

A RIGOROUS DERIVATION OF A BOLTZMANN SYSTEM FOR A MIXTURE OF HARD-SPHERE GASES*

IOAKEIM AMPATZOGLOU[†], JOSEPH K. MILLER[‡], AND NATAŠA PAVLOVIĆ[‡]

Abstract. In this paper, we rigorously derive a Boltzmann equation for mixtures from the many body dynamics of two types of hard-sphere gases. We prove that the microscopic dynamics of two gases with different masses and diameters is well defined, and we introduce the concept of a two parameter BBGKY hierarchy to handle the nonsymmetric interaction of these gases. As a corollary of the derivation, we prove Boltzmann’s *propagation of chaos* assumption for the case of a mixture of gases.

Key words. Boltzmann, derivation, mathematical physics, analysis of PDEs, mixture, BBGKY

AMS subject classifications. 35A01, 35A02, 70F45, 82C22, 82C40

DOI. 10.1137/21M1424779

1. Introduction. Much effort has been put into studying the dynamics of a collection of interacting gases. Gas mixtures such as helium and xenon were studied as a possible coolant for nuclear reactors in spacecraft [21]. Similar mixtures have also undergone extensive analysis as possible coolants for thermoacoustic refrigerators [13]. Sound propagation in binary mixtures [16] and hypersonic shockwave analysis for aerospace applications [30, 2] have also been studied.

In the case of two gases evolving in $\mathbb{R}^d$, the phase space of a single particle takes the form $\mathbb{R}^d \times \mathbb{R}^d$. Call one gas type A and the other type B. If $g_0(x, v) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is an initial density distribution on phase space of the type A gas, and $h_0(x, v) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ the distribution of the type B gas, the evolution of the two gases is modeled by the Boltzmann system for mixtures:

$$(1.1) \quad \begin{cases} \partial_t g + v \cdot \nabla_x g = c_1 Q_{1,1}(g, g) + c_{1,2} Q_{1,2}(g, h), \\ \partial_t h + v \cdot \nabla_x h = c_2 Q_{2,2}(h, h) + c_{2,1} Q_{2,1}(h, g), \\ g(0, x, v) = g_0(x, v), \quad h(0, x, v) = h_0(x, v). \end{cases}$$

Here, $Q_{i,j}$ are integral, bilinear operators called collision kernels, and $c_1, c_2, c_{2,1}, c_{1,2} \in (0, \infty)$ are constants. They encode all the information about the possible collisions between two particles. This system of equations was initially studied starting in the 1950s by Chapman and Cowling [14] and later by Hamel in the 1960s [22]. The system (1.1) can be seen as a generalization of the standard Boltzmann equation, which has its roots in the works of Maxwell in 1867 [27] and Boltzmann in 1872 [11]. It is

*Received by the editors June 4, 2021; accepted for publication (in revised form) January 31, 2022; published electronically April 19, 2022.

<https://doi.org/10.1137/21M1424779>

Funding: This work was supported by National Science Foundation grants DMS-1840314 and DMS-2009549. The work of the first author was also supported by the Simons Collaboration on Wave Turbulence. The work of the second author was also supported by the Provost’s Graduate Excellence Fellowship at The University of Texas at Austin. The work of the third author was also supported by NSF grant DMS-2052789.

[†]Department of Mathematics, New York University, New York, NY 10012 USA (ioakampa@nyu.edu).

[‡]Department of Mathematics, University of Texas at Austin, Austin, TX 78712 USA (jkmiller@utexas.edu, natasa@math.utexas.edu).

interesting to note that although Maxwell did not study the Boltzmann system (1.1) explicitly, he did consider collisions between particles of disparate masses [27].

Much work has been done mathematically to study the solutions of (1.1). Global well-posedness for mild solutions has been proven in the case of inverse power molecular interactions and small data [20] and in the case of hard potentials also with small data [19]. Stability in various formulations has been extensively studied [12, 20, 19] in addition to numerous numerical schemes [24, 30, 8]. Furthermore, rigorous connections between (1.1) and the compressible Navier–Stokes equations for mixtures of fluids have been established [12]. Explicit solutions to the space homogeneous variant of (1.1) have been studied [10] in addition to recent proofs of global well-posedness and propagation of moments [18, 15].

However, despite the mathematical progress on the subject of the system (1.1), no work has been done on rigorously deriving the system from a system of hard spheres. Mathematical derivation results for the single type hard sphere system trace back to the pioneering work of Alexander [3, 4], Lanford [26], and King [23]. Gallagher, Saint-Raymond, and Texier refined and extended the derivation of a single gas Boltzmann equation for hard spheres and short range potentials [17]. More recently, the first and last authors of this paper considered more complex interactions which model dense gases with ternary interactions [6] and binary-ternary interactions [7]. It is of relevance to note that each of these derivation results considers a Boltzmann like equation for a single type of particle. In addition to the theoretical framework developed above for a single type gas, new techniques for multiple gases are needed to keep track of the evolution and correlation of one type of gas to the other. This is exactly what we do in this paper.

More precisely, in this paper, we derive the Boltzmann system (1.1) from a mixture of finitely many hard spheres. In order to do this, we consider N_1 hard spheres of mass M_1 and diameter ϵ_1 mixed with a system of N_2 hard spheres of mass M_2 and diameter ϵ_2 . For the i th particle of mass M_1 , we denote its center by x_i and its velocity by v_i . Similarly, for the i th particle of mass M_2 , we denote its center by y_i and its velocity by w_i . For notational simplicity, we will write the vector of all positions and velocities by

$$Z_{(N_1, N_2)} = (x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2}, v_1, \dots, v_{N_1}, w_1, \dots, w_{N_2}).$$

The natural phase space for this collection of particles is

$$\mathcal{D}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)} := \left\{ Z_{(N_1, N_2)} : \begin{array}{l} \forall i \neq j, |x_i - x_j| \geq \epsilon_1, |y_i - y_j| \geq \epsilon_2, \\ \forall i, j, |x_i - y_j| \geq \frac{\epsilon_1 + \epsilon_2}{2} \end{array} \right\}.$$

To model a mixture of gases, we assume that each particle is a hard sphere that evolves according to Newton's laws. That is, if noncollisional, we assume that the particles perform rectilinear motion, i.e.,

$$(1.2) \quad \begin{cases} \dot{x}_i = v_i, & \dot{y}_j = w_j, \\ \dot{v}_i = 0, & \dot{w}_j = 0, \\ (x_i(0), v_i(0)) = (x_{i,0}, v_{i,0}), & (y_i(0), w_i(0)) = (y_{i,0}, w_{i,0}). \end{cases}$$

If there exists exactly one collisional pair of particles, we assume the collision is completely elastic, i.e., energy and momentum are conserved under collisions. Consequently, we have the following collisional laws:

1. If (x_i, v_i) and (x_j, v_j) are such that $|x_i - x_j| = \epsilon_1$, then the precollisional velocities (v_i, v_j) give rise to the postcollisional velocities (v_i^*, v_j^*) by

$$(1.3) \quad v_i^* = v_i - \left((v_i - v_j) \cdot \frac{(x_i - x_j)}{\|x_i - x_j\|} \right) \frac{(x_i - x_j)}{\|x_i - x_j\|},$$

$$(1.4) \quad v_j^* = v_j + \left((v_i - v_j) \cdot \frac{(x_i - x_j)}{\|x_i - x_j\|} \right) \frac{(x_i - x_j)}{\|x_i - x_j\|}.$$

2. If (y_i, w_i) and (y_j, w_j) are such that $|y_i - y_j| = \epsilon_2$, then the precollisional velocities (w_i, w_j) give rise to the postcollisional velocities (w_i^*, w_j^*) by

$$(1.5) \quad w_i^* = w_i - \left((w_i - w_j) \cdot \frac{(y_i - y_j)}{\|y_i - y_j\|} \right) \frac{(y_i - y_j)}{\|y_i - y_j\|},$$

$$(1.6) \quad w_j^* = w_j + \left((w_i - w_j) \cdot \frac{(y_i - y_j)}{\|y_i - y_j\|} \right) \frac{(y_i - y_j)}{\|y_i - y_j\|}.$$

3. If (x_i, v_i) and (y_j, w_j) are such that $|x_i - y_j| = (\epsilon_1 + \epsilon_2)/2$, then the precollisional velocities (v_i, w_j) give rise to the postcollisional velocities (v_i^*, w_j^*) by

$$(1.7) \quad v_i^* = v_i - \frac{2M_2}{M_1 + M_2} \left((v_i - w_j) \cdot \frac{(x_i - y_j)}{\|x_i - y_j\|} \right) \frac{(x_i - y_j)}{\|x_i - y_j\|},$$

$$(1.8) \quad w_j^* = w_j + \frac{2M_1}{M_1 + M_2} \left((v_i - w_j) \cdot \frac{(x_i - y_j)}{\|x_i - y_j\|} \right) \frac{(x_i - y_j)}{\|x_i - y_j\|}.$$

For convenience, we call the above system which satisfies (1.2) and (1.3)–(1.8) an $N_1, N_2, \epsilon_1, \epsilon_2$ particle system. This particle system describes a deterministic, i.e., pointwise defined, evolution on the phase space $\mathcal{D}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)}$. In section 3, we prove that for almost every initial configuration in phase space, the above flow is well defined and measure preserving. Consequently, an initial density $f_{(N_1, N_2), 0}$ on the phase space $\mathcal{D}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)}$ evolves according to the following Liouville equation:

$$(1.9) \quad \begin{cases} \partial_t f_{(N_1, N_2)} + \sum_{k=1}^{N_1} v_k \cdot \nabla_{x_k} f_{(N_1, N_2)} + \sum_{k=1}^{N_2} w_k \cdot \nabla_{y_k} f_{(N_1, N_2)} = 0 & \text{on } \mathring{\mathcal{D}}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)}, \\ f_{(N_1, N_2)}(Z_{(N_1, N_2)}^*) = f_{(N_1, N_2)}(Z_{(N_1, N_2)}) & \text{on } \partial \mathcal{D}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)}, \\ f_{(N_1, N_2)}(0) = f_{(N_1, N_2), 0}, \end{cases}$$

where $Z_{(N_1, N_2)}^*$ is the postcollisional configuration related to the precollisional configuration $Z_{(N_1, N_2)}$ by the collisional laws given in (1.3)–(1.8).¹ We note that the above boundary condition is defined for a full surface measure subset of the boundary (see section 3).

¹For a precise definition, see Definition 3.2.

In order to understand the statistical behavior of the $N_1, N_2, \epsilon_1, \epsilon_2$ particle system, we require each of the particles of the same mass to behave identically. This manifests as the condition that $f_{(N_1, N_2)}$ is invariant under permutations of the (x_i, v_i) variables and invariant under permutations of the (y_j, w_j) variables. We call this condition the *identical particles* assumption.² Note that we do not require that $f_{(N_1, N_2)}$ is invariant under interchanging (x_i, v_i) and (y_j, w_j) variables for any i, j . This lack of symmetry forces $f_{(N_1, N_2)}$ to take into account the behavior of both types of particles, but places the $N_1, N_2, \epsilon_1, \epsilon_2$ particle system outside of the standard framework developed for hard-sphere systems of a single type, such as those in [17]. In order to handle this asymmetry, we introduce the following definition, which will be given more precisely in Definition 2.1.

DEFINITION 1.1. For each $s \in \{1, \dots, N_1 - 1\}$ and $\ell \in \{1, \dots, N_2 - 1\}$, we define the mixed marginal of $f_{(N_1, N_2)}$ to be

$$f_{(N_1, N_2)}^{(s, \ell)} := \int \mathbf{1}_{\mathcal{D}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)}} f_{(N_1, N_2)} dx_{s+1} \dots dx_{N_1} dv_{s+1} \dots dv_{N_1} dy_{\ell+1} \dots dy_{N_2} dw_{\ell+1} \dots dw_{N_2}.$$

This concept of a mixed marginal is key to our analysis and allows us to distinguish the behavior of both types of particles.

Integrating by parts equation (1.9) and using the identical particles assumption, we will derive in section 2 an evolution system for $f_{(N_1, N_2)}^{(s, \ell)}$ which will be given as a two parameter hierarchy of equations called the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy:

$$(1.10) \quad \left(\frac{D}{dt} \right)_{(s, \ell)} f_{(N_1, N_2)}^{(s, \ell)} = \mathcal{C}_{(s, \ell), (s+1, \ell)}^{(N_1, N_2)} f_{(N_1, N_2)}^{(s+1, \ell)} + \mathcal{C}_{(s, \ell), (s, \ell+1)}^{(N_1, N_2)} f_{(N_1, N_2)}^{(s, \ell+1)}.$$

Here, the operators $\mathcal{C}_{(s, \ell), (s+1, \ell)}^{(N_1, N_2)}, \mathcal{C}_{(s, \ell), (s, \ell+1)}^{(N_1, N_2)}$ are integral operators given explicitly in subsection 2.4 and

$$\left(\frac{D}{dt} \right)_{(s, \ell)} := \partial_t + \sum_{k=1}^s v_k \cdot \nabla_{x_k} + \sum_{k=1}^{\ell} w_k \cdot \nabla_{y_k}.$$

Solutions $f_{(N_1, N_2)}^{(s, \ell)}$ to the BBGKY hierarchy represent the densities of subsystems to the $N_1, N_2, \epsilon_1, \epsilon_2$ particle system. Taking $N_1, N_2 \rightarrow \infty$ and $\epsilon_1, \epsilon_2 \rightarrow 0$ appropriately, we formally obtain an infinite two parameter hierarchy of equations called the Boltzmann hierarchy:

$$(1.11) \quad \left(\frac{D}{dt} \right)_{(s, \ell)} f^{(s, \ell)} = \mathcal{C}_{(s, \ell), (s+1, \ell)} f^{(s+1, \ell)} + \mathcal{C}_{(s, \ell), (s, \ell+1)} f^{(s, \ell+1)}.$$

Here, the operators $\mathcal{C}_{(s, \ell), (s+1, \ell)}, \mathcal{C}_{(s, \ell), (s, \ell+1)}$ are integral operators given explicitly in subsection 2.4. Solutions $f^{(s, \ell)}$ to the Boltzmann hierarchy correspond to the densities of finite subsystems to a mixture of two gases.

We are now ready to give an informal statement of our main theorem. The rigorous statements are given in Theorems 5.1 and 5.2.

²The density $f_{(N_1, N_2)}$ obeying (1.9) and the identical particles assumption represents the *statistical ensemble* for the gas mixture of the $N_1, N_2, \epsilon_1, \epsilon_2$ particle system.

STATEMENT OF MAIN RESULT. Let $f_0^{(s,\ell)}$ be a sequence of initial data for the Boltzmann hierarchy (1.11), and for each $N_1, N_2 \in \mathbb{N}_+$, let $f_{(N_1, N_2), 0}^{(s,\ell)}$ be an approximating sequence of data for the BBGKY hierarchy. Furthermore, let $f^{(s,\ell)}$ solve (1.11) with initial data $f_0^{(s,\ell)}$ and let $f_{(N_1, N_2)}^{(s,\ell)}$ solve (1.10) with initial data $f_{(N_1, N_2), 0}^{(s,\ell)}$. Then, for fixed $c_1, c_2, b \in \mathbb{R}_+$ in the mixed Boltzmann-Grad scalings

$$N_1 c_1^{d-1} \equiv c_1, \quad N_2 c_2^{d-1} \equiv c_2, \quad \epsilon_1 \equiv b c_2,$$

we have that $f_{(N_1, N_2)}^{(s,\ell)}$ converges to $f^{(s,\ell)}$ in the sense of observables³ as $N_1, N_2 \rightarrow \infty$ and $\epsilon_1, \epsilon_2 \rightarrow 0$.

Furthermore, if $f_0^{(s,\ell)}$ is a tensor of the form $g_0^{\otimes s} \otimes h_0^{\otimes \ell}$, then $f_{(N_1, N_2)}^{(s,\ell)}$ converges to $g^{\otimes s} \otimes h^{\otimes \ell}$ in the sense of observables as $N_1, N_2 \rightarrow \infty$ and $\epsilon_1, \epsilon_2 \rightarrow 0$, where (g, h) is a solution of (1.1) with (g_0, h_0) as initial data.⁴ The constants $c_{1,2}, c_{2,1}$ are given by

$$(1.12) \quad c_{1,2} = c_2 \left(\frac{1+b}{2} \right)^{d-1}, \quad c_{2,1} = c_1 \left(\frac{1+b^{-1}}{2} \right)^{d-1}.$$

We note that these constants $c_1, c_2, c_{1,2}, c_{2,1}$ cannot be picked arbitrarily. In particular, we have that the constants must lie in a three dimensional subset of $(0, \infty)^4$ which is parametrized by c_1, c_2 , and b . Crucially, the constants $c_{1,2}$ and $c_{2,1}$ in (1.1) which describe the strength of the interaction between the two gases can be calculated from the inverse mean free paths c_1, c_2 and the ratio b of the diameters of the spheres. Moreover, (1.12) shows that as b grows large, the constant $c_{1,2}$ grows large while the constant $c_{2,1}$ becomes small. This agrees with physical intuition that if one gas is comprised of larger particles than the other, it has a larger effect on the system as a whole. See (2.27)–(2.29) for the computation of these constants.

REMARK 1.1. As a result of the above statement, Boltzmann's propagation of chaos assumption is rigorously verified. That is, in the infinite particle limit, the joint density $f_{(N_1, N_2)}^{(s,\ell)}$ factors as $g^{\otimes s} \otimes h^{\otimes \ell}$ as $N_1, N_2 \rightarrow \infty$. It is worthwhile to note that while the density $g^{\otimes s} \otimes h^{\otimes \ell}$ indeed factors on $(\mathbb{R}^d \times \mathbb{R}^d)^{(s+\ell)}$, the evolution of g still depends on the evolution of h via (1.1) and vice versa.

While we are inspired by the program for deriving the Boltzmann equation, introduced by Lanford [26] and recently revisited by Gallagher, Saint-Raymond, and Texier [17], in this work we needed to introduce the instruments that can help us track simultaneously more than one type of gas. In particular, the main novel contribution of this work is to develop a theoretical framework which can handle multiple types of collisions. After proving the mixed particle dynamics are well defined in Theorem 3.1, we introduce a notion of identical particles (2.13) and mixed marginals in Definition 2.1. Using this initial framework, we formally derive a two parameter hierarchy of equations in (2.26) and (2.36) that to the best of our knowledge had not been studied in the kinetic context before. This two parameter hierarchy generates a quartic tree of interactions between the different types of particles, as can be seen in (4.5) and (4.17). In the proof of Theorem 5.1, we establish measure estimates in Proposition 7.2 which crucially depend on the masses of the particles. We also note that the framework introduced below can be adapted to any number k of different types of

³Convergence in observables is defined explicitly in Definition 5.4.

⁴Here, we use the convention that $g^{\otimes s} \otimes h^{\otimes \ell}(Z_{(s,\ell)}) = \prod_{i=1}^s g(x_i, v_i) \prod_{i=1}^\ell h(y_i, w_i)$.

gases. The resulting marginals would be indexed by an element of $\mathbb{N}^k$. For the sake of clarity, we will present only the result for the mixture of two gases.

We structure the paper as follows. In section 2, we introduce the concept of a *mixed marginal* and derive a hierarchy of equations which relate mixed marginals. In section 3, we prove the existence almost everywhere of a global mixed particle flow. Next, section 4 covers the well-posedness theory for the hierarchies in addition to the well-posedness theory for the Boltzmann system (1.1). Section 5 gives the precise statements of the main results of the paper. The following sections are devoted to proving these results. First, a series of approximations are proved in section 6 which allow us to handle the observables. Next, an adjunction lemma is proved in section 7, which enables us to add particles to our system while keeping track of our global flow. Section 8 uses this control to obtain a formulation of the observables in terms of specific *pseudo-trajectories*. The final section (section 9) pieces all of the previous approximations together to prove the main theorem.

2. Vocabulary of the paper.

2.1. Definitions. In this section, we consider two types of hard spheres evolving in $\mathbb{R}^d$. We call one group of particles *type (1,0) particles* and the other *type (0,1) particles*. We assume that all particles perform rectilinear motion in $\mathbb{R}^d$, until they undergo a collision with another particle of either type. The collisions occurring are assumed to be perfectly elastic and instantaneous. Our goal is to keep track of both gases separately, extracting some qualitative information about the evolution of their probability distributions. To do this, we establish some new notational conventions simplifying the combinatorics involved. While the below notations are cumbersome, they allow us to carry out proofs without repetitious arguments and easily generalize to k many types of particles.

- We define the set

$$(2.1) \quad \mathcal{T} := \{\alpha \in \mathbb{N}^2 : |\alpha| = 1\}$$

to be the set of all types of particles. Greek indices such as α, β , or σ in $\mathcal{T}$ will designate the type of particle we are considering. We introduce an ordering on the set $\mathcal{T}$ by simply declaring $(1,0) < (0,1)$.

- For each $\alpha \in \mathcal{T}$, denote the number of type α particles by N_α , their diameter by ϵ_α , and their mass by M_α . For $i \in \{1, \dots, N_\alpha\}$, $x_i^\alpha \in \mathbb{R}^d$ will denote the position vector of the i th type α particle, and v_i^α will denote its velocity. For a more compact notation, we will write $X_{N_\alpha}^\alpha = (x_1^\alpha, \dots, x_{N_\alpha}^\alpha)$, $V_{N_\alpha}^\alpha = (v_1^\alpha, \dots, v_{N_\alpha}^\alpha)$, and $Z_{N_\alpha}^\alpha = (X_{N_\alpha}^\alpha, V_{N_\alpha}^\alpha)$.
- For each $\alpha, \beta \in \mathcal{T}$, we define the *interaction distance* between a particle of type α and a particle of type β to be the quantity

$$\epsilon_{(\alpha, \beta)} = \frac{\epsilon_\alpha + \epsilon_\beta}{2}.$$

Additionally, we define the index set of *interacting pairs* to be

$$(2.2) \quad \mathcal{I}^{(\alpha, \beta)} = \mathcal{I}_{(N_\alpha, N_\beta)}^{(\alpha, \beta)} := \begin{cases} \{1, \dots, N_\alpha\} \times \{1, \dots, N_\beta\}, & \alpha \neq \beta, \\ \{(i, j) \in \{1, \dots, N_\alpha\}^2 : i < j\}, & \alpha = \beta. \end{cases}$$

The set of indices $(i, j) \in \mathcal{I}^{(\alpha, \beta)}$ above is exactly the indices i of type α particles and indices j of type β particles which are interacting each other (the ordering $i < j$ excludes double counting for particles of the same type).

- For convenience, let us write the full vector of positions and velocities as

$$(2.3) \quad Z = Z_{(N_{(1,0)}, N_{(0,1)})} := \left(X_{N_{(1,0)}}^{(1,0)}, X_{N_{(0,1)}}^{(0,1)}, V_{N_{(1,0)}}^{(1,0)}, V_{N_{(0,1)}}^{(0,1)} \right).$$

The spacial variables of $Z_{(N_{(1,0)}, N_{(0,1)})}$ will be written as $X_{(N_{(1,0)}, N_{(0,1)})} = (X_{N_{(1,0)}}^{(1,0)}, X_{N_{(0,1)}}^{(0,1)})$, and the velocity variables of $Z_{(N_{(1,0)}, N_{(0,1)})}$ will be written as $V_{(N_{(1,0)}, N_{(0,1)})} = (V_{N_{(1,0)}}^{(1,0)}, V_{N_{(0,1)}}^{(0,1)})$. We will sometimes abuse notation slightly and write $x_i^\alpha(\cdot)$ or $v_i^\alpha(\cdot)$ to denote the projection onto the correct component:

$$(2.4) \quad x_i^\alpha(Z) = x_i^\alpha, \quad v_i^\alpha(Z) = v_i^\alpha.$$

- We will often use the convention that $N_1 = N_{(1,0)}, N_2 = N_{(0,1)}, \epsilon_1 = \epsilon_{(1,0)}, \epsilon_2 = \epsilon_{(0,1)}, M_1 = M_{(1,0)},$ and $M_2 = M_{(0,1)}$.

With these notational conventions, our phase space is given by

$$(2.5) \quad \mathcal{D} = \mathcal{D}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)} := \bigcap_{\alpha \leq \beta} \bigcap_{(i,j) \in \mathcal{I}^{(\alpha, \beta)}} \{Z : |x_i^\alpha - x_j^\beta| \geq \epsilon_{(\alpha, \beta)}\},$$

where the outer intersection is taken over all $\alpha, \beta \in \mathcal{T}$ and $\alpha \leq \beta$. The condition that $\alpha \leq \beta$ excludes double counting (since $|x_i^\alpha - x_j^\beta| \geq \epsilon_{(\alpha, \beta)}$ implies $|x_j^\beta - x_i^\alpha| \geq \epsilon_{(\alpha, \beta)}$).

REMARK 2.1. *We note that while these notational conventions are unwieldy, their use is necessary for the derivation procedure. In particular, in section 6 we expand solutions of the BBGKY and Boltzmann hierarchies in terms of their “collision histories.” By using multi-indices, keeping track of this collision history is simplified.*

We also remark that another advantage of the above notation is that it can be extended in a natural way to k types of particles by defining the set of types to be

$$\mathcal{T}_k = \{\alpha \in \mathbb{N}^k : |\alpha| = 1\}.$$

For simplicity, we will only present the case $k = 2$.

2.2. Dynamics and Liouville’s equation. We assume that particles perform free motion as long as there is no collision, i.e., for each $\alpha \in \mathcal{T}$,

$$(2.6) \quad \dot{x}_i^\alpha = v_i^\alpha, \quad \dot{v}_i^\alpha = 0 \quad \forall i \in \{1, \dots, N_\alpha\}.$$

When two particles collide, we assume that they behave like hard spheres. That is, if for some fixed $\alpha, \beta \in \mathcal{T}$ and $(i, j) \in \mathcal{I}^{(\alpha, \beta)}$ we have $|x_i^\alpha - x_j^\beta| = \epsilon_{(\alpha, \beta)}$, then the precollisional velocities v_i^α, v_j^β are instantaneously changed to the postcollisional velocities $(v_i^\alpha)^*, (v_j^\beta)^*$ by

$$(2.7) \quad (v_i^\alpha)^* = v_i^\alpha - \frac{2M_\beta}{M_\alpha + M_\beta} \left((v_i^\alpha - v_j^\beta) \cdot \frac{x_i^\alpha - x_j^\beta}{\|x_i^\alpha - x_j^\beta\|} \right) \frac{x_i^\alpha - x_j^\beta}{\|x_i^\alpha - x_j^\beta\|},$$

$$(2.8) \quad (v_j^\beta)^* = v_j^\beta + \frac{2M_\alpha}{M_\alpha + M_\beta} \left((v_i^\alpha - v_j^\beta) \cdot \frac{x_i^\alpha - x_j^\beta}{\|x_i^\alpha - x_j^\beta\|} \right) \frac{x_i^\alpha - x_j^\beta}{\|x_i^\alpha - x_j^\beta\|}.$$

Equations (2.7) and (2.8) are consequences of the conservation of energy and the conservation of momentum:

$$(2.9) \quad M_\alpha v_i^\alpha + M_\beta v_j^\beta = M_\alpha (v_i^\alpha)^* + M_\beta (v_j^\beta)^*,$$

$$(2.10) \quad M_\alpha |v_i^\alpha|^2 + M_\beta |v_j^\beta|^2 = M_\alpha |(v_i^\alpha)^*|^2 + M_\beta |(v_j^\beta)^*|^2.$$

In section 3, we show that these conditions give dynamics which is globally in time defined for almost every initial configuration.

We now consider probability densities which are constant along the above dynamics. Given an initial probability density $f_{(N_1, N_2), 0}$, let

$$f_{(N_1, N_2)}(t, Z) = f_{(N_1, N_2), 0}(\Phi_{(N_1, N_2)}^{-t}(Z)),$$

where $\Phi_{(N_1, N_2)}^\tau$ is the mixed hard-sphere flow as introduced in Theorem 3.1. Since $\Phi_{(N_1, N_2)}^\tau$ is measure preserving, $f_{(N_1, N_2)}(t)$ is a probability density for all $t > 0$. Formally assuming sufficient regularity for our calculations to make sense, the generated flow yields that the probability density $f_{(N_1, N_2)}$ of the full particle system satisfies the Liouville equation:

$$(2.11) \quad \partial_t f_{(N_1, N_2)} + \sum_{\alpha \in \mathcal{T}} \sum_{i=1}^{N_\alpha} v_i^\alpha \cdot \nabla_{x_i^\alpha} f_{(N_1, N_2)} = 0 \quad \text{on } [0, T] \times \mathring{\mathcal{D}},$$

where $\mathring{\mathcal{D}}$ is the interior of the phase space $\mathcal{D}$ defined in (2.5). This is accompanied by the boundary condition

$$(2.12) \quad f_{(N_1, N_2)}(t, Z) = f_{(N_1, N_2)}(t, Z^*) \quad \forall (t, Z) \in [0, T] \times \partial \mathcal{D}.$$

We define Z^* in the following way. If $Z \in \partial \mathcal{D}$ is such that there exists exactly one pair $\alpha, \beta \in \mathcal{T}$ and exactly one pair $(i, j) \in \mathcal{I}^{(\alpha, \beta)}$ such that $|x_i^\alpha - x_j^\beta| = \epsilon_{(\alpha, \beta)}$, then Z^* is the vector Z with the v_i^α, v_j^β components replaced by $(v_i^\alpha)^*, (v_j^\beta)^*$ as defined in (2.7), (2.8). It can be shown (see section 3) that the set of all such $Z \in \partial \mathcal{D}$ fills a full surface measure subset of $\partial \mathcal{D}$, and so the boundary equation (2.12) is defined almost everywhere.

2.3. Symmetry with respect to same type particles. In order to observe the statistical behavior of a mixture of gases, we require all particles of the same type to behave identically. For this reason, we assume that the joint probability density $f_{(N_1, N_2)}$ is invariant under permutations among the same type of particles. Mathematically, we assume that for any $\sigma \in S(N_1)$ and $\sigma' \in S(N_2)$,⁵

$$(2.13) \quad f_{(N_1, N_2)}(t, Z) = f_{(N_1, N_2)}(t, \sigma \oplus \sigma'(Z)),$$

where we are defining

$$\sigma \oplus \sigma'(Z) = (x_{\sigma(1)}^{(1,0)}, \dots, x_{\sigma(N_1)}^{(1,0)}, x_{\sigma'(1)}^{(0,1)}, \dots, x_{\sigma'(N_2)}^{(0,1)}, v_{\sigma(1)}^{(1,0)}, \dots, v_{\sigma(N_1)}^{(1,0)}, v_{\sigma'(1)}^{(0,1)}, \dots, v_{\sigma'(N_2)}^{(0,1)}).$$

That is, we require that $f_{(N_1, N_2)}$ is invariant under the above action by the group $S(N_1) \oplus S(N_2)$.

2.4. Mixed marginals and BBGKY hierarchies. Since Liouville's equation (2.11) is a linear transport equation, it yields a complete description of the mixed particle system. However, since the number of particles is extremely large, efficiently solving it is almost impossible. As mentioned in the introduction, we wish to extract a statistical description with the hope that qualitative properties of the gas mixture will

⁵Here, we let $S(N)$ denote the symmetric group on $\{1, \dots, N\}$.

be revealed as $N_1, N_2 \rightarrow \infty$ and $\epsilon_1, \epsilon_2 \rightarrow 0$. In the case of a single gas of hard spheres, this description is the Boltzmann equation. Rigorous derivation of the Boltzmann equation was made by Lanford [26] and later revisited by Gallagher, Saint-Raymond, and Texier [17]. Their key idea is to derive a hierarchy of equations for the marginal densities of the density function. In the mixture case, though, it is impossible to keep track of the two gases separately. The reason is that by using ordinary marginals, we would view the mixture as a uniform gas of $N_1 + N_2$ particles. To overcome this problem, we introduce the notion of *mixed marginals*.

DEFINITION 2.1. For each $\alpha \in \mathcal{T}$, let $s_\alpha \in \{1, \dots, N_\alpha - 1\}$ and let $s_1 := s_{(1,0)}$, $s_2 := s_{(0,1)}$. We use the notation

$$Z_{(s_1, s_2)} := (x_1^{(1,0)}, \dots, x_{s_1}^{(1,0)}, x_1^{(0,1)}, \dots, x_{s_2}^{(0,1)}, v_1^{(1,0)}, \dots, v_{s_1}^{(1,0)}, v_1^{(0,1)}, \dots, v_{s_2}^{(0,1)}),$$

$$Z_{(N_1, N_2)}^{(s_1, s_2)} := (x_{s_1+1}^{(1,0)}, \dots, x_{N_1}^{(1,0)}, x_{s_2+1}^{(0,1)}, \dots, x_{N_2}^{(0,1)}, v_{s_1+1}^{(1,0)}, \dots, v_{N_1}^{(1,0)}, v_{s_2+1}^{(0,1)}, \dots, v_{N_2}^{(0,1)}).$$

We also define

$$Z_{(N_1, N_2)}^{(s_1, N_2)} := (x_{s_1+1}^{(1,0)}, \dots, x_{N_1}^{(1,0)}, v_{s_1+1}^{(1,0)}, \dots, v_{N_1}^{(1,0)}),$$

$$Z_{(N_1, N_2)}^{(N_1, s_2)} := (x_{s_2+1}^{(0,1)}, \dots, x_{N_2}^{(0,1)}, v_{s_2+1}^{(0,1)}, \dots, v_{N_2}^{(0,1)}).$$

For $(s_1, s_2) \neq (N_1, N_2)$, we define the (s_1, s_2) mixed marginal of $f_{(N_1, N_2)}$ to be the function

$$(2.14) \quad f_{(N_1, N_2)}^{(s_1, s_2)}(t, Z_{(s_1, s_2)}) := \int_{\mathcal{D}(Z_{(s_1, s_2)})} f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dZ_{(N_1, N_2)}^{(s_1, s_2)},$$

where the integral is taken over the set

$$(2.15) \quad \mathcal{D}(Z_{(s_1, s_2)}) = \left\{ Z_{(N_1, N_2)}^{(s_1, s_2)} : Z_{(N_1, N_2)} \in \mathcal{D}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)} \right\}.$$

Additionally, we will define $f_{(N_1, N_2)}^{(N_1, N_2)} := f_{(N_1, N_2)}$.

Using Liouville's equation (2.11), the boundary condition (2.12), and the symmetry condition (2.13), we may formally derive a relation between these mixed marginals.

Let us assume that $f_{(N_1, N_2)} \in \mathcal{C}_c^\infty([0, T] \times \mathcal{D})$ and calculate by definition (2.14) that

$$\partial_t f_{(N_1, N_2)}^{(s_1, s_2)}(t, Z_{(s_1, s_2)}) = \int_{\mathcal{D}(Z_{(s_1, s_2)})} \partial_t f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dZ_{(N_1, N_2)}^{(s_1, s_2)}.$$

Now fix $\alpha \in \mathcal{T}$ and $i \in \{1, \dots, s_\alpha\}$ and compute, using (2.14) again, that

$$\begin{aligned} v_i^\alpha \cdot \nabla_{x_i^\alpha} f_{(N_1, N_2)}^{(s_1, s_2)}(t, Z_{(s_1, s_2)}) &= \int_{\mathcal{D}(Z_{(s_1, s_2)})} v_i^\alpha \cdot \nabla_{x_i^\alpha} f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dZ_{(N_1, N_2)}^{(s_1, s_2)} \\ &\quad + \int_{\partial \mathcal{D}(Z_{(s_1, s_2)}; i, \alpha)} v_i^\alpha \cdot n f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dS, \end{aligned}$$

where n is the outward normal vector to the surface

$$(2.16) \quad \partial \mathcal{D}(Z_{(s_1, s_2)}; i, \alpha) := \bigcup_{\beta} \bigcup_{j=s_\beta+1}^{N_\beta} \left\{ Z_{(N_1, N_2)}^{(s_1, s_2)} \in \mathcal{D}(Z_{(s_1, s_2)}) : |x_i^\alpha - x_j^\beta| = \epsilon_{(\alpha, \beta)} \right\}.$$

Summing over all i, α and using (2.11), we obtain

$$\begin{aligned} & \left(\partial_t + \sum_{\alpha} \sum_{i=1}^{s_{\alpha}} v_i^{\alpha} \cdot \nabla_{x_i^{\alpha}} \right) f_{(N_1, N_2)}^{(s_1, s_2)}(t, Z_{(s_1, s_2)}) \\ &= - \sum_{\alpha} \sum_{i=s_{\alpha}+1}^{N_{\alpha}} \int_{\mathcal{D}(Z_{(s_1, s_2)})} v_i^{\alpha} \cdot \nabla_{x_i^{\alpha}} f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dZ_{(N_1, N_2)}^{(s_1, s_2)} \\ & \quad + \sum_{\alpha} \sum_{i=1}^{s_{\alpha}} \int_{\partial \mathcal{D}(Z_{(s_1, s_2)}; i, \alpha)} v_i^{\alpha} \cdot n f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dS. \end{aligned}$$

Now, integrating by parts the first term on the right-hand side results in only the boundary terms

$$\begin{aligned} & \int_{\mathcal{D}(Z_{(s_1, s_2)})} v_i^{\alpha} \cdot \nabla_{x_i^{\alpha}} f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dZ_{(N_1, N_2)}^{(s_1, s_2)} \\ &= \int_{\partial^* \mathcal{D}(Z_{(s_1, s_2)}; i, \alpha)} v_i^{\alpha} \cdot n f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dS(x_i^{\alpha}) dv_i^{\alpha} dZ_{(N_1, N_2)}^{(s_1, s_2), i, \alpha}, \end{aligned}$$

where $Z_{(N_1, N_2)}^{(s_1, s_2), i, \alpha}$ is the $Z_{(N_1, N_2)}^{(s_1, s_2)}$ variable without the $x_i^{\alpha}, v_i^{\alpha}$ components and

$$(2.17) \quad \partial^* \mathcal{D}(Z_{(s_1, s_2)}; i, \alpha) := \bigcup_{\beta} \bigcup_{j=1}^{s_{\beta}} \left\{ Z_{(N_1, N_2)}^{(s_1, s_2)} \in \mathcal{D}(Z_{(s_1, s_2)}) : |x_j^{\beta} - x_i^{\alpha}| = \epsilon_{(\alpha, \beta)} \right\}.$$

Combining the above expressions together,⁶ we obtain

$$\begin{aligned} & \left(\partial_t + \sum_{\alpha} \sum_{i=1}^{s_{\alpha}} v_i^{\alpha} \cdot \nabla_{x_i^{\alpha}} \right) f_{(N_1, N_2)}^{(s_1, s_2)}(t, Z_{(s_1, s_2)}) = \\ (2.18) \quad & - \sum_{\alpha} \sum_{i=s_{\alpha}+1}^{N_{\alpha}} \sum_{\beta} \sum_{j=1}^{s_{\beta}} \int_{\mathcal{D}(Z_{(s_1, s_2)} + \alpha)} \int_{\mathbb{R}^d} \int_{\partial B_{\epsilon_{(\alpha, \beta)}}(x_j^{\beta})} v_i^{\alpha} \cdot n f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dS(x_i^{\alpha}) dv_i^{\alpha} dZ_{(N_1, N_2)}^{(s_1, s_2), i, \alpha} \\ (2.19) \quad & + \sum_{\alpha} \sum_{i=1}^{s_{\alpha}} \sum_{\beta} \sum_{j=s_{\beta}+1}^{N_{\beta}} \int_{\mathcal{D}(Z_{(s_1, s_2)} + \beta)} \int_{\mathbb{R}^d} \int_{\partial B_{\epsilon_{(\alpha, \beta)}}(x_i^{\alpha})} v_i^{\alpha} \cdot n f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dS(x_j^{\beta}) dv_j^{\beta} dZ_{(N_1, N_2)}^{(s_1, s_2), j, \beta}, \end{aligned}$$

where here we are taking n to be the inward pointing normal. Relabeling (2.18) by reversing the roles of (j, β) and (i, α) , we can combine (2.18) and (2.19) to produce

$$\begin{aligned} & \left(\partial_t + \sum_{\alpha} \sum_{i=1}^{s_{\alpha}} v_i^{\alpha} \cdot \nabla_{x_i^{\alpha}} \right) f_{(N_1, N_2)}^{(s_1, s_2)}(t, Z_{(s_1, s_2)}) \\ &= \sum_{\alpha} \sum_{i=1}^{s_{\alpha}} \sum_{\beta} \sum_{j=s_{\beta}+1}^{N_{\beta}} \int_{\mathcal{D}(Z_{(s_1, s_2)} + \beta)} \int_{\mathbb{R}^d} \int_{\partial B_{\epsilon_{(\alpha, \beta)}}(x_i^{\alpha})} (-v_j^{\beta} + v_i^{\alpha}) \cdot n f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dS(x_j^{\beta}) dv_j^{\beta} dZ_{(N_1, N_2)}^{(s_1, s_2), j, \beta}. \end{aligned}$$

By our symmetry assumption (2.13) and recalling Definition 2.1, we may simplify the each of the above integrals into

$$\begin{aligned} & \int_{\mathcal{D}(Z_{(s_1, s_2)} + \beta)} \int_{\mathbb{R}^d} \int_{\partial B_{\epsilon_{(\alpha, \beta)}}(x_i^{\alpha})} (-v_j^{\beta} + v_i^{\alpha}) \cdot n f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dS(x_j^{\beta}) dv_j^{\beta} dZ_{(N_1, N_2)}^{(s_1, s_2), j, \beta} \\ &= \int_{\mathcal{D}(Z_{(s_1, s_2)} + \beta)} \int_{\mathbb{R}^d} \int_{\partial B_{\epsilon_{(\alpha, \beta)}}(x_i^{\alpha})} (-v_{s_{\beta}+1}^{\beta} + v_i^{\alpha}) \cdot n f_{(N_1, N_2)}(t, Z_{(N_1, N_2)}) dS(x_{s_{\beta}+1}^{\beta}) dv_{s_{\beta}+1}^{\beta} dZ_{(N_1, N_2)}^{(s_1, s_2) + \beta} \\ &= \int_{\mathbb{R}^d} \int_{\partial B_{\epsilon_{(\alpha, \beta)}}(x_i^{\alpha})} (-v_{s_{\beta}+1}^{\beta} + v_i^{\alpha}) \cdot n f_{(N_1, N_2)}^{(s_1, s_2) + \beta}(t, Z_{(s_1, s_2)} + \beta) dS(x_{s_{\beta}+1}^{\beta}) dv_{s_{\beta}+1}^{\beta}. \end{aligned}$$

⁶Note that the boundary sets (2.16) and (2.17) each consist of a union of sets which are pairwise disjoint except for a set of surface measure zero. See Remark 3.4.

For each of these integrals, change variables so that we integrate over $\mathbb{S}^{d-1}$ instead of $\partial B_{\epsilon(\alpha,\beta)}(x_i^\alpha)$ to obtain

$$(2.20) \quad \epsilon_{(\alpha,\beta)}^{d-1} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (v_{s_\beta+1}^\beta - v_i^\alpha) \cdot \theta f_{(N_1,N_2)}^{(s_1,s_2)+\beta}(t, Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,+}) d\theta dv_{s_\beta+1}^\beta.$$

Here, we are using the notation that $Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,+} \in \mathcal{D}_{(\epsilon_1,\epsilon_2)}^{(s_1,s_2)+\beta}$ is the vector

$$(2.21) \quad Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,+} := \begin{cases} \left(X_{s_1}^{(1,0)}, x_i^\alpha + \epsilon_{(\alpha,\beta)}\theta, X_{s_2}^{(0,1)}, V_{s_1}^{(1,0)}, v_{s_1+1}^{(1,0)}, V_{s_2}^{(0,1)} \right), & \beta = (1,0), \\ \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, x_i^\alpha + \epsilon_{(\alpha,\beta)}\theta, V_{s_1}^{(1,0)}, V_{s_2}^{(0,1)}, v_{s_2+1}^{(0,1)} \right), & \beta = (0,1). \end{cases}$$

Next, break (2.20) into parts $S_+ := \{\theta \in \mathbb{S}^{d-1} : (v_{s_\beta+1}^\beta - v_i^\alpha) \cdot \theta > 0\}$ and $S_- := \{\theta \in \mathbb{S}^{d-1} : (v_{s_\beta+1}^\beta - v_i^\alpha) \cdot \theta < 0\}$. On S_- we use the change of variables $\theta \mapsto -\theta$ and on S_+ we use the boundary condition (2.12) to obtain

$$(2.22) \quad \left(\partial_t + \sum_{\alpha} \sum_{i=1}^{s_\alpha} v_i^\alpha \cdot \nabla_{x_i^\alpha} \right) f_{(N_1,N_2)}^{(s_1,s_2)}(t, Z_{(s_1,s_2)}) = \sum_{\beta,\alpha} (N_\beta - s_\beta) \epsilon_{(\alpha,\beta)}^{d-1} \sum_{i=1}^{s_\alpha},$$

$$(2.23) \quad \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [(v_{s_\beta+1}^\beta - v_i^\alpha) \cdot \theta]_+ \left(f_{(N_1,N_2)}^{(s_1,s_2)+\beta}(t, Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,*}) - f_{(N_1,N_2)}^{(s_1,s_2)+\beta}(t, Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,+}) \right) d\theta dv_{s_\beta+1}^\beta,$$

where we are using the notations

$$(2.24) \quad Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,*} := (Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,+})^*,$$

$$(2.25) \quad Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha} := \begin{cases} \left(X_{s_1}^{(1,0)}, x_i^\alpha - \epsilon_{(\alpha,\beta)}\theta, X_{s_2}^{(0,1)}, V_{s_1}^{(1,0)}, v_{s_1+1}^{(1,0)}, V_{s_2}^{(0,1)} \right), & \beta = (1,0), \\ \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, x_i^\alpha - \epsilon_{(\alpha,\beta)}\theta, V_{s_1}^{(1,0)}, V_{s_2}^{(0,1)}, v_{s_2+1}^{(0,1)} \right), & \beta = (0,1). \end{cases}$$

From these calculations, we are led to the following definition.

DEFINITION 2.2. For each $\alpha, \beta \in \mathcal{T}$ and $s_1, s_2 \in \mathbb{N}$, we define the collision operator

$$\mathcal{C}_{(s_1,s_2),(s_1,s_2)+\beta}^\alpha := \mathcal{C}_{(s_1,s_2),(s_1,s_2)+\beta}^{(N_1,N_2),\alpha} : \mathcal{C}_c^\infty \left(\mathcal{D}_{(\epsilon_1,\epsilon_2)}^{(s_1,s_2)+\beta} \right) \rightarrow \mathcal{C}_c^\infty \left(\mathcal{D}_{(\epsilon_1,\epsilon_2)}^{(s_1,s_2)} \right)$$

by the expression

$$\mathcal{C}_{(s_1,s_2),(s_1,s_2)+\beta}^\alpha f_{(N_1,N_2)}^{(s_1,s_2)+\beta}(Z_{(s_1,s_2)}) := (N_\beta - s_\beta) \epsilon_{(\alpha,\beta)}^{d-1} \sum_{i=1}^{s_\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [(v_{s_\beta+1}^\beta - v_i^\alpha) \cdot \theta]_+ \cdot \left(f_{(N_1,N_2)}^{(s_1,s_2)+\beta}(Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,*}) - f_{(N_1,N_2)}^{(s_1,s_2)+\beta}(Z_{(s_1,s_2)+\beta,\epsilon}^{i,\alpha,+}) \right) d\theta dv_{s_\beta+1}^\beta.$$

The collisional operator $\mathcal{C}_{(s_1,s_2),(s_1,s_2)+\beta}^\alpha$ can be viewed as counting the effects of colliding a new type β particle with existing particles of type α . As with the Boltzmann equation, the above equations represent the effects of a collision between two particles and naturally split into *gain* and *loss* terms.

Summarizing the discussion above, if $f_{(N_1,N_2)} \in \mathcal{C}_c^\infty([0,T] \times \mathcal{D})$ satisfies Liouville's equation (2.11), the boundary condition (2.12), and the identical particles assumption

(2.13), then we have for each $\alpha \in \mathcal{T}$ and $s_\alpha \in \{1, \dots, N_\alpha\}$ that the mixed marginals satisfy

$$(2.26) \quad \left(\partial_t + \sum_{\alpha} \sum_{i=1}^{s_\alpha} v_i^\alpha \cdot \nabla_{x_i^\alpha} \right) f_{(N_1, N_2)}^{(s_1, s_2)}(t, Z_{(s_1, s_2)}) = \sum_{\beta, \alpha \in \mathcal{T}} \mathcal{C}_{(s_1, s_2), (s_1, s_2) + \beta}^\alpha f_{(N_1, N_2)}^{(s_1, s_2) + \beta}(t, Z_{(s_1, s_2)}).$$

We call the set of $N_1 N_2$ coupled equations in (2.26) the BBGKY hierarchy. Solutions to this hierarchy will be our main object of study. We will give the function spaces and definition of solutions to the BBGKY hierarchy in section 4.

2.5. Scalings and the Boltzmann hierarchy. Recall that our goal is the asymptotic behavior of the system, i.e., $N_1, N_2 \rightarrow \infty$ and $\epsilon_1, \epsilon_2 \rightarrow 0$. The only possible scaling to make this feasible is dictated by Definition 2.2, and so we assume that $N_\beta \epsilon_{(\alpha, \beta)}^{d-1} \approx 1$, where

$$(2.27) \quad N_\beta \epsilon_{(\alpha, \beta)}^{d-1} = \begin{cases} N_1 \epsilon_1^{d-1}, & \alpha = \beta = (1, 0), \\ N_2 \epsilon_2^{d-1}, & \alpha = \beta = (0, 1), \\ N_1 \left(\frac{\epsilon_1 + \epsilon_2}{2} \right)^{d-1}, & \alpha = (0, 1), \beta = (1, 0), \\ N_2 \left(\frac{\epsilon_1 + \epsilon_2}{2} \right)^{d-1}, & \alpha = (1, 0), \beta = (0, 1). \end{cases}$$

Note that the terms where $\alpha \neq \beta$ in (2.27) imply $N_1 \approx N_2$. This fact, combined with the terms in (2.27) where $\alpha = \beta$, implies $\epsilon_1 \approx \epsilon_2$. Hence, we will explicitly require the following scalings:

$$(2.28) \quad N_1 \epsilon_1^{d-1} \equiv c_1, \quad N_2 \epsilon_2^{d-1} \equiv c_2, \quad \epsilon_1 \equiv b \epsilon_2.$$

A simple calculation then yields

$$(2.29) \quad N_1 \left(\frac{\epsilon_1 + \epsilon_2}{2} \right)^{d-1} = c_1 \left(\frac{1 + b^{-1}}{2} \right)^{d-1}, \quad N_2 \left(\frac{\epsilon_1 + \epsilon_2}{2} \right)^{d-1} = c_2 \left(\frac{1 + b}{2} \right)^{d-1},$$

$$(2.30) \quad N_1 = \left(\frac{c_1}{c_2} b^{1-d} \right) N_2.$$

Here, the constants $c_1, c_2 > 0$ describe the mean free path density of the different types of gases, and b is the ratio of the type $(1, 0)$ particle diameter to the type $(0, 1)$ particle diameter. For the sake of simplicity, we assume that the constant $c_1 b^{1-d}/c_2$ is a rational number. While this condition is not strictly needed, it simplifies the relation N_1 has with N_2 in (2.30). Assuming that our scalings (2.28) hold, we can now obtain a formal limit of the above collisional operators by taking $N_1, N_2 \rightarrow \infty$ and $\epsilon_1, \epsilon_2 \rightarrow 0$.

DEFINITION 2.3. For $s_1, s_2 \in \mathbb{N}_+$, define for each $\alpha, \beta \in \mathcal{T}$ the operators

$$(2.31) \quad \mathcal{C}_{(s_1, s_2), (s_1, s_2) + \beta}^\alpha : \mathcal{C}_c^\infty \left(\mathbb{R}^{d(s_1 + s_2 + 1)} \right) \rightarrow \mathcal{C}_c^\infty \left(\mathbb{R}^{d(s_1 + s_2)} \right)$$

given by

$$(2.32) \quad \mathcal{C}_{(s_1, s_2), (s_1, s_2) + \beta}^\alpha f^{(s_1, s_2) + \beta}(Z_{(s_1, s_2)}) \\ := A_{\beta}^{\alpha} \sum_{i=1}^{s_{\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [(v_{s_{\beta}+1}^{\beta} - v_i^{\alpha}) \cdot \theta]_+ \left(f^{(s_1, s_2) + \beta}(Z_{(s_1, s_2) + \beta}^{i, \alpha, *}) - f^{(s_1, s_2) + \beta}(Z_{(s_1, s_2) + \beta}^{i, \alpha}) \right) d\theta dv_{s_{\beta}+1}^{\beta}.$$

Here, we are using the notation that

$$(2.33) \quad Z_{(s_1, s_2) + \beta}^{i, \alpha} := \begin{cases} \left(X_{s_1}^{(1,0)}, x_i^\alpha, X_{s_2}^{(0,1)}, V_{s_1}^{(1,0)}, v_{s_1+1}^{(1,0)}, V_{s_2}^{(0,1)} \right), & \beta = (1, 0), \\ \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, x_i^\alpha, V_{s_1}^{(1,0)}, V_{s_2}^{(0,1)}, v_{s_2+1}^{(0,1)} \right), & \beta = (0, 1), \end{cases}$$

and $Z_{(s_1, s_2) + \beta}^{i, \alpha, *}$ is the vector $Z_{(s_1, s_2) + \beta}^{i, \alpha}$ whose components $v_i^\alpha, v_{s_\beta+1}^\beta$ are replaced with $(v_i^\alpha)^*, (v_{s_\beta+1}^\beta)^*$ given by the collisional laws

$$(2.34) \quad \begin{aligned} (v_i^\alpha)^* &= v_i^\alpha - \frac{2M_\beta}{M_\alpha + M_\beta} \left((v_i^\alpha - v_{s_\beta+1}^\beta) \cdot \theta \right) \theta, \\ (v_{s_\beta+1}^\beta)^* &= v_{s_\beta+1}^\beta + \frac{2M_\alpha}{M_\alpha + M_\beta} \left((v_i^\alpha - v_{s_\beta+1}^\beta) \cdot \theta \right) \theta. \end{aligned}$$

The constants A_β^α in (2.32) are given by

$$(2.35) \quad A_\beta^\alpha = \begin{cases} c_1, & \alpha = \beta = (1, 0), \\ c_2, & \alpha = \beta = (0, 1), \\ c_1 \left(\frac{1+b^{-1}}{2} \right)^{d-1}, & \alpha = (0, 1), \beta = (1, 0), \\ c_2 \left(\frac{1+b}{2} \right)^{d-1}, & \alpha = (1, 0), \beta = (0, 1). \end{cases}$$

The associated limiting differential equation of (2.26) is given by

$$(2.36) \quad \left(\partial_t + \sum_{\alpha \in \mathcal{T}} \sum_{k=1}^{s_\alpha} v_k^\alpha \cdot \nabla_{x_k^\alpha} \right) f^{(s_1, s_2)} = \sum_{\alpha, \beta \in \mathcal{T}} \mathcal{C}_{(s_1, s_2), (s_1, s_2) + \beta}^\alpha f^{(s_1, s_2) + \beta}.$$

We call the infinite set of coupled equations (2.36) the Boltzmann hierarchy. The exact definition of the functional spaces on which we consider solutions is given in section 4.2. Formal solutions to this hierarchy which are tensors of the form

$$(2.37) \quad f^{(s_1, s_2)}(t, Z_{(s_1, s_2)}) = \prod_{\gamma \in \mathcal{T}} \prod_{k=1}^{s_\gamma} f_\gamma(t, x_k^\gamma, v_k^\gamma)$$

for some functions $f_{(1,0)}, f_{(0,1)}$ are intimately related to the Boltzmann equation for mixtures. Namely, (2.37) solves the Boltzmann hierarchy if for each $\alpha \in \mathcal{T}$

$$(2.38) \quad [\partial_t + v_k^\alpha \cdot \nabla_{x_k^\alpha}] f_\alpha(t, x_k^\alpha, v_k^\alpha) = \sum_{\beta \in \mathcal{T}} A_\beta^\alpha Q_\beta^\alpha(f_\alpha, f_\beta)(t, x_k^\alpha, v_k^\alpha).$$

The collision operators Q_β^α above are given by the following definition.

DEFINITION 2.4. For $G, H : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $\alpha, \beta \in \mathcal{T}$, we define the bilinear forms

$$\begin{aligned} Q_\beta^\alpha(G, H)(t, x^\alpha, v^\alpha) \\ := \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} ((v^\beta - v^\alpha) \cdot \theta)_+ \left[G(t, x^\alpha, (v^\alpha)^*) H(t, x^\alpha, (v^\beta)^*) - G(t, x^\alpha, v^\alpha) H(t, x^\alpha, v^\beta) \right] d\theta dv^\beta, \end{aligned}$$

where $(v^\alpha)^*, (v^\beta)^*$ are given by the collisional laws

$$(v^\alpha)^* = v^\alpha - \frac{2M_\beta}{M_\alpha + M_\beta} ((v_i^\alpha - v^\beta) \cdot \theta) \theta, \quad (v^\beta)^* = v^\beta + \frac{2M_\alpha}{M_\alpha + M_\beta} ((v_i^\alpha - v^\beta) \cdot \theta) \theta.$$

These operators Q_β^α are related to the operators $Q_{i,j}$ in the Boltzmann system for mixtures (1.1) by

$$(2.39) \quad Q_\beta^\alpha = \begin{cases} Q_{1,1}, & \alpha = \beta = (1, 0), \\ Q_{2,2}, & \alpha = \beta = (0, 1), \\ Q_{1,2}, & \alpha = (1, 0), \beta = (0, 1), \\ Q_{2,1}, & \alpha = (0, 1), \beta = (1, 0). \end{cases}$$

3. Dynamics of mixed particles. In this section, we rigorously show that a measure preserving global in time flow for almost every initial configuration can be defined for the mixture of two hard-sphere gases such that the first gas consists of N_1 identical particles of mass M_1 and diameter ϵ_1 and the second gas consists of N_2 identical particles of mass M_2 and diameter ϵ_2 . We assume that both types of particles perform rectilinear motion, until they run into a binary collision with a particle of either type. Depending on the type of particles colliding, velocities instantaneously transform according to (2.7), (2.8). However, since the exchange of velocities is not smooth in time, it is not obvious that a global dynamics can be defined. In particular, the system might run into pathological trajectories (multiple collisions of particles, grazing collisions, infinitely many collisions in a finite time).

In the case of a single gas of identical hard sphere, pathologies might arise as well, and existence of a global flow was established by Alexander [3]. Inspired by the ideas of [3, 4], Ampatzoglou and Pavlović [7] constructed a global flow for a system of particles performing ternary interactions and later [6] for particles performing both binary and ternary interactions.

Our case does not directly follow from [3] because our mixture consists of gases of different masses, and therefore we have to construct the dynamics from the very beginning. To do this, we adapt ideas from [6] in the case of mixture of gases. The crucial lemma required to pass from the local flow to the global flow is to note that once a collision occurs, subsequent collisions cannot involve the same particles. This observation allows us to remove a set of measure zero leading to pathological configurations while having well-defined trajectories on the complement.

3.1. Mixed particle notation. Recall our phase space $\mathcal{D} = \mathcal{D}_{(\epsilon_1, \epsilon_2)}^{(N_1, N_2)}$ is given by (2.5) and our set of interacting pairs $\mathcal{I}^{(\alpha, \beta)}$ is given by (2.2). We denote the interior of $\mathcal{D}$ by $\mathring{\mathcal{D}}$, and we write its boundary as

$$(3.1) \quad \partial\mathcal{D} = \bigcup_{\alpha \leq \beta} \bigcup_{(i,j) \in \mathcal{I}^{(\alpha, \beta)}} \Sigma_{(i,j)}^{(\alpha, \beta)}, \quad \Sigma_{(i,j)}^{(\alpha, \beta)} := \{Z : |x_i^\alpha - x_j^\beta| = \epsilon_{(\alpha, \beta)}\}.$$

These parts $\Sigma_{(i,j)}^{(\alpha, \beta)}$ of the boundary are not disjoint, and the intersection of two non-identical parts forms sets where three or more particles are colliding. The subset of the boundary where the only particles in collision are exactly the i th α -particle and j th β -particle will be denoted by

$$(3.2) \quad \Sigma_{(i,j), sc}^{(\alpha, \beta)} := \{Z \in \partial\mathcal{D} : Z \in \Sigma_{(i,j)}^{(\alpha, \beta)} \text{ for a unique 4-tuple } (\alpha, \beta, i, j) \text{ with } \alpha \leq \beta \text{ and } (i, j) \in \mathcal{I}^{(\alpha, \beta)}\}.$$

The set where exactly two particles are colliding is called the *simple collisional subset of the boundary* and will be given by the disjoint union

$$(3.3) \quad \partial_{sc}\mathcal{D} := \bigcup_{\alpha \leq \beta} \bigcup_{(i,j) \in \mathcal{I}^{(\alpha, \beta)}} \Sigma_{(i,j), sc}^{(\alpha, \beta)}.$$

The subset of the boundary not in a simple collision is said to be in *multiple collisions* and is denoted by

$$(3.4) \quad \partial_{mc}\mathcal{D} := \partial\mathcal{D} \setminus \partial_{sc}\mathcal{D}.$$

In the following definition, we further classify $\Sigma_{(i,j),sc}^{(\alpha,\beta)}$ into parts where we have *grazing* or *nongrazing* collisions.

DEFINITION 3.1. Let $\alpha \leq \beta \in \mathcal{T}$. For $(i, j) \in \mathcal{I}^{(\alpha,\beta)}$ and $Z \in \Sigma_{(i,j),sc}^{(\alpha,\beta)}$, we define the following collision types:

$$\begin{cases} \text{precollisional:} & (x_i^\alpha - x_j^\beta) \cdot (v_i^\alpha - v_j^\beta) < 0, \\ \text{postcollisional:} & (x_i^\alpha - x_j^\beta) \cdot (v_i^\alpha - v_j^\beta) > 0, \\ \text{grazing:} & (x_i^\alpha - x_j^\beta) \cdot (v_i^\alpha - v_j^\beta) = 0. \end{cases}$$

This leads to the definition of the simple collisional grazing and nongrazing sets

$$\Sigma_{(i,j),sc,g}^{(\alpha,\beta)} := \{Z \in \Sigma_{(i,j),sc}^{(\alpha,\beta)} : Z \text{ is grazing}\}, \quad \Sigma_{(i,j),sc,ng}^{(\alpha,\beta)} := \Sigma_{(i,j),sc}^{(\alpha,\beta)} \setminus \Sigma_{(i,j),sc,g}^{(\alpha,\beta)}.$$

Moreover, we have the decomposition

$$\partial_{sc}\mathcal{D} = \partial_{sc,ng}\mathcal{D} \cup \partial_{sc,g}\mathcal{D},$$

where we define the simple collisional grazing and nongrazing sets

$$\partial_{sc,ng}\mathcal{D} := \{Z \in \partial_{sc}\mathcal{D} : Z \text{ is not grazing}\}, \quad \partial_{sc,g}\mathcal{D} := \{Z \in \partial_{sc}\mathcal{D} : Z \text{ is grazing}\}.$$

On these nongrazing sets $\Sigma_{(i,j),sc,ng}^{(\alpha,\beta)}$ we have a natural impact operator.

DEFINITION 3.2. Let $\alpha \leq \beta \in \mathcal{T}$. Then for $(i, j) \in \mathcal{I}^{(\alpha,\beta)}$, define $T_{(i,j)}^{(\alpha,\beta)} : \Sigma_{(i,j),sc,ng}^{(\alpha,\beta)} \rightarrow \Sigma_{(i,j),sc,ng}^{(\alpha,\beta)}$ by

$$T_{(i,j)}^{(\alpha,\beta)}(Z) := \begin{cases} \left(X_{N_1}^{(1,0)}, X_{N_2}^{(0,1)}, V_{N_1}^{(1,0)}, v_1^{(0,1)}, \dots, (v_i^{(0,1)})^*, \dots, (v_j^{(0,1)})^*, \dots, v_{N_2}^{(0,1)} \right), & \alpha = \beta = (0, 1), \\ \left(X_{N_1}^{(1,0)}, X_{N_2}^{(0,1)}, v_1^{(1,0)}, \dots, (v_i^{(1,0)})^*, \dots, v_{N_1}^{(1,0)}, v_1^{(0,1)}, \dots, (v_j^{(0,1)})^*, \dots, v_{N_2}^{(0,1)} \right), & \alpha = (1, 0), \beta = (0, 1), \\ \left(X_{N_1}^{(1,0)}, X_{N_2}^{(0,1)}, v_1^{(1,0)}, \dots, (v_i^{(1,0)})^*, \dots, (v_j^{(1,0)})^*, \dots, v_{N_1}^{(1,0)}, V_{N_2}^{(0,1)} \right), & \alpha = \beta = (1, 0), \end{cases}$$

where the $(v_i^\alpha)^*, (v_j^\beta)^*$ are given by (2.7), (2.8). Since the sets $\Sigma_{(i,j),sc,ng}^{(\alpha,\beta)}$ are disjoint for $\alpha \leq \beta$ and $(i, j) \in \mathcal{I}^{(\alpha,\beta)}$, the above operators define an operator $T : \partial_{sc,ng}\mathcal{D} \rightarrow \partial_{sc,ng}\mathcal{D}$. For notational convenience, we will often write

$$(3.5) \quad Z^* := T(Z).$$

DEFINITION 3.3. We define the energy of the configuration $Z \in \mathcal{D}$ to be

$$(3.6) \quad E(Z) := \sum_{i=1}^{N_1} M_1 |v_i^{(1,0)}|^2 + \sum_{i=1}^{N_2} M_2 |v_i^{(0,1)}|^2.$$

We conclude this section with the following remarks.

REMARK 3.1. *It is clear that T is an involution. Moreover, it satisfies that $Z \in \partial_{sc,ng}\mathcal{D}$ is precollisional (postcollisional) if and only if $T(Z)$ is postcollisional (precollisional).*

REMARK 3.2. *One can check by a simple computation that the conservation of energy holds under the action of the operator T . That is, for every $Z \in \partial_{sc,ng}\mathcal{D}$, we have that $E(T(Z)) = E(Z)$.*

REMARK 3.3. *The operator $T : \partial_{sc,ng}\mathcal{D} \rightarrow \partial_{sc,ng}\mathcal{D}$ leaves positions invariant. That is, $x_i^\alpha \circ T = x_i^\alpha$, where here x_i^α is the projection operator given by (2.4).*

3.2. Mixed particle local flow. In this section, we construct our mixed particle flow up to the time of the first collision. First, let us define the *refined phase space*

$$(3.7) \quad \mathcal{D}^* := \partial_{sc,ng}\mathcal{D} \cup \mathring{\mathcal{D}}.$$

REMARK 3.4. *The sets $\partial_{mc}\mathcal{D}$ and $\partial_{sc,g}\mathcal{D}$ have zero Hausdorff $2d(N_1 + N_2) - 1$ measure. Hence, the set $\mathcal{D}^*$ differs from the full phase space $\mathcal{D}$ only up to a set of surface measure zero.*

To discuss the propagation of the particles in the direction of their velocities, we define the velocity propagator

$$(3.8) \quad P : \mathcal{D} \rightarrow \mathbb{R}^{2d(N_1+N_2)}, \quad P(Z) := \left(V_{N_1}^{(1,0)}, V_{N_2}^{(0,1)}, 0, 0 \right).$$

This vector $P(Z)$ allows us to keep track of the velocities of our particles and formulate the following lemma compactly.

LEMMA 3.1. *Let $N_1, N_2 \geq 2$ and $Z \in \mathcal{D}^*$, where $\mathcal{D}$ is defined in (3.7). Then, there is a time $\tau^1 = \tau_Z^1 \in (0, \infty]$ such that the function*

$$(3.9) \quad Z : [0, \tau^1] \rightarrow \mathcal{D}, \quad \text{given by} \quad Z(t) := \begin{cases} Z + tP(Z) & \text{if } Z \text{ is non- or postcollisional,} \\ T(Z) + tP \circ T(Z) & \text{if } Z \text{ is precollisional,} \end{cases}$$

with T given by Definition 3.2 satisfies the following conditions:

- $Z(t) \in \mathring{\mathcal{D}}$ for $t \in (0, \tau^1)$.
- If $\tau^1 < \infty$, then $Z(\tau^1) \in \partial\mathcal{D}$.
- If $Z \in \Sigma_{(i,j),ng}^{(\alpha,\beta)}$ for some $\alpha \leq \beta$ and $(i,j) \in \mathcal{I}^{(\alpha,\beta)}$ and $\tau^1 < \infty$, then $Z(\tau^1) \notin \Sigma_{(i,j)}^{(\alpha,\beta)}$.

Proof. This is a standard *stopping time* construction. Note that $Z(\cdot)$ given above is a well defined-function from $[0, \infty) \rightarrow \mathbb{R}^{2d(N_1+N_2)}$. In order to keep its range within the domain $\mathcal{D}$, we define

$$\tau^1 = \tau_Z^1 := \inf\{t > 0 / Z(t) \in \partial\mathcal{D}\}.$$

First note that if $\tau_Z^1 = \infty$, then $Z(t) \notin \partial\mathcal{D}$ for all times $t > 0$. So we can assume that $\tau_Z^1 < \infty$. Since $Z \in \mathcal{D}^*$, it is either an interior point of $\mathcal{D}$ or it is in the simple, nongrazing subset of the boundary. If it is an interior point, by continuity $\tau^1 > 0$ and by the definition of the stopping time we have that the conclusions of the lemma are satisfied.

Next, assume that $Z \in \Sigma_{(i,j),sc,ng}^{(\alpha,\beta)}$ for some $\alpha \leq \beta$ and index $(i,j) \in \mathcal{I}^{(\alpha,\beta)}$. We consider the two cases: where Z is postcollisional and where Z is precollisional. For

the postcollisional case, we calculate for $t > 0$

$$\begin{aligned} |x_i^\alpha + tv_i^\alpha - (x_j^\beta + tv_j^\beta)|^2 &= |x_i^\alpha - x_j^\beta + t(v_i^\alpha - v_j^\beta)|^2 \\ &\geq |x_i^\alpha - x_j^\beta|^2 + 2t\langle x_i^\alpha - x_j^\beta, v_i^\alpha - v_j^\beta \rangle \\ &> |x_i^\alpha - x_j^\beta|^2 = \epsilon_{(\alpha,\beta)}^2. \end{aligned}$$

This strict inequality shows that x_i^α does not collide with x_j^β for small, positive times t . Moreover, since we are assuming a simple collision, no other particles are colliding at time $t = 0$ apart from x_i^α, x_j^β . By using continuity, this fact combined with the strict inequality above shows that $\tau^1 > 0$. By the definition of the stopping time, the first two conditions of the lemma are also immediate. Moreover, the inequality directly above also shows that the last condition of the lemma is true.

Now, for the precollisional case, apply the above argument to the postcollisional configuration $T(Z)$. This completes the proof in all cases. $\square$

3.3. Mixed particle global flow. To define global flow, we inductively apply the above construction. We will use the notation $\tau^1 = \tau_Z^1$ as in Lemma 3.1 above, and if $Z(\tau^1) \in \partial_{sc,ng}\mathcal{D}$, then we define

$$(3.10) \quad \tau_Z^2 := \tau_{Z(\tau^1)}^1.$$

That is, τ_Z^2 is the stopping time of the process started at $Z(\tau^1)$ which is guaranteed to exist by Lemma 3.1. We then can define a process $Z(\cdot) : [0, \tau^2] \rightarrow \mathcal{D}$ given by

$$(3.11) \quad \begin{cases} Z(\cdot) : [0, \tau^1] \rightarrow \mathcal{D} & \text{on } [0, \tau^1], \\ Z(\cdot) : (\tau^1, \tau^2] \rightarrow \mathcal{D} & \text{on } (\tau^1, \tau^2]. \end{cases}$$

We wish to continue this local construction of the flow $Z(\cdot)$ for arbitrarily large times for initial configurations Z outside a set of measure zero in $\mathcal{D}$.

Now, to analyze the measure of the sets in which this inductive construction is well-defined, we introduce a time truncation parameter $\delta > 0$ and a velocity truncation parameter $R > 0$ that satisfy

$$(3.12) \quad 0 < \delta R \ll 1 \ll R.$$

Given $\rho > 0$, we define

$$(3.13) \quad B_\rho^x := \{(X_{N_1}^{(1,0)}, X_{N_2}^{(0,1)}) \in \mathbb{R}^{d(N_1+N_2)} / |(X_{N_1}^{(1,0)}, X_{N_2}^{(0,1)})| \leq \rho\}.$$

We additionally define

$$(3.14) \quad B_R^v := \{(V_{N_1}^{(1,0)}, V_{N_2}^{(0,1)}) \in \mathbb{R}^{d(N_1+N_2)} / |(V_{N_1}^{(1,0)}, V_{N_2}^{(0,1)})| \leq R\}.$$

Now, define the *truncated, refined phase space* as

$$(3.15) \quad \mathcal{D}(\rho, R) := \mathcal{D}^* \cap (B_\rho^x \times B_R^v),$$

where $\mathcal{D}^*$ is given by (3.7). We now decompose the truncated refined phase space into five parts:

$$(3.16) \quad I_{free} := \{Z \in \mathcal{D}(\rho, R) / \tau_Z^1 > \delta\},$$

$$(3.17) \quad I_{sc,ng}^1 := \{Z \in \mathcal{D}(\rho, R) / \tau_Z^1 \leq \delta, Z(\tau^1) \in \partial_{sc,ng} \mathcal{D}, \text{ and } \tau_Z^2 > \delta\},$$

$$(3.18) \quad I_{sc,ng}^2 := \{Z \in \mathcal{D}(\rho, R) / \tau_Z^1 \leq \delta, Z(\tau^1) \in \partial_{sc,ng} \mathcal{D}, \text{ and } \tau_Z^2 \leq \delta\},$$

$$(3.19) \quad I_{sc,g}^1 := \{Z \in \mathcal{D}(\rho, R) / \tau_Z^1 \leq \delta, Z(\tau^1) \in \partial_{sc,g} \mathcal{D}\},$$

$$(3.20) \quad I_{mc}^1 := \{Z \in \mathcal{D}(\rho, R) / \tau_Z^1 \leq \delta, Z(\tau^1) \in \partial_{mc} \mathcal{D}\}.$$

Clearly, we have that

$$\mathcal{D}(\rho, R) = I_{free} \cup I_{sc,ng}^1 \cup I_{sc,g}^1 \cup I_{mc}^1 \cup I_{sc,ng}^2.$$

Here, the *good* sets are I_{free} and $I_{sc,ng}^1$, for they allow us to continue our inductive construction without issue. The *bad* sets for which we seek measure estimates are $I_{sc,g}^1$, I_{mc}^1 , and $I_{sc,ng}^2$. To handle the sets $I_{sc,ng}^2$, I_{mc}^1 , we first show they have a particular covering by intersections of orthogonal annuli. In particular, for $\alpha \leq \beta$ and $(i, j) \in \mathcal{I}^{(\alpha, \beta)}$ we define the annuli to be

$$(3.21) \quad U_{(i,j)}^{(\alpha, \beta)} := \{Z \in B_\rho^x \times B_R^v / \epsilon_{(\alpha, \beta)} \leq |x_i^\alpha - x_j^\beta| \leq \epsilon_{(\alpha, \beta)} + 2C\delta R\},$$

where $C = C(M_1, M_2) > 1$ is a constant depending only on the masses of the particles. We have the following lemma.

LEMMA 3.2. *We have*

$$(3.22) \quad I_{sc,ng}^2 \cup I_{mc}^1 \subset \bigcup_{\mathcal{J}} U_{(i,j)}^{(\alpha, \beta)} \cap U_{(i',j')}^{(\alpha', \beta')},$$

where the index set $\mathcal{J}$ is defined to be the set of pairs of 4-tuples $(\alpha, \beta, i, j), (\alpha', \beta', i', j')$ such that $\alpha \leq \beta, \alpha' \leq \beta', (i, j) \in \mathcal{I}^{(\alpha, \beta)}, (i', j') \in \mathcal{I}^{(\alpha', \beta')}$, and

$$(3.23) \quad (\alpha, \beta, i, j) \neq (\alpha', \beta', i', j').$$

Proof. We prove that the set $I_{sc,ng}^2$ is contained in the union. The proof for the set I_{mc}^1 is carried out similarly.

Let $Z(\cdot) : [0, \tau^2] \rightarrow \mathcal{D}$ be our constructed flow, and assume that $Z \in I_{sc,ng}^2$. Then by definition we have $\tau^1, \tau^2 \leq \delta$, and

$$Z(\tau^1) \in \Sigma_{(i,j),sc,ng}^{(\alpha, \beta)} \quad \text{and} \quad Z(\tau^2) \in \Sigma_{(i',j')}^{(\alpha', \beta')}$$

for some $\alpha \leq \beta, \alpha' \leq \beta'$ and indices $(i, j) \in \mathcal{I}^{(\alpha, \beta)}, (i', j') \in \mathcal{I}^{(\alpha', \beta')}$. Note that by the last condition of Lemma 3.1, we *cannot* have that $(\alpha, \beta) = (\alpha', \beta')$ and $(i, j) = (i', j')$.

Recall from (2.4) the projection operators. We consider two cases.

- Assume that Z is noncollisional or postcollisional. Then we can write

$$\begin{aligned} \epsilon_{(\alpha, \beta)} &\leq |x_i^\alpha - x_j^\beta| \\ &\leq |x_i^\alpha + \tau^1 v_i^\alpha - (x_j^\beta + \tau^1 v_j^\beta)| + \tau^1 |v_i^\alpha - v_j^\beta| \\ &= |x_i^\alpha(Z(\tau^1)) - x_j^\beta(Z(\tau^1))| + \tau^1 |v_i^\alpha - v_j^\beta| \\ &= \epsilon_{(\alpha, \beta)} + \tau^1 |v_i^\alpha - v_j^\beta|. \end{aligned}$$

Noting that $|v_i^\alpha|, |v_j^\beta| \leq R$ and $\tau^1 \leq \delta$, we obtain that $\epsilon_\alpha \leq |x_i^\alpha - x_j^\beta| \leq \epsilon_{(\alpha,\beta)} + 2R\delta$.

• When Z is precollisional, apply the above argument to the precollisional $T(Z)$. Note that since $Z \in B_\rho^x \times B_R^v$, we have $T(Z) \in B_\rho^x \times B_{CR}^v$ by Remarks 3.2 and 3.3 and a comparison of $E(Z)$ with the Euclidean norm on $\mathbb{R}^{d(N_1+N_2)}$. The constant $C > 1$ above can explicitly be computed in terms of M_1, M_2 . We thus obtain $\epsilon_{(\alpha,\beta)} \leq |x_i^\alpha(T(Z)) - x_j^\beta(T(Z))| \leq \epsilon_{(\alpha,\beta)} + 2CR\delta$. Since T leaves the positions unaffected, by Remark 3.3 we have that $\epsilon_{(\alpha,\beta)} \leq |x_i^\alpha - x_j^\beta| \leq \epsilon_{(\alpha,\beta)} + 2CR\delta$ in this case also. So in all cases, $Z \in U_{(i,j)}^{(\alpha,\beta)}$.

Next, let us show that $Z \in U_{(i',j')}^{(\alpha',\beta')}$. We can calculate, by Remark 3.3,

$$\begin{aligned} \epsilon_{(\alpha',\beta')} &\leq |x_{i'}^{\alpha'} - x_{j'}^{\beta'}| \\ &= |x_{i'}^{\alpha'}(T(Z)) - x_{j'}^{\beta'}(T(Z))| \\ &\leq |x_{i'}^{\alpha'}(T(Z)) + \tau^2 v_{i'}^{\alpha'}(T(Z)) - (x_{j'}^{\beta'}(T(Z)) \\ &\quad + \tau^2 v_{j'}^{\beta'}(T(Z)))| + \tau^2 |v_{i'}^{\alpha'}(T(Z)) - v_{j'}^{\beta'}(T(Z))|. \end{aligned}$$

Now, since there are no collisions in the interval (τ^1, τ^2) , we have

$$x_{i'}^{\alpha'}(T(Z)) + \tau^2 v_{i'}^{\alpha'}(T(Z)) - (x_{j'}^{\beta'}(T(Z)) + \tau^2 v_{j'}^{\beta'}(T(Z))) = x_{i'}^{\alpha'}(T(Z(\tau^2))) - x_{j'}^{\beta'}(T(Z(\tau^2))).$$

Hence, we can conclude that

$$\begin{aligned} \epsilon_{(\alpha',\beta')} &\leq |x_{i'}^{\alpha'} - x_{j'}^{\beta'}| \\ &\leq |x_{i'}^{\alpha'}(T(Z(\tau^2))) - x_{j'}^{\beta'}(T(Z(\tau^2)))| + \tau^2 |v_{i'}^{\alpha'}(T(Z)) - v_{j'}^{\beta'}(T(Z))| \\ &= \epsilon_{(\alpha',\beta')} + \tau^2 |v_{i'}^{\alpha'}(T(Z)) - v_{j'}^{\beta'}(T(Z))|. \end{aligned}$$

Again, by Remark 3.2 and a comparison of $E(Z)$ with the Euclidean norm on $\mathbb{R}^{d(N_1+N_2)}$ we obtain that $|v_{i'}^{\alpha'}(T(Z))|, |v_{j'}^{\beta'}(T(Z))| \leq CR$. Hence, since $\tau^2 \leq \delta$, we obtain $\epsilon_{(\alpha',\beta')} \leq |x_{i'}^{\alpha'} - x_{j'}^{\beta'}| \leq \epsilon_{(\alpha',\beta')} + 2CR\delta$. This shows that $Z \in U_{(i,j)}^{(\alpha,\beta)} \cap U_{(i',j')}^{(\alpha',\beta')}$ and $I_{sc,ng}^2$ is contained in the claimed union. $\square$

REMARK 3.5. One can show that the Hausdorff measure $\mathcal{H}^{2d(N_1+N_2)}(I_{sc,g}^1) = 0$.

Next, we estimate the measure of our “bad” sets using Lemma 3.2 and Remark 3.5.

LEMMA 3.3. Assume that $0 < \epsilon_1, \epsilon_2 < 1$. We have the measure estimate

$$\mathcal{H}^{2d(N_1+N_2)}(I_{sc,g}^1 \cup I_{mc}^1 \cup I_{sc,ng}^2) \leq C(N_1, N_2, d, R) \rho^{d(N_1+N_2-2)} \delta^2.$$

Proof. By Lemma 3.2 and Remark 3.5, it suffices to uniformly estimate $U_{(i,j)}^{(\alpha,\beta)} \cap U_{(i',j')}^{(\alpha',\beta')}$ for the indices $\alpha \leq \beta$, $\alpha' \leq \beta'$, $(i, j) \in \mathcal{I}^{(\alpha,\beta)}$, and $(i', j') \in \mathcal{I}^{(\alpha',\beta')}$ such that either $(\alpha, \beta) \neq (\alpha', \beta')$ or $(i, j) \neq (i', j')$. Note that this implies we have only three possibilities for the coordinates $x_i^\alpha, x_j^\beta, x_{i'}^{\alpha'}, x_{j'}^{\beta'}$:

1. $x_i^\alpha = x_{i'}^{\alpha'}$ and the rest are distinct.
2. $x_j^\beta = x_{j'}^{\beta'}$ and the rest are distinct.
3. All coordinates are distinct.

By symmetry, cases 1 and 2 are identical, so it suffices to prove the bounds for cases 1 and 3 only.

1. Assume $x_i^\alpha = x_{i'}^{\alpha'}$ and the rest are distinct. Then, we have that

$$\int_{\mathbb{R}^{2d(N_1+N_2)}} \mathbf{1}_{U_{(i,j)}^{(\alpha,\beta)} \cap U_{(i',j')}^{(\alpha',\beta')}}(Z) dZ = |B(0, \rho)|_d^{N_1+N_2-3} |B(0, R)|_d^{N_1+N_2} \int_{B(0,\rho)^3} \mathbf{1}_{S_{(i,j,j')}^{(\alpha,\beta,\beta')}} dx_i^\alpha dx_j^\beta dx_{j'}^{\beta'},$$

where

$$S_{(i,j,j')}^{(\alpha,\beta,\beta')} := \{(x_i^\alpha, x_j^\beta, x_{j'}^{\beta'}) / \epsilon_{(\alpha,\beta)} \leq |x_i^\alpha - x_j^\beta| \leq \epsilon_{(\alpha,\beta)} + 2C\delta R, \epsilon_{(\alpha',\beta')} \leq |x_i^\alpha - x_{j'}^{\beta'}| \leq \epsilon_{(\alpha',\beta')} + 2C\delta R\}.$$

We have

$$\begin{aligned} \int_{B(0,\rho)^3} \mathbf{1}_{S_{(i,j,j')}^{(\alpha,\beta,\beta')}} dx_i^\alpha dx_j^\beta dx_{j'}^{\beta'} &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{B(0,\rho)} \mathbf{1}_{S_{(i,j,j')}^{(\alpha,\beta,\beta')}} dx_i^\alpha dx_j^\beta dx_{j'}^{\beta'} \\ &\leq C \int_{B(0,\rho)} \left((2C\delta R + \epsilon_{(\alpha,\beta)})^d - \epsilon_{(\alpha,\beta)}^d \right) \left((2C\delta R + \epsilon_{(\alpha',\beta')})^d - \epsilon_{(\alpha',\beta')}^d \right) dx. \end{aligned}$$

Since we picked δ, R such that $\delta R \ll 1$, this is bounded above by $C\rho^d R^2 \delta^2$. Hence, we get

$$\int_{\mathbb{R}^{2d(N_1+N_2)}} \mathbf{1}_{U_{(i,j)}^{\alpha} \cap U_{(i',j')}^{\alpha'}}(Z) dZ \leq C\rho^{d(N_1+N_2-2)} R^{d(N_1+N_2)+2} \delta^2.$$

This completes this case.

3. Assume that all coordinates $x_i^\alpha, x_j^\beta, x_{i'}^{\alpha'}, x_{j'}^{\beta'}$ are distinct. In this case, we have

$$\begin{aligned} &\int_{\mathbb{R}^{2d(N_1+N_2)}} \mathbf{1}_{U_{(i,j)}^{(\alpha,\beta)} \cap U_{(i',j')}^{(\alpha',\beta')}}(Z) dZ \\ &= |B(0, \rho)|_d^{N_1+N_2-4} |B(0, R)|_d^{N_1+N_2} \int_{B(0,\rho)^4} \mathbf{1}_{S_{(i,j,i',j')}^{(\alpha,\beta,\alpha',\beta')}} dx_i^\alpha dx_j^\beta dx_{i'}^{\alpha'} dx_{j'}^{\beta'}, \end{aligned}$$

where here we are defining

$$\begin{aligned} S_{(i,j)}^{(\alpha,\beta)} &:= \{(x_i^\alpha, x_j^\beta, x_{i'}^{\alpha'}, x_{j'}^{\beta'}) / \epsilon_{(\alpha,\beta)} \leq |x_i^\alpha - x_j^\beta| \leq \epsilon_{(\alpha,\beta)} + 2C\delta R\}, \\ S_{(i',j')}^{(\alpha',\beta')} &:= \{(x_i^\alpha, x_j^\beta, x_{i'}^{\alpha'}, x_{j'}^{\beta'}) / \epsilon_{(\alpha',\beta')} \leq |x_{i'}^{\alpha'} - x_{j'}^{\beta'}| \leq \epsilon_{(\alpha',\beta')} + 2C\delta R\}. \end{aligned}$$

Again, we can estimate the above integral by Fubini:

$$\begin{aligned} &\int_{B(0,\rho)^4} \mathbf{1}_{S_{(i,j)}^{(\alpha,\beta)} \cap S_{(i',j')}^{(\alpha',\beta')}} dx_i^\alpha dx_j^\beta dx_{i'}^{\alpha'} dx_{j'}^{\beta'} \\ &\leq \int_{\mathbb{R}^d} \int_{B(0,\rho)} \int_{\mathbb{R}^d} \int_{B(0,\rho)} \mathbf{1}_{S_{(i,j)}^{(\alpha,\beta)} \cap S_{(i',j')}^{(\alpha',\beta')}} dx_i^\alpha dx_j^\beta dx_{i'}^{\alpha'} dx_{j'}^{\beta'} \\ &= C\rho^d \left((2C\delta R + \epsilon_{(\alpha,\beta)})^d - \epsilon_{(\alpha,\beta)}^d \right) \rho^d \left((2C\delta R + \epsilon_{(\alpha',\beta')})^d - \epsilon_{(\alpha',\beta')}^d \right) \\ &\leq C\rho^{2d} R^2 \delta^2. \end{aligned} \quad \square$$

Hence, we obtain

$$\int_{\mathbb{R}^{2d(N_1+N_2)}} \mathbf{1}_{U_{(i,j)}^{\alpha} \cap U_{(i',j')}^{\alpha'}}(Z) dZ \leq C\rho^{d(N_1+N_2-2)} R^{d(N_1+N_2)+2} \delta^2.$$

After having established the measure estimates of Lemma 3.3, we are able to construct a global in time measure preserving flow by iteration. We will omit the proof since it follows the same arguments presented in detail in, e.g., [3, 5]. However, let us outline the main idea: First, we introduce a small time parameter δ and truncate the phase space by a large parameter R . By Lemma 3.3, outside of a small measure set (in terms of δ and R) of initial configurations, we can define a measure preserving flow in $[0, \delta]$. Then, we inductively repeat the argument and use conservation of measure by the flow to reach arbitrarily large times, outside of a small set of initial data. In the end, it can be shown that as $\delta \rightarrow 0$ and $R \rightarrow \infty$, the measure preserving flow can be defined almost everywhere in the phase space for arbitrarily large times. More specifically, we obtain the following result.

THEOREM 3.1. *Let $N_1, N_2 \in \mathbb{N}_+$ and $\epsilon_1, \epsilon_2 > 0$, and recall the definition of the energy (3.6). Then there exists a family of measure preserving maps $(\Psi_{(N_1, N_2)}^t)_{t \in \mathbb{R}} : \mathcal{D} \rightarrow \mathcal{D}$ such that for all $t \in \mathbb{R}^+$ the following hold:*

1. $\Psi_{(N_1, N_2)}^{t+s}(Z) = \Psi_{(N_1, N_2)}^t(Z) \circ \Psi_{(N_1, N_2)}^s(Z) = \Psi_{(N_1, N_2)}^s(Z) \circ \Psi_{(N_1, N_2)}^t(Z)$ for almost every $Z \in \mathcal{D}$.
2. $E(\Psi_{(N_1, N_2)}^t(Z)) = E(Z)$ for almost every $Z \in \mathcal{D}$.
3. We have $\Psi_{(N_1, N_2)}^t(T(Z)) = \Psi_{(N_1, N_2)}^t(Z)$ for $\mathcal{H}^{2d(N_1+N_2)-1}$ almost every $Z \in \partial_{sc, ng} \mathcal{D}$.

4. Local well-posedness. In this section, we show local in time well-posedness of the BBGKY hierarchy, the Boltzmann hierarchy, and the Boltzmann system for mixtures in their mild forms. The proofs for these three results are carried out in a similar spirit, and we will present it only in the BBGKY setting. We remark that throughout this section we will maintain the superscript notation $g_{\mathbf{N}}^{(s)}$ to denote sequences of functions, even though we do not assume that they come from mixed marginals of $g_{\mathbf{N}}$ as in Definition 2.1.

4.1. Well-posedness of BBGKY hierarchy. Recall that $\mathcal{T}$ is the set of types of particles, as given in (2.1). Let $\epsilon = (\epsilon_{(1,0)}, \epsilon_{(0,1)})$ record the diameters of the particles of each type, and let $\mathbf{N} = (N_{(1,0)}, N_{(0,1)})$ record the total number of particles of each type. Throughout this section, we will assume the Boltzmann-Grad scaling (2.28) holds. We first define the Maxwellian weighted spaces in which we will be working.

DEFINITION 4.1. *Recall the definition of the phase space (2.5) and the energy (3.6). Fix $\gamma > 0$, and let $\mu \in \mathbb{R}$. Define the index sets*

$$(4.1) \quad [\mathbf{N}] := \{1, \dots, N_{(1,0)}\} \times \{1, \dots, N_{(0,1)}\}$$

and

$$(4.2) \quad [\mathbf{N} - 1] := \{1, \dots, N_{(1,0)} - 1\} \times \{1, \dots, N_{(0,1)} - 1\}.$$

- For $\mathbf{s} \in [\mathbf{N}]$ and $g_{\mathbf{N}}^{(\mathbf{s})} \in L^\infty(\mathcal{D}_\epsilon^{\mathbf{s}}; \mathbb{R})$, define the Banach space

$$(4.3) \quad \begin{aligned} X_{\mathbf{s}, \epsilon, \gamma} &:= \left\{ g_{\mathbf{N}}^{(\mathbf{s})} \in L^\infty(\mathcal{D}_\epsilon^{\mathbf{s}}; \mathbb{R}) \mid |g_{\mathbf{N}}^{(\mathbf{s})}|_{\mathbf{s}, \epsilon, \gamma} < \infty \right\}, \\ \text{with } |g_{\mathbf{N}}^{(\mathbf{s})}|_{\mathbf{s}, \epsilon, \gamma} &:= \left\| e^{\gamma E(\cdot)} g_{\mathbf{N}}^{(\mathbf{s})} \right\|_{L^\infty(\mathcal{D}_\epsilon^{\mathbf{s}})}. \end{aligned}$$

- Define for $G_{\mathbf{N}} = (g_{\mathbf{N}}^{(\mathbf{s})})_{\mathbf{s} \in [\mathbf{N}]}$ with $g_{\mathbf{N}}^{(\mathbf{s})} \in L^\infty(\mathcal{D}_\epsilon^{\mathbf{s}}; \mathbb{R})$ the norm and the Banach

space

$$\mathbf{X}_{\epsilon, \gamma, \mu}^N := \left\{ G_N = (g_N^{(s)})_{s \in [N]} \mid g_N^{(s)} \in L^\infty(\mathcal{D}_\epsilon^s; \mathbb{R}), \quad \|G_N\|_{\epsilon, \gamma, \mu} < \infty \right\},$$

$$\|G_N\|_{\epsilon, \gamma, \mu} := \sup_{s \in [N]} \left(e^{\mu|s|} |g_N^{(s)}|_{s, \epsilon, \gamma} \right).$$

Let $T > 0$, $\gamma_0 > 0$, $\mu_0 \in \mathbb{R}$. Consider $\gamma : [0, T] \rightarrow (0, \infty)$, and let $\mu : [0, T] \rightarrow \mathbb{R}$ be nonincreasing functions with $\gamma(0) = \gamma_0$ and $\mu(0) = \mu_0$. Define $\mathbf{X}_{\epsilon, \gamma, \mu}^N([0, T])$ to be the set of all continuous mappings $G_N : [0, T] \rightarrow \mathbf{X}_{\epsilon, \gamma(t), \mu(t)}^N$ such that the following norm is finite:

$$\|G_N\|_{\epsilon, \gamma, \mu} := \sup_{0 \leq t \leq T} \|G_N(t)\|_{\epsilon, \gamma(t), \mu(t)}.$$

To state the mild formulation of the BBGKY hierarchy, we define for each $s \in [N]$ and $t \in \mathbb{R}$

$$(4.4) \quad T_{s, \epsilon}^t : X_{s, \epsilon, \gamma} \rightarrow X_{s, \epsilon, \gamma}, \quad \text{given by} \quad T_{s, \epsilon}^t g_N^{(s)}(Z_s) := g_N^{(s)}(\Psi_{s, \epsilon}^{-t}(Z_s)),$$

where $\Psi_{s, \epsilon}$ is the s particle flow given by Theorem 3.1. Conservation of energy and invariance of the flow under particle collisions as proven in Theorem 3.1 imply that $T_{s, \epsilon}^t$ is an isometry of $X_{s, \epsilon, \gamma}$. Also, $T_{s, \epsilon}$ is the semigroup which generates the left-hand side of (2.26). With this in mind, we define the following mild formulation of the BBGKY hierarchy.

DEFINITION 4.2. Let $T > 0$, $\gamma_0 > 0$, $\mu_0 \in \mathbb{R}$. Consider $\gamma : [0, T] \rightarrow (0, \infty)$, and let $\mu : [0, T] \rightarrow \mathbb{R}$ be nonincreasing functions with $\gamma(0) = \gamma_0$ and $\mu(0) = \mu_0$. Given $F_{N, 0} \in \mathbf{X}_{\epsilon, \gamma_0, \mu_0}^N$, we say that

$$F_N = (f_N^{(s)})_{s \in [N]} \in \mathbf{X}_{\epsilon, \gamma, \mu}^N([0, T])$$

is a mild solution to the BBGKY hierarchy with initial data $F_{N, 0}$ if for each $t \in [0, T]$, we have

$$(4.5) \quad F_N(t) = T_\epsilon^t F_{N, 0} + \sum_{\alpha, \beta \in \mathcal{T}} \int_0^t T_\epsilon^{t-\tau} \mathcal{C}_\beta^\alpha F_N(\tau) d\tau,$$

where for each $\alpha, \beta \in \mathcal{T}$, we define the operators $\mathcal{C}_\beta^\alpha, T_\epsilon^t$ by their action on each component:

$$(4.6) \quad \mathcal{C}_\beta^\alpha G_N := \left(\mathcal{C}_{s, s+\beta}^\alpha g_N^{(s+\beta)} \right)_{s \in [N]}, \quad T_\epsilon^t G_N := \left(T_{s, \epsilon}^t g_N^{(s)} \right)_{s \in [N]}.$$

Here, the operators $\mathcal{C}_{s, s+\beta}^\alpha$ are given in Definition 2.2 and $T_{s, \epsilon}^t$ is given by (4.4).

REMARK 4.1. We note that the above collision operators $\mathcal{C}_{s, s+\beta}^\alpha$ are ill-defined on L^∞ since they involve integration over a set of measure zero (the sphere $\mathbb{S}^{d-1}$). However, one can overcome this technical problem by defining the filtered BBGKY hierarchy as

$$(4.7) \quad G_N(t) = G_{N, 0} + \sum_{\alpha, \beta \in \mathcal{T}} \int_0^t T_\epsilon^{t-\tau} \mathcal{C}_\beta^\alpha T_\epsilon^\tau G_N(\tau) d\tau.$$

Although (4.7) is just a filtered hierarchy by the transport version of (4.5), it enjoys much better regularity properties, allowing us to define the collision integrals in L^∞ outside of some measure zero sets. This has been done in detail in the erratum of [17] for the derivation of the classical Boltzmann equation and can be naturally extended in the case of mixtures. The idea is that upon combining $\mathcal{C}_{\mathbf{s}, \mathbf{s}+\beta}^\alpha$ with $T_{\mathbf{s}+\beta, \epsilon}^\tau$, time can be seen as the missing coordinate in the direction orthogonal to the boundary of the phase space, and that allows the collision integral to be defined. See [28] for a different approach which avoids this issue by working with measures on the phase space. To avoid further technical complications, throughout this work, we will be using the mild formulation (4.5), assuming that the collision integrals are well-defined. However, we should note that the same estimates and results we prove hold true for the filtered hierarchy (4.7) as well.

LEMMA 4.1. Assume that we have the Boltzmann-Grad scalings (2.28). For all $\mathbf{s} = (s_{(1,0)}, s_{(0,1)}) \in [\mathbf{N} - \mathbf{1}]$, all $\alpha, \beta \in \mathcal{T}$, and all $Z_{\mathbf{s}} \in \mathcal{D}_\epsilon^{\mathbf{s}}$, we have

$$(4.8) \quad \left| \mathcal{C}_{\mathbf{s}, \mathbf{s}+\beta}^\alpha g^{(\mathbf{s}+\beta)}(Z_{\mathbf{s}}) \right| \leq C \gamma^{-d/2} \left[s_\alpha \gamma^{-1/2} + \sum_{k=1}^{s_\alpha} |v_k^\alpha| \right] e^{-\gamma E(Z_{\mathbf{s}})} |g^{(\mathbf{s}+\beta)}|_{\mathbf{s}+\beta, \epsilon, \gamma}.$$

The constant C above depends only on d, c_1, c_2, a , and b as in the Boltzmann-Grad scalings (2.28) and the masses M_1, M_2 of the particles.

Proof. Fix α, β , and $Z_{\mathbf{s}}$. Using the definition of the operator, the triangle inequality, the definition of the norm (4.3), and the conservation of energy, we obtain

$$\begin{aligned} & |\mathcal{C}_{\mathbf{s}, \mathbf{s}+\beta}^\alpha g^{(\mathbf{s}+\beta)}(Z_{\mathbf{s}})| \\ & \leq C \sum_{k=1}^{s_\alpha} \int_{\mathbb{R}^d} (|v_{s_\beta+1}^\beta| + |v_k^\alpha|) e^{-\gamma E(Z_{\mathbf{s}})} e^{-\gamma M_\beta |v_{s_\beta+1}^\beta|^2} |g^{(\mathbf{s}+\beta)}|_{\mathbf{s}+\beta, \epsilon, \gamma} dv_{s_\beta+1}^\beta \\ & = C e^{-\gamma E(Z_{\mathbf{s}})} |g^{(\mathbf{s}+\beta)}|_{\mathbf{s}+\beta, \epsilon, \gamma} \\ & \quad \times \left[s_\alpha \int_{\mathbb{R}^d} |v_{s_\beta+1}^\beta| e^{-\gamma M_\beta |v_{s_\beta+1}^\beta|^2} dv_{s_\beta+1}^\beta + \sum_{k=1}^{s_\alpha} |v_k^\alpha| \int_{\mathbb{R}^d} e^{-\gamma M_\beta |v_{s_\beta+1}^\beta|^2} dv_{s_\beta+1}^\beta \right]. \end{aligned}$$

Computing these Gaussian integrals results in the desired bounds. $\square$

LEMMA 4.2. Fix $\mathbf{N} \in \mathbb{N}_+^2$ and $\epsilon \in (0, \infty)^2$ to agree with the mixed Boltzmann-Grad scaling (2.28). Let $\gamma_0 > 0$, $\mu_0 \in \mathbb{R}$, and $T, \lambda > 0$ such that $T\lambda < \gamma$. Then, the following bound holds on $\mathbf{X}_{\epsilon, \gamma, \mu}^{\mathbf{N}}([0, T])$:

$$(4.9) \quad \left\| \int_0^t \mathbf{T}_\epsilon^{t-\tau} \mathcal{C}_\beta^\alpha G_{\mathbf{N}}(\tau) d\tau \right\|_{\epsilon, \gamma, \mu} \leq c \|G_{\mathbf{N}}\|_{\epsilon, \gamma, \mu} \quad \text{for all } \alpha, \beta \in \mathcal{T},$$

where $\gamma(t) = \gamma_0 - \lambda t$, $\mu(t) = \mu_0 - \lambda t$, and $c = c(d, c_1, c_2, a, b, M_1, M_2, \gamma_0, \mu_0, \lambda, T)$. We can pick $T > 0$ and a $\lambda \in (0, \gamma_0/T)$ independent of $\mathbf{N}, \epsilon$ such that $c < 1/8$.

Proof. This is a standard argument which follows from carefully lifting the estimate in Lemma 4.1 to the space $\mathbf{X}_{\epsilon, \gamma(t), \mu(t)}^{\mathbf{N}}$ for each $t > 0$ and then estimating a time integral. Details can be found in [17]. $\square$

THEOREM 4.1. Fix $\mathbf{N} \in \mathbb{N}_+^2$ and $\epsilon \in (0, \infty)^2$ to agree with the Boltzmann-Grad scaling (2.28). Let $\mu_0 \in \mathbb{R}$ and $\gamma_0 > 0$ be given. Then there exists $T, \lambda > 0$ independent of $\mathbf{N}, \epsilon$ such that for any $F_{\mathbf{N}, 0} \in X_{\epsilon, \gamma_0, \mu_0}^{\mathbf{N}}$ we have a unique mild solution $F_{\mathbf{N}} \in$

$\mathbf{X}_{\epsilon, \gamma, \mu}^N([0, T])$ to the BBGKY hierarchy (4.5) with $\gamma(t) = \gamma_0 - \lambda t$ and $\mu(t) = \mu_0 - \lambda t$. Moreover, this unique solution satisfies

$$(4.10) \quad \|F_N\|_{\epsilon, \gamma, \mu} \leq 2\|F_{N,0}\|_{\epsilon, \gamma_0, \mu_0}.$$

Additionally, for any $G_N \in \mathbf{X}_{\epsilon, \gamma, \mu}^N([0, T])$, we have

$$(4.11) \quad \left\| \int_0^t \mathbf{T}_\epsilon^{t-\tau} \mathbf{C}_\beta^\alpha G_N(\tau) d\tau \right\|_{\epsilon, \gamma, \mu} \leq 1/8 \|G_N\|_{\epsilon, \gamma, \mu} \quad \text{for all } \alpha, \beta \in \mathcal{T}.$$

Proof. The proof follows by applying a fixed point argument which can be done thanks to Lemma 4.2. $\square$

4.2. Well-posedness of the Boltzmann hierarchy. This subsection is devoted to proving the local well-posedness of the Boltzmann hierarchy. The estimates and proofs essentially mirror those of the previous subsection 4.1, with appropriate adjustment of the functional spaces. We begin by introducing the relevant Maxwellian weighted spaces.

DEFINITION 4.3. Recall the definition (3.6) of the energy $E(\cdot)$. Fix $\gamma > 0$, and let $\mu \in \mathbb{R}$.

- For $\mathbf{s} \in \mathbb{N}_+^2$ and $g^{(\mathbf{s})} \in L^\infty(\mathbb{R}^{d|\mathbf{s}|}; \mathbb{R})$, define the Banach space

$$(4.12) \quad \begin{aligned} |g^{(\mathbf{s})}|_{\mathbf{s}, 0, \gamma} &:= \left\| e^{\gamma E(\cdot)} g^{(\mathbf{s})} \right\|_{L^\infty(\mathbb{R}^{d|\mathbf{s}|})}, \\ X_{\mathbf{s}, 0, \gamma} &:= \left\{ g^{(\mathbf{s})} \in L^\infty(\mathbb{R}^{d|\mathbf{s}|}; \mathbb{R}) \mid |g^{(\mathbf{s})}|_{\mathbf{s}, 0, \gamma} < \infty \right\}. \end{aligned}$$

- Define for $G = (g^{(\mathbf{s})})_{\mathbf{s} \in \mathbb{N}_+^2}$ with $g^{(\mathbf{s})} \in L^\infty(\mathbb{R}^{d|\mathbf{s}|}; \mathbb{R})$ the Banach space

$$(4.13) \quad \mathbf{X}_{0, \gamma, \mu}^\infty := \left\{ G = (g^{(\mathbf{s})})_{\mathbf{s} \in \mathbb{N}_+^2} \mid g^{(\mathbf{s})} \in L^\infty(\mathbb{R}^{d|\mathbf{s}|}; \mathbb{R}), \quad \|G\|_{0, \gamma, \mu} < \infty \right\},$$

$$(4.14) \quad \|G\|_{0, \gamma, \mu} := \sup_{\mathbf{s} \in \mathbb{N}_+^2} \left(e^{\mu|\mathbf{s}|} |g^{(\mathbf{s})}|_{\mathbf{s}, 0, \gamma} \right).$$

- Let $T > 0$, $\gamma_0 > 0$, $\mu_0 \in \mathbb{R}$. Consider $\gamma : [0, T] \rightarrow (0, \infty)$, and let $\mu : [0, T] \rightarrow \mathbb{R}$ be nonincreasing functions with $\gamma(0) = \gamma_0$ and $\mu(0) = \mu_0$. Define $\mathbf{X}_{0, \gamma, \mu}^\infty([0, T])$ to be the set of all continuous mappings $G : [0, T] \rightarrow \mathbf{X}_{0, \gamma(t), \mu(t)}^\infty$ such that the following norm is finite:

$$\|G\|_{0, \gamma, \mu} := \sup_{0 \leq t \leq T} \|G(t)\|_{0, \gamma(t), \mu(t)}.$$

In order to state the mild formulation of the Boltzmann hierarchy, we define for each $\mathbf{s} \in \mathbb{N}_+^2$ and $t \in \mathbb{R}$ the free flow propagator

$$(4.15) \quad S_{\mathbf{s}}^t : X_{\mathbf{s}, 0, \gamma} \rightarrow X_{\mathbf{s}, 0, \gamma}, \quad S_{\mathbf{s}}^t g^{(\mathbf{s})}(Z_{\mathbf{s}}) := g^{(\mathbf{s})}(\Phi_{\mathbf{s}}^{-t}(Z_{\mathbf{s}})),$$

where $\Phi_{\mathbf{s}}$ is the $\mathbf{s}$ particle free flow given by

$$(4.16) \quad \Phi_{\mathbf{s}}^t(X_{\mathbf{s}}, V_{\mathbf{s}}) = (X_{\mathbf{s}} + tV_{\mathbf{s}}, V_{\mathbf{s}}).$$

It follows directly from the definitions that $S_{\mathbf{s}}$ is an isometry of $X_{\mathbf{s}, 0, \gamma}$. Moreover, one can check that $S_{\mathbf{s}}$ is the semigroup whose generator is the left-hand side of (2.36). With these definitions in hand, we define a solution to the Boltzmann hierarchy as follows.

DEFINITION 4.4. Let $T > 0$, $\gamma_0 > 0$, $\mu_0 \in \mathbb{R}$. Consider $\gamma : [0, T] \rightarrow (0, \infty)$, and let $\mu : [0, T] \rightarrow \mathbb{R}$ be nonincreasing functions with $\gamma(0) = \gamma_0$ and $\mu(0) = \mu_0$. Given $F_0 \in \mathbf{X}_{0, \gamma_0, \mu_0}^\infty$, we say that

$$F = \left(f^{(s)} \right)_{s \in \mathbb{N}_+^2} \in \mathbf{X}_{0, \gamma, \mu}^\infty([0, T])$$

is a mild solution to the Boltzmann hierarchy with initial data F_0 if for each $t \in [0, T]$ we have

$$(4.17) \quad F(t) = \mathbf{S}^t F_0 + \sum_{\alpha, \beta \in \mathcal{T}} \int_0^t \mathbf{S}^{t-\tau} \mathfrak{C}_\beta^\alpha F(\tau) d\tau,$$

where for each $\alpha, \beta \in \mathcal{T}$ we define the operators $\mathfrak{C}_\beta^\alpha, \mathbf{S}^t$ by their action on each component:

$$(4.18) \quad \mathfrak{C}_\beta^\alpha G := \left(\mathcal{C}_{s, s+\beta}^\alpha g^{(s+\beta)} \right)_{s \in \mathbb{N}_+^2}, \quad \mathbf{S}^t G := \left(S_s^t g^{(s)} \right)_{s \in \mathbb{N}_+^2}.$$

Here, the operators $\mathcal{C}_{s, s+\beta}^\alpha$ are given in (2.32) and S_s is given by (4.15).

REMARK 4.2. As in Remark 4.1, the operators $\mathcal{C}_{s, s+\beta}^\alpha$ are ill-defined on L^∞ (here, one could consider continuous functions, though). In the same spirit as in the BBGKY hierarchy case, one can define the filtered Boltzmann hierarchy as

$$(4.19) \quad G(t) = G_0 + \sum_{\alpha, \beta \in \mathcal{T}} \int_0^t \mathbf{S}^{-\tau} \mathfrak{C}_\beta^\alpha \mathbf{S}^\tau G(\tau) d\tau$$

and make sense of the collision integrals in L^∞ . To avoid further technical complications, throughout this work, we will be using the mild formulation (4.17), assuming that the collision integrals are well-defined. However, we should note that the same estimates and results we prove hold true for the filtered hierarchy (4.19) as well.

THEOREM 4.2. Let $\mu_0 \in \mathbb{R}$ and $\gamma_0 > 0$ be given. Then there exists $T, \lambda > 0$ such that for any $F_0 \in \mathbf{X}_{0, \gamma_0, \mu_0}^\infty$ we have a unique mild solution $F \in \mathbf{X}_{0, \gamma, \mu}^\infty([0, T])$ to the Boltzmann hierarchy (4.17) with $\gamma(t) = \gamma_0 - \lambda t$ and $\mu(t) = \mu_0 - \lambda t$. Moreover, this unique solution satisfies

$$(4.20) \quad \|F\|_{0, \gamma, \mu} \leq 2 \|F_0\|_{0, \gamma_0, \mu_0}.$$

Additionally, for any $G \in \mathbf{X}_{0, \gamma, \mu}^\infty([0, T])$ we have

$$(4.21) \quad \left\| \int_0^t \mathbf{S}^{t-\tau} \mathfrak{C}_\beta^\alpha G(\tau) d\tau \right\|_{0, \gamma, \mu} \leq 1/8 \|G\|_{0, \gamma, \mu} \quad \text{for all } \alpha, \beta \in \mathcal{T}.$$

4.3. Well-posedness of the Boltzmann equation for mixtures. Recall from section 2.4 the Boltzmann system for mixtures given by (2.38). Recall also that the set of types $\mathcal{T}$ is given by (2.1). We begin our analysis by defining the appropriate function spaces.

DEFINITION 4.5. Let $\gamma > 0$ and $\mu \in \mathbb{R}$. Define for each $\alpha \in \mathcal{T}$ the one particle space

$$X_{\alpha, \gamma, \mu} := \{g_\alpha \mid g_\alpha \in L^\infty(\mathbb{R}^{2d}), |g_\alpha|_{\alpha, \gamma, \mu} < \infty\},$$

$$|g_\alpha|_{\alpha, \gamma, \mu} = \left\| e^{\mu + \gamma M_\alpha |v|^2} g_\alpha(x, v) \right\|_{L^\infty(\mathbb{R}^{2d})},$$

where M_α is the mass of the type α particle. We also define the two particle space given by $X_{\gamma,\mu} := \prod_{\alpha \in \mathcal{T}} X_{\alpha,\gamma,\mu}$ with the induced ℓ^1 product norm $|G|_{\gamma,\mu} := \sum_{\alpha \in \mathcal{T}} |g_\alpha|_{\alpha,\gamma,\mu}$, where $G = (g_\alpha)_{\alpha \in \mathcal{T}}$.

Given $T > 0$ and nonincreasing functions $\gamma : [0, T] \rightarrow \mathbb{R}_+$, $\mu : [0, T] \rightarrow \mathbb{R}$, we define

$$X_{\gamma,\mu}([0, T]) := \{G = (g_\alpha)_{\alpha \in \mathcal{T}} \mid g_\alpha \in \mathcal{C}^0([0, T]; L^\infty(\mathbb{R}^{2d})) \text{ and } \|G\|_{\gamma,\mu} < \infty\},$$

where the norm is given by

$$\|G\|_{\gamma,\mu} := \sup_{t \in [0, T]} |G(t, \cdot)|_{\gamma(t), \mu(t)}.$$

It is clear that the space $(X_{\gamma,\mu}([0, T]), \|\cdot\|_{\gamma,\mu})$ is a complete metric space. We will study the local well-posedness of the Boltzmann equation for mixtures on the above spaces. We begin with defining our notion of solution. First, fix $T > 0$, $\gamma_0 > 0$, and $\mu_0 \in \mathbb{R}$ and let $\gamma : [0, T] \rightarrow \mathbb{R}_+$, $\mu : [0, T] \rightarrow \mathbb{R}$ be nonincreasing functions with $\gamma(0) = \gamma_0$, $\mu(0) = \mu_0$. Recalling the constants (2.35) and collision operators given in Definition 2.4, introduce the nonlinear operator $\mathcal{N}$ on $X_{\gamma,\mu}([0, T])$ for each component $\alpha \in \mathcal{T}$ by

$$[\mathcal{N}(G)]_\alpha := \sum_{\beta \in \mathcal{T}} A_\beta^\alpha Q_\beta^\alpha(g_\alpha, g_\beta), \quad \text{where } G = (g_\alpha)_{\alpha \in \mathcal{T}}.$$

Recall the free flow operator $S_{(1,1)}^t$ from (4.15). In this section, we will set $S^t = S_{(1,1)}^t$ to reduce the notation. As in the case for the BBGKY and Boltzmann hierarchies, we consider mild formulations.

DEFINITION 4.6. Let $T > 0$, $\gamma_0 > 0$, $\mu_0 \in \mathbb{R}$. Consider $\gamma : [0, T] \rightarrow (0, \infty)$, and let $\mu : [0, T] \rightarrow \mathbb{R}$ be nonincreasing functions with $\gamma(0) = \gamma_0$ and $\mu(0) = \mu_0$. Given $F_0 = (f_{\alpha,0})_{\alpha \in \mathcal{T}} \in X_{\gamma_0,\mu_0}$, we say that $F = (f_\alpha)_{\alpha \in \mathcal{T}} \in X_{\gamma,\mu}([0, T])$ is a solution of the Boltzmann equation with initial data $F_0 = (f_{\alpha,0})_{\alpha \in \mathcal{T}}$ if

$$(4.22) \quad F(t) = S^t F_0 + \int_0^t S^{t-\tau} \mathcal{N} F(\tau) d\tau \quad \text{for every } t \in [0, T].$$

REMARK 4.3. As in Remarks 4.1 and 4.2, the operator $\mathcal{N}$ is ill-defined on L^∞ (one could consider continuous functions, though). In the same spirit as in the BBGKY and Boltzmann hierarchies, one can define the filtered Boltzmann equation as

$$(4.23) \quad G(t) = G_0 + \int_0^t S^{-\tau} \mathcal{N} S^\tau G(\tau) d\tau$$

and make sense of the collision integrals in L^∞ . To avoid technical complications, throughout this work, we will be using the mild form (4.22), assuming that the collision integrals are well-defined. However, we should note that the same estimates and results we prove hold true for the filtered equation (4.23) as well.

As in the previous subsections, we first prove some estimates on the nonlinearity $\mathcal{N}$ and use this to set up a contraction mapping.

LEMMA 4.3. Fix $\gamma > 0$, $\mu \in \mathbb{R}$, and $G = (g_\alpha)_{\alpha \in \mathcal{T}}$, $G' = (g'_\alpha)_{\alpha \in \mathcal{T}} \in X_{\gamma,\mu}$. Then, we obtain the pointwise estimates for all $x, v \in \mathbb{R}^d$:

$$\begin{aligned} |Q_\beta^\alpha(g_\alpha, g_\beta) - Q_\beta^\alpha(g'_\alpha, g'_\beta)|(x, v) &\leq C e^{-2\mu - \gamma(M_\alpha + M_\beta)|v|^2/2} \gamma^{-d/2} (\gamma^{-1/2} + |v|) \\ &\times \begin{cases} (|g_\alpha|_{\alpha,\gamma,\mu} + |g'_\alpha|_{\alpha,\gamma,\mu}) |g_\alpha - g'_\alpha|_{\alpha,\gamma,\mu}, & \alpha = \beta, \\ |g_\alpha|_{\alpha,\gamma,\mu} |g_\beta - g'_\beta|_{\beta,\gamma,\mu} + |g'_\beta|_{\beta,\gamma,\mu} |g_\alpha - g'_\alpha|_{\alpha,\gamma,\mu}, & \alpha \neq \beta. \end{cases} \end{aligned}$$

Here, the constant C depends only on the dimension d , the masses M_α, M_β , and the constants c_1, c_2, a, b in the Boltzmann-Grad scaling (2.28). Also note that these inequalities are invariant under the action of S^t .

Proof. This follows from applications of the triangle inequality and the conservation of energy and momentum for the collision laws stated in (2.32). $\square$

THEOREM 4.3. *Let $\gamma_0 > 0$ and $\mu_0 \in \mathbb{R}$ be given. Then there exists $T, \lambda > 0$ depending only on γ_0, μ_0 and the Boltzmann-Grad scalings as in (2.28) such that given any $F_0 = (f_{\alpha,0})_{\alpha \in \mathcal{T}} \in X_{\gamma_0, \mu_0}$ with $|F_0|_{\gamma_0, \mu_0} \leq 1$ there exists a unique solution $F \in X_{\gamma, \mu}([0, T])$ of (4.22) that satisfies*

$$\|F\|_{\gamma, \mu} \leq 2|F_0|_{\gamma_0, \mu_0},$$

with $\gamma(t) = \gamma_0 - \lambda t$ and $\mu(t) = \mu_0 - \lambda t$.

Proof. This proof follows again from setting up a contraction mapping using Lemma 4.3. $\square$

REMARK 4.4. *The local time of existence T and the constant $\lambda > 0$ in Theorems 4.1, 4.2, and 4.3 can all be taken to be the same, which we assume throughout the rest of the paper. Crucially, they only depend on the parameters γ_0, μ_0 , the masses M_1, M_2 , the Boltzmann-Grad scaling (2.28), the dimension d , and universal constants.*

5. Statement of the main theorem. In this section, we define the appropriate notion of convergence and state the main theorem of the paper. Throughout, we will be using the convention that $\mathbf{N} = (N_1, N_2)$ and $\epsilon = (\epsilon_1, \epsilon_2)$. Moreover, $\mathbf{N}$ and ϵ are related by the Boltzmann-Grad scalings (2.28). We now give some notation and define approximate Boltzmann initial data.

5.1. Approximation of Boltzmann initial data.

DEFINITION 5.1 (joint limit in N_1, N_2). *We say a doubly indexed sequence of real numbers $(A_{\mathbf{N}})_{\mathbf{N} \in \mathbb{N}_+^2}$ converges to a real number A with respect to the mixed Boltzmann-Grad scalings if the following condition holds: For every $\zeta > 0$, there exists $N_1^*, N_2^* \in \mathbb{N}$ such that for all $N_i \geq N_i^*$, $i = 1, 2$, which satisfy the scalings (2.30), we have*

$$|A_{\mathbf{N}} - A| < \zeta.$$

We denote this type of convergence as $\lim_{\mathbf{N} \rightarrow \infty} A_{\mathbf{N}} = A$ or $A_{\mathbf{N}} \rightarrow A$.

For each $\theta > 0$ and $\mathbf{m} \in \mathbb{N}_+^2$, we define the set of θ -well separated configurations as

$$(5.1) \quad \Delta_{\mathbf{m}}(\theta) := \{Z_{\mathbf{m}} : \forall \alpha, \beta \in \mathcal{T}, \forall (i, j) \in \mathcal{I}_{\mathbf{m}}^{(\alpha, \beta)}, |x_i^\alpha - x_j^\beta| > \theta\},$$

where we recall the definition of the impact operator T given in Definition 3.2 and the set $\mathcal{I}_{(i,j)}^{(\alpha, \beta)}$ given by (2.2). We also define the spatial components of this set as

$$(5.2) \quad \Delta_{\mathbf{m}}^X(\theta) := \{X_{\mathbf{m}} : \forall \alpha, \beta \in \mathcal{T}, \forall (i, j) \in \mathcal{I}_{\mathbf{m}}^{(\alpha, \beta)}, |x_i^\alpha - x_j^\beta| > \theta\}.$$

In analogy with [7], we introduce an approximating sequence for the BBGKY initial data.

DEFINITION 5.2. *Let $\mu_0 \in \mathbb{R}$ and $\gamma_0 > 0$. Given $F_0 = (f_0^{(\mathbf{s})}) \in \mathbf{X}_{0, \gamma_0, \mu_0}^\infty$, we say a sequence $F_{\mathbf{N}, 0} = (f_{\mathbf{N}, 0}^{(\mathbf{s})})_{\mathbf{s} \in [\mathbf{N}]}$ is a BBGKY hierarchy sequence approximating F_0 if the following conditions hold:*

- We have $\sup_{\mathbf{N}} \|F_{\mathbf{N},0}\|_{\mathbf{X}_{\epsilon,\gamma_0,\mu_0}} < \infty$, where the supremum is taken over $(\mathbf{N}, \epsilon)$ obeying the mixed Boltzmann-Grad scalings (2.28).
- We have for each $\mathbf{s} \in \mathbb{N}_+^2$ and $\theta > 0$ that $\lim_{\mathbf{N} \rightarrow \infty} \|f_0^{(\mathbf{s})} - f_{\mathbf{N},0}^{(\mathbf{s})}\|_{L^\infty(\Delta_{\mathbf{s}}(\theta))} = 0$.

REMARK 5.1. For any initial data $F_0 \in \mathbf{X}_{0,\gamma_0,\mu_0}^\infty$, there always exists at least one approximating BBGKY hierarchy sequence, although it may not be a probability measure. A simple example of such a sequence is given by $f_{\mathbf{N},0}^{(\mathbf{s})} = \mathbb{1}_{\Delta_{\mathbf{s}}(\max(\epsilon_1, \epsilon_2))} f_0^{(\mathbf{s})}$. The uniform upper bound for $F_{\mathbf{N},0} = (f_{\mathbf{N},0}^{(\mathbf{s})})_{\mathbf{s} \in [\mathbf{N}]}$ follows by the definition of the function spaces, and the convergence follows from the fact that $\max(\epsilon_1, \epsilon_2) \rightarrow 0$ as $\mathbf{N} \rightarrow \infty$ in the mixed Boltzmann-Grad scaling (2.28).

REMARK 5.2. Consider initial data $u_0 = (f_{0,\alpha})_{\alpha \in \mathcal{T}} \in X_{\gamma_0,\mu_0+1}$ for the Boltzmann equation for mixtures with $\|f_0\|_{\gamma_0,\mu_0+1} \leq 1/2$. Assume that $\int_{\mathbb{R}^{2d}} f_{0,\alpha}(x,v) dx dv = 1$ and $f_{0,\alpha} > 0$ almost everywhere for each $\alpha \in \mathcal{T}$. Define

$$F_0 := (f_0^{(\mathbf{s})})_{\mathbf{s} \in \mathbb{N}^2}, \quad f_0^{(\mathbf{s})} := f_{0,(1,0)}^{\otimes s_1} \otimes f_{0,(0,1)}^{\otimes s_2}.$$

This tensor corresponds to an initially chaotic configuration of the Boltzmann hierarchy. From this data, we may form the conditioned BBGKY initial data defined by

$$\begin{aligned} f_{\mathbf{N},0}^{(\mathbf{s})}(Z_{\mathbf{s}}) &:= Z_{\mathbf{N}}^{-1} \int_{\mathbb{R}^{2d|\mathbf{N}-\mathbf{s}|}} \mathbb{1}_{\mathcal{D}_{\epsilon}^{\mathbf{N}}} f_{0,(1,0)}^{\otimes N_1} \otimes f_{0,(0,1)}^{\otimes N_2}(Z_{\mathbf{N}}) dx_{s_1+1}^{(1,0)} dv_{s_1+1}^{(1,0)} \\ (5.3) \quad &\dots dx_{N_1}^{(1,0)} dv_{N_1}^{(1,0)} dx_{s_2+1}^{(0,1)} dv_{s_2+1}^{(0,1)} \dots dx_{N_2}^{(0,1)} dv_{N_2}^{(0,1)}, \end{aligned}$$

where $Z_{\mathbf{N}}$ is a normalization factor which makes $f_{\mathbf{N},0}^{(\mathbf{s})}$ a probability measure given by

$$Z_{\mathbf{N}} := \int_{\mathcal{D}_{\epsilon}^{\mathbf{N}}} f_{0,(1,0)}^{\otimes N_1} \otimes f_{0,(0,1)}^{\otimes N_2}(Z_{\mathbf{N}}) dZ_{\mathbf{N}}.$$

This sequence $F_{\mathbf{N},0} := (f_{\mathbf{N},0}^{(\mathbf{s})})_{\mathbf{s} \in [\mathbf{N}]}$ is a BBGKY hierarchy sequence approximating F_0 . In fact, it can be shown (see Chapter 6 of [17]) that we get explicit rates for any $\mathbf{s} \in \mathbb{N}^2$ and $\theta > 0$:

$$(5.4) \quad \|f_{\mathbf{N},0}^{(\mathbf{s})} - f_0^{(\mathbf{s})}\|_{L^\infty(\Delta_{\mathbf{s}}(\theta))} \leq C|\mathbf{s}| \max(\epsilon_1, \epsilon_2)^d |\mathbf{N}| \|F_0\|_{0,\gamma_0,\mu_0} \leq C_{\mathbf{s}} \max(\epsilon_1, \epsilon_2) \|F_0\|_{0,\gamma_0,\mu_0},$$

where in the last inequality we have crucially used the scalings (2.28).

5.2. Convergence in observables. In this subsection, we define the notion of convergence of observables. As usual, we let $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ and denote the space of test functions $\mathcal{C}_c(\mathbb{R}^{d|\mathbf{s}|})$ to be the space of continuous compactly supported functions.

DEFINITION 5.3. Let $T > 0$, $\mathbf{s} \in \mathbb{N}_+^2$, and $f^{(\mathbf{s})} \in L^\infty([0, T]; L^\infty(\mathbb{R}^{2d|\mathbf{s}|}))$. Given a test function $\phi_{\mathbf{s}} \in \mathcal{C}_c(\mathbb{R}^{d|\mathbf{s}|})$, we define the $\mathbf{s}$ -observable function as

$$I_{\phi_{\mathbf{s}}}(f^{(\mathbf{s})})(t, X_{\mathbf{s}}) := \int_{\mathbb{R}^{d|\mathbf{s}|}} \phi_{\mathbf{s}}(V_{\mathbf{s}}) f^{(\mathbf{s})}(t, X_{\mathbf{s}}, V_{\mathbf{s}}) dV_{\mathbf{s}}.$$

With this definition in hand, we can now define the notion of convergence in observables.

DEFINITION 5.4 (convergence in observables). Let $T > 0$, $\gamma > 0$, and $\mu \in \mathbb{R}$. Given a sequence $(F_{\mathbf{N}})_{\mathbf{N} \in \mathbb{N}_+^2}$ such that

$$F_{\mathbf{N}} = (f_{\mathbf{N}}^{(\mathbf{s})})_{\mathbf{s} \in [\mathbf{N}]} \in \mathbf{X}_{\epsilon,\gamma,\mu}^{\mathbf{N}} \quad \text{and} \quad F = (f^{(\mathbf{s})})_{\mathbf{s} \in \mathbb{N}_+^2} \in \mathbf{X}_{0,\gamma,\mu}^\infty,$$

we say that F_N converges to F if for any $\mathbf{s} \in \mathbb{N}_+^2$, any $\theta > 0$, and any $\phi_{\mathbf{s}} \in \mathcal{C}_c(\mathbb{R}^{d|\mathbf{s}|})$ we have

$$\lim_{N \rightarrow \infty} \|I_{\phi_{\mathbf{s}}}(f_N^{(\mathbf{s})})(t) - I_{\phi_{\mathbf{s}}}(f^{(\mathbf{s})})(t)\|_{L^\infty(\Delta_{\mathbf{s}}^X(\theta))} = 0 \quad \text{uniformly in } [0, T].$$

We denote this type of convergence by $F_N \xrightarrow{\sim} F$.

REMARK 5.3. We note that the above convergence of the functionals in $L^\infty(\Delta_{\mathbf{s}}^X(\sigma))$ for each $\sigma > 0$ fixed implies locally uniform convergence on $\mathbb{R}^{d|\mathbf{s}|} \setminus \{X_{\mathbf{s}} \in \mathbb{R}^{d|\mathbf{s}|} : \forall i \neq j, x_i \neq x_j\}$.

5.3. Statement of the main theorem. We are now ready to state the main results of this paper. We start with the most general theorem.

THEOREM 5.1. Let $\gamma_0 > 0$, let $\mu_0 \in \mathbb{R}$, and let $T > 0$ be the existence time for the BBGKY and Boltzmann hierarchies as found in Theorems 4.1 and 4.2. Let $F_0 = (f_0^{(\mathbf{s})})_{\mathbf{s} \in \mathbb{N}_+^2} \in \mathbf{X}_{0, \gamma_0, \mu_0}^\infty$ be Boltzmann initial data and $(F_{N,0})_{N \in \mathbb{N}_+^2}$ be an approximating BBGKY hierarchy sequence as given in Definition 5.2. Assume the following:

- For each $N \in \mathbb{N}_+^2$, $F_N \in \mathbf{X}_{\epsilon, \gamma, \mu}^N([0, T])$ is a solution to the BBGKY hierarchy with initial data $F_{N,0}$ (as in Theorem 4.1). Here, ϵ is related to N by (2.28), and the functions $\gamma, \mu : [0, T] \rightarrow \mathbb{R}$ are given as in Theorem 4.1.
- $F \in \mathbf{X}_{0, \gamma, \mu}^\infty([0, T])$ solves the Boltzmann hierarchy (as in Theorem 4.2), where $\gamma, \mu : [0, T] \rightarrow \mathbb{R}$ are given as in Theorem 4.2.
- The initial data F_0 satisfies the following uniform continuity condition: There exists a constant C such that for all $\zeta > 0$ there exists a $q = q(\zeta)$ such that for all $\mathbf{s} \in \mathbb{N}_+^2$ and $Z_{\mathbf{s}}, Z'_{\mathbf{s}} \in \mathbb{R}^{2d|\mathbf{s}|}$ with $|Z_{\mathbf{s}} - Z'_{\mathbf{s}}| < q(\zeta)$ we have

$$(5.5) \quad |f_0^{(\mathbf{s})}(Z_{\mathbf{s}}) - f_0^{(\mathbf{s})}(Z'_{\mathbf{s}})| < C^{|\mathbf{s}|-1}\zeta.$$

Then, F_N converges to F in the sense of observables.

REMARK 5.4. Using Definition 5.4, the convergence in observables of Theorem 5.1 is equivalent to showing that for every $\theta > 0$, $\mathbf{s} \in \mathbb{N}_+^2$, and every $\phi_{\mathbf{s}} \in \mathcal{C}_c(\mathbb{R}^{d|\mathbf{s}|})$ we have

$$\lim_{N \rightarrow \infty} \|I_{\mathbf{s}}^\infty - I_{\mathbf{s}}^N\|_{L^\infty(\Delta^X(\theta))} = 0 \quad \text{uniformly in } [0, T],$$

where we define

$$(5.6) \quad I_{\mathbf{s}}^N(X_{\mathbf{s}}) := \int_{\mathbb{R}^{d|\mathbf{s}|}} \phi_{\mathbf{s}}(V_{\mathbf{s}}) f_N^{(\mathbf{s})}(X_{\mathbf{s}}, V_{\mathbf{s}}) dV_{\mathbf{s}}, \quad I_{\mathbf{s}}^\infty(X_{\mathbf{s}}) := \int_{\mathbb{R}^{d|\mathbf{s}|}} \phi_{\mathbf{s}}(V_{\mathbf{s}}) f^{(\mathbf{s})}(X_{\mathbf{s}}, V_{\mathbf{s}}) dV_{\mathbf{s}}.$$

REMARK 5.5. The condition (5.5) is a uniform continuity condition on the initial data with uniform growth as the number of particles becomes large. Indeed, it can be easily seen by induction that in the special case of tensorized Hölder continuous initial data, i.e., $F_0 := (g_0^{\otimes s_1} \otimes h_0^{\otimes s_2})_{s_1, s_2 \in \mathbb{N}_+}$, where $g_0, h_0 \in C^{0, \lambda}$, $0 < \lambda \leq 1$, condition (5.5) is satisfied. As we will see in Theorem 5.2, in the case of Hölder continuous, tensorized initial data, we obtain explicit rates of convergence as well.

REMARK 5.6. The convergence in observables above can be upgraded to another type of weak convergence by a density argument. In particular, we can show using Theorem 5.1 and the continuity of the BBGKY and Boltzmann hierarchy solution

mappings that for any ϕ in the exponentially weighted space $L^1_\omega(\mathbb{R}^{2d|s|})$ with $\omega(V_s) = e^{-\gamma(T)E(Z_s)}$

$$(5.7) \quad \int_{\mathbb{R}^{2d|s|}} \phi f_N^{(s)} dX_s dV_s \rightarrow \int_{\mathbb{R}^{2d|s|}} \phi f^{(s)} dX_s dV_s$$

uniformly for all $t \in [0, T]$. Here we are extending $f_N^{(s)}$ to be zero outside of $\mathcal{D}_\epsilon^N$.

In statistical mechanics [25], the integrals (5.7) above correspond to the expected value of the function ϕ of the system. For example, if $A \subset \mathbb{R}^{2d}$ is measurable and $|A| < \infty$, define the function $\phi(X_{(1,1)}, V_{(1,1)}) = \mathbb{1}_A(X_{(1,1)})v_1^{(1,0)}$. The integral of ϕ against $f_N^{(1,1)}$ corresponds to the mean velocity of the type $(1, 0)$ particle in the region A of space.

We end this section with the theorem showing the propagation of chaos and the relation between the finite BBGKY hierarchy and solutions of the Boltzmann equation for mixtures.

THEOREM 5.2. Let $\gamma_0 > 0$, let $\mu_0 \in \mathbb{R}$, and let $T > 0$ be the existence time for the BBGKY hierarchy (4.5), the Boltzmann hierarchy (4.17), and the Boltzmann equation for mixtures (4.22) obtained in Theorems 4.1–4.3. Additionally, define γ, μ to be as in those theorems. Also let $u_0 = (g_0, h_0) \in X_{\gamma_0, \mu_0+1}$ be Hölder $\mathcal{C}^{0, \lambda}$ initial data with $0 < \lambda \leq 1$ and $|u_0|_{\gamma_0, \mu_0+1} \leq 1/2$. Let $(g, h) \in \mathbf{X}_{\gamma, \mu}([0, T])$ be the unique solution to the Boltzmann equation (4.22) as given by Theorem 4.3. Define

$$F_0 := (g_0^{\otimes s_1} \otimes h_0^{\otimes s_2})_{s_1, s_2 \in \mathbb{N}_+} \quad \text{and} \quad F := (g^{\otimes s_1} \otimes h^{\otimes s_2})_{s_1, s_2 \in \mathbb{N}_+}.$$

Then, $F_0 \in \mathbf{X}_{0, \gamma_0, \mu_0}^\infty$, $F \in \mathbf{X}_{\gamma, \mu}^\infty([0, T])$, and F solves the Boltzmann hierarchy with initial data F_0 . Moreover, if $(F_{N,0})_{N \in \mathbb{N}_+^2}$ is the conditioned BBGKY hierarchy initial data as given by (5.3) and $F_N = (f_N^{(s)})_{s \in [N]} \in \mathbf{X}_{\epsilon, \gamma, \mu}^N([0, T])$ is the unique solution to the BBGKY hierarchy with initial data $F_{N,0}$, then we have explicit rates of convergence for each $\phi \in \mathcal{C}_c(\mathbb{R}^{d|s|})$ given by

$$(5.8) \quad \|I_\phi(f_N^{(s)})(t) - I_\phi(g^{\otimes s_1} \otimes h^{\otimes s_2})(t)\|_{L^\infty(\Delta_s^X(\sigma))} = O(\epsilon^r), \quad \epsilon := \max_{\alpha \in \mathcal{T}} \epsilon_\alpha$$

for any $r < \lambda$ and uniformly in $t \in [0, T]$.

6. Reduction to term-by-term convergence.

6.1. Initial expansion and notation. In this section, we show that solutions to the BBGKY and Boltzmann hierarchies can be expanded into a series depending only on initial data. In the BBGKY hierarchy, this series expansion terminates. In the Boltzmann hierarchy, the series expansion converges in the sense of observables. For notational convenience, we will continue to use our convention that $\mathbf{N} = (N_{(1,0)}, N_{(0,1)}) = (N_1, N_2)$ in addition to $\mathbf{s} = (s_{(1,0)}, s_{(0,1)})$. We note that the combinatorial complexity of this series expansion motivated us to introduce the vector notation for particle numbers and particle types, which in turn simplifies the record-keeping and proofs. In particular, the notation allows the quartic tree generated by the above series expansions to be dealt with.

Recall that $F_N = (f_N^{(s)})_{s \in [N]}$ with initial data $F_{N,0} = (f_{N,0}^{(s)})_{s \in [N]}$ (where $[N]$ is given by (4.1)) solves the BBGKY hierarchy (4.5) if it satisfies, for every $\mathbf{s} \in [N]$,

$$f_N^{(\mathbf{s})}(t) = T_{\mathbf{s}, \epsilon}^t f_{N,0}^{(\mathbf{s})} + \sum_{\alpha_1, \beta_1 \in \mathcal{T}} \int_0^t T_{\mathbf{s}, \epsilon}^{t-\tau_1} C_{\mathbf{s}, \mathbf{s}+\beta_1}^{\alpha_1} f_N^{(\mathbf{s}+\beta_1)}(\tau_1) d\tau_1.$$

Here, we recall that $T_{\mathbf{s},\epsilon}$ is the particle flow operator given in (4.4), $\mathcal{T}$ is the set of types given in (2.1), and $\mathcal{C}_{\mathbf{s},\mathbf{s}+\beta}^\alpha$ is the collisional operator given in (2.2). Using this representation for $f_{\mathbf{N}}^{(\mathbf{s}+\beta)}$ for each $\beta \in \mathcal{T}$, we may expand the above expression in terms of the initial data $f_{\mathbf{N},0}^{(\mathbf{s}+\beta)}$ and a time dependent remainder. For clarity, we do this for the first term below:

$$\begin{aligned} f_{\mathbf{N}}^{(\mathbf{s})}(t) &= T_{\mathbf{s},\epsilon}^t f_{\mathbf{N},0}^{(\mathbf{s})} + \sum_{\alpha,\beta \in \mathcal{T}} \int_0^t T_{\mathbf{s},\epsilon}^{t-\tau} \mathcal{C}_{\mathbf{s},\mathbf{s}+\beta}^\alpha T_{\mathbf{s}+\beta,\epsilon}^\tau f_{\mathbf{N},0}^{(\mathbf{s}+\beta)} d\tau \\ &\quad + \sum_{\alpha_1,\beta_1 \in \mathcal{T}} \sum_{\alpha_2,\beta_2 \in \mathcal{T}} \int_0^t \int_0^{\tau_1} T_{\mathbf{s},\epsilon}^{t-\tau_1} \mathcal{C}_{\mathbf{s},\mathbf{s}+\beta_1}^{\alpha_1} T_{\mathbf{s}+\beta_1,\epsilon}^{\tau_1-\tau_2} \mathcal{C}_{\mathbf{s}+\beta_1,\mathbf{s}+\beta_1+\beta_2}^{\alpha_2} f_{\mathbf{N}}^{(\mathbf{s}+\beta_1+\beta_2)}(\tau_2) d\tau_2 d\tau_1, \end{aligned}$$

where we have used the continuity and linearity of the operators to commute the integrals and operators. This can be simplified notationally. Let us define

$$(6.1) \quad S_k := \{\boldsymbol{\beta} = (\beta_1, \dots, \beta_k) \mid \forall i, \beta_i \in \mathcal{T}\}.$$

Then, for each $1 \leq l \leq k$ and $\boldsymbol{\beta} \in S_k$, define the quantities

$$(6.2) \quad \tilde{\boldsymbol{\beta}}_l := \sum_{i=1}^l \beta_i \in \mathbb{N}^2 \quad \text{and} \quad \tilde{\boldsymbol{\beta}}_l = (\tilde{\beta}_l^{(1,0)}, \tilde{\beta}_l^{(0,1)}).$$

Also define the sequence of times

$$(6.3) \quad \mathcal{T}_k(t) := \{(t_1, t_2, \dots, t_k) \in \mathbb{R}_+^k : 0 \leq t_k \leq \dots \leq t_1 \leq t\}.$$

Now, for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k$ introduce the function

$$\begin{aligned} f_{\mathbf{N},(\boldsymbol{\alpha},\boldsymbol{\beta})}^{(\mathbf{s})}(t) &:= \int_{\mathcal{T}_k(t)} T_{\mathbf{s},\epsilon}^{t-\tau_1} \mathcal{C}_{\mathbf{s},\mathbf{s}+\tilde{\boldsymbol{\beta}}_1}^{\alpha_1} T_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_1,\epsilon}^{\tau_1-\tau_2} \mathcal{C}_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_1,\mathbf{s}+\tilde{\boldsymbol{\beta}}_2}^{\alpha_2} \\ (6.4) \quad &\dots T_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{k-1},\epsilon}^{\tau_{k-1}-\tau_k} \mathcal{C}_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{k-1},\mathbf{s}+\tilde{\boldsymbol{\beta}}_k}^{\alpha_k} T_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_k,\epsilon}^{\tau_k} f_{\mathbf{N},0}^{(\mathbf{s}+\tilde{\boldsymbol{\beta}}_k)} d\tau_k \dots d\tau_1. \end{aligned}$$

For $k=0$, define $S_0 := \{(0,0)\}$ and write for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_0$

$$(6.5) \quad f_{\mathbf{N},(\boldsymbol{\alpha},\boldsymbol{\beta})}^{(\mathbf{s})}(t) := T_{\mathbf{s},\epsilon}^t f_{\mathbf{N},0}^{(\mathbf{s})}.$$

These are the terms in the expansion of $f_{\mathbf{N}}^{(\mathbf{s})}$ which only depend on initial data. The rest of the terms are considered as a remainder, which we define as

$$\begin{aligned} R_{\mathbf{N},(\boldsymbol{\alpha},\boldsymbol{\beta})}^{(\mathbf{s})}(t) &:= \int_{\mathcal{T}_k(t)} T_{\mathbf{s},\epsilon}^{t-\tau_1} \mathcal{C}_{\mathbf{s},\mathbf{s}+\tilde{\boldsymbol{\beta}}_1}^{\alpha_1} T_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_1,\epsilon}^{\tau_1-\tau_2} \mathcal{C}_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_1,\mathbf{s}+\tilde{\boldsymbol{\beta}}_2}^{\alpha_2} \\ &\dots T_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{k-1},\epsilon}^{\tau_{k-1}-\tau_k} \mathcal{C}_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{k-1},\mathbf{s}+\tilde{\boldsymbol{\beta}}_k}^{\alpha_k} f_{\mathbf{N}}^{(\mathbf{s}+\tilde{\boldsymbol{\beta}}_k)}(\tau_k) d\tau_k \dots d\tau_1. \end{aligned}$$

By induction, the solution $F_{\mathbf{N}} = (f_{\mathbf{N}}^{(\mathbf{s})})_{\mathbf{s} \in [N]}$ of the BBGKY hierarchy (4.5) can be written as

$$(6.6) \quad f_{\mathbf{N}}^{(\mathbf{s})}(t) = \sum_{k=0}^n \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k} f_{\mathbf{N},(\boldsymbol{\alpha},\boldsymbol{\beta})}^{(\mathbf{s})}(t) + \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_{n+1}} R_{\mathbf{N},(\boldsymbol{\alpha},\boldsymbol{\beta})}^{(\mathbf{s})}(t), \quad \text{where } n < \min_{\alpha \in \mathcal{T}} (N_\alpha - s_\alpha).$$

We can proceed in a similar manner for the Boltzmann hierarchy (4.17), where $F = (f^{(s)})_{s \in \mathbb{N}_+^2}$ is a solution to the Boltzmann hierarchy. Define the following expression for $\alpha, \beta \in S_k$:

$$(6.7) \quad \begin{aligned} f_{(\alpha, \beta)}^{(s)}(t) &:= \int_{\mathcal{T}_k(t)} S_s^{t-\tau_1} \mathcal{C}_{s, s+\tilde{\beta}_1}^{\alpha_1} S_{s+\tilde{\beta}_1}^{\tau_1-\tau_2} \mathcal{C}_{s+\tilde{\beta}_1, s+\tilde{\beta}_2}^{\alpha_2} \\ &\dots S_{s+\tilde{\beta}_{k-1}}^{\tau_{k-1}-\tau_k} \mathcal{C}_{s+\tilde{\beta}_{k-1}, s+\tilde{\beta}_k}^{\alpha_k} S_{s+\tilde{\beta}_k}^{\tau_k} f_0^{(s+\tilde{\beta}_k)} d\tau_k \dots d\tau_1. \end{aligned}$$

Here, we recall that S_s^t is free transport as given in (4.15), and $\mathcal{C}_{s, s+\beta}^\alpha$ is the collision operator as given in (2.32). As above, we define for $\alpha, \beta \in S_0$

$$(6.8) \quad f_{(\alpha, \beta)}^{(s)}(t) := S_s^t f_0^{(s)}.$$

Next, define the remainder term:

$$\begin{aligned} R_{(\alpha, \beta)}^{(s)}(t) &:= \int_{\mathcal{T}_k(t)} S_s^{t-\tau_1} \mathcal{C}_{s, s+\tilde{\beta}_1}^{\alpha_1} S_{s+\tilde{\beta}_1}^{\tau_1-\tau_2} \mathcal{C}_{s+\tilde{\beta}_1, s+\tilde{\beta}_2}^{\alpha_2} \\ &\dots S_{s+\tilde{\beta}_{k-1}}^{\tau_{k-1}-\tau_k} \mathcal{C}_{s+\tilde{\beta}_{k-1}, s+\tilde{\beta}_k}^{\alpha_k} f^{(s+\tilde{\beta}_k)}(\tau_k) d\tau_k \dots d\tau_1. \end{aligned}$$

Then, as above, we have for any $n \in \mathbb{N}_+$

$$(6.9) \quad \boxed{f^{(s)}(t) = \sum_{k=0}^n \sum_{\alpha, \beta \in S_k} f_{(\alpha, \beta)}^{(s)}(t) + \sum_{\alpha, \beta \in S_{n+1}} R_{(\alpha, \beta)}^{(s)}(t).}$$

6.2. Reduction to finitely many terms. Here, we reduce the convergence proof to term by term convergence of terms with bounded energy and separated collision times. Recalling (3.6), given $R > 0$, $\ell \in \mathbb{N}_+^2$, and $\alpha, \beta \in \mathcal{T}$ types as given in (2.1), we define the energy truncated operators

$$(6.10) \quad \begin{aligned} \mathcal{C}_{\ell, \ell+\beta, R}^\alpha g_N^{(\ell+\beta)} &:= \mathcal{C}_{\ell, \ell+\beta}^\alpha \left(g_N^{(\ell+\beta)} \mathbf{1}_{[E \leq B_R^{d|\ell+\beta|}]} \right), \\ \mathcal{C}_{\ell, \ell+\beta, R}^\alpha g^{(\ell+\beta)} &:= \mathcal{C}_{\ell, \ell+\beta}^\alpha \left(g^{(\ell+\beta)} \mathbf{1}_{[E \leq B_R^{d|\ell+\beta|}]} \right). \end{aligned}$$

Consider $\delta > 0$. Given $t \geq 0$ and $k \in \mathbb{N}$, we define the separated collision times

$$(6.11) \quad \mathcal{T}_{k, \delta}(t) := \{(t_1, \dots, t_k) \in \mathcal{T}_k(t) : 0 \leq t_{i+1} \leq t_i - \delta \quad \forall i \in [0, k]\}, \quad t_{k+1} := 0, \quad t_0 := t.$$

For the BBGKY hierarchy, we define for $k \in \mathbb{N}_+$ and $\alpha, \beta \in S_k$

$$(6.12) \quad \begin{aligned} f_{N, (\alpha, \beta), R, \delta}^{(s)}(t) &:= \int_{\mathcal{T}_{k, \delta}(t)} T_{s, \epsilon}^{t-\tau_1} \mathcal{C}_{s, s+\tilde{\beta}_1, R}^{\alpha_1} T_{s+\tilde{\beta}_1, \epsilon}^{\tau_1-\tau_2} \mathcal{C}_{s+\tilde{\beta}_1, s+\tilde{\beta}_2, R}^{\alpha_2} \\ &\dots T_{s+\tilde{\beta}_{k-1}, \epsilon}^{\tau_{k-1}-\tau_k} \mathcal{C}_{s+\tilde{\beta}_{k-1}, s+\tilde{\beta}_k, R}^{\alpha_k} T_{s+\tilde{\beta}_k, \epsilon}^{\tau_k} f_{N, 0}^{(s+\tilde{\beta}_k)} d\tau_k \dots d\tau_1, \end{aligned}$$

and for $k = 0$ we define $f_{N, (0, 0), R, \delta}^{(s)}(t, Z_s) := T_s^t (f_{N, 0}^{(s)} \mathbf{1}_{[E \leq B_R^{d|s|}]}) (Z_s)$.

For the Boltzmann hierarchy, we define for $k \in \mathbb{N}_+$ and $\alpha, \beta \in S_k$

$$(6.13) \quad \begin{aligned} f_{(\alpha, \beta), R, \delta}^{(s)}(t) &:= \int_{\mathcal{T}_{k, \delta}(t)} S_s^{t-\tau_1} \mathcal{C}_{s, s+\tilde{\beta}_1, R}^{\alpha_1} S_{s+\tilde{\beta}_1}^{\tau_1-\tau_2} \mathcal{C}_{s+\tilde{\beta}_1, s+\tilde{\beta}_2, R}^{\alpha_2} \\ &\dots S_{s+\tilde{\beta}_{k-1}}^{\tau_{k-1}-\tau_k} \mathcal{C}_{s+\tilde{\beta}_{k-1}, s+\tilde{\beta}_k, R}^{\alpha_k} S_{s+\tilde{\beta}_k}^{\tau_k} f_0^{(s+\tilde{\beta}_k)} d\tau_k \dots d\tau_1, \end{aligned}$$

and for $k = 0$ we define $f_{(0,0),R,\delta}^{(s)}(t, Z_s) := S_s^t(f_0 \mathbb{1}_{[E \leq B_R^{d|s|}]})(Z_s)$.

Given $\phi_s \in C_c(\mathbb{R}^{d|s|})$, $k \in \mathbb{N}$, and $\alpha, \beta \in S_k$, let us write

$$(6.14) \quad I_{s,k,R,\delta}^N(t)(X_s) := \sum_{\alpha, \beta \in S_k} \int_{E \leq B_R^{d|s|}} \phi_s(V_s) f_{N,(\alpha,\beta),R,\delta}^{(s)}(t, X_s, V_s) dV_s,$$

$$(6.15) \quad I_{s,k,R,\delta}^\infty(t)(X_s) := \sum_{\alpha, \beta \in S_k} \int_{E \leq B_R^{d|s|}} \phi_s(V_s) f_{(\alpha,\beta),R,\delta}^{(s)}(t, X_s, V_s) dV_s.$$

The following estimates show that the observables I_s^N , I_s^∞ defined in (5.6) can be approximated by the truncated observables (6.14), (6.15) for small δ and large n and R .

PROPOSITION 6.1. *For any $s \in \mathbb{N}_+^2$, $n \in \mathbb{N}$, $R > 1$, $\delta > 0$, and $t \in [0, T]$, the following estimates hold:*

$$\left\| I_s^N(t) - \sum_{k=0}^n I_{s,k,R,\delta}^N(t) \right\|_{L_{X_s}^\infty} \leq C_{s,\gamma_0,\mu_0,T} \|\phi_s\|_{L_{V_s}^\infty} \left(2^{-n} + e^{-\frac{\gamma_0}{3} R^2} + \delta C_{d,s,\gamma_0,\mu_0,T}^n \right) \|F_{N,0}\|_{N,\gamma_0,\mu_0},$$

$$\left\| I_s^\infty(t) - \sum_{k=0}^n I_{s,k,R,\delta}^\infty(t) \right\|_{L_{X_s}^\infty} \leq C_{s,\gamma_0,\mu_0,T} \|\phi_s\|_{L_{V_s}^\infty} \left(2^{-n} + e^{-\frac{\gamma_0}{3} R^2} + \delta C_{d,s,\gamma_0,\mu_0,T}^n \right) \|F_0\|_{\infty,\gamma_0,\mu_0}.$$

Proof. For the proof, one needs to use the a priori bounds of section 4 to perform successive reductions to finitely many terms, bounded energies, and separated collision times, respectively, and connect these estimates via the triangle inequality. The proof is similar to the corresponding reductions in the binary case [17]. For more details on the strategy of the proof, see [5], where related reductions were made for the case of ternary interactions. $\square$

7. Good configurations and stability.

7.1. Construction of good sets and notation. Let $\mathbf{m} = (m_1, m_2) \in \mathbb{N}_+^2$ be the number of particles of each type, let $\epsilon = (\epsilon_1, \epsilon_2) \in (0, \infty)^2$ be their diameters, and recall from section 2.1 that we denote the vectors of all positions and velocities by

$$X_{\mathbf{m}} = \left(X_{m_1}^{(1,0)}, X_{m_2}^{(0,1)} \right) \quad \text{and} \quad V_{\mathbf{m}} = \left(V_{m_1}^{(1,0)}, V_{m_2}^{(0,1)} \right).$$

The full phase space vector is similarly given by $Z_{\mathbf{m}} := (X_{\mathbf{m}}, V_{\mathbf{m}})$. For each $\theta > 0$ and $\mathbf{m} \in \mathbb{N}_+^2$, recall from (5.1)–(5.2) the set of θ -well separated configurations

$$\Delta_{\mathbf{m}}(\theta) := \{Z_{\mathbf{m}} : \forall \alpha, \beta \in \mathcal{T}, \forall (i, j) \in \mathcal{I}_{\mathbf{m}}^{(\alpha,\beta)}, |x_i^\alpha - x_j^\beta| > \theta\}$$

and its spatial components

$$\Delta_{\mathbf{m}}^X(\theta) := \{X_{\mathbf{m}} : \forall \alpha, \beta \in \mathcal{T}, \forall (i, j) \in \mathcal{I}_{\mathbf{m}}^{(\alpha,\beta)}, |x_i^\alpha - x_j^\beta| > \theta\}.$$

Now, let $Z_{\mathbf{m}}(t)$ be the backwards particle flow given by

$$(7.1) \quad Z_{\mathbf{m}}(t) = \Psi_{\mathbf{m},\epsilon}^{-t}(Z_{\mathbf{m}}),$$

where $\Psi_{\mathbf{m},\epsilon}$ is given in Theorem 3.1. Define a (θ, t_0) -good configuration to be

$$(7.2) \quad G_{\mathbf{m}}(\theta, t_0) := \{Z_{\mathbf{m}} : Z_{\mathbf{m}}(t) \in \Delta_{\mathbf{m}}(\theta) \forall t \geq t_0\}.$$

Define additionally

$$(7.3) \quad (\mathbb{S}_1^{d-1} \times B_R^d)^+(v) := \{(\omega_1, v_1) \in \mathbb{S}_1^{d-1} \times B_R^d : \omega_1 \cdot (v_1 - v) > 0\}.$$

The “plus” is meant to indicate postcollisional configurations. Now, as in [26, 17], we wish to exclude trajectories on which recollisions occur in the backwards flow. The strategy is to construct a “bad set” whose complement is exactly the set of initial configurations of an adjoined particle which do not run into collisions under backwards flow.

We will now fix parameters $\gamma, \epsilon_0, R, \eta, \delta$ (to be chosen later) which are related by

$$(7.4) \quad \max(\epsilon_1, \epsilon_2) \ll \gamma \ll \epsilon_0 \ll \eta\delta, \quad R\gamma \ll \eta\epsilon_0.$$

PROPOSITION 7.1. Fix $\mathbf{m} = (m_{(1,0)}, m_{(0,1)}) \in \mathbb{N}_+^2$, and recall that $\mathcal{T}$ is the set given by (2.1). Let $\bar{\mathbf{Z}}_{\mathbf{m}} = (\bar{X}_{\mathbf{m}}, \bar{V}_{\mathbf{m}}) \in G_{\mathbf{m}}(\epsilon_0, 0)$, and assume that $E(\bar{\mathbf{Z}}_{\mathbf{m}}) \leq R^2$, where $E(\cdot)$ is given by (3.6). For each $\alpha \in \mathcal{T}$ and $\ell \in \{1, \dots, m_\alpha\}$, there exists a bad set $\mathcal{B}_{\ell, \alpha}(\bar{\mathbf{Z}}_{\mathbf{m}}) \subset (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell^\alpha)$ such that for any $X_{\mathbf{m}} \in B_{\gamma/2}^d(\bar{X}_{\mathbf{m}})$ we have the following:

1. For all $\beta \in \mathcal{T}$ and $(\omega_1, v_{m_\beta+1}^\beta) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell^\alpha) \setminus \mathcal{B}_{\ell, \alpha}(\bar{\mathbf{Z}}_{\mathbf{m}})$, we have

- (a) $Z_{\mathbf{m}+\beta}(t) \in \dot{\mathcal{D}}_{\epsilon}^{\mathbf{m}+\beta}$ for all $t \geq 0$,
- (b) $Z_{\mathbf{m}+\beta} \in G_{\mathbf{m}+\beta}(\epsilon_0/2, \delta)$, and
- (c) $\bar{Z}_{\mathbf{m}+\beta} \in G_{\mathbf{m}+\beta}(\epsilon_0, \delta)$,

where we have defined $Z_{\mathbf{m}+\beta}(t)$ via (7.1) and

$$Z_{\mathbf{m}+\beta} := \begin{cases} (X_{m_{(1,0)}}^{(1,0)}, x_{m_{(1,0)}+1}^{(1,0)}, X_{m_{(0,1)}}^{(0,1)}, \bar{V}_{m_{(1,0)}}^{(1,0)}, v_{m_{(1,0)}+1}^{(1,0)}, \bar{V}_{m_{(0,1)}}^{(0,1)}), & \beta = (1, 0), \\ (X_{m_{(1,0)}}^{(1,0)}, X_{m_{(0,1)}}^{(0,1)}, x_{m_{(0,1)}+1}^{(0,1)}, \bar{V}_{m_{(1,0)}}^{(1,0)}, \bar{V}_{m_{(0,1)}}^{(0,1)}, v_{m_{(0,1)}+1}^{(0,1)}), & \beta = (0, 1), \end{cases}$$

$$\bar{Z}_{\mathbf{m}+\beta} := \begin{cases} (\bar{X}_{m_{(1,0)}}^{(1,0)}, x_{m_{(1,0)}+1}^{(1,0)}, \bar{X}_{m_{(0,1)}}^{(0,1)}, \bar{V}_{m_{(1,0)}}^{(1,0)}, v_{m_{(1,0)}+1}^{(1,0)}, \bar{V}_{m_{(0,1)}}^{(0,1)}), & \beta = (1, 0), \\ (\bar{X}_{m_{(1,0)}}^{(1,0)}, \bar{X}_{m_{(0,1)}}^{(0,1)}, x_{m_{(0,1)}+1}^{(0,1)}, \bar{V}_{m_{(1,0)}}^{(1,0)}, \bar{V}_{m_{(0,1)}}^{(0,1)}, v_{m_{(0,1)}+1}^{(0,1)}), & \beta = (0, 1), \end{cases}$$

$$x_{m_\beta+1}^\beta = x_\ell^\alpha - \epsilon_{(\alpha, \beta)} \omega_1.$$

2. For all $\beta \in \mathcal{T}$ and $(\omega_1, v_{m_\beta+1}^\beta) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell^\alpha) \setminus \mathcal{B}_{\ell, \alpha}(\bar{\mathbf{Z}}_{\mathbf{m}})$, we have

- (a) $Z_{\mathbf{m}+\beta}^*(t) \in \dot{\mathcal{D}}_{\epsilon}^{\mathbf{m}+\beta}$ for all $t \geq 0$,
- (b) $Z_{\mathbf{m}+\beta}^* \in G_{\mathbf{m}+\beta}(\epsilon_0/2, \delta)$, and
- (c) $\bar{Z}_{\mathbf{m}+\beta}^* \in G_{\mathbf{m}+\beta}(\epsilon_0, \delta)$,

where we have defined the backwards flow via (7.1), $Z_{\mathbf{m}+\beta}^* := T(Z_{\mathbf{m}+\beta}^+)$,

$\bar{Z}_{\mathbf{m}+\beta}^* := T(\bar{Z}_{\mathbf{m}+\beta}^+)$, where T is given by Definition 3.2, and

$$Z_{\mathbf{m}+\beta}^+ := \begin{cases} (X_{m_{(1,0)}}^{(1,0)}, (x_{m_{(1,0)}+1}^{(1,0)})^+, X_{m_{(0,1)}}^{(0,1)}, \bar{V}_{m_{(1,0)}}^{(1,0)}, v_{m_{(1,0)}+1}^{(1,0)}, \bar{V}_{m_{(0,1)}}^{(0,1)}), & \beta = (1, 0), \\ (X_{m_{(1,0)}}^{(1,0)}, X_{m_{(0,1)}}^{(0,1)}, (x_{m_{(0,1)}+1}^{(0,1)})^+, \bar{V}_{m_{(1,0)}}^{(1,0)}, \bar{V}_{m_{(0,1)}}^{(0,1)}, v_{m_{(0,1)}+1}^{(0,1)}), & \beta = (0, 1), \end{cases}$$

$$\bar{Z}_{\mathbf{m}+\beta}^+ := \begin{cases} (\bar{X}_{m_{(1,0)}}^{(1,0)}, (x_{m_{(1,0)}+1}^{(1,0)})^+, \bar{X}_{m_{(0,1)}}^{(0,1)}, \bar{V}_{m_{(1,0)}}^{(1,0)}, v_{m_{(1,0)}+1}^{(1,0)}, \bar{V}_{m_{(0,1)}}^{(0,1)}), & \beta = (1, 0), \\ (\bar{X}_{m_{(1,0)}}^{(1,0)}, \bar{X}_{m_{(0,1)}}^{(0,1)}, (x_{m_{(0,1)}+1}^{(0,1)})^+, \bar{V}_{m_{(1,0)}}^{(1,0)}, \bar{V}_{m_{(0,1)}}^{(0,1)}, v_{m_{(0,1)}+1}^{(0,1)}), & \beta = (0, 1), \end{cases}$$

$$(x_{m_\beta+1}^\beta)^+ = x_\ell^\alpha + \epsilon_{(\alpha, \beta)} \omega_1.$$

We now require a few lemmas in order to prove this proposition. The first lemma shows that in the simple case of two particles, one can exclude a small cylinder to obtain a noncollisional trajectory.

LEMMA 7.1. *Let the parameters $\gamma, \epsilon_0, R, \eta, \delta$ be related by (7.4). Let $\bar{y}_1, \bar{y}_2 \in \mathbb{R}^d$ with $|\bar{y}_1 - \bar{y}_2| > \epsilon_0$. Also let $v_1 \in B_R^d$. Then there exists a d -dimensional cylinder K_η^d such that for any $v_2 \in B_R^d \setminus K_\eta^d$, $y_1 \in B_\gamma^d(\bar{y}_1)$, and $y_2 \in B_\gamma^d(\bar{y}_2)$ we have the following:*

1. *For all $t \geq 0$, $|(y_1 - tv_1) - (y_2 - tv_2)| \geq \epsilon_0/2$.*
2. *For all $t \geq \delta$, $|(y_1 - tv_1) - (y_2 - tv_2)| \geq \epsilon_0$.*

Proof. The proof of this lemma can be found in Chapter 12 of [17]. $\square$

For the next few lemmas, we construct a “bad set” such that outside of this set, the new particle is noncollisional for its whole trajectory. We compare each pair of particles considering three cases:

1. Compare $z_i^\sigma, z_j^{\sigma'}$ for $i \in \{1, \dots, m_\sigma\}$ and $j \in \{1, \dots, m_{\sigma'}\}$. We will call this the *existing particle case*.
2. Compare the adjoined particle $z_{m_\beta+1}^\beta$ with z_i^σ for $(i, \sigma) \neq (\ell, \alpha)$. We will call this the *existing particle and adjoined particle case*.
3. Compare the adjoined particle $z_{m_\beta+1}^\beta$ with the particle z_ℓ^α which is adjoined close to. This case is the only case which uses the collisional laws, and hence we call it the *collisional law case*.

For the first lemma, we find a bad set $\mathcal{B}_{\ell, \alpha}^{0, -}(\bar{Z}_m)$ such that for all time $t \geq 0$, the initial configurations in the complement do not encounter collisions.

LEMMA 7.2. *Let us be in the same scenario as in Proposition 7.1. Then, there exists a set $\mathcal{B}_{\ell, \alpha}^{0, -}(\bar{Z}_m)$ such that for all $(\omega_1, v_{m_\beta+1}^\beta) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+ (\bar{v}_\ell^\alpha) \setminus \mathcal{B}_{\ell, \alpha}^{0, -}(\bar{Z}_m)$, we have $Z_{m+\beta}(t) \in \dot{\mathcal{D}}_\epsilon^{m+\beta}$ for all $t \geq 0$.*

Proof. This proof follows by applying the same arguments as found in [17, 6, 7] to the three cases above.

1. (*Existing Particle Case*) We first show that for $\sigma, \sigma' \in \mathcal{T}$, $i \in \{1, \dots, m_\sigma\}$, and $j \in \{1, \dots, m_{\sigma'}\}$, we have $|x_i^\sigma(t) - x_j^{\sigma'}(t)| > \epsilon_{(\sigma, \sigma')}$ for all $t \geq 0$. Since $\bar{Z}_m \in G_m(\epsilon_0, 0)$, we have by definition that

$$(7.5) \quad |\bar{x}_i^\sigma(t) - \bar{x}_j^{\sigma'}(t)| > \epsilon_0 \quad \forall t \geq 0.$$

Therefore, by the reverse triangle inequality

$$(7.6) \quad \begin{aligned} |x_i^\sigma(t) - x_j^{\sigma'}(t)| &= |x_i^\sigma - x_j^{\sigma'} - t(\bar{v}_i^\sigma - \bar{v}_j^{\sigma'})| \\ &\geq |\bar{x}_i^\sigma - \bar{x}_j^{\sigma'} - t(\bar{v}_i^\sigma - \bar{v}_j^{\sigma'})| - |\bar{x}_i^\sigma - \bar{x}_j^{\sigma'} - (x_i^\sigma - x_j^{\sigma'})| \\ &\geq \epsilon_0 - \gamma \geq \epsilon_{(\sigma, \sigma')}. \end{aligned}$$

We have used in the first inequality in (7.6) equation (7.5) and the fact that $X_m \in B_{\gamma/2}^{d|m|}(\bar{X}_m)$. For the second inequality in (7.6), we have used that $\epsilon_{(\sigma, \sigma')} \ll \gamma \ll \epsilon_0$ by the scaling (7.4).

2. (*Existing Particle and Adjoined Particle Case*) Next, we show that for all $\sigma \in \mathcal{T}$ and $i \in \{1, \dots, m_\sigma\}$ with $(i, \sigma) \neq (\ell, \alpha)$ we have $|x_i^\sigma(t) - x_{m_\beta+1}^\beta(t)| > \epsilon_{(\sigma, \beta)}$ for $(\omega_1, v_{m_\beta+1}^\beta)$ outside a specific set. Again using that $\bar{Z}_m \in G_m(\epsilon_0, 0)$, we have $|\bar{x}_i^\sigma - \bar{x}_\ell^\alpha| > \epsilon_0$. Since $X_m \in B_{\gamma/2}^{d|m|}(\bar{X}_m)$, we also have that

$$|x_i^\sigma - \bar{x}_i^\sigma| < \gamma/2 < \gamma,$$

$$|\bar{x}_\ell^\alpha - x_{m_\beta+1}^\beta| = |\bar{x}_\ell^\alpha - x_\ell^\alpha + \epsilon_{(\alpha, \beta)} \omega_1| \leq \frac{\gamma}{2} + \epsilon_{(\alpha, \beta)} < \gamma,$$

where in the last line we have used $\epsilon_{(\alpha,\beta)} \ll \gamma$. Applying the first part of Lemma 7.1 with $\bar{y}_1 = \bar{x}_i^\sigma$, $\bar{y}_2 = \bar{x}_\ell^\alpha$, $y_1 = x_i^\sigma$, and $y_2 = x_{m_\beta+1}^\beta$, we obtain a cylinder $K_{\eta,i,\sigma}^d$ such that for any $v_{m_\beta+1}^\beta \in B_R^d \setminus K_{\eta,i,\sigma}^d$ and any $\omega_1 \in \mathbb{S}_1^{d-1}$,

$$|x_i^\sigma(t) - x_{m_\beta+1}^\beta(t)| \geq \epsilon_0/2 > \epsilon_{(\sigma,\beta)} \quad \forall t \geq 0,$$

where we have used that $\epsilon_{(\sigma,\beta)} \ll \epsilon_0$.

3. (*Collisional Law Case*) Finally, let's show that for any $(\omega_1, v_{m_\beta+1}^\beta) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell^\alpha)$ we have $|x_\ell^\alpha(t) - x_{m_\beta+1}^\beta(t)| > \epsilon_{(\alpha,\beta)}$ for all $t \geq 0$. First, note that

(7.7)

$$|x_\ell^\alpha(t) - x_{m_\beta+1}^\beta(t)|^2 = |x_\ell^\alpha - t\bar{v}_\ell^\alpha - (x_\ell^\alpha - \epsilon_{(\alpha,\beta)}\omega_1 - tv_{m_\beta+1}^\beta)|^2$$

(7.8)

$$= |\epsilon_{(\alpha,\beta)}\omega_1 + t(v_{m_\beta+1}^\beta - \bar{v}_\ell^\alpha)|^2$$

(7.9)

$$= \epsilon_{(\alpha,\beta)}^2 + t^2|v_{m_\beta+1}^\beta - \bar{v}_\ell^\alpha|^2 + 2t\epsilon_{(\alpha,\beta)}\omega_1 \cdot (v_{m_\beta+1}^\beta - \bar{v}_\ell^\alpha)$$

(7.10)

$$> \epsilon_{(\alpha,\beta)}^2,$$

where in (7.10) we used that $(\omega_1, v_{m_\beta+1}^\beta) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell^\alpha)$.

Combining cases (1)–(3) together, set $U_{m_\beta+1,i,\sigma} := \mathbb{S}_1^{d-1} \times K_{\eta,i,\sigma}^d$. Then, the set $\mathcal{B}_{\ell,\alpha}^{0,-}(\bar{Z}_m) := \bigcup_{(i,\sigma) \neq (\ell,\alpha)} U_{m_\beta+1,i,\sigma}$, satisfies the desired properties. $\square$

For the next lemma, we obtain a bad set $\mathcal{B}_{\ell,\alpha}^{\delta,-}(\bar{Z}_m)$ outside of which we are in a well separated configuration for the precollisional trajectory for all times $t \geq \delta$.

LEMMA 7.3. *Let us be in the same scenario as in Proposition 7.1. Then, there exists a set $\mathcal{B}_{\ell,\alpha}^{\delta,-}(\bar{Z}_m)$ such that for $(\omega_1, v_{m_\beta+1}^\beta) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell^\alpha) \setminus \mathcal{B}_{\ell,\alpha}^{\delta,-}(\bar{Z}_m)$ we have $Z_{m+\beta} \in G_{m+\beta}(\epsilon_0/2, \delta)$ and $\bar{Z}_{m+\beta} \in G_{m+\beta}(\epsilon_0, \delta)$.*

Proof. By considering the three cases found in the proof of Lemma 7.2, we show

$$(7.11) \quad \mathcal{B}_{\ell,\alpha}^{\delta,-}(\bar{Z}_m) := (\mathbb{S}_1^{d-1} \times B_\eta^d(\bar{v}_\ell^\alpha)) \cup \mathcal{B}_{\ell,\alpha}^{0,-}(\bar{Z}_m),$$

with $\mathcal{B}_{\ell,\alpha}^{0,-}$ given in Lemma 7.2 satisfying our desired properties. $\square$

Lemmas 7.4 and 7.5 below are the postcollisional analogues of Lemmas 7.2 and 7.3. Their proofs are similar after taking preimages under the collisional law.

LEMMA 7.4. *Let us be in the same scenario as in Proposition 7.1. Then, there exists a bad set $\mathcal{B}_{\ell,\alpha}^{0,+}(\bar{Z}_m)$ such that for all $(\omega_1, v_{m_\beta+1}^\beta) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell^\alpha) \setminus \mathcal{B}_{\ell,\alpha}^{0,+}(\bar{Z}_m)$, $Z_{m+\beta}^*(t) \in \mathring{\mathcal{D}}_\epsilon^{m+\beta}$ for all $t \geq 0$.*

Proof. By applying Lemma 7.2 to $Z_{m+\beta}^*$, the set

(7.12)

$$\mathcal{B}_{\ell,\alpha}^{0,+}(\bar{Z}_m) := \bigcup_{(i,\sigma) \neq (\ell,\alpha)} U_{m_\beta+1,i,\sigma}^*, \quad U_{m_\beta+1,i,\sigma}^* := \left\{ (\omega_1, v_{m_\beta+1}^\beta) : (v_{m_\beta+1}^\beta)^* \in K_{\eta,i,\sigma}^d \right\}$$

satisfies the desired properties, where $(v_{m_\beta+1}^\beta)^*$ is given by the collisional law

$$(v_{m_\beta+1}^\beta)^* = v_j^\beta + \frac{2M_\alpha}{M_\alpha + M_\beta}((\bar{v}_\ell^\alpha - v_j^\beta) \cdot \omega_1)\omega_1. \quad \square$$

LEMMA 7.5. *Let us be in the same scenario as in Proposition 7.1. Then, there exists a set $\mathcal{B}_{\ell,\alpha}^{\delta,+}(\bar{Z}_{\mathbf{m}})$ such that for all $(\omega_1, v_{m_{\beta}+1}^{\beta}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_{\ell}^{\alpha}) \setminus \mathcal{B}_{\ell,\alpha}^{\delta,+}(\bar{Z}_{\mathbf{m}})$ we have $Z_{\mathbf{m}+\beta}^* \in G_{\mathbf{m}+\beta}(\epsilon_0/2, \delta)$ and $\bar{Z}_{\mathbf{m}+\beta}^* \in G_{\mathbf{m}+\beta}(\epsilon_0, \delta)$.*

Proof. We apply Lemma 7.3 to $Z_{\mathbf{m}+\beta}^*$. For $\eta > 0$ satisfying (7.4), define the set

(7.13)

$$\tilde{U}_{m_{\beta}+1,\ell,\alpha}^* = \left\{ (\omega_1, v_{m_{\beta}+1}^{\beta}) : |(v_{m_{\beta}+1}^{\beta})^* - (\bar{v}_{\ell}^{\alpha})^*| < \eta \right\} = \{ (\omega_1, v_{m_{\beta}+1}^{\beta}) : v_{m_{\beta}+1}^{\beta} \in B_{\eta}(\bar{v}_{\ell}^{\alpha}) \}.$$

The last equality follows from conservation of momentum and energy. One is then able to verify that the set $\mathcal{B}_{\ell,\alpha}^{\delta,+}(\bar{Z}_{\mathbf{m}}) := \tilde{U}_{m_{\beta}+1,\ell,\alpha}^* \cup \mathcal{B}_{\ell,\alpha}^{0,+}(\bar{Z}_{\mathbf{m}})$ satisfies the desired properties. $\square$

Proof of Proposition 7.1. Combining the sets found in Lemmas 7.2 through 7.5, define

$$\mathcal{B}_{\ell,\alpha}(\bar{Z}_{\mathbf{m}}) := \mathcal{B}_{\ell,\alpha}^{0,-}(\bar{Z}_{\mathbf{m}}) \cup \mathcal{B}_{\ell,\alpha}^{0,+}(\bar{Z}_{\mathbf{m}}) \cup \mathcal{B}_{\ell,\alpha}^{\delta,-}(\bar{Z}_{\mathbf{m}}) \cup \mathcal{B}_{\ell,\alpha}^{\delta,+}(\bar{Z}_{\mathbf{m}}).$$

This set satisfies all of the desired properties. $\square$

7.2. Measure estimates. We will now provide a measure estimate of the set constructed in Proposition 7.1 with respect to the truncation parameters. We will rely on the geometric estimates for binary interactions as presented in [6]. However, our collision law may involve particles of different masses, which we treat using the corresponding transition maps (operators reducing the postcollisional case to the pre-collisional case) for each type of collision.

PROPOSITION 7.2. *Let $\gamma, \epsilon_0, \epsilon_{(1,0)}, \epsilon_{(0,1)}, R, \eta, \delta$ be related by (7.4). Fix $\mathbf{m} = (m_{(1,0)}, m_{(0,1)}) \in \mathbb{N}_+^2$, and $\alpha, \beta \in \mathcal{T}$, where $\mathcal{T}$ is the set of types (2.1). Let $\bar{Z}_{\mathbf{m}} \in G_{\mathbf{m}}(\epsilon_0, 0)$, and assume that $E(\bar{Z}_{\mathbf{m}}) \leq R^2$, where $E(\cdot)$ is given by (3.6). Let $\mathcal{B}_{\ell,\alpha}(\bar{Z}_{\mathbf{m}})$ be the set found in Proposition 7.1. We have*

$$(7.14) \quad |\mathcal{B}_{\ell,\alpha}(\bar{Z}_{\mathbf{m}})| \lesssim (m_{(1,0)} + m_{(0,1)}) R^d \eta^{\frac{d-1}{2d+2}},$$

where $|\cdot|$ denotes the product measure on $\mathbb{S}_1^{d-1} \times B_R^d$.

REMARK 7.1. *The constants hidden in (7.14) only depend on dimension and the ratio of the masses of the type (1, 0) and type (0, 1) particles.*

Proof. As in [6], it suffices to bound each of the terms in

$$(7.15) \quad \left[(\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_{\ell}^{\alpha}) \right] \cap \left[\mathcal{B}_{\ell,\alpha}^{0,-}(\bar{Z}_{\mathbf{m}}) \cup \mathcal{B}_{\ell,\alpha}^{0,+}(\bar{Z}_{\mathbf{m}}) \cup \mathcal{B}_{\ell,\alpha}^{\delta,-}(\bar{Z}_{\mathbf{m}}) \cup \mathcal{B}_{\ell,\alpha}^{\delta,+}(\bar{Z}_{\mathbf{m}}) \right],$$

where we recall (7.3). We will only prove the estimate on the $\mathcal{B}_{\ell,\alpha}^{0,+}$ term since it is the most delicate and uses the collisional law.

Estimate of $\mathcal{B}_{\ell,\alpha}^{0,+}(\bar{Z}_{\mathbf{m}})$. Recall the definition of $\mathcal{B}_{\ell,\alpha}^{0,+}$ as given in (7.12). Fix $(i, \sigma) \neq (\ell, \alpha)$, and define

$$\begin{aligned} \mathcal{S}^+(\bar{v}_{\ell}^{\alpha}, v_{m_{\beta}+1}^{\beta}) &:= \{ \omega_1 \in \mathbb{S}_1^{d-1} : \omega_1 \cdot (v_{m_{\beta}+1}^{\beta} - \bar{v}_{\ell}^{\alpha}) > 0 \} \\ &= \{ \omega_1 \in \mathbb{S}_1^{d-1} : (\omega_1, v_{m_{\beta}+1}^{\beta}) \in (\mathbb{S}_1^d \times B_R^d)^+(\bar{v}_{\ell}^{\alpha}) \}. \end{aligned}$$

Using radial coordinates with integration in $v_{m_{\beta}+1}^{\beta}$ centered at $\bar{v}_{\ell}^{\alpha}$, we estimate

(7.16)

$$|(\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_{\ell}^{\alpha}) \cap U_{m_{\beta}+1,i,\sigma}^*| \leq \int_0^{2R} \int_{\partial B_r(\bar{v}_{\ell}^{\alpha})} \int_{\mathcal{S}^+(\bar{v}_{\ell}^{\alpha}, v_{m_{\beta}+1}^{\beta})} \mathbb{1}_{U_{m_{\beta}+1,i,\sigma}^*} d\omega_1 d\sigma_r dr,$$

where σ_r is the surface measure on $\partial B_r(\bar{v}_\ell^\alpha)$. Now, fix $r \in [0, 2R]$ and introduce a parameter $\theta \in (0, 1)$ to decompose $\mathcal{S}^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta)$ into two parts:

$$(7.17) \quad \mathcal{S}_1^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta) := \{\omega_1 \in \mathbb{S}_1^{d-1} : \omega_1 \cdot (v_{m_\beta+1}^\beta - \bar{v}_\ell^\alpha) > \theta |v_{m_\beta+1}^\beta - \bar{v}_\ell^\alpha|\},$$

$$(7.18) \quad \mathcal{S}_2^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta) := \{\omega_1 \in \mathbb{S}_1^{d-1} : \omega_1 \cdot (v_{m_\beta+1}^\beta - \bar{v}_\ell^\alpha) \leq \theta |v_{m_\beta+1}^\beta - \bar{v}_\ell^\alpha|\}.$$

It is clear that $\mathcal{S}_2^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta)$ is contained in a spherical cap of direction $v_{m_\beta+1}^\beta - \bar{v}_\ell^\alpha$ and angle $\arccos \theta$. Hence, by integration in spherical coordinates we have (see, e.g., [5])

$$(7.19) \quad |\mathcal{S}_2^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta)|_{\mathbb{S}_1^d} \lesssim \arcsin \theta.$$

The other term $\mathcal{S}_1^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta)$ is more difficult to handle. Motivated by the binary transition map from [6], we introduce the following transition map tailored to the masses of the particles:

$$(7.20) \quad \mathcal{J}_{\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta} : \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}, \quad \mathcal{J}_{\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta}(\omega) = r^{-1} \left((\bar{v}_\ell^\alpha)^* - (v_{m_\beta+1}^\beta)^* \right).$$

Here, we define

$$(v_{m_\beta+1}^\beta)^* = v_j^\beta + \frac{2M_\alpha}{M_\alpha + M_\beta} ((\bar{v}_\ell^\alpha - v_j^\beta) \cdot \omega) \omega, \quad (\bar{v}_\ell^\alpha)^* = \bar{v}_\ell^\alpha + \frac{2M_\beta}{M_\alpha + M_\beta} ((\bar{v}_\ell^\alpha - v_j^\beta) \cdot \omega) \omega,$$

and $r = |\bar{v}_\ell^\alpha - v_{m_\beta+1}^\beta|$. Set $\nu := \mathcal{J}_{\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta}(\omega_1)$. We can check that

$$(7.21) \quad \begin{aligned} (v_{m_\beta+1}^\beta)^* &= C_{(\alpha, \beta)} v_{m_\beta+1}^\beta + C_{(\beta, \alpha)} \bar{v}_\ell^\alpha - C_{(\beta, \alpha)} r \nu, \\ (\bar{v}_\ell^\alpha)^* &= C_{(\alpha, \beta)} v_{m_\beta+1}^\beta + C_{(\beta, \alpha)} \bar{v}_\ell^\alpha + C_{(\alpha, \beta)} r \nu, \end{aligned}$$

where we define

$$(7.22) \quad C_{(\sigma, \sigma')} := \frac{M_{\sigma'}}{M_{\sigma'} + M_\sigma}, \quad M_\sigma = \text{“mass of the } \sigma\text{-type particle.”}$$

Using (7.21) and recalling the cylinder $K_{\eta, i, \sigma}$ from (7.12), note that

$$(7.23) \quad (v_{m_\beta+1}^\beta)^* \in K_{\eta, i, \sigma}^d \Leftrightarrow \nu \in K_{\eta/rC_{(\beta, \alpha)}, i, \sigma}^d,$$

where $K_{\eta/rC_{(\beta, \alpha)}, i, \sigma}^d$ is a cylinder of radius $\eta/rC_{(\beta, \alpha)}$. Using this fact, we write

$$(7.24) \quad \int_{\mathcal{S}_1^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta)} \mathbb{1}_{U_{m_\beta+1, i, \sigma}^*}(\omega_1) d\omega_1 = \int_{\mathcal{S}_1^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta)} \mathbb{1}_{K_{\eta/rC_{(\beta, \alpha)}, i, \sigma}^d} \circ \mathcal{J}_{\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta}(\omega_1) d\omega_1.$$

Now, using a change of variables and a Jacobian estimate similar to Proposition 12.2 in [6], we estimate

$$(7.25) \quad \int_{\mathcal{S}_1^+(\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta)} \mathbb{1}_{K_{\eta/rC_{(\beta, \alpha)}, i, \sigma}^d} \circ \mathcal{J}_{\bar{v}_\ell^\alpha, v_{m_\beta+1}^\beta}(\omega_1) d\omega_1 \lesssim \theta^{-d} \min \left\{ 1, \left(\frac{\eta}{rC_{(\beta, \alpha)}} \right)^{\frac{d-1}{2}} \right\}.$$

Now, putting (7.16), (7.19), and (7.25) together, we obtain

$$(7.26) \quad |(\mathbb{S}_1^{d-1} \times B_R^d)^+ (\bar{v}_\ell^\alpha) \cap U_{m_\beta+1, i, \sigma}^*| \lesssim \int_0^{2R} \int_{\partial B_r(\bar{v}_\ell^\alpha)} \left(\arcsin \theta + \theta^{-d} \min \left\{ 1, \left(\frac{\eta}{r C_{(\beta, \alpha)}} \right)^{\frac{d-1}{2}} \right\} \right) d\sigma dr.$$

Estimating this integral and minimizing over allowable θ (see, e.g., [5]), we obtain

$$(7.27) \quad |(\mathbb{S}_1^{d-1} \times B_R^d)^+ (\bar{v}_\ell^\alpha) \cap U_{m_\beta+1, i, \sigma}^*| \lesssim (\max(1, C_{(\beta, \alpha)}))^{\frac{1-d}{2}} R^d \eta^{\frac{d-1}{2d+2}}.$$

Summing over indices $(i, \sigma) \neq (\ell, \alpha)$, of which there are less than $m_{(1,0)} + m_{(0,1)}$, subadditivity gives the estimate. $\square$

8. Elimination of recollisions for mixtures. In this section, we reduce the convergence proof to comparing truncated elementary observables. We first restrict ourselves to good configurations and provide the corresponding measure estimate.

8.1. Restriction to good configurations. Inductively using Proposition 7.2 we are able to reduce the convergence proof to good configurations, up to a small measure set. The measure of the complement will be negligible in the limit. Throughout this subsection, $\mathbf{s} = (s_{(1,0)}, s_{(0,1)}) \in \mathbb{N}_+^2$ will be fixed, and $\mathbf{N} = (N_{(1,0)}, N_{(0,1)})$, $\boldsymbol{\epsilon} = (\epsilon_{(1,0)}, \epsilon_{(0,1)})$ will be given the Boltzmann-Grad scaling as in (2.28). The parameters $R, \epsilon_0, \gamma, \eta, \delta$ satisfy (7.4).

Given $\mathbf{m} \in \mathbb{N}_+^2$, and recalling (7.2), let us define the set

$$(8.1) \quad G_{\mathbf{m}}(\boldsymbol{\epsilon}, \epsilon_0, \delta) := G_{\mathbf{m}}(\epsilon_{(1,0)}, 0) \cap G_{\mathbf{m}}(\epsilon_{(0,1)}, 0) \cap G_{\mathbf{m}}(\epsilon_0, \delta).$$

Let us also recall from (5.1)–(5.2) the set $\Delta_{\mathbf{s}}^X(\epsilon_0)$ of well separated spatial configurations. The following lemma can be found in [5].

LEMMA 8.1. *Let $\mathbf{s} \in \mathbb{N}_+^2$, $\gamma, \epsilon_0, R, \eta, \delta$ be parameters as in (7.4). Then, for any $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$, there is a subset of velocities $\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}}) \subseteq B_R^{d|\mathbf{s}|}$ of measure*

$$(8.2) \quad |\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})|_{d|\mathbf{s}|} \leq C_{d,\mathbf{s}} R^{d|\mathbf{s}|} \eta^{\frac{d-1}{2}},$$

such that for any $V_{\mathbf{s}} \in B_R^{d|\mathbf{s}|} \setminus \mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})$ we have

$$(8.3) \quad Z_{\mathbf{s}} := (X_{\mathbf{s}}, V_{\mathbf{s}}) \in G_{\mathbf{s}}(\boldsymbol{\epsilon}, \epsilon_0, \delta).$$

For each $\mathbf{s} \in \mathbb{N}_+^2$ and $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$, let us denote $\mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}}) := B_R^{d|\mathbf{s}|} \setminus \mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})$. Consider $1 \leq k \leq n$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k$, where S_k is given in (6.1). Let us recall the observables $I_{\mathbf{s},k,R,\delta}^{\mathbf{N}}, I_{\mathbf{s},k,R,\delta}^{\infty}$ defined in (6.14), (6.15). We will restrict the domain of integration to velocities giving good configurations. In particular, we define

$$(8.4) \quad \tilde{I}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t)(X_{\mathbf{s}}) := \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k} \int_{\mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})} \phi_{\mathbf{s}}(V_{\mathbf{s}}) f_{\mathbf{N},(\boldsymbol{\alpha}, \boldsymbol{\beta}), R, \delta}^{(\mathbf{s})}(X_{\mathbf{s}}, V_{\mathbf{s}}) dV_{\mathbf{s}},$$

$$(8.5) \quad \tilde{I}_{\mathbf{s},k,R,\delta}^{\infty}(t)(X_{\mathbf{s}}) := \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k} \int_{\mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})} \phi_{\mathbf{s}}(V_{\mathbf{s}}) f_{(\boldsymbol{\alpha}, \boldsymbol{\beta}), R, \delta}^{(\mathbf{s})}(X_{\mathbf{s}}, V_{\mathbf{s}}) dV_{\mathbf{s}},$$

where we recall that $f_{\mathbf{N},(\boldsymbol{\alpha}, \boldsymbol{\beta}), R, \delta}^{(\mathbf{s})}$ and $f_{(\boldsymbol{\alpha}, \boldsymbol{\beta}), R, \delta}^{(\mathbf{s})}$ are defined in (6.12), (6.13). Let us apply Lemma 8.1 to restrict ourselves to initially good configurations.

PROPOSITION 8.1. Let $n \in \mathbb{N}$, let $\mathbf{s} \in \mathbb{N}_+^2$, and let $\gamma, \epsilon_0, R, \eta, \delta$ be parameters as in (7.4). Then, the following estimates hold for all $t \in [0, T]$ and uniformly in $\mathbf{N}$:

$$\begin{aligned} \sum_{k=0}^n \|I_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t) - \tilde{I}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t)\|_{L^\infty(\Delta_{\mathbf{s}}^X(\epsilon_0))} &\leq C_{d,\mathbf{s},\mu_0,T} R^{d|\mathbf{s}|} \eta^{\frac{d-1}{2}} \|F_{\mathbf{N},0}\|_{\mathbf{N},\gamma_0,\mu_0}, \\ \sum_{k=0}^n \|I_{\mathbf{s},k,R,\delta}^\infty(t) - \tilde{I}_{\mathbf{s},k,R,\delta}^\infty(t)\|_{L^\infty(\Delta_{\mathbf{s}}^X(\epsilon_0))} &\leq C_{d,\mathbf{s},\mu_0,T} R^{d|\mathbf{s}|} \eta^{\frac{d-1}{2}} \|F_0\|_{\infty,\gamma_0,\mu_0}. \end{aligned}$$

Proof. We present the proof for the BBGKY hierarchy case only. The proof for the Boltzmann hierarchy case is similar. Let us fix $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$ and $k \in \{1, \dots, n\}$. Applying k times Lemma 4.2, we obtain

$$\begin{aligned} &|I_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t)(X_{\mathbf{s}}) - \tilde{I}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t)(X_{\mathbf{s}})| \\ &\leq \sum_{\alpha,\beta \in S_k} \int_{\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})} |\phi_{\mathbf{s}}(V_{\mathbf{s}}) f_{\mathbf{N},(\alpha,\beta),R,\delta}^{(\mathbf{s})}(t, X_{\mathbf{s}}, V_{\mathbf{s}})| dV_{\mathbf{s}} \\ &\leq \sum_{\alpha,\beta \in S_k} \|\phi_{\mathbf{s}}\|_{L^\infty} e^{-|\mathbf{s}|\mu(T)} 8^{-k} \|F_{\mathbf{N},0}\|_{\mathbf{N},\gamma_0,\mu_0} \int_{\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})} e^{-\gamma(T)E_{\mathbf{s}}(Z_{\mathbf{s}})} dV_{\mathbf{s}} \\ (8.6) \quad &\leq \sum_{\alpha,\beta \in S_k} \|\phi_{\mathbf{s}}\|_{L^\infty} e^{-|\mathbf{s}|\mu(T)} 8^{-k} |\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})|_{d|\mathbf{s}|} \|F_{\mathbf{N},0}\|_{\mathbf{N},\gamma_0,\mu_0}. \end{aligned}$$

For $k = 0$, recall that the map $\mathbf{T}_\epsilon^t$ defined in (4.2) is an isometry on the space $\mathbf{X}_{\epsilon,\gamma,\mu}^{\mathbf{N}}([0, T])$. An application of the triangle inequality thus implies

$$(8.7) \quad |I_{\mathbf{s},0,R,\delta}^{\mathbf{N}}(t)(X_{\mathbf{s}}) - \tilde{I}_{\mathbf{s},0,R,\delta}^{\mathbf{N}}(t)(X_{\mathbf{s}})| \leq \|\phi_{\mathbf{s}}\|_{L^\infty} e^{-|\mathbf{s}|\mu(T)} |\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})|_{d|\mathbf{s}|} \|F_{\mathbf{N},0}\|_{\mathbf{N},\gamma_0,\mu_0}.$$

We now sum the estimates (8.6)–(8.7) over $k = 0, \dots, n$ and apply the measure estimate of Lemma 8.1. $\square$

REMARK 8.1. Given $\mathbf{s} \in \mathbb{N}_+^2$ and $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$, the definition of $\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})$ implies that $\tilde{I}_{\mathbf{s},0,R,\delta}^{\mathbf{N}}(t)(X_{\mathbf{s}}) = \tilde{I}_{\mathbf{s},0,R,\delta}^\infty(t)(X_{\mathbf{s}})$. Therefore, Proposition 8.1 allows us to reduce the convergence to controlling the differences $\tilde{I}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t) - \tilde{I}_{\mathbf{s},k,R,\delta}^\infty(t)$, for $k = 1, \dots, n$, in the scaled limit.

8.2. Reduction to elementary observables. In this subsection, given $\mathbf{s} \in \mathbb{N}^2$ and $1 \leq k \leq n$, we express the observables $\tilde{I}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t)$ and $\tilde{I}_{\mathbf{s},k,R,\delta}^\infty(t)$, defined in (8.4), (8.5), as a superposition of elementary observables.

For this purpose, given $\ell := (\ell_{(1,0)}, \ell_{(0,1)}) \in \mathbb{N}^2$, and $\alpha, \beta \in \mathcal{T}$, and recalling the truncated collision operators (6.10), we decompose the BBGKY hierarchy collisional operators in the following way:

$$\mathcal{C}_{\ell,\ell+\beta,R}^\alpha = \sum_{i=1}^{\ell_\alpha} \mathcal{C}_{\ell,\ell+\beta,R}^{\alpha,i,+} - \sum_{i=1}^{\ell_\alpha} \mathcal{C}_{\ell,\ell+\beta,R}^{\alpha,i,-},$$

where we define

$$(8.8) \quad \mathcal{C}_{\ell,\ell+\beta,R}^{\alpha,i,+} g^{(\ell+\beta)}(Z_\ell) = A_{\epsilon,\ell}^{\mathbf{N},(\alpha,\beta)} \int_{\mathbb{S}_1^{d-1} \times B_R^d} (\omega_1 \cdot (v_{\ell_\beta+1}^\beta - v_i^\alpha))_+ g^{(\ell+\beta)}(Z_{\ell+\beta,\epsilon}^{i,\alpha,*}) d\omega_1 dv_{\ell_\beta+1}^\beta,$$

(8.9)

$$\mathcal{C}_{\ell, \ell+\beta, R}^{\alpha, i, -} g^{(\ell+\beta)}(Z_\ell) = A_{\epsilon, \ell}^{N, (\alpha, \beta)} \int_{\mathbb{S}_1^{d-1} \times B_R^d} (\omega_1 \cdot (v_{\ell\beta+1}^\beta - v_i^\alpha))_+ g^{(\ell+\beta)}(Z_{\ell+\beta, \epsilon}^{i, \alpha}) d\omega_1 dv_{\ell\beta+1}^\beta.$$

Here, we let $A_{\epsilon, \ell}^{N, (\alpha, \beta)} = (N_\beta - \ell_\beta) \epsilon_{(\alpha, \beta)}^{d-1}$ and let $Z_{\ell+\beta, \epsilon}^{i, \alpha, *}$, $Z_{\ell+\beta, \epsilon}^{i, \alpha}$ be as defined in (2.21), (2.24), and (2.25). This process of splitting the collision operators can be viewed as isolating the types of interactions being summed over. Given $1 \leq k \leq n$, $\alpha, \beta \in S_k$ and recalling (6.2), let us denote

(8.10)

$$\mathcal{J}_{s, k} = \{J = (j_1, \dots, j_k) : j_i \in \{+, -\} \quad \forall i \in \{1, \dots, k\}\},$$

(8.11)

$$\mathcal{M}_{s, k, \beta} = \left\{M = (m_1, \dots, m_k) \in \mathbb{N}^k : m_i \in \{1, \dots, s_{\beta_i} + \tilde{\beta}_{i-1}^{\beta_i}\} \quad \forall i \in \{1, \dots, k\}\right\},$$

(8.12)

$$\mathcal{U}_{s, k, \beta} = \mathcal{J}_{s, k} \times \mathcal{M}_{s, k, \beta}.$$

Here, the number $s_{\beta_i} + \tilde{\beta}_{i-1}^{\beta_i}$ is exactly the number of β_i type particles in the system after adding particles of types $\beta_1, \dots, \beta_{i-1}$ to the system of s particles. Under this notation, the BBGKY hierarchy observable functional $\tilde{I}_{s, k, R, \delta}^N(t)$ defined in (8.4) can be expressed, for $1 \leq k \leq n$, as a superposition of elementary observables

$$(8.13) \quad \tilde{I}_{s, k, R, \delta}^N(t)(X_s) = \sum_{\alpha, \beta \in S_k} \sum_{(J, M) \in \mathcal{U}_{s, k, \beta}} \left(\prod_{i=1}^k j_i \right) \tilde{I}_{s, k, R, \delta}^N(t, \alpha, \beta, J, M)(X_s),$$

where the elementary observables are defined by

(8.14)

$$\begin{aligned} \tilde{I}_{s, k, R, \delta}^N(t, \alpha, \beta, J, M)(X_s) &= \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k, \delta}(t)} T_{s, \epsilon}^{t-t_1} \mathcal{C}_{s, s+\beta_1, R}^{\alpha_1, m_1, j_1} T_{s+\beta_1, \epsilon}^{t_1-t_2} \\ &\quad \dots T_{s+\tilde{\beta}_{k-1}, \epsilon}^{t_{k-1}-t_k} \mathcal{C}_{s+\tilde{\beta}_{k-1}, s+\tilde{\beta}_k, R}^{\alpha_k, m_k, j_k} T_{s+\tilde{\beta}_k, \epsilon}^{t_k} f_{N, 0}^{(s+\tilde{\beta}_k)}(Z_s) dt_k \dots dt_1 dV_s. \end{aligned}$$

Similarly, given $\ell = (\ell_{(1,0)}, \ell_{(0,1)}) \in \mathbb{N}_+^2$, $\alpha, \beta \in \mathcal{T}$, and recalling the truncated Boltzmann collision operator (6.10), we decompose the Boltzmann collisional operators as

$$\mathcal{C}_{\ell, \ell+\beta, R}^\alpha = \sum_{i=1}^{\ell_\alpha} \mathcal{C}_{\ell, \ell+\beta, R}^{\alpha, i, +} - \sum_{i=1}^{\ell_\alpha} \mathcal{C}_{\ell, \ell+\beta, R}^{\alpha, i, -},$$

where we define

(8.15)

$$\mathcal{C}_{\ell, \ell+\beta, R}^{\alpha, i, +} g^{(\ell+\beta)}(Z_\ell) = A_\beta^\alpha \int_{\mathbb{S}_1^{d-1} \times B_R^d} (\omega_1 \cdot (v_{\ell\beta+1}^\beta - v_i^\alpha))_+ g^{(\ell+\beta)}(Z_{\ell+\beta}^{i, \alpha, *}) d\omega_1 dv_{\ell\beta+1}^\beta,$$

(8.16)

$$\mathcal{C}_{\ell, \ell+\beta, R}^{\alpha, i, -} g^{(\ell+\beta)}(Z_\ell) = A_\beta^\alpha \int_{\mathbb{S}_1^{d-1} \times B_R^d} (\omega_1 \cdot (v_{\ell\beta+1}^\beta - v_i^\alpha))_+ g^{(\ell+\beta)}(Z_{\ell+\beta}^{i, \alpha}) d\omega_1 dv_{\ell\beta+1}^\beta.$$

Here, A_β^α is given as in (2.35) and $Z_{\ell+\beta}^{i, \alpha, *}$, $Z_{\ell+\beta}^{i, \alpha}$ are given as in Definition 2.3. Under this notation, the Boltzmann hierarchy observable functional $\tilde{I}_{s, k, R, \delta}^\infty(t)$ defined in

(8.5) can be expressed, for $1 \leq k \leq n$, as a superposition of elementary observables

$$(8.17) \quad \tilde{I}_{s,k,R,\delta}^\infty(t)(X_s) = \sum_{\alpha, \beta \in S_k} \sum_{(J,M) \in \mathcal{U}_{s,k,\beta}} \left(\prod_{i=1}^k j_i \right) \tilde{I}_{s,k,R,\delta}^\infty(t, \alpha, \beta, J, M)(X_s),$$

where the elementary observables are defined by

$$(8.18) \quad \tilde{I}_{s,k,R,\delta}^\infty(t, \alpha, \beta, J, M)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} S_s^{t-t_1} \mathcal{C}_{s, s+\tilde{\beta}_1, R}^{\alpha_1, m_1, j_1} S_s^{t_1-t_2} \dots S_{s+\tilde{\beta}_{k-1}}^{t_{k-1}-t_k} \mathcal{C}_{s+\tilde{\beta}_{k-1}, s+\tilde{\beta}_k, R}^{\alpha_k, m_k, j_k} S_{s+\tilde{\beta}_k}^{t_k} f_0^{(s+\tilde{\beta}_k)}(Z_s) dt_k \dots dt_1 dV_s.$$

8.3. Boltzmann hierarchy pseudo-trajectories. We introduce the following notation, which we will be constantly using from now on. Let $s = (s_{(1,0)}, s_{(0,1)}) = (s_1, s_2) \in \mathbb{N}_+^2$, $Z_s \in \mathbb{R}^{2d|s|}$, $1 \leq k \leq n$, $\alpha, \beta \in S_k$, and $t \in [0, T]$. Let us recall the set $\mathcal{T}_k(t)$ defined in (6.3).

Consider $(t_1, \dots, t_k) \in \mathcal{T}_k(t)$, $J = (j_1, \dots, j_k)$, $M = (m_1, \dots, m_k)$, $(J, M) \in \mathcal{U}_{s,k,\beta}$ given in (8.12). We inductively define the Boltzmann hierarchy pseudo-trajectory of Z_s . Roughly speaking, the Boltzmann hierarchy pseudo-trajectory forms the configurations on which particles are adjusted during backwards in time evolution.

For instance, assume we are given a configuration $Z_s \in \mathbb{R}^{2d|s|}$ at time $t_0 = t$. The dynamics of $Z_s(\cdot)$ evolves under backwards free flow until the time t_1 where the configuration $(\omega_1, v_{s_{\alpha_1+1}}^{\alpha_1}) \in \mathbb{S}_1^{d-1} \times B_R^d$ is added, neglecting positions, to the m_1 th particle of type α_1 , the adjunction being precollisional if $j_1 = -1$ and postcollisional if $j_1 = 1$. We then form an $(s + \alpha_1)$ -configuration and continue this process inductively until time $t_{k+1} = 0$. In order to concisely write the adjunction process in the Boltzmann pseudo-trajectory, we introduce the following definitions.

DEFINITION 8.1 (Boltzmann particle adjunction). *Let $s = (s_1, s_2) \in \mathbb{N}_+^2$, $\alpha, \beta \in \mathcal{T}$, and $m \in \{1, \dots, s_\alpha\}$. Then, for any $Z_s = (X_s, V_s) \in \mathbb{R}^{2d|s|}$ and $v \in \mathbb{R}^d$ define*

$$(8.19) \quad Z_s \oplus_\beta^{\alpha, m} v := \begin{cases} \left(X_{s_1}^{(1,0)}, x_m^\alpha, X_{s_2}^{(0,1)}, V_{s_1}^{(1,0)}, v, V_{s_2}^{(0,1)} \right), & \beta = (1, 0), \\ \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, x_m^\alpha, V_{s_1}^{(1,0)}, V_{s_2}^{(0,1)}, v \right), & \beta = (0, 1). \end{cases}$$

That is, the vector $Z_s \oplus_\beta^{\alpha, m} v$ is the result of a new particle to the configuration Z_s that is of type β with position x_m^α and velocity v .

DEFINITION 8.2 (ω -impact operator). *Let $s = (s_1, s_2) \in \mathbb{N}_+^2$, and $\alpha, \beta \in \mathcal{T}$, and let $(m, \ell) \in \{1, \dots, s_\alpha\} \times \{1, \dots, s_\beta\}$. For any $Z_s \in \mathbb{R}^{2d|s|}$ and $\omega \in \mathbb{S}^{d-1}$, define $T_{(m,\ell),\omega}^{(\alpha,\beta)}(Z_s) \in \mathbb{R}^{2d|s|}$ to be the vector whose components are given by*

$$T_{(m,\ell),\omega}^{(\alpha,\beta)}(Z_s) := \begin{cases} \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, V_{s_1}^{(1,0)}, v_1^{(0,1)}, \dots, (v_m^{(1,0)})^{\omega*}, \dots, (v_\ell^{(0,1)})^{\omega*}, \dots, v_{s_2}^{(0,1)} \right), & \alpha = \beta = (0, 1), \\ \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, v_1^{(1,0)}, \dots, (v_m^{(1,0)})^{\omega*}, \dots, v_{s_1}^{(1,0)}, v_1^{(0,1)}, \dots, (v_\ell^{(0,1)})^{\omega*}, \dots, v_{s_2}^{(0,1)} \right), & \alpha = (1, 0), \beta = (0, 1), \\ \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, v_1^{(1,0)}, \dots, (v_\ell^{(1,0)})^{\omega*}, \dots, v_{s_1}^{(1,0)}, v_1^{(0,1)}, \dots, (v_m^{(0,1)})^{\omega*}, \dots, v_{s_2}^{(0,1)} \right), & \alpha = (0, 1), \beta = (1, 0), \\ \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, v_1^{(1,0)}, \dots, (v_m^{(1,0)})^{\omega*}, \dots, (v_\ell^{(1,0)})^{\omega*}, \dots, v_{s_1}^{(0,1)}, V_{s_2}^{(0,1)} \right), & \alpha = \beta = (1, 0), \end{cases}$$

where we define

$$(8.20) \quad (v_m^\alpha)^{\omega*} = v_m^\alpha - \frac{2M_\beta}{M_\alpha + M_\beta} \left((v_m^\alpha - v_\ell^\beta) \cdot \omega \right) \omega,$$

$$(8.21) \quad (v_\ell^\beta)^{\omega*} = v_\ell^\beta + \frac{2M_\alpha}{M_\alpha + M_\beta} \left((v_m^\alpha - v_\ell^\beta) \cdot \omega \right) \omega.$$

That is, $T_{(m,\ell),\omega}^{(\alpha,\beta)}(Z_s)$ takes the configuration Z_s and collides v_m^α and v_ℓ^β with relative impact direction ω .

Now, we inductively construct the Boltzmann hierarchy pseudo-trajectory of $Z_s = (X_s, V_s) \in \mathbb{R}^{2d|s|}$ as follows.

Time $t_0 = t$. We initially define

$$Z_s^\infty(t_0^-) := Z_s.$$

We will use the following notation for the components of $Z_s^\infty(t_0^-)$:

$$(8.22) \quad Z_s^\infty(t_0^-) = (X_s^\infty(t_0^-), V_s^\infty(t_0^-)).$$

Time t_i with $i \in \{1, \dots, k\}$. Consider $i \in \{1, \dots, k\}$, and assume we know $(Z_{s+\tilde{\beta}_{i-1}})^\infty(t_{i-1}^-)$. We define $(Z_{s+\tilde{\beta}_{i-1}})^\infty(t_i^+)$ as follows:

$$(Z_{s+\tilde{\beta}_{i-1}})^\infty(t_i^+) := \left((X_{s+\tilde{\beta}_{i-1}})^\infty(t_{i-1}^-) - (t_{i-1} - t_i)(V_{s+\tilde{\beta}_{i-1}})^\infty(t_{i-1}^-), (V_{s+\tilde{\beta}_{i-1}})^\infty(t_{i-1}^-) \right).$$

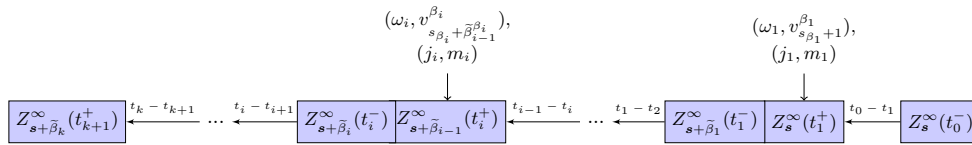
We also define $(Z_{s+\tilde{\beta}_i})^\infty(t_i^-)$ depending on the sign of j_i as

$$(8.23) \quad (Z_{s+\tilde{\beta}_i})^\infty(t_i^-) := \begin{cases} (Z_{s+\tilde{\beta}_{i-1}})^\infty(t_i^+) \oplus_{\beta_i}^{\alpha_i, m_i} v_{s_{\beta_i} + \tilde{\beta}_{i-1}}^{\beta_i}, & j_i = -1, \\ T_{(m_i, s_{\beta_i} + \tilde{\beta}_{i-1}), \omega_i}^{(\alpha_i, \beta_i)} \left((Z_{s+\tilde{\beta}_{i-1}})^\infty(t_i^+) \oplus_{\beta_i}^{\alpha_i, m_i} v_{s_{\beta_i} + \tilde{\beta}_{i-1}}^{\beta_i} \right), & j_i = 1. \end{cases}$$

Time $t_{k+1} = 0$. We finally obtain

$$Z_{s+\tilde{\beta}_k}^\infty(0^+) = Z_{s+\tilde{\beta}_k}^\infty(t_{k+1}^+) = \left(X_{s+\tilde{\beta}_k}^\infty(t_k^-) - t_k V_{s+\tilde{\beta}_k}^\infty(t_k^-), V_{s+\tilde{\beta}_k}^\infty(t_k^-) \right).$$

The process is illustrated in the following diagram:



DEFINITION 8.3. Let $Z_s = (X_s, V_s) \in \mathbb{R}^{2d|s|}$, $k \in \mathbb{N}$, $(t_1, \dots, t_k) \in \mathcal{T}_k(t)$, $\alpha, \beta \in S_k$, $J = (j_1, \dots, j_k)$, $M = (m_1, \dots, m_k)$, $(J, M) \in \mathcal{U}_{s,k,\beta}$, and for each $i = 1, \dots, k$, we consider $(\omega_i, v_{s_{\beta_i} + \tilde{\beta}_{i-1}}^{\beta_i}) \in \mathbb{S}_1^{d-1} \times B_R^d$. The sequence $\{Z_{s+\tilde{\beta}_{i-1}}^\infty(t_i^+)\}_{i=0,\dots,k+1}$ constructed above is called the Boltzmann hierarchy pseudo-trajectory of Z_s .

8.4. Reduction to truncated elementary observables. We will now use the Boltzmann hierarchy pseudo-trajectory to define the BBGKY hierarchy and Boltzmann hierarchy truncated observables. The convergence proof will then be reduced to the convergence of the corresponding truncated elementary observables. Given $\ell \in \mathbb{N}^2$, recall the notation from (8.1):

$$G_\ell(\epsilon_{(1,0)}, \epsilon_{(0,1)}, \epsilon_0, \delta) := G_\ell(\epsilon_{(1,0)}, 0) \cap G_\ell(\epsilon_{(0,1)}, 0) \cap G_\ell(\epsilon_0, \delta).$$

Fix $\mathbf{s} = (s_{(1,0)}, s_{(0,1)}) \in \mathbb{N}^2$, and let $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$, $1 \leq k \leq n$, $\alpha, \beta \in S_k$, $t \in [0, T]$, and $(J, M) \in \mathcal{U}_{s,k,\beta}$. Also fix $(t_1, \dots, t_k) \in \mathcal{T}_{k,\delta}(t)$. By Lemma 8.1, for any $V_{\mathbf{s}} \in \mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})$, we have

$$Z_{\mathbf{s}} = (X_{\mathbf{s}}, V_{\mathbf{s}}) \in G_{\mathbf{s}}(\epsilon_{(1,0)}, \epsilon_{(0,1)}, \epsilon_0, \delta).$$

This implies that by construction, since $t_0 - t_1 \geq \delta$, we obtain $Z_{\mathbf{s}}^\infty(t_1^+) \in G_{\mathbf{s}}(\epsilon_0, 0)$. We will inductively apply part 1(c) of Proposition 7.1. Given $i \in \{1, \dots, k\}$, assume that

$$(8.24) \quad Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+) \in G_{\mathbf{s}+\tilde{\beta}_{i-1}}(\epsilon_0, 0).$$

Then, there exists a set $\mathcal{B}_{m_i, \alpha_i}(Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+)) \subseteq \mathbb{S}_1^{d-1} \times B_R^d$ such that

$$(8.25) \quad Z_{\mathbf{s}+\tilde{\beta}_i}^\infty(t_{i+1}^+) \in G_{\mathbf{s}+\tilde{\beta}_i}(\epsilon_0, 0) \quad \forall (\omega_i, v_{\mathbf{s}\beta_i+\tilde{\beta}_{i-1}+1}^{\beta_i}) \in \mathcal{B}_{m_i, \alpha_i}^\infty(Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+)),$$

where

$$\mathcal{B}_{m_i, \alpha_i}^\infty(Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+)) := (\mathbb{S}_1^{d-1} \times B_R^d)^+((v_{m_i}^{\alpha_i})^\infty(t_i^+)) \setminus \mathcal{B}_{m_i, \alpha_i}(Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+)).$$

After completing this procedure, we finally obtain $Z_{\mathbf{s}+\tilde{\beta}_k}^\infty(0^+) \in G_{\mathbf{s}+\tilde{\beta}_k}(\epsilon_0, 0)$.

Let us now define the truncated elementary observables. Heuristically we will truncate the domains of adjusted particles in the definition of the observables $\tilde{I}_{\mathbf{s},k,R,\delta}^N$, $\tilde{I}_{\mathbf{s},k,R,\delta}^\infty$, defined in (8.4)–(8.5). More precisely, consider $1 \leq k \leq n$, $\alpha, \beta \in S_k$, $(J, M) \in \mathcal{U}_{s,k,\beta}$, and $t \in [0, T]$. For $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$, Lemma 8.1 implies there is a set of velocities $\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}}) \subseteq B_R^{d|s|}$ such that $Z_{\mathbf{s}} = (X_{\mathbf{s}}, V_{\mathbf{s}}) \in G_{\mathbf{s}}(\epsilon_{(1,0)}, \epsilon_{(0,1)}, \epsilon_0, \delta)$ for all $V_{\mathbf{s}} \in \mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})$. Now we define the BBGKY hierarchy truncated observables as

$$(8.26) \quad J_{\mathbf{s},k,R,\delta}^N(t, \alpha, \beta, J, M)(X_{\mathbf{s}}) = \int_{\mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})} \phi_{\mathbf{s}}(V_{\mathbf{s}}) \int_{\mathcal{T}_{k,\delta}(t)} T_{\mathbf{s}}^{t-t_1} \tilde{\mathcal{C}}_{\mathbf{s},\mathbf{s}+\tilde{\beta}_1,R}^{\alpha_1, m_1, j_1} T_{\mathbf{s}+\tilde{\beta}_1}^{t_1-t_2} \dots \tilde{\mathcal{C}}_{\mathbf{s}+\tilde{\beta}_{k-1}, \mathbf{s}+\tilde{\beta}_k, R}^{\alpha_k, m_k, j_k} T_{\mathbf{s}+\tilde{\beta}_k}^{t_k} f_{N,0}^{(\mathbf{s}+\tilde{\beta}_k)}(Z_{\mathbf{s}}) dt_k \dots dt_1 dV_{\mathbf{s}},$$

where for each $i = 1, \dots, k$ we recall the sets (8.25) and define

$$\tilde{\mathcal{C}}_{\mathbf{s}+\tilde{\beta}_{i-1}, \mathbf{s}+\tilde{\beta}_i, R}^{\alpha_i, m_i, j_i}(g_{\mathbf{s}+\tilde{\beta}_i}) = \mathcal{C}_{\mathbf{s}+\tilde{\beta}_{i-1}, \mathbf{s}+\tilde{\beta}_i, R}^{\alpha_i, m_i, j_i} \left[g_{\mathbf{s}+\tilde{\beta}_i} \mathbb{1}_{\mathcal{B}_{m_i, \alpha_i}^\infty(Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+))} \right].$$

In the same spirit, for $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$, we define the Boltzmann hierarchy truncated elementary observables:

$$(8.27) \quad J_{\mathbf{s},k,R,\delta}^\infty(t, \alpha, \beta, J, M)(X_{\mathbf{s}}) = \int_{\mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})} \phi_{\mathbf{s}}(V_{\mathbf{s}}) \int_{\mathcal{T}_{k,\delta}(t)} S_{\mathbf{s}}^{t-t_1} \tilde{\mathcal{C}}_{\mathbf{s},\mathbf{s}+\tilde{\beta}_1,R}^{\alpha_1, m_1, j_1} S_{\mathbf{s}+\tilde{\beta}_1}^{t_1-t_2} \dots \tilde{\mathcal{C}}_{\mathbf{s}+\tilde{\beta}_{k-1}, \mathbf{s}+\tilde{\beta}_k, R}^{\alpha_k, m_k, j_k} S_{\mathbf{s}+\tilde{\beta}_k}^{t_k} f_0^{(\mathbf{s}+\tilde{\beta}_k)}(Z_{\mathbf{s}}) dt_k \dots dt_1 dV_{\mathbf{s}},$$

where for each $i = 1, \dots, k$ we recall the sets (8.25) and define

$$\tilde{\mathcal{C}}_{\mathbf{s}+\tilde{\beta}_{i-1}, \mathbf{s}+\tilde{\beta}_i, R}^{\alpha_i, m_i, j_i}(g_{\mathbf{s}+\tilde{\beta}_i}) = \mathcal{C}_{\mathbf{s}+\tilde{\beta}_{i-1}, \mathbf{s}+\tilde{\beta}_i, R}^{\alpha_i, m_i, j_i} \left[g_{\mathbf{s}+\tilde{\beta}_i} \mathbb{1}_{\mathcal{B}_{m_i, \alpha_i}^\infty(Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+))} \right].$$

Recalling the observables $\tilde{I}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}$, $\tilde{I}_{\mathbf{s},k,R,\delta}^{\infty}$ defined in (8.4)–(8.5) and using Proposition 7.2, we obtain the following proposition.

PROPOSITION 8.2. *Let $n \in \mathbb{N}$, let $\mathbf{s} \in \mathbb{N}_+^2$, and let $\gamma, \epsilon_0, R, \eta, \delta$ be parameters as in (7.4). Additionally, let $(\mathbf{N}, \epsilon)$ be as in the scaling (2.28). Then, for all $t \in [0, T]$ the following estimates hold uniformly in $\mathbf{N}$:*

$$\begin{aligned} \sum_{k=1}^n \sum_{\alpha, \beta \in S_k} \sum_{(J, M) \in \mathcal{U}_{\mathbf{s}, k, \beta}} & \|\tilde{I}_{\mathbf{s}, k, R, \delta}^{\mathbf{N}}(t, \alpha, \beta, J, M) - J_{\mathbf{s}, k, R, \delta}^{\mathbf{N}}(t, \alpha, \beta, J, M)\|_{L^\infty(\Delta_{\mathbf{s}}^X(\epsilon_0))} \\ & \leq C_{d, \mathbf{s}, \mu_0, T}^n \|\phi_{\mathbf{s}}\|_{L_{\tilde{V}_{\mathbf{s}}}^\infty} R^{d(|\mathbf{s}|+2n)} \eta^{\frac{(d-1)(d+2)}{2d+2}} \|F_{\mathbf{N}, 0}\|_{\epsilon, \gamma_0, \mu_0}, \\ \sum_{k=1}^n \sum_{\alpha, \beta \in S_k} \sum_{(J, M) \in \mathcal{U}_{\mathbf{s}, k, \beta}} & \|\tilde{I}_{\mathbf{s}, k, R, \delta}^{\infty}(t, \alpha, \beta, J, M) - J_{\mathbf{s}, k, R, \delta}^{\infty}(t, \alpha, \beta, J, M)\|_{L^\infty(\Delta_{\mathbf{s}}^X(\epsilon_0))} \\ & \leq C_{d, \mathbf{s}, \mu_0, T}^n \|\phi_{\mathbf{s}}\|_{L_{\tilde{V}_{\mathbf{s}}}^\infty} R^{d(|\mathbf{s}|+2n)} \eta^{\frac{(d-1)(d+2)}{2d+2}} \|F_0\|_{0, \gamma_0, \mu_0}. \end{aligned}$$

Proof. As usual, it suffices to prove the estimate for the BBGKY hierarchy case and the Boltzmann hierarchy case follows similarly. Fix $k \in \{1, \dots, n\}$, $\alpha, \beta \in S_k$, and $(J, M) \in \mathcal{U}_{\mathbf{s}, k, \beta}$. We will bound the norm of the summand

$$(8.28) \quad \tilde{I}_{\mathbf{s}, k, R, \delta}^{\mathbf{N}}(t, \alpha, \beta, J, M) - J_{\mathbf{s}, k, R, \delta}^{\mathbf{N}}(t, \alpha, \beta, J, M)$$

and then use some combinatorial estimates to evaluate a bound on the whole sum. To bound this single term, note first that the Cauchy–Schwarz inequality and triangle inequality imply

$$(8.29) \quad |\langle \omega_1, v_1 - v \rangle| \leq 2R \quad \forall \omega_1 \in \mathbb{S}_1^{d-1}, \quad \forall v, v_1 \in B_R^d.$$

Therefore, we have for large R ,

$$(8.30) \quad \int_{\mathbb{S}^{d-1} \times B_R^d} |\langle \omega, v_1 - v \rangle| d\omega dv_1 \leq C_d R^{d+1} \quad \forall v \in B_R^d.$$

In order to estimate the iterated integrals (8.28), we must integrate over at least one of the sets $\mathcal{B}_{m_i, \alpha_i}(Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+))$ for some $i \in \{1, \dots, k\}$. By Proposition 7.2 and (8.29), we may estimate

$$(8.31) \quad \int_{\mathcal{B}_{m_i, \alpha_i}(Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+))} |\langle \omega_1, v_1 - v \rangle| d\omega_1 dv_1 \leq C_d |\mathbf{s} + \tilde{\beta}_{i-1}| R^{d+1} \eta^{\frac{d-1}{2d+2}} \quad \forall v \in B_R^d.$$

Moreover, we have the elementary inequalities

$$(8.32) \quad \|f_{\mathbf{N}, 0}^{(\mathbf{s}+\tilde{\beta}_k)}\|_{L^\infty} \leq e^{-(|\mathbf{s}|+k)\mu_0} \|F_{\mathbf{N}, 0}\|_{\epsilon, \gamma_0, \mu_0},$$

$$(8.33) \quad \int_{\mathcal{T}_{k, \delta}(t)} dt_1 \dots dt_k \leq \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} dt_1 \dots dt_k = \frac{t^k}{k!} \leq \frac{T^k}{k!},$$

and by Lemma 8.1 we have the estimate

$$(8.34) \quad |\mathcal{M}_{\mathbf{s}}(X_{\mathbf{s}})| \leq C_{d, \mathbf{s}} R^{d|\mathbf{s}|} \eta^{\frac{d-1}{2}}.$$

Therefore, (8.29)–(8.34) imply that for some $i \in \{1, \dots, k\}$ and sufficiently large R, n ,

$$\begin{aligned} & \left| \tilde{I}_{\mathbf{s},k,R,\delta}^N(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M) - J_{\mathbf{s},k,R,\delta}^N(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M) \right| \leq \|\phi_{\mathbf{s}}\|_{L^\infty} e^{-(|\mathbf{s}|+k)\mu_0} \|F_{N,0}\|_{\epsilon, \gamma_0, \mu_0} \\ & \quad \times C_{d,\mathbf{s}} R^{d|\mathbf{s}|} \eta^{\frac{d-1}{2}} C_d^{k-1} R^{(d+1)(k-1)} C_d(|\mathbf{s} + \tilde{\beta}_{i-1}|) R^{d+1} \eta^{\frac{d-1}{2d+2}} \frac{T^k}{k!} \\ & \leq C_{d,\mathbf{s},T,\mu_0}^k \|\phi_{\mathbf{s}}\|_{L^\infty} \|F_{N,0}\|_{\epsilon, \gamma_0, \mu_0} \frac{(|\mathbf{s}|+k)}{k!} R^{d(|\mathbf{s}|+2n)} \eta^{\frac{(d-1)(d+2)}{2d+2}}. \end{aligned}$$

Adding for all $(J, M) \in \mathcal{U}_{\mathbf{s},k,\beta}$ we have $2^k |\mathbf{s}|(|\mathbf{s}|+1) \dots (|\mathbf{s}|+k) \leq 2^k (|\mathbf{s}|+k)^k$ contributions, and thus

$$\begin{aligned} & \sum_{(J,M) \in \mathcal{U}_{\mathbf{s},k,\beta}} \|\tilde{I}_{\mathbf{s},k,R,\delta}^N(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M) - J_{\mathbf{s},k,R,\delta}^N(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \\ & \leq C_{d,\mathbf{s},\mu_0,T}^k \frac{2^k (|\mathbf{s}|+k)^{k+1}}{k!} \|\phi_{\mathbf{s}}\|_{L_{V_s}^\infty} \|F_{N,0}\|_{\epsilon, \gamma_0, \mu_0} R^{d(|\mathbf{s}|+2n)} \eta^{\frac{(d-1)(d+2)}{2d+2}} \\ & \leq C_{d,\mathbf{s},\mu_0,T}^k \|\phi_{\mathbf{s}}\|_{L_{V_s}^\infty} R^{d(|\mathbf{s}|+2n)} \eta^{\frac{(d-1)(d+2)}{2d+2}} \|F_{N,0}\|_{\epsilon, \gamma_0, \mu_0} \end{aligned}$$

since $\frac{2^k (|\mathbf{s}|+k)^{k+1}}{k!} \leq C_{\mathbf{s}}^k$. Summing over $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k$, $k = 1, \dots, n$, we gain a factor of $\sum_{k=1}^n 4^k \leq n4^n \leq C^n$ in the full sum but still obtain the required estimate. $\square$

9. Convergence proof. In order to conclude the convergence proof, we will estimate the differences of truncated elementary observables for the BBGKY and Boltzmann hierarchies in the scaled limit.

9.1. BBGKY pseudo-trajectories and proximity to Boltzmann pseudo-trajectories. Let $\mathbf{s} = (s_{(1,0)}, s_{(0,1)}) = (s_1, s_2) \in \mathbb{N}^2$, $Z_{\mathbf{s}} \in \mathbb{R}^{2d|\mathbf{s}|}$, $1 \leq k \leq n$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k$, and $t \in [0, T]$. Moreover, fix (N, ϵ) to obey the Boltzmann-Grad scaling (2.28). Let us recall from the set $\mathcal{T}_k(t)$ defined in (6.3). Consider $(t_1, \dots, t_k) \in \mathcal{T}_k(t)$, $J = (j_1, \dots, j_k)$, $M = (m_1, \dots, m_k)$ with $(J, M) \in \mathcal{U}_{\mathbf{s},k,\beta}$. We define the BBGKY hierarchy pseudo-trajectory of $Z_{\mathbf{s}}$ in an inductive manner similar to that of the Boltzmann pseudo-trajectory, with the appropriate modifications to account for the positive diameter of the particles.

DEFINITION 9.1 (BBGKY particle adjunction). *Let $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$, $\alpha, \beta \in \mathcal{T}$, and $m \in \{1, \dots, s_\alpha\}$. Then, for any $Z_{\mathbf{s}} = (X_{\mathbf{s}}, V_{\mathbf{s}}) \in \mathbb{R}^{2d|\mathbf{s}|}$ and $(\omega, v) \in \mathbb{R}^d \times \mathbb{R}^d$ define*

$$(9.1) \quad Z_{\mathbf{s}} \oplus_{\beta}^{\alpha, m}(\omega, v) := \begin{cases} \left(X_{s_1}^{(1,0)}, x_m^{\alpha} + \omega, X_{s_2}^{(0,1)}, V_{s_1}^{(1,0)}, v, V_{s_2}^{(0,1)} \right), & \beta = (1, 0), \\ \left(X_{s_1}^{(1,0)}, X_{s_2}^{(0,1)}, x_m^{\alpha} + \omega, V_{s_1}^{(1,0)}, V_{s_2}^{(0,1)}, v \right), & \beta = (0, 1). \end{cases}$$

That is, the vector $Z_{\mathbf{s}} \oplus_{\beta}^{\alpha, m}(\omega, v)$ is the result of adding a new particle to the configuration $Z_{\mathbf{s}}$ that is of type β with position $x_m^{\alpha} + \omega$ and velocity v .

Time $t_0 = t$. We initially define $Z_{\mathbf{s}}^N(t_0^-) := Z_{\mathbf{s}}$. We will denote the components of $Z_{\mathbf{s}}^N(t_0^-)$ by

$$(9.2) \quad Z_{\mathbf{s}}^N(t_0^-) = (X_{\mathbf{s}}^N(t_0^-), V_{\mathbf{s}}^N(t_0^-)).$$

Individual components of the vectors $X_{\mathbf{s}}^N(t_0^-), V_{\mathbf{s}}^N(t_0^-)$ will simply be written as $(x_i^{\alpha})^N(t_0^-)$ or $(v_i^{\alpha})^N(t_0^-)$, respectively.

Time t_i with $i \in \{1, \dots, k\}$. Consider $i \in \{1, \dots, k\}$, and assume we know $(Z_{\mathbf{s}+\tilde{\beta}_{i-1}})^{\mathbf{N}}(t_{i-1}^-)$. We define $(Z_{\mathbf{s}+\tilde{\beta}_{i-1}})^{\mathbf{N}}(t_i^+)$ as follows:

$$(Z_{\mathbf{s}+\tilde{\beta}_{i-1}})^{\mathbf{N}}(t_i^+) := \left((X_{\mathbf{s}+\tilde{\beta}_{i-1}})^{\mathbf{N}}(t_{i-1}^-) - (t_{i-1} - t_i)(V_{\mathbf{s}+\tilde{\beta}_{i-1}})^{\mathbf{N}}(t_{i-1}^-), (V_{\mathbf{s}+\tilde{\beta}_{i-1}})^{\mathbf{N}}(t_{i-1}^-) \right).$$

We also define $(Z_{\mathbf{s}+\tilde{\beta}_i})^{\mathbf{N}}(t_i^-)$ depending on the sign of j_i as

$$(9.3) \quad (Z_{\mathbf{s}+\tilde{\beta}_i})^{\mathbf{N}}(t_i^-) := \begin{cases} (Z_{\mathbf{s}+\tilde{\beta}_{i-1}})^{\mathbf{N}}(t_i^+) \oplus_{\beta_i}^{\alpha_i, m_i} (j_i \epsilon_{(\alpha_i, \beta_i)} \omega_i, v_{s_{\beta_i} + \tilde{\beta}_{i-1}}^{\beta_i}), & j_i = -1, \\ T_{(m_i, s_{\beta_i} + \tilde{\beta}_{i-1}}^{(\alpha_i, \beta_i)} \left((Z_{\mathbf{s}+\tilde{\beta}_{i-1}})^{\mathbf{N}}(t_i^+) \oplus_{\beta_i}^{\alpha_i, m_i} (j_i \epsilon_{(\alpha_i, \beta_i)} \omega_i, v_{s_{\beta_i} + \tilde{\beta}_{i-1}}^{\beta_i}) \right), & j_i = 1, \end{cases}$$

where $T_{(m_i, s_{\beta_i} + \tilde{\beta}_{i-1})}^{(\alpha_i, \beta_i)}$ is the impact operator given in Definition 3.2.

Time $t_{k+1} = 0$. We finally obtain

$$Z_{\mathbf{s}+\tilde{\beta}_k}^{\mathbf{N}}(0^+) = Z_{\mathbf{s}+\tilde{\beta}_k}^{\mathbf{N}}(t_{k+1}^+) = \left(X_{\mathbf{s}+\tilde{\beta}_k}^{\mathbf{N}}(t_k^-) - t_k V_{\mathbf{s}+\tilde{\beta}_k}^{\mathbf{N}}(t_k^-), V_{\mathbf{s}+\tilde{\beta}_k}^{\mathbf{N}}(t_k^-) \right).$$

DEFINITION 9.2. Let $t > 0$, $Z_{\mathbf{s}} \in \mathbb{R}^{2d(s_1+s_2)}$, $(t_1, \dots, t_k) \in \mathcal{T}_k(t)$, $1 \leq k \leq n$, $\alpha, \beta \in S_k$, $(J, M) \in \mathcal{U}_{\mathbf{s}, k, \beta}$, and for each $i = 1, \dots, k$ let $(\omega_i, v_{s_{\beta_i} + \tilde{\beta}_{i-1} + 1}^{\beta_i}) \in \mathbb{S}_1^{d-1} \times B_R^d$.

We call the sequence $\{Z_{\mathbf{s}+\tilde{\beta}_{i-1}}^{\mathbf{N}}(t_i^+)\}_{i=1}^k$ defined above the BBGKY pseudo-trajectory.

LEMMA 9.1. Fix $\mathbf{s} = (s_{(1,0)}, s_{(0,1)}) = (s_1, s_2) \in \mathbb{N}^2$, $n \in \mathbb{N}_+$, and $(\mathbf{N}, \epsilon)$. Fix $t \in [0, T]$, $(t_1, \dots, t_k) \in \mathcal{T}_k(t)$, $1 \leq k \leq n$, $\alpha, \beta \in S_k$, $(J, M) \in \mathcal{U}_{\mathbf{s}, k, \beta}$, and for each $i = 1, \dots, k$ let $(\omega_i, v_{s_{\beta_i} + \tilde{\beta}_{i-1} + 1}^{\beta_i}) \in \mathbb{S}_1^{d-1} \times B_R^d$. Then, for each $i = 1, \dots, k+1$ and each $\sigma \in \mathcal{T}$, we have

$$(9.4) \quad |(x_\ell^\sigma)^\infty(t_i^+) - (x_\ell^\sigma)^{\mathbf{N}}(t_i^+)| \leq \sqrt{2}(i-1) \max_{\alpha \in \mathcal{T}} \epsilon_\alpha, \quad (v_\ell^\sigma)^\infty(t_i^+) = (v_\ell^\sigma)^{\mathbf{N}}(t_i^+)$$

for each⁷ $\ell = 1, \dots, s_\sigma + \tilde{\beta}_{i-1}^\sigma$. In particular, for $s_1, s_2 < n$ and $i = 1, \dots, k+1$, we have

$$(9.5) \quad |X_{\mathbf{s}+\tilde{\beta}_{i-1}}^{\mathbf{N}}(t_i^+) - X_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+)| \leq \sqrt{8}n^2 \max_{\alpha \in \mathcal{T}} \epsilon_\alpha.$$

Proof. The statement (9.4) follows from a simple inductive argument. See [17] for details. For the uniform bound (9.5), fix $s_1, s_2 < n$, $1 \leq k \leq n$, and $1 \leq i \leq k+1$. Now apply (9.4) to calculate

$$\left| X_{\mathbf{s}+\tilde{\beta}_{i-1}}^{\mathbf{N}}(t_i^+) - X_{\mathbf{s}+\tilde{\beta}_{i-1}}^\infty(t_i^+) \right|^2 = \sum_{\sigma \in \mathcal{T}} \sum_{\ell=1}^{s_\sigma + \tilde{\beta}_{i-1}^\sigma} |(x_\ell^\sigma)^\infty(t_i^+) - (x_\ell^\sigma)^{\mathbf{N}}(t_i^+)|^2 \leq 8n^4 \max_{\alpha \in \mathcal{T}} (\epsilon_\alpha)^2.$$

Taking square roots proves (9.5). $\square$

9.2. Truncated observables in terms of pseudo-trajectories. We will now write the truncated observables coming from the Boltzmann hierarchy in terms of

⁷Here, the number $s_\sigma + \tilde{\beta}_{i-1}^\sigma$ comes from the total number of particles in the system plus the number of particles we have added in constructing the BBGKY pseudo-trajectory.

the Boltzmann pseudo-trajectories. By Definition 8.3 of the Boltzmann pseudo-trajectories, we may rewrite the truncated observables given in (8.27) as

$$\begin{aligned}
 & J_{\mathbf{s},k,R,\delta}^\infty(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M)(X_{\mathbf{s}}) \\
 &= A_{\boldsymbol{\alpha}, \boldsymbol{\beta}, k}^\infty \int_{\mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})} \phi_{\mathbf{s}}(V_{\mathbf{s}}) \int_{\mathcal{T}_{k,\delta}(t)} \int_{B_1^c} \cdots \int_{B_k^c} f_0^{(\mathbf{s}+\tilde{\boldsymbol{\beta}}_k)} \left(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_k}^\infty(0^+) \right) \\
 & \quad \prod_{i=1}^k j_i \langle \omega_i, (v_{s_{\beta_i} + \tilde{\beta}_{i-1} + 1}^{\beta_i})^\infty(t_i^+) - v_{m_i}^{\alpha_i} \rangle_+ d\omega_k dv_{s_{\beta_k} + \tilde{\beta}_{k-1} + 1}^{\beta_k} \\
 (9.6) \quad & \dots d\omega_1 dv_{s_{\beta_1} + 1}^{\beta_1} dt_k \dots dt_1 dt dV_{\mathbf{s}},
 \end{aligned}$$

where we recall the sets given in (8.25) and define $B_i^c := \mathcal{B}_{m_i, \alpha_i}^c(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{i-1}}^\infty(t_i^+))$ and $A_{\boldsymbol{\alpha}, \boldsymbol{\beta}, k}^\infty := \prod_{i=1}^k A_{\beta_i}^{\alpha_i}$. The constants A_{β}^{α} are determined by our scaling (2.28) and are explicitly given in (2.35).

Since the ϵ particle flow may include recollisions, it is not immediately clear that we can perform the same expansion for the BBGKY truncated observables $J_{\mathbf{s},k,R,\delta}^N(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M)(X_{\mathbf{s}})$ as given in (8.26) in terms of the BBGKY pseudo-trajectories given in Definition 9.2. However, due to the angle and velocity sets which we excluded in constructing $J_{\mathbf{s},k,R,\delta}^N(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M)(X_{\mathbf{s}})$ from $\tilde{I}_{\mathbf{s},k,R,\delta}^N(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M)(X_{\mathbf{s}})$ (as given in (8.14)), we claim the relevant ϵ flows will not experience recollisions. To show this claim, fix initial positions $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$, $1 \leq k \leq n$, $(J, M) \in \mathcal{U}_{\mathbf{s},k,\boldsymbol{\beta}}$, and $(t_1, \dots, t_k) \in \mathcal{T}_{k,\delta}(t)$ for $t \in [0, T]$. Consider $(\mathbf{N}, \epsilon)$ that obey the scalings (2.28) and, moreover, satisfy

$$(9.7) \quad n^2 \max_{\alpha \in \mathcal{T}} \epsilon_{\alpha} \ll \gamma,$$

where the implicit constants depend only on universal constants. Additionally assume that $s_1, s_2 < n$. Then, given $V_{\mathbf{s}} \in \mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})$, we have by Lemma 8.1 that

$$Z_{\mathbf{s}} = (X_{\mathbf{s}}, V_{\mathbf{s}}) \in G_{\mathbf{s}}(\epsilon_{(1,0)}, \epsilon_{(0,1)}, \epsilon_0, \delta),$$

where $G_{\mathbf{s}}(\epsilon_{(1,0)}, \epsilon_{(0,1)}, \epsilon_0, \delta)$ is defined in (8.1). Recalling the operators $\Psi_{\mathbf{s},\epsilon}^\tau, \Phi_{\mathbf{s}}^\tau$ given in (3.1), (4.16), we obtain

$$(9.8) \quad \Psi_{\mathbf{s},\epsilon}^{t_1-t_0}(Z_{\mathbf{s}}) = \Phi_{\mathbf{s}}^{t_1-t_0}(Z_{\mathbf{s}}) = Z_{\mathbf{s}}^N(t_1^+)$$

since $t_0 - t_1 \geq \delta$. Moreover, recall that by (8.25) we have by construction that for each $i \in \{1, \dots, k\}$,

$$Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i}^\infty(t_{i+1}^+) \in G_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i}(\epsilon_0, 0) \quad \forall (\omega_i, v_{s_{\beta_i} + \tilde{\beta}_{i-1} + 1}^{\beta_i}) \in \mathcal{B}_{m_i, \alpha_i}^c(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{i-1}}^\infty(t_i^+)).$$

Since we have $s_1, s_2 < n$, we obtain by Lemma 9.1, recalling (9.7), that

$$(9.9) \quad |X_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{i-1}}^N(t_i^+) - X_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{i-1}}^\infty(t_i^+)| \leq \sqrt{8}n^2 \max_{\alpha \in \mathcal{T}} \epsilon_{\alpha} \leq \frac{\gamma}{2}.$$

Now, note by parts 1(a) and 2(a) of Proposition 7.1 that for each $i \in \{1, \dots, k\}$ we have picked the set $\mathcal{B}_{m_i, \alpha_i}^c(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{i-1}}^\infty(t_i^+))$ so that by (9.9) we have that $Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i}^N(t_i^-)$ lies in the interior of the phase space $\mathcal{D}_{\epsilon}^{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i}$. Therefore, we have

$$(9.10) \quad \Psi_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i, \epsilon}^{t_{i+1}-t_i}(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i}^N(t_i^-)) = \Phi_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i}^{t_{i+1}-t_i}(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i}^N(t_i^-)) = Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i}^N(t_{i+1}^+).$$

Combining (9.8) and (9.10) together, we obtain the following expansion for the truncated observable:

$$\begin{aligned} J_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M)(X_{\mathbf{s}}) &= A_{\mathbf{s},\boldsymbol{\alpha},\boldsymbol{\beta},k}^{\mathbf{N}} \int_{\mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})} \phi_{\mathbf{s}}(V_{\mathbf{s}}) \int_{\mathcal{T}_{k,\delta}} \int_{B_1^c} \\ &\cdots \int_{B_k^c} f_{\mathbf{N},0}^{(\mathbf{s}+\tilde{\boldsymbol{\beta}}_k)} \left(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_k}^{\mathbf{N}}(0^+) \right) \\ &\prod_{i=1}^k j_i \langle \omega_i, (v_{s_{\beta_i}+\tilde{\beta}_{i-1}+1}^{\beta_i})^{\mathbf{N}}(t_i^+) - v_{m_i}^{\alpha_i} \rangle_+ d\omega_k dv_{s_{\beta_k}+\tilde{\beta}_{k-1}+1}^{\beta_k} \\ &\cdots d\omega_1 dv_{s_{\beta_1}+1}^{\beta_1} dt_k \cdots dt_1 dt dV_{\mathbf{s}}, \end{aligned}$$

where we are defining $B_i^c := \mathcal{B}_{m_i, \alpha_i}^c(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_{i-1}}^{\infty}(t_i^+))$ as in (8.25). The constant $A_{\mathbf{s},\boldsymbol{\alpha},\boldsymbol{\beta},k}^{\mathbf{N}}$ is given by the following formula:

$$A_{\mathbf{s},\boldsymbol{\alpha},\boldsymbol{\beta},k}^{\mathbf{N}} := \prod_{i=1}^k A_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i, (\alpha_i, \beta_i)}^{\mathbf{N}}, \quad \text{where} \quad A_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_i, (\alpha_i, \beta_i)}^{\mathbf{N}} := (N_{\beta_i} - s_{\beta_i} - \tilde{\beta}_{i-1}^{\beta_i}) \epsilon_{(\alpha_i, \beta_i)}^{d-1}.$$

REMARK 9.1. Note that as $\mathbf{N} \rightarrow \infty$ and $\epsilon \rightarrow 0$ according to the scalings (2.28), we have for fixed $\mathbf{s} \in \mathbb{N}^2$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k$, and $k \in \mathbb{N}_+$ that $A_{\mathbf{s},\boldsymbol{\alpha},\boldsymbol{\beta},k}^{\mathbf{N}} \nearrow A_{\boldsymbol{\alpha},\boldsymbol{\beta},k}^{\infty}$. Moreover, we have the trivial bound

$$(9.11) \quad 0 < 1 - (A_{\boldsymbol{\alpha},\boldsymbol{\beta},k}^{\infty})^{-1} A_{\mathbf{s},\boldsymbol{\alpha},\boldsymbol{\beta},k}^{\mathbf{N}} \leq C_{\mathbf{s}} \max_{\alpha \in \mathcal{T}} \epsilon_{\alpha}^{d-1}.$$

Let us now approximate the BBGKY truncated observables in terms of the Boltzmann initial data and the kernel coming from the Boltzmann pseudo-trajectories. Let $\mathbf{s} = (s_{(1,0)}, s_{(0,1)}) \in \mathbb{N}_+^2$, $X_{\mathbf{s}} \in \Delta_{\mathbf{s}}^X(\epsilon_0)$, $1 \leq k \leq n$, $(J, M) \in \mathcal{U}_{\mathbf{s},k,\boldsymbol{\beta}}$, and $(t_1, \dots, t_k) \in \mathcal{T}_{k,\delta}(t)$ for $t \in [0, T]$. Define for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k$ the functional

$$\begin{aligned} \hat{J}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M)(X_{\mathbf{s}}) \\ &:= A_{\boldsymbol{\alpha},\boldsymbol{\beta},k}^{\infty} \int_{\mathcal{M}_{\mathbf{s}}^c(X_{\mathbf{s}})} \phi_{\mathbf{s}}(V_{\mathbf{s}}) \int_{\mathcal{T}_{k,\delta}} \int_{B_1^c} \cdots \int_{B_k^c} f_0^{(\mathbf{s}+\tilde{\boldsymbol{\beta}}_k)} \left(Z_{\mathbf{s}+\tilde{\boldsymbol{\beta}}_k}^{\mathbf{N}}(0^+) \right) \\ &\prod_{i=1}^k j_i \langle \omega_i, (v_{s_{\beta_i}+\tilde{\beta}_{i-1}+1}^{\beta_i})^{\mathbf{N}}(t_i^+) - v_{m_i}^{\alpha_i} \rangle_+ d\omega_k dv_{s_{\beta_k}+\tilde{\beta}_{k-1}+1}^{\beta_k} \\ &\cdots d\omega_1 dv_{s_{\beta_1}+1}^{\beta_1} dt_k \cdots dt_1 dt dV_{\mathbf{s}}. \end{aligned}$$

We can now approximate the functional $J_{\mathbf{s},k,R,\delta}^{\mathbf{N}}$ in terms of the functional $\hat{J}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}$.

PROPOSITION 9.1. Let $n \in \mathbb{N}$, and let $\mathbf{s} := (s_1, s_2) \in \mathbb{N}_+^2$ such that $s_1, s_2 < n$. Fix parameters $\gamma, \epsilon_0, R, \eta, \delta$ as in (7.4), and let $t \in [0, T]$. Then, given $\zeta > 0$, there exists a pair $(N_1^*, N_2^*) \in \mathbb{N}^2$ with $N_i^* = N_i^*(\zeta, n, \gamma, \eta, \epsilon_0)$ such that for all $N_i \geq N_i^*$, $i = 1, 2$, for which $(\mathbf{N}, \boldsymbol{\epsilon}) = (N_1, N_2, \epsilon_1, \epsilon_2)$ obey the scalings of (2.28) we have

$$\begin{aligned} &\sum_{k=1}^n \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_k} \sum_{(J, M) \in \mathcal{U}_{\mathbf{s},k,\boldsymbol{\beta}}} \|J_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M) - \hat{J}_{\mathbf{s},k,R,\delta}^{\mathbf{N}}(t, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, M)\|_{L_{\mathbf{s}}^{\infty}(\Delta_{\mathbf{s}}^X(\epsilon_0))} \\ &\leq C_{d,\mathbf{s},\mu_0,T}^n \|\phi_{\mathbf{s}}\|_{L^{\infty}} R^{d(|\mathbf{s}|+2n)} \zeta^2. \end{aligned}$$

In the case of conditioned tensorized initial data as in Theorem 5.2, we have the upgraded bound

$$\begin{aligned} & \sum_{k=1}^n \sum_{\alpha, \beta \in S_k} \sum_{(J, M) \in \mathcal{U}_{s, k, \beta}} \|J_{s, k, R, \delta}^N(t, \alpha, \beta, J, M) - \widehat{J}_{s, k, R, \delta}^N(t, \alpha, \beta, J, M)\|_{L_s^\infty(\Delta_s^X(\epsilon_0))} \\ & \leq C_{d, s, \mu_0, T}^n \|\phi_s\|_{L^\infty} R^{d(|s|+2n)} \max_{\alpha \in \mathcal{T}} \epsilon_\alpha. \end{aligned}$$

Proof. Fix $1 \leq k \leq n$, $\alpha, \beta \in S_k$, and $(J, M) \in \mathcal{U}_{s, k, \beta}$. Assume $(N_1, N_2, \epsilon_1, \epsilon_2)$ obey the scalings (2.28). Then, for N_i large enough, the scaling assumption implies that

$$(9.12) \quad n^2 \max(\epsilon_1, \epsilon_2) \ll \gamma,$$

where the implicit constants are universal. By an argument similar to that of [6], we can bound

$$\begin{aligned} & \|J_{s, k, R, \delta}^N(t, \alpha, \beta, J, M) - \widehat{J}_{s, k, R, \delta}^N(t, \alpha, \beta, J, M)\|_{L_s^\infty(\Delta_s^X(\epsilon_0))} \leq \frac{C_{d, T, \mu_0}^k}{k!} \|\phi_s\|_{L^\infty} R^{d(|s|+2k)} \\ & \times \left(\|f_{N, 0}^{(s+\tilde{\beta}_k)} - f_0^{(s+\tilde{\beta}_k)}\|_{L^\infty(\Delta_{s+\tilde{\beta}_k}(\epsilon_0/2))} + |(A_{\alpha, \beta}^\infty)^{-1} A_{s, \alpha, \beta, k}^N - 1| \|F_0\|_{0, \gamma_0, \mu_0} \right). \end{aligned}$$

Summing the above estimate over all α, β, J, M , and k , we use the elementary inequality

$$\sum_{k=1}^n \frac{C_{d, T, \mu_0}^k}{k!} 4^k 2^k |s|(|s|+1) \dots (|s|+k) \leq C_{d, T, \mu_0, s}^m.$$

Using Definition 5.2, Remark 9.1, and the case of conditioned tensorized initial data, the estimate (5.4) finishes the proof. $\square$

Next, we compare the functionals $\widehat{J}_{s, k, R, \delta}^N(t, \alpha, \beta, J, M)$ to $J_{s, k, R, \delta}^\infty(t, \alpha, \beta, J, M)$. The following proposition crucially uses the continuity assumption on our initial data F_0 .

PROPOSITION 9.2. *Let $n \in \mathbb{N}$, and let $s_1, s_2 < n$. Fix parameters $\gamma, \epsilon_0, R, \eta, \delta$ as in (7.4), and let $t \in [0, T]$. Then, given $\zeta > 0$, there exists a pair $(N_1^{**}, N_2^{**}) \in \mathbb{N}^2$ with $N_i^{**} = N_i^{**}(\zeta, n, c_1, c_2, a, b)$ such that for all $N_i \geq N_i^{**}$, $i = 1, 2$, for which $(N_1, N_2, \epsilon_1, \epsilon_2)$ obey the scalings of (2.28) we have*

$$\begin{aligned} & \sum_{k=1}^n \sum_{\alpha, \beta \in S_k} \sum_{(J, M) \in \mathcal{U}_{s, k, \beta}} \|\widehat{J}_{s, k, R, \delta}^N(t, \alpha, \beta, J, M) - J_{s, k, R, \delta}^\infty(t, \alpha, \beta, J, M)\|_{L_s^\infty(\Delta_s^X(\epsilon_0))} \\ & \leq C_{d, s, \mu_0, T}^n \|\phi_s\|_{L^\infty} R^{d(|s|+2n)} \zeta^2. \end{aligned}$$

For the case of conditioned, tensorized, and Hölder initial data in $\mathcal{C}^{0, \lambda}$, we have the improved estimate

$$\begin{aligned} & \sum_{k=1}^n \sum_{\alpha, \beta \in S_k} \sum_{(J, M) \in \mathcal{U}_{s, k, \beta}} \|\widehat{J}_{s, k, R, \delta}^N(t, \alpha, \beta, J, M) - J_{s, k, R, \delta}^\infty(t, \alpha, \beta, J, M)\|_{L_s^\infty(\Delta_s^X(\epsilon_0))} \\ & \leq C_{d, s, \mu_0, T}^n \|\phi_s\|_{L^\infty} R^{d(|s|+2n)} (\max_{\alpha \in \mathcal{T}} \epsilon_\alpha)^\lambda. \end{aligned}$$

Proof. Let $\zeta > 0$ be given. Fix $1 \leq k \leq n$, $\alpha, \beta \in S_k$, and $(J, M) \in \mathcal{U}_{s,k,\beta}$. Then, since $s_1, s_2 < n$, we may apply Lemma 9.1 to obtain

$$(9.13) \quad |Z_{s+\tilde{\beta}_k}^N(0^+) - Z_{s+\tilde{\beta}_k}^\infty(0^+)| \leq \sqrt{8}n^2 \max(\epsilon_1, \epsilon_2).$$

According to (2.28), there exists $(N_1^{**}, N_2^{**}) \in \mathbb{N}^2$ with $N_i^{**} = N_i^{**}(\zeta, n, c_1, c_2, a, b)$ such that for all $N_i \geq N_i^{**}$, the right-hand side of (9.13) is so small that by the continuity condition (5.5) we have

$$|f_0^{(s+\tilde{\beta}_k)}(Z_{s+\tilde{\beta}_k}^N(0^+)) - f_0^{(s+\tilde{\beta}_k)}(Z_{s+\tilde{\beta}_k}^\infty(0^+))| \leq C^{|s|+k-1}\zeta^2$$

for all $Z_s \in \mathbb{R}^{2d(s_1+s_2)}$. As in the proof of Proposition 9.1, summing this inequality over $1 \leq k \leq n$, $\alpha, \beta \in S_k$, and $(J, M) \in \mathcal{U}_{s,k,\beta}$ proves the first part of the proposition. For the second estimate, note that for any $Z_\ell, Z'_\ell \in \mathbb{R}^{2d|\ell|}$ we have by induction for $\ell = (\ell_1, \ell_2) \in \mathbb{N}_+^2$

$$(9.14) \quad |g_0^{\otimes \ell_1} \otimes h_0^{\otimes \ell_2}(Z_\ell) - g_0^{\otimes \ell_1} \otimes h_0^{\otimes \ell_2}(Z'_\ell)| \leq C_{d,\ell} |Z_\ell - Z'_\ell|^\lambda.$$

Applying this estimate with (9.13), the proof is complete. $\square$

9.3. Proof of the main theorem.

Proof of Theorem 5.1. Here, we choose parameters $n \in \mathbb{N}$, $\delta, \eta, \gamma, \epsilon_0 > 0$ and $R > 1$ to show convergence in Theorem 5.1. First, fix $s = (s_1, s_2) \in \mathbb{N}^2$ and $\phi_s \in C_c(\mathbb{R}^{d|s|})$. Define the constant $C_{s,\gamma_0,\mu_0,T} > 1$ to be the maximum of all of the constants found in Propositions 6.1, 8.1, 8.2, 9.1, and 9.2. Then, define the constant

$$C := C_{s,\gamma_0,\mu_0,T} \|\phi_s\|_{L_s^\infty} \max(1, \|F_0\|_{\infty,\gamma_0,\mu_0}) > 1.$$

Then, let $\sigma > 0$ and $0 < \zeta < 1$ be sufficiently small so that

$$(9.15) \quad \zeta e^{\gamma_0 \zeta^{-1/3}} > C.$$

We pick parameters so that for all sufficiently large N_1, N_2 and sufficiently small ϵ_1, ϵ_2 which obey (2.28) we have for all $t \in [0, T]$ that $\|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\sigma))} \lesssim \zeta$. We choose these parameters in the following order:

- (P₁) Pick $n \in \mathbb{N}$ such that $n > \max(s_1, s_2, \log_4(C\zeta^{-1}))$. This implies that $s_1, s_2 < n$ and $C4^{-n} < \zeta$.
- (P₂) Pick $\delta > 0$ such that $\delta < \zeta C^{-(n+1)}$. This implies that $\delta C^{n+1} < \zeta$.
- (P₃) Pick $\eta > 0$ such that $\eta < \zeta^{4/(d-1)}$. This implies that $\eta^{(d-1)/2} < \zeta^2$.
- (P₄) Pick $R > 1$ such that $\max(1, \sqrt{3}\gamma_0^{-1/2} \log^{1/2}(C\zeta^{-1})) < R < \zeta^{-1/(4dn)} C^{-1/(4d)}$. This implies that $\zeta^2 R^{4dn} C^n < \zeta$ and $Ce^{-\gamma_0 R^2/3} < \zeta$.
- (P₅) Pick $\epsilon_0 > 0$ such that $\epsilon_0 \ll \eta\delta$ as in (7.4), and let $\epsilon_0 < \sigma$.
- (P₆) Pick $\gamma > 0$ such that $\gamma \ll \epsilon_0$ and $\gamma \ll \epsilon_0 R^{-1}\eta$ as in (7.4).

Note that (P₁) implies that $s_1, s_2 < n$, so the fact that $R > 1$ implies $R^{d|s|}, R^{d(|s|+2n)} \leq R^{4dn}$. Moreover, since $\frac{d-1}{2} < \frac{(d-1)(d+2)}{2d+2}$ and $\eta < 1$, we get $\eta^{(d-1)/2} > \eta^{(d-1)(d+2)/(2d+2)}$. Hence, we have

$$C^n (R^{d|s|} \eta^{(d-1)/2} + R^{d(|s|+2n)} \eta^{(d-1)(d+2)/(2d+2)}) \leq 2C^n R^{4dn} \eta^{(d-1)/2},$$

$$C^n R^{d(|s|+2n)} \zeta \leq C^n R^{4dn} \zeta.$$

Now note that for $i = 1, 2$, if $N_i \geq \max(N_i^*, N_i^{**})$ and N_i so large that $\max(\epsilon_1, \epsilon_2) \ll \gamma$ as in (7.4), our parameter choices (P_1) – (P_6) imply that the conditions on our parameters from Propositions 6.1, 8.1, 8.2, 9.1, and 9.2 hold. Moreover, we have that

(9.16)

$$\begin{aligned} \|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} &\leq C(4^{-n} + e^{-\gamma_0 R^2/3} + \delta C^n) \\ &\quad + C^n(R^{d|s|}\eta^{(d-1)/2} + R^{d(|s|+2n)}\eta^{(d-1)(d+2)/(2d+2)}) \\ &\quad + C^n R^{d(|s|+2n)}\zeta^2 \end{aligned}$$

(9.17)

$$\leq 6\zeta$$

for all $N_i \geq \max(N_i^*, N_i^{**})$, $(N_1, N_2, \epsilon_1, \epsilon_2)$ that satisfy the mixed Boltzmann-Grad scalings (2.28). Because of our choice (P_5) , we have $\epsilon_0 < \sigma$, which implies $\Delta_s^X(\sigma) \subset \Delta_s^X(\epsilon_0)$. This, combined with the bound (9.17), concludes the proof. $\square$

Proof of Theorem 5.2. The inclusion of F_0 and F in the correct spaces follows from the definition of the norms and the continuity estimate of Theorem 4.3. It follows by a computation that F indeed solves the Boltzmann hierarchy. By Theorem 5.1 it thus suffices to check the continuity estimate. This follows from an induction using the bound proved in Theorem 4.3 and our assumption that $|u_0|_{\gamma_0, \mu_0+1} \leq 1/2$. Using the upgraded estimates in Propositions 9.1 and 9.2, we obtain as above

$$\begin{aligned} \|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} &\leq C(4^{-n} + e^{-\gamma_0 R^2/3} + \delta C^n) \\ &\quad + C^n(R^{d|s|}\eta^{(d-1)/2} + R^{d(|s|+2n)}\eta^{(d-1)(d+2)/(2d+2)}) \\ &\quad + C^n R^{d(|s|+2n)} \max_{\alpha \in \mathcal{T}}(\epsilon_\alpha)^\lambda. \end{aligned}$$

Picking parameters $n, \delta, \eta, R, \epsilon_0$, and γ in a manner similar to (P_1) – (P_6) above, we obtain the claimed convergence rate in $\epsilon = \max_{\alpha \in \mathcal{T}} \epsilon_\alpha$. For details of a related calculation, see, e.g., [7]. $\square$

Acknowledgments. The authors are thankful to Thomas Chen, Erica de la Canal, and Irene M. Gamba for helpful discussions regarding physical and mathematical aspects of the problem. The authors also thank the reviewers for carefully reading the manuscript and providing helpful comments.

REFERENCES

- [1] R. K. AGARWAL, R. CHEN, AND F. G. TCHEREMISINE, *Computation of hypersonic flow of a diatomic gas in rotational non-equilibrium past a blunt body using the generalized Boltzmann equation*, in Parallel Computational Fluid Dynamics 2007, Lect. Notes Comput. Sci. Eng. 67, Springer, Berlin, Heidelberg, 2009, pp. 115–122.
- [2] R. K. AGARWAL AND F. G. TCHEREMISINE, *Computation of Hypersonic Shock Wave Flows of Diatomic Gases and Gas Mixtures Using the Generalized Boltzmann Equation*, in 48th AIAA Aerospace Sciences Meeting Including the New Horizons Forum and Aerospace Exposition, 2010.
- [3] R. ALEXANDER, *The Infinite Hard Sphere System*, Ph.D. thesis, University of California, Berkeley, Berkeley, CA, 1975.
- [4] R. ALEXANDER, *Time evolution for infinitely many hard spheres*, Comm. Math. Phys., 49 (1976), pp. 217–232.
- [5] I. AMPATZOGLOU, *Higher Order Extensions of the Boltzmann Equation*, Ph.D. thesis, University of Texas at Austin, Austin, TX, 2020.
- [6] I. AMPATZOGLOU AND N. PAVLOVIĆ, *Rigorous Derivation of a Binary-Ternary Boltzmann Equation for a Dense Gas of Hard Spheres*, preprint, <https://arxiv.org/abs/2007.00446>, 2019.

- [7] I. AMPATZOGLOU AND N. PAVLOVIĆ, *Rigorous derivation of a ternary Boltzmann equation for a classical system of particles*, Comm. Math. Phys., 387 (2021), pp. 793–863.
- [8] Y. ANIKIN, O. DODULAD, Y. KLOSS, AND F. TCHEREMISSINE, *Method of calculating the collision integral and solution of the Boltzmann kinetic equation for simple gases, gas mixtures and gases with rotational degrees of freedom*, Int. J. Comput. Math., 92 (2015), pp. 1775–1789.
- [9] C. BIANCA AND C. DOGBE, *On the Boltzmann gas mixture equation: Linking the kinetic and fluid regimes*, Commun. Nonlinear Sci. Numer. Simul., 29 (2015), pp. 240–256.
- [10] A. BOBYLEV AND I. GAMBA, *Boltzmann equations for mixtures of Maxwell gases: Exact solutions and power like tails*, J. Stat. Phys., 124 (2006), pp. 497–516.
- [11] L. E. BOLTZMANN, *Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen*, Sitzungsberichte Akad. Wiss., Vienna (II), 66 (1872), pp. 275–370.
- [12] M. BRIANT AND E. DAUS, *The Boltzmann equation for a multi-species mixture close to global equilibrium*, Arch. Ration. Mech. Anal., 222 (2016), pp. 1367–1443.
- [13] A. CAMPO, M. M. PAPARI, AND E. ABU-NADA, *Estimation of the minimum Prandtl number for binary gas mixtures formed with light helium and certain heavier gases: Application to thermoacoustic refrigerators*, Appl. Therm. Eng., 31 (2011), pp. 3142–3246.
- [14] S. CHAPMAN AND T. COWLING, *The Mathematical Theory of Non-uniform Gases*, Cambridge University Press, London, 1952.
- [15] E. DE LA CANAL, I. M. GAMBA, AND M. PAVIĆ-ČOLIĆ, *Propagation of L^p_β -Norm, $1 < p \leq \infty$, for the System of Boltzmann Equations for Monatomic Gas Mixtures*, preprint, <https://arxiv.org/abs/2001.09204>, 2020.
- [16] A. FERNANDES AND W. MARQUES, JR., *Sound propagation in binary gas mixtures from a kinetic model of the Boltzmann equation*, Phys. A: Stat. Mech. Appl., 332 (2004), pp. 29–46.
- [17] I. GALLAGHER, L. SAINT-REYMOND, AND B. TEXIER, *From Newton to Boltzmann: Hard Spheres and Short-Range Potentials*, Zur. Lect. Adv. Math., European Mathematical Society, Zürich, Switzerland, 2013.
- [18] I. M. GAMBA AND M. PAVIĆ-ČOLIĆ, *On existence and uniqueness to homogeneous Boltzmann flows of monatomic gas mixtures*, Arch. Ration. Mech. Anal., 235 (2020), pp. 723–781.
- [19] S. HA AND S. E. NOH, *Global existence and stability of mild solutions to the inelastic Boltzmann system for gas mixtures*, Quart. Appl. Math., 68 (2010), pp. 671–699.
- [20] S. HA, S. E. NOH, AND S. B. YUN, *Global existence and stability of mild solutions to the Boltzmann system for gas mixtures*, Quart. Appl. Math., 65 (2007), pp. 757–779.
- [21] M. A. HAIRE AND D. D. VARGO, *Review of helium and xenon pure component and mixture transport properties and recommendation of estimating approach for project Prometheus (viscosity and thermal conductivity)*, in AIP Conference Proceedings, Vol. 880, 2007.
- [22] B. B. HAMEL, *Kinetic model for binary gas mixtures*, Phys. Fluids, 8 (1965), pp. 418–425.
- [23] F. KING, *BBGKY Hierarchy for Positive Potentials*, Ph.D. thesis, University of California, Berkeley, Berkeley, CA, 1975.
- [24] S. KOSUGE, *Model Boltzmann equation for gas mixtures: Construction and numerical comparison*, Eur. J. Mech. B Fluids, 28 (2009), pp. 170–184.
- [25] L. LANDAU AND E. LIFSHITZ, *Statistical Physics, Part 1*, 1st ed., Course Theoret. Phys. 5, Addison-Wesley, Reading, MA, 1958.
- [26] O. LANFORD, *Time evolution of large classical systems*, in Dynamical Systems, Theory and Applications, Lecture Notes in Phys. 38, Springer, Berlin, 1975, pp. 1–111.
- [27] J. C. MAXWELL, IV, *On the Dynamical Theory of Gases*, Royal Society, London, 1867.
- [28] S. SIMONELLA, *Evolution of correlation functions in the hard sphere dynamics*, J. Stat. Phys., 155 (2014), pp. 1191–1221.
- [29] A. SOTIROV AND S. H. YU, *On the solution of a Boltzmann system for gas mixtures*, Arch. Ration. Mech. Anal., 195 (2010), pp. 675–700.
- [30] F. G. TCHEREMISSINE AND R. K. AGARWAL, *A Conservative Numerical Method for Solving the Generalized Boltzmann Equation for an Inert Mixture of Diatomic Gases*, American Institute of Aeronautics and Astronautics, Reston, VA, 2009.