COMMUN. MATH. SCI.
Vol. 20, No. 8, pp. 2315–2365

EXISTENCE OF SMOOTH SOLUTIONS TO THE LANDAU-FERMI-DIRAC EQUATION WITH COULOMB POTENTIAL*

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Abstract. In this paper, we prove global-in-time existence and uniqueness of smooth solutions to the homogeneous Landau-Fermi-Dirac equation with Coulomb potential. The initial conditions are nonnegative, bounded and integrable. We also show that any weak solution converges towards the steady state given by the Fermi-Dirac statistics. Furthermore, the convergence is algebraic, provided that the initial datum is close to the steady state in a suitable weighted Lebesgue norm.

Keywords. Landau-Fermi-Dirac equation; existence and uniqueness; regularity; coercivity; dissipation; long-time behavior; algebraic decay; H-theorem.

AMS subject classifications. 35BXX; 35K59; 35K55; 35P15; 35Q84; 82C40; 82D10.

1. Introduction

We consider the homogeneous Landau-Fermi-Dirac equation with Coulomb potential

$$\partial_t f = \frac{1}{8\pi}\mathrm{div}_v \int_{\mathbb{R}^3} \frac{\Pi(v-v_*)}{|v-v_*|}\left[f(v_*)(1-\varepsilon f(v_*))\nabla_v f - f(v)(1-\varepsilon f(v))\nabla_{v_*} f(v_*)\right] dv_*, \tag{1.1}$$

where $\Pi(z)$ is the standard projection matrix

$$\Pi(z) = Id - \frac{z\otimes z}{|z|^2}.$$

The function $f(v,t)$ models the distribution of velocities within a single species quantum gas. The particles considered here are fermions (e.g. electrons) interacting in a grazing collision regime [1]. The parameter ε quantifies the strength of the quantum effects of the system for the particular species considered and depends on Planck's constant, the mass of the species, and the number of independent quantum weights of the species. In particular, we notice that in the case $\varepsilon = 0$ Equation (1.1) reduces to the classical Landau equation. The Pauli exclusion principle implies that f satisfies the a priori bound

$$0 \le f \le \frac{1}{\varepsilon}.$$

This bound is the key ingredient in our proof. See also [21] for a discussion on the Boltzmann equation with Fermi-Dirac statistic.

*Received: October 20, 2021; Accepted (in revised form): March 21, 2022. Communicated by Pierre-Emmanuel Jabin.

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WG is partially supported by the NSF-DMS grant 1840314.
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MG is partially supported by the DMS-NSF 2019335 and would like to thank NCTS Mathematical Division of Taipei for their kind hospitality.
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NZ acknowledges support from the Alexander von Humboldt Foundation (AvH) and from the Austrian Science Foundation (FWF), grants P30000, P33010.

Equation (1.1) is well-understood for the cases of moderately soft and hard potentials, namely when the kernel $\frac{1}{|v-v^*|}$ is replaced by $\frac{1}{|v-v^*|^{-\gamma-2}}$ for $\gamma \geq -2$. In [2], the authors consider the moderately soft potentials case ($-2 \leq \gamma \leq 0$) and show algebraic convergence of non-degenerate solutions towards equilibrium for initial data satisfying a suitable non-saturation condition. Existence and uniqueness of weak solution for hard potentials ($\gamma \geq 0$) are shown in [7], regularity and smoothing effects are studied in [14, 15], and exponential convergence towards equilibrium in [4]. In [3], the authors present fundamental properties of the entropy and entropy production functional for hard and moderately soft potentials. The existence of nondegenerate stationary solutions for any potential is shown in [8].

The Landau-Fermi-Dirac equation shares several properties with the classical Landau equation. Multiplying (1.1) by a test function ϕ, integrating by parts, and applying a straightforward symmetry argument, one obtains

$$\int_{\mathbb{R}^3} \partial_t f \phi \, dv = -\frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\Pi(v-v_*)}{|v-v_*|} [f(1-\varepsilon f)\nabla_{v_*} f - f_*(1-\varepsilon f_*)\nabla_v f] \cdot \\ \cdot [\nabla \phi_* - \nabla \phi] \, dv_* dv. \tag{1.2}$$

Conservation of mass, momentum and energy follows from (1.2) by choosing $\phi(v) \in \{1, v, |v|^2\}$. A version of the H-theorem for (1.1) is also available: with

$$\phi = \ln\left(\frac{\varepsilon f}{1-\varepsilon f}\right)$$

in (1.2), one obtains that

$$\frac{d}{dt} H_\varepsilon[f](t) = -\frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(1-\varepsilon f) f_*(1-\varepsilon f_*) \cdot \\ \cdot \frac{\Pi(v-v_*)}{|v-v_*|} \left[\frac{\nabla_{v_*} f_*}{f_*(1-\varepsilon f_*)} - \frac{\nabla_v f}{f(1-\varepsilon f)}\right]^2 dv_* dv \leq 0, \tag{1.3}$$

where

$$H_\varepsilon[f] := \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \varepsilon f \ln(\varepsilon f) + (1-\varepsilon f) \ln(1-\varepsilon f) \, dv.$$

Equation (1.3) is the *entropy balance equation* associated to (1.1), with $-H_\varepsilon$ being the (physical) Fermi-Dirac entropy functional. The only smooth function that nullifies the entropy production, $\frac{d}{dt} H_\varepsilon[f] = 0$, is the *Fermi-Dirac equilibrium distribution*

$$\mathcal{M}_\varepsilon(v) := \frac{a e^{-b|v-u|^2}}{1 + \varepsilon a e^{-b|v-u|^2}}, \tag{1.4}$$

which is also the *only smooth minimizer* of H_ε under the constraints of given mass, momentum, and energy [8]. The constants $a > 0$, $b > 0$, and $u \in \mathbb{R}^3$ are determined by the mass, first and second moment of the initial data

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathcal{M}_\varepsilon(v) dv = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(v,t) dv = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(0,v) dv.$$

There are other nonsmooth distributions of the form

$$\mathcal{F}_\varepsilon(v) := \varepsilon^{-1} \chi_\Omega,$$

with Ω of $\mathbb{R}^3$ a measurable subset, that satisfies (formally) $H_\varepsilon[\mathcal{F}_\varepsilon] = \frac{d}{dt}H_\varepsilon[\mathcal{F}_\varepsilon] = 0$ and solves (1.1). These particular stationary solutions are called *saturated Fermi-Dirac states*. As such, any solution to (1.1) with general initial data could approach, as time grows, such saturated states. However, given an initial data with mass ρ, momentum u and energy E, there exists only one value of ε, uniquely determined by ρ, u and E, for which $\mathcal{F}_\varepsilon(v)$ is an admissible stationary solution. For ε below such value, the only steady-state is $\mathcal{M}_\varepsilon$.

Taking the formal limit $\varepsilon \to 0$ in Equation (1.1), one obtains the classical Landau equation. Furthermore, $H_\varepsilon[f] \to H[f] = \int_{\mathbb{R}^3} f \ln f \, dv$ as $\varepsilon \to 0$ modulus a multiple of the mass $\int_{\mathbb{R}^3} f \, dv$:

$$H_\varepsilon[f] - (\ln\varepsilon - 1)\int_{\mathbb{R}^3} f \, dv \to H[f] \quad \text{as } \varepsilon \to 0.$$

The addition of a multiple of the mass to $H_\varepsilon[f]$ does not change the entropy balance Equation (1.3) nor the form of the equilibrium distribution (1.4), thanks to the conservation of mass property. The equilibrium distribution $\mathcal{M}_\varepsilon(v)$ also converges towards the classical Maxwellian distribution $M(v) = ae^{-b|v-u|^2}$ as $\varepsilon \to 0$. Finally, strictly related to the limits $H_\varepsilon[f] \to H[f]$ and $\mathcal{M}_\varepsilon \to M$ is the fact that the relative entropy

$$\begin{aligned} H_\varepsilon[f|\mathcal{M}_\varepsilon] := &\int_{\mathbb{R}^3} \mathcal{M}_\varepsilon \left[\frac{f}{\mathcal{M}_\varepsilon}\ln\left(\frac{f}{\mathcal{M}_\varepsilon}\right) - \frac{f}{\mathcal{M}_\varepsilon} + 1\right] \\ &+ \frac{1}{\varepsilon}\int_{\mathbb{R}^3}(1-\varepsilon\mathcal{M}_\varepsilon)\left[\frac{1-\varepsilon f}{1-\varepsilon\mathcal{M}_\varepsilon}\ln\left(\frac{1-\varepsilon f}{1-\varepsilon\mathcal{M}_\varepsilon}\right) - \frac{1-\varepsilon f}{1-\varepsilon\mathcal{M}_\varepsilon} + 1\right] dv \end{aligned}$$

converges to the relative entropy of the classical Landau equation

$$H[f|M] := \int_{\mathbb{R}^3} M\left[\frac{f}{M}\ln\left(\frac{f}{M}\right) - \frac{f}{M} + 1\right] dv.$$

The next observation concerns the structure of the collision operator. For a smooth f, the interaction term can be expressed as a second order elliptic nonlinear operator with non-local coefficients:

$$\mathrm{div}_v\left(A[f(1-\varepsilon f)]\nabla f - f(1-\varepsilon f)\nabla a[f]\right),$$

where the matrix $A[f(1-\varepsilon f)]$ is defined through the map

$$A : g \mapsto A[g],$$

with

$$A[g] := \frac{1}{8\pi}\int_{\mathbb{R}^3}\frac{\Pi(v-v_*)}{|v-v_*|}g(v_*)\,dv_*, \quad a[f] := \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{f(v_*)}{|v-v_*|}\,dv_*. \tag{1.5}$$

1.1. Main results. Our first result concerns existence of smooth solutions to (1.1). Unlike in the case of the classical Landau equation, we are able to show global-in-time existence of smooth solutions for a general class of initial datum. Our regularity estimates depend on the quantum parameter. At the present moment, it seems out of reach to obtain similar results uniformly with respect to ε. Therefore, in the rest of the manuscript we set $\varepsilon = 1$.

THEOREM 1.1. *Suppose $f_{in} : \mathbb{R}^3 \to \mathbb{R}$ satisfies $0 \le f_{in} \le 1$, $(1+|v|^3)f_{in} \in L^1(\mathbb{R}^3)$, and $H_1(f_{in}) < 0$. Then, there is a solution $f : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ with $f \in C([0,\infty); L^2(\mathbb{R}^3))$ such* *that* $f(0)=f_{in}$, $0\leq f\leq 1$, $f\in L^\infty([0,\infty);L^p(\mathbb{R}^3))\cap L^2([0,T];H^1(\mathbb{R}^3))$ *for each* $1\leq p\leq\infty$, *and for each* $T>0$, *and* $\varphi\in L^2([0,T];H^1(\mathbb{R}^3))$,

$$\int_0^T \langle\varphi,\partial_t f\rangle_{H^1,H^{-1}}\,dt=-\int_0^T\int_{\mathbb{R}^3}(A[f(1-f)]\nabla f-\nabla a[f]f(1-f))\cdot\nabla\varphi\,dvdt. \tag{1.6}$$

Moreover, f has decreasing (Fermi-Dirac) entropy and satisfies conservation of mass, energy, and momentum.

If the initial data has moments $(1+|v|^m)f_{in}\in L^1(\mathbb{R}^3)$ *with* $m>9$, *the solution is unique.*

By a simple time rescaling, we obtain global-in-time existence and uniqueness for any quantum parameter:

Corollary 1.1. *Fix* $\varepsilon>0$ *and let* $f_{in}:\mathbb{R}^3\to\mathbb{R}$ *satisfies* $0\leq f_{in}\leq\varepsilon^{-1}$, $(1+|v|^3)f_{in}\in L^1(\mathbb{R}^3)$, *and* $H_\varepsilon(f_{in})<0$. *Then, there is a unique* $f:[0,\infty)\times\mathbb{R}^3\to\mathbb{R}$ *with* $f\in C([0,\infty);L^2(\mathbb{R}^3))$ *such that* $f(0)=f_{in}$, $0\leq f\leq\varepsilon^{-1}$, $f\in L^\infty([0,\infty);L^p(\mathbb{R}^3)\cap L^2([0,T];H^1(\mathbb{R}^3))$ *for each* $1\leq p\leq\infty$, *and for each* $T>0$, *and* $\varphi\in L^2([0,T];H^1(\mathbb{R}^3))$,

$$\int_0^T \langle\varphi,\partial_t f\rangle_{H^1,H^{-1}}\,dt=-\int_0^T\int_{\mathbb{R}^3}(A[f(1-\varepsilon f)]\nabla f-\nabla a[f]f(1-\varepsilon f))\cdot\nabla\varphi\,dvdt.$$

Moreover, f has decreasing (Fermi-Dirac) entropy and satisfies conservation of mass, energy, and momentum.

Theorem 1.1 is proved in several steps. First, we approximate the problem by discretizing the time variable and adding suitable regularizing terms. The approximating problem is well-posed thanks to suitable fixed-point arguments. We use uniform L^2 and entropy inequalities to take limits as our regularizing terms vanish. A crucial ingredient is the uniform positive lower bound for the diffusion matrix $A[f(1-f)]$, which follows from the boundedness of the second moment of f and a uniform negative upper bound for the Fermi-Dirac entropy. This guarantees that Equation (1.1) remains uniformly parabolic during the evolution of the system.

The weak solutions from Theorem 1.1 are, in fact, smooth solutions, provided the initial data has high enough moments:

Theorem 1.2. *Let f be a weak solution as in Theorem 1.1. If the initial data f_{in} is, in addition, such that* $(1+|v|^{12})\in L^1(\mathbb{R}^3)$ *then* $f\in C^\infty((0,T];C^\infty(\mathbb{R}^3))$.

The higher regularity of the solution is obtained thanks to parabolic regularity arguments, Morrey's inequality and Schauder estimates. The parabolic regularity argument yields estimates for f in $W^{1,\infty}(0,T;W^{-1,p})\cap L^\infty(0,T;W^{1,p}))$ for any $p\in[2,6]$. Via interpolation between Sobolev spaces, we obtain a bound for f in a fractional Sobolev space. From here, we deduce, the Hölder continuity of f via Morrey's inequality. A standard parabolic bootstrap argument yields $f\in C^\infty((0,T];C^\infty(\mathbb{R}^3))$.

Our regularity results do not hold in the limit $\varepsilon\to 0$, since they heavily rely on the bound $f\leq\frac{1}{\varepsilon}$. For the classical Landau equation, the Cauchy problem has been understood only for weak solutions [1, 6, 18, 29, 35]. Recently, in [28] and [34], the authors showed that, for a short time, weak solutions become instantaneously regular and smooth. The time asymptotic for weak solutions has been studied in [13] and [12]. However, the question of whether solutions stay smooth for all time or become unbounded after a finite time is still open. Recent research has produced several conditional results regarding this inquiry. These results show regularity properties of solutions that already possess some basic properties (yet to be verified). They include (i) conditional uniqueness [16, 23], and (ii) conditional smoothness for solutions in $L^\infty(0,T,L^p(\mathbb{R}^d))$ with $p>\frac{d}{2}$ [28, 34]. In a very recent manuscript [19], the authors studied behavior of solutions in the space $L^\infty(0,T,\dot{H}^1(\mathbb{R}^3))$. They show that for general initial data there exists a time T^* after which the weak solutions belong to $L^\infty((T^*,+\infty),H^1(\mathbb{R}^3))$. This result agrees with the one in [25], in which the authors showed that the set of singular times for weak solutions has Hausdorff dimension at most $\frac{1}{2}$. In [9], the authors show that self-similar blow-up of type I cannot occur for solutions to the Landau equation.

The second result of this paper concerns the convergence towards the steady state as the time approaches infinity. We show that the convergence is algebraic, provided that the initial datum f_{in} is close to the steady state $\mathcal{M}_\varepsilon$ in a suitable weighted Lebesgue norm. Hereafter, we denote with $\mathcal{M}$ the function defined in (1.4) with $\varepsilon=1$.

THEOREM 1.3. *Given any initial datum $f_{in}:\mathbb{R}^3\to[0,1]$, $f_{in}\in L^1_2$, such that $H_1[f_{in}]<0$, the solution f to the initial value problem associated to (1.1) converges strongly in L^1 as $t\to\infty$ to the Fermi-Dirac distribution $\mathcal{M}$ with same mass, momentum and energy as f_{in}.*

Furthermore, there exists a constant $\ell>0$ such that, if

$$\int_{\mathbb{R}^3}(f_{in}-\mathcal{M})^2\mathcal{M}^{-1}(1-\mathcal{M})^{-1}dv<\ell,$$

$$\int_{\mathbb{R}^3}(f_{in}-\mathcal{M})^2\mathcal{M}^{-1}(1-\mathcal{M})^{-1}(1+|v|^2)^{N/2}dv<\infty,\quad \textit{for some}\, N\geq 1,$$

then

$$\int_{\mathbb{R}^3}\frac{(f(t)-\mathcal{M})^2}{\mathcal{M}(1-\mathcal{M})}dv\lesssim(1+t)^{-N},\qquad t>0.$$

The unconditional convergence (without rate) towards the steady state is obtained from the entropy balance equation in the following way. We integrate the balance equation in time and use the ellipticity properties of the entropy dissipation to deduce that $f(t_n)\to\mathcal{M}$ along a suitable sequence of time instants $t_n\to\infty$. The monotonicity in time of the relative entropy yields that $f(t)\to\mathcal{M}$ strongly in L^1 as $t\to\infty$.

The algebraic convergence for initial data close to the steady state is achieved by linearizing (1.1) around $\mathcal{M}$. First, we show existence of a spectral gap for the linearized Landau-Fermi-Dirac operator using two different weighted Lebesgue spaces. Precisely, such relation has the structure

$$-(Lh,h)_{E_1}\geq\lambda\|h\|_{E_2},\qquad h\in D(L)\cap N(L)^\perp,$$

with E_2 not included in E_1. This latter fact is the reason why we are not able to obtain exponential convergence towards equilibrium, but only algebraic. We derive a uniform bound for some moment of the solution to the linearized equation in a weighted Lebesgue space. In the last step, we bound the contributions of the nonlinear corrections, and derive a differential inequality for the weighted L^2-norm of the perturbation

$$h:=\frac{f-\mathcal{M}}{\mathcal{M}(1-\mathcal{M})}.$$

An elementary argument of ordinary differential equations' theory yields algebraic convergence to zero with rate N for $\|h\|_{L^2(m)}$, provided that, at initial time, the latter is small enough and $\|h\|_{L^2(m(1+|v|^2)^{N/2})}<\infty$.

1.2. Notations. Here we list some of the notation conventions adopted throughout the manuscript:

- Universal constants that may change from line to line are denoted C or $C(A,B)$ if the constant is allowed to depend on the quantities A and B.
- We write $A \lesssim B$ to mean there is a universal constant C such that $A \le CB$. Similarly, we write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. If we write $A \lesssim_\Lambda B$, the implicit constant C is allowed to depend on Λ.
- We write $L^p([0,T];X)$ for $T>0$ and X a Banach space to denote the space of strongly measurable X-valued functions satisfying

$$\int_0^T \|f(t)\|_X^p \, dt < \infty.$$

When we write L^p without specifying the measure space, we mean $L^p(\mathbb{R}^3)$.
- We use the bracket notation $\langle v\rangle := (1+|v|^2)^{1/2}$. Given $p \in [1,\infty]$, we denote with L^p_m the space of functions that have the following norm

$$\|f\|^p_{L^p_m} := \int_{\mathbb{R}^3} |f|^p \langle v\rangle^m \, dv,$$

finite. We denote with $\|f\|_{L^p}$ the $L^p(\mathbb{R}^3)$ norm of f.
- Given $p\in[1,\infty]$ we denote with $p^* \in [1,\infty]$ the conjugate exponent of p, $p^* := \frac{p}{p-1}$.

In Section 2, we recall some useful estimates for the coefficients $A[f]$, $a[f]$ appearing in (1.1). In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.3.

2. Coefficient bounds

The following standard bounds will be used throughout our proofs.

LEMMA 2.1. *Any $f(v)$ such that $0 \le f(v) \le \frac{1}{\varepsilon}$, $\int_{\mathbb{R}^3} f(1+|v|^2) = E_0$, and $H_\varepsilon[f] \le H_0 < 0$ satisfies*

$$\langle A[f(1-\varepsilon f)]\xi,\xi\rangle \ge \frac{C}{1+|v|^3}|\xi|^2, \quad \forall \xi \in \mathbb{R}^3,$$

where C depends on E_0 and H_0.

Proof. We begin by quoting a known result (see [20] Lemma 6 and Proposition 4, or [34] Lemma 3.2 and Lemma 3.3) that says that for any nonnegative function φ with mass, second momentum and entropy bounded, for all $v\in\mathbb{R}^3$:

$$\int_{\mathbb{R}^3} \frac{\Pi(v-v_*)}{|v-v_*|}\varphi(v_*)\,dv_* \ge \frac{C}{1+|v|^3}\mathbb{I},$$

where the constant C depends only on the quantities

$$\int \varphi(v)\,dv, \quad \int \varphi(v)|v|^2\,dv, \quad \int \varphi(v)|\ln\varphi(v)|\,dv.$$

In light of this inequality, we need only show that

$$\int f(1-\varepsilon f)|\ln f(1-\varepsilon f)|\,dv < +\infty, \tag{2.1}$$

and that there exists a strictly positive constant m_0 such that

$$\int f(1-\varepsilon f)\,dv \geq m_0. \tag{2.2}$$

The proof of (2.1) and (2.2) can be found in [7] in Lemma 3.1. ◻

The previous lemma together with (1.3) show that, as long as the initial data has strictly negative entropy, our equation is *uniformly* parabolic, and saturated-Fermi-Dirac-distributions are not admissible solutions.

Lemma 2.2 (Upper Bound on $A[f]$). *For $A[f]$ defined in (1.5), and for any $f\in L^p\cap L^q$ with $1\leq p<3/2<q\leq\infty$, we have*

$$\|A[f]\|_{L^\infty}\leq C(p,q)\|f\|_{L^q}^{1-\alpha}\|f\|_{L^p}^{\alpha}, \tag{2.3}$$

where $\alpha=\frac{\frac{1}{3}-\frac{1}{p^*}}{\frac{1}{q^*}-\frac{1}{p^*}}\in(0,1)$. Furthermore, $\nabla\cdot A[f]=\nabla a[f]$.

Proof. For $R>0$ arbitrary,

$$\begin{aligned}
|A[f]| &\leq \int_{|x-y|\leq R}\frac{|f(y)|}{|x-y|}\,dy+\int_{|x-y|\geq R}\frac{|f(y)|}{|x-y|}\,dy\\
&\leq \|f\|_{L^q}\left(\int_{|x-y|\leq R}\frac{1}{|x-y|^{q^*}}\,dy\right)^{1/q^*}+\|f\|_{L^p}\left(\int_{|x-y|\geq R}\frac{1}{|x-y|^{p^*}}\,dy\right)^{1/p^*}\\
&\lesssim_{p,q}\|f\|_{L^q}R^{3/q^*-1}+\|f\|_{L^p}R^{3/p^*-1},
\end{aligned}$$

provided $q^*<3$ and $p^*>3$. Optimizing in R yields $R\approx\|f\|_{L^q}^{-\beta}\|f\|_{L^p}^{\beta}$ and the bound,

$$\|f\|_{L^q}R^{\frac{3(q-1)}{q}-1}+\|f\|_{L^p}R^{\frac{3(p-1)}{p}-1}\lesssim\|f\|_{L^q}^{\alpha}\|f\|_{L^p}^{1-\alpha},$$

for $\beta=\frac{1}{3\left(\frac{1}{q^*}-\frac{1}{p^*}\right)}$, and $\alpha=\beta\left(1-\frac{3}{p^*}\right)>0$. Note that $0<\alpha<1$.

Finally, notice that

$$\operatorname{div}A[f]=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}f(y)\,dy=\frac{1}{4\pi}\nabla\int_{\mathbb{R}^3}\frac{f(y)}{|x-y|}f(y)\,dy=\nabla a[f].$$

◻

Lemma 2.3 (Upper Bound on $\nabla a[f]$). *For $a[f]$ defined in (1.5), we have*

$$\|\nabla a[f]\|_{L^2}\leq C\|f\|_{L^{6/5}},$$

and

$$\|\nabla a[f]\|_{L^\infty}\leq C(p,q)\|f\|_{L^p}^{\alpha}\|f\|_{L^q}^{1-\alpha},$$

for any $1\leq p<3<q\leq\infty$, and some $\alpha\in(0,1)$.

Proof. The Hardy-Littlewood-Sobolev inequality (in $\mathbb{R}^3$) states that

$$\left\|\frac{1}{|x|^\lambda}*f\right\|_{L^q}\lesssim_{\alpha,p,q}\|f\|_{L^p}$$

provided $1<p,q,\frac{3}{\lambda}<\infty$ and $\frac{1}{p}+\frac{\lambda}{3}=1+\frac{1}{q}$ (see [32]). The kernel $K(x)$ for ∇a satisfies $K(x)\sim|x|^{-2}$ and the L^2 estimate follows immediately. The L^∞-bound follows the same steps as in Lemma 2.2. ◻

3. Existence of bounded weak solutions

In order to find weak solutions to (1.1), we first introduce an extra dissipative term $\delta_1\Delta f$ to counter the degenerate ellipticity of $A[f(1-f)]$ (see Lemma 2.1) and study the approximating problems

$$\partial_t f = \nabla\cdot(A[f(1-f)]\nabla f - \nabla a[f]f(1-f)) + \delta_1\Delta f. \tag{3.1}$$

We will first prove there exist solutions to (3.1), then taking δ_1,we recover global-in-time weak solutions to (1.1). To this end, we introduce an auxiliary equation,

$$\begin{aligned}\frac{(f_k-f_{k-1})}{\tau} &= \nabla\cdot(A_{k-1}\nabla f_k - \nabla a_{k-1}z_+(1-z)_+ + \delta_1\nabla f_k) - \delta_2|v|^m f_k,\\ A_{k-1} &:= A[f_{k-1}(1-f_{k-1})] \qquad \text{and} \qquad a_{k-1} := a[f_{k-1}],\end{aligned} \tag{3.2}$$

obtained by dividing the time interval $[0,T]$ into N subintervals, each of length τ, linearizing (3.1) around a measurable function z, and adding an additional localizing term, $\delta_2|v|^m f$. In the first step of our construction, we use the Lax-Milgram Theorem to find unique weak solutions to (3.2) and prove the following proposition:

PROPOSITION 3.1. *Let $f_{k-1}\in L^1$ with $0\le f_{k-1}\le 1$, z be a measurable function, and $m\ge 0$. Then, there is a unique $f_k\in H^1\cap L^2_m$ that satisfies*

$$\begin{aligned}&\int \frac{(f_k-f_{k-1})}{\tau}\varphi - \nabla a_{k-1}z_+(1-z)_+\cdot\nabla\varphi\,dv\\ &= -\int \nabla\varphi\cdot A_{k-1}\nabla f_k\,dv - \delta_1\int\nabla\varphi\cdot\nabla f_k - \delta_2\int|v|^m\varphi f_k\,dv,\end{aligned} \tag{3.3}$$

for any $\varphi\in H^1\cap L^2_m$.

For a fixed k, Proposition 3.1 defines a solution operator Φ to (3.2) via $\Phi(z) = f_k$. In the second step of our construction, we seek solutions f_k to the nonlinear equation:

$$\frac{(f_k-f_{k-1})}{\tau} = \nabla\cdot(A_{k-1}\nabla f_k - \nabla a_{k-1}f_k(1-f_k) + \delta_1\nabla f_k) - \delta_2|v|^m f_k, \tag{3.4}$$

for given f_{k-1} and fixed $\delta_1,\delta_2,\tau,m>0$. To this end, we show that $\Phi: L^2\to L^2$ is continuous and compact, and the set $\{z \mid z = t\Phi(z), \text{ for some } t\in[0,1]\}$ is bounded in L^2. Therefore, we apply the Schaeffer Fixed Point Theorem to conclude the following proposition:

PROPOSITION 3.2. *Suppose $f_0\in L^1$ with $0\le f_0\le 1$. Then, there is a family of functions $\{f_k\}_{k=0}^N$ such that $f_k\in L^2_m\cap H^1$ and $\{f_k\}$ solve (3.4). That is, for $k\ge 1$, f_k satisfies that for any $\varphi\in H^1\cap L^2_m$,*

$$\begin{aligned}&\int \frac{(f_k-f_{k-1})}{\tau}\varphi - \nabla a_{k-1}f_k(1-f_k)\cdot\nabla\varphi\,dv\\ &= -\int A_{k-1}\nabla f_k\cdot\nabla\varphi\,dv - \delta_1\int\nabla f_k\cdot\nabla\varphi\,dv - \delta_2\int|v|^m f_k\varphi\,dv.\end{aligned} \tag{3.5}$$

Furthermore, for each $k\ge 1$, $f_k\in L^1$ and $0\le f_k\le 1$.

In the third step of our construction, we seek a weak solution to the auxiliary equation (3.1) on a time interval $[0,T]$. To this end, we divide $[0,T]$ into N pieces of size τ_N and from Proposition 3.2, for an initial datum f_{in}, we may define

$$f^{(N)}(v,t)=f_{in}(v)\chi_0(t)+\sum_{k=1}^{N} f_k(v)\chi_{(t_{k-1},t_k]}(t),$$

where $\{f_k\}_{0\le k\le N}$ solves (3.4) with parameters $\tau=\delta_2=\tau_n$. We show propagation of L^1 moments and use a variant of the Aubin-Lions Lemma to conclude the following proposition:

PROPOSITION 3.3. *Suppose $f_{in}\in L^1$, $|v|^2 f_{in}\in L^1$, and $0\le f_{in}\le 1$ and $\delta_1>0$. Then, for any $T>0$, there is an $f:[0,T]\times\mathbb{R}^3\to\mathbb{R}$ with $0\le f\le 1$ such that for each $1\le p<\infty$, $f\in L^\infty([0,T];L^p)$, $f\in C([0,T];L^2)$, $f\in L^2([0,T];H^1)$, $f\in L^\infty([0,T];L^1_2)$ and f satisfies (3.1) in the form,*

$$\begin{aligned}&\int_{\mathbb{R}^3} f_{in}\varphi(0)\,dv-\int_0^T\int_{\mathbb{R}^3} f\partial_t\varphi\,dvdt\\ &=-\int_0^T\int_{\mathbb{R}^3}(A[f(1-f)]\nabla f-\nabla a[f]f(1-f))\cdot\nabla\varphi\,dvdt-\delta_1\int_0^T\int_{\mathbb{R}^3}\nabla f\cdot\nabla\varphi\,dvdt,\end{aligned}\tag{3.6}$$

for each $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^3)$ and

$$\begin{aligned}\int_0^T\langle\partial_t f,\Phi\rangle_{H^1,H^{-1}}\,dt=&-\int_0^T\int_{\mathbb{R}^3}(A[f(1-f)]\nabla f-\nabla a[f]f(1-f))\cdot\nabla\Phi\,dvdt\\ &-\delta_1\int_0^T\int_{\mathbb{R}^3}\nabla f\cdot\nabla\Phi\,dvdt,\end{aligned}\tag{3.7}$$

for each $\Phi\in L^2([0,T];H^1)$. Furthermore, f conserves mass and satisfies the bound

$$\|f\|_{L^\infty([0,T];L^1_2)}+\delta_1\|\nabla f\|_{L^2([0,T];L^2)}+\delta_1\|\partial_t f\|_{L^2([0,T];H^{-1})}\le C(\|f_{in}\|_{L^1_2},T).$$

Finally, in the fourth step of our construction, we conclude the proof of Theorem 1.1. From Proposition 3.3, for an initial datum f_{in} and a sequence $\delta_n\to 0^+$, we obtain a family of solutions $\{f_n\}$ to the Equation (3.1) with parameters $\delta_1=\delta_n$ on the interval $[0,T]$. We show propagation of higher L^1 moments and an H-Theorem for the Equation (3.1). Combined with Lemma 2.1, this implies a uniform lower bound on the coefficients $A[f_n(1-f_n)]$, which is sufficient to gain compactness as $n\to\infty$.

3.1. Step 1: Existence and uniqueness of solutions to (3.2). In this step, we use the Lax-Milgram Theorem to prove Proposition 3.1. We recall that in this step, we construct weak solutions f_k to

$$\begin{aligned}&\frac{(f_k-f_{k-1})}{\tau}=\nabla\cdot(A_{k-1}\nabla f_k-\nabla a_{k-1}z_+(1-z)_++\delta_1\nabla f_k)-\delta_2|v|^m f_k,\\ &A_{k-1}:=A[f_{k-1}(1-f_{k-1})]\qquad\text{and}\qquad a_{k-1}:=a[f_{k-1}],\end{aligned}\tag{3.8}$$

where f_{k-1}, z, τ, δ_1, δ_2, and m are fixed.

Proof. (**Proof of Proposition 3.1.**)
We define

$$B[u,\varphi]=\int A_{k-1}\nabla u\cdot\nabla\varphi+\delta_1\nabla u\cdot\nabla\varphi+\tau^{-1}u\varphi+\delta_2|v|^m u\varphi\,dv,$$

$$L[\varphi]=\int \tau^{-1}f_{k-1}\varphi+\nabla a_{k-1}z_+(1-z)_+\cdot\nabla\varphi\,dv.$$

Since $0\le f_{k-1}\le 1$, $A_{k-1}\ge 0$ and we have

$$B[u,u]\ge\int \delta_1|\nabla u|^2+\tau^{-1}u^2+\delta_2|v|^m u^2\,dv\gtrsim_{\delta_1,\delta_2,\tau}\|u\|_{H^1}^2+\|u\|_{L^2_m}^2.$$

Therefore, $B[u,\varphi]$ is coercive on $H^1\cap L^2_m$. Moreover, B is bounded on $H^1\cap L^2_m$ thanks to Lemma 2.2 and $0\le f_{k-1}\le 1$ as

$$\begin{aligned}|B[u,\varphi]|&\le(\|A_{k-1}\|_{L^\infty}+\delta_1)\|\nabla u\|_{L^2}\|\nabla\varphi\|_{L^2}+\tau^{-1}\|u\|_{L^2}\|\varphi\|_{L^2}+\delta_2\|u\|_{L^2_m}\|\varphi\|_{L^2_m}\\&\lesssim_{\delta_1,\delta_2,\tau,\|f_{k-1}\|_{L^1}}\|u\|_{H^1\cap L^2_m}\|\varphi\|_{H^1\cap L^2_m}.\end{aligned}$$

Also, L is bounded on $H^1\cap L^2_m$ by the Cauchy-Schwarz Inequality and Lemma 2.3,

$$\begin{aligned}|L(\varphi)|&\le\tau^{-1}\|f_{k-1}\|_{L^2}\|\varphi\|_{L^2}+\|\nabla a_{k-1}\|_{L^2}\|z_+(1-z)_+\|_{L^\infty}\|\nabla\varphi\|_{L^2}\\&\lesssim_{\tau,\|f_{k-1}\|_{L^2\cap L^{6/5}}}\|\varphi\|_{H^1}.\end{aligned}$$

We conclude, using the Lax-Milgram Theorem on $H^1\cap L^2_m$, that there is a unique $f_k\in H^1\cap L^2_m$ satisfying the weak formulation (3.3). ◻

3.2. Step 2: Existence of solutions to (3.4). In this step, we use a fixed-point argument to prove Proposition 3.2. We show that the nonlinear, semi-discretized equation,

$$\frac{(f_k-f_{k-1})}{\tau}=\nabla\cdot(A[(1-f_{k-1})f_{k-1}]\nabla f_k-\nabla a[f_{k-1}]f_k(1-f_k)+\delta_1\nabla f_k)-\delta_2|v|^m f_k,$$

has a solution f_k provided f_{k-1} is known and satisfies $0\le f_{k-1}\le 1$ and $f_{k-1}\in L^1$. Moreover, we show these assumptions are propagated, so that for a fixed $f_0=f_{in}$, we have the existence of a family $\{f_k\}$ for $k=0,1,\dots,N$ for any N.

We begin by showing the existence of solutions f_k to the nonlinear weak formulation (3.5) provided f_{k-1} is known and satisfies $f_{k-1}\in L^1$ and $0\le f_{k-1}\le 1$. To this end, we fix k and define $\Phi:X\to X$ with $\Phi(z)=f_k$, where f_k is the unique solution to (3.3) given z (and fixed $\delta_1,\delta_2,\tau,m,f_{k-1}$). We also fix X to be L^2. We would like to apply the Schaeffer Fixed Point Theorem [24, Theorem 11.3] to $\Phi:X\to X$ to conclude that there exists a fixed point for Φ in X. To apply Schaeffer's Theorem we need to verify the following conditions:

- The map Φ maps X into X, i.e. if $z\in L^2$, the weak solution f_k to (3.3) also satisfies $f_k\in L^2$. This is done in Lemma 3.1 via an L^2 estimate.
- The set of approximate fixed points,

  $$\{z\mid z=t\Phi(z)\text{ for some }0\le t\le 1\}$$

  should be bounded in X. This is done in Lemma 3.2.
- The map Φ is compact. This is done in Lemma 3.3 via the compact embedding $\Phi(X)\subset H^1\cap L^2_m\hookrightarrow L^2$.
- The map $\Phi:X\to X$ is continuous. This is done in Lemma 3.4 by showing that if $z_k\to z$, the corresponding weak solutions $\Phi(z_k)$ converge to the unique weak solution $\Phi(z)$ of (3.3).

To this end, we have our first a priori bound:

LEMMA 3.1. *For $f_{k-1}\in L^1$ with $0\le f_{k-1}\le 1$, let f_k be the unique solution to (3.3). Then, $f_k\in H^1\cap L^2_m$ and satisfies the estimate*

$$\|f_k\|_{L^2}^2+\tau\delta_1\|\nabla f_k\|_{L^2}^2+2\tau\delta_2\||v|^{m/2}f_k\|_{L^2}^2\le\|f_{k-1}\|_{L^2}^2+C\frac{\tau}{\delta_1}\|f_{k-1}\|_{L^{6/5}}^2. \tag{3.9}$$

Proof. We test (3.3) with $\varphi=f_k$. Using $A_{k-1}\ge 0$, we obtain

$$\tau^{-1}\|f_k\|_{L^2}^2+\delta_1\|\nabla f_k\|_{L^2}^2+\delta_2\||v|^{m/2}f_k\|_{L^2}^2\le\int\tau^{-1}f_{k-1}f_k+\nabla a_{k-1}(z)_+(1-z)_+\cdot\nabla f_k\,dv.$$

We bound the first term on the right-hand side with Young's inequality as

$$\tau^{-1}\int f_{k-1}f_k\,dv\le\frac{\tau^{-1}}{2}\|f_{k-1}\|_{L^2}^2+\frac{\tau^{-1}}{2}\|f_k\|_{L^2}^2,$$

and the second term via Young's inequality and Lemma 2.3 as

$$\int\nabla a_{k-1}(z)_+(1-z)_+\cdot\nabla f_k\,dv\le C\delta_1^{-1}\|f_{k-1}\|_{L^{6/5}}^2+\frac{\delta_1}{2}\|\nabla f_k\|_{L^2}^2.$$

Rearranging terms and combining bounds yield (3.9). ∎

We note that the preceding lemma immediately implies the following result:

LEMMA 3.2 (A priori bounds on approximate fixed points). *Let f_k be the unique solution to (3.3) with $f_{k-1}\in L^1$ with $0\le f_{k-1}\le 1$ and $X:=L^2$. The map $\Phi:X\to X$ defined as $z\mapsto f_k$ is such that $A:=\{z\in X\mid t\Phi(z)=z$ for some $0\le t\le 1\}$ is a bounded subset of X.*

Proof. Suppose $z\in A$. Then, we note by Lemma 3.1,

$$\|z\|_{L^2}^2\le\|\Phi(z)\|_{L^2}^2\le\|f_{k-1}\|_{L^2}^2+C\frac{\tau}{\delta_1}\|f_{k-1}\|_{L^{6/5}}^2,$$

which completes the proof. ∎

LEMMA 3.3 (Compactness). *For Φ and X as in Lemma 3.2, $\Phi(X)$ is pre-compact as a subset of $L^q(\mathbb{R}^3)$ for any $2\le q<6$.*

Proof. Fix any such q. Then, we note that Lemma 3.1 guarantees that $\Phi(z)$ is bounded in $H^1\cap L^2_m$, uniformly in z measurable. We claim that $L^2_m\cap H^1$ embeds compactly in L^q for $2\le q<6$ provided $m>0$.

Indeed, fix g_n a sequence uniformly bounded in L^2_m and H^1, so that $\|g_n\|_{L^2_m\cap H^1}\le M$. Then, use Rellich-Kondrachov Theorem and a diagonalization argument to extract a subsequence g_{n_k} for which $g_{n_k}\to g$ in $L^2(K)\cap L^q(K)$ for every $K\subset\mathbb{R}^3$ compact. We will show $g_{n_k}\to g$ in L^q. Fix $\varepsilon>0$. Then, decompose the norm into two parts via,

$$\|g_{n_k}-g\|_{L^q}^q=\int_{B_R(0)}|g_{n_k}-g|^q\,dx+\int_{\mathbb{R}^3\setminus B_R(0)}|g_{n_k}-g|^q\,dx. \tag{3.10}$$

The first term converges to 0 for any fixed R. For the second, we interpolate between L^2 and L^6 and use the Sobolev embedding $H^1\hookrightarrow L^6$ to guarantee the L^6 norm is uniformly bounded in k. Thus,

$$\begin{aligned}\left(\int_{\mathbb{R}^3\setminus B(0,R)}|g_{n_k}-g|^q\,dx\right)^{1/q}&\lesssim M^{1-\alpha}\left(\int_{\mathbb{R}^3\setminus B(0,R)}|g_{n_k}-g|^2\,dx\right)^{\alpha/2}\\&\lesssim M^{1-\alpha}R^{-m\alpha/2}\|g_{n_k}-g\|_{L^2_m}^{\alpha}\lesssim MR^{-m\alpha/2},\end{aligned} \tag{3.11}$$

where $\frac{1}{q}=\frac{\alpha}{2}+\frac{1-\alpha}{6}$, i.e. $\alpha=\frac{6-q}{2q}$. So for $m>0$ and $2\le q<6$, this converges to 0 as $R\to\infty$ uniformly in k. Thus, first pick R sufficiently large that the second term of (3.10) is less than $\varepsilon/2$ for all k. Then, pick k sufficiently large such that first term of (3.10) is less than $\varepsilon/2$. ∎

Lemma 3.4 (Continuity). *Let Φ be defined as in Lemma 3.2. Suppose $z_n\to z$ strongly in X. Then, $\Phi(z_n)\to\Phi(z)$ strongly in X.*

Proof. Suppose $z_n\to z$ in $X=L^2$. Combining the a priori bound from Lemma 3.1 and compactness from Lemma 3.2, $\Phi(z_n)$ is uniformly bounded in $L^2_m\cap H^1$ and compact in $X=L^2$. Therefore, by extracting subsequences, it suffices to show that if $z_n\to z$ in X and $\Phi(z_n)\to y$ in X and $\Phi(z_n)\rightharpoonup y$ in $H^1\cap L^2_m$, then $y=\Phi(z)$. Finally, since Proposition 3.1 guarantees uniqueness of solutions to (3.3), it suffices to show

$$\begin{aligned}&\int\frac{(y-f_{k-1})}{\tau}\varphi-\nabla a_{k-1}z_+(1-z)_+\cdot\nabla\varphi\,dv\\&=-\int\nabla\varphi\cdot A_{k-1}\nabla y\,dv-\delta_1\int\nabla\varphi\cdot\nabla y+\delta_2|v|^m\varphi y\,dv.\end{aligned}\tag{3.12}$$

Since $\Phi(z_n)$ solves (3.3) with coefficients z_n, we know

$$\begin{aligned}\int\frac{\Phi(z_n)-f_{k-1}}{\tau}\varphi\,dv=&\int\nabla a_{k-1}(z_n)_+(1-z_n)_+\cdot\nabla\varphi-\nabla\varphi\cdot A_{k-1}\nabla\Phi(z_n)\,dv\\&-\delta_1\int\nabla\varphi\cdot\nabla\Phi(z_n)+\delta_2|v|^m\varphi\Phi(z_n)\,dv.\end{aligned}$$

The weak convergence $\Phi(z_n)\rightharpoonup y$ in $L^2_m\cap H^1$ is sufficient to pass to the limit $n\to\infty$ in each term, except in the term containing $(z_n)_+(1-z_n)_+$. For this term, we first observe that

$$\begin{aligned}&\int|[(z_n)_+(1-z_n)_+-z_+(1-z)_+]\nabla a_{k-1}\cdot\nabla\varphi|\,dv\\&\lesssim\|f_{k-1}\|_{L^1}\|\nabla\varphi\|_{L^2}\times\left(\int|(z_n)_+(1-z_n)_+-z_+(1-z)_+|^2\,dv\right)^{1/2}.\end{aligned}$$

Since the function $\varphi(x)=x_+(1-x)_+$ is Lipschitz, we get

$$\left(\int|(z_n)_+(1-z_n)_+-z_+(1-z)_+|^2\,dv\right)^{1/2}\lesssim\|z_n-z\|_{L^2}\to0,$$

since $z_n\to z$ in X. Therefore, we obtain (3.12) and the proof is complete. ∎

The following lemma states that any fixed point f_k of Φ is also a solution to (3.5). Note, this is not immediate because (3.5) does not contain any positive part operators, while (3.3) does.

Lemma 3.5. *Suppose $f_k\in H^1\cap L^2_m$ satisfies $\Phi(f_k)=f_k$ with $f_{k-1}\in L^1$ and $0\le f_{k-1}\le 1$. Then, $0\le f_k\le1$ and consequently f_k solves (3.5).*

Proof. The idea is to test the weak formulation (3.3) with $(f_k)_-$ and $(1-f_k)_-$ and show that both are identically 0:

$$\tau^{-1}\int_{\{f_k\le0\}}f_k^2-f_{k-1}f_k\,dv+\int_{\{f_k\le0\}}\nabla f_k\cdot(A_{k-1}\nabla f_k+\delta_1\nabla f_k)\,dv+\delta_2\int_{\{f_k\le0\}}|v|^mf_k^2\,dv=0.$$

Since each term is positive, all are 0 and we conclude $f_k = 0$ on $\{f_k \le 0\}$, i.e. $f_k \ge 0$. Similarly,

$$\tau^{-1}\int_{\{f_k\ge 1\}} (f_k - f_{k-1})(1-f_k)\,dv - \int_{\{f_k \ge 1\}} \nabla f_k\cdot(A_{k-1}\nabla f_k + \delta\nabla f_k)\,dv \\ + \delta_2 \int_{\{f_k\ge 1\}} |v|^m f_k(1-f_k)\,dv = 0.$$

Now, because $f_{k-1}\le 1$, $(f_k - f_{k-1})\chi_{\{f_k\ge 1\}} \ge (f_k - 1)\chi_{\{f_k \ge 1\}} \ge 0$. Thus, each term is negative and we conclude $f_k \le 1$. □

Next, we show the assumption that $f_{k-1}\in L^1$ is propagated. That is, if $f_{k-1}\in L^1$, then $f_k \in L^1$ and therefore, we may iterate the fixed-point argument to construct a family $\{f_k\}$ solving (3.5).

Lemma 3.6. *Suppose $f_k \in H^1 \cap L^2_m$ satisfies $f_k = \Phi(f_k)$ with $f_{k-1}\in L^1$ and $0 \le f_{k-1} \le 1$. Then, f_k satisfies the estimate*

$$\|f_k\|_{L^1} + \tau\delta_2 \|f_k |v|^m\|_{L^1} = \|f_{k-1}\|_{L^1}. \tag{3.13}$$

Proof. Let $\varphi_R(v)$ be a cutoff function in $C_c^\infty(\mathbb{R}^3)$ such that

$$\begin{cases} 0\le \varphi_R \le 1, \\ \varphi_R(v) = 1 \quad \text{if} \quad |v|\le R, \\ \varphi_R(v) = 0 \quad \text{if} \quad |v| \ge 2R, \\ |\nabla\varphi_R| \le \frac{C}{R}, \quad |\nabla^2 \varphi_R| \le \frac{C}{R^2}. \end{cases}$$

Then, we test (3.3) with φ_R to obtain

$$\begin{aligned} &\int \left[\frac{(f_k - f_{k-1})}{\tau} + \delta_2 |v|^m f_k\right]\varphi_R\,dv \\ &= -\int A_{k-1}\nabla f_k\cdot\nabla\varphi_R\,dv - \int \nabla a_{k-1} f_k(1-f_k)\cdot\nabla\varphi_R\,dv - \delta_1\int \nabla f_k\cdot\nabla\varphi_R\,dv \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

First, we claim that the right-hand side converges to 0 as $R\to\infty$. Indeed, we bound each term separately, beginning with I_3 as,

$$\begin{aligned} \delta_1 \int \nabla f_k\cdot\nabla\varphi_R\,dv &\le \frac{C\delta_1}{R^2}\int_{\{R\le |v|\le 2R\}} f_k\,dv \\ &\le \frac{C\delta_1}{R^2}\|f_k\|_{L^2}|\{R\le |v|\le 2R\}|^{1/2} \le \frac{C\delta_1}{R^{1/2}}. \end{aligned}$$

Next, we bound I_2 using $0\le f_k\le 1$ and Lemma 2.3 to obtain

$$\int (\nabla a_{k-1} f_k(1-f_k))\cdot\nabla\varphi_R\,dv \le \frac{C\|\nabla a_{k-1}\|_{L^2}\|f_k\|_{L^2}}{R} \le \frac{C\|f_{k-1}\|_{L^{6/5}}\|f_k\|_{L^2}}{R}.$$

For I_1, we integrate by parts to obtain

$$\begin{aligned} -\int A_{k-1}\nabla f_k\cdot\nabla\varphi_R\,dv &= \int f_k(\nabla\cdot A_{k-1})\cdot\nabla\varphi_R\,dv - \int \nabla\cdot(A_{k-1}f_k)\cdot\nabla\varphi_R\,dv \\ &= \int f_k(\nabla\cdot A_{k-1})\cdot\nabla\varphi_R\,dv + \int \operatorname{tr}\left(f_k A_{k-1}\nabla^2\varphi_R\right)dv \\ &=: I_1^1 + I_1^2. \end{aligned}$$

Now, I_1^1 vanishes by a similar estimate, using Lemma 2.2. Finally I_1^2 vanishes by the estimate

$$\int \operatorname{tr}\left(f_k A_{k-1}\nabla^2\varphi_R\right) dv \le \|A_{k-1}\|_{L^\infty}\|f_k\|_{L^2}\|\nabla^2\varphi_R\|_{L^2} \le \frac{C\|f_{k-1}\|_{L^1}\|f_k\|_{L^2}}{R^{1/2}}.$$

Thus, piecing together all the above estimates, we conclude that $I_1+I_2+I_3$ vanishes as $R\to\infty$. Second, taking $R_n\to\infty$ sufficiently fast so that φ_{R_n} are increasing to 1, the monotone convergence theorem yields

$$\int f_k\,dv+\tau\delta_2\int f_k|v|^m\,dv=\int f_{k-1}\,dv.$$

By Lemma 3.5, $0\le f_k\le 1$ and the proof is complete. ∎

Proof. **(Proof of Proposition 3.2.)** Fix $f_0=f_{in}$ as in the statement of Proposition 3.2. Suppose moreover that $f_1,f_2,\dots,f_{k-1}$ have been constructed so that $0\le f_i\le 1$ and $f_i\in L^1$ for $0\le i\le k-1$ and $\{f_i\}_{i=0}^{k-1}$ satisfies (3.5). We will now construct f_k. Indeed, fix $X=L^2$ and Φ the solution map to (3.3) with f_{k-1} fixed.

As stated at the beginning of this step, the role of Lemmas 3.1, 3.2, 3.3 and 3.4 is to verify the hypotheses of the Schaeffer Fixed Point Theorem for $\Phi:X\to X$.

- Lemma 3.1 implies Φ maps X to itself;
- Lemma 3.2 implies that approximate fixed points of Φ are bounded in X;
- Lemma 3.3 implies Φ is a compact map;
- Lemma 3.4 implies $\Phi:X\to X$ is a continuous (nonlinear) map.

Therefore, the Schaeffer Fixed Point Theorem (see [24, Theorem 11.3] for a precise statement) yields a (not necessarily unique) fixed point f_k of the map $z\mapsto\Phi(z)$. Because $\Phi(X)\subset L^2_m\cap H^1$, $f_k\in H^1\cap L^2_m$. As $\Phi(f_k)=f_k$, f_k solves

$$\int \frac{(f_k-f_{k-1})}{\tau}\varphi-\nabla a_{k-1}(f_k)_+(1-f_k)_+\cdot\nabla\varphi\,dv=-\int A_{k-1}\nabla f_k\cdot\nabla\varphi\,dv -\delta_1\int\nabla f_k\cdot\nabla\varphi\,dv-\delta_2\int|v|^m f_k\varphi\,dv.$$

However, since $0\le f_{k-1}\le 1$ by Lemma 3.5, $0\le f_k\le 1$, and we may remove the positive parts to conclude f_k solves the desired weak formulation, namely (3.5). Finally, Lemma 3.6 implies $f_k\in L^1$. By induction, the proof is complete. ∎

3.3. Step 3: Existence of solutions to (3.1). In this step we construct weak solutions $f:[0,T]\times\mathbb{R}^3\to\mathbb{R}$ to the nonlinear, continuous time equation,

$$\partial_t f=\nabla\cdot(A[f(1-f)]\nabla f-\nabla a[f]f(1-f))+\delta_1\Delta f, \tag{3.14}$$

on an arbitrary fixed time interval $[0,T]$ for any fixed $\delta_1>0$ and for fixed initial data f_{in}, where $f_{in}\in L^1$ and $0\le f_{in}\le 1$. We first prove uniform in τ (the time mesh) and δ_2 (the strength of the added localization) estimates on solutions to Equation (3.4). For all $T>0$, let $N=\frac{T}{\tau}$. Define the piecewise interpolant of $\{f_k\}$ as

$$f^{(N)}(v,t)=f_{in}(v)\chi_0(t)+\sum_{k=1}^{N}f_k(v)\chi_{(t_{k-1},t_k]}(t), \tag{3.15}$$

and the backward finite difference operator D_τ as

$$D_\tau f(t) := \frac{f(t)-f(t-\tau)}{\tau}.$$

We also introduce the shift operator

$$\sigma_N(f^{(N)})(\cdot,t) = f_{k-1} \quad \text{for} \quad t \in (t_{k-1}, t_k].$$

With this new notation, we can rewrite (3.5) as

$$\begin{aligned}\int_0^T \int D_\tau f^{(N)} \varphi - \nabla a_N f^{(N)}(1-f^{(N)}) \cdot \nabla \varphi \, dv dt \\ = -\int_0^T \int A_N \nabla f^{(N)} \cdot \nabla \varphi - \delta_1 \nabla f^{(N)} \cdot \nabla \varphi - \delta_2 |v|^m f^{(N)} \varphi \, dv dt,\end{aligned} \tag{3.16}$$

where

$$A_N = A[\sigma_N(f^{(N)})(1-f^{(N)})] \qquad \text{and} \qquad a_N = a[\sigma_N(f^{(N)})].$$

For strong compactness, we need propagation of moments (shown in Lemma 3.8) in the form of

$$\|f^{(N)}\|_{L^\infty([0,T];L^1_2)} \lesssim_T \|f_{in}\|_{L^1_2},$$

and a variation of the Aubin-Lions Lemma for piecewise constant functions, which requires an estimate (shown in Lemma 3.9) of the form

$$\left\| D_\tau f^{(N)} \right\|_{L^2([0,T];H^{-1})} + \|f^{(N)}\|_{L^2([0,T];H^1 \cap L^1_2)} \lesssim \|f_{in}\|_{L^1_2}.$$

We begin with L^1 and L^2 estimates, which are continuous-time analogous of Lemma 3.6 and Lemma 3.1, respectively.

Lemma 3.7 (L^1 and L^2 Estimates). *Suppose $f_{in} \in L^1$ and $0 \le f_{in} \le 1$. Then, the following estimates hold:*

$$\|f^{(N)}\|^2_{L^\infty([0,T];L^2)} + 2\delta_1 \|\nabla f^{(N)}\|^2_{L^2([0,T];L^2)} \le \|f_{in}\|^2_{L^2} + T\|f_{in}\|^{3/2}_{L^1}, \tag{3.17}$$

and

$$\|f^{(N)}\|_{L^\infty([0,T];L^1)} \le \|f_{in}\|_{L^1}. \tag{3.18}$$

Proof. Inequality (3.18) follows by iterating Lemma 3.6. Next, we estimate the L^2 norm of $f^{(N)}$ by testing (3.5) with f_k, using Young's inequality and $A_{k-1} \ge 0$ to obtain

$$\frac{1}{2}\|f_k\|^2_{L^2} + \tau\delta_1 \|\nabla f_k\|^2_{L^2} \le \frac{1}{2}\|f_{k-1}\|^2_{L^2} + \tau \int \nabla a_{k-1} f_k(1-f_k) \cdot \nabla f_k \, dv.$$

For the last integral, we integrate by parts, using $-\Delta a_{k-1} = f_{k-1}$ and get

$$\begin{aligned} 2\int \nabla a_{k-1} f_k (1-f_k) \cdot \nabla f_k \, dv &= 2\int \nabla a_{k-1} \cdot \nabla \left[\frac{1}{2} f_k^2 - \frac{1}{3} f_k^3\right] dv \\ &= 2\int f_{k-1} \left[\frac{1}{2}(f_k)^2 - \frac{1}{3}(f_k)^3\right] dv. \end{aligned}$$

Since $0 \le f_k \le 1$, $\left[\frac{1}{2}(f_k)^2 - \frac{1}{3}(f_k)^3\right] \ge 0$. Therefore, using $0 \le f_{k-1} \le 1$, the interpolation inequality $\|g\|_{L^2} \le \|g\|_{L^1}^{1/4} \|g\|_{L^3}^{3/4}$, and Young's inequality, we have

$$\begin{aligned} 2\int \nabla a_{k-1} f_k(1-f_k)\cdot\nabla f_k\,dv &\le \|f_k\|_{L^2}^2 - \frac{2}{3}\|f_k\|_{L^3}^3 \\ &\le \|f_k\|_{L^1}^{1/2}\|f_k\|_{L^3}^{3/2} - \frac{2}{3}\|f_k\|_{L^3}^3 \\ &\le \frac{3}{8}\|f_k\|_{L^1}. \end{aligned}$$

Using Lemma 3.6, we obtain

$$\|f_k\|_{L^2}^2 + 2\tau\delta_1\|\nabla f_k\|_{L^2}^2 \le \|f_{k-1}\|_{L^2}^2 + \tau\|f_{k-1}\|_{L^1}^{3/2},$$

which implies, recursively,

$$\begin{aligned} \sup_{0\le j\le k}\|f_j\|_{L^2}^2 + 2\delta_1\left(\sum_{j=1}^{k}\tau\|\nabla f_j\|_{L^2}^2\right) &\le \|f_{in}\|_{L^2}^2 + \tau\sum_{j=0}^{k-1}\|f_j\|_{L^1}^{3/2} \\ &\le \|f_{in}\|_{L^2}^2 + \tau\sum_{j=0}^{k-1}\|f_{in}\|_{L^1}^{3/2} \\ &\le \|f_{in}\|_{L^2}^2 + k\tau\|f_{in}\|_{L^1}^{3/2}. \end{aligned}$$

Taking $k = N$ and recalling the definition of $f^{(N)}$ in (3.15) finish the proof of the lemma. ∎

LEMMA 3.8 (Propagation of Moments). *Suppose $f_{in} \in L^1_2$ and $0 \le f_{in} \le 1$. Then, the following estimates hold:*

$$\|f^{(N)}\|_{L^\infty([0,T];L^1_1)} \le \|f_{in}\|_{L^1_1} + C(\|f_{in}\|_{L^1})T, \tag{3.19}$$

and

$$\|f^{(N)}\|_{L^\infty([0,T];L^1_2)} \le \|f_{in}\|_{L^1_2} + C(\|f_{in}\|_{L^1_1})T, \tag{3.20}$$

where the implicit constants are independent of τ_n, δ_1, and δ_2.

Proof. Let $\varphi_R \in C_c^\infty(\mathbb{R}^3)$ be as in Lemma 3.6. Then, we test (3.5) with $\langle v\rangle\varphi_R(v)$ to obtain

$$\begin{aligned} &\int \varphi_R\langle v\rangle f_k dv + \tau\delta_2\int\varphi_R\langle v\rangle|v|^m f_k\,dv \\ &= \int\varphi_R\langle v\rangle f_{k-1}\,dv - \tau\int A_{k-1}\nabla f_k\cdot\nabla(\langle v\rangle\varphi_R)\,dv \\ &\quad + \tau\int\nabla a_{k-1}f_k(1-f_k)\cdot\nabla(\langle v\rangle\varphi_R)\,dv - \tau\delta_1\int\nabla f_k\cdot\nabla(\langle v\rangle\varphi_R)\,dv \\ &=: \int\varphi_R\langle v\rangle f_{k-1}\,dv - \tau(I_1 - I_2 + \delta_1 I_3). \end{aligned}$$

We bound I_3 using $|\Delta(\langle v\rangle\varphi_R)| \le C$ to obtain

$$|I_3| = \left|\int f_k\Delta(\langle v\rangle\varphi_R)\,dv\right| \le \|\Delta(\langle v\rangle\varphi_R)\|_{L^\infty}\|f_k\|_{L^1} \lesssim \|f_k\|_{L^1}.$$

For I_2 we use Lemma 2.3, $0\le f_k\le 1$, and $|\nabla(\langle v\rangle\varphi_R)|\le C$:

$$|I_2|\le \|\nabla a_{k-1}\|_{L^\infty}\|f_k\|_{L^1}\|\nabla(\langle v\rangle\varphi_R)\|_{L^\infty}\lesssim (\|f_{k-1}\|_{L^1}+\|f_{k-1}\|_{L^\infty})\|f_k\|_{L^1}.$$

For I_1, we integrate by parts twice to get

$$\begin{aligned} I_1 &= \int \nabla\cdot(A_{k-1}f_k)\cdot\nabla(\langle v\rangle\varphi_R) - (\nabla\cdot A_{k-1})f_k\cdot\nabla(\langle v\rangle\varphi_R)\,dv \\ &= -\int \operatorname{tr}(A_{k-1}f_k\nabla^2(\langle v\rangle\varphi_R))\,dv - \int(\nabla\cdot A_{k-1})f_k\cdot\nabla(\langle v\rangle\varphi_R)\,dv \\ &=: -I_{1,1}-I_{1,2}. \end{aligned}$$

Lemma 2.2 and $|\nabla^2(\langle v\rangle\varphi_R)|\le C$ yield

$$\begin{aligned} |I_{1,1}| &\le \|A_{k-1}\|_{L^\infty}\|f_k\|_{L^1}\|\nabla^2(\langle v\rangle\varphi_R)\|_{L^\infty} \\ &\lesssim (\|f_{k-1}\|_{L^1}+\|f_{k-1}\|_{L^\infty})\|f_k\|_{L^1}. \end{aligned}$$

Lemma 2.3 and $|\nabla(\langle v\rangle\varphi_R)|\le C$ yield

$$\begin{aligned} |I_{1,2}| &\le \|\nabla\cdot A_{k-1}\|_{L^\infty}\|f_k\|_{L^1}\|\nabla(\langle v\rangle\varphi_R)\|_{L^\infty} \\ &\lesssim (\|f_{k-1}\|_{L^1}+\|f_{k-1}\|_{L^\infty})\|f_k\|_{L^1}. \end{aligned}$$

Combining all above estimates we obtain

$$\begin{aligned} \sup_{0\le j\le k}\int \varphi_R\langle v\rangle f_j\,dv &\le \int \varphi_R\langle v\rangle f_{in}\,dv + C\sum_{j=1}^k \tau\left(\|f_{j-1}\|_{L^1}+\|f_{j-1}\|_{L^1}^2\right) \\ &\le \int \langle v\rangle f_{in}\,dv + Ck\tau\left(\|f_{in}\|_{L^1}+\|f_{in}\|_{L^1}^2\right). \end{aligned}$$

Now, taking $k=N$, recalling the definition of $f^{(N)}$ in (3.15) and letting $R\to\infty$, the monotone convergence theorem implies (3.19).

The proof of (3.20) proceeds nearly identically after testing with $\langle v\rangle^2\varphi_R$. ∎

The bounds in Lemmas 3.7 and 3.8 are sufficient for weak or weak star compactness. For strong compactness, we will use the version of the Aubin-Lions Lemma for piecewise constant functions [22, Theorem 1].

LEMMA 3.9. *For any $T>0$, $f_{in}\in L^1_2$, and $0\le f_{in}\le 1$, for $f^{(N)}$ defined above with $0<m<1$,*

$$\|D_\tau f^{(N)}\|_{L^2([0,T];H^{-1})}\le C(\|f_{in}\|_{L^1_2},T,\delta_1). \tag{3.21}$$

Moreover, the family $\{f^{(N)}\}$ is compact in $L^2([0,T];L^q)$, provided $1\le q<6$.

Proof. Let us define the triple $X:=L^1_2\cap H^1$, $Y:=L^q\cap L^2$ and $Z:=H^{-1}$ for a fixed $1\le q<6$. Following the proof of Lemma 3.3, the embedding $X\hookrightarrow Y$ is compact for $1\le q<6$. Certainly, $Y\hookrightarrow Z$ continuously for this range of q. Moreover, we have shown in Lemmas 3.7 and 3.8,

$$\|f^{(N)}\|_{L^2([0,T];X)}\le C(\delta_1,\|f_{in}\|_{L^1},\|f_{in}\|_{L^2},T), \tag{3.22}$$

where the constant on the right-hand side is independent of δ_2 and τ. To obtain (3.21) we first consider

$$\left|\int_0^T \int_{\mathbb{R}^3} D_\tau f^{(N)} \varphi \, dv dt\right|$$
$$\leq \left|\int_0^T \int_{\mathbb{R}^3} A_N \nabla f^{(N)} \cdot \nabla\varphi \, dv dt\right| + \left|\int_0^T \int_{\mathbb{R}^3} \nabla a_N f^{(N)}(1-f^{(N)}) \cdot \nabla\varphi \, dv dt\right|$$
$$+\delta_1 \left|\int_0^T \int_{\mathbb{R}^3} \nabla f^{(N)} \cdot \nabla\varphi \, dv dt\right| + \delta_2 \left|\int_0^T \int_{\mathbb{R}^3} f^{(N)} \varphi |v|^m \, dv dt\right| =: I_1 + \ldots + I_4.$$

For $\varphi \in L^2([0,T];H^1)$, thanks to Lemma 2.2, one gets

$$I_1 \lesssim \int_0^T \|\varphi(t)\|_{H^1} \|\nabla f^{(N)}\|_{L^2} \|A_N\|_{L^\infty} \, dt$$
$$\lesssim \|\varphi\|_{L^2(0,T;H^1)} \|\nabla f^{(N)}\|_{L^2(0,T;L^2)} (\|f_{in}\|_{L^1} + 1),$$

and, using Lemma 2.3,

$$I_2 \lesssim \int_0^T \|\varphi(t)\|_{H^1} \|f^{(N)}(1-f^{(N)})\|_{L^\infty} \|\nabla a_N\|_{L^2} \, dt$$
$$\lesssim \|\varphi\|_{L^2(0,T;H^1)} \|f^{(N)}\|_{L^2(0,T;L^{6/5})}.$$

Finally,

$$I_3 \lesssim \|\varphi\|_{L^2(0,T;H^1)} \|\nabla f^{(N)}\|_{L^2(0,T;L^2)},$$

and, since $2m < 2$,

$$I_4 \lesssim \|\varphi\|_{L^2(0,T;L^2)} \|f^{(N)}\|_{L^1(0,T;L^1_{2m})} \leq C(T, f_{in}) \|\varphi\|_{L^2(0,T;L^2)},$$

using Lemma 3.8. We note $\|\nabla f^{(N)}\|_{L^2([0,T];L^2)}$, $\|f^{(N)}\|_{L^\infty([0,T];L^1)}$, and $\|f^{(N)}\|_{L^\infty([0,T];L^1_2)}$ are uniformly bounded in δ_2 and N (but not in δ_1) by Lemmas 3.7 and 3.8. Thus (3.21) follows. Theorem 1 in [22], (3.21) and (3.22) yield the desired compactness. □

We are now ready to prove Proposition 3.3:

Proof. **(Proof of Proposition 3.3.)** Let $\delta_2 = \tau$, fix some $0 < m < 1$, and $\{f^{(N)}\}_{N \in \mathbb{N}}$ be the corresponding sequence of piecewise constant solutions to (3.16). Thanks to the estimates from Lemma 3.7, Lemma 3.8, and Lemma 3.9, we may assume that $f^{(N)}$ converges to f, as $\tau \to 0$, in the following topologies:

- Weak star in $L^\infty([0,T] \times \mathbb{R}^3)$,
- Weakly in $L^2([0,T];H^1)$,
- Weak star in $L^\infty([0,T];L^2)$,
- Strongly in $L^p([0,T];L^q)$ for $1 \leq p \leq 2$ and $1 \leq q < 6$.

Moreover, by taking a further subsequence, we will also have that $f^{(N)} \to f$ pointwise almost everywhere. Therefore, thanks to Fatou's lemma

$$\|f\|_{L^\infty([0,T];L^1_2)} + \delta_1 \|\nabla f\|_{L^2([0,T];L^2)} \leq C(\|f_{in}\|_{L^1_2}, T).$$

All these convergences are enough to pass to the limit $N\to+\infty$ in (3.16). We briefly highlight the convergence in the nonlinear terms. Let us first consider $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^3)$. We have

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^3}\left[\nabla a_N f^{(N)}(1-f^{(N)})-\nabla a[f]f(1-f)\right]\cdot\nabla\varphi\,dvdt\right|\\
&\leq\left|\int_0^T\int_{\mathbb{R}^3}(\nabla a_N-\nabla a[f])\,f^{(N)}(1-f^{(N)})\cdot\nabla\varphi\,dvdt\right|\\
&\quad+\left|\int_0^T\int_{\mathbb{R}^3}\nabla a[f]\left[f(1-f)-f^{(N)}(1-f^{(N)})\right]\cdot\nabla\varphi\,dvdt\right|\\
&=:I_1+I_2.
\end{aligned}$$

We estimate I_1 using Hölder's inequality and $\|f^{(N)}(1-f^{(N)})\|_{L^\infty}\leq 1$:

$$\begin{aligned}
I_1&\leq\int_0^T\|\nabla a[\sigma_N f^{(N)}-f]\|_{L^2}\|\nabla\varphi\|_{L^2}\,dt\\
&\lesssim\int_0^T\|\sigma_N f^{(N)}-f]\|_{L^{6/5}}\|\nabla\varphi\|_{L^2}\,dt\to 0,
\end{aligned}$$

thanks to the strong convergence, and, similarly, using Lemma 2.3,

$$I_2\leq 2\int_0^T\|\nabla a[f]\|_{L^\infty}\|f^{(N)}-f\|_{L^2}\|\nabla\varphi\|_{L^2}\,dt\to 0.$$

Next, we handle the nonlinear term involving A_N, which we decompose as

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^3}\left[A_N\nabla f^{(N)}-A[f(1-f)]\nabla f\right]\cdot\nabla\varphi\,dvdt\right|\\
&\leq\left|\int_0^T\int_{\mathbb{R}^3}(A_N-A[f(1-f)])\,\nabla f^{(N)}\cdot\nabla\varphi\,dvdt\right|\\
&\quad+\left|\int_0^T\int_{\mathbb{R}^3}A[f(1-f)]\,(\nabla f-\nabla f^{(N)})\cdot\nabla\varphi\,dvdt\right|\\
&=:J_1+J_2.
\end{aligned}$$

The term J_2 converges to zero thanks to the weak convergence of $f^{(N)}$ in $H^1(\mathbb{R}^3)$ and Lemma 2.2. For J_1, we use Hölder's inequality, Lemma 2.2, estimate (3.17) and the strong convergence in $L^2([0,T];L^2)$ to obtain $J_1\to 0$, since

$$\begin{aligned}
J_1\lesssim_T&\|\nabla f^{(N)}\|_{L^2([0,T];L^2)}\|\varphi\|_{L^\infty([0,T];H^1)}\cdot\\
&\cdot\|\sigma_N(f^{(N)})-f\|_{L^\infty([0,T];L^1)}^{4/3}\|\sigma_N(f^{(N)})-f\|_{L^2([0,T];L^2)}^{2/3}.
\end{aligned}$$

We treat the left-hand side of (3.16) by integrating by parts,

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^3}D_\tau f^{(N)}\varphi\,dvdt=&-\int_0^{T-\tau}D_{-\tau}\varphi f^{(N)}(t)\,dvdt\\
&+\frac{1}{\tau}\int_{T-\tau}^T\int_{\mathbb{R}^3}f^{(N)}(t)\varphi(t)\,dvdt-\frac{1}{\tau}\int_{-\tau}^0\int_{\mathbb{R}^3}f^{(N)}(t)\varphi(t+\tau)\,dvdt.
\end{aligned}$$

For N sufficiently large,

$$\frac{1}{\tau}\int_{T-\tau}^{T}\int_{\mathbb{R}^3} f^{(N)}(t)\varphi(t)\,dvdt=0,$$

as φ is compactly supported in $[0,T)\times\mathbb{R}^3$. Moreover, for $0\le t<\tau$, $f^{(N)}(t)=f_{in}$ so that

$$\frac{1}{\tau}\int_{-\tau}^{0}\int_{\mathbb{R}^3} f^{(N)}(t)\varphi(t+\tau)\,dvdt=\frac{1}{\tau}\int_{-\tau}^{0}\int_{\mathbb{R}^3} f_{in}\varphi(t+\tau)\,dvdt.$$

Since φ is smooth, the right-hand side converges to $\int_{\mathbb{R}^3}\varphi(0,v)f_{in}(v)$ as $N\to\infty$. Finally, since φ is smooth and $f^{(N)}$ are uniformly bounded in $L^2([0,T];L^2)$, we have

$$-\int_0^{T-\tau} D_{-\tau}\varphi f^{(N)}(t)\,dvdt\to-\int_0^T\int_{\mathbb{R}^3} f(v,t)\partial_t\varphi(v,t)\,dvdt. \tag{3.23}$$

This concludes the proof of (3.6). Lemma 3.9 implies that, for some $g\in L^2([0,T];H^{-1})$,

$$\int_0^T\int_{\mathbb{R}^3} D_\tau f^{(N)}\varphi\,dvdt\to\int_0^T\int_{\mathbb{R}^3} g\varphi\,dvdt,$$

for every $\varphi\in L^2([0,T];H^1)$. Hence, (3.23) yields $g=\partial_t f$. The distributional formulation implies

$$\begin{aligned}\int_0^T\langle\varphi,\partial_t f\rangle_{H^1\times H^{-1}}\,dt=&-\int_0^T\int_{\mathbb{R}^3}(A[f(1-f)]\nabla f-\nabla a[f]f(1-f))\cdot\nabla\varphi\,dvdt\\&-\delta_1\int_0^T\int_{\mathbb{R}^3}\nabla f\cdot\nabla\varphi\,dvdt,\end{aligned}$$

for each $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^3)$. Now, fix $\Phi\in L^2([0,T];H^1)$ and let $\varphi_\varepsilon\in C_c^\infty([0,T)\times\mathbb{R}^3)$ such that $\|\Phi-\varphi_\varepsilon\|_{L^2([0,T];H^1)}\le\varepsilon$. Then, substituting φ_ε into the above weak formulation, and passing to the limit $\varepsilon\to0$, we obtain (3.7).

Finally, we note that because $f\in L^2([0,T];H^1)$ and $\partial_t f\in L^2([0,T];H^{-1})$, $f\in C([0,T];L^2)$ and therefore (3.7) implies $f(t)\to f_{in}$ strongly in L^2 as $t\to0^+$. Moreover, repeating the proof of Lemma 3.6, the additional $\delta_2\|f|v|^m\|_{L^1}$ term disappears thanks to the uniform bound from Lemma 3.8, and we obtain conservation of mass. ∎

3.4. Step 4: Proof of Theorem 1.1. We conclude the proof of Theorem 1.1 by showing compactness in δ_1 for solutions to (3.1). We already have uniform in δ_1 bounds of the form,

$$\|f_{\delta_1}\|_{L^\infty([0,T];L^1_2)}+\|f_{\delta_1}\|_{L^\infty([0,T]\times\mathbb{R}^3)}\le C(\|f_{in}\|_{L^1_2},T). \tag{3.24}$$

Thus, to gain strong compactness as $\delta_1\to0^+$, we will show (in Lemma 3.12) the estimate

$$\|f_{\delta_1}\|_{L^2([0,T];H^1)}+\|\partial_t f_{\delta_1}\|_{L^2([0,T];H^{-1})}\le C(\|f_{\delta_1}\|_{L^\infty([0,T];L^1_3)},T), \tag{3.25}$$

by leveraging the degenerate dissipation present in (1.1) (see Lemma 2.1), which up to this point, we have neglected. However, we do not have control over L^1_3 and therefore, we also show propagation of higher moments in Lemma 3.10.

To this end, we recall the dependence of our solutions on the parameter δ_1. Throughout this section, we will write $f_\delta:[0,T]\times\mathbb{R}^3\to\mathbb{R}$ to denote the solution f_δ to (3.1) on $[0,T]$ constructed in Proposition 3.3 with parameter $\delta_1=\delta$. Let us begin with a propagation of higher moments estimate that is uniform in δ:

LEMMA 3.10. *Suppose $f_{in}\in L^1_s$ for some $s>2$ and $T>0$. Then, the family $\{f_\delta\}_{0<\delta<1}$ satisfies the uniform in δ estimate,*

$$\|f_\delta\|_{L^\infty([0,T];L^1_s)}\le C(\|f_{in}\|_{L^1_s},T). \tag{3.26}$$

Proof. We note that the propagation of moments for $0\le s\le 2$ follows directly from Proposition 3.3. We will prove the rest of them by induction on the integer part of s. Indeed, fix some $2\le n<s\le n+1$ and suppose that (3.26) holds for any $0\le s\le n$. Then, we will test (3.7) with $\Psi(v,t)=\varphi_R(v)|v|^s\chi_{[0,t_0]}(t)$ where $\varphi_R\in C^\infty_c(\mathbb{R}^3)$ is as in Lemma 3.6. We obtain

$$\begin{aligned}\int_{\mathbb{R}^3}(f_\delta(t_0)-f_{in})\varphi_R|v|^s\,dv=&-\int_0^{t_0}\int_{\mathbb{R}^3}A[f_\delta(1-f_\delta)]\nabla f_\delta\cdot\nabla(|v|^s\varphi_R)\,dvdt\\&+\int_0^{t_0}\int_{\mathbb{R}^3}(\nabla a[f_\delta]f_\delta(1-f_\delta)-\delta\nabla f_\delta)\cdot\nabla(|v|^s\varphi_R)\,dvdt.\end{aligned}$$

We estimate the right-hand side by decomposing into multiple parts:

$$\begin{aligned}&-\int_0^{t_0}\int_{\mathbb{R}^3}\Big(A[f_\delta(1-f_\delta)]\nabla f_\delta-\nabla a[f_\delta]f_\delta(1-f_\delta)+\delta\nabla f_\delta\Big)\cdot\nabla(|v|^s\varphi_R)\,dvdt\\&=-\int_0^{t_0}\int_{\mathbb{R}^3}\nabla\cdot(A[f_\delta(1-f_\delta)]f_\delta)\cdot\nabla(|v|^s\varphi_R)\,dvdt\\&\quad+\int_0^{t_0}\int_{\mathbb{R}^3}f_\delta(\nabla\cdot A)[f_\delta(1-f_\delta)]\cdot\nabla(|v|^s\varphi_R)\,dvdt\\&\quad+\int_0^{t_0}\int_{\mathbb{R}^3}(\nabla a[f_\delta]f_\delta(1-f_\delta))\cdot\nabla(|v|^s\varphi_R)\,dvdt-\delta\int_0^{t_0}\int_{\mathbb{R}^3}\nabla f_\delta\cdot\nabla(|v|^s\varphi_R)\,dvdt\\&=-I_1+I_2+I_3-\delta I_4.\end{aligned}$$

For I_1, after integrating by parts, thanks to Lemma 2.2 and $|\nabla^2(|v|^s\varphi_R)|\le C|v|^{s-2}$ we obtain

$$\begin{aligned}|I_1|&\lesssim\big(\|f_\delta\|_{L^1([0,T];L^\infty)}+\|f_\delta\|_{L^1([0,T];L^1)}\big)\,\|f_\delta|v|^{s-2}\|_{L^\infty([0,T];L^1)}\\&\le C(\|f_{in}\|_{L^1_{s-2}},T),\end{aligned}$$

where in the last line we used the induction hypothesis. For I_2, we use Lemma 2.2 and $|\nabla(|v|^s\varphi_R)|\le C|v|^{s-1}$ to obtain

$$|I_2|\le\|f_\delta\nabla(|v|^s\varphi_R)\|_{L^\infty([0,T];L^1)}\|\nabla\cdot A[f_\delta(1-f_\delta)]\|_{L^1([0,T];L^\infty)}\le C(\|f_{in}\|_{L^1_{s-1}},T).$$

Similarly, for I_3, we use Lemma 2.3 and $0\le f_\delta\le 1$ to obtain

$$|I_3|\le\|f_\delta\nabla(|v|^s\varphi_R)\|_{L^\infty([0,T];L^1)}\|\nabla a[f_\delta]\|_{L^1([0,T];L^\infty)}\le C(\|f_{in}\|_{L^1_{s-1}},T).$$

Finally, for I_4, integration by parts yields

$$|I_4|\le T\|f\Delta(|v|^s\varphi_R)\|_{L^\infty([0,T];L^1}\le C(\|f_{in}\|_{L^1_{s-2}},T).$$

Combining all above estimates, we prove (3.26) for any $s\in(n,n+1]$. The proof is complete. ∎

The following lemma, combined with Lemma 2.1, gives a quantitative lower bound on the ellipticity of $A[f_\delta(1-f_\delta)]$. This will allow us to gain some control over ∇f_δ uniformly in δ.

Lemma 3.11. *Suppose $0\le f_{in}\le 1$, $f_{in}\in L^1_2$, $H_1(f_{in})<0$, and $T>0$. Then, f_δ has decreasing entropy, i.e. for almost every $0\le t_1<t_2\le T$,*

$$H_1(f_\delta(t_2))\le H_1(f_\delta(t_1)). \tag{3.27}$$

Moreover, the dissipative coefficients $A[f_\delta(1-f_\delta)]$ are bounded uniformly from below:

$$A[f_\delta(1-f_\delta)]\ge \frac{C(\|f_{in}\|_{L^1_2},H_1(f_{in}),T)}{1+|v|^3}. \tag{3.28}$$

Proof. By Lemma 2.1, (3.28) is a consequence of (3.27) and

$$\|f_\delta(t)\|_{L^1_2}\le C(\|f_{in}\|_{L^1_2}),\quad \text{for all}\, t>0. \tag{3.29}$$

The energy bound (3.29) is shown in Lemma 3.10. It remains to estimate the entropy and obtain (3.27). We test (3.7) with

$$\psi_\eta:=\log(f_\delta+\eta)-\log(\eta)-\log(1-f_\delta+\eta)+\log(1+\eta),\quad \eta>0.$$

We have

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\psi_\eta\partial_t f_\delta\,dvdt=&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\psi_\eta\cdot A[f_\delta(1-f_\delta)]\nabla f_\delta\,dvdt\\&+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\psi_\eta\cdot\nabla a[f_\delta](f_\delta)(1-f_\delta)\,dvdt-\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\psi_\eta\cdot\nabla f_\delta\,dvdt\\=&:-I_1(\eta)+I_2(\eta)-I_3(\eta).\end{aligned}$$

We now take $\eta\to 0^+$. For the left-hand side, we use conservation of mass from Proposition 3.3 to obtain:

$$\begin{aligned}\lim_{\eta\to0^+}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\psi_\eta\partial_t f_\delta\,dvdt=&\lim_{\eta\to0^+}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\log(f_\delta+\eta)\partial_t f_\delta-\log(1-f_\delta+\eta)\partial_t f_\delta\,dvdt\\=&H_1(f_\delta(t_2))-H_1(f_\delta(t_1)).\end{aligned}$$

By the monotone convergence theorem,

$$\begin{aligned}\lim_{\eta\to0^+}I_1(\eta)=&\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left[(f_\delta)^{-1}+(1-f_\delta)^{-1}\right]\nabla f_\delta\cdot A[f_\delta(1-f_\delta)]\nabla f_\delta\,dvdt\\=&\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\left[\log(f_\delta)-\log(1-f_\delta)\right]\cdot A[f_\delta(1-f_\delta)]\nabla f_\delta\,dvdt.\end{aligned}$$

Next, for I_2, we decompose further as

$$\begin{aligned}I_2(\eta)=&\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left[(f_\delta+\eta)^{-1}+(1-f_\delta+\eta)^{-1}\right]\nabla f_\delta\cdot\nabla a[f_\delta](f_\delta)(1-f_\delta)\,dvdt\\=&(1+2\eta)\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla f_\delta\cdot\nabla a[f_\delta]\,dvdt\\&+\eta(1-\eta)\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left[(f_\delta+\eta)^{-1}+(1-f_\delta+\eta)^{-1}\right]\nabla f_\delta\cdot\nabla a[f_\delta]\,dvdt\\=&I_2^1(\eta)+I_2^2(\eta).\end{aligned}$$

For I_2^1, we integrate by parts to obtain

$$\lim_{\delta\to 0} I_2^1(\eta) = \int_{t_1}^{t_2}\int_{\mathbb{R}^3} f_\delta^2\, dvdt.$$

For I_2^2, we use $0 \le f_\delta \le 1$ with $f_\delta \in L^\infty([0,T];L^1)$ and $\eta\log\eta \to 0$ as $\eta \to 0^+$ to obtain

$$\begin{aligned}|I_2^2(\eta)| &= \eta(1-\eta)\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^3} f_\delta\left[\log(f_\delta+\eta) - \log(1-f_\delta+\eta)\right] dvdt\right| \\ &\le 2|\log(\eta)|\eta(1-\eta)\int_{t_1}^{t_2}\int_{\mathbb{R}^3} f_\delta\, dvdt \to 0.\end{aligned}$$

Finally, we note for I_3 that

$$I_3(\eta) = \delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left[(f_\delta+\eta)^{-1} + (1-f_\delta+\eta)^{-1}\right]|\nabla f_\delta|^2\, dvdt \ge 0.$$

Thus, combining our estimates, we have shown

$$H_1(f_\delta(t_2)) - H_1(f_\delta(t_1)) \le -\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left(\nabla[\log(f_\delta/(1-f_\delta))]\cdot A[f_\delta(1-f_\delta)]\nabla f_\delta - f_\delta^2\right) dvdt.$$

We conclude by noticing that

$$\begin{aligned}&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left(\nabla[\log(f_\delta) - \log(1-f_\delta)]\cdot A[f_\delta(1-f_\delta)]\nabla f_\delta - f_\delta^2\right) dvdt \\ &= -\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\delta f_\delta^*(1-f_\delta)(1-f_\delta^*) \\ &\quad\times\left(\frac{\Pi(v-v^*)}{|v-v^*|}\left[\frac{\nabla f^*}{f^*(1-f^*)} - \frac{\nabla f}{f(1-f)}\right]\cdot\left[\frac{\nabla f^*}{f^*(1-f^*)} - \frac{\nabla f}{f(1-f)}\right]\right) dvdv^*dt \\ &\le 0.\end{aligned}$$

□

The next lemma contains the coercive estimate we need to pass to the limit $\delta \to 0$.

Lemma 3.12 (L^2 Estimate). *Suppose $f_{in} \in L_3^1$ with $H_1(f_{in}) < 0$ and $T > 0$. Then, the family $\{f_\delta\}_{0<\delta<1}$ satisfies the estimate*

$$\|\partial_t f_\delta\|_{L^2([0,T];H^{-1})} + \|f_\delta\|_{L^\infty([0,T];L_3^1)} + \|\nabla f_\delta\|_{L^2([0,T];L^2)} \le C(\|f_{in}\|_{L_3^1}, H_1(f_{in}), T).$$

Proof. We test (3.7), with $\Psi(v,t) = f_\delta(v,t)\varphi_R(v)\langle v\rangle^3\chi_{[0,t_0]}(t)$ where $\varphi_R \in C_c^\infty(\mathbb{R}^3)$ is a cutoff function as in Lemma 3.6. We obtain,

$$\begin{aligned}&\int_{\mathbb{R}^3} f_\delta^2(t_0)\varphi_R\langle v\rangle^3 - f_{in}^2\varphi_R\langle v\rangle^3\, dv \\ &= -\int_0^{t_0}\int_{\mathbb{R}^3}\left(A[f_\delta(1-f_\delta)]\nabla f_\delta - \nabla a[f_\delta]f_\delta(1-f_\delta)\right)\cdot\nabla\Psi\, dvdt \\ &\quad -\delta\int_0^{t_0}\int_{\mathbb{R}^3}\nabla f_\delta\cdot\nabla\Psi\, dvdt.\end{aligned}$$

We expand the right-hand side as

$$
\begin{aligned}
&-\int_0^{t_0}\int_{\mathbb{R}^3}(A[f_\delta(1-f_\delta)]\nabla f_\delta-\nabla a[f_\delta]f_\delta(1-f_\delta))\cdot\nabla\Psi\,dvdt-\delta\int_0^{t_0}\int_{\mathbb{R}^3}\nabla f_\delta\cdot\nabla\Psi\,dvdt\\
&=-\int_0^{t_0}\int_{\mathbb{R}^3}\varphi_R\langle v\rangle^3 A[f_\delta(1-f_\delta)]\nabla f_\delta\cdot\nabla f_\delta\,dvdt\\
&\quad-\int_0^{t_0}\int_{\mathbb{R}^3} f_\delta A[f_\delta(1-f_\delta)]\nabla f_\delta\cdot\nabla(\varphi_R\langle v\rangle^3)\,dvdt\\
&\quad+\int_0^{t_0}\int_{\mathbb{R}^3}\nabla a[f_\delta]f_\delta(1-f_\delta)\cdot f_\delta\nabla(\varphi_R\langle v\rangle^3)\,dvdt\\
&\quad+\int_0^{t_0}\int_{\mathbb{R}^3}\nabla a[f_\delta](f_\delta-f_\delta^2)\cdot\varphi_R\langle v\rangle^3\nabla f_\delta\,dvdt\\
&\quad-\delta\int_0^{t_0}\int_{\mathbb{R}^3}\nabla f_\delta\cdot f_\delta\nabla(\varphi_R\langle v\rangle^3)\,dvdt-\delta\int_0^{t_0}\int_{\mathbb{R}^3}\nabla f_\delta\cdot\varphi_R\langle v\rangle^3\nabla f_\delta\,dvdt\\
&=:-I_1-I_2+I_3+I_4-\delta I_5-\delta I_6.
\end{aligned}
$$

We bound I_j for $2\le j\le 5$ using the propagation of moments from Lemma 3.10 and the upper bounds on the coefficients A and ∇a from Lemmas 2.2 and 2.3. We will lower bound I_1 using Lemma 3.11. We begin to bound I_2 by decomposing further:

$$
\begin{aligned}
I_2&=\frac{1}{2}\int_0^{t_0}\int_{\mathbb{R}^3}A[f_\delta(1-f_\delta)]\nabla f_\delta^2\cdot\nabla(\varphi_R\langle v\rangle^3)\,dvdt\\
&=\frac{1}{2}\int_0^{t_0}\int_{\mathbb{R}^3}\nabla\cdot\left(A[f_\delta(1-f_\delta)]f_\delta^2\right)\cdot\nabla(\varphi_R\langle v\rangle^3)\,dvdt\\
&\quad-\frac{1}{2}\int_0^{t_0}\int_{\mathbb{R}^3}(\nabla\cdot A[f_\delta(1-f_\delta)])f_\delta^2\cdot\nabla(\varphi_R\langle v\rangle^3)\,dvdt\\
&=:I_2^1-I_2^2.
\end{aligned}
$$

For I_2^1, integration by parts, Lemma 2.2, and $|\nabla^2(\varphi_R\langle v\rangle^3)|\lesssim\langle v\rangle$ imply

$$|I_2^1|\le\|A[f_\delta(1-f_\delta)\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|f_\delta\nabla^2(\varphi_R\langle v\rangle^3)\|_{L^1([0,T]\times\mathbb{R}^3)}\lesssim C(\|f_{in}\|_{L^1_1},T).$$

For I_2^2, obtain by Lemma 2.2, and $|\nabla(\varphi_R\langle v\rangle^3)|\lesssim\langle v\rangle^2$,

$$|I_2^2|\le\|\nabla\cdot A[f_\delta(1-f_\delta)\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|f_\delta\nabla(\varphi_R\langle v\rangle^3)\|_{L^1([0,T]\times\mathbb{R}^3)}\lesssim C(\|f_{in}\|_{L^1_2},T).$$

Piecing together, we obtain

$$|I_2|\le C(\|f_{in}\|_{L^1_2},T).$$

For I_3, we directly use the estimates from Lemmas 3.10, 2.2, and 2.3 and $|\nabla(\varphi_R\langle v\rangle^3)|\lesssim\langle v\rangle^2$ to obtain

$$|I_3|\le\|\nabla a[f_\delta\|_{L^\infty([0,T]\times\mathbb{R}^3)}\|f_\delta\nabla(\varphi_R\langle v\rangle^3)\|_{L^1([0,T];L^1)}\le C(\|f_{in}\|_{L^1_2},T).$$

Next, for I_4, we integrate by parts and recall $-\Delta a[f]=f$ to decompose further:

$$
\begin{aligned}
I_4&=\int_0^{t_0}\int_{\mathbb{R}^3}\nabla a[f_\delta]\cdot\varphi_R\langle v\rangle^3\nabla\left(\frac{f_\delta^2}{2}-\frac{f_\delta^3}{3}\right)dvdt\\
&=-\int_0^{t_0}\int_{\mathbb{R}^3}\nabla a[f_\delta]\cdot\nabla(\varphi_R\langle v\rangle^3)\left(\frac{f_\delta^2}{2}-\frac{f_\delta^3}{3}\right)dvdt+\int_0^{t_0}\int_{\mathbb{R}^3}\varphi_R\langle v\rangle^3\left(\frac{f_\delta^3}{2}-\frac{f_\delta^4}{3}\right)dvdt\\
&=-I_4^1+I_4^2.
\end{aligned}
$$

We bound I_4^1 using Lemma 2.3 and $|\nabla(\varphi_R(v)\langle v\rangle^3)| \lesssim \langle v\rangle^2$, to obtain

$$|I_4^1| \leq \|\nabla a[f_\delta]\|_{L^\infty([0,T]\times\mathbb{R}^3)} \|f_\delta \nabla(\varphi_R\langle v\rangle^3)\|_{L^1([0,T]\times\mathbb{R}^3)} \leq C(\|f_{in}\|_{L^1_2}, T).$$

For I_4^2, we bound using $0 \leq f_\delta \leq 1$ so that

$$|I_4^2| \leq \|\varphi_R\langle v\rangle^3 f_\delta\|_{L^1([0,T]\times\mathbb{R}^3)} \|f_\delta^2/2 - f_\delta^3/3\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq C(\|f_{in}\|_{L^1_3}, T).$$

Hence,

$$|I_4| \leq C(\|f_{in}\|_{L^1_3}, T).$$

Using $|\nabla^2(\varphi_R(v)\langle v\rangle^3)| \lesssim \langle v\rangle$, integration by parts yields

$$|I_5| \leq \|f_\delta\|_{L^\infty([0,T]\times\mathbb{R}^3)} \|f_\delta \nabla^2(\varphi_R(v)\langle v\rangle^3)\|_{L^1([0,T]\times\mathbb{R}^3)} \leq C(\|f_{in}\|_{L^1_1}, T).$$

Finally, we note $I_6 \geq 0$ and by Lemma 3.11,

$$I_1 \geq C(\|f_{in}\|_{L^1_2}, H_1(f_{in}), T) \int_0^{t_0} \int_{\mathbb{R}^3} \frac{\varphi_R\langle v\rangle^3}{1+|v|^3} |\nabla f_\delta|^2 \, dvdt.$$

Summarizing, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi_R\langle v\rangle^3 f_\delta(t_0)\, dv + C(\|f_{in}\|_{L^1_2}, H_1(f_{in}), T) \int_0^{t_0} \int_{\mathbb{R}^3} \varphi_R |\nabla f_\delta|^2 \, dvdt \\
&\leq \int_{\mathbb{R}^3} \varphi_R\langle v\rangle^3 f_{in}\, dv + C(\|f_{in}\|_{L^1_3}, T).
\end{aligned}$$

Letting $R \to \infty$, applying the monotone convergence theorem, and taking a supremum over $t_0 \in [0,T]$ yield the desired bound on ∇f_δ.

Next, we test (3.7) with an arbitrary test function $\Phi \in L^2([0,T]; H^1)$ and, by duality, obtain a bound on $\partial_t f$. In particular, we have

$$\|\partial_t f_\delta\|_{L^2([0,T];H^{-1})} = \sup \int_0^T \int_{\mathbb{R}^3} (A[f_\delta(1-f_\delta)]\nabla f_\delta - \nabla a[f_\delta] f_\delta(1-f_\delta) + \delta \nabla f_\delta) \cdot \nabla\Phi \, dvdt,$$

where the supremum is taken over all the functions Φ such that $\|\Phi\|_{L^2([0,T];H^1)} \leq 1$. Since

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} A[f_\delta(1-f_\delta)]\nabla f_\delta \cdot \nabla\Phi \, dvdt \\
&\leq \|A[f_\delta(1-f_\delta)]\|_{L^\infty([0,T]\times\mathbb{R}^3)} \cdot \|\nabla f\|_{L^2([0,T];L^2)} \|\nabla\Phi\|_{L^2([0,T];L^2)},
\end{aligned}$$

and

$$\int_0^T \int_{\mathbb{R}^3} \nabla a[f_\delta] f_\delta(1-f_\delta) \cdot \nabla\Phi \, dvdt \leq \|\nabla a[f_\delta]\|_{L^2([0,T];L^2)} \|\nabla\Phi\|_{L^2([0,T];L^2)},$$

we conclude

$$\begin{aligned}
|\partial_t f_\delta\|_{L^2([0,T];H^{-1})} &\leq C(\|f_{in}\|_{L^1}) \sup_{\|\Phi\|_{L^2([0,T];H^1)} \leq 1} \|f\|_{L^2([0,T];H^1)} \|\nabla\Phi\|_{L^2([0,T];L^2)} \\
&\leq C(\|f_{in}\|_{L^1_3}, H_1(f_{in}), T).
\end{aligned}$$

This completes the proof. ∎

In the next lemma we state a weighted L^2 estimate, proved via a slight modification to the L^2-estimate in Lemma 3.12.

LEMMA 3.13. *Let f be any weak solution to (1.6) as in Theorem 1.1 with initial data f_{in} as described above. Then, for every $m \geq 3$,*

$$\sup_{(0,T)} \|f(\cdot)\|_{L^2_m}^2 + \int_0^T \|\nabla f(t)\|_{L^2_{m-3}}^2 \, dt \leq C(f_{in}, m, T).$$

Proof. We test (1.6) with $\varphi = \langle v \rangle^m f$, and estimate the resulting terms as in the proof of Lemma 3.12. ∎

Proof. **(Proof of Theorem 1.1.)** Fix $T>0$, f_{in} with $0 \leq f_{in} \leq 1$, and $f_{in} \in L^1_3$ and fix some sequence $\delta_n \to 0^+$ and let $f_n(v,t)$ be the solutions with $\delta = \delta_n$ constructed in Proposition 3.3. Then, the uniform-in-δ estimates from Lemma 3.10 and Lemma 3.12 together with Aubin-Lions Lemma imply that we may assume that $f_n \to f$ for some limit f in the following topologies:

- Weak star in $L^\infty([0,T] \times \mathbb{R}^3)$,
- Weakly in $L^2([0,T]; H^1)$,
- Weak star in $L^\infty([0,T]; L^2)$,
- Strongly in $L^p([0,T]; L^q)$ for any $1 \leq p \leq 2$ and $1 \leq q < 6$.

Furthermore, we may also assume $f_n \to f$ pointwise almost everywhere on $[0,T] \times \mathbb{R}^3$. Therefore, by Fatou's lemma, it follows that for almost every $t \in [0,T]$,

$$\|f(t)\|_{L^1_3} \leq C(\|f_{in}\|_{L^1_3}, T). \tag{3.30}$$

Note also that the weak star convergence in L^∞ is sufficient to guarantee $0 \leq f \leq 1$.

Next, since each f_n solves (3.1) on the time interval $[0,T]$, for any $\Phi \in C_c^\infty([0,T) \times \mathbb{R}^3)$, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} f_n \partial_t \Phi \, dv dt - \int_{\mathbb{R}^3} f_{in} \Phi(0) \, dv \\ &= -\int_0^T \int_{\mathbb{R}^3} (A[f_n(1-f_n)] \nabla f_n - \nabla a[f_n] f_n (1-f_n) + \delta_n \nabla f_n) \cdot \nabla \Phi \, dv dt. \end{aligned}$$

We conclude the proof of existence by following the same steps as in the proof of Proposition 3.3.

To show uniqueness of solution, we assume by contradiction that there exist two solutions f and g. Their difference

$$w := f - g,$$

is identically zero at $t=0$ and solves the following weak formulation:

$$\begin{aligned} &\int_0^T \langle \varphi, \partial_t w \rangle_{H^1, H^{-1}} \, dt \\ &= -\int_0^T \int_{\mathbb{R}^3} A[f(1-f)] \nabla w \cdot \nabla \varphi \, dv dt + \int_0^T \int_{\mathbb{R}^3} A[fw + (g-1)w] \nabla g \cdot \nabla \varphi \, dv dt \\ &\quad + \int_0^T \int_{\mathbb{R}^3} f(1-f) \nabla a[w] \cdot \nabla \varphi \, dv dt - \int_0^T \int_{\mathbb{R}^3} (fw + (g-1)w) \nabla a[g] \cdot \nabla \varphi \, dv dt. \end{aligned}$$

We consider $\varphi = w\langle v\rangle^{2m}$ for some $m>\frac{3}{2}$, and get

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3} w^2(t)\langle v\rangle^{2m}\,dv =& -\int_0^T\int_{\mathbb{R}^3} A[f(1-f)]\nabla w\cdot\nabla(w\langle v\rangle^{2m})\,dvdt\\ &+\int_0^T\int_{\mathbb{R}^3} A[fw+(g-1)w]\nabla g\cdot\nabla(w\langle v\rangle^{2m})\,dvdt\\ &+\int_0^T\int_{\mathbb{R}^3} f(1-f)\nabla a[w]\cdot\nabla(w\langle v\rangle^{2m})\,dvdt\\ &-\int_0^T\int_{\mathbb{R}^3} (fw+(g-1)w)\nabla a[g]\cdot\nabla(w\langle v\rangle^{2m})\,dvdt\\ =&: I_1+I_2+I_3+I_4.\end{aligned}$$

The term I_1 is estimated with Young's inequality:

$$\begin{aligned}I_1 =& -\int_0^T\int_{\mathbb{R}^3}\frac{A[f(1-f)]}{\langle v\rangle^{2m}}|\nabla w\langle v\rangle^{2m}|^2\,dvdt\\ &+\int_0^T\int_{\mathbb{R}^3}\frac{A[f(1-f)]w}{\langle v\rangle^{2m}}\nabla w\langle v\rangle^{2m}\cdot\nabla\langle v\rangle^{2m}\,dvdt\\ \le& -(1-\delta)\int_0^T\int_{\mathbb{R}^3}\frac{A[f(1-f)]}{\langle v\rangle^{2m}}|\nabla w\langle v\rangle^{2m}|^2\,dvdt\\ &+C(m,\delta,f_{in})\int_0^T\int_{\mathbb{R}^3}|w|^2\langle v\rangle^{2m}\,dvdt.\end{aligned}$$

Similarly,

$$\begin{aligned}I_2 \le& C(f_{in})\int_0^T\int_{\mathbb{R}^3} A[w]\nabla g\cdot\nabla w\langle v\rangle^{2m}\,dvdt\\ \le& \delta\int_0^T\int_{\mathbb{R}^3}\frac{|\nabla w\langle v\rangle^{2m}|^2}{\langle v\rangle^{2m}(1+|v|)^3}\,dvdt+\frac{1}{\delta}\int_0^T\int_{\mathbb{R}^3}\langle v\rangle^{2m}(1+|v|)^3|A[w]|^2|\nabla g|^2\,dvdt\\ \le& \delta\int_0^T\int_{\mathbb{R}^3}\frac{|A[f(1-f)]|}{\langle v\rangle^{2m}}|\nabla w\langle v\rangle^{2m}|^2\,dvdt+\frac{1}{\delta}\int_0^T\|w\|^2_{L^2_{2m}}\int_{\mathbb{R}^3}(1+|v|)^{3+2m}|\nabla g|^2\,dvdt,\end{aligned}$$

using the bound from below for $A[\cdot]$ and the bound from above in Lemma 2.2:

$$\|A[h]\|_{L^\infty}\le C\|h\|_{L^1}^{2/3}\|h\|_{L^2}^{1/3}\le C\|h\|_{L^2_{2m}},\quad \text{for all}\, m>\frac{3}{2}.$$

Hölder and Hardy-Littlewood-Sobolev inequalities applied to I_3 lead to

$$\begin{aligned}I_3 \le& \int_0^T\|\nabla a[w]\|_{L^6}\left(\int_{\mathbb{R}^3} f^{6/5}|\nabla w\langle v\rangle^{2m}|^{6/5}\,dv\right)^{\frac{5}{6}}dt\\ \le& \frac{1}{\delta}\int_0^T\|w\|^2_{L^2}\,dt+\delta\int_0^T\left(\int_{\mathbb{R}^3} f(1+|v|)^{9/2+3m}\,dv\right)^{2/3}\int_{\mathbb{R}^3}\frac{|\nabla w\langle v\rangle^{2m}|^2}{\langle v\rangle^{2m}(1+|v|)^3}\,dvdt\\ \le& \frac{1}{\delta}\int_0^T\|w\|^2_{L^2_{2m}}\,dt+\tilde\delta\int_0^T\int_{\mathbb{R}^3}\frac{A[f(1-f)]}{\langle v\rangle^{2m}}|\nabla w\langle v\rangle^{2m}|^2\,dvdt,\end{aligned}$$

using Lemma 3.13 to bound the $9/2+3m$ moments of f (uniformly in time). Similarly,

$$I_4\le\frac{1}{\delta}\|\nabla a[g](1+|v|)^{3/2}\|^2_{L^\infty_{v,t}}\int_0^T\int_{\mathbb{R}^3} w^2\langle v\rangle^{2m}\,dvdt+\delta\int_0^T\int_{\mathbb{R}^3}\frac{|\nabla w\langle v\rangle^{2m}|^2}{\langle v\rangle^{2m}(1+|v|)^3}\,dvdt.$$

We briefly show how the term $\|\nabla a[g](1+|v|)^{3/2}\|^2_{L^\infty_{v,t}}$ is uniformly bounded. Let $|v|$ be large enough. For $s>3$, Hölder inequality yields:

$$
\begin{aligned}
|\nabla a[f]| &\le \int_{\mathbb{R}^3} \frac{f(y)}{|v-y|^2}\,dy \le |v|^{\frac{3-2s'}{s'}} \left(\int_{B_{\frac{|v|}{2}}(|v|)} f^s\,dy\right)^{1/s} + \frac{1}{|v|^2}\|f\|_{L^1(\mathbb{R}^3)} \\
&\le c\frac{|v|^{\frac{3-2s'}{s'}}}{(1+|v|)^{\lambda/s}} \left(\int_{\mathbb{R}^3} f^s(1+|y|)^\lambda\,dy\right)^{1/s} + \frac{1}{|v|^2}\|f\|_{L^1(\mathbb{R}^3)},
\end{aligned}
$$

with $\frac{1}{s}+\frac{1}{s'}=1$ and $s'<3/2$. The choice of λ so that $\frac{3-s'}{s'}-\frac{\lambda}{s}=-2$ leads to the desired estimate.

Combining the estimates for $I_1,\dots,I_4$ and choosing δ small enough, one gets

$$
\frac{1}{2}\int_{\mathbb{R}^3} w^2(T)\langle v\rangle^{2m}\,dv \le C\int_0^T \|w\|^2_{L^2_{2m}}\left(1+\int_{\mathbb{R}^3}(1+|v|)^{3+2m}|\nabla g|^2\,dv\right)dt.
$$

Since $\int_0^T\int_{\mathbb{R}^3}(1+|v|)^{3+2m}|\nabla g|^2\,dvdt \le C$ thanks to Lemma 3.13, Gronwall's inequality implies that $w(t)=0$ for all $t\ge 0$. This concludes the proof of the theorem. ∎

4. Regularity of weak solutions

In this section, we prove Theorem 1.2. Throughout this section we consider initial data f_{in} such that $0\le f_{in}\le 1$, $\|f_{in}\|_{L^1\cap L^2_m}<\infty$ for a general $m\ge 3$, and $H_1(f_{in})<0$. The exact value of m needed for Theorem 1.2 is determined in Lemma 4.3.

As a first step, we use Lemma 3.13 to show that weak solutions of (1.1) instantaneously regularize and belong to weighted $L^\infty(t,T,H^1)$ and weighted $L^2(t,T,H^2)$ for any $t>0$.

LEMMA 4.1. *Let f be any weak solution to (1.6) as in Theorem 1.1. For any $t>0$ and $m\ge 3$, we have*

$$
\sup_{(t,T)}\|\nabla f(\cdot)\|^2_{L^2_m} + \int_t^T \|\nabla^2 f(s)\|^2_{L^2_{m-3}}\,ds \le C(f_{in})\left(1+\frac{1}{t}\right).
$$

Proof. Fix $i=1,2,3$ arbitrary. We first recall the notation for divided differences,

$$
\tilde{\partial}_h g := \frac{g(v+e_i h)-g(v)}{h},
$$

for which the following discrete product formula holds,

$$
\tilde{\partial}_h(fg) = g\tilde{\partial}_h f + f\tilde{\partial}_h g + h\tilde{\partial}_h f\tilde{\partial}_h g.
$$

We test (1.6) with $\psi(v,t) = -\chi_{[t_1,t_2]}(t)\tilde{\partial}_{-h}\left(\langle v\rangle^m\tilde{\partial}_h f\right)$ and obtain

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \partial_t f\tilde{\partial}_{-h}\left(\langle v\rangle^m\tilde{\partial}_h f\right)dvdt \\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla\tilde{\partial}_{-h}\left(\langle v\rangle^m\tilde{\partial}_h f\right)\cdot\left(A[f(1-f)]\nabla f - \nabla a[f]f(1-f)\right)dvdt.
\end{aligned}
$$

On the left-hand side, we perform a discrete integration by parts:

$$
\begin{aligned}
-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \partial_t f\tilde{\partial}_{-h}\left(\langle v\rangle^m\tilde{\partial}_h f\right)dvdt &= \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^m\partial_t\left[\tilde{\partial}_h f\right]^2 dvdt \\
&= \frac{1}{2}\|\langle v\rangle^m\tilde{\partial}_h f(t_2)\|^2_{L^2} - \frac{1}{2}\|\langle v\rangle^m\tilde{\partial}_h f(t_1)\|^2_{L^2}.
\end{aligned}
$$

We decompose the right-hand side as

$$
\begin{aligned}
RHS = &-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla\tilde\partial_h f\cdot A[f(1-f)]\nabla\tilde\partial_h f - \langle v\rangle^m\nabla\tilde\partial_h f\cdot\Big(\tilde\partial_h A[f(1-f)]\nabla f\Big)\,dvdt\\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla\tilde\partial_h f\cdot\Big(h\tilde\partial_h A[f(1-f)]\nabla\tilde\partial_h f\Big) + \langle v\rangle^m\nabla\tilde\partial_h f\cdot\tilde\partial_h\nabla a[f]f(1-f)\,dvdt\\
&+\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla\tilde\partial_h f\cdot\Big(\nabla a[f]\tilde\partial_h(f(1-f))\Big) + \langle v\rangle^m\nabla\tilde\partial_h f\cdot h\tilde\partial_h\nabla a[f]\tilde\partial_h[f(1-f)]\,dvdt\\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla\langle v\rangle^m \tilde\partial_h f\cdot\Big(\tilde\partial_h A[f(1-f)]\nabla f\Big) - \nabla\langle v\rangle^m\tilde\partial_h f\cdot A[f(1-f)]\nabla\tilde\partial_h f\,dvdt\\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla\langle v\rangle^m \tilde\partial_h f\cdot\Big(h\tilde\partial_h A[f(1-f)]\nabla\tilde\partial_h f\Big) + \nabla\langle v\rangle^m\tilde\partial_h f\cdot\tilde\partial_h\nabla a[f]f(1-f)\,dvdt\\
&+\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla\langle v\rangle^m \tilde\partial_h f\cdot\Big(\nabla a[f]\tilde\partial_h(f(1-f))\Big) + \nabla\langle v\rangle^m\tilde\partial_h f\cdot h\tilde\partial_h\nabla a[f]\tilde\partial_h(f(1-f))\,dvdt\\
=:&\sum_{j=1}^{12} I_j.
\end{aligned}
$$

For I_1 we use Lemma 2.1 to obtain

$$
I_1 \le -C(f_{in})\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde\partial_h f|^2\,dvdt.
$$

Next, for any $\delta>0$, we upper bound I_2 using $\|\nabla A[f(1-f)\|_{L^\infty}\le C(f_{in})$ and Young's inequality,

$$
\begin{aligned}
|I_2| &\lesssim \delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde\partial_h f|^2\,dvdt + \delta^{-1}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m+3}|\nabla f|^2\,dvdt\\
&\lesssim \delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde\partial_h f|^2\,dvdt + \delta^{-1}\|\nabla f\|^2_{L^2([t_1,t_2];L^2_{m+3})}.
\end{aligned}
$$

In the same way, we bound I_3. We bound I_4 using $\|\nabla^2 a[f]\|_{L^p}\lesssim\|f\|_{L^p}$ for $1<p<\infty$, by the Calderón-Zygmund Lemma (Chapter 9.4 in [24]), and Young's inequality:

$$
\begin{aligned}
|I_4| \lesssim& \delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde\partial_h f|^2\,dvdt + \delta^{-1}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m+3}|\tilde\partial_h\nabla a[f]|^2 f^2\,dvdt\\
\lesssim& \delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde\partial_h f|^2\,dvdt\\
&+\delta^{-1}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\Big(|\tilde\partial_h\nabla a[f]|^{\frac{2(m+6)}{3}} + f^{\frac{2(m+6)}{m+3}}\langle v\rangle^{m+6}\Big)\,dvdt\\
\lesssim& \delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde\partial_h f|^2\,dvdt + \delta^{-1}\|f\|^2_{L^2([t_1,t_2];L^2_{m+6})}.
\end{aligned}
$$

For $\delta>0$, we bound I_5, using Young's inequality and Lemma 2.3, as

$$
\begin{aligned}
|I_5| &\lesssim \delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde\partial_h f|^2\,dvdt + \delta^{-1}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m+3}|\tilde\partial_h f(1-f)|^2\,dvdt\\
&\lesssim \delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde\partial_h f|^2\,dvdt + \delta^{-1}\|\tilde\partial_h f\|^2_{L^2([t_1,t_2];L^2_{m+3})}.
\end{aligned}
$$

Again, we bound I_6 in a similar manner to I_4 and I_5. For I_7, we use $|\nabla\langle v\rangle^m)|\lesssim\langle v\rangle^{m-1}$, Lemma 2.2, and $\|\tilde{\partial}_h f\|_{L^2_m}\lesssim\|\nabla f\|_{L^2_m}$ (via a simple modification to Proposition IX.9(iii) in [10]), to obtain

$$|I_7|\lesssim\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-1}|\nabla f|^2\,dvdt.$$

Next, for I_8, we integrate by parts and use Lemma 2.2 and $|\nabla^2\varphi_R(|v|)\langle v\rangle^m|\lesssim\langle v\rangle^{m-2}$ to obtain

$$\begin{aligned}I_8=&-\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\langle v\rangle^m\cdot A[f(1-f)]\nabla(\tilde{\partial}_h f)^2\,dvdt\\
=&\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\langle v\rangle^m\cdot(\nabla\cdot A)[f(1-f)](\tilde{\partial}_h f)^2\,dvdt\\
&+\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla^2\langle v\rangle^m : A[f(1-f)](\tilde{\partial}_h f)^2\,dvdt\\
\lesssim&\|\nabla f\|^2_{L^2([t_1,t_2];L^2_{m-1})}+\|\nabla f\|^2_{L^2([t_1,t_2];L^2_{m-2})}.\end{aligned}$$

We bound I_9 in a similar manner to I_7 and I_8. For I_{10}, we use Young's inequality and the Calderon-Zygmund inequality, to obtain

$$\begin{aligned}|I_{10}|\lesssim&\|\tilde{\partial}_h f\|^2_{L^2([t_1,t_2];L^2_{m-1})}+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-1}|\tilde{\partial}_h\nabla a[f]|^2 f^2\,dvdt\\
\lesssim&\|\tilde{\partial}_h f\|^2_{L^2([t_1,t_2];L^2_{m-1})}+\|f\|^2_{L^2([t_1,t_2];L^2_m)}.\end{aligned}$$

For I_{11}, we use $\nabla a[f]\in L^\infty([0,T]\times\mathbb{R}^3)$ from Lemma 2.3 to obtain

$$|I_{11}|\lesssim\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-1}|\tilde{\partial}_h f|^2\,dvdt\lesssim\|\tilde{\partial}_h f\|^2_{L^2([t_1,t_2];L^2_{m-1})}.$$

Finally, I_{12} is bounded similarly to I_{10} and I_{11}. Thus, we have shown (using once more that $\|\tilde{\partial}_h f\|_{L^2_m}\lesssim\|\nabla f\|_{L^2_m})$),

$$\begin{aligned}&\int_{\mathbb{R}^3}\varphi_R(|v|)\langle v\rangle^m\left[\tilde{\partial}_h f(t_2)^2-\tilde{\partial}_h f(t_1)^2\right]dv+(C-\delta)\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\varphi_R(|v|)\langle v\rangle^{m-3}|\tilde{\partial}_h\nabla f|^2\,dvdt\\
&\lesssim(1+\delta^{-1})\left(\|\nabla f\|^2_{L^2([t_1,t_2];L^2_{m-2}\cap L^2_{m+3})}+\|f\|^2_{L^2([t_1,t_2];L^2_m\cap L^2_{m+6})}\right),\end{aligned}$$

where the implicit constants depend only on f_{in}, T, and m. Now, taking $\delta<C/2$ and taking $h\to 0^+$, $R\to\infty$, we see ∇f is weakly differentiable and

$$\begin{aligned}&\int_{\mathbb{R}^3}\langle v\rangle^m|\nabla f(t_2)|^2\,dv+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla^2 f|^2\,dvdt\\
&\lesssim\int_{\mathbb{R}^3}\langle v\rangle^m|\nabla f(t_1)|^2\,dv+\left(\|\nabla f\|^2_{L^2([t_1,t_2];L^2_{m-2}\cap L^2_{m+3})}+\|f\|^2_{L^2([t_1,t_2];L^2_m\cap L^2_{m+6})}\right).\end{aligned}$$

Next, taking a supremum over t_2 in $[t,T]$ and an average over $t_1\in[0,t]$, and applying

Lemma 3.13, we get

$$\begin{aligned}
&\sup_{(t,T)} \|\nabla f(\cdot)\|^2_{L^2_m} + \int_t^T \|\nabla^2 f(s)\|^2_{L^2_{m-3}}\, ds \\
&\lesssim \frac{1}{t} C(f_{in},T,m) + \|\nabla f\|^2_{L^2([0,T];L^2\cap L^2_{m+3})} + \|f\|^2_{L^2([0,T];L^2\cap L^2_{m+6})} \\
&\le C(f_{in},T,m)\left(1+\frac{1}{t}\right).
\end{aligned}$$

This concludes the proof of the lemma. ◻

Next, we show how to control the $L^\infty(t,T,H^2)\cap L^2(t,T,H^3)$-regularity of f:

Lemma 4.2. *Let f be any weak solution to (1.6) as in Theorem 1.1 with initial data as described at the beginning of this section. For any $t>0$ and $m\ge 3$, we have*

$$\sup_{(t,T)} \|\nabla^2 f(\cdot)\|^2_{L^2_m} + \int_t^T \|\nabla^3 f(s)\|^2_{L^2_{m-3}}\, ds \le C(f_{in})\left(1+\frac{1}{t^2}\right).$$

Proof. Thanks to Lemma 4.1, we can take

$$\psi(v,t) ;= \chi_{[t_1,t_2]} \tilde\partial_{-h} \partial_{v_i} \left(\langle v\rangle^m \tilde\partial_h f_{v_i}\right),$$

as test function for (1.6), and obtain

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \partial_t f \partial_{v_i} \tilde\partial_{-h}\left(\langle v\rangle^m \tilde\partial_h f_{v_i}\right) dv dt \\
&= -\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla \partial_{v_i} \tilde\partial_{-h}\left(\langle v\rangle^m \tilde\partial_h f_{v_i}\right) \cdot \left(A[f(1-f)]\nabla f - \nabla a[f] f(1-f)\right) dv dt.
\end{aligned}$$

On the left-hand side, we perform one discrete integration by parts and one standard integration by parts and get

$$LHS = \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \partial_t \left[\tilde\partial_h f_{v_i}\right]^2 dv dt = \frac{1}{2}\|\langle v\rangle^m \tilde\partial_h f_{v_i}(t_2)\|^2_{L^2} - \frac{1}{2}\|\langle v\rangle^m \tilde\partial_h f_{v_i}(t_1)\|^2_{L^2}.$$

We also perform discrete and standard integration by parts to decompose the right-hand side as

$$\begin{aligned}
RHS = &-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla \tilde\partial_h f_{v_i} \cdot A[f(1-f)] \nabla \tilde\partial_h f_{v_i}\, dv dt \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla \tilde\partial_h f_{v_i} \cdot \left(\tilde\partial_h A[f(1-f)]\nabla f_{v_i} + \partial_{v_i} A[f(1-f)]\tilde\partial_h \nabla f\right) dv dt \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla \tilde\partial_h f_{v_i} \cdot \left(\partial_{v_i}\tilde\partial_h A[f(1-f)]\nabla f\right) + \langle v\rangle^m \nabla \tilde\partial_h f_{v_i} \cdot \left(\tilde\partial_h \partial_{v_i} \nabla a[f] f(1-f)\right) dv dt \\
&+\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla \tilde\partial_h f_{v_i} \cdot \left(\partial_{v_i}\nabla a[f]\tilde\partial_h (f(1-f)) + \tilde\partial_h \nabla a[f] \partial_{v_i}(f(1-f))\right) dv dt \\
&+\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla \tilde\partial_h f_{v_i} \cdot \left(\nabla a[f]\partial_{v_i}\tilde\partial_h [f(1-f)]\right) - \nabla\langle v\rangle^m \tilde\partial_h f_{v_i} \cdot \left(A[f(1-f)]\nabla \tilde\partial_h f_{v_i}\right) dv dt \\
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla\langle v\rangle^m \tilde\partial_h f_{v_i} \cdot \left(\tilde\partial_h A[f(1-f)]\nabla f_{v_i} + \partial_{v_i} A[f(1-f)]\tilde\partial_h \nabla f\right) dv dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\langle v\rangle^m\tilde{\partial}_h f_{v_i}\cdot\Big(\partial_{v_i}\tilde{\partial}_h A[f(1-f)]\nabla f\Big)+\nabla\langle v\rangle^m\tilde{\partial}_h f_{v_i}\cdot\Big(\tilde{\partial}_h\partial_{v_i}\nabla a[f]f(1-f)\Big)\,dvdt\\
&+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\langle v\rangle^m\tilde{\partial}_h f_{v_i}\cdot\Big(\partial_{v_i}\nabla a[f]\tilde{\partial}_h(f(1-f))+\tilde{\partial}_h\nabla a[f]\partial_{v_i}(f(1-f))\Big)\,dvdt\\
&+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla\langle v\rangle^m\tilde{\partial}_h f_{v_i}\cdot\Big(\nabla a[f]\partial_{v_i}\tilde{\partial}_h[f(1-f)]\Big)\,dvdt+\mathcal{E}\\
&:=\sum_{j=1}^{12}I_j+\mathcal{E},
\end{aligned}
$$

where $\mathcal{E}$ denotes the error terms, which originate from the discrepancy between the product rules for $\tilde{\partial}_h$ and ∂_{v_i}. These terms are bounded identically to the others and so we omit the bound on $\mathcal{E}$. For I_1, our coercive term, we use Lemma 2.1 to obtain

$$
I_1\le -C(f_{in})\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2.
$$

For I_3, when two derivatives land on the kernel $A[f]$, we use Young's inequality, Hölder's inequality in space, the Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, and the Calderón-Zygmund Lemma, to obtain for any $\delta>0$, the estimate

$$
\begin{aligned}
|I_3|&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2+\delta^{-1}\langle v\rangle^{m+3}|\partial_{v_i}\tilde{\partial}_h A[f(1-f)]|^2|\nabla f|^2\,dvdt\\
&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2\,dvdt+\delta^{-1}\int_{t_1}^{t_2}\|f\|_{L^3}\|\langle v\rangle^{m+3}|\nabla f|^2\|_{L^3}\,dt\\
&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2\,dvdt+\delta^{-1}\int_{t_1}^{t_2}\|\langle v\rangle^{\frac{m+3}{2}}\nabla f\|_{L^6}^2\,dt\\
&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2\,dvdt\\
&\quad+\delta^{-1}\int_{t_1}^{t_2}\Big(\|\langle v\rangle^{\frac{m+1}{2}}\nabla f)\|_{L^2}^2+\|\langle v\rangle^{\frac{m+3}{2}}\nabla^2 f\|_{L^2}^2\Big)\,dt\\
&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2\,dvdt\\
&\quad+\delta^{-1}\Big(\|\nabla f\|^2_{L^2\left([t_1,t_2];L^2_{m+1}\right)}+\|\nabla^2 f\|^2_{L^2\left([t_1,t_2];L^2_{m+3}\right)}\Big).
\end{aligned}
$$

Similarly, for I_4, when two derivatives land on the kernel $\nabla a[f]$, we use Young's inequality, Hölder's inequality, the Calderón-Zygmund Lemma, Lebesgue interpolation, the Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, and Lemma 3.13 to obtain for any $\delta>0$, the estimate

$$
\begin{aligned}
|I_4|&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2\,dvdt+\frac{1}{\delta}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m+3}|\tilde{\partial}_h\partial_{v_i}\nabla a[f]|^2 f^2\,dvdt\\
&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2\,dvdt+\frac{1}{\delta}\int_{t_1}^{t_2}\||\partial_{v_i}\tilde{\partial}_h a[\nabla f]|^2\|_{L^3}\|f^2\langle v\rangle^{m+3}\|_{L^{3/2}}\,dt\\
&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2\,dvdt+\frac{1}{\delta}\int_{t_1}^{t_2}\|\partial_{v_i}\tilde{\partial}_h a[\nabla f]\|_{L^6}^2\|f\langle v\rangle^{\frac{m+3}{2}}\|_{L^3}^2\,dt\\
&\lesssim\delta\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{m-3}|\nabla\tilde{\partial}_h f_{v_i}|^2\,dvdt+\frac{1}{\delta}\int_{t_1}^{t_2}\|\nabla f\|_{L^6}^2\|f\langle v\rangle^{\frac{m+3}{2}}\|_{L^2}\|f\langle v\rangle^{\frac{m+3}{2}}\|_{L^6}\,dt
\end{aligned}
$$

$$\lesssim \delta \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \langle v \rangle^{m-3} |\nabla \tilde{\partial}_h f_{v_i}|^2 \, dv dt + \frac{1}{\delta} \int_{t_1}^{t_2} \|\nabla^2 f\|_{L^2}^2 \|f\|_{L^2_{m+3}} \left\| \nabla \left(f \langle v \rangle^{\frac{m+3}{2}} \right) \right\|_{L^2} dt$$
$$\lesssim \delta \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \langle v \rangle^{m-3} |\nabla \tilde{\partial}_h f_{v_i}|^2 \, dv dt + \frac{1}{\delta} \int_{t_1}^{t_2} \|\nabla^2 f\|_{L^2}^2 \|f\|_{L^2_{m+3}} (\|\nabla f\|_{L^2_{m+3}} + \|f\|_{L^2_{m+1}}) \, dt$$
$$\lesssim \delta \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \langle v \rangle^{m-3} |\nabla \tilde{\partial}_h f_{v_i}|^2 \, dv dt + \frac{1}{\delta} \|\nabla^2 f\|_{L^2([t_1,t_2];L^2)}^2 (\|\nabla f\|_{L^\infty([t_1,t_2];L^2_{m+3})} + 1).$$

To bound the remaining terms I_2 and $I_5, \cdots, I_{12}$, we modify the arguments from Lemma 4.1 in a similar fashion, using the additional tool of the Sobolev embedding as necessary, to obtain

$$\int_{\mathbb{R}^3} \langle v \rangle^m |\tilde{\partial}_h f_{v_i}(t_2)|^2 \, dv + (C - \delta) \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \langle v \rangle^{m-3} |\nabla \tilde{\partial}_h f_{v_i}|^2 \, dv dt$$
$$\lesssim 1 + \int_{\mathbb{R}^3} \langle v \rangle^m |\tilde{\partial}_h f_{v_i}|^2 \, dv + (1 + \delta^{-1}) \left(\|\nabla f\|_{L^\infty([t_1,t_2];L^2_{m+6})}^4 + \|\nabla^2 f\|_{L^2([t_1,t_2];L^2_{m+3})}^4 \right).$$

Thus, taking δ sufficiently small and taking the limit $h \to 0^+$, we conclude that $\nabla^2 f$ is weakly differentiable and we obtain

$$\|\nabla^2 f(t_2)\|_{L^2_m}^2 + \int_{t_1}^{t_2} \|\nabla^3 f\|_{L^2_{m-3}}^2 \, ds$$
$$\lesssim 1 + \|\nabla^2 f(t_1)\|_{L^2_m}^2 + \|\nabla f\|_{L^\infty([t_1,t_2];L^2_{m+6})}^4 + \|\nabla^2 f\|_{L^2([t_1,t_2];L^2_{m+3})}^4.$$

Taking a supremum over $t_2 \in [2t, T]$ and an average over $t_1 \in [t, 2t]$, and applying Lemma 4.1, we obtain

$$\sup_{(2t,T)} \|\nabla^2 f(\cdot)\|_{L^2_m}^2 + \int_{2t}^T \|\nabla^3 f\|_{L^2_{m-3}}^2 \, ds$$
$$\lesssim \frac{1}{t} \int_t^{T_2} \|\nabla^2 f\|_{L^2_m}^2 \, ds + \left(\int_t^T \|\nabla^2 f\|_{L^2_{m+3}}^2 \, ds \right)^2 + \left(\sup_{(t,T)} \|\nabla f\|_{L^2_{m+6}}^2 \right)^2$$
$$\lesssim \left(1 + \frac{1}{t^2} \right).$$

□

Remark 4.1. From Lemma 4.2, one can continue to bootstrap spatial regularity, and obtain the corresponding higher regularity estimates, that provided $f_{in} \in L^2_{m+6k}$, for each $0 < t_0 < T$, $\nabla^k f \in L^\infty([t_0, T]; L^2_m)$, and moreover,

$$\sup_{t_0 < t < T} \|\nabla^k f(\cdot)\|_{L^2_m} + \left(\int_{t_0}^T \|\nabla^{k+1} f(s)\|_{L^2_{m-3}}^2 \, ds \right)^{1/2} \lesssim_{f_{in},k,m,T} \left(1 + \frac{1}{t_0} \right)^{k/2}.$$

If f_{in} is rapidly decaying, i.e. $f_{in} \in L^2_m$ for each $m \geq 0$, then f is Schwartz class in space. That is, $f \in L^\infty([t, T]; \mathcal{S}(\mathbb{R}^3))$ for each $t > 0$.

Instead of bootstrapping spatial regularity and deducing the corresponding time regularity from the equation, we use Lemma 4.2 to conclude Hölder regularity of f. Combined with the parabolic divergence structure of (1.1), we deduce spatial and temporal regularity simultaneously via classical Schauder estimates. As the initial step, we have the following lemma:

LEMMA 4.3. *Let f be any weak solution to (1.6) as in Theorem 1.1, with $f_{in} \in L^1_3 \cap L^2(12)$. Then, $f \in C^{\alpha/2}((0,T];C^\alpha(\mathbb{R}^3))$ for some $\alpha \in (0,1)$.*

Proof. By Lemma 4.2, we conclude $f \in L^\infty((0,T];W^{1,p})$ for each $2 \le p \le 6$. Therefore, by a duality argument, $\partial_t f$ belongs to $L^\infty((0,T];W^{-1,p})$ for $2 \le p \le 6$. By a (real) interpolation of the Sobolev spaces $L^\infty((0,T);W^{1,p})$ and $W^{1,\infty}((0,T);W^{-1,p})$, we obtain $f \in W^{s_1,6}((0,T);W^{s_2,6})$ for s_2 strictly less, but as close as one wishes, than $1-2\theta$, and s_1 strictly less, but as close as one wishes, than θ, for any $\theta \in (0,1)$ (see Theorem 3.1 in [5]). Hence, choosing $\theta < \frac{1}{4}$, Morrey's inequality implies $f \in C^{0,\alpha/2}((0,T);C^{0,\alpha}(\mathbb{R}^3))$, for some $\alpha > 0$. ∎

Now, we are ready to apply a standard bootstrapping argument and conclude f is smooth:

Proof. **(Proof of Theorem 1.2.)** By Lemma 4.3, we conclude $f \in C^{\alpha/2}((0,T);C^\alpha(\mathbb{R}^3))$ for some $\alpha \in (0,1)$ and f solves the divergence form parabolic equation,

$$\partial_t f = \nabla \cdot (A[f(1-f)]\nabla f - \nabla a[f](1-f)f), \tag{4.1}$$

in the weak sense. Hence, Lemma 4.7 in [27] shows that $\nabla a[f]$ and $A[f(1-f)]$ belong to $C^{0,\eta/2}((0,T];C^{0,\eta})$. Thus, f satisfies a divergence-form parabolic equation with Hölder continuous coefficients. By Theorem 12.1 from Chapter 3 in [31], we conclude $f \in C^{1,\mu/2}((0,T];C^{1,\mu})$. Bootstrapping the argument, we obtain higher regularity of the coefficients $A[f(1-f)]$ and $\nabla a[f]$, from which $f \in C^\infty((0,T];C^\infty)$ follows, as desired. ∎

5. Long time behavior

In this section we prove Theorem. 1.3. Without loss of generality, we can assume that $\varepsilon = 1$. We first rewrite the initial value problem associated to (1.1) in the following compact form

$$\begin{cases} \partial_t f + \mathcal{T}[f] = 0 & v \in \mathbb{R}^3, \quad t > 0, \\ f(0,v) = f_{in} & v \in \mathbb{R}^3, \end{cases} \tag{5.1}$$

where the Landau-Fermi-Dirac operator is defined by

$$\begin{aligned} \mathcal{T}[f](v) &= -\nabla \cdot \int_{\mathbb{R}^3} \frac{\Pi(v-v^*)}{|v-v^*|} (f^*(1-f^*)\nabla f - f(1-f)\nabla f^*)\, dv^* \\ &= -\nabla \cdot (A[f(1-f)]\nabla f - f(1-f)\nabla a[f]), \end{aligned} \tag{5.2}$$

and the quantities $A[\cdot]$, $a[\cdot]$ are defined in (1.5).

We first show unconditional convergence without rate towards the steady state for (1.1), which is the first part of Theorem 1.3.

PROPOSITION 5.1 (Convergence to the steady state). *Given any initial datum $f_{in}: \mathbb{R}^3 \to [0,1]$, $f_{in} \in L^1_2$, such that $H_1[f_{in}] < 0$, the solution f to (5.1) tends to the Fermi-Dirac distribution $\mathcal{M}$ with same mass, momentum and energy as f_{in} when $t \to \infty$.*

Proof. We recall that f satisfies a bound in $L^1 \cap L^\infty$ uniformly in time, and therefore $\sup_{t \ge 0} \|f(t)\|_{L^2} < \infty$. In what follows we will often make use of this relation without mentioning it.

Integrating the entropy balance equation in time yields

$$H_1[f(t)] + \int_0^t D[f(\tau)]d\tau \le H_1[f_{in}], \qquad t > 0,$$

with

$$D[f] = \int_{\mathbb{R}^3} \frac{A[f(1-f)]}{f(1-f)} \nabla f \cdot \nabla f dv - 8\pi \int_{\mathbb{R}^3} f^2 dv \geq 0.$$

Since $D[f(\cdot)] \in L^1(0,\infty)$, there exists a sequence $t_n \to \infty$ such that $D[f(t_n)] \to 0$ as $n \to \infty$. Define $f_n = f(t_n)$. Given the lower bound for A we deduce

$$\int_{\mathbb{R}^3} |\nabla f_n|^2 \langle v \rangle^{-3} dv \lesssim \int_{\mathbb{R}^3} \frac{A[f_n(1-f_n)]}{f_n(1-f_n)} \nabla f_n \cdot \nabla f_n dv \lesssim \int_{\mathbb{R}^3} f_n^2 dv + D[f_n] \lesssim 1.$$

Therefore $\langle v \rangle^{-3/2} \nabla f_n$ is bounded in L^2. However $f_n \nabla \langle v \rangle^{-3/2}$ is bounded in L^2, so the product $f_n \langle v \rangle^{-3/2}$ is bounded in H^1. Furthermore $f_n \langle v \rangle^2$ is bounded in L^1. We deduce via Sobolev embedding that f_n is relatively compact in L^2, and more in general (via the L^∞ bounds and the bound on the second moment of f_n) in L^p for every $p \in [1,\infty)$. Let us denote with f_∞ its limit. We have that $\langle v \rangle^{-3/2} \nabla f_n \rightharpoonup \langle v \rangle^{3/2} \nabla f_\infty$ weakly in L^2. This is enough to deduce via a generalized Fatou argument [11, Lemma A.4] that

$$D_\delta[f_\infty] \leq \liminf_{n\to\infty} D_\delta[f_n] \leq \liminf_{n\to\infty} D[f_n] = 0,$$

with

$$D_\delta[f] := \int_{B_{1/\delta}} \frac{A[f(1-f)]}{f(1-f)+\delta} \nabla f \cdot \nabla f dv - 8\pi \int_{\mathbb{R}^3} f^2 dv,$$

and $\delta > 0$ is arbitrary. Via monotone convergence we deduce

$$0 \leq D[f_\infty] = \lim_{\delta \to 0} D_\delta[f_\infty] \leq 0.$$

It follows that $D[f_\infty] = 0$. Since we know that $\int_{\mathbb{R}^3} f_n(1-f_n) dv \geq c > 0$, it follows [8] that $f_\infty = \mathcal{M}$. This means that $f_n = f(t_n) \to \mathcal{M}$ strongly in L^p for $p \in [1,\infty)$. In particular the relative Fermi-Dirac entropy $H_1[f(t_n)|\mathcal{M}] = H_1[f(t_n)] - H_1[\mathcal{M}] \to 0$ as $n \to \infty$. On the other hand, we know that $t \mapsto H_1[f(t)|\mathcal{M}]$ is non-increasing, so it must hold $\lim_{t\to\infty} H_1[f(t)|\mathcal{M}] = 0$. This easily implies the strong convergence $f(t) \to \mathcal{M}$ as $t \to \infty$ in L^1. This finishes the proof. ∎

Our next goal is to prove exponential convergence of the solution $f(t)$ to (1.1) towards the steady state $\mathcal{M}$ in case the initial datum f_{in} is close enough to $\mathcal{M}$ in the norm $L^2(m)$. This is in the second part of Theorem 1.3. We linearize our equation around the steady state $\mathcal{M}$. We will work in weighted Lebesgue spaces with weight m defined by

$$m := \mathcal{M}(1-\mathcal{M}), \tag{5.3}$$

where $\mathcal{M}$ is the Fermi-Dirac distribution defined in (1.4). Writing

$$h := \frac{f-\mathcal{M}}{m}, \qquad \text{and} \qquad -\frac{1}{m}\mathcal{T}[f] =: Lh + \Gamma_2[h,h] + \Gamma_3[h,h,h] \tag{5.4}$$

it defines the linearized operator L and the quadratic and cubic perturbations Γ_2, Γ_3, respectively.

Via straightforward computations [4] one finds

$$(Lh)(v) = \frac{1}{m(v)} \nabla \cdot \int_{\mathbb{R}^3} \frac{m(v^*)m(v)}{|v-v^*|} \Pi(v-v^*)(\nabla h(v) - \nabla h(v^*)) dv^*, \tag{5.5}$$

$$\Gamma_2[h,h](v) = \frac{1}{m(v)}\nabla\cdot\big(A[(1-2\mathcal{M})mh]\nabla(mh) - A[m^2h^2]\nabla\mathcal{M} \\ -(1-2\mathcal{M})mh\nabla a[mh] + m^2h^2\nabla a[\mathcal{M}]\big), \tag{5.6}$$

$$\Gamma_3[h,h,h](v) = \frac{1}{m(v)}\nabla\cdot\big(-A[m^2h^2)]\nabla(mh) + m^2h^2\nabla a[mh]\big). \tag{5.7}$$

Define the spaces

$$L^2(m) := L^2(\mathbb{R}^3, m(v)dv), \quad H^1(m) := H^1(\mathbb{R}^3, m(v)dv),$$

and recall that $\langle v\rangle = (1+|v|^2)^{1/2}$.

Our goal is to prove a spectral gap estimate for the linearized operator L. We will apply [17, Lemma 10]. In order to do so, we adapt the latter result's framework and therefore define for $k\geq 0$ the following Hilbert spaces

$$\mathcal{H}_0^k = L^2(m\langle v\rangle^{k-1}),$$
$$\mathcal{H}^k = \left\{h\in\mathcal{H}_0^k \;:\; \|h\|^2_{\mathcal{H}^k} \equiv \|h\|^2_{\mathcal{H}_0^k} + \int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv < \infty\right\}.$$

Clearly $\mathcal{H}^k \hookrightarrow \mathcal{H}_0^k$ with continuous embedding.

We split then the linearized operator L into two contributions, in the following fashion:

$$L = \mathcal{K}_k - \Lambda_k, \tag{5.8}$$

$$(\Lambda_k h)(v) := -\frac{1}{m(v)}\nabla\cdot\left[\left(\int_{\mathbb{R}^3}\frac{m(v^*)m(v)}{|v-v^*|}\Pi(v-v^*)dv^*\right)\nabla h(v)\right] \\ -8\pi m(v)h(v) + \xi\int_{\mathbb{R}^3} mh\langle v\rangle^k dv, \tag{5.9}$$

$$(\mathcal{K}_k h)(v) := -\frac{1}{m(v)}\nabla\cdot\int_{\mathbb{R}^3}\frac{m(v^*)m(v)}{|v-v^*|}\Pi(v-v^*)\nabla h(v^*)dv^* \\ -8\pi m(v)h(v) + \xi\int_{\mathbb{R}^3} mh\langle v\rangle^k dv, \tag{5.10}$$

where $\xi>0$ is an arbitrary constant, to be specified later. We also recall the definition of the Maxwellian M:

$$M(v) = e^{-b|v-u|^2}, \qquad v\in\mathbb{R}^3,$$

and point out that $M\sim m$ (via direct computations).

We prove now the following coercivity estimate for Λ_k.

LEMMA 5.1. *$\Lambda_k : \mathcal{H}^k \to (\mathcal{H}^k)'$ is bounded and $(\Lambda_k h, h)_{L^2(m\langle v\rangle^k)} \gtrsim \|h\|^2_{\mathcal{H}^k}$ for every $h\in\mathcal{H}^k$, provided that $\xi>0$ is large enough.*

Proof. From (5.9) and the Definition (1.5) of A it follows, via an integration by parts,

$$(\Lambda_k h_1, h_2)_{L^2(m\langle v\rangle^k)} = \int_{\mathbb{R}^3}\nabla(\langle v\rangle^k h_2(v))\cdot\left(\int_{\mathbb{R}^3}\frac{m(v^*)}{|v-v^*|}\Pi(v-v^*)dv^*\right)\nabla h_1(v)m(v)dv \\ -8\pi\int_{\mathbb{R}^3} h_1h_2\langle v\rangle^k m^2 dv + \xi\left(\int_{\mathbb{R}^3} h_1\langle v\rangle^k m dv\right)\left(\int_{\mathbb{R}^3} h_2\langle v\rangle^k m dv\right)$$

$$
\begin{aligned}
&=8\pi\int_{\mathbb{R}^3}\nabla h_2(v)\cdot A[m]\nabla h_1(v)\langle v\rangle^k m(v)dv\\
&\quad+8k\pi\int_{\mathbb{R}^3}\langle v\rangle^{k-2}h_2(v)v\cdot A[m]\nabla h_1(v)m(v)dv\\
&\quad-8\pi\int_{\mathbb{R}^3}h_1h_2m^2\langle v\rangle^k dv+\xi\left(\int_{\mathbb{R}^3}h_1m\langle v\rangle^k dv\right)\left(\int_{\mathbb{R}^3}h_2m\langle v\rangle^k dv\right),
\end{aligned}
\tag{5.11}
$$

for $h_1,h_2\in\mathcal{H}^k$. Since $A[m](v)$ is symmetric and positive definite for $v\in\mathbb{R}^3$, Cauchy-Schwartz yields

$$
\begin{aligned}
&\left|(\Lambda_k h_1,h_2)_{L^2(\langle v\rangle^k m)}\right|\\
&\lesssim\int_{\mathbb{R}^3}(\nabla h_2(v)\cdot A[m]\nabla h_2(v))^{1/2}(\nabla h_1(v)\cdot A[m]\nabla h_1(v))^{1/2}\langle v\rangle^k m(v)dv\\
&\quad+\int_{\mathbb{R}^3}(\nabla h_1(v)\cdot A[m]\nabla h_1(v))^{1/2}\left(h_2(v)^2\langle v\rangle^{-4}v\cdot A[m]v\right)^{1/2}\langle v\rangle^k m(v)dv\\
&\quad+\|h_1\|_{\mathcal{H}_0}\|h_2\|_{\mathcal{H}_0}\\
&\lesssim\left(\int_{\mathbb{R}^3}\nabla h_2\cdot A[m]\nabla h_2\langle v\rangle^k mdv\right)^{\frac12}\left(\int_{\mathbb{R}^3}\nabla h_1\cdot A[m]\nabla h_1\langle v\rangle^k mdv\right)^{\frac12}\\
&\quad+\left(\int_{\mathbb{R}^3}\nabla h_1(v)\cdot A[m]\nabla h_1(v)\langle v\rangle^k m(v)dv\right)^{\frac12}\left(\int_{\mathbb{R}^3}h_2(v)^2\langle v\rangle^{k-4}v\cdot A[m]vmdv\right)^{\frac12}\\
&\quad+\|h_1\|_{\mathcal{H}_0^k}\|h_2\|_{\mathcal{H}_0^k}.
\end{aligned}
$$

Therefore

$$
\left|(\Lambda_k h_1,h_2)_{L^2(m\langle v\rangle^k)}\right|\lesssim\|h_1\|_{\mathcal{H}^k}\|h_2\|_{\mathcal{H}^k}.
$$

Via a duality argument it follows that Λ is bounded as an operator $\mathcal{H}^k\to(\mathcal{H}^k)'$.

Choosing $h_1=h_2=h$ in (5.11) yields

$$
\begin{aligned}
(\Lambda_k h,h)_{L^2(m\langle v\rangle^k)}=&\,8\pi\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k mdv\\
&-8\pi\int_{\mathbb{R}^3}h^2\langle v\rangle^k m^2dv+\xi\left(\int_{\mathbb{R}^3}h\langle v\rangle^k mdv\right)^2\\
&+8k\pi\int_{\mathbb{R}^3}\langle v\rangle^{k-2}h(v)v\cdot A[m]\nabla h(v)m(v)dv.
\end{aligned}
\tag{5.12}
$$

The last integral can be estimated via the Cauchy-Schwarz inequality:

$$
\begin{aligned}
(\Lambda_k h,h)_{L^2(m\langle v\rangle^k)}\ge&\,4\pi\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k mdv\\
&-8\pi\int_{\mathbb{R}^3}h^2\langle v\rangle^k m^2dv+\xi\left(\int_{\mathbb{R}^3}h\langle v\rangle^k mdv\right)^2\\
&-C\int_{\mathbb{R}^3}h^2\langle v\rangle^{k-5}mdv.
\end{aligned}
\tag{5.13}
$$

Let us focus on the first integral on the right-hand side of (5.13). Lemma 2.1 and the fact that $H_1[\mathcal{M}]<0$ lead to

$$
\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k mdv\gtrsim\int_{\mathbb{R}^3}|\nabla h|^2m(v)\langle v\rangle^{k-3}dv.
$$

For every $R>0$, since $m(v)\langle v\rangle^{k-3}$ is uniformly positive on B_R (with an $R-$dependent lower bound), it follows via (the standard) Sobolev's embedding and Poincaré's Lemma

$$\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv\geq c_R\int_{B_R}|\nabla h|^2dv\geq c_R\left\|h-\frac{1}{|B_R|}\int_{B_R}hdv\right\|^2_{L^6(B_R)}$$
$$\geq c_R\|h\|^2_{L^6(B_R)}-c_R'\left(\int_{B_R}hdv\right)^2$$
$$\geq c_R\|h\|^2_{L^6(B_R)}-c_R''\int_{B_R}h^2dv.$$

From (5.13) and the above inequality we deduce

$$\begin{aligned}(\Lambda_k h,h)_{L^2(m\langle v\rangle^k)}\geq&\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv+c_R\|h\|^2_{L^6(B_R)}-c_R''\int_{B_R}h^2dv\\&-8\pi\int_{B_R}h^2\langle v\rangle^k m^2dv-8\pi\int_{\mathbb{R}^3\setminus B_R}h^2\langle v\rangle^k m^2dv+\xi\left(\int_{\mathbb{R}^3}h\langle v\rangle^k mdv\right)^2\\&-C\int_{B_R}h^2\langle v\rangle^{k-5}mdv-C\int_{\mathbb{R}^3\setminus B_R}h^2\langle v\rangle^{k-5}mdv\\\geq&\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv+c_R\|h\|^2_{L^6(B_R)}-c_R'''\|h\|^2_{L^2(B_R)}\\&-8\pi\int_{\mathbb{R}^3\setminus B_R}h^2\langle v\rangle^k m^2dv+\xi\tilde{c}_R\|h\|^2_{L^1(B_R)}-C\int_{\mathbb{R}^3\setminus B_R}h^2\langle v\rangle^{k-5}mdv.\end{aligned}\tag{5.14}$$

Let us now consider

$$\|h\|^2_{\mathcal{H}^k_0}=\int_{\mathbb{R}^3}h^2\langle v\rangle^{k-1}m(v)dv\lesssim\int_{\mathbb{R}^3}\langle v\rangle^{k-3}h^2\langle v\rangle^2M(v)dv.$$

Young's inequality with the convex conjugated functions $s\mapsto\frac{s}{\delta}\log\frac{s}{\eta}-\frac{s}{\delta}$, $s\mapsto\eta\delta^{-1}e^{\delta s}$ (with $\eta>0$ arbitrary and $\delta>0$ fixed small enough such that $\int_{\mathbb{R}^3}e^{\delta|v|^2}M(v)dv<\infty$) leads to

$$\|h\|^2_{\mathcal{H}^k_0}\lesssim\int_{\mathbb{R}^3}[\delta^{-1}\langle v\rangle^{k-3}h^2\log(\eta^{-1}\langle v\rangle^{k-3}h^2)-\delta^{-1}\langle v\rangle^{k-3}h^2]M(v)dv$$
$$+\eta\delta^{-1}\int_{\mathbb{R}^3}e^{\delta\langle v\rangle^2}M(v)dv.$$

By defining $u=\langle v\rangle^{(k-3)/2}h$ and rescaling $\eta\mapsto\|u\|^2_{L^2(G)}\eta$, the above inequality can be rewritten as

$$\|h\|^2_{\mathcal{H}^k_0}\lesssim\int_{\mathbb{R}^3}u^2\log\frac{u^2}{\|u\|^2_{L^2(M)}}M(v)dv-c\|u\|^2_{L^2(M)}(1+\log\eta)+\eta\|u\|^2_{L^2(M)}.$$

By employing the log-Sobolev's inequality with Gaussian weight [26] one obtains

$$\|h\|^2_{\mathcal{H}^k_0}\lesssim\|\nabla u\|^2_{L^2(M)}+\|u\|^2_{L^2(M)}(\eta-c-c\log\eta).$$

Replacing u with $\langle v\rangle^{(k-3)/2}h$ and choosing $\eta>0$ the minimum point of $\eta-c-c\log\eta$, one finds

$$\|h\|^2_{\mathcal{H}^k_0}\lesssim\int_{\mathbb{R}^3}|\nabla h|^2\langle v\rangle^{k-3}M(v)dv+\int_{\mathbb{R}^3}h^2\langle v\rangle^{k-3}M(v)dv. \tag{5.15}$$

Lemma 2.1, relation $m\sim M$ and (5.15) yield

$$\|h\|^2_{\mathcal{H}_0}\lesssim\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv+\int_{\mathbb{R}^3}h^2\langle v\rangle^{k-3}m dv. \tag{5.16}$$

At this point, (5.14) and (5.16) yield

$$\begin{aligned}(\Lambda_k h,h)_{L^2(m\langle v\rangle^k)}\gtrsim&\|h\|^2_{\mathcal{H}^k_0}+\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv+c_R\|h\|^2_{L^6(B_R)}-c'''_R\|h\|^2_{L^2(B_R)}\\&-8\pi\int_{\mathbb{R}^3\setminus B_R}h^2\langle v\rangle^k m^2dv+\xi\tilde{c}_R\|h\|^2_{L^1(B_R)}-C\int_{\mathbb{R}^3}h^2\langle v\rangle^{k-3}m dv\\&-C\int_{\mathbb{R}^3\setminus B_R}h^2\langle v\rangle^{k-5}m dv,\end{aligned}$$

which implies, given that $m\lesssim\langle v\rangle^{-3}$,

$$\begin{aligned}(\Lambda_k h,h)_{L^2(m\langle v\rangle^k)}\gtrsim&\|h\|^2_{\mathcal{H}^k_0}+\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv-C\int_{\mathbb{R}^3\setminus B_R}h^2 m\langle v\rangle^{k-3}dv\\&+c_R\|h\|^2_{L^6(B_R)}-c'''_R\|h\|^2_{L^2(B_R)}+\xi\tilde{c}_R\|h\|^2_{L^1(B_R)}.\end{aligned} \tag{5.17}$$

Choosing $R>0$, we absorb the third integral on the right-hand side of (5.17) via $\|h\|^2_{\mathcal{H}_0}$, yielding

$$\begin{aligned}(\Lambda_k h,h)_{L^2(m\langle v\rangle^k)}\geq&\|h\|^2_{\mathcal{H}^k_0}+\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv\\&+\|h\|^2_{L^6(B_R)}-K\|h\|^2_{L^2(B_R)}+\xi\|h\|^2_{L^1(B_R)}.\end{aligned} \tag{5.18}$$

By interpolating L^2 between L^1 and L^6 and applying Young's inequality one finds

$$K\|h\|^2_{L^2(B_R)}\leq K\|h\|^{4/5}_{L^1(B_R)}\|h\|^{6/5}_{L^6(B_R)}\leq\frac{2}{5}\xi\|h\|^2_{L^1(B_R)}+\frac{3}{5}K^{5/3}\xi^{-2/3}\|h\|^2_{L^6(B_R)}.$$

Therefore, for $\xi>0$ large enough, it holds $\|h\|^2_{L^6(B_R)}-K\|h\|^2_{L^2(B_R)}+\xi\|h\|^2_{L^1(B_R)}\geq0$. We conclude

$$(\Lambda_k h,h)_{L^2(m\langle v\rangle^k)}\geq\|h\|^2_{\mathcal{H}^k_0}+\int_{\mathbb{R}^3}\nabla h\cdot A[m]\nabla h\langle v\rangle^k m dv=\|h\|^2_{\mathcal{H}^k}.$$

This finishes the proof. ∎

Concerning $\mathcal{K}_k$, we are going to prove the following result:

LEMMA 5.2. *For $k\geq0$ it holds*

$$(\mathcal{K}_k h)(v)=\frac{\nabla m(v)}{m(v)}\cdot\Big(\tilde{\mathcal{K}}*(h\nabla m)-\nabla\tilde{\mathcal{K}}*(hm)\Big)+\nabla\tilde{\mathcal{K}}*(h\nabla m)+\xi\int_{\mathbb{R}^3}mh\langle v\rangle^k dv, \tag{5.19}$$

with

$$\tilde{\mathcal{K}}(v)=\frac{\Pi(v)}{|v|}.$$

Furthermore $\mathcal{K}_k:\mathcal{H}_0^k\to\mathcal{H}_0^k$ *is a compact operator and the following bound holds for* $k\geq 0$

$$|(\mathcal{K}_k h,h)_{L^2(m\langle v\rangle^k)}|\lesssim \|h\|^2_{L^2(m\langle v\rangle^{k-2})}. \tag{5.20}$$

Proof. Integration by parts yields

$$\begin{aligned}(\mathcal{K}_k h)(v)=&-\frac{\nabla m(v)}{m(v)}\cdot\int_{\mathbb{R}^3}\frac{m(v^*)}{|v-v^*|}\Pi(v-v^*)\nabla h(v^*)dv^*\\&-\nabla\cdot\int_{\mathbb{R}^3}\frac{m(v^*)}{|v-v^*|}\Pi(v-v^*)\nabla h(v^*)dv^*-8\pi m(v)h(v)+\xi\int_{\mathbb{R}^3}mh\langle v\rangle^k dv\\=&\frac{\nabla m(v)}{m(v)}\cdot\int_{\mathbb{R}^3}\frac{\nabla m(v^*)}{|v-v^*|}\Pi(v-v^*)h(v^*)dv^*\\&+\frac{\nabla m(v)}{m(v)}\cdot\int_{\mathbb{R}^3}h(v^*)m(v^*)\nabla_{v^*}\left[\frac{\Pi(v-v^*)}{|v-v^*|}\right]dv^*\\&+\nabla\cdot\int_{\mathbb{R}^3}\frac{h(v^*)}{|v-v^*|}\Pi(v-v^*)\nabla m(v^*)dv^*\\&-\nabla\cdot\int_{\mathbb{R}^3}\frac{\Pi(v-v^*)}{|v-v^*|}\nabla[m(v^*)h(v^*)]dv^*-8\pi m(v)h(v)+\xi\int_{\mathbb{R}^3}mh\langle v\rangle^k dv.\end{aligned}$$

Since

$$-\nabla\cdot\int_{\mathbb{R}^3}\frac{\Pi(v-v^*)}{|v-v^*|}\nabla f(v^*)dv^*=8\pi f(v)\qquad\forall f\in C_c^\infty(\mathbb{R}^3),$$

we deduce that (5.19) holds.

Let now $(h_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}_0^k=L^2(m\langle v\rangle^{k-1})$. For $1<p\leq 2$, $s\in\mathbb{R}$, we have

$$\|h_n\nabla m\|^p_{L^p(\mathbb{R}^3)}\lesssim\int_{\mathbb{R}^3}|h_n|^p|\nabla m^{1-1/p}|^p\langle v\rangle^{-s}\ m\langle v\rangle^s dv\lesssim\int_{\mathbb{R}^3}|h_n|^p m\langle v\rangle^s dv.$$

Hölder's inequality yields

$$\|h_n\nabla m\|_{L^p(\mathbb{R}^3)}\lesssim\|h_n\|_{L^2(m\langle v\rangle^s)},\qquad 1<p\leq 2,\quad s\in\mathbb{R}. \tag{5.21}$$

In a similar way, one shows

$$\|h_n m\|_{L^p(\mathbb{R}^3)}\lesssim\|h_n\|_{L^2(m\langle v\rangle^s)},\qquad 1<p\leq 2,\quad s\in\mathbb{R}. \tag{5.22}$$

This means that $h_n\nabla m$, $h_n m$ are bounded in $L^p(\mathbb{R}^3)$ for $1<p\leq 2$. Let us now consider, for $R>0$ arbitrary,

$$\nabla\tilde{\mathcal{K}}*(h_n\nabla m)=(\mathbf{1}_{B_R}\nabla\tilde{\mathcal{K}})*(h_n\nabla m)+(\mathbf{1}_{\mathbb{R}^3\setminus B_R}\nabla\tilde{\mathcal{K}})*(h_n\nabla m).$$

Given that $\nabla\tilde{\mathcal{K}}\in L^1(B_R)$, from [10, Corollary 4.28] it follows that $(\mathbf{1}_{B_R}\nabla\tilde{\mathcal{K}})*(h_n\nabla m)$ is relatively compact in $L^2(\Omega)$ for every measurable set Ω with finite measure. A Cantor diagonal argument yields the existence of a subsequence of h_n (not relabeled) such that $(\mathbf{1}_{B_R}\nabla\tilde{\mathcal{K}})*(h_n\nabla m)$ is strongly convergent in $L^2(B_r)$ for every $r\in\mathbb{N}$. Given that $\int_{\mathbb{R}^3} m\langle v\rangle^{k-1}dv<\infty$, it is easily seen that

$$(\mathbf{1}_{B_R}\nabla\tilde{\mathcal{K}})*(h_n\nabla m)\to(\mathbf{1}_{B_R}\nabla\tilde{\mathcal{K}})*(h\nabla m) \quad \text{strongly in } L^2(m\langle v\rangle^{k-1})=\mathcal{H}_0^k. \tag{5.23}$$

On the other hand, Young's inequality for convolutions yields

$$\|(\mathbf{1}_{\mathbb{R}^3\setminus B_R}\nabla\tilde{\mathcal{K}})*(h_n\nabla m)\|_{L^2(\mathbb{R}^3)}\le\|\nabla\tilde{\mathcal{K}}\|_{L^q(\mathbb{R}^3\setminus B_R)}\|h_n\nabla m\|_{L^p(\mathbb{R}^3)},\quad \frac{3}{2}=\frac{1}{p}+\frac{1}{q}, 1<p<\frac{6}{5}.$$

Since $q>3/2$ then $\|\nabla\tilde{\mathcal{K}}\|_{L^q(\mathbb{R}^3\setminus B_R)}\to 0$ as $R\to\infty$, while $\|h_n\nabla m\|_{L^p(\mathbb{R}^3)}\lesssim 1$ for $1<p<6/5$. From this fact and (5.23) we obtain

$$\nabla\tilde{\mathcal{K}}*(h_n\nabla m)\to\nabla\tilde{\mathcal{K}}*(h\nabla m) \quad \text{strongly in } \mathcal{H}_0^k. \tag{5.24}$$

In a similar way one shows that

$$\frac{\nabla m}{m}\cdot\nabla\tilde{\mathcal{K}}*(h_n m)\to\frac{\nabla m}{m}\cdot\nabla\tilde{\mathcal{K}}*(hm) \quad \text{strongly in } \mathcal{H}_0^k. \tag{5.25}$$

Let us now deal with $\tilde{\mathcal{K}}*(h_n\nabla m)$. One can prove, via a similar argument as the one employed to show (5.23), that (up to subsequences)

$$(\mathbf{1}_{B_R}\tilde{\mathcal{K}})*(h_n\nabla m)\to(\mathbf{1}_{B_R}\tilde{\mathcal{K}})*(h\nabla m) \quad \text{strongly in } \mathcal{H}_0^k. \tag{5.26}$$

On the other hand, for $\zeta\in(0,1)$,

$$|((\mathbf{1}_{\mathbb{R}^3\setminus B_R}\tilde{\mathcal{K}})*(h_n\nabla m))(v)|\le R^{-\zeta}|(|\cdot|^{\zeta-1}*|h_n\nabla m|)(v)|, \qquad v\in\mathbb{R}^3,$$

so Hardy-Littlewood-Sobolev's inequality yields

$$\|((\mathbf{1}_{\mathbb{R}^3\setminus B_R}\tilde{\mathcal{K}})*(h_n\nabla m))\|_{L^q(\mathbb{R}^3)}\lesssim R^{-\zeta}\|h_n\nabla m\|_{L^p(\mathbb{R}^3)}, \qquad \frac{1}{p}+\frac{1-\zeta}{3}=1+\frac{1}{q}, \quad 1<p\le 2.$$

This means that

$$\lim_{R\to\infty}\sup_{n\in\mathbb{N}}\|(\mathbf{1}_{\mathbb{R}^3\setminus B_R}\tilde{\mathcal{K}})*(h_n\nabla m)\|_{L^q(\mathbb{R}^3)}=0.$$

Putting the above relation and (5.26) together yields the strong convergence of $\tilde{\mathcal{K}}*(h_n\nabla m)$ in $\mathcal{H}_0^k$. Finally, $\int_{\mathbb{R}^3} mh_n dv$ is obviously relatively compact in $\mathcal{H}_0$. Thus $\mathcal{K}_k:\mathcal{H}_0^k\to\mathcal{H}_0^k$ is a compact operator for every $k\ge 0$. Bound (5.20) is a straightforward byproduct of the previous computations and of estimates (5.21), (5.22). This finishes the proof. ∎

We now want to prove the following theorem:

Theorem 5.1 (Spectral gap for L). *There exists a constant $C_L>0$ such that*

$$-(Lh,h)_{L^2(m)}\ge C_L\left(\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h m dv+\|h\|^2_{L^2(m\langle v\rangle^{-1})}\right), \quad \forall h\in D(L)\cap N(L)^\perp. \tag{5.27}$$

Proof. From [8] we know that for all $h\in\mathcal{H}^0$

$$(Lh,h)_{L^2(m)}\le 0,$$

and equality holds if and only if $h\in N(L)$. This fact, Lemmas 5.1, 5.2, and [17, Lemma 10] yield (5.27). ∎

REMARK 5.1. The constant C_L appearing in the statement of Theorem 5.1 is not explicit. It is a consequence of [17, Lemma 10], whose proof is non-constructive. A similar estimate already appeared in [30,33] for the classical Landau equation.

Next, we show some bounds for A and ∇a. Define preliminarily for $p,q\geq 1$ and $g:\mathbb{R}^3\to\mathbb{R}$ arbitrary measurable function

$$\begin{aligned}
\mathcal{E}^{\perp}_{p,q}[g] &= \int_{\mathbb{R}^3}|g(w)|dw + \left(\int_{\mathbb{R}^3}|w|^p|g(w)|^p dw\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^3}|w|^q|g(w)|^q dw\right)^{\frac{1}{q}},\\
\mathcal{E}^{\parallel}_{p,q}[g] &= \int_{\mathbb{R}^3}|w|^2|g(w)|dw + \left(\int_{\mathbb{R}^3}|w|^{3p}|g(w)|^p dw\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^3}|w|^{3q}|g(w)|^q dw\right)^{\frac{1}{q}},\\
\mathcal{E}_{p,q}[g] &= \mathcal{E}^{\perp}_{p,q}[g] + \mathcal{E}^{\parallel}_{p,q}[g],\\
\widetilde{\mathcal{E}}^{\perp}_{p,q}[g] &= \left(\int_{\mathbb{R}^3}|g(w)|^p dw\right)^{1/p} + \left(\int_{\mathbb{R}^3}|g(w)|^q dw\right)^{1/q}\\
&\quad + \left(\int_{\mathbb{R}^3}|w|^{2p}|g(w)|^{2p} dw\right)^{1/2p} + \left(\int_{\mathbb{R}^3}|w|^{2q}|g(w)|^{2q} dw\right)^{1/2q},\\
\widetilde{\mathcal{E}}^{\parallel}_{p,q}[g] &= \int_{\mathbb{R}^3}|g(w)|dw + \left(\int_{\mathbb{R}^3}|w|^p|g(w)|^p dw\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^3}|w|^q|g(w)|^q dw\right)^{\frac{1}{q}}\\
&\quad + \left(\int_{\mathbb{R}^3}|w|^{4p}|g(w)|^{2p} dw\right)^{1/2p} + \left(\int_{\mathbb{R}^3}|w|^{4q}|g(w)|^{2q} dw\right)^{1/2q},\\
\widetilde{\mathcal{E}}_{p,q}[g] &= \widetilde{\mathcal{E}}^{\perp}_{p,q}[g] + \widetilde{\mathcal{E}}^{\parallel}_{p,q}[g].
\end{aligned}$$

The following result holds.

LEMMA 5.3 (Bounds for A). *For every $p,q\in[1,\infty)$, $p<\frac{3}{2}<q$, and every $z\in\mathbb{R}^3$,*

$$z\cdot A[g](v)z \lesssim_{p,q} \frac{\mathcal{E}^{\perp}_{p,q}[g]}{|v|}|z^{\perp}|^2 + \frac{\mathcal{E}^{\parallel}_{p,q}[g]}{|v|^3}|z^{\parallel}|^2 \lesssim_{p,q} \mathcal{E}_{p,q}[g]\, z\cdot A[m](v)z, \tag{5.28}$$

$$|\Pi(v)\nabla a[g]| \lesssim_{p,q} \widetilde{\mathcal{E}}^{\perp}_{p,q}[g]|v|^{-1}, \qquad \left|\frac{v}{|v|}\cdot\nabla a[g]\right| \lesssim_{p,q} \widetilde{\mathcal{E}}^{\parallel}_{p,q}[g]|v|^{-2}, \tag{5.29}$$

with $z^{\parallel}=|v|^{-2}(v\cdot z)v$, $z^{\perp}=z-z^{\parallel}=\Pi(v)z$, for every $g\in L^1(\mathbb{R}^d)$ such that $\mathcal{E}^{\perp}_{p,q}[g]<\infty$, $\mathcal{E}^{\parallel}_{p,q}[g]<\infty$, $\widetilde{\mathcal{E}}^{\perp}_{p,q}[g]<\infty$, $\widetilde{\mathcal{E}}^{\parallel}_{p,q}[g]<\infty$.

Proof. The upper bound in (5.28) is already known since m can be estimated from below via the Maxwell-Boltzmann distribution. Therefore we only prove the lower bound.

We first observe that it is enough to prove the statement for $z=z^{\parallel}$ and $z=z^{\perp}$, since via Cauchy-Schwarz and Young's inequalities it holds (remember that $A[g]$ is symmetric and positive definite)

$$z^{\perp}\cdot A[g]z^{\parallel} \leq \left(z^{\perp}\cdot A[g]z^{\perp}\right)^{1/2}\left(z^{\parallel}\cdot A[g]z^{\parallel}\right)^{1/2} \leq \frac{1}{2}z^{\perp}\cdot A[g]z^{\perp} + \frac{1}{2}z^{\parallel}\cdot A[g]z^{\parallel}.$$

Let us now deal with the case $z=z^{\parallel}$. We start by considering $z=v$. It holds

$$v\cdot A[g](v)v = \int_{\mathbb{R}^3}\frac{g(w)}{|v-w|}v\cdot\Pi(v-w)v\,dw$$

$$= \int_{\mathbb{R}^3} \frac{g(w)}{|v-w|} w \cdot \Pi(v-w) w \, dw$$
$$\leq \int_{\mathbb{R}^3} \frac{|w|^2 |g(w)|}{|v-w|} dw.$$

Let us now consider

$$|v| \int_{\mathbb{R}^3} \frac{|w|^2 |g(w)|}{|v-w|} dw \leq \int_{\mathbb{R}^3} (|v-w| + |w|) \frac{|w|^2 |g(w)|}{|v-w|} dw$$
$$\leq \int_{\mathbb{R}^3} |w|^2 |g(w)| dw + \int_{\mathbb{R}^3} \frac{|w|^3 |g(w)|}{|v-w|} dw.$$

Since

$$\int_{\mathbb{R}^3} \frac{|w|^3 |g(w)|}{|v-w|} dw = \int_{B_1(v)} \frac{|w|^3 |g(w)|}{|v-w|} dw + \int_{\mathbb{R}^3 \setminus B_1(v)} \frac{|w|^3 |g(w)|}{|v-w|} dw,$$

Hölder's inequality yields

$$\int_{\mathbb{R}^3} \frac{|w|^3 |g(w)|}{|v-w|} dw \lesssim_{\ell_1, \ell_2} \| |\cdot|^3 g \|_{3/2+\ell_1} + \| |\cdot|^3 g \|_{3/2-\ell_2}, \quad \forall \ell_1 > 0, \quad \forall \ell_2 \in \left(0, \frac{1}{2}\right].$$

It follows

$$\frac{v}{|v|} \cdot A[g](v) \frac{v}{|v|} \lesssim_{p,q} \mathcal{E}^{\|}_{p,q}[g] \, |v|^{-3}, \qquad 1 \leq p < \frac{3}{2} < q.$$

Let us now consider, for $z = z^{\perp}$, $|z| = 1$,

$$|v| z \cdot A[g](v) z = \int_{\mathbb{R}^3} \frac{|v| g(w)}{|v-w|} z \cdot \Pi(v-w) z \, dw$$
$$\leq \int_{\mathbb{R}^3} \frac{|v||g(w)|}{|v-w|} dw \leq \int_{\mathbb{R}^3} |g(w)| dw + \int_{\mathbb{R}^3} \frac{|w||g(w)|}{|v-w|} dw$$
$$\lesssim_{\ell_1, \ell_2} \|g\|_1 + \| |\cdot| g \|_{3/2+\ell_1} + \| |\cdot| g \|_{3/2-\ell_2}, \quad \forall \ell_1 > 0, \quad \forall \ell_2 \in \left(0, \frac{1}{2}\right].$$

It follows

$$z \cdot A[g](v) z \lesssim_{p,q} \mathcal{E}^{\perp}_{p,q}[g] \, |v|^{-1}, \qquad 1 \leq p < \frac{3}{2} < q.$$

Hence (5.28) holds.

Let us now prove (5.29). Using a Young's inequality for convolutions, we get

$$|v| |\nabla a[g](v)| \leq \int_{\mathbb{R}^3} \frac{|v-w| + |w|}{|v-w|^2} |g(w)| dw$$
$$= \int_{\mathbb{R}^3} \frac{|g(w)|}{|v-w|} dw + \int_{\mathbb{R}^3} \frac{|w||g(w)|}{|v-w|^2} dw$$
$$\leq \widetilde{\mathcal{E}}^{\perp}_{p,q}[g],$$

while

$$|v|^2 \left| \frac{v}{|v|} \cdot \nabla a[g] \right| = |v| \left| \int_{\mathbb{R}^3} g(w) \frac{(v-w) \cdot v}{|v-w|^3} dw \right|$$

$$\begin{aligned}&\le |v|\int_{\mathbb{R}^3}\frac{|g(w)|}{|v-w|}dw+|v|\int_{\mathbb{R}^3}\frac{|g(w)||w|}{|v-w|^2}dw\\&\le\int_{\mathbb{R}^3}|g(w)|dw+2\int_{\mathbb{R}^3}\frac{|g(w)||w|}{|v-w|}dw+\int_{\mathbb{R}^3}\frac{|g(w)||w|^2}{|v-w|^2}dw\\&\le\widetilde{\mathcal{E}}^{\|}_{p,q}[g].\end{aligned}$$

This finishes the proof. ∎

The next lemma deals with the nonlinear contributions Γ_2 and Γ_3.

LEMMA 5.4 (Bounds for nonlinear terms). *For every $p,q\ge 1$, $p<\frac{3}{2}<q$, $k\ge 0$ it holds*

$$\begin{aligned}(\Gamma_2(h,h),h)_{L^2(m\langle v\rangle^k)}\lesssim_{p,q}&\left[\rho(\mathcal{E}_{p,q}[mh]+\widetilde{\mathcal{E}}_{p,q}[mh])+\rho\|m^{1/2}h\|_2^{2/3}+\rho\|m^{1/2}h\|_2^{4/3}+\rho^{-1}\right]\\&\left(\int_{\mathbb{R}^3}A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv+\int_{\mathbb{R}^3}h^2\langle v\rangle^{k-1}m dv\right),\end{aligned}\tag{5.30}$$

$$\begin{aligned}&(\Gamma_3[h,h,h],h)_{L^2(m\langle v\rangle^k)}\\&\lesssim_{p,q}\rho(\mathcal{E}_{p,q}[mh]+\widetilde{\mathcal{E}}_{p,q}[mh])\left(\int_{\mathbb{R}^3}A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv+\int_{\mathbb{R}^3}h^2\langle v\rangle^{k-1}m dv\right)\\&\quad+\rho^{-1}\int_{\mathbb{R}^3}A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv,\end{aligned}\tag{5.31}$$

for every $\rho>0$.

Proof. Let us first consider the contribution of the quadratic terms.

$$\begin{aligned}(\Gamma_2(h,h),h)_{L^2(m\langle v\rangle^k)}=&-\int_{\mathbb{R}^3}\langle v\rangle^k\nabla h\cdot\big(A[(1-2\mathcal{M})mh]\nabla(mh)-A[m^2h^2]\nabla\mathcal{M}\\&\qquad-(1-2\mathcal{M})mh\nabla a[mh]+m^2h^2\nabla a[\mathcal{M}]\big)\,dv\\&-\int_{\mathbb{R}^3}h\nabla\langle v\rangle^k\cdot\big(A[(1-2\mathcal{M})mh]\nabla(mh)-A[m^2h^2]\nabla\mathcal{M}\\&\qquad-(1-2\mathcal{M})mh\nabla a[mh]+m^2h^2\nabla a[\mathcal{M}]\big)\,dv,\end{aligned}$$

that can be rewritten as

$$\begin{aligned}\langle\Gamma_2(h,h),h\rangle_{L^2(m)}&=\sum_{j=1}^5 I_j+\sum_{j=1}^5 I_j',\\I_1&:=-\int_{\mathbb{R}^3}A[(1-2\mathcal{M})mh]\nabla h\cdot\nabla h\langle v\rangle^k m dv,\\I_2&:=-\int_{\mathbb{R}^3}A[(1-2\mathcal{M})mh]\nabla h\cdot\nabla m\langle v\rangle^k h dv,\\I_3&:=+\int_{\mathbb{R}^3}\nabla h\cdot A[m^2h^2]\nabla\mathcal{M}\langle v\rangle^k dv,\\I_4&:=+\int_{\mathbb{R}^3}\nabla h\cdot(1-2\mathcal{M})mh\nabla a[mh]\langle v\rangle^k dv,\\I_5&:=-\int_{\mathbb{R}^3}\nabla h\cdot m^2h^2\nabla a[\mathcal{M}]\langle v\rangle^k dv,\end{aligned}$$

$$I_1' := -\int_{\mathbb{R}^3} A[(1-2\mathcal{M})mh]\nabla h \cdot \nabla \langle v\rangle^k hm dv,$$
$$I_2' := -\int_{\mathbb{R}^3} A[(1-2\mathcal{M})mh]\nabla \langle v\rangle^k \cdot \nabla m h^2 dv,$$
$$I_3' := +\int_{\mathbb{R}^3} \nabla \langle v\rangle^k \cdot A[m^2h^2]\nabla \mathcal{M} h dv,$$
$$I_4' := +\int_{\mathbb{R}^3} \nabla \langle v\rangle^k \cdot (1-2\mathcal{M})mh \nabla a[mh] h dv,$$
$$I_5' := -\int_{\mathbb{R}^3} \nabla \langle v\rangle^k \cdot m^2 h^3 \nabla a[\mathcal{M}] dv,$$

For every $1 \le p < \frac{3}{2} < q$, thanks to (5.28), we get

$$I_1 \lesssim_{p,q} \mathcal{E}_{p,q}[mh] \int_{\mathbb{R}^3} A[m]\nabla h \cdot \nabla h m \langle v\rangle^k dv,$$

while Cauchy-Schwarz inequality and Young's inequalities lead to

$$I_2 \lesssim \int_{\mathbb{R}^3} A[|mh|]\nabla h \cdot \nabla h \langle v\rangle^k m dv + \int_{\mathbb{R}^3} A[|mh|]\nabla \log m \cdot \nabla \log m h^2 \langle v\rangle^k m dv.$$

However, it is easy to see (via direct computation) that

$$\nabla \log m(v) = b\frac{1-ae^{-b|u-v|^2/2}}{1+ae^{-b|u-v|^2/2}}(u-v),$$

so, using (5.28), we obtain

$$A[|mh|]\nabla \log m \cdot \nabla \log m \lesssim_{p,q} \mathcal{E}_{p,q}[mh]\langle v\rangle^{-1},$$

which implies

$$I_2 \lesssim_{p,q} \int_{\mathbb{R}^3} A[|mh|]\nabla h \cdot \nabla h \langle v\rangle^k m dv + \mathcal{E}_{p,q}[mh] \int_{\mathbb{R}^3} h^2 \langle v\rangle^{k-1} m dv.$$

Applying (5.28) once again leads to

$$I_2 \lesssim_{p,q} \mathcal{E}_{p,q}[mh] \left(\int_{\mathbb{R}^3} A[m]\nabla h \cdot \nabla h \langle v\rangle^k m dv + \int_{\mathbb{R}^3} h^2 \langle v\rangle^{k-1} m dv \right).$$

Let us now consider, for arbitrary $\rho > 0$,

$$\begin{aligned} I_3 \le & \rho \int_{\mathbb{R}^3} A[m^2h^2]\nabla h \cdot \nabla h \langle v\rangle^k m dv \\ & + \rho^{-1} \int_{\mathbb{R}^3} A[m^2h^2]\nabla \log\left(\frac{\mathcal{M}}{1-\mathcal{M}}\right) \cdot \nabla \log\left(\frac{\mathcal{M}}{1-\mathcal{M}}\right) \langle v\rangle^k m dv \\ = & \rho \int_{\mathbb{R}^3} A[m^2h^2]\nabla h \cdot \nabla h \langle v\rangle^k m dv + \rho^{-1} b^2 \int_{\mathbb{R}^3} A[m^2h^2](u-v)\cdot(u-v)\langle v\rangle^k m dv. \end{aligned}$$

It is quite easy to see that

$$\int_{\mathbb{R}^3} A[m^2h^2](u-v)\cdot(u-v)\langle v\rangle^k m dv \le \int_{\mathbb{R}^3} m^2(w)h^2(w) \int_{\mathbb{R}^3} \frac{m(v)|u-v|^2}{|v-w|} \langle v\rangle^k dv dw$$

$$\lesssim \int_{\mathbb{R}^3} m^2 h^2 dv \lesssim \int_{\mathbb{R}^3} m h^2 \langle v\rangle^{k-1} dv,$$

while, on the other hand,

$$\int_{\mathbb{R}^3} A[m^2h^2]\nabla h\cdot\nabla h\,\langle v\rangle^k m dv \lesssim_{p,q} \mathcal{E}_{p,q}[m^2h^2]\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h\,\langle v\rangle^k m dv.$$

Since $|mh| = |f-\mathcal{M}| \le 1$, it follows

$$I_3 \lesssim_{p,q} \rho^{-1}\int_{\mathbb{R}^3} mh^2\langle v\rangle^{k-1}dv + \rho\mathcal{E}_{p,q}[mh]\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h\,\langle v\rangle^k m dv.$$

Let us now deal with I_4. Young's inequality yields

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^3} \nabla h\cdot(1-2\mathcal{M})mh\nabla a[mh]\,\langle v\rangle^k dv\\
&= \int_{\mathbb{R}^3} \Pi(v)\nabla h\cdot(1-2\mathcal{M})mh\Pi(v)\nabla a[mh]\,\langle v\rangle^k dv\\
&\quad + \int_{\mathbb{R}^3} \frac{v\otimes v}{|v|^2}\nabla h\cdot(1-2\mathcal{M})mh\frac{v\otimes v}{|v|^2}\nabla a[mh]\,\langle v\rangle^k dv\\
&\lesssim \frac{1}{\rho}\int_{\mathbb{R}^3} |\Pi(v)\nabla h|^2\langle v\rangle^{k-1} m dv + \rho\int_{\mathbb{R}^3} h^2|\Pi(v)\nabla a[mh]|^2\langle v\rangle^{k+1} m dv\\
&\quad + \frac{1}{\rho}\int_{\mathbb{R}^3}\left|\frac{v\otimes v}{|v|^2}\nabla h\right|^2\langle v\rangle^{k-3} m dv + \rho\int_{\mathbb{R}^3} h^2\left|\frac{v\otimes v}{|v|^2}\nabla a[mh]\right|^2\langle v\rangle^{k+3} m dv.
\end{aligned}$$

From (5.28), (5.29) it follows

$$I_4 \lesssim \frac{1}{\rho}\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv + \rho\widetilde{\mathcal{E}}_{p,q}[mh]\int_{\mathbb{R}^3} mh^2\langle v\rangle^{k-1}dv.$$

Finally, let us consider, for a generic $0<\eta<1/3$,

$$\begin{aligned}
I_5 &= -\int_{\mathbb{R}^3} \nabla h\cdot m^2h^2\nabla a[\mathcal{M}]\,\langle v\rangle^k dv\\
&\lesssim \int_{\mathbb{R}^3} m^{1/2+\eta}|\nabla h| m^{3/2-\eta}|h|^2\,\langle v\rangle^k dv\\
&\lesssim \int_{\mathbb{R}^3} m^{1/2+\eta}|\nabla h| m^{7/6-\eta}|h|^{5/3}\,\langle v\rangle^k dv,
\end{aligned}$$

where the last inequality holds because $|mh|^{1/3} = |f-\mathcal{M}|^{1/3} \le 1$. It follows via the Cauchy-Schwarz inequality

$$I_5 \lesssim \|\langle v\rangle^k m^{1/2+\eta}|\nabla h|\|_2 \|m^{7/6-\eta}|h|^{5/3}\|_2 \lesssim_\eta \|\langle v\rangle^{(k-3)/2} m^{1/2}|\nabla h|\|_2 \|m^{7/10-3\eta/5}|h|\|_{10/3}^{5/3}.$$

The Gagliardo-Nirenberg inequality leads to

$$\begin{aligned}
I_5 &\lesssim_\eta \|\langle v\rangle^{(k-3)/2} m^{1/2}|\nabla h|\|_2 \|m^{7/10-3\eta/5}h\|_2^{2/3}\|\nabla(m^{7/10-3\eta/5}h)\|_2\\
&\lesssim_\eta \|\langle v\rangle^{(k-3)/2} m^{1/2}|\nabla h|\|_2 \|m^{7/10-3\eta/5}h\|_2^{2/3}\cdot\\
&\qquad\cdot\Big(\|m^{7/10-3\eta/5}\nabla h\|_2 + \|h\nabla(m^{7/10-3\eta/5})\|_2\Big)
\end{aligned}$$

$$\lesssim_\eta \|\langle v\rangle^{(k-3)/2} m^{1/2}|\nabla h|\|_2^2 \|m^{7/10-3\eta/5}h\|_2^{2/3} \\ + \|\langle v\rangle^{(k-3)/2} m^{1/2}|\nabla h|\|_2 \|m^{7/10-3\eta/5}h\|_2^{2/3} \|h m^{7/10-3\eta/5}\nabla \log m\|_2.$$

Choosing $\eta = 1/6$ yields

$$I_5 \lesssim \|\langle v\rangle^{(k-3)/2} m^{1/2}|\nabla h|\|_2^2 \|m^{1/2}h\|_2^{2/3} \\ + \|\langle v\rangle^{(k-3)/2} m^{1/2}|\nabla h|\|_2 \|\langle v\rangle^{(k-1)/2} m^{1/2} h\|_2 \|m^{1/2}h\|_2^{2/3} \\ \lesssim \rho(\|m^{1/2}h\|_2^{2/3} + \|m^{1/2}h\|_2^{4/3}) \|\langle v\rangle^{(k-3)/2} m^{1/2}|\nabla h|\|_2^2 + \rho^{-1}\|\langle v\rangle^{(k-1)/2} m^{1/2}h\|_2^2.$$

From (5.28) we conclude

$$I_5 \lesssim \rho(\|m^{1/2}h\|_2^{2/3} + \|m^{1/2}h\|_2^{4/3}) \int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h \langle v\rangle^k m dv + \rho^{-1}\int_{\mathbb{R}^3} h^2 \langle v\rangle^{k-1} m dv.$$

Since $|\nabla\langle v\rangle^k| \lesssim \langle v\rangle^{k-1}$, the terms $I_1',\dots,I_5'$ can be estimated in a similar way as the terms $I_1,\dots,I_5$. Therefore we deduce that (5.30) holds.

Next, we deal with the contributions from the cubic terms:

$$\begin{aligned}
\langle \Gamma_3[h,h,h],h\rangle_{L^2(m)} =& \int_{\mathbb{R}^3} \nabla h\cdot\left(A[m^2h^2]\nabla(mh) - m^2h^2\nabla a[mh]\right)\langle v\rangle^k dv \\
&+ \int_{\mathbb{R}^3} \nabla\langle v\rangle^k\cdot\left(A[m^2h^2]\nabla(mh) - m^2h^2\nabla a[mh]\right) h dv \\
=& I_6 + I_7 + I_8 + I_6' + I_7' + I_8', \\
I_6 :=& \int_{\mathbb{R}^3} \nabla h\cdot A[m^2h^2]\nabla h \langle v\rangle^k m dv, \\
I_7 :=& \int_{\mathbb{R}^3} \nabla h\cdot A[m^2h^2]\nabla m \langle v\rangle^k h dv, \\
I_8 :=& -\int_{\mathbb{R}^3} \nabla h\cdot m^2h^2\nabla a[mh]\langle v\rangle^k dv, \\
I_6' :=& \int_{\mathbb{R}^3} \nabla\langle v\rangle^k\cdot A[m^2h^2]\nabla h h m dv, \\
I_7' :=& \int_{\mathbb{R}^3} \nabla\langle v\rangle^k\cdot A[m^2h^2]\nabla m h^2 dv, \\
I_8' :=& -\int_{\mathbb{R}^3} \nabla\langle v\rangle^k\cdot m^2h^3\nabla a[mh] dv.
\end{aligned}$$

From (5.28) and relation $|mh|\le 1$ it follows

$$I_6 \lesssim_{p,q} \mathcal{E}_{p,q}[mh]\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv.$$

The term I_7 can be estimated like I_2 to obtain

$$I_7 \lesssim_{p,q} \mathcal{E}_{p,q}[mh]\left(\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv + \int_{\mathbb{R}^3} h^2\langle v\rangle^{k-1} m dv\right).$$

The term I_8 can be estimated like I_4 to obtain

$$I_8 \lesssim \frac{1}{\rho}\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv + \rho\widetilde{\mathcal{E}}_{p,q}[mh]\int_{\mathbb{R}^3} m^3h^4\langle v\rangle^{k-1} dv,$$

but, given that $m^2h^2 \le 1$, it follows

$$I_8 \lesssim \frac{1}{\rho}\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv + \rho\widetilde{\mathcal{E}}_{p,q}[mh]\int_{\mathbb{R}^3} mh^2\langle v\rangle^{k-1}dv.$$

Finally, since $|\nabla\langle v\rangle^k| \lesssim \langle v\rangle^{k-1}$, the terms $I_6',\dots,I_8'$ can be estimated in a similar way as the terms $I_6,\dots,I_8$. Therefore we deduce that (5.31) holds. This finishes the proof. ∎

We are now ready to prove the conditional algebraic convergence result, thereby concluding the proof of Theorem 1.3.

LEMMA 5.5 (Algebraic rate of convergence for initial data close to equilibrium). *There exists a constant $\ell > 0$ such that, if $\int_{\mathbb{R}^3}(f_{in}-\mathcal{M})^2 m^{-1}dv < \ell$, and if $\int_{\mathbb{R}^3}(f_{in}-\mathcal{M})^2\langle v\rangle^N m^{-1}dv < \infty$ for some $N \ge 1$, then*

$$\int_{\mathbb{R}^3}(f(t)-\mathcal{M})^2 m^{-1}dv \lesssim (1+t)^{-N}, \qquad t>0.$$

Proof. From (1.1), (5.4) it follows that the perturbation $h=(f-\mathcal{M})/m$ satisfies the equation

$$\partial_t h = Lh + \Gamma_2[h,h] + \Gamma_3[h,h,h]. \tag{5.32}$$

Testing the above equation against h in the sense of $L^2(m)$ yields

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^3} h^2 m dv = \langle Lh,h\rangle_{L^2(m)} + \langle\Gamma_2[h,h],h\rangle_{L^2(m)} + \langle\Gamma_3[h,h,h],h\rangle_{L^2(m)}.$$

From (5.27), (5.30), (5.31) it follows that a suitable constant $C(p,q)>0$ exists such that

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^3} h^2 m dv\\ &\le \Big[\rho C(p,q)(\mathcal{E}_{p,q}[mh]+\widetilde{\mathcal{E}}_{p,q}[mh]) + \rho\|m^{1/2}h\|_2^{2/3} + \rho\|m^{1/2}h\|_2^{4/3} + \rho^{-1} - C_L\Big]\\ &\quad\cdot\left(\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h m dv + \int_{\mathbb{R}^3} h^2 m\langle v\rangle^{-1}dv\right).\end{aligned}$$

We are now going to prove that

$$\exists\alpha_1>0:\quad \mathcal{E}_{p,q}[mh]+\widetilde{\mathcal{E}}_{p,q}[mh] \lesssim_{p,q} \rho^{-2}+\rho^{\alpha_1}\|m^{1/2}h\|_2. \tag{5.33}$$

Indeed, the left-hand side of (5.33) is a sum of terms having the form

$$J_{k,s} = \left(\int_{\mathbb{R}^3}|v|^k|mh|^s dv\right)^{1/s}, \quad k\ge 0, \quad s\ge 1.$$

If $s\ge 2$ then from the property $|mh|\le 1$ and the fact that $|v|^k\sqrt{m(v)}$ is bounded in $\mathbb{R}^3$ for every $k\ge 0$ it follows immediately that

$$J_{k,s} \le \left(\int_{\mathbb{R}^3}|v|^k|mh|^2 dv\right)^{1/s} \lesssim_k \left(\int_{\mathbb{R}^3} mh^2 dv\right)^{1/s},$$

so via Young's inequality

$$J_{k,s} \lesssim_{s,k} \rho^{-2} + \rho^{s-2}\|m^{1/2}h\|_2.$$

If $1 \le s < 2$, it suffices to notice that

$$J_{k,s} = \|m^{1/2}h\|_{L^s(\mathbb{R}^3, |v|^k m^{s/2}(v)dv)}.$$

Since $|v|^k m^{s/2} \in L^1 \cap L^\infty(\mathbb{R}^3)$, Jensen's inequality yields

$$J_{k,s} \lesssim_{k,s} \|m^{1/2}h\|_{L^2(\mathbb{R}^3, |v|^k m^{s/2}(v)dv)} \lesssim_{k,s} \|m^{1/2}h\|_2.$$

Therefore (5.33) holds. We therefore conclude that, for some suitable constant $C'(p,q) > 0$ and $\alpha > 1$,

$$\frac{d}{dt}\int_{\mathbb{R}^3} h^2 m dv \le \Big[C'(p,q)(\rho^\alpha \|m^{1/2}h\|_2^2 + \rho^{-1}) - C_L\Big] \cdot \cdot \left(\int_{\mathbb{R}^3} A[m]\nabla h \cdot \nabla h m dv + \int_{\mathbb{R}^3} h^2 m \langle v\rangle^{-1} dv\right),$$

for every $\rho > 1$. We point out that

$$C'(p,q)(\rho^\alpha \|m^{1/2}h\|_2^2 + \rho^{-1}) - C_L = C'(p,q)\rho^\alpha \left(\|m^{1/2}h\|_2^2 - \tilde{\ell}(\rho)\right),$$
$$\tilde{\ell}(\rho) = \left(\frac{C_L}{C'(p,q)} - \rho^{-1}\right)\rho^{-\alpha}.$$

The maximum of $\tilde{\ell}(\rho)$ is achieved for $\rho = \frac{1+\alpha}{\alpha C_L} C'(p,q)$. Choosing ρ in this way yields

$$\frac{d}{dt}\int_{\mathbb{R}^3} h^2 m dv \le C''(p,q)\left[\|m^{1/2}h\|_2^2 - \ell\right]\left(\int_{\mathbb{R}^3} A[m]\nabla h \cdot \nabla h m dv + \int_{\mathbb{R}^3} h^2 m \langle v\rangle^{-1} dv\right),$$

for $C''(p,q) = \left(\frac{1+\alpha}{\alpha C_L}\right)^\alpha C'(p,q)^{1+\alpha}$ and

$$\ell := \frac{C_L}{(1+\alpha)C'(p,q)}\left(\frac{1+\alpha}{\alpha C_L}C'(p,q)\right)^{-\alpha} = \left(\frac{C_L}{C'(p,q)}\right)^{1+\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}.$$

Since $\|m^{1/2}h(\cdot,0)\|_2^2 - \ell < 0$ by assumption on the initial data, we deduce that $\|m^{1/2}h(\cdot,t)\|_2 \le \|m^{1/2}h(\cdot,0)\|_2$ for all $t > 0$. It follows that, for some $\lambda > 0$,

$$\frac{d}{dt}\int_{\mathbb{R}^3} h^2 m dv \le -\lambda\left(\int_{\mathbb{R}^3} A[m]\nabla h \cdot \nabla h m dv + \int_{\mathbb{R}^3} h^2 m \langle v\rangle^{-1} dv\right). \tag{5.34}$$

Integrating (5.34) in time yields

$$\sup_{t>0}\int_{\mathbb{R}^3} h^2 m dv + \lambda\int_0^\infty\int_{\mathbb{R}^3} h^2 m\langle v\rangle^{-1} dv dt \le \int_{\mathbb{R}^3} h(\cdot,0)^2 m dv. \tag{5.35}$$

We will now show that $\sup_{t>0}\int_{\mathbb{R}^3} h^2 m\langle v\rangle^N dv < \infty$. We proceed iteratively, proving that

$$\sup_{t>0}\int_{\mathbb{R}^3} h^2 m\langle v\rangle^j dv + \int_0^\infty\int_{\mathbb{R}^3} h^2 m\langle v\rangle^{j-1} dv dt < \infty, \tag{5.36}$$

for $j = 0, \dots, \lfloor N \rfloor$. We argue by induction on j. Estimate (5.35) and the assumption on the initial datum imply that (5.36) holds for $j = 0$. Let us now assume that (5.36) holds for $j = 0, \dots, k-1$, $1 \le k \le \lfloor N \rfloor$ generic. By testing (5.32) against h in the sense of

$L^2(m\langle v\rangle^k)$, exploiting Lemma 5.1 and bound (5.20) and proceeding like in the proof of (5.34) one finds

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} h^2 m\langle v\rangle^k dv \leq &-\lambda_k\left(\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h\langle v\rangle^k m dv + \int_{\mathbb{R}^3} h^2 m\langle v\rangle^{k-1}dv\right)\\ &+\mu_k\int_{\mathbb{R}^3} h^2 m\langle v\rangle^{k-2}dv,\end{aligned} \tag{5.37}$$

for some $\lambda_k,\ \mu_k>0$. By integrating (5.37) in time we get

$$\begin{aligned}&\sup_{t>0}\int_{\mathbb{R}^3} h^2 m\langle v\rangle^k dv + \lambda_k\int_0^\infty\int_{\mathbb{R}^3} h^2 m\langle v\rangle^{k-1}dvdt\\ \leq &\mu_k\int_0^\infty\int_{\mathbb{R}^3} h^2 m\langle v\rangle^{k-2}dvdt + \int_{\mathbb{R}^3} h(\cdot,0)^2 m\langle v\rangle^k dv.\end{aligned} \tag{5.38}$$

From the assumption that $\int_{\mathbb{R}^3} h(\cdot,0)^2 m\langle v\rangle^k dv<\infty$ for $k\leq N$ as well as the inductive hypothesis it follows that the right-hand side of (5.38) is finite, meaning that (5.36) holds for $j=k$. Via the induction principle we deduce that (5.36) holds for $j=0,\dots,\lfloor N\rfloor$. Choosing $k=N$ in (5.38) and exploiting (5.36) for $j=\lfloor N\rfloor$ yields (5.36) for $k=N$. In particular

$$\sup_{t>0}\int_{\mathbb{R}^3} h^2 m\langle v\rangle^N dv<\infty.$$

Therefore via Hölder's inequality

$$\int_{\mathbb{R}^3} h^2 m dv \leq \left(\int_{\mathbb{R}^3} h^2 m\langle v\rangle^{-1}dv\right)^{\frac{N}{N+1}}\left(\int_{\mathbb{R}^3} h^2 m\langle v\rangle^N dv\right)^{\frac{1}{N+1}} \lesssim \left(\int_{\mathbb{R}^3} h^2 m\langle v\rangle^{-1}dv\right)^{\frac{N}{N+1}},$$

so from (5.34) it follows

$$\frac{d}{dt}\int_{\mathbb{R}^3} h^2 m dv \leq -\lambda\left(\int_{\mathbb{R}^3} A[m]\nabla h\cdot\nabla h m dv + \left(\int_{\mathbb{R}^3} h^2 m dv\right)^{\frac{N+1}{N}}\right).$$

This (via Gronwall's inequality) finishes the proof of the lemma, and of Theorem 1.3. □

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