that contradicts the following: $d(f_u, f) \leq d(f_u, u) + d(u, v) + d(v, f) \leq r_1 + \alpha_1 r_1 + r_1 = 4r_1$. \square

³ Recall, $\hat{\text{cov}}(v) = \hat{\text{cov}}_1(v) + \hat{\text{cov}}_2(v)$.

Next, we can prove that overall, the leaves saved by T capture the whole set of points covered by S . Recall that $\mathcal{C}(T) = \{v \in L_2 : v \in T_2 \vee p(v) \in T_1\}$ is the set of leaves saved by T . For $v \in X$ let $\mathcal{F}(v) := \mathcal{F}_1(v) \cup \mathcal{F}_2(v)$ be the set of solutions that cover v .

Proposition 4 *Take 2-FF solution T corresponding to Well-Separated Robust 2-NUkC solution S as described earlier. We have:*

$$\sum_{v \in L_2 : S \in \mathcal{F}(v)} w(v) \leq w(\mathcal{C}(T)).$$

That is, the total w of the points covered by S is at most $w(\mathcal{C}(T))$.

Proof The leaves saved by T are saved either by T_1 or T_2 . Thus, we get

$$w(\mathcal{C}(T)) = \sum_{u \in T_1} \sum_{v \in \text{Leaf}(u)} w(v) + \sum_{u \notin T_1} \sum_{v \in \text{Leaf}(u) : v \in T_2} w(v). \quad (1)$$

The first of these terms can be lower-bounded as

$$\sum_{u \in T_1} \sum_{v \in \text{Leaf}(u)} w(v) \geq \sum_{u \in T_1} \sum_{v \in \text{Leaf}(u) : S \in \mathcal{F}(v)} w(v).$$

that is, we only consider the leaves v of $u \in T_1$ which are covered by the Robust 2-NUkC solution S . The second term, by definition of T_2 is

$$\sum_{u \notin T_1} \sum_{v \in \text{Leaf}(u) : v \in T_2} w(v) = \sum_{u \notin T_1} \sum_{v \in \text{Leaf}(u) : S \in \mathcal{F}_2(v)} w(v) = \sum_{u \notin T_1} \sum_{v \in \text{Leaf}(u) : S \in \mathcal{F}(v)} w(v).$$

where the last equality uses Proposition 3 which implies for $u \notin T_1$ and $v \in \text{Leaf}(u)$, $d(v, S_1) > r_1$. Thus, the solution S covers v iff S_2 covers v . Plugging back in (1), we complete the proof.

The proof of Lemma 4 now follows from the fact that T is not valuable thus $w(\mathcal{C}(T)) \leq m - 1$ and therefore, for any $S \in \mathcal{F}$ we have $\sum_{v \in L_2 : S \in \mathcal{F}(v)} w(v) \leq m - 1$. So we have:

$$\begin{aligned} \sum_{v \in L_2} w(v) \text{cov}(v) &=_{(\text{PI}_{\text{cov},1})} \sum_{v \in L_2} w(v) \sum_{S \in \mathcal{F}(v)} z_S = \sum_{S \in \mathcal{F}} z_S \sum_{\substack{v \in L_2 : \\ S \in \mathcal{F}(v)}} w(v) \\ &\leq (m - 1) \sum_{S \in \mathcal{F}} z_S =_{(\text{PI}_{\text{cov},2})} m - 1. \end{aligned}$$

□

4 The main algorithm: proof of theorem 1

As mentioned in Sect. 2, we focus on the feasibility version of the problem: given an instance \mathbb{I} of Robust 2-NUkC we either want to prove it is infeasible, that is, there are no subsets $S_1, S_2 \subseteq X$ with (a) $|S_i| \leq k_i$ and (b) $|\bigcup_i \bigcup_{u \in S_i} B(u, r_i)| \geq m$, or give a 10-approximation that is, open subsets S_1, S_2 that satisfy (a) and $|\bigcup_i \bigcup_{u \in S_i} B(u, 10r_i)| \geq m$. To this end, we apply the round-or-cut methodology on $P_{\text{cov}}^{\mathbb{I}}$. Given a purported $\hat{\text{cov}} := (\hat{\text{cov}}_1(v), \hat{\text{cov}}_2(v) : v \in X)$ we want to either use it to get a 10-approximate solution, or find a hyperplane separating it from $P_{\text{cov}}^{\mathbb{I}}$. Furthermore, we want the coefficients in the hyperplane to be poly-bounded. Using the ellipsoid method we indeed get a polynomial time algorithm thereby proving Theorem 1.

Upon receiving $\hat{\text{cov}}$, we first check whether $\hat{\text{cov}}(X) \geq m$ or not, and if not that will be the separating hyperplane. Henceforth, we assume this holds. Then, we run CGK Algorithm 2 with $\alpha_1 = 8$ and $\alpha_2 = 2$ to get 2-FF instance $\mathbb{T} = ((L_1, L_2, \text{Leaf}, w), (k_1, k_2))$. Let $\{y_v : v \in L_1 \cup L_2\}$ be the solution described in Proposition 1. Next, we check if $\hat{\text{cov}}_i(L_i) = y(L_i) \leq k_i$ for both $i \in \{1, 2\}$; if not, by Proposition 1 that hyperplane would separate $\hat{\text{cov}}$ from $P_{\text{cov}}^{\mathbb{I}}$ (and even Robust 2-NUkC LP in fact). The algorithm then branches into two cases.

Case 1 $y(L_1) \leq k_1 - 2$. In this case, we assert that \mathbb{T} is valuable, and therefore by Lemma 1 we get an $\alpha_1 + \alpha_2 = 10$ -approximate solution for \mathbb{I} via Lemma 1, and we are done.

Proposition 5 *If $y(L_1) \leq k_1 - 2$, then there is an integral solution T for \mathbb{T} with $w(\mathcal{C}(T)) \geq m$.*

Proof Since $y(L_1) \leq k_1 - 2$, we see that there is a feasible solution to the slightly revised LP below.

$$\begin{aligned} \max \sum_{v \in L_2} w(v)Y(v) : \quad & \sum_{u \in L_1} y_u \leq k_1 - 2, \quad \sum_{u \in L_2} y_u \leq k_2, \\ & Y(v) := y_{p(v)} + y_v \leq 1, \quad \forall v \in L_2 \end{aligned}$$

Consider a basic feasible solution $\{y'_v : v \in L_1 \cup L_2\}$ for this LP, and let $T_1 := \{v \in L_1 : y'_v > 0\}$. By definition $y'(T_1) = y'(L_1) \leq k_1 - 2$. According to Lemma 2, there are at most 2 loose variables in y' . So there are at most 2 fractional vertices in T_1 . This implies $|T_1| \leq k_1$. Let U be the set of leaves that are not saved by T_1 , that is, $U := \{v \in L_2 : p(v) \notin T_1\}$. Let T_2 be the top k_2 members of U according to decreasing w order. We return $T = (T_1, T_2)$.

We claim T has value at least m , that is, $w(\mathcal{C}(T)) \geq m$. Note that $w(\mathcal{C}(T)) = w(T_2) + \sum_{u \in T_1} w(\text{Leaf}(u))$. By the greedy choice of T_2 , $w(T_2) \geq \sum_{v \in U} w(v)y'_v$. Since $y'_{p(v)} = 0$ for any $v \in U$, we have $w(T_2) \geq \sum_{v \in U} w(v)y'_v = \sum_{v \in U} w(v)Y'(v)$.

Furthermore, by definition, $\sum_{u \in T_1} w(\text{Leaf}(u)) = \sum_{v \in L_2 \setminus U} w(v)$ which in turn is at least $\sum_{v \in L_2 \setminus U} w(v)Y'(v)$. Adding up proves the claim as the objective value is at

least m .

$$w(\mathcal{C}(T)) \geq \sum_{v \in U} w(v)Y'(v) + \sum_{v \in L_2 \setminus U} w(v)Y'_v = \sum_{v \in L_2} w(v)Y'(v) \geq m.$$

Case II $y(L_1) > k_1 - 2$. In this case, we either get an 8-approximation or prove that the following is a valid inequality which will serve as the separating hyperplane (Recall $\text{cov}_1(L_1) = y(L_1)$).

$$\text{cov}_1(L_1) \leq k_1 - 2. \quad (2)$$

To do so, we need the following proposition which formalizes the idea stated in Sect. 2.2 that in case II, we can enumerate over $O(|X|)$ many *well-separated instances*.

Proposition 6 Let $(\text{cov}_1, \text{cov}_2) \in \mathbb{P}_{\text{cov}}^{\mathbb{I}}$ be fractional coverages and suppose there is a subset $Y \subseteq X$ with $d(u, v) > 8r_1$ for all $u, v \in Y$. Then either $\text{cov}_1(Y) \leq k_1 - 2$, or at least one of the following Well-Separated Robust 2-NUkC instances are feasible

$$\mathbb{I}_{\emptyset} := ((X, d), (2r_1, r_2), (k_1, k_2), Y, m)$$

$$\mathbb{I}_q := ((X \setminus B(q, r_1), d), (2r_1, r_2), (k_1 - 1, k_2), Y, m - |B(q, r_1)|) \quad \forall q \in X : d(q, Y) > r_1.$$

Before proving the above proposition, let us use it to complete the proof of Theorem 1. We let $Y := L_1$, and obtain the instances \mathbb{I}_{\emptyset} and \mathbb{I}_q 's as mentioned in the proposition. We apply the algorithm in Theorem 2 on each of them. If any of them returns a solution, then we have an 8-approximation. More precisely, if \mathbb{I}_{\emptyset} is feasible, Theorem 2 gives a 4-approximation for it which is indeed an 8-approximation for \mathbb{I} (the extra factor 2 is because \mathbb{I}_{\emptyset} uses $2r_1$ as its largest radius). If \mathbb{I}_q is feasible for some $q \in X$ and Theorem 2 gives us a 4-approximate solution $S' = (S'_1, S'_2)$ for it and $S = (S'_1 \cup \{q\}, S'_2)$ is an 8-approximation for \mathbb{I} . If none of them are feasible, then we see that $\text{cov}_1(L_1) \leq k_1 - 2$ indeed serves as a separating hyperplane between cov and $\mathbb{P}_{\text{cov}}^{\mathbb{I}}$. This ends the proof of Theorem 1.

Proof (Proof of Proposition 6) Let us assume $\text{cov}_1(Y) > k_1 - 2$, and prove that one of the proposed Well-Separated Robust 2-NUkC instances are feasible. First of all, note that the described Well-Separated Robust 2-NUkC instances indeed satisfy the definition: Y is separated enough for radius $2r_1$ and by definition of q , $Y \subseteq (X \setminus B(q, r_1))$.

Suppose, for the sake of contradiction, none of the described Well-Separated Robust 2-NUkC instances are feasible. Since $\text{cov}_1(Y) > k_1 - 2$ and $\text{cov} \in \mathbb{P}_{\text{cov}}^{\mathbb{I}}$, there has to be some $S = (S_1, S_2) \in \mathcal{F}$ such that S_1 covers strictly more than $k_1 - 2$ points in Y . Take any such S . There are two types of centers in S_1 , the ones that do contribute to $\text{cov}_1(Y)$, and the ones that do not. The former is $A := \{f \in S_1 : d(f, Y) \leq r_1\}$ and the latter is $B := \{f \in S_1 : d(f, Y) > r_1\}$. By our assumption of $\text{cov}_1(Y) > k_1 - 2$ and the fact that Y points are more than $2r_1$ apart, $|A| > k_1 - 2$. This leaves us with $|B| \leq 1$.

Let \mathbb{I}_B be the Well-Separated Robust 2-NUkC instance corresponding to this B (i.e. $\mathbb{I}_B = \mathbb{I}_{\emptyset}$ if $B = \emptyset$ and $\mathbb{I}_B = \mathbb{I}_q$ if $B = \{q\}$). We construct a feasible solution S'

for \mathcal{I}_B which contradicts our assumption. By definition of \mathcal{I}_B , S' only needs to cover as many points as (A, S_2) covers in \mathcal{I} . That is, m points if $B = \emptyset$ and $m - |B(q, r_1)|$ points if $B = \{q\}$. Setting $S'_2 := S_2$ and $S'_1 := \{f \in Y : d(f, A) \leq 1\}$ does the trick: S' balls with radii $2r_1$ and r_2 , cover all the elements in X that are covered by (A, S_2) in \mathcal{I} . Also note that S' satisfies the budget constraints. \square

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Adjiashvili, D., Baggio, A., Zenklusen, R.: Firefighting on trees beyond integrality gaps. *ACM Trans. Algorithms (TALG)* **15**(2), 20 (2018)
2. An, H., Singh, M., Svensson, O.: LP-based algorithms for capacitated facility location. *SIAM J. Comput. (SICOMP)* **46**(1), 272–306 (2017)
3. Anegg, G., Angelidakis, H., Kurpisz, A., Zenklusen, R.: A technique for obtaining true approximations for k -center with covering constraints. In: *Proceedings, MPS Conference on Integer Programming and Combinatorial Optimization (IPCO)*, pp. 52–65 (2020)
4. Angelidakis, H., Makarychev, K., Makarychev, Y.: Algorithms for stable and perturbation-resilient problems. In: *Proceedings, ACM Symposium on Theory of Computing (STOC)*, pp. 438–451 (2017)
5. Awasthi, P., Blum, A., Sheffet, O.: Center-based clustering under perturbation stability. *Inform. Process. Lett.* **112**(1–2), 49–54 (2012)
6. Bandyapadhyay, S.: On Perturbation resilience of non-uniform k -center. In: *Proceedings, International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)* (2020)
7. Bandyapadhyay, S., Inamdar, T., Pai, S., Varadarajan, K.R.: A constant approximation for colorful k -center. In: *Proceedings, European Symposium on Algorithms (ESA)*, pp. 12:1–12:14 (2019)
8. Carr, R.D., Fleischer, L.K., Leung, V.J., Phillips, C.A.: Strengthening integrality gaps for capacitated network design and covering problems. In: *Proceedings, ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 106–115 (2000)
9. Chakrabarty, D., Chekuri, C., Khanna, S., Korula, N.: Approximability of capacitated network design. *Algorithmica* **72**(2), 493–514 (2015)
10. Chakrabarty, D., Goyal, P., Krishnaswamy, R.: The non-uniform k -center problem. *ACM Trans. Algorithms (TALG)* **16**(4), 1–19 (2020)
11. Chakrabarty, D., Negahbani, M.: Generalized center problems with outliers. *ACM Trans. Algorithms (TALG)* **15**(3), 1–14 (2019)
12. Chalermsook, P., Chuzhoy, J.: Resource minimization for fire containment. In: *Proceedings, ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 1334–1349 (2010)
13. Charikar, M., Khuller, S., Mount, D.M., Narasimhan, G.: Algorithms for facility location problems with outliers. In: *Proceedings, ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 642–651 (2001)
14. Chekuri, C.S., Gupta, S.: Perturbation resilient clustering for k -center and related problems via LP relaxations. In: *Proceedings, International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, pp. 9 (2018)
15. Finbow, S., King, A., MacGillivray, G., Rizzi, R.: The firefighter problem for graphs of maximum degree three. *Discret. Math.* **307**(16), 2094–2105 (2007)
16. Gonzalez, T.F.: Clustering to minimize the maximum intercluster distance. *Theor. Comput. Sci.* **38**, 293–306 (1985)
17. Hochbaum, D.S., Shmoys, D.B.: A best possible heuristic for the k -center problem. *Math. Oper. Res.* **10**, 180–184 (1985)

18. Hochbaum, D.S., Shmoys, D.B.: A unified approach to approximation algorithms for bottleneck problems. *J. ACM* **33**(3), 533–550 (1986)
19. Jia, X., Rohwedder, L., Sheth, K., Svensson, O.: Towards non-uniform k -center with constant types of radii. In: *Proceedings, ACM-SIAM Symposium on Simplicity in Algorithms (SOSA)* (2022)
20. Jia, X., Sheth, K., Svensson, O.: Fair colorful k -center clustering. In: *Proceedings, MPS Conference on Integer Programming and Combinatorial Optimization (IPCO)*, pp. 209–222 (2020)
21. Kaibel, V.: Extended formulations in combinatorial optimization. *Optima* **85**, 2–7 (2011)
22. Li, S.: On uniform capacitated k -median beyond the natural LP relaxation. In: *Proceedings, ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 696–707 (2015)
23. Li, S.: Approximating capacitated k -median with $(1 + \varepsilon)k$ open facilities. In: *Proceedings, ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 786–796 (2016)
24. Rahgoshay, M., Salavatipour, M.R.: Asymptotic quasi-polynomial time approximation scheme for resource minimization for fire containment. In: *Proceedings, International Symposium on Theoretical Aspects of Computer Science (STACS)* (2020)

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