



A smectic liquid crystal model in the periodic setting

Michael Novack ^{a,*}, Xiaodong Yan ^b^a Department of Mathematics, The University of Texas at Austin, Austin, TX, USA^b Department of Mathematics, The University of Connecticut, Storrs, CT, USA

ARTICLE INFO

Article history:

Received 3 May 2022

Accepted 22 November 2022

Communicated by Lia Bronsard

Keywords:

Liquid crystal

Smectics

Fourier

ABSTRACT

We consider the asymptotic behavior as ε goes to zero of the 2D smectics model in the periodic setting given by

$$\mathcal{E}_\varepsilon(w) = \frac{1}{2} \int_{\mathbb{T}^2} \frac{1}{\varepsilon} \left(|\partial_1|^{-1} \left(\partial_2 w - \partial_1 \frac{1}{2} w^2 \right) \right)^2 + \varepsilon (\partial_1 w)^2 \, dx.$$

We show that the energy $\mathcal{E}_\varepsilon(w)$ controls suitable L^p and Besov norms of w and use this to demonstrate the existence of minimizers for $\mathcal{E}_\varepsilon(w)$, which has not been proved for this smectics model before, and compactness in L^p for an energy-bounded sequence. We also prove an asymptotic lower bound for $\mathcal{E}_\varepsilon(w)$ as $\varepsilon \rightarrow 0$ by means of an entropy argument.

© 2022 Elsevier Ltd. All rights reserved.

1. Introduction

We consider the variational model

$$\mathcal{E}_\varepsilon(w) = \frac{1}{2} \int_{\mathbb{T}^2} \frac{1}{\varepsilon} \left(|\partial_1|^{-1} \left(\partial_2 w - \partial_1 \frac{1}{2} w^2 \right) \right)^2 + \varepsilon (\partial_1 w)^2 \, dx, \quad (1.1)$$

where $w : \mathbb{T}^2 \rightarrow \mathbb{R}$ is a periodic function with vanishing mean in x_1 , that is¹

$$\int_0^1 w(x_1, x_2) \, dx_1 = 0 \quad \text{for any } x_2 \in [0, 1]. \quad (1.2)$$

Here $|\partial_1|^{-1}$ is defined via its Fourier coefficients

$$\widehat{|\partial_1|^{-1} f}(k) = |k_1|^{-1} \widehat{f}(k) \quad \text{for } k \in (2\pi\mathbb{Z})^2,$$

and is well defined when (1.2) holds.

* Corresponding author.

E-mail addresses: michael.novack@austin.utexas.edu (M. Novack), xiaodong.yan@uconn.edu (X. Yan).

¹ More generally, a periodic distribution f on \mathbb{T}^2 has “vanishing mean in x_1 ” if for all $(k_1, k_2) = k \in (2\pi\mathbb{Z})^2$ with $k_1 = 0$, $\widehat{f}(k) = 0$. If f corresponds to an L^p function, $p \in [1, \infty)$, this is equivalent to the existence of a sequence $\{\varphi_k\}$ of smooth, periodic functions satisfying (1.2) that converges in L^p to f .

This model is motivated by a nonlinear approximate model of smectic liquid crystals. The following functional has been proposed as an approximate model for smectic liquid crystals [4,14,24,29,30] in two space dimensions:

$$E_\varepsilon(u) = \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon} \left(\partial_2 u - \frac{1}{2} (\partial_1 u)^2 \right)^2 + \varepsilon (\partial_1 u)^2 dx, \quad (1.3)$$

where u is the Eulerian deviation from the ground state $\Phi(x) = x_2$ and ε is the characteristic length scale. The first term represents the compression energy and the second term represents the bending energy. For further background on the model, we refer to [24,25] and the references contained therein. The 3D version of (1.3), which we analyzed in [25] but do not consider further here, is also used for example in the mathematical description of nuclear pasta in neutron stars [6]. Assuming that u is periodic on the torus $\mathbb{T}^2 = \Omega$ and setting $w = \partial_1 u$, (1.3) becomes

$$E_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{T}^2} \frac{1}{\varepsilon} \left(|\partial_1|^{-1} \left(\partial_2 w - \partial_1 \frac{1}{2} w^2 \right) \right)^2 + \varepsilon (\partial_1 w)^2 dx.$$

Finally, a similar model to (1.1) with $|\partial_1|^{-1/2}$ replacing $|\partial_1|^{-1}$ has been derived in the context of micromagnetics [12]; see also [31].

The asymptotic behavior of (1.3) as ε goes to zero was studied in [24]. Given $\varepsilon_n \rightarrow 0$ and a sequence $\{u_n\}$ with bounded energies $E_{\varepsilon_n}(u_n)$, the authors proved pre-compactness of $\{\partial_1 u_n\}$ in L^q for any $1 \leq q < p$ and pre-compactness of $\{\partial_2 u_n\}$ in L^2 under the additional assumption $\|\partial_1 u_n\|_{L^p} \leq C$ for some $p > 6$. The compactness proof in [24] uses a compensated compactness argument based on entropies, following the work of Tartar [32–34] and Murat [21–23]. In addition, a lower bound on E_ε and a matching upper bound corresponding to a 1D ansatz was obtained as $\varepsilon \rightarrow 0$ under the assumption that the limiting function u satisfies $\nabla u \in (L^\infty \cap BV)(\Omega)$.

In this paper, we approach the compactness via a different argument in the periodic setting. Our proof is motivated by recent work on related variational models in the periodic setting [5,10,13,26,27,36] where strong convergence of a weakly convergent L^2 sequence is proved via estimates on Fourier series. Given a sequence u_ε weakly converging in $L^2(\mathbb{T}^2)$, to prove strong convergence of u_ε in L^2 , it is sufficient to show that there is no concentration in the high frequencies. The center piece of this approach relies on the estimates for solutions to Burgers equation

$$-\partial_1 \frac{1}{2} w^2 + \partial_2 w = \eta$$

in suitable Besov spaces. This type of compactness argument also applies to a sequence $\{w_n\}$ with $\mathcal{E}_\varepsilon(w_n) \leq C$ for any fixed ε . As a direct corollary, we obtain the existence of minimizers of E_ε in $W^{1,2}(\mathbb{T}^2)$ (see Corollary 2.12) for any fixed ε . We observe that to the best of our knowledge, the existence of minimizers of E_ε in any setting was not known due to the lack of compactness for sequence $\{u_n\}$ satisfying $E_\varepsilon(u_n) \leq C$ with fixed ε .

To further understand the minimization of \mathcal{E}_ε , we are also interested in a sharp lower bound for the asymptotic limit of \mathcal{E}_ε as ε approaches zero. In the literature for such problems (see for example [1,3,11,16]), one useful technique in achieving such a bound is an “entropy” argument, in which the entropy production $\int \operatorname{div} \Sigma(w)$ of a vector field $\Sigma(w)$ is used to bound the energy \mathcal{E}_ε from below. For the 2D Aviles–Giga functional

$$\frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon} (|\nabla u|^2 - 1)^2 + \varepsilon |\nabla^2 u|^2 dx, \quad (1.4)$$

such vector fields were introduced in [8,16]. In [24,25], the analogue for the smectic energy, in 2D and 3D respectively, of the Jin–Kohn entropies from [16] were used to prove a sharp lower bound which can be matched by a construction similar to [7,28]. In this paper, we use the vector field

$$\Sigma(w) = \left(-\frac{1}{3} w^3, \frac{1}{2} w^2 \right) \quad (1.5)$$

which is $(-(\partial_1 u)^3/3, (\partial_1 u)^2/2)$ in terms of u , to prove a sharp lower bound. As $\varepsilon \rightarrow 0$, entropy production concentrates along curves and approximates the total variation of the distributional divergence of a BV vector field. An interesting open direction which motivates studying (1.5) is utilizing the correct version of (1.5) (or the entropies from [8,9]) in 3D, for example in a compactness argument.

The paper is organized as follows. The pre-compactness of a sequence of functions with bounded energy is proved in Section 2, for both fixed ε and $\varepsilon \rightarrow 0$. The lower bound is established in Section 3.

2. Compactness of a sequence with bounded energy

2.1. Preliminaries

Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ be unit vectors in \mathbb{R}^2 . We recall some definitions from [13]. For $f : \mathbb{T}^2 \rightarrow \mathbb{R}$, we write

$$\partial_j^h f(x) = f(x + h\mathbf{e}_j) - f(x) \quad x \in \mathbb{T}^2, h \in \mathbb{R}.$$

Definition 2.1. Given $f : \mathbb{T}^2 \rightarrow \mathbb{R}$, $j \in \{1, 2\}$, $s \in (0, 1]$, and $p \in [1, \infty)$, the directional Besov seminorm is defined as

$$\|f\|_{\mathcal{B}_{p;j}^{s,p}} = \sup_{h \in (0,1]} \frac{1}{h^s} \left(\int_{\mathbb{T}^2} |\partial_j^h f|^p dx \right)^{\frac{1}{p}}$$

Remark 2.2. This is the $\mathcal{B}^{s;p,\infty}$ seminorm defined in each direction separately.

Remark 2.3. For $p = 2$ and $s \in (0, 1)$, given $s' \in (s, 1)$, the following inequality holds ([13, Equation (2.2)]):

$$\int_{\mathbb{T}^2} |\partial_j^s f|^2 = \sum |k_j|^{2s} |\widehat{f}(k)|^2 = c_s \int_{\mathbb{R}} \frac{1}{|h|^{2s}} \int_{\mathbb{T}^2} |\partial_j^h f|^2 dx \frac{dh}{|h|} \leq C(s, s') \|f\|_{\mathcal{B}_{2;j}^{s,p}}^2.$$

We quote two results from [13].

Lemma 2.4 ([13, Proposition B.9]). *For every $p \in (1, \infty]$ and $q \in [1, p]$ with $(p, q) \neq (\infty, 1)$, there exists a constant $C(p, q) > 0$ such that for every periodic function $f : [0, 1] \rightarrow \mathbb{R}$ with vanishing mean,*

$$\left(\int_0^1 |f(z)|^p dz \right)^{\frac{1}{p}} \leq C(p, q) \int_0^1 \frac{1}{h^{\frac{1}{q} - \frac{1}{p}}} \left(\int_0^1 |\partial_1^h f(z)|^q dz \right)^{\frac{1}{q}} \frac{dh}{h}, \quad (2.1)$$

with the usual interpretation for $p = \infty$ or $q = \infty$.

The following estimate was derived in the proof of Lemma B.10 in [13].

Lemma 2.5 ([13, In the Proof of Lemma B.10]). *For every $p \in [1, \infty)$ and every periodic function $f : [0, 1] \rightarrow \mathbb{R}$, $h \in (0, 1]$, the following estimate holds.*

$$\left(\int_0^1 |\partial_1^h f(z)|^p dz \right)^{\frac{1}{p}} \leq 2 \left(\frac{1}{h} \int_0^h \int_0^1 |\partial_1^{h'} f(z)|^p dz dh' \right)^{\frac{1}{p}}. \quad (2.2)$$

We define $\eta_w = \partial_2 w - \partial_1 \frac{1}{2} w^2$, and thus (1.1) can be written as

$$\mathcal{E}_\varepsilon(w) = \frac{1}{2} \int_{\mathbb{T}^2} \frac{1}{\varepsilon} (|\partial_1|^{-1} \eta_w)^2 + \varepsilon (\partial_1 w)^2 dx. \quad (2.3)$$

Finally, we introduce the ε -independent energy

$$\mathcal{E}(w) = \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1} \eta_w \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} (\partial_1 w)^2 dx \right)^{\frac{1}{2}}, \quad (2.4)$$

and note that

$$\mathcal{E}(w) \leq \mathcal{E}_\varepsilon(w) \quad \text{for all } \varepsilon > 0. \quad (2.5)$$

2.2. Besov and L^p estimates

We obtain the following estimates. The proofs follow closely those in [13, Propositions 2.3-2.4].

Lemma 2.6. *There exists a universal constant $C_1 > 0$ such that if $w \in L^2(\mathbb{T}^2)$ and has vanishing mean in x_1 and $h \in (0, 1]$, then*

$$\int_{\mathbb{T}^2} |\partial_1^h w|^3 dx \leq C_1 h \mathcal{E}(w) \quad (2.6)$$

and

$$\sup_{x_2 \in [0, 1)} \int_0^h \int_0^1 \left| \partial_1^{h'} w(x_1, x_2) \right|^2 dx_1 dh' \leq C_1 \left(h \mathcal{E}(w) + h^{\frac{5}{3}} \mathcal{E}^{\frac{2}{3}}(w) \right). \quad (2.7)$$

Proof. Throughout the proof, we assume that w is smooth; once the estimates hold for smooth w , they hold in generality by approximation. The constant C_1 may change from line to line. Following [13, Equations (2.5)–(2.6)], we apply the modified Howarth–Kármán–Monin identities for the Burgers operator. For every $h' \in (0, 1]$, we have

$$\partial_2 \frac{1}{2} \int_0^1 \left| \partial_1^{h'} w \right| \partial_1^{h'} w dx_1 - \frac{1}{6} \partial_{h'} \int_0^1 \left| \partial_1^{h'} w \right|^3 dx_1 = \int_0^1 \partial_1^{h'} \eta_w \left| \partial_1^{h'} w \right| dx_1, \quad (2.8)$$

$$\partial_2 \frac{1}{2} \int_0^1 \left(\partial_1^{h'} w \right)^2 dx_1 - \frac{1}{6} \partial_{h'} \int_0^1 \left(\partial_1^{h'} w \right)^3 dx_1 = \int_0^1 \partial_1^{h'} \eta_w \partial_1^{h'} w dx_1. \quad (2.9)$$

Integrating (2.8) over x_2 and using the periodicity of w yields

$$\begin{aligned} \partial_{h'} \int_{\mathbb{T}^2} \left| \partial_1^{h'} w \right|^3 dx &= -6 \int_{\mathbb{T}^2} \partial_1^{h'} \eta_w \left| \partial_1^{h'} w \right| dx \\ &= -6 \int_{\mathbb{T}^2} \eta_w \partial_1^{-h'} \left| \partial_1^{h'} w \right| dx. \end{aligned} \quad (2.10)$$

Now

$$\begin{aligned} \left| \int_{\mathbb{T}^2} \eta_w \partial_1^{-h'} \left| \partial_1^{h'} w \right| dx \right| &\leq \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1} \eta_w \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} \left(\partial_1 \partial_1^{-h'} \left| \partial_1^{h'} w \right| \right)^2 dx \right)^{\frac{1}{2}} \\ &\leq C_1 \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1} \eta_w \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} (\partial_1 w)^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

so that integrating (2.10) from 0 to h and using $\partial_1^0 w = 0$, we have

$$\int_{\mathbb{T}^2} |\partial_1^h w|^3 dx \leq C_1 \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1} \eta_w \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} (\partial_1 w)^2 dx \right)^{\frac{1}{2}} h \leq C_1 h \mathcal{E}(w).$$

To prove (2.7), we integrate (2.9) from 0 to h and again utilize $\partial_1^0 w = 0$ to obtain

$$\partial_2 \frac{1}{2} \int_0^h \int_0^1 \left(\partial_1^{h'} w \right)^2 dx_1 dh' - \frac{1}{6} \int_0^h \int_0^1 \left(\partial_1^{h'} w \right)^3 dx_1 dh' = \int_0^h \int_0^1 \partial_1^{h'} \eta_w \partial_1^{h'} w dx_1 dh'. \quad (2.11)$$

We set

$$f(x_2) = \int_0^h \int_0^1 \left(\partial_1^{h'} w \right)^2 dx_1 dh',$$

and recall the Sobolev embedding inequality for $W^{1,1}(\mathbb{T}) \subset L^\infty(\mathbb{T})$:

$$\sup_{z \in \mathbb{T}} |f(z)| \leq \int_{\mathbb{T}} |f(y)| dy + \int_{\mathbb{T}} |f'(y)| dy.$$

Then applying this to $f(x_2)$ and referring to (2.11), we have

$$\begin{aligned} & \sup_{x_2 \in [0,1]} \int_0^h \int_0^1 \left(\partial_1^{h'} w \right)^2 dx_1 dh' \\ & \leq \int_0^h \int_{\mathbb{T}^2} \left(\partial_1^{h'} w \right)^2 dx dh' \\ & \quad + \frac{1}{3} \int_{\mathbb{T}^2} \left| \partial_1^h w \right|^3 dx + 2 \int_0^h \int_0^1 \left| \int_0^1 \eta_w \partial_1^{-h'} \left| \partial_1^{h'} w \right| dx_1 \right| x_2 dh'. \end{aligned} \quad (2.12)$$

Since

$$\int_{\mathbb{T}^2} \left(\partial_1^{h'} w \right)^2 dx \leq \left(\int_{\mathbb{T}^2} \left| \partial_1^{h'} w \right|^3 dx \right)^{\frac{2}{3}} \leq C_1 (h' \mathcal{E}(w))^{\frac{2}{3}},$$

and

$$\begin{aligned} & \int_0^1 \left| \int_0^1 \eta_w \partial_1^{-h'} \left| \partial_1^{h'} w \right| dx_1 \right| x_2 \\ & \leq \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1} \eta_w \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} \left(\partial_1 \partial_1^{-h'} \left| \partial_1^{h'} w \right| \right)^2 dx \right)^{\frac{1}{2}} \\ & \leq C_1 \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1} \eta_w \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} (\partial_1 w)^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

(2.12) therefore implies

$$\sup_{x_2 \in [0,1]} \int_0^h \int_0^1 \left(\partial_1^{h'} w \right)^2 dx_1 dh' \leq C_1 \left(h^{\frac{5}{3}} \mathcal{E}^{\frac{2}{3}}(w) + h \mathcal{E}(w) \right),$$

which is (2.7). \square

Lemma 2.7. *If $w \in L^2(\mathbb{T}^2)$ and has vanishing mean in x_1 , then the following estimates hold:*

$$\|w\|_{\mathcal{B}_{3;1}^s} \leq C_1 \mathcal{E}^{\frac{1}{3}}(w), \quad \text{for every } s \in \left(0, \frac{1}{3} \right], \quad (2.13)$$

where C_1 is as in Lemma 2.6;

$$\|w\|_{L^p(\mathbb{T}^2)} \leq C_2(p) \mathcal{E}^{\frac{2}{3\alpha}}(w) (\mathcal{E}(w) + \mathcal{E}^{\frac{2}{3}}(w))^{\frac{\alpha-2}{2\alpha}}, \quad (2.14)$$

for every $1 \leq p < \frac{10}{3}$, where $\alpha = \max\{2, p\}$; and

$$\|w\|_{L^p(\mathbb{T}^2)} \leq C_2(p) \varepsilon^{-\frac{1}{\alpha}} \mathcal{E}_\varepsilon^{\frac{1}{\alpha}}(w) (\mathcal{E}_\varepsilon(w) + \mathcal{E}_\varepsilon^{\frac{2}{3}}(w))^{\frac{\alpha-2}{2\alpha}} \quad (2.15)$$

for every $\varepsilon > 0$ and $1 \leq p < 6$, where again $\alpha = \max\{2, p\}$.

Proof. The estimate (2.13) follows from (2.6) and the definition of $\|\cdot\|_{\mathcal{B}_{3;1}^s}$. Turning to (2.14)–(2.15), we first prove a preliminary estimate. We fix $x_2 \in [0, 1]$ and apply Lemma 2.4 to $f(z) = w(z, x_2)$ with $q = 2$, $p > 2$ to deduce

$$\left(\int_0^1 |w(x_1, x_2)|^p dx_1 \right)^{\frac{1}{p}} \leq C_2(p) \int_0^1 \frac{1}{h^{\frac{1}{2} - \frac{1}{p}}} \left(\int_0^1 |\partial_1^h w(x_1, x_2)|^2 dx_1 \right)^{\frac{1}{2}} \frac{dh}{h}.$$

Integrating over x_2 , we thus have by Minkowski's integral inequality

$$\begin{aligned}
\|w\|_{L^p(\mathbb{T}^2)} &= \left(\int_0^1 \int_0^1 |w(x_1, x_2)|^p dx_1 dx_2 \right)^{\frac{1}{p}} \\
&\leq C_2(p) \left(\int_0^1 \left[\int_0^1 h^{\frac{1}{p}-\frac{3}{2}} \left(\int_0^1 |\partial_1^h w(x_1, x_2)|^2 dx_1 \right)^{\frac{1}{2}} dh \right]^p dx_2 \right)^{\frac{1}{p}} \\
&\leq C_2(p) \int_0^1 h^{\frac{1}{p}-\frac{3}{2}} \left[\int_0^1 \left(\int_0^1 |\partial_1^h w(x_1, x_2)|^2 dx_1 \right)^{\frac{p}{2}} dx_2 \right]^{\frac{1}{p}} dh \\
&\leq C_2(p) \int_0^1 h^{\frac{1}{p}-\frac{3}{2}} \sup_{x_2 \in [0,1)} \left(\int_0^1 |\partial_1^h w(x_1, x_2)|^2 dx_1 \right)^{\frac{p-2}{2p}} \cdot \left(\int_{\mathbb{T}^2} |\partial_1^h w(x)|^2 dx \right)^{\frac{1}{p}} dh.
\end{aligned}$$

The first term in the integrand can be estimated using (2.2) and (2.7), which gives

$$\begin{aligned}
\sup_{x_2 \in [0,1)} \left(\int_0^1 |\partial_1^h w(x_1, x_2)|^2 dx_1 \right)^{\frac{p-2}{2p}} &\leq \sup_{x_2 \in [0,1)} \left(\frac{4}{h} \int_0^h \int_0^1 |\partial_1^{h'} w(x_1, x_2)|^2 dx_1 dh' \right)^{\frac{p-2}{2p}} \\
&\leq C_1 \left(\mathcal{E}(w) + h^{\frac{2}{3}} \mathcal{E}^{\frac{2}{3}}(w) \right)^{\frac{p-2}{2p}},
\end{aligned}$$

and therefore

$$\|w\|_{L^p(\mathbb{T}^2)} \leq C_2(p) \left(\mathcal{E}(w) + \mathcal{E}^{\frac{2}{3}}(w) \right)^{\frac{p-2}{2p}} \int_0^1 h^{\frac{1}{p}-\frac{3}{2}} \left(\int_{\mathbb{T}^2} |\partial_1^h w(x)|^2 dx \right)^{\frac{1}{p}} dh. \quad (2.16)$$

To prove (2.14) and (2.15) we estimate the h -integrand in two different fashions before integrating. For (2.14), using Hölder's inequality and (2.6), we have the upper bound

$$\left(\int_{\mathbb{T}^2} |\partial_1^h w(x)|^2 dx \right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{T}^2} |\partial_1^h w(x)|^3 dx \right)^{\frac{2}{3p}} \leq C_1 h^{\frac{2}{3p}} \mathcal{E}^{\frac{2}{3p}}(w).$$

Inserting this into (2.16) and using $p \in (2, 10/3)$ yields

$$\begin{aligned}
\|w\|_{L^p(\mathbb{T}^2)} &\leq C_2(p) \mathcal{E}^{\frac{2}{3p}}(w) \left(\mathcal{E}(w) + \mathcal{E}^{\frac{2}{3}}(w) \right)^{\frac{p-2}{2p}} \int_0^1 h^{\frac{5}{3p}-\frac{3}{2}} dh \\
&= C_2(p) \mathcal{E}^{\frac{2}{3p}}(w) \left(\mathcal{E}(w) + \mathcal{E}^{\frac{2}{3}}(w) \right)^{\frac{p-2}{2p}},
\end{aligned}$$

which is (2.14) when $p > 2$. For $p \leq 2$, we apply (2.14) with $p' > 2$, use the fact that $\|w\|_{L^p} \leq \|w\|_{L^{p'}}$, and let $p' \searrow 2$. Now for (2.15), we instead use the fundamental theorem of calculus and Jensen's inequality to estimate

$$\begin{aligned}
\left(\int_{\mathbb{T}^2} |\partial_1^h w(x)|^2 dx \right)^{\frac{1}{p}} &\leq \left(h^2 \int_{\mathbb{T}^2} (\partial_1 w(x))^2 dx \right)^{\frac{1}{p}} \\
&\leq h^{\frac{2}{p}} \varepsilon^{-\frac{1}{p}} \mathcal{E}_{\varepsilon}^{\frac{1}{p}}(w).
\end{aligned}$$

When plugged into (2.16) and combined with (2.5), this implies

$$\begin{aligned}
\|w\|_{L^p(\mathbb{T}^2)} &\leq C_2(p) \varepsilon^{-\frac{1}{p}} \mathcal{E}_{\varepsilon}^{\frac{1}{p}}(w) \left(\mathcal{E}_{\varepsilon}(w) + \mathcal{E}_{\varepsilon}^{\frac{2}{3}}(w) \right)^{\frac{p-2}{2p}} \int_0^1 h^{\frac{3}{p}-\frac{3}{2}} dh \\
&= C_2(p) \varepsilon^{-\frac{1}{p}} \mathcal{E}_{\varepsilon}^{\frac{1}{p}}(w) \left(\mathcal{E}_{\varepsilon}(w) + \mathcal{E}_{\varepsilon}^{\frac{2}{3}}(w) \right)^{\frac{p-2}{2p}}
\end{aligned}$$

for $p \in (2, 6)$. The case $p \in [1, 2)$ is handled similarly as in (2.14). \square

Remark 2.8. Generalizing the previous argument to the 3D smectics model from [25] is open. An intermediate step would be analyzing the Aviles–Giga model (which is a special case of the energy in [25]) on \mathbb{T}^2 using these type of ideas.

2.3. Compactness and existence

We prove compactness and existence theorems in this section. First we define the admissible sets

$$\mathcal{A}_\varepsilon = \left\{ w \in L^2(\mathbb{T}^2) : \int_0^1 w(x_1, x_2) dx_1 = 0 \text{ for each } x_2 \in [0, 1] \text{ and } \mathcal{E}_\varepsilon(w) < \infty \right\}$$

and

$$\mathcal{A} = \left\{ w \in L^2(\mathbb{T}^2) : \int_0^1 w(x_1, x_2) dx_1 = 0 \text{ for each } x_2 \in [0, 1] \text{ and } \mathcal{E}(w) < \infty \right\}.$$

Note that for any positive $\varepsilon > 0$, (2.5) implies that $\mathcal{A}_\varepsilon \subset \mathcal{A}$. We prove the following compactness result.

Proposition 2.9. *If $\{w_n\} \subset \mathcal{A}$ satisfy $\mathcal{E}_{\varepsilon_n}(w_n) \leq C_3 < \infty$ and $\sup_n |\varepsilon_n| \leq \varepsilon_0$, then $\{w_n\}$ is precompact in $L^2(\mathbb{T}^2)$.*

Proof. By (2.14),

$$\|w_n\|_{L^2(\mathbb{T}^2)} \leq C_2(p) \mathcal{E}^{\frac{2}{3\alpha}}(w) (\mathcal{E}(w) + \mathcal{E}^{\frac{2}{3}}(w))^{\frac{\alpha-2}{2\alpha}},$$

and thus, by (2.5) (that is, $\mathcal{E}(w) \leq \mathcal{E}_\varepsilon(w)$), $\|w_n\|_{L^2(\mathbb{T}^2)} \leq C_4$ depending on p and C_3 . As a consequence, we can find $w_0 \in L^2(\mathbb{T}^2)$ such that up to a subsequence, $w_n \rightharpoonup w_0$ weakly in $L^2(\mathbb{T}^2)$. Therefore, for each $k \in (2\pi\mathbb{Z})^2$,

$$\widehat{w_n}(k) \rightarrow \widehat{w_0}(k), |\widehat{w_n}(k)| \leq \left(\int_{\mathbb{T}^2} w_n^2 \right)^{\frac{1}{2}} \leq C_4, \text{ and } |\widehat{w_n^2}(k)| \leq \int_{\mathbb{T}^2} w_n^2 \leq C_4^2. \quad (2.17)$$

We therefore know that for any fixed $N \in \mathbb{N}$,

$$\sum_{\substack{|k_1| \leq 2\pi N, \\ |k_2| \leq 2\pi N}} |\widehat{w_n}(k) - \widehat{w_0}(k)|^2 \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and so the strong convergence of $w_n \rightarrow w_0$ would follow if

$$\sum_{\substack{|k_1| > 2\pi N \\ \text{or} \\ |k_2| > 2\pi N}} |\widehat{w_n}(k)|^2 \rightarrow 0 \text{ uniformly in } n \text{ as } N \rightarrow \infty. \quad (2.18)$$

The rest of the proof is dedicated to showing (2.18).

We fix $0 < s < 1/3$ and appeal to Remark 2.3 and (2.13) to calculate

$$\begin{aligned} \int_{\mathbb{T}^2} |\partial_1|^s w_n|^2 &= \sum |k_1|^{2s} |\widehat{w_n}(k)|^2 \leq C(s, 1/3) \|w_n\|_{\mathcal{B}_{2;1}^{1/3}}^2 \\ &\leq C(s, 1/3) \|w_n\|_{\mathcal{B}_{3;1}^{1/3}}^2 \leq C(s, 1/3) C_1 \mathcal{E}^{\frac{2}{3}}(w_n) \leq C_5, \end{aligned} \quad (2.19)$$

for suitable C_5 . We recall the formula

$$\eta_w = \partial_2 w - \partial_1 \frac{1}{2} w^2,$$

which, in terms of Fourier coefficients, reads

$$\widehat{\eta_w}(k) = -ik_2 \widehat{w}(k) + \frac{1}{2} ik_1 \widehat{w^2}(k).$$

For $M_1, M_2 \in \mathbb{N}$ to be chosen momentarily, we combine this with (2.17) and then (2.19) to find

$$\begin{aligned}
& \sum_{\substack{|k_1| > 2\pi M_1 \\ \text{or} \\ |k_2| > 2\pi M_2}} |\widehat{w_n}(k)|^2 \\
& \leq \sum_{|k_1| > 2\pi M_1} |\widehat{w_n}(k)|^2 + \sum_{\substack{|k_1| \leq 2\pi M_1 \\ |k_2| > 2\pi M_2}} |\widehat{w_n}(k)|^2 \\
& \leq (2\pi M_1)^{-2s} \sum_{|k_1| > 2\pi M_1} |k_1|^{2s} |\widehat{w_n}(k)|^2 + 2 \sum_{\substack{|k_1| \leq 2\pi M_1 \\ |k_2| > 2\pi M_2}} \frac{1}{|k_2|^2} |\widehat{\eta_{w_n}}(k)|^2 \\
& \quad + \frac{1}{2} \sum_{\substack{|k_1| \leq 2\pi M_1 \\ |k_2| > 2\pi M_2}} \frac{|k_1|^2}{|k_2|^2} |\widehat{w_n^2}(k)| \\
& \leq (2\pi M_1)^{-2s} \sum_{|k_1| > 2\pi M_1} |k_1|^{2s} |\widehat{w_n}(k)|^2 + \frac{2M_1^2}{M_2^2} \sum_{\substack{|k_1| \leq 2\pi M_1 \\ |k_2| > 2\pi M_2}} \frac{1}{|k_1|^2} |\widehat{\eta_{w_n}}(k)|^2 + \frac{C_4^2}{2} \sum_{\substack{|k_1| \leq 2\pi M_1 \\ |k_2| > 2\pi M_2}} \frac{|k_1|^2}{|k_2|^2} \\
& \leq (2\pi M_1)^{-2s} C_5 + \frac{2M_1^2}{M_2^2} \times \varepsilon_0 \mathcal{E}_{\varepsilon_n}(w_n) + \frac{C_4^2}{2} \times 2(2\pi M_1)^3 \times \frac{1}{\pi M_2}.
\end{aligned}$$

Taking $M_1 = M \in \mathbb{N}$ and $M_2 = M^4$, we find that

$$\sum_{\substack{|k_1| > 2\pi M \\ \text{or} \\ |k_2| > 2\pi M^4}} |\widehat{w_n}(k)|^2 \rightarrow 0 \text{ uniformly in } n \text{ as } M \rightarrow \infty,$$

which concludes the proof of (2.18). \square

Corollary 2.10. *If $\{w_n\} \subset \mathcal{A}$ satisfy $\mathcal{E}_{\varepsilon_n}(w_n) \leq C < \infty$ and $\sup_n |\varepsilon_n| \leq \varepsilon_0$, then $\{w_n\}$ is precompact in $L^p(\mathbb{T}^2)$ for any $p \in [1, \frac{10}{3})$.*

Proof. The conclusion follows from the precompactness of $\{w_n\}$ in $L^2(\mathbb{T}^2)$, the bound (2.14) from Lemma 2.7, and interpolation. \square

Corollary 2.11. *If $\{w_n\} \subset \mathcal{A}$ satisfy $\mathcal{E}_\varepsilon(w_n) \leq C < \infty$ for a fixed ε , then $\{w_n\}$ is precompact in $L^p(\mathbb{T}^2)$ for any $p \in [1, 6)$.*

Proof. We again appeal to the precompactness of w_n in $L^2(\mathbb{T}^2)$ (taking $\varepsilon_n = \varepsilon$ in Proposition 2.9), but instead use the bound (2.15) from Lemma 2.7 before interpolating. \square

As a direct application of Corollary 2.11, we can prove an existence theorem for the original smectic energy E_ε defined in (1.3). For any periodic $g : \mathbb{T}^1 \rightarrow \mathbb{R}$, we define

$$\widetilde{\mathcal{A}}_{\varepsilon, g} = \left\{ u \in W^{1,2}(\mathbb{T}^2) : E_\varepsilon(u) < \infty, \int_0^1 u(x_1, x_2) dx_1 = g(x_2) \text{ for a.e. } x_2 \in [0, 1] \right\}.$$

We note that $\widetilde{\mathcal{A}}_{\varepsilon, g}$ is non-empty for example when g is smooth.

Corollary 2.12. *For fixed $\varepsilon > 0$, if $\widetilde{\mathcal{A}}_{\varepsilon, g}$ is non-empty, then there exists $u_\varepsilon \in \widetilde{\mathcal{A}}_{\varepsilon, g}$ such that $E_\varepsilon(u_\varepsilon) = \inf_{u \in \widetilde{\mathcal{A}}_{\varepsilon, g}} E_\varepsilon(u)$.*

Proof. Since admissible class is non-empty, we can let u_n be a minimizing sequence for

$$E_\varepsilon(u) = \frac{1}{2} \int_{\Omega} \frac{1}{\varepsilon} \left(\partial_2 u - \frac{1}{2} (\partial_1 u)^2 \right)^2 + \varepsilon (\partial_{11} u)^2 dx;$$

in particular, the energies are uniformly bounded. By [Corollary 2.11](#), we have, up to a subsequence that we do not relabel,

$$\partial_1 u_n \rightarrow \partial_1 u_0 \quad \text{in } L^4(\mathbb{T}^2) \quad (2.20)$$

for some u_0 . Since u_n is a minimizing sequence, the first term in E_ε combined with the L^4 -convergence of $\partial_1 u_n$ implies that $\{\partial_2 u_n\}$ are uniformly bounded in $L^2(\mathbb{T}^2)$. Thus, up to a further subsequence which we do not note, there exists $v_0 \in L^2$ such that $\partial_2 u_n \rightharpoonup v_0$ weakly in $L^2(\mathbb{T}^2)$. Furthermore, by the uniqueness of weak limits, it must be that $v_0 = \partial_2 u_0$, so $u_0 \in W^{1,2}(\mathbb{T}^2)$. Expanding

$$\int_{\Omega} \left(\partial_2 u_n - \frac{(\partial_1 u_n)^2}{2} \right)^2 dx = \int_{\Omega} \left[(\partial_2 u_n)^2 - (\partial_1 u_n)^2 \partial_2 u_n + \frac{1}{4} (\partial_1 u_n)^4 \right] dx,$$

we see that by [\(2.20\)](#), the lower semicontinuity of the L^2 -norm under weak convergence, and the fact that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}^2} (\partial_1 u_n)^2 \partial_2 u_n dx = \int_{\mathbb{T}^2} (\partial_1 u_0)^2 \partial_2 u_0 dx,$$

we have

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{T}^2} \left(\partial_2 u_n - \frac{(\partial_1 u_n)^2}{2} \right)^2 dx \geq \int_{\mathbb{T}^2} \left(\partial_2 u - \frac{(\partial_1 u)^2}{2} \right)^2 dx. \quad (2.21)$$

Also, the uniform L^2 -bound on $\partial_{11} u$ and the uniqueness of limits implies that, up to a subsequence, $\partial_{11} u_n \rightharpoonup \partial_{11} u_0$ weakly in $L^2(\mathbb{T}^2)$, and thus

$$\liminf_{n \rightarrow \infty} \int_{\Omega} (\partial_{11} u_n)^2 dx \geq \int_{\mathbb{T}^2} (\partial_{11} u_0)^2 dx. \quad (2.22)$$

Putting together [\(2.21\)](#)–[\(2.22\)](#), we conclude

$$\inf_{\mathcal{A}_{\varepsilon,g}} E_\varepsilon = \liminf_{n \rightarrow \infty} E_\varepsilon(u_n) \geq E_\varepsilon(u_0).$$

Finally, by Poincaré's inequality and the weak convergence of ∇u_n to ∇u_0 in $L^2(\mathbb{T}^2)$, we conclude that u_n converges to u_0 strongly in $L^2(\mathbb{T}^2)$. Hence

$$\int_0^1 u(x_1, x_2) dx_1 = \lim_{n \rightarrow \infty} \int_0^1 u_n(x_1, x_2) dx_1 = g(x_2) \text{ for a.e. } x_2 \in [0, 1],$$

therefore u_0 belongs to $\mathcal{A}_{\varepsilon,g}$ and is a minimizer. \square

3. Lower bound

We consider the question of finding a limiting functional as a lower bound for E_ε as ε goes to zero. Given a sequence $\{w_\varepsilon\}$ with $E_\varepsilon(w_\varepsilon) \leq C$ and $\varepsilon \rightarrow 0$, then

$$\int_{\mathbb{T}^2} (|\partial_1|^{-1} \eta_{w_\varepsilon})^2 dx \rightarrow 0. \quad (3.1)$$

Therefore $\eta_{w_\varepsilon} \rightarrow 0$ distributionally and the natural function space for the limiting problem is

$$\mathcal{A}_0 = \{w \in L^2(\mathbb{T}^2) : \eta_w = -\partial_1 \frac{1}{2} w^2 + \partial_2 w = 0 \text{ in } \mathcal{D}'\}.$$

3.1. Properties of BV functions

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. We first recall the BV structure theorem. For $v \in [BV(\Omega)]^2$, the Radon measure Dv can be decomposed as

$$Dv = D^a v + D^c v + D^j v$$

where $D^a v$ is the absolutely continuous part of Dv with respect to Lebesgue measure \mathcal{L}^2 and $D^c v$, $D^j v$ are the Cantor part and the jump part, respectively. All three measures are mutually singular. Furthermore, $D^a v = \nabla v \mathcal{L}^2 \llcorner \Omega$ where ∇v is the approximate differential of v ; $D^c v = D_s v \llcorner (\Omega \setminus S_v)$ and $D^j v = D_s v \llcorner J_v$, where $D_s v$ is the singular part of Dv with respect to \mathcal{L}^2 , S_v is the set of approximate discontinuity points of v , and J_v is the jump set of v . Since J_v is countably \mathcal{H}^1 -rectifiable, $D^j v$ can be expressed as

$$(v^+ - v^-) \otimes \nu \mathcal{H}^1 \llcorner J_v,$$

where ν is orthogonal to the approximate tangent space at each point of J_v and v^+ , v^- are the traces of v from either side of J_v .

Next we quote the following general chain rule formula for BV functions.

Theorem 3.1 ([2, Theorem 3.96]). *Let $w \in [BV(\Omega)]^2$, $\Omega \subset \mathbb{R}^2$, and $f \in [C^1(\mathbb{R}^2)]^2$ be a Lipschitz function satisfying $f(0) = 0$ if $|\Omega| = \infty$. Then $v = f \circ w$ belongs to $[BV(\Omega)]^2$ and*

$$Dv = \nabla f(w) \nabla w \mathcal{L}^2 \llcorner \Omega + \nabla f(\tilde{w}) D^c w + (f(w^+) - f(w^-)) \otimes \nu_w \mathcal{H}^1 \llcorner J_w. \quad (3.2)$$

Here $\tilde{w}(x)$ is the approximate limit of w at x and is defined on $\Omega \setminus J_w$.

In what follows, we will use [Theorem 3.1](#) to compute the distributional divergence of such $f \circ w$ as the trace of the measure (3.2), that is

$$\operatorname{div}(f \circ w) = \operatorname{tr}(\nabla f(w) \nabla w) \mathcal{L}^2 \llcorner \Omega + \operatorname{tr}(\nabla f(\tilde{w}) D^c w) + (f(w^+) - f(w^-)) \cdot \nu_w \mathcal{H}^1 \llcorner J_w \quad (3.3)$$

as measures.

Lemma 3.2. *If $w \in \mathcal{A}_0 \cap (BV \cap L^\infty)(\mathbb{T}^2)$, then denoting by $D_i^a w$ and $D_i^c w$ the i th components of the measures $D^a w$ and $D^c w$, we have*

$$(-w D_1^a w + D_2^a w) = 0 \quad \text{and} \quad (-\tilde{w} D_1^c w + D_2^c w) = 0$$

as measures, and, setting $\sigma(w) = (-w^2/2, w)$,

$$[\sigma(w^+) - \sigma(w^-)] \cdot \nu_w = 0 \quad \mathcal{H}^1 \text{-a.e. on } J_w. \quad (3.4)$$

Proof. Let $\sigma(w) = (-w^2/2, w)$. By virtue of $w \in \mathcal{A}_0 \cap (BV \cap L^\infty)(\mathbb{T}^2)$ and (3.3), we know that, in the sense of distributions,

$$\begin{aligned} 0 &= -\partial_1 \frac{1}{2} w^2 + \partial_2 w \\ &= \operatorname{div} \sigma(w) \\ &= (-w D_1^a w + D_2^a w) + (-\tilde{w} D_1^c w + D_2^c w) + (\sigma(w^+) - \sigma(w^-)) \cdot \nu_w \mathcal{H}^1 \llcorner J_w. \end{aligned} \quad (3.5)$$

But the measures $D^a w$, $D^c w$, and $D^j w$ are mutually singular, which implies that each individual term in (3.5) is the zero measure. The lemma immediately follows. \square

3.2. Limiting functional and the proof of the lower bound

Let

$$\Sigma(w) = \left(-\frac{1}{3}w^3, \frac{1}{2}w^2 \right).$$

If $w \in \mathcal{A}_0 \cap (BV \cap L^\infty)(\mathbb{T}^2)$, we can apply the chain rule (3.2) and Lemma 3.2 to $\Sigma(w)$, yielding

$$\begin{aligned} \operatorname{div} \Sigma(w) &= w(-wD_1^a w + D_2^a w) \mathcal{L}^2 + \tilde{w}(-\tilde{w}D_1^c w + D_2^c w) \\ &\quad + (\Sigma(w^+) - \Sigma(w^-)) \cdot \nu_w \mathcal{H}^1 \llcorner J_w \\ &= (\Sigma(w^+) - \Sigma(w^-)) \cdot \nu_w \mathcal{H}^1 \llcorner J_w. \end{aligned} \quad (3.6)$$

Remark 3.3. Observe if $w = u_x$ and $u_z = \frac{1}{2}u_x^2$, the entropy $\Sigma(w)$ here is exactly the entropy $\tilde{\Sigma}(\nabla u) = -(u_x u_z - \frac{1}{6}u_x^3, \frac{1}{2}u_x^2)$, which we used in the lower bound estimates in [24]. In fact, the argument below also gives a proof of the lower bound on any domain $\Omega \subset \mathbb{R}^2$; the only necessary modification of the proof presented above is that one does not use $|\partial_1|^{-1}\eta_w$ to represent the compression energy, but rather the original expression from (1.3).

Theorem 3.4. Let $\varepsilon_n \searrow 0$, $\{w_n\} \subset L^2(\mathbb{T}^2)$ with $\partial_1 w_n \in L^2(\mathbb{T}^2)$ such that

$$w_n \rightarrow w \text{ in } L^3(\mathbb{T}^2), \quad (3.7)$$

for some $w \in (BV \cap L^\infty)(\mathbb{T}^2)$. Then

$$\liminf_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(w_n) \geq \int_{J_w} \frac{|w^+ - w^-|^3}{12\sqrt{1 + \frac{1}{4}(w^+ + w^-)^2}} d\mathcal{H}^1. \quad (3.8)$$

Remark 3.5. Due to recent progress on the rectifiability for the defect set to certain solutions of Burgers equation [20], the lower bound should in fact be valid among a larger class of limiting functions. Specifically, if $w \in \mathcal{A}_0 \cap L^\infty(\mathbb{T}^2)$ and for every smooth convex entropy $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ and corresponding entropy flux $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ with $\Psi'(v) = -\Phi'(v)v$,

$$\partial_1 \Psi(w) + \partial_2 \Phi(w) \text{ is a finite Radon measure,} \quad (3.9)$$

then there exists an H^1 -rectifiable set J_w with strong traces on either side such that

$$|\operatorname{div} \Sigma(w)| = \frac{|w^+ - w^-|^3}{12\sqrt{1 + \frac{1}{4}(w^+ + w^-)^2}} \mathcal{H}^1 \llcorner J_w. \quad (3.10)$$

In particular, by substituting any entropy/entropy flux pair for Σ in the argument below, one finds that for an energy bounded sequence, any limiting function w satisfies (3.9) and thus (3.10). Technically, applying the results of [20] to deduce (3.10) would require extending the arguments there from $[0, T] \times \mathbb{R}$ to the bounded domain \mathbb{T}^2 as in [17, 18] and proving that (3.9) implies that $w \in C^0([0, 1]; L^1(\mathbb{T}^1))$ (the continuous in time dependence being a technical assumption in [20, Definition 1.1]). Regarding the regularity assumption, it is known (see e.g. [19, Remark 5.2], [15, pg. 191]) that the argument of Vasseur [35] applies in this context and gives a representative of w belonging to $C^0([0, 1]; L^1(\mathbb{T}^1))$. The extension of [20] to a bounded domain should not present serious difficulties, although we have not pursued the details further. The concentration of the entropy measures on an \mathcal{H}^1 -rectifiable jump set should be a key step in obtaining the full Γ -convergence of E_ε in (1.3) to the limiting energy (3.10). The remaining obstacles to such a result are the construction of a recovery sequence for functions that with gradients that do not belong to $BV \cap L^\infty$ (as the existing technology from [7, 28] uses both those assumptions) and the strengthening of the results of [20] to include functions which do not belong to L^∞ .

Proof of Theorem 3.4. Without loss of generality, we assume $\liminf_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(w_n) < \infty$, so that $w \in \mathcal{A}_0$ by (3.1). Now for any smooth v , direct calculation shows

$$\begin{aligned} \operatorname{div} \Sigma(v) &= \partial_1\left(-\frac{1}{3}v^3\right) + \partial_2\left(\frac{1}{2}v^2\right) \\ &= v(\partial_2 v - v\partial_1 v) = v\eta_v. \end{aligned} \quad (3.11)$$

On the other hand, we can bound $\mathcal{E}_{\varepsilon}$ from below as follows:

$$\begin{aligned} \mathcal{E}_{\varepsilon}(v) &= \frac{1}{2} \int_{\mathbb{T}^2} \frac{1}{\varepsilon} \left(|\partial_1|^{-1} \left(\partial_2 v - \partial_1 \frac{1}{2}v^2 \right) \right)^2 + \varepsilon(\partial_1 v)^2 dx \\ &= \frac{1}{2\varepsilon} \left\| |\partial_1|^{-1} \eta_v \right\|_{L^2(\mathbb{T}^2)}^2 + \frac{\varepsilon}{2} \left\| \partial_1 v \right\|_{L^2(\mathbb{T}^2)}^2 \\ &\geq \left\| |\partial_1|^{-1} \eta_v \right\|_{L^2(\mathbb{T}^2)} \left\| \partial_1 v \right\|_{L^2(\mathbb{T}^2)}. \end{aligned} \quad (3.12)$$

From (3.11) and (3.12), given any smooth periodic function ϕ , for any smooth v , we have

$$\begin{aligned} \left| - \int_{\mathbb{T}^2} \Sigma(v) \cdot \nabla \phi dx \right| &= \left| \int_{\mathbb{T}^2} \operatorname{div} \Sigma(v) \phi dx \right| \\ &\leq \left(\int_{\mathbb{T}^2} \left\| |\partial_1|^{-1} \eta_v \right\|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} |\partial_1(v\phi)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left\| |\partial_1|^{-1} \eta_v \right\|_{L^2(\mathbb{T}^2)} \left\| \partial_1 v \right\|_{L^2(\mathbb{T}^2)} \left\| \phi \right\|_{L^\infty(\mathbb{T}^2)} \\ &\quad + \left\| |\partial_1|^{-1} \eta_v \right\|_{L^2(\mathbb{T}^2)} \left\| v \right\|_{L^2(\mathbb{T}^2)} \left\| \partial_1 \phi \right\|_{L^\infty(\mathbb{T}^2)} \\ &\leq \mathcal{E}_{\varepsilon}(v) \left\| \phi \right\|_{L^\infty(\mathbb{T}^2)} + C\sqrt{\varepsilon} \mathcal{E}_{\varepsilon}(v)^{\frac{1}{2}} \left\| v \right\|_{L^2(\mathbb{T}^2)} \left\| \partial_1 \phi \right\|_{L^\infty(\mathbb{T}^2)}. \end{aligned} \quad (3.13)$$

By the density of smooth functions in $L^2(\mathbb{T}^2)$, (3.13) holds for any $v \in L^2(\mathbb{T}^2)$ with $|\partial_1|^{-1} \eta_v, \partial_1 v \in L^2(\mathbb{T}^2)$. Thus

$$\begin{aligned} \left| - \int_{\mathbb{T}^2} \Sigma(w_n) \cdot \nabla \phi dx \right| \\ \leq \mathcal{E}_{\varepsilon_n}(w_n) \left\| \phi \right\|_{L^\infty(\mathbb{T}^2)} + C\sqrt{\varepsilon_n} \mathcal{E}_{\varepsilon_n}(w_n)^{\frac{1}{2}} \left\| w_n \right\|_{L^2(\mathbb{T}^2)} \left\| \partial_1 \phi \right\|_{L^\infty(\mathbb{T}^2)}. \end{aligned} \quad (3.14)$$

Letting $n \rightarrow \infty$, by the strong convergence of w_n in $L^3(\mathbb{T}^2)$, we have $\Sigma(w_n) \rightarrow \Sigma(w)$ in $L^1(\mathbb{T}^2)$, so that

$$\begin{aligned} - \int_{\mathbb{T}^2} \Sigma(w) \cdot \nabla \phi dx &= - \lim_{n \rightarrow \infty} \int_{\mathbb{T}^2} \Sigma(w_n) \cdot \nabla \phi dx \\ &\leq \liminf_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(w_n) \left\| \phi \right\|_{L^\infty(\mathbb{T}^2)}. \end{aligned} \quad (3.15)$$

By taking the supremum over all smooth test functions ϕ with $\left\| \phi \right\|_{L^\infty} \leq 1$ in (3.15), we see that $|\operatorname{div} \Sigma(w)|(\mathbb{T}^2)$ is a lower bound for the energies. To derive the explicit expression for this measure, we note that since $w \in \mathcal{A}_0 \cap (BV \cap L^\infty)(\mathbb{T}^2)$, (3.4) and (3.6) apply, so that

$$|\operatorname{div} \Sigma(w)|(\mathbb{T}^2) = \left| [\Sigma(w^+) - \Sigma(w^-)] \cdot \frac{(\sigma(w^+) - \sigma(w^-))^\perp}{|\sigma(w^+) - \sigma(w^-)|} \right| \mathcal{H}^1 \llcorner J_w.$$

The right hand side of this equation can be calculated directly from the formulas for $\sigma(w)$ and $\Sigma(w)$ and simplifies to (3.8) (see [24, Proof of Lemma 4.1, Equation (6.3)]). \square

Remark 3.6. When comparing with the lower bound proof from [24], this proof requires an extra integration by parts, as it does not rely on a pointwise lower bound on the energy density (see e.g. [24, Equation (4.11)]). The relationship between these two entropies and the structure of the corresponding arguments is exactly mirrored in the entropies devised in [8, 16] for the Aviles–Giga problem — they are equal on the zero set of the potential term, and both give lower bounds, with only one of them [16] bounding the energy density from below pointwise.

Acknowledgments

We thank both referees for useful comments that improved the article. M.N. also thanks Elio Marconi for helpful discussions regarding [20]. M.N.'s research is supported by NSF grant RTG-DMS 1840314. X.Y.'s research is supported by Simons Collaboration Grant #947054, together with a Research Excellence Grant and a College of Liberal Arts and Sciences Dean's summer research grant from University of Connecticut.

References

- [1] L. Ambrosio, C. De Lellis, C. Mantegazza, Line energies for gradient vector fields in the plane, *Calc. Var. Partial Differential Equations* 9 (4) (1999) 327–355.
- [2] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, in: Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000, p. xviii+434, 1857292.
- [3] P. Aviles, Y. Giga, On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields, *Proc. Roy. Soc. Edinburgh Sect. A* 129 (1) (1999) 1–17.
- [4] E.A. Brener, V.I. Marchenko, Nonlinear theory of dislocations in smectic crystals: An exact solution, *Phys. Rev. E* 59 (1999) R4752–R4753.
- [5] R. Cantero-Álvarez, F. Otto, J. Steiner, The concertina pattern: A bifurcation in ferromagnetic thin films, *J. Nonlinear Sci.* 17 (1) (2007) 221–281.
- [6] M.E. Caplan, A.S. Schneider, C.J. Horowitz, Elasticity of nuclear pasta, *Phys. Rev. Lett.* 121 (2018) 132701.
- [7] S. Conti, C. De Lellis, Sharp upper bounds for a variational problem with singular perturbation, *Math. Ann.* 338 (1) (2007) 119–146.
- [8] A. DeSimone, S. Müller, R.V. Kohn, F. Otto, A compactness result in the gradient theory of phase transitions, *Proc. Roy. Soc. Edinburgh Sect. A* 131 (4) (2001) 833–844.
- [9] F. Ghiraldin, X. Lamy, Optimal besov differentiability for entropy solutions of the eikonal equation, *Comm. Pure Appl. Math.* 73 (2) (2020) 317–349.
- [10] M. Goldman, M. Josien, F. Otto, New bounds for the inhomogenous Burgers and the Kuramoto-Sivashinsky equations, *Comm. Partial Differential Equations* 40 (12) (2015) 2237–2265.
- [11] R. Ignat, B. Merlet, Entropy method for line-energies, *Calc. Var. Partial Differential Equations* 44 (3–4) (2012) 375–418.
- [12] R. Ignat, F. Otto, The magnetization ripple: a nonlocal stochastic PDE perspective, *J. Math. Pures Appl.* (9) 130 (2019) 157–199.
- [13] R. Ignat, F. Otto, T. Reid, P. Tsatsoulis, Variational methods for a singular SPDE yielding the universality of the magnetization ripple. URL <https://arxiv.org/pdf/2010.13123.pdf>.
- [14] T. Ishikawa, O.D. Lavrentovich, Dislocation profile in cholesteric finger texture, *Phys. Rev. E* 60 (1999) R5037–R5039.
- [15] P.-E. Jabin, F. Otto, B. Perthame, Line-energy Ginzburg-Landau models: zero-energy states, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 1 (1) (2002) 187–202.
- [16] W. Jin, R.V. Kohn, Singular perturbation and the energy of folds, *J. Nonlinear Sci.* 10 (3) (2000) 355–390.
- [17] E. Marconi, Characterization of minimizers of Aviles-Giga functionals in special domains, *Arch. Ration. Mech. Anal.* 242 (2) (2021) 1289–1316.
- [18] E. Marconi, Rectifiability of entropy defect measures in a micromagnetics model, *Adv. Calc. Var.* (2021) 000010151520210012.
- [19] E. Marconi, On the structure of weak solutions to scalar conservation laws with finite entropy production, *Cal. Var. PDE* 61 (1) (2022) 30 pp, aper No. 32.
- [20] E. Marconi, The rectifiability of the entropy defect measure for Burgers equation, *J. Funct. Anal.* 283 (6) (2022) 19, Paper No. 109568.
- [21] F. Murat, Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 5 (3) (1978) 489–507.
- [22] F. Murat, Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 8 (1) (1981) 69–102.
- [23] F. Murat, L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl.* (9) 60 (3) (1981) 309–322.
- [24] M. Novack, X. Yan, Compactness and sharp lower bound for a 2D smectics model, *J. Nonlinear Sci.* 31 (3) (2021) 26, Paper No. 60.
- [25] M. Novack, X. Yan, Nonlinear approximation of 3D smectic liquid crystals: sharp lower bound and compactness, *Calc. Var. Partial Differential Equations* 61 (4) (2022) 29, Paper No. 157.
- [26] F. Otto, Optimal bounds on the kuramoto-Sivashinsky equation, *J. Funct. Anal.* 257 (2009) 2188–2245.
- [27] F. Otto, J. Steiner, The concertina pattern, *Calc. Var. Partial Differential Equations* 39 (1–2) (2010) 139–181.
- [28] A. Poliakovsky, Upper bounds for singular perturbation problems involving gradient fields, *J. Eur. Math. Soc. (JEMS)* 9 (1) (2007) 1–43.
- [29] C.D. Santangelo, Geometry and the nonlinear elasticity of defects in smectic liquid crystals, *Liq. Cryst. Today* 15 (3) (2006) 11–18.
- [30] C.D. Santangelo, R.D. Kamien, Bogomol'nyi, Prasad, and Sommerfield configurations in smectics, *Phys. Rev. Lett.* 91 (2003) 045506.

- [31] J. Steiner, The Formation of the Concertina Pattern : Experiments, Analysis, and Numerical Simulations (Ph.D. thesis), 2011.
- [32] L. Tartar, Compensated compactness and applications to partial differential equations, in: Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV, in: Res. Notes in Math., vol. 39, Pitman, Boston, Mass.-London, 1979, pp. 136–212.
- [33] L. Tartar, The compensated compactness method applied to systems of conservation laws, in: Systems of Nonlinear Partial Differential Equations (Oxford, 1982), in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 111, Reidel, Dordrecht, 1983, pp. 263–285.
- [34] L. Tartar, Compensation effects in partial differential equations, *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.* (5) 29 (2005) 395–453.
- [35] A. Vasseur, Strong traces for solutions of multidimensional scalar conservation laws, *Arch. Ration. Mech. Anal.* 160 (3) (2001) 181–193.
- [36] R. Venkatraman, V. Dabade, R.D. James, Bounds on the energy of a soft cubic ferromagnet with large magnetostriction, *J. Nonlinear Sci.* 30 (6) (2020) 3367–3388.