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RESEARCH ARTICLE

K-stability of Fano varieties via admissible flags

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Abstract

We develop a general approach to prove K-stability of Fano varieties. The new theory is used to (a) prove the existence of Kähler-Einstein metrics on all smooth Fano hypersurfaces of Fano index two, (b) compute the stability thresholds for hypersurfaces at generalised Eckardt points and for cubic surfaces at all points, and (c) provide a new algebraic proof of Tian's criterion for K-stability, amongst other applications.

Contents

1. Introduction

Introduced by Tian [47] and reformulated algebraically by Donaldson [18], K-stability is an algebro-geometric property of Fano varieties that detects the existence of Kähler-Einstein metrics. By the celebrated works of Chen-Donaldson-Sun [14] and Tian [48], a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable. However, it is in general very hard to verify the K-stability of a given Fano variety. Tian's criterion, introduced in [46], provides a sufficient condition for K-stability and has arguably become the most famous validity criterion for K-stability. There are also a few variants [15, 21, 41] of Tian's criterion, and a notable application is the K-stability of smooth hypersurfaces of Fano index one [21]. More recently, [44] discovered another K-stability criterion in the particular case of birationally superrigid Fano varieties; and as an application, [51] proved that Fano complete intersections of index one and large dimension are K-stable. However, both criteria apply exclusively to certain Fano varieties of index one, and except in a few sporadic cases, it is unclear how to attack the problem when the required conditions in neither criterion are satisfied; see for example [1, 16, 38, 43].

The purpose of this paper is to develop a systematic approach for proving the K-stability of Fano varieties. As a major application, we confirm the K-stability of all smooth hypersurfaces of Fano index two.

Theorem 1.1 (=Theorem 4.12). *Let $X = X_n \subseteq \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of degree $n \geq 3$. Then X is uniformly K-stable.*

In particular, this generalises the work of [38] on K-stability of smooth cubic threefolds, although our argument is completely different.

As another application, we prove the following K-stability criterion, giving a unified treatment for several Fano manifolds that are previously known to be K-(semi)stable; see Definition 2.3 for the definition of $\beta_X(E)$ in the statement.

Theorem 1.2 (=Corollary 4.4). *Let X be a Fano manifold of dimension n. Assume that there exists an ample line bundle L on X such that*

1. *$-K_X \sim_{\mathbb{Q}} rL$ for some $r \in \mathbb{Q}$ with $(L^n) \leq \frac{n+1}{r}$; and*
2. *for every $x \in X$, there exists $H_1, \dots, H_{n-1} \in |L|$ containing x such that $H_1 \cap \cdots \cap H_{n-1}$ is an integral curve that is smooth at x.*

Then X is K-semistable. If it is not uniformly K-stable, then $(L^n) = \frac{n+1}{r}$ and there exists some prime divisor $E \subseteq X$ such that $\beta_X(E) = 0$.

For instance, this applies to projective spaces, hypersurfaces of Fano index one and double covers of $\mathbb{P}^n$ branched along a hypersurface of degree at least $n+1$. We refer to Corollary 4.5 for a more exhaustive list. While Tian's criterion or the criterion from [44] apply to some of them, the conditions in Theorem 1.2 are usually easier to check; indeed, we never use Tian's criterion or the criterion from [44] in this paper, as most varieties considered here are of higher Fano index. On the other hand, it may be worth pointing out that our general approach also leads to a new proof of these two criteria; see Subsection 4.1.

Before we state further applications, let us recall that by [2, 24], K-stability of a Fano variety X can be characterised by its stability threshold $\delta(X)$, defined via log canonical thresholds of anti-canonical $\mathbb{Q}$-divisors of basis type; see Subsection 2.2. For example, X is K-semistable if and only if $\delta(X) \geq 1$. One can also define local stability thresholds $\delta_x(X)$ at some $x \in X$ by taking log canonical thresholds around the point x so that the global invariant $\delta(X)$ is the minimum of the local ones $\delta_x(X)$; see Subsection 2.2. It is again a challenging problem to find the precise value of these invariants, unless the variety has a large group of automorphisms [2, 25, 50]; see [13, 42] for some estimates on del Pezzo surfaces.

We also compute these invariants in some nontrivial cases. As a first example, we study the local stability thresholds of hypersurfaces at generalised Eckardt points.

Theorem 1.3 (=Corollary 4.10). *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of degree d, and let $x \in X$ be a generalised Eckardt point (the tangent hyperplane section at x is the cone over a hypersurface*

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$Y \subseteq \mathbb{P}^{n-1}$ of degree d). Assume that Y is K-semistable when $d \leq n-1$ (i.e., when it is Fano). Then $\delta_x(X) = \frac{n(n+1)}{(n-1+d)(n+2-d)}$, and it is computed by the ordinary blowup of x.

Since on smooth quadric hypersurfaces every closed point is a generalised Eckardt point, by induction on dimension, we obtain an algebraic proof of their K-semistability. In general, we get $\delta_x(X) \geq 1$ as long as Y is K-semistable. We expect that if X has a generalised Eckardt point x, then $\delta(X) = \delta_x(X)$, and smooth Fano hypersurfaces of degree d with smallest stability thresholds are those with generalised Eckardt points (see Theorem 4.6 and Corollary 4.7 for some evidence on cubic surfaces). Thus the above theorem suggests a possible inductive approach to the K-stability of Fano hypersurfaces.

As a second example, we calculate the local stability thresholds of all cubic surfaces, from which we derive the following consequences.

Theorem 1.4 (see Theorems 4.6 and 4.8). *Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface. Then there exists some boundary divisor Δ such that (X, Δ) is log Fano and $\delta(X, \Delta) = \frac{9}{25-8\sqrt{6}} \notin \mathbb{Q}$. Moreover, there exists $C \in |-K_X|$ such that (X, C) is log canonical and some valuation v that is an lc place of (X, C) such that the associated graded ring*

$$\mathrm{gr}_v R := \oplus_{m,\lambda} \mathrm{Gr}^{\lambda}_{\mathcal{F}_v} H^0(X, -mK_X)$$

is not finitely generated, where $\mathcal{F}_v$ is the filtration induced by v.

This is somewhat surprising as, by [4, Theorem 1.4], the global stability thresholds $\delta(X)$ are always rational on Fano manifolds that are not K-stable. Moreover, graded rings associated to lc places of $\mathbb{Q}$-complement as in the above statement are usually expected to be finitely generated; see, for example, [35, Conjecture 1.2]. Thus our example shows that the situation is more complicated in general.

1.1. Overview of the proof

We now describe our approach to proving the K-stability of Fano varieties. In general, one would like to estimate, or perhaps calculate, the stability threshold of a Fano variety. A priori, we need to consider log canonical thresholds of all anti-canonical basis type divisors. Our first observation is that it suffices to consider a smaller class of them: that is, those that are compatible with a given divisor over the Fano variety.

Definition 1.5. Let X be a Fano variety, and let E be a divisor over X: that is, a prime divisor on some birational model of X. Let $m \in \mathbb{N}$, and let D be an m-basis type $\mathbb{Q}$-divisor on X: that is, there exists a basis $s_1, \cdots, s_{N_m}$ of $V_m = H^0(X, -mK_X)$, where $N_m = h^0(X, -mK_X)$, such that

$$D = \frac{1}{mN_m} \sum_{i=1}^{N_m} \{s_i = 0\}.$$

We say that D is compatible with E if for every $j \in \mathbb{N}$, the subspace

$$\mathcal{F}^j_E V_m := \{s \in V_m \mid \mathrm{ord}_E(s) \geq j\} \subseteq V_m$$

is spanned by some s_i.

We may then define the stability threshold $\delta(X; \mathcal{F}_E)$ of X with respect to E by restricting to basis type divisors that are compatible with E. It turns out that

Proposition 1.6 (see Proposition 3.1). $\delta(X) = \delta(X; \mathcal{F}_E)$.

In other words, only basis type divisors compatible with E are relevant when computing stability thresholds. While basis type divisors can be hard to study in general, those compatible with a given divisor E are often concentrated around E, making it convenient to apply the inversion of adjunction. As an illustration, we consider the example of projective spaces.

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Example 1.1. Let $X = \mathbb{P}^n$, and let E be a hyperplane. Then asymptotically, basis type $\mathbb{Q}$-divisors D on X that are compatible with E can be written as $D = E + D_0$, where D_0 does not contain E in its support and $D_0|_E$ is a convex linear combination of basis type $\mathbb{Q}$-divisors on $E \cong \mathbb{P}^{n-1}$.

By induction and inversion of adjunction, this easily implies that $\delta(\mathbb{P}^n) \geq 1$ and thus gives an algebraic proof of the K-semistability of $\mathbb{P}^n$. In general, there is a lot of flexibility in the choice of the auxiliary divisors E, leading to various applications. In fact, as we will show in Subsection 4.1, both Tian's criterion and the criterion from [44] are implied by taking E to be a general member of $|-mK_X|$ for some sufficiently divisible integer m. On explicitly given Fano varieties, however, the geometry usually suggests more natural choices of E; sometimes we can even start with the optimal one – that is, a divisor that computes $\delta(X)$, as in Example 1.1 – and no information will be lost in the process. This is exactly how we compute the stability thresholds in Theorems 1.3 and 1.4.

More generally, instead of using an auxiliary divisor to refine the class of basis type divisors, we can also use an admissible flag, which is an important tool in the construction of Okounkov bodies of line bundles; see, for example, [33]. Indeed, in the inductive proof of the K-semistability of $\mathbb{P}^n$ as outlined above, we already implicitly use the full flags of linear subspaces. One can similarly define the compatibility of a basis type divisor with an admissible flag and show that to compute the stability threshold, it suffices to consider basis type divisors compatible with a chosen flag; see Section 3 for details. To prove the K-stability of a Fano variety, it is often enough to carefully choose the auxiliary divisor or admissible flag and analyse the corresponding compatible basis type divisors through inversion of adjunction. In particular, the proofs of Theorems 1.1 and 1.2 are obtained this way and involve several different auxiliary divisors and admissible flags.

1.2. Structure of the paper

This paper is organised as follows. In Subsections 2.2–2.4, we put together various preliminary materials. As we apply inversion of adjunction to basis type divisors compatible with an admissible flag, we get basis type divisors of some filtered multigraded linear series in a natural way. We define and study the invariants associated to such linear series in Subsections 2.5 and 2.6. In Section 3, we develop the framework to study stability thresholds of Fano varieties or, more generally, δ-invariants of big line bundles and derive a few inversion-of-adjunction type results for stability thresholds. The applications are presented in Section 4: in Subsection 4.1, we give a new proof of Tian's criterion and the criterion from [44]; in Subsection 4.2, we study K-stability of Fano manifolds of small degree and prove Theorem 1.2; in Subsection 4.3, we explain how to compute stability thresholds of log del Pezzo surfaces almost in complete generality, and in particular we prove Theorem 1.4; in Subsection 4.4, we prove Theorem 1.3; and finally, Theorem 1.1 is proved in Subsection 4.5.

2. Preliminaries

2.1. Notation and conventions

We work over $\mathbb{C}$. Unless otherwise specified, all varieties are assumed to be normal and projective. A pair (X, Δ) consists of a variety X and an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The notions of klt and lc singularities are defined as in [31, Definition 2.8]. The non-lc centre $\mathrm{Nlc}(X, \Delta)$ of a pair (X, Δ) is the set of closed points $x \in X$ such that (X, Δ) is not lc at x. If $\pi : Y \to X$ is a projective birational morphism and E is a prime divisor on Y, then we say E is a divisor over X. A valuation on X will mean a valuation $v \colon \mathbb{C}(X)^* \to \mathbb{R}$ that is trivial on $\mathbb{C}^*$. We write $C_X(E)$ (respectively, $C_X(v)$) for the centre of a divisor (respectively, valuation) and $A_{X,\Delta}(E)$ (respectively, $A_{X,\Delta}(v)$) for the log discrepancy of the divisor E (respectively, the valuation v) with respect to the pair (X, Δ) (see [6, 27]). We write Val_X^* for the set of nontrivial valuations. Let (X, Δ) be a klt pair, $Z \subseteq X$ a closed subset (may be reducible) and D an effective divisor on X; we denote by $\mathrm{lct}_Z(X, \Delta; D)$ the largest number $\lambda \geq 0$

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such that $\mathrm{Nlc}(X, \Delta + \lambda D)$ does not contain Z. Given a $\mathbb{Q}$-divisor D on X, we set

$$H^0(X, D) := \{0 \neq s \in \mathbb{C}(X) \,|\, \mathrm{div}(s) + D \geq 0\} \cup \{0\}$$

whose members can be viewed as effective $\mathbb{Q}$-divisors that are $\mathbb{Z}$-linearly equivalent to D. In particular, if D is $\mathbb{Q}$-Cartier, then $\mathrm{ord}_E(s) := \mathrm{ord}_E(\mathrm{div}(s) + D)$ is well-defined for any $0 \neq s \in H^0(X, D)$ and any divisor E over X. We also define the sheaf $\mathcal{O}_X(D)$ by localising the above construction.

2.2. K-stability and stability thresholds

Let (X, Δ) be a projective pair, and let L be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. We denote by $M(L)$ the set of integers $m \in \mathbb{N}_+$ such that $H^0(X, mL) \neq \{0\}$.

Definition 2.1. Notation as above. Let $m \in M(L)$, and let $V \subseteq H^0(X, mL)$ be a linear series. We say that D is a basis type divisor of V if $D = \sum_{i=1}^{N} \{s_i = 0\}$ for some basis $s_1, \cdots, s_N$ of V (where, by abuse of notation, $\{s_i = 0\}$ refers to the $\mathbb{Q}$-divisor $\mathrm{div}(s_i) + mL$). By convention, this means $D = 0$ if $V = \{0\}$. We say that D is an m-basis type $\mathbb{Q}$-divisor of L if $D = \frac{1}{m \cdot h^0(X, mL)} D_0$ for some basis type divisor D_0 of $H^0(X, mL)$ (in particular, $D \sim_{\mathbb{Q}} L$).

Definition 2.2. Let $m \in M(L)$, and let $v \in \mathrm{Val}_X^*$. In the above notation, we set

$$S_m(L; v) = \sup_{D \sim_{\mathbb{Q}} L,\, m\text{-basis type}} v(D),$$

where the supremum runs over all m-basis type $\mathbb{Q}$-divisor of L. We define $S(L; v)$ to be the limit $\lim_{m \to \infty} S_m(L; v)$, which exists by [2, 8]. We also define the pseudo-effective threshold as

$$T(L; v) = \sup\{\lambda \geq 0 \,|\, \mathrm{vol}(L; v \geq t) > 0\},$$

where

$$\mathrm{vol}(L; v \geq t) = \lim_{m \to \infty} \frac{\dim\{s \in H^0(X, mL) \,|\, v(s) \geq mt\}}{m^{\dim X}/(\dim X)!}.$$

We say that v is of linear growth if $T(L; v) < \infty$ (e.g., when v is divisorial or has finite discrepancy; see [11, Section 2.3] and [2, Section 3.1]). By [2, Theorem 3.3], for any valuation v of linear growth, we have

$$S(L; v) = \frac{1}{\mathrm{vol}(L)} \int_0^\infty \mathrm{vol}(L; v \geq t) \mathrm{d}t,$$

where $\mathrm{vol}(L)$ denotes the volume of the divisor L (see, for example, [32, Section 2.2.C]). If E is a divisor over X, we put $S(L; E) = S(L; \mathrm{ord}_E)$ and $T(L; E) = T(L; \mathrm{ord}_E)$. We will simply write $S_m(E)$, $S(E)$, and so on, if the divisor L is clear from the context.

Definition 2.3. Let (X, Δ) be a log Fano pair: that is, (X, Δ) is klt and $-(K_X + \Delta)$ is ample. We say (X, Δ) is K-semistable (respectively, K-stable) if

$$\beta_{X,\Delta}(E) := A_{X,\Delta}(E) - S(-K_X - \Delta; E) \geq 0$$

(respectively, $\beta_{X,\Delta}(E) > 0$) for all divisors E over X. We say that (X, Δ) is uniformly K-stable if

$$\beta_{X,\Delta}(v) := A_{X,\Delta}(v) - S(-K_X - \Delta; v) > 0$$

for all $v \in \mathrm{Val}_X^*$ such that $A_{X,\Delta}(v) < \infty$.

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By [2, 5, 23, 34], this is equivalent to the original definition [10, 17, 18, 47] of K-stability notions in terms of test configurations.

Definition 2.4. Let (X, Δ) be a klt pair, and let L be a $\mathbb{Q}$-Cartier big divisor on X. The (adjoint) stability threshold (or δ-invariant) of L is defined as

$$\delta(L) = \delta(X, \Delta; L) = \inf_E \frac{A_{X,\Delta}(E)}{S(L; E)}, \tag{2.1}$$

where the infimum runs over all divisors E over X. Equivalently [2], it can also be defined as the limit $\delta(L) = \lim_{m\to\infty} \delta_m(L)$, where

$$\delta_m(L) = \sup\{\lambda \geq 0 \,|\, (X, \Delta + \lambda D) \text{ is lc for all } m\text{-basis type } \mathbb{Q}\text{-divisors } D \sim_{\mathbb{Q}} L\}. \tag{2.2}$$

We say that a divisor E over X computes $\delta(L)$ if it achieves the infimum in equation (2.1). When (X, Δ) is log Fano, we write $\delta(X, \Delta)$ (or $\delta(X)$ when $\Delta = 0$) for $\delta(-K_X - \Delta)$.

We also introduce a local version of stability thresholds.

Definition 2.5. Let (X, Δ) be a klt pair, and let L be a $\mathbb{Q}$-Cartier big divisor on X. Let Z be a closed subset of X. We set

$$\delta_{Z,m}(L) = \sup\{\lambda \geq 0 \,|\, Z \nsubseteq \mathrm{Nlc}(X, \Delta + \lambda D) \text{ for all } m\text{-basis type } \mathbb{Q}\text{-divisors } D \sim_{\mathbb{Q}} L\}$$

and define the (adjoint) stability threshold of L along Z as $\delta_Z(L) = \limsup_{m\to\infty} \delta_{Z,m}(L)$. When Z is irreducible, it is not hard to see (by an argument similar to that in [2, §4]; see also Lemma 2.9) that the above limsup is a limit, and we have

$$\delta_Z(L) = \inf_{E, Z\subseteq C_X(E)} \frac{A_{X,\Delta}(E)}{S(L; E)} = \inf_{v, Z\subseteq C_X(v)} \frac{A_{X,\Delta}(v)}{S(L; v)},$$

where the first infimum runs over all divisors E over X whose centre contains Z and the second infimum runs over all valuations $v \in \mathrm{Val}_X^*$ such that $A_{X,\Delta}(v) < \infty$ and $Z \subseteq C_X(v)$. If in addition L is ample, then the second infimum is a minimum by (the same proof of) [2, Theorem E]. As in the global case, we then say that E (respectively, v) computes $\delta_Z(L)$ if it achieves the above infimum. When (X, Δ) is log Fano, we also write $\delta_Z(X, \Delta)$ (or $\delta_Z(X)$ when $\Delta = 0$) for $\delta_Z(-K_X - \Delta)$.

2.3. Plt-type divisors

Definition 2.6. Let (X, Δ) be a pair, and let F be a divisor over X. When F is a divisor on X, we write $\Delta = \Delta_1 + aF$, where $F \notin \mathrm{Supp}(\Delta_1)$; otherwise let $\Delta_1 = \Delta$.

1. F is said to be primitive over X if there exists a projective birational morphism $\pi : Y \to X$ such that Y is normal, F is a prime divisor on Y and $-F$ is a π-ample $\mathbb{Q}$-Cartier divisor. We call $\pi : Y \to X$ the associated prime blowup (it is uniquely determined by F).
2. F is said to be of plt type if it is primitive over X and the pair $(Y, \Delta_Y + F)$ is plt in a neighbourhood of F, where $\pi : Y \to X$ is the associated prime blowup and Δ_Y is the strict transform of Δ_1 on Y. When (X, Δ) is klt and F is exceptional over X, π is called a plt blowup over X.

Lemma 2.1. *Let $(Y, F + \Delta)$ be a plt pair with $\lfloor F + \Delta \rfloor = F$. Then for any $\mathbb{Q}$-Cartier Weil divisor D on Y, there exists a uniquely determined $\mathbb{Q}$-divisor class (i.e., $\mathbb{Q}$-divisor up to $\mathbb{Z}$-linear equivalence) $D|_F$ on F and a canonical isomorphism*

$$\mathcal{O}_Y(D)/\mathcal{O}_Y(D - F) \cong \mathcal{O}_F(D|_F).$$

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Proof. The $\mathbb{Q}$-divisor class $D|_F$ is defined in [26, Definition A.2 and A.4] by localising at every codimension 1 point of F, and the isomorphism is established by [26, Lemma A.3]. □

2.4. Filtrations and admissible flags

We recall the notation of filtrations as well as some constructions from the study of Okounkov bodies.

Definition 2.7. Let V be a finite dimensional vector space. A filtration $\mathcal{F}$ on V is given by a collection of subspaces $\mathcal{F}^\lambda V$ indexed by a totally ordered abelian monoid Λ (in which case we also call the filtration a Λ-filtration) such that $\mathcal{F}^{\lambda_0} V = V$, $\mathcal{F}^{\lambda_1} V = 0$ for some $\lambda_0, \lambda_1 \in \Lambda$ and $\mathcal{F}^\lambda V \subseteq \mathcal{F}^{\lambda'} V$ whenever $\lambda \geq \lambda'$. When $\Lambda = \mathbb{R}$, we will also require that the filtration is left continuous: that is, for any $\lambda \in \mathbb{R}$, we have $\mathcal{F}^{\lambda-\varepsilon} V = \mathcal{F}^\lambda V$ for all $0 < \varepsilon \ll 1$. For each $\lambda \in \Lambda$, we set $\mathrm{Gr}^\lambda_{\mathcal{F}} V = \mathcal{F}^\lambda V / \cup_{\mu>\lambda} \mathcal{F}^\mu V$. A basis $s_1, \cdots, s_N$ (where $N = \dim V$) of V is said to be compatible with $\mathcal{F}$ if every $\mathcal{F}^\lambda V$ is the span of some s_i.

Most filtrations we use are induced by a divisor or an admissible flag.

Example 2.2. Let L be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, and let $V \subseteq H^0(X, L)$ be a subspace. Let E be a divisor over X. Then it induces an $\mathbb{R}$-filtration $\mathcal{F}_E$ on V by setting

$$\mathcal{F}^\lambda_E V := \{s \in V \,|\, \mathrm{ord}_E(s) \geq \lambda\}.$$

More generally, every valuation v on X induces a filtration $\mathcal{F}_v$ on V with $\mathcal{F}^\lambda_v V := \{s \in V \,|\, v(s) \geq \lambda\}$.

Definition 2.8 [33]. Let X be a variety. An admissible flag $Y_\bullet$ over X of length $\ell \leq \dim X$ is defined as a flag of subvarieties

$$Y_\bullet \quad : \quad Y = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_\ell$$

on some projective birational model $\pi : Y \to X$ of X, where each Y_i is an (irreducible) subvariety of codimension i in Y that is smooth at the generic point of Y_ℓ.

Given an admissible flag $Y_\bullet$ over X as above and a $\mathbb{Q}$-divisor L on X that is Cartier at the generic point of Y_ℓ, one can define a valuation-like function

$$\nu = \nu_{Y_\bullet} = \nu_{Y_\bullet, X} : \Big(H^0(X, L) \setminus \{0\}\Big) \to \mathbb{N}^\ell, \quad s \mapsto \nu(s) = (\nu_1(s), \cdots, \nu_\ell(s)) \tag{2.3}$$

as follows. First, $\nu_1 = \nu_1(s) = \mathrm{ord}_{Y_1}(s)$; over an open neighbourhood $U \subseteq Y$ of the generic point of Y_ℓ, s naturally determines a section $\tilde{s}_1 \in H^0(U, \mathcal{O}_U(\pi^* L - \nu_1 Y_1))$ that restricts to a nonzero section $s_1 \in H^0(Y_1 \cap U, \mathcal{O}_{Y_1 \cap U}(\pi^* L - \nu_1 Y_1))$. We set $\nu_2(s) = \mathrm{ord}_{Y_2}(s_1)$ and continue in this way to define the remaining $\nu_i(s)$ inductively. Via the lexicographic ordering on $\mathbb{Z}^\ell$, every flag $Y_\bullet$ over X induces a filtration $\mathcal{F}_{Y_\bullet}$ (indexed by $\mathbb{N}^\ell$) on $V = H^0(X, L)$ by setting

$$\mathcal{F}^\lambda_{Y_\bullet} V = \{s \in V \,|\, \nu(s) \geq \lambda\}.$$

We also define the graded semigroup of L (with respect to $Y_\bullet$) as the subsemigroup

$$\Gamma(L) = \Gamma_{Y_\bullet}(L) = \{(m, \nu_{Y_\bullet}(s)) \,|\, m \in \mathbb{N}, 0 \neq s \in H^0(X, mL)\}$$

of $\mathbb{N} \times \mathbb{N}^\ell = \mathbb{N}^{\ell+1}$. The Okounkov body $\Delta(L) = \Delta_{Y_\bullet}(L)$ of L is then the base of the closed convex cone $\Sigma(L) = \Sigma_{Y_\bullet}(L) \subseteq \mathbb{R}^{\ell+1}$ spanned by $\Gamma(L)$: that is, $\Delta(L) = \Sigma(L) \cap (\{1\} \times \mathbb{R}^\ell)$.

For later use, we introduce some more notation. For a subspace $V \subseteq H^0(X, L)$ and an effective Weil divisor E on some birational model $\pi : Y \to X$ of X, we set $V(-E) := V \cap H^0(Y, \pi^* L(-E)) \subseteq H^0(X, L)$. Let $Y_\bullet$ is an admissible flag over X of length r. Assume that L is Cartier and that each Y_i in the flag is a

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Cartier divisor in Y_{i-1}. Then for every s-tuple $(1 \leq s \leq \ell)$ of integers $\vec{a} = (a_1, \cdots, a_s) \in \mathbb{N}^s$, following [28], we define

$$V(\vec{a}) \subseteq H^0(Y_s, L \otimes \mathcal{O}_{Y_s}(-a_1Y_1 - a_2Y_2 - \cdots - a_sY_s))$$

inductively so that $V(a_1) = V(-a_1Y_1)|_{Y_1}$ and

$$V(a_1, \cdots, a_s) = V(a_1, \cdots, a_{s-1})(-a_sY_s)|_{Y_s} \ (2 \leq s \leq \ell).$$

Note that $\mathcal{F}_{Y_\bullet}$ induces a filtration on $V(a_1, \cdots, a_s)$ indexed by $\mathbb{N}^{\ell-s}$.

2.5. Multigraded linear series

Definition 2.9 [33, §4.3]. Let $L_1, \cdots, L_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. An $\mathbb{N}^r$-graded linear series $W_{\vec{\bullet}}$ on X associated to the L_i's consists of finite-dimensional subspaces

$$W_{\vec{a}} \subseteq H^0(X, \mathcal{O}_X(a_1L_1 + \cdots + a_rL_r))$$

for each $\vec{a} \in \mathbb{N}^r$ such that $W_{\vec{0}} = \mathbb{C}$ and $W_{\vec{a}_1} \cdot W_{\vec{a}_2} \subseteq W_{\vec{a}_1+\vec{a}_2}$ for all $\vec{a}_1, \vec{a}_2 \in \mathbb{N}^r$. The support $\mathrm{Supp}(W_{\vec{\bullet}}) \subseteq \mathbb{R}^r$ of $W_{\vec{\bullet}}$ is defined as the closed convex cone spanned by all $\vec{a} \in \mathbb{N}^r$ such that $W_{\vec{a}} \neq 0$. We say that $W_{\vec{\bullet}}$ has bounded support if $\mathrm{Supp}(W_{\vec{\bullet}}) \cap (\{1\} \times \mathbb{R}^{r-1})$ is bounded. For such $W_{\vec{\bullet}}$, we set

$$h^0(W_{m,\vec{\bullet}}) := \sum_{\vec{a} \in \mathbb{N}^{r-1}} \dim(W_{m,\vec{a}})$$

for each $m \in \mathbb{N}$ (it is a finite sum when $W_{\vec{\bullet}}$ has bounded support) and define the volume of $W_{\vec{\bullet}}$ as (where $n = \dim X$)

$$\mathrm{vol}(W_{\vec{\bullet}}) := \limsup_{m \to \infty} \frac{h^0(W_{m,\vec{\bullet}})}{m^{n+r-1}/(n+r-1)!}.$$

We say that $W_{\vec{\bullet}}$ contains an ample series if the following conditions are satisfied:

1. $\mathrm{Supp}(W_{\vec{\bullet}}) \subseteq \mathbb{R}^r$ contains a nonempty interior;
2. for any $\vec{a} \in \mathrm{int}(\mathrm{Supp}(W_{\vec{\bullet}})) \cap \mathbb{N}^r$, $W_{k\vec{a}} \neq 0$ for $k \gg 0$;
3. there exists some $\vec{a}_0 \in \mathrm{int}(\mathrm{Supp}(W_{\vec{\bullet}})) \cap \mathbb{N}^r$ and a decomposition $\vec{a}_0 \cdot \vec{L} = A + E$ (where $\vec{L} = (L_1, \cdots, L_r)$) with A an ample $\mathbb{Q}$-line bundle and E an effective $\mathbb{Q}$-divisor such that $H^0(X, mA) \subseteq W_{m\vec{a}_0}$ for all sufficiently divisible m.

If $Y_\bullet$ is an admissible flag of length ℓ over X such that $L_1, \cdots, L_r$ are *Cartier* at the generic point of Y_ℓ, the multigraded semigroup of $W_{\vec{\bullet}}$ with respect to $Y_\bullet$ is defined to be

$$\Gamma(W_{\vec{\bullet}}) = \Gamma_{Y_\bullet}(W_{\vec{\bullet}}) := \{(\vec{a}, \nu(s)) \mid 0 \neq s \in W_{\vec{a}}\} \subseteq \mathbb{N}^r \times \mathbb{N}^\ell = \mathbb{N}^{r+\ell}.$$

Remark 2.3. Note that the above definition is slightly more general than [33] since we allow divisors L_i that may not be Cartier or integral. However, most results of [33, §4.3] carry over to our setting. In particular, when $W_{\vec{\bullet}}$ contains an ample series, one can verify as in [33, Lemma 4.20] that $\Gamma(W_{\vec{\bullet}})$ generates $\mathbb{Z}^{r+\ell}$ as a group. If in addition W has bounded support, then we can define the associated Okounkov body $\Delta(W_{\vec{\bullet}}) = \Delta_{Y_\bullet}(W_{\vec{\bullet}})$ as $\Sigma(W_{\vec{\bullet}}) \cap (\{1\} \times \mathbb{R}^{r-1+\ell})$, where $\Sigma(W_{\vec{\bullet}})$ is the closed convex cone spanned by $\Gamma(W_{\vec{\bullet}})$. When $\ell = n = \dim X$, we let $\Gamma_m = \Gamma(W_{\vec{\bullet}}) \cap (\{m\} \times \mathbb{N}^{r-1+n})$ and let

$$\rho_m = \frac{1}{m^{r-1+n}} \sum_{a \in \Gamma_m} \delta_{m^{-1}a} \tag{2.4}$$

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be the atomic positive measure on $\Delta(W_\bullet)$. Then by [7, Théorème 1.12], ρ_m converges weakly as $m \to \infty$ to the Lebesgue measure on $\Delta(W_\bullet)$. In particular, we have $\mathrm{vol}(W_\bullet) = (n+r-1)! \cdot \mathrm{vol}(\Delta(W_\bullet))$ as in [33, Theorem 2.13]. By [33, Corollary 4.22], there is also a continuous function

$$\mathrm{vol}_{W_\bullet} : \mathrm{int}(\mathrm{Supp}(W_\bullet)) \to \mathbb{R} \tag{2.5}$$

such that for any integer vector $\vec{a} \in \mathrm{int}(\mathrm{Supp}(W_\bullet))$, $\mathrm{vol}_{W_\bullet}(\vec{a})$ equals the volume of the graded linear series $\{W_{m\vec{a}}\}_{m\in\mathbb{N}}$.

We give some examples of multigraded linear series that naturally arise in our later analysis (i.e., when applying inversion of adjunction to basis type divisors compatible with a given divisor or admissible flag). The following lemma ensures that the graded linear series we construct contains an ample series.

Lemma 2.4. *Let $W_\bullet$ be an $\mathbb{N}^r$-graded linear series on X with bounded support and containing an ample series. Then for any admissible flag $Y_\bullet$ of length ℓ over X such that $L_1, \cdots, L_r$ are Cartier at the generic point of Y_ℓ and any $\gamma \in \mathrm{int}(\Sigma(W_\bullet)) \cap \mathbb{N}^{r+\ell}$, we have $k\gamma \in \Gamma(W_\bullet)$ when $k \gg 0$.*

Proof. By [33, Lemma 4.20] as in the previous remark, the semigroup $\Gamma(W_\bullet)$ generates $\mathbb{Z}^{r+\ell}$ as a group. Let $\Gamma \subseteq \Gamma(W_\bullet)$ be a finitely generated subsemigroup that still generates $\mathbb{Z}^{r+\ell}$ and such that $\gamma \in \mathrm{int}(\Sigma)$, where $\Sigma \subseteq \Sigma(W_\bullet)$ is the subcone generated by Γ. By [29, Proposition 3], there exists some $\gamma_0 \in \Gamma$ such that

$$(\Sigma + \gamma_0) \cap \mathbb{N}^{r+\ell} \subseteq \Gamma \subseteq \Gamma(W_\bullet).$$

As $\gamma \in \mathrm{int}(\Sigma)$, we have $k\gamma \in \Sigma + \gamma_0$ when $k \gg 0$, and thus the lemma follows. □

Example 2.5. Let L be a big line bundle on X. The *complete linear series associated to* L is the $\mathbb{N}$-graded linear series $V_\bullet$ on X defined by $V_m = H^0(X, mL)$. It is clear that $V_\bullet$ has bounded support and contains an ample series.

Example 2.6. Let $L_1, \cdots, L_r$ be *Cartier* divisors on X, and let $V_\bullet$ be an $\mathbb{N}^r$-graded linear series associated to the L_is. Denote $\vec{L} = (L_1, \cdots, L_r)$. Let F be a primitive divisor over X with associated prime blowup $\pi : Y \to X$, and let $\mathcal{F}$ be the induced filtration on $V_\bullet$ (see Example 2.2). Assume that F is either Cartier on Y or of plt type. In the latter case, we define $F|_F$ as the $\mathbb{Q}$-divisor class given by Lemma 2.1. Then in both cases,

$$W_{\vec{a},j} = \mathcal{F}^j V_{\vec{a}} / \mathcal{F}^{j+1} V_{\vec{a}}$$

can be naturally identified with the image of $\mathcal{F}^j V_{\vec{a}}$ under the composition

$$\mathcal{F}^j V_{\vec{a}} \to H^0(Y, \pi^*(\vec{a} \cdot \vec{L}) - jF) \to H^0(F, \pi^*(\vec{a} \cdot \vec{L})|_F - jF|_F)$$

(this is clear if F is Cartier on Y; when F is of plt type, we use Lemma 2.1). It follows that $W_\bullet$ is an $\mathbb{N}^{r+1}$-graded linear series on F (associated to the divisors $\pi^*L_1|_F, \cdots, \pi^*L_r|_F$ and $-F|_F$), called the *refinement* of $V_\bullet$ by F. It is not hard to see that $W_\bullet$ has bounded support if $V_\bullet$ does (see, for example, [33, Remark 1.12]). We show that $W_\bullet$ contains an ample series if $V_\bullet$ does. Indeed, condition (1) and (3) are easy to verify as $V_\bullet$ contains an ample series. For condition (2), consider the admissible flag $Y_0 = Y$, $Y_1 = F$; then we see that $W_{\vec{a},j} \neq 0$ if and only if $(\vec{a}, j) \in \Gamma_{Y_\bullet}(V_\bullet)$, and hence condition (2) follows from Lemma 2.4.

Example 2.7. More generally, let $L_1, \cdots, L_r$ be Cartier divisors on X, let $V_\bullet$ be an $\mathbb{N}^r$-graded linear series associated to the L_i's, and let $Y_\bullet$ be an admissible flag of length ℓ over X. Assume that each Y_i in the flag is a Cartier divisor in Y_{i-1}. Then in the notation of Section 2.4,

$$W_{\vec{a},b_1,\cdots,b_\ell} = V_{\vec{a}}(b_1, \cdots, b_\ell)$$

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defines an $\mathbb{N}^{r+\ell}$-graded linear series on Y_ℓ. We call it the *refinement* of $V_\bullet$ by $Y_\bullet$. As in the previous example, one can check that $W_\bullet$ has bounded support (respectively, contains an ample series) if $V_\bullet$ does.

2.6. Invariants associated to filtered multigraded linear series

Definition 2.10. Let $W_\bullet$ be an $\mathbb{N}^r$-graded linear series. A filtration $\mathcal{F}$ on $W_\bullet$ (indexed by Λ) is given by a filtration on each $W_{\vec{a}}$ ($\vec{a} \in \mathbb{N}^r$) such that $\mathcal{F}^{\lambda_1} W_{\vec{a}_1} \cdot \mathcal{F}^{\lambda_2} W_{\vec{a}_2} \subseteq \mathcal{F}^{\lambda_1+\lambda_2} W_{\vec{a}_1+\vec{a}_2}$ for all $\lambda_i \in \Lambda$ and all $\vec{a}_i \in \mathbb{N}^r$. If $\Lambda \subseteq \mathbb{R}$, we say the filtration $\mathcal{F}$ is linearly bounded if there exist constants C_1 and C_2 such that $\mathcal{F}^\lambda W_{\vec{a}} = W_{\vec{a}}$ for all $\lambda < C_1|\vec{a}|$ and $\mathcal{F}^\lambda W_{\vec{a}} = 0$ for all $\lambda > C_2|\vec{a}|$.

One can generalise the definition of basis type divisors, S-invariants and stability thresholds to filtered multigraded linear series.

Definition 2.11. Let $W_\bullet$ be an $\mathbb{N} \times \mathbb{N}^r$-graded linear series with bounded support. Let $M(W_\bullet)$ be the set of $m \in \mathbb{N}_+$ such that $W_{m,\vec{a}} \neq 0$ for some $\vec{a} \in \mathbb{N}^r$. Let $m \in M(W_\bullet)$, and let $N_m = h^0(W_{m,\bullet})$. We say that D is an m-basis type divisor (respectively, $\mathbb{Q}$-divisor) of $W_\bullet$ if there exist basis type divisors $D_{\vec{a}}$ of $W_{m,\vec{a}}$ for each $\vec{a} \in \mathbb{N}^r$ such that

$$D = \sum_{\vec{a} \in \mathbb{N}^r} D_{\vec{a}} \quad \text{respectively, } D = \frac{1}{mN_m} \sum_{\vec{a} \in \mathbb{N}^r} D_{\vec{a}}.$$

When $r = 0$ and $W_\bullet$ is the complete linear series associated to L, this reduces to the usual definition of m-basis type ($\mathbb{Q}$-)divisors of L (compare to Section 2.2). Let $\mathcal{F}$ be a filtration on $W_\bullet$, and let D be an m-basis type ($\mathbb{Q}$-)divisor of $W_\bullet$. We say that D is compatible with $\mathcal{F}$ if all the $D_{\vec{a}}$ above has the form $D_{\vec{a}} = \sum_{i=1}^N \{s_i = 0\}$ for some basis s_i ($i = 1, \cdots, N$) of $W_{m,\vec{a}}$ that is compatible with $\mathcal{F}$. In particular, we say that D is compatible with a divisor E (respectively, an admissible flag $Y_\bullet$) if it is compatible with the filtration induced by E (respectively, $Y_\bullet$). Note that the divisor class $c_1(D) \in \mathrm{Cl}(X)_\mathbb{Q}$ of an m-basis type divisor does not depend on the choice of D. We denote it by $c_1(W_{m,\bullet})$.

Definition 2.12. Let (X, Δ) be a klt pair, and let Z be a closed subset of X. Let $W_\bullet$ be an $\mathbb{N} \times \mathbb{N}^r$-graded linear series on X with bounded support, let $\mathcal{F}, \mathcal{G}$ be filtrations on $W_\bullet$, and let $v \in \mathrm{Val}_X^*$ be a valuation on X. Assume that $\mathcal{G}$ is a linearly bounded, left continuous $\mathbb{R}$-filtration and $A_{X,\Delta}(v) < \infty$. Associated to $\mathcal{G}$, we have a valuation-like function $v_\mathcal{G} \colon W_\bullet \to \mathbb{R}$ given by

$$s \in W_{\vec{a}} \mapsto \sup\{\lambda \in \mathbb{R} \mid s \in \mathcal{G}^\lambda W_{\vec{a}}\}.$$

If $D = \frac{1}{mN_m} \sum_{i=1}^{N_m} \{s_i = 0\}$ is an m-basis type $\mathbb{Q}$-divisor D of $W_\bullet$, where each $s_i \in W_{m,\vec{a}}$ for some $\vec{a} \in \mathbb{N}^r$, then we define

$$v_\mathcal{G}(D) = \frac{1}{mN_m} \sum_{i=1}^{N_m} v_\mathcal{G}(s_i).$$

Clearly $v_\mathcal{G} = v$ if $\mathcal{G} = \mathcal{F}_v$ is the filtration induced by the valuation v. Similar to Section 2.2, for each $m \in M(W_\bullet)$, we set

$$S_m(W_\bullet, \mathcal{F}; \mathcal{G}) = \sup_D v_\mathcal{G}(D), \quad S_m(W_\bullet, \mathcal{F}; v) = S_m(W_\bullet, \mathcal{F}; \mathcal{F}_v) = \sup_D v(D),$$

where the supremum runs over all m-basis type $\mathbb{Q}$-divisors D of $W_\bullet$ that are compatible with $\mathcal{F}$. We also set

$$\delta_m(W_\bullet, \mathcal{F}) = \delta_m(X, \Delta; W_\bullet, \mathcal{F}) = \inf_D \mathrm{lct}(X, \Delta; D)$$
$$\delta_{Z,m}(W_\bullet, \mathcal{F}) = \delta_{Z,m}(X, \Delta; W_\bullet, \mathcal{F}) = \inf_D \mathrm{lct}_Z(X, \Delta; D),$$

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where the infimum runs over all m-basis type $\mathbb{Q}$-divisors D of $W_{\bullet}$ that are compatible with $\mathcal{F}$. We then define

$$S(W_{\bullet};\mathcal{F};\mathcal{G}) = \limsup_{m\to\infty} S_m(W_{\bullet};\mathcal{F};\mathcal{G}), \quad S(W_{\bullet};\mathcal{F};v) = S(W_{\bullet};\mathcal{F};\mathcal{F}_v)$$

and similarly the (adjoint) stability thresholds $\delta(W_{\bullet},\mathcal{F})$ (respectively, $\delta_Z(W_{\bullet},\mathcal{F})$) of a filtered multigraded linear series $W_{\bullet}$. If E is a divisor over X, we set $S(W_{\bullet};\mathcal{F};E) = S(W_{\bullet};\mathcal{F};\mathrm{ord}_E)$ and $S_m(W_{\bullet};\mathcal{F};E) = S_m(W_{\bullet};\mathcal{F};\mathrm{ord}_E)$. When the filtration $\mathcal{F}$ is trivial (i.e., $\mathcal{F}^{\lambda}W_{\vec{a}}$ equals $W_{\vec{a}}$ when $\lambda \leq 0$ and is 0 when $\lambda > 0$), we simply write $S(W_{\bullet};\mathcal{G})$, $\delta(W_{\bullet})$, $\delta_Z(W_{\bullet})$, and so on.

Remark 2.8. When L is a big line bundle on X and $W_{\bullet}$ is the complete linear series associated to L, we have $S(W_{\bullet};v) = S(L;v)$ for any valuation v on X; similarly, $\delta(W_{\bullet}) = \delta(L)$ and $\delta_Z(W_{\bullet}) = \delta_Z(L)$ for any closed subset $Z \subseteq X$.

The following statement is the direct generalisation of [2] to multigraded linear series.

Lemma 2.9. *Let (X,Δ) be a klt pair, and let $Z \subseteq X$ be a subvariety. Let $W_{\bullet}$ be an $\mathbb{N}\times\mathbb{N}^r$-graded linear series with bounded support that contains an ample series. Then $S(W_{\bullet};\mathcal{F}) = \lim_{m\to\infty} S_m(W_{\bullet};\mathcal{F})$ for any linearly bounded, left continuous $\mathbb{R}$-filtration $\mathcal{F}$ on $W_{\bullet}$, and we have*

$$\delta(W_{\bullet}) = \inf_E \frac{A_{X,\Delta}(E)}{S(W_{\bullet};E)} = \inf_v \frac{A_{X,\Delta}(v)}{S(W_{\bullet};v)} \quad \textit{respectively,}$$
$$\delta_Z(W_{\bullet}) = \inf_{E,Z\subseteq C_X(E)} \frac{A_{X,\Delta}(E)}{S(W_{\bullet};E)} = \inf_{v,Z\subseteq C_X(v)} \frac{A_{X,\Delta}(v)}{S(W_{\bullet};v)},$$

where the first infimum runs over all divisors E over X (respectively, all divisors E over X whose centre contains Z) and the second infimum runs over all valuations $v \in \mathrm{Val}_X^*$ (respectively, all valuations $v \in \mathrm{Val}_X^*$ whose centre contains Z) such that $A_{X,\Delta}(v) < \infty$. Moreover, it holds that

$$\delta(W_{\bullet}) = \lim_{m\to\infty} \delta_m(W_{\bullet}) \quad \textit{and} \quad \delta_Z(W_{\bullet}) = \lim_{m\to\infty} \delta_{Z,m}(W_{\bullet}).$$

In view of this lemma, we say that a divisor E over X (or a valuation $v \in \mathrm{Val}_X^*$) computes $\delta(W_{\bullet})$ (respectively, $\delta_Z(W_{\bullet})$) if it achieves the above infimum.

Proof. The argument is almost identical to those in [2] (which is in turn based on [8]). Using the filtration $\mathcal{F}$, we define a family $W_{\bullet}^t$ of multigraded linear series on X (indexed by $t \in \mathbb{R}$) where $W^t_{m,\vec{a}} = \mathcal{F}^{mt}W_{m,\vec{a}}$. Set

$$T_m(W_{\bullet};\mathcal{F}) = \max\{j \in \mathbb{N} \mid \mathcal{F}^j W_{m,\vec{a}} \neq 0 \text{ for some } \vec{a}\}.$$

It is easy to see that the sequence $T_m(W_{\bullet};\mathcal{F})$ is super-additive, and we set

$$T(W_{\bullet};\mathcal{F}) = \lim_{m\to\infty} \frac{T_m(W_{\bullet};\mathcal{F})}{m} = \sup_{m\in\mathbb{N}} \frac{T_m(W_{\bullet};\mathcal{F})}{m}.$$

One can check as in [8, Lemma 1.6] that for any $t < T(W_{\bullet};\mathcal{F})$, the multigraded linear series $W_{\bullet}^t$ contains an ample series. Therefore, for any fixed admissible flag $Y_{\bullet}$ of length $n = \dim X$ centered at a general point of X, we have the associated Okounkov bodies $\Delta^t = \Delta_{Y_{\bullet}}(W_{\bullet}^t)$ $(t \in \mathbb{R})$. The result is now simply a consequence of properties of Okounkov bodies. More precisely, consider the function $G\colon \Delta := \Delta^0 \to [0, T(W_{\bullet};\mathcal{F})]$ given by

$$G(\gamma) = \sup\{t \in \mathbb{R} \mid \gamma \in \Delta^t\}.$$

It is straightforward to check that G is concave and hence continuous in the interior of Δ. By the exact same proof of [2, Lemma 2.9] (using [8, Theorem 1.11]), we get the equality (where ρ is the

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Lebesgue measure on Δ^0)

$$S(W_{\bullet,\bullet};\mathcal{F}) = \frac{1}{\mathrm{vol}(\Delta)}\int_\Delta G\mathrm{d}\rho = \lim_{m\to\infty} S_m(W_{\bullet,\bullet};\mathcal{F})$$

and an estimate

$$S_m(W_{\bullet,\bullet};\mathcal{F}) \le \frac{m^{n+r}}{h^0(W_{m,\vec{\bullet}})}\int_\Delta G\mathrm{d}\rho_m,$$

where ρ_m is as in equation (2.4) (note that $\Delta = \Delta(W_{\bullet,\bullet})$). Applied to $\mathcal{F} = \mathcal{F}_v$, the argument of [2, Lemma 2.2 and Corollary 2.10] then implies that for any $\epsilon > 0$, there exists some $m_0 = m_0(\epsilon)$ such that $S_m(W_{\bullet,\bullet};v) \le (1+\epsilon)S(W_{\bullet,\bullet};v)$ for any valuation $v \in \mathrm{Val}_X^*$ with $A_{X,\Delta}(v) < \infty$ and any $m \ge m_0$ (the key point is that m_0 doesn't depend on v). The remaining equalities in the lemma now follow from the exact same proof of [2, Theorem 4.4]. □

The above proof also gives a formula for the S-invariants of multigraded linear series, similar to the one in Definition 2.2.

Corollary 2.13. *Notation as above. Then* $S(W_{\bullet,\bullet};\mathcal{F}) = \frac{1}{\mathrm{vol}(W_{\bullet,\bullet})}\int_0^\infty \mathrm{vol}(W^t_{\bullet,\bullet})\mathrm{d}t$.

Proof. We already have $S(W_{\bullet,\bullet};\mathcal{F}) = \frac{1}{\mathrm{vol}(\Delta)}\int_\Delta G\mathrm{d}\rho$. It is not hard to see that $\int_\Delta G\mathrm{d}\rho = \int_0^\infty \mathrm{vol}(\Delta^t)\mathrm{d}t$. Since $\mathrm{vol}(W_{\bullet,\bullet}) = (n+r)!\cdot\mathrm{vol}(\Delta)$ and $\mathrm{vol}(W^t_{\bullet,\bullet}) = (n+r)!\cdot\mathrm{vol}(\Delta^t)$ for all $t \ge 0$ (see Remark 2.3), the result follows. □

We also provide a more explicit formula for the volumes $\mathrm{vol}(W^t_{\bullet,\bullet})$. To this end, let $W_{\bullet,\bullet}$ and $\mathcal{F}$ be as in Lemma 2.9, let $\Delta^t_{\mathrm{supp}} = \mathrm{Supp}(W^t_{\bullet,\bullet}) \cap (\{1\}\times\mathbb{R}^r)$, and let

$$\mathrm{vol}_{W^t_{\bullet,\bullet}} : \mathrm{int}(\Delta^t_{\mathrm{supp}}) \to \mathbb{R}$$

be the volume function as in equation (2.5). Then we have

Lemma 2.10. $\mathrm{vol}(W^t_{\bullet,\bullet}) = \frac{(n+r)!}{n!}\int_{\Delta^t_{\mathrm{supp}}} \mathrm{vol}_{W^t_{\bullet,\bullet}}(\gamma)\mathrm{d}\gamma$.

Proof. Let $\mathrm{pr}\colon \mathbb{R}^{r+1+n} \to \mathbb{R}^{r+1}$ be the projection to the first $r+1$ coordinates that induces a map $p\colon \Delta^t \to \Delta^t_{\mathrm{supp}}$. By [33, Theorem 2.13 and 4.21], we know that $\mathrm{vol}(W^t_{\bullet,\bullet}) = (n+r)!\cdot\mathrm{vol}(\Delta^t)$ and $\mathrm{vol}_{W^t_{\bullet,\bullet}}(\gamma) = n!\cdot\mathrm{vol}(p^{-1}(\gamma))$ for all $\gamma \in \mathrm{int}(\Delta^t_{\mathrm{supp}})$. The lemma then follows from the obvious identity $\mathrm{vol}(\Delta^t) = \int_{\Delta^t_{\mathrm{supp}}} \mathrm{vol}(p^{-1}(\gamma))\mathrm{d}\gamma$. □

Recall that for any $\mathbb{Q}$-Cartier big divisor L on X and any integer $k > 0$, we have $\delta(kL) = \frac{1}{k}\delta(L)$. This can be generalised to multigraded linear series as follows. Let $L_1,\cdots,L_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X, and let $W_{\bullet,\bullet}$ be an $\mathbb{N}^r$-graded linear series associated to them. Let $k > 0$ be an integer such that kL_i is Cartier for all $1 \le i \le r$. Set $W'_{\vec{a}} = W_{k\vec{a}}$ ($\vec{a} \in \mathbb{N}^r$); then $W'_{\bullet,\bullet}$ is an $\mathbb{N}^r$-graded linear series associated to $kL_1,\cdots,kL_r$.

Lemma 2.11. *In the above notation, assume that $W_{\bullet,\bullet}$ contains an ample series and has bounded support. Then*

1. $S(W'_{\bullet,\bullet};v) = k\cdot S(W_{\bullet,\bullet};v)$ *for any valuation v on X;*
2. $\delta(W_{\bullet,\bullet}) = k\cdot\delta(W'_{\bullet,\bullet})$ *and* $\delta_Z(W_{\bullet,\bullet}) = k\cdot\delta_Z(W'_{\bullet,\bullet})$ *for any subvariety Z of X.*

In particular, this implies that for the calculation of stability thresholds, we only need to consider multigraded linear series associated to *Cartier divisors*.

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Proof. We use the same notation as in the proof of Lemma 2.9 and let Δ', G', and so on be the counterparts on $W'_\bullet$. Let $f\colon \mathbb{R}^{r+n} \to \mathbb{R}^{r+n}$ be given by

$$(x_1, \cdots, x_{r+n}) \mapsto (kx_1, \cdots, kx_r, x_{r+1}, \cdots, x_{r+n}).$$

We claim that

$$\Sigma(W_\bullet) = f(\Sigma(W'_\bullet)). \tag{2.6}$$

Indeed, it is clear from the construction that $f(\Gamma(W'_\bullet)) \subseteq \Gamma(W_\bullet)$, hence $f(\Sigma(W'_\bullet)) \subseteq \Sigma(W_\bullet)$. On the other hand, from the proof of Lemma 2.4, we know that there exists some $\gamma_0 \in \Gamma(W_\bullet)$ such that

$$(\Sigma(W_\bullet) + \gamma_0) \cap \mathbb{N}^{r+n} \subseteq \Gamma(W_\bullet),$$

hence as $f(\Gamma(W'_\bullet)) = f(\mathbb{N}^{r+n}) \cap \Gamma(W_\bullet)$, we have $(\Sigma(W_\bullet) + \gamma_0) \cap f(\mathbb{N}^{r+n}) \subseteq f(\Gamma(W'_\bullet))$ and therefore $\Sigma(W_\bullet) \subseteq f(\Sigma(W'_\bullet))$, which proves the claim.

It follows from equation (2.6) that $\Delta(W_\bullet) = \frac{1}{k} f(\Delta(W'_\bullet))$ (recall that we identify $\Delta(W_\bullet)$ as a subset of $\{1\} \times \mathbb{R}^{r-1+n}$). Replace $W_\bullet$ with $W_\bullet^{t/k}$, noting that $W''_{m,\vec{a}} := \mathcal{F}^{mt} W'_{m,\vec{a}} = W^{t/k}_{km,k\vec{a}}$, and we deduce $\Delta^{t/k} = \frac{1}{k} f(\Delta'')$. Hence $\Delta = \frac{1}{k} f(\Delta')$ and

$$G\left(\frac{f(\gamma)}{k}\right) = \frac{G'(\gamma)}{k} \tag{2.7}$$

for any $\gamma \in \Delta'$. Substitute it into the equality $S(W_\bullet; v) = \frac{1}{\mathrm{vol}(\Delta)} \int_\Delta G \mathrm{d}\rho$ from the proof of Lemma 2.9, we obtain $S(W_\bullet; v) = \frac{1}{k} S(W'_\bullet; v)$. The remaining parts of the lemma now follow immediately from Lemma 2.9. □

To further analyse basis type divisors of $W_\bullet$, for each $\vec{a} \in \mathbb{N}^{r+1}$ with $W_{\vec{a}} \neq 0$, we let $M_{\vec{a}}$ (respectively, $F_{\vec{a}}$) be the movable (respectively, fixed) part of the linear system $|W_{\vec{a}}|$. Thus we have a decomposition $|W_{\vec{a}}| = |M_{\vec{a}}| + F_{\vec{a}}$. For each $m \in M(W_\bullet)$, let

$$F_m = F_m(W_\bullet) := \frac{1}{m \cdot h^0(W_{m,\bullet})} \sum_{\vec{a} \in \mathbb{N}^r} \dim(W_{m,\vec{a}}) \cdot F_{m,\vec{a}}.$$

Then it is clear that every m-basis type $\mathbb{Q}$-divisor D of $W_\bullet$ can be decomposed as $D = D' + F_m$, where D' is an m-basis type $\mathbb{Q}$-divisor of $M_\bullet$ (the definition of basis type divisors works for any collection of linear series indexed by $\mathbb{N} \times \mathbb{N}^r$). We next study the asymptotic behaviour of D' and F_m.

Lemma-Definition 2.14. *Let $L_0, \cdots, L_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X, and let $W_\bullet$ be an associated $\mathbb{N} \times \mathbb{N}^r$-graded linear series that has bounded support and contains an ample series. Then in the notation of Definition 2.11, the limit*

$$c_1(W_\bullet) := \lim_{m \to \infty} \frac{c_1(W_{m,\bullet})}{m \cdot h^0(W_{m,\bullet})}$$

exists in $\mathrm{Pic}(X)_\mathbb{R}$. *Similarly,* $\lim_{m\to\infty} \mathrm{ord}_D F_m$ *exists for any prime divisor $D \subseteq X$. We will formally write*

$$F(W_\bullet) := \sum_D (\lim_{m \to \infty} \mathrm{ord}_D(F_m)) \cdot D.$$

When this is a finite sum, we set $c_1(M_\bullet) := c_1(W_\bullet) - F(W_\bullet) \in \mathrm{Cl}(X)_\mathbb{R}$.

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Proof. Let $\vec{L} = (L_1, \cdots, L_r)$. In the notation of Definitions 2.9 and 2.11, we have

$$\frac{c_1(W_{m,\vec{\bullet}})}{m \cdot h^0(W_{m,\vec{\bullet}})} = L_0 + \frac{\sum_{\vec{a}\in\mathbb{N}^r} h^0(W_{m,\vec{a}}) \cdot (\vec{a} \cdot \vec{L})}{m \cdot h^0(W_{m,\vec{\bullet}})}.$$

Thus for $c_1(W_{\vec{\bullet}})$, it suffices to show that $\lim_{m\to\infty} \frac{\sum_{\vec{a}\in\mathbb{N}^r} h^0(W_{m,\vec{a}})\cdot a_i}{m\cdot h^0(W_{m,\vec{\bullet}})}$ exists for each $1 \le i \le r$. In the notation of Remark 2.3, we have

$$\frac{\sum_{\vec{a}\in\mathbb{N}^r} h^0(W_{m,\vec{a}}) \cdot a_i}{m \cdot h^0(W_{m,\vec{\bullet}})} = \frac{\int x_i \mathrm{d}\rho_m}{\int \mathrm{d}\rho_m},$$

where x_i denotes the ith entry of an element of $\mathbb{R}^{r+n}$. Hence by [7, Théorème 1.12], the limit exists and equals $\frac{1}{\mathrm{vol}(\Delta)} \int_\Delta x_i \mathrm{d}\rho$, where $\Delta = \Delta(W_{\vec{\bullet}})$.

For $F(W_{\vec{\bullet}})$, it suffices to show that $\lim_{m\to\infty} \mathrm{ord}_D(F_m)$ exists for any prime divisor D. First note that since $W_{\vec{\bullet}}$ has bounded support, there exists some constant $C_1 > 0$ such that $|\vec{a}| \le C_1 m$ for any $\vec{a} \in \mathbb{N}^r$ with $W_{m,\vec{a}} \neq 0$. Thus as $mL_0 + \vec{a} \cdot \vec{L} - F_{m,\vec{a}}$ is effective, we further deduce $\mathrm{ord}_D(F_{m,\vec{a}}) \le Tm$ for some absolute constant T. Let $\Delta_0 := \mathrm{int}(\mathrm{Supp}(W_{\vec{\bullet}})) \cap (\{1\} \times \mathbb{R}^r) \subseteq \mathbb{R}^r$. Since $W_{\vec{a}} \cdot W_{\vec{a}'} \subseteq W_{\vec{a}+\vec{a}'}$, we have $F_{\vec{a}} + F_{\vec{a}'} \ge F_{\vec{a}+\vec{a}'}$ (whenever $W_{\vec{a}}, W_{\vec{a}'} \neq 0$); thus if we let

$$f_{W_{\vec{\bullet}},D}(\gamma) := \inf_m \frac{\mathrm{ord}_D(F_{m,m\vec{\gamma}})}{m} = \lim_{m\to\infty} \frac{\mathrm{ord}_D(F_{m,m\vec{\gamma}})}{m}$$

for $\gamma \in \Delta_0 \cap \mathbb{Q}^r$, where the infimum and limit are taken over sufficiently divisible integers m, then $f_{W_{\vec{\bullet}},D}(t\gamma_1 + (1-t)\gamma_2) \le t f_{W_{\vec{\bullet}},D}(\gamma_1) + (1-t) f_{W_{\vec{\bullet}},D}(\gamma_2)$ for any $\gamma_1, \gamma_2 \in \Delta_0$. Therefore it naturally extends to a convex (and hence continuous) function $f_{W_{\vec{\bullet}},D}$ on Δ_0. For simplicity, we denote $f_{W_{\vec{\bullet}},D}$ by f. By the previous discussion, $f(\gamma) \le T$ for all $\gamma \in \Delta_0$.

We claim that $f(\gamma) = \lim_{m\to\infty} f_m(\gamma)$ for any $\gamma \in \Delta_0$ where

$$f_m(\gamma) := \begin{cases} \frac{1}{m}\mathrm{ord}_D(F_{m,\lfloor m\vec{\gamma}\rfloor}) & \text{if } W_{m,\lfloor m\vec{\gamma}\rfloor} \neq 0 \\ T & \text{if } W_{m,\lfloor m\vec{\gamma}\rfloor} = 0. \end{cases}$$

Indeed, as $f_m(\gamma) \ge f(\frac{\lfloor m\vec{\gamma}\rfloor}{m})$ $(m \gg 0)$ by definition, we have

$$\liminf_{m\to\infty} f_m(\gamma) \ge \lim_{m\to\infty} f(\frac{\lfloor m\vec{\gamma}\rfloor}{m}) = f(\gamma).$$

To get the reverse direction, let $\varepsilon > 0$ and choose $\gamma_i \in \Delta_0 \cap \mathbb{Q}^r$ $(i = 0, \cdots, r)$ that are sufficiently close to γ such that their convex hull contains γ in the interior and $f(\gamma_i) < f(\gamma) + \varepsilon$. Then we may choose some sufficiently divisible $m_0 \in \mathbb{N}$ such that $f_{m_0}(\gamma_i) < f(\gamma) + \varepsilon$. Let $\Pi \subseteq \mathbb{R}^{r+1}$ be the cone spanned by all the γ_is. From the proof of Lemma 2.4, we know that there exists some $\vec{a}_0 \in \mathbb{N}^{r+1}$ such that $W_{\vec{a}} \neq 0$ for all $\vec{a} \in (\Pi + \vec{a}_0) \cap \mathbb{N}^{r+1}$ (consider the semigroup $\{\vec{a} \mid W_{\vec{a}} \neq 0\} \subseteq \mathbb{N}^{r+1}$, choose a finitely generated subsemigroup that generates $\mathbb{Z}^{r+1}$ such that the cone it spans contains Π, and apply [29, Proposition 3]). Then one can verify that there exists some constant $C > 0$ such that for all $m \gg 0$, we have

$$(m, \lfloor m\vec{\gamma}\rfloor) = \vec{a} + \sum_{i=0}^{r} k_i(m_0, m_0\gamma_i)$$

for some $k_i \in \mathbb{N}$ and some $\vec{a} \in \mathbb{N}^{r+1}$ satisfying $W_{\vec{a}} \neq 0$ and $|\vec{a}| \le C$. In particular, $|m - m_0 \sum k_i| \le C$ and $\mathrm{ord}_D(W_{\vec{a}}) \le CT$. It follows that

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$$\begin{aligned}\mathrm{ord}_D(F_{m,\lfloor m\vec{\gamma}\rfloor}) &\le \mathrm{ord}_D(W_{\vec{a}}) + \sum_{i=0}^{r} k_i \mathrm{ord}_D(F_{m_0,m_0\gamma_i}) = \mathrm{ord}_D(W_{\vec{a}}) + \sum_{i=0}^{r} k_i m_0 f_{m_0}(\gamma_i)\\ &\le \mathrm{ord}_D(W_{\vec{a}}) + (m-C)(f(\gamma)+\varepsilon) \le CT + (m-C)(f(\gamma)+\varepsilon).\end{aligned}$$

Hence $\limsup_{m\to\infty} f_m(\gamma) \le f(\gamma)+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\limsup_{m\to\infty} f_m(\gamma) \le f(\gamma)$, and this proves the claim. Note that the argument also shows $W_{m,\lfloor m\vec{\gamma}\rfloor} \neq 0$ for $m \gg 0$.

It is clear that

$$\mathrm{ord}_D(F_m) = \frac{\int (f_m \circ p)\mathrm{d}\rho_m}{\int \mathrm{d}\rho_m},$$

where $p\colon \Delta = \Delta(W_\bullet) \to \Delta_0$ is the natural projection. By dominated convergence and the above claim, the latter limit exists and equals $\frac{1}{\mathrm{vol}(\Delta)}\int_\Delta (f\circ p)\mathrm{d}\rho$. □

For later calculations, we extract a formula for $F(W_\bullet)$ from the above proof.

Corollary 2.15. *Let $W_\bullet$ be an $\mathbb{N}\times\mathbb{N}^r$-graded linear series on X that has bounded support and contains an ample series, and let D be a prime divisor. Then*

$$\mathrm{ord}_D(F(W_\bullet)) = \frac{(n+r)!}{n!}\cdot\frac{1}{\mathrm{vol}(W_\bullet)}\int_{\Delta_{\mathrm{supp}}} f(\gamma)\mathrm{vol}_{W_\bullet}(\gamma)\mathrm{d}\gamma,$$

where $\Delta_{\mathrm{supp}} = \mathrm{Supp}(W_\bullet)\cap(\{1\}\times\mathbb{R}^r)$, $f(\gamma) = f_{W_\bullet,D}(\gamma) := \lim_{m\to\infty}\frac{1}{m}\mathrm{ord}_D(F_{m,\lfloor m\vec{\gamma}\rfloor})$, $n = \dim X$ and $\mathrm{vol}_{W_\bullet}(\cdot)$ is as in equation (2.5).

Proof. The above proof gives $\mathrm{ord}_D(F(W_\bullet)) = \frac{1}{\mathrm{vol}(\Delta)}\int_\Delta (f\circ p)\mathrm{d}\rho$. We have $\mathrm{vol}(W_\bullet) = (n+r)!\cdot\mathrm{vol}(\Delta)$, $p(\Delta) = \Delta_{\mathrm{supp}}$ and $\mathrm{vol}_{W_\bullet}(\gamma) = n!\cdot\mathrm{vol}(p^{-1}(\gamma))$ for any $\gamma\in\mathrm{int}(\Delta_{\mathrm{supp}})$. These together imply the given formula. □

Most multigraded linear series considered in this paper come from the refinement of some complete linear series by a divisor or a flag. To simplify computations, we often carefully choose the divisor (or flag) so that the corresponding multigraded linear series behaves like complete linear systems associated to multiples of a fixed line bundle.

Definition 2.16. Let L be a big line bundle on X, and let $W_\bullet$ be an $\mathbb{N}\times\mathbb{N}^r$-graded linear series. We say that $W_\bullet$ is *almost complete* (with respect to L) if the following two conditions are both satisfied:

1. there are at most finitely many prime divisors $D\subseteq X$ with $\mathrm{ord}_D(F(W_\bullet)) > 0$ (so that $F(W_\bullet)$ is an $\mathbb{R}$-divisor);
2. for every $\vec{\gamma}\in\mathbb{Q}^r$ in the interior of $\Delta_{\mathrm{supp}} := \mathrm{Supp}(W_\bullet)\cap(\{1\}\times\mathbb{R}^r)$ and all sufficiently divisible integers m (depending on $\vec{\gamma}$), we have $|M_{m,m\vec{\gamma}}|\subseteq|L_{m,\vec{\gamma}}|$ for some $L_{m,\vec{\gamma}}\equiv\ell_{m,\vec{\gamma}}L$ and some $\ell_{m,\vec{\gamma}}\in\mathbb{N}$ (where $M_\bullet$ is the movable part of $W_\bullet$) such that

$$\frac{h^0(W_{m,m\vec{\gamma}})}{h^0(X,\ell_{m,\vec{\gamma}}L)} = \frac{h^0(M_{m,m\vec{\gamma}})}{h^0(X,\ell_{m,\vec{\gamma}}L)} \to 1$$

as $m\to\infty$.

Example 2.12. Let L be an ample line bundle on X, and let $H\in|L|$. Assume that H is irreducible and reduced. Let $V_\bullet$ be the complete linear series associated to rL for some positive integer r, and let $W_\bullet$ be its refinement by H (Example 2.6). Then the $\mathbb{N}^2$-graded linear series $W_\bullet$ is almost complete. Indeed, we have $W_{m,j} = |(mr-j)L|_H$; but since L is ample, the natural restriction $H^0(X,kL)\to H^0(H,kL|_H)$ is surjective when $k\gg 0$, hence $W_{m,j} = |(mr-j)L_0|$ (where $L_0 = L|_H$) and $F_{m,j} = 0$ when $mr-j\gg 0$, so the conditions of Definition 2.16 are satisfied and $F(W_\bullet) = 0$. More generally, if $Y_\bullet$ is an admissible

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flag *on* X (i.e., $Y_0 = X$) such that each Y_i is Cartier on Y_{i-1} and $Y_i \sim m_i L|_{Y_{i-1}}$ for some $m_i \in \mathbb{N}$, then the refinement of $V_{\bullet}$ by $Y_{\bullet}$ (Example 2.7) is almost complete as well.

Lemma 2.13. *Let L be a big line bundle on X, and let $W_{\bullet}$ be an $\mathbb{N} \times \mathbb{N}^r$-graded linear series. Assume that $W_{\bullet}$ has bounded support, contains an ample series and is almost complete with respect to L. Then*

1. *$F(W_{\bullet})$ is $\mathbb{R}$-Cartier (i.e., it is an $\mathbb{R}$-linear combination of Cartier divisors);*
2. *there exists a constant $\mu = \mu(X, L, W_{\bullet})$ such that $c_1(M_{\bullet}) = \mu L$ in $\mathrm{NS}(X)_{\mathbb{R}}$ and*

$$S(W_{\bullet}; v) = \mu \cdot S(L; v) + v(F(W_{\bullet})) \tag{2.8}$$

for all valuations $v \in \mathrm{Val}_X^*$ of linear growth.

Proof. Let $M(\gamma) := \lim_{m\to\infty} \frac{1}{m} c_1(M_{m,\lfloor m\vec{\gamma}\rfloor}) \in \mathrm{Cl}(X)_{\mathbb{R}}$ and $F(\gamma) = \lim_{m\to\infty} \frac{1}{m} F_{m,\lfloor m\vec{\gamma}\rfloor}$ for $\gamma \in \mathrm{int}(\Delta_{\mathrm{supp}})$. As in the previous proof, the limit exists: $M(\gamma) = \vec{\gamma} \cdot \vec{L} - \sum_D f_{W_{\bullet},D}(\gamma) \cdot D$ and $F(\gamma) = \sum_D f_{W_{\bullet},D}(\gamma) \cdot D$ in the notation of Corollary 2.15. Moreover, M is continuous, and we have

$$c_1(M_{\bullet}) = \frac{1}{\mathrm{vol}(\Delta)} \int_{\Delta} (M \circ p) \mathrm{d}\rho,$$

where $\Delta = \Delta(W_{\bullet})$ and $p \colon \Delta \to \Delta_{\mathrm{supp}}$ is the natural projection. Since $W_{\bullet}$ is almost complete, we see that $M(\gamma)$ is $\mathbb{R}$-Cartier and $M(\gamma) \equiv g(\gamma)L$ for some $g(\gamma) \in \mathbb{R}$. It follows that $c_1(M_{\bullet})$ is also $\mathbb{R}$-Cartier and $c_1(M_{\bullet}) = \mu L$ in $\mathrm{NS}(X)_{\mathbb{R}}$, where

$$\mu = \frac{1}{\mathrm{vol}(\Delta)} \int_{\Delta} (g \circ p) \mathrm{d}\rho = \frac{1}{\mathrm{vol}(\Delta)} \int_{\Delta_{\mathrm{supp}}} \mathrm{vol}(p^{-1}(\gamma)) \cdot g(\gamma) \mathrm{d}\gamma.$$

Since $F(W_{\bullet}) \sim_{\mathbb{R}} c_1(W_{\bullet}) - c_1(M_{\bullet})$, we also see that $F(W_{\bullet})$ is $\mathbb{R}$-Cartier. It remains to prove equation (2.8).

As $F(\gamma) \sim_{\mathbb{R}} \vec{\gamma} \cdot \vec{L} - M(\gamma)$ is also $\mathbb{R}$-Cartier, we may define $h(\gamma) = v(F(\gamma))$; and as in the proof of Corollary 2.15, we have

$$v(F(W_{\bullet})) = \frac{1}{\mathrm{vol}(\Delta)} \int_{\Delta} (h \circ p) \mathrm{d}\rho.$$

We claim that

$$\mathrm{vol}_{W_{\bullet}^t}(\gamma) = \mathrm{vol}(g(\gamma)L; v \geq t - h(\gamma)) \tag{2.9}$$

in the notation of Corollary 2.13 and Lemma 2.10. For this, we may assume that $\gamma \in \mathbb{Q}^r$. Let $\mathcal{F}$ be the filtration induced by v, and let m be a sufficiently divisible integer. From the exact sequence

$$0 \to \mathcal{F}^{\lambda} M_{m,m\vec{\gamma}} \to \mathcal{F}^{\lambda} H^0(X, L_{m,\vec{\gamma}}) \to H^0(X, L_{m,\vec{\gamma}})/M_{m,m\vec{\gamma}}$$

and the obvious equality

$$|\mathcal{F}^{mt} W_{m,m\vec{\gamma}}| = |\mathcal{F}^{mt - v(F_{m,m\vec{\gamma}})} M_{m,m\vec{\gamma}}| + F_{m,m\vec{\gamma}},$$

we deduce that

$$\left|\dim(\mathcal{F}^{mt} W_{m,m\vec{\gamma}}) - \dim(\mathcal{F}^{mt - v(F_{m,m\vec{\gamma}})} H^0(X, L_{m,\vec{\gamma}}))\right| \leq h^0(X, L_{m,\vec{\gamma}}) - h^0(M_{m,m\vec{\gamma}}). \tag{2.10}$$

By [32, Lemma 2.2.42], there exists a fixed effective divisor N on X such that $N \pm (L_{m,\vec{\gamma}} - \ell_{m,\vec{\gamma}} L)$ is effective. In particular, we have the inclusions

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$$H^0(X, \ell_{m,\vec{\gamma}}L - N) \hookrightarrow H^0(X, L_{m,\vec{\gamma}}) \hookrightarrow H^0(X, \ell_{m,\vec{\gamma}}L + N),$$

$$H^0(X, \ell_{m,\vec{\gamma}}L - N) \hookrightarrow H^0(X, \ell_{m,\vec{\gamma}}L) \hookrightarrow H^0(X, \ell_{m,\vec{\gamma}}L + N),$$

which implies

$$\lim_{m\to\infty} \frac{\dim(\mathcal{F}^{mt-v(F_{m,m\vec{\gamma}})}H^0(X, L_{m,\vec{\gamma}}))}{m^n/n!} = \lim_{m\to\infty} \frac{\dim(\mathcal{F}^{mt-v(F_{m,m\vec{\gamma}})}H^0(X, \ell_{m,\vec{\gamma}}L))}{m^n/n!}.$$

Thus as we divide equation (2.10) by $m^n/n!$, and letting $m \to \infty$, the right side of the inequality becomes 0 by the definition of almost completeness, and the equality equation (2.9) follows as $g(\gamma) = \lim_{m\to\infty} \frac{1}{m}\ell_{m,\vec{\gamma}}$.

By Corollary 2.13 and Lemma 2.10, we have

$$S(W_{\vec{\bullet}}; v) = \frac{1}{n!\mathrm{vol}(\Delta)} \iint_{\Delta_{\mathrm{supp}}\times\mathbb{R}_+} \mathrm{vol}_{W^t_{\vec{\bullet}}}(\gamma)\mathrm{d}t\mathrm{d}\gamma.$$

Combined with equation (2.9), we then obtain

$$\begin{aligned} S(W_{\vec{\bullet}}; v) &= \frac{1}{n!\mathrm{vol}(\Delta)} \iint_{\Delta_{\mathrm{supp}}\times\mathbb{R}_+} \mathrm{vol}(g(\gamma)L; v \geq t - h(\gamma))\mathrm{d}t\mathrm{d}\gamma \\ &= \frac{1}{n!\mathrm{vol}(\Delta)} \int_{\Delta_{\mathrm{supp}}} \left(\int_0^{h(\gamma)} + \int_{h(\gamma)}^{\infty}\right) \mathrm{vol}(g(\gamma)L; v \geq t - h(\gamma))\mathrm{d}t\mathrm{d}\gamma \\ &= \frac{1}{n!\mathrm{vol}(\Delta)} \int_{\Delta_{\mathrm{supp}}} \left(h(\gamma)\cdot g(\gamma)^n\mathrm{vol}(L) + \int_0^{\infty} g(\gamma)^{n+1}\mathrm{vol}(L; v \geq t)\mathrm{d}t\right)\mathrm{d}\gamma. \end{aligned}$$

Notice that $\mathrm{vol}_{W_{\vec{\bullet}}}(\gamma) = \mathrm{vol}_{W^0_{\vec{\bullet}}}(\gamma) = g(\gamma)^n\mathrm{vol}(L)$ by equation (2.9); thus we deduce that

$$\begin{aligned} S(W_{\vec{\bullet}}; v) &= \frac{1}{n!\mathrm{vol}(\Delta)} \int_{\Delta_{\mathrm{supp}}} \mathrm{vol}_{W_{\vec{\bullet}}}(\gamma)\cdot(h(\gamma) + g(\gamma)S(L; v))\mathrm{d}\gamma \\ &= \frac{1}{\mathrm{vol}(\Delta)} \int_{\Delta_{\mathrm{supp}}} \mathrm{vol}(p^{-1}(\gamma))\cdot(h(\gamma) + g(\gamma)S(L; v))\mathrm{d}\gamma \\ &= v(F(W_{\vec{\bullet}})) + \mu\cdot S(L; v). \end{aligned}$$

This finishes the proof. □

Corollary 2.17. *Let C be a smooth curve, and let $W_{\vec{\bullet}}$ be an almost complete multigraded linear series on C that has bounded support and contains an ample series. Then*

$$\delta_P(C; W_{\vec{\bullet}}) = \frac{2}{\deg(c_1(W_{\vec{\bullet}}) - F(W_{\vec{\bullet}})) + 2\cdot\mathrm{mult}_P F(W_{\vec{\bullet}})}$$

for all closed point $P \in C$. In particular, $\delta(C; W_{\vec{\bullet}}) = \frac{2}{\deg c_1(W_{\vec{\bullet}})}$ if $F(W_{\vec{\bullet}}) = 0$.

Proof. We have $S(L; P) = \frac{1}{2}\deg L$ for any ample line bundle L and any closed point P on C. Combining with Lemma 2.13, we see that $S(W_{\vec{\bullet}}; P) = S(c_1(W_{\vec{\bullet}}) - F(W_{\vec{\bullet}}); P) + \mathrm{mult}_P F(W_{\vec{\bullet}}) = \frac{1}{2}\deg(c_1(W_{\vec{\bullet}}) - F(W_{\vec{\bullet}})) + \mathrm{mult}_P F(W_{\vec{\bullet}})$. Since $\delta_P(C; W_{\vec{\bullet}}) = \frac{1}{S(W_{\vec{\bullet}}; P)}$ and $\delta(C; W_{\vec{\bullet}}) = \inf_{P\in C}\delta_P(C; W_{\vec{\bullet}})$ by definition, the result follows. □

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3. Adjunction for stability thresholds

In this section, we develop a framework to estimate stability thresholds. The starting point is the following elementary observation (compare to [9, Proposition 1.14]).

Lemma 3.1. *Let V be a finite-dimensional vector space, and let $\mathcal{F}$, $\mathcal{G}$ be two filtrations on V. Then there exists some basis $s_1, \cdots, s_N$ of V that is compatible with both $\mathcal{F}$ and $\mathcal{G}$.*

Proof. By enumerating all different subspaces $\mathcal{F}^\lambda V$ and $\mathcal{G}^\mu V$, we may assume that $\mathcal{F}$ and $\mathcal{G}$ are both $\mathbb{N}$-filtrations. Note that $\mathcal{F}$ (respectively, $\mathcal{G}$) induces a filtration (which is also denoted by $\mathcal{F}$, respectively $\mathcal{G}$) on each graded quotient $\mathrm{Gr}^i_{\mathcal{G}} V$ (respectively, $\mathrm{Gr}^j_{\mathcal{F}} V$). It is not hard to check that

$$\mathrm{Gr}^j_{\mathcal{F}}\mathrm{Gr}^i_{\mathcal{G}} V \cong (\mathcal{F}^j V \cap \mathcal{G}^i V)/(\mathcal{F}^{j+1}V \cap \mathcal{G}^i V + \mathcal{F}^j V \cap \mathcal{G}^{i+1}V) \cong \mathrm{Gr}^i_{\mathcal{G}}\mathrm{Gr}^j_{\mathcal{F}} V$$

for each $i, j \in \mathbb{N}$. To construct a basis of V that is compatible with $\mathcal{F}$, it suffices to lift a basis of each $\mathrm{Gr}^i_{\mathcal{F}} V$ to $\mathcal{F}^i V$ and take their union. In particular, we may lift bases of $\mathrm{Gr}^i_{\mathcal{F}} V$ that are compatible with the induced filtration $\mathcal{G}$. By the above isomorphism, such bases can be obtained by lifting a basis of $(\mathcal{F}^j V \cap \mathcal{G}^i V)/(\mathcal{F}^{j+1}V \cap \mathcal{G}^i V + \mathcal{F}^j V \cap \mathcal{G}^{i+1}V)$ to $\mathcal{F}^j V \cap \mathcal{G}^i V$ (for each $i, j \in \mathbb{N}$) and then taking the union. But since the construction is symmetric in $\mathcal{F}$ and $\mathcal{G}$, it follows that the basis obtained in this way is also compatible with $\mathcal{G}$. □

As an immediate consequence, we have

Proposition 3.1. *Let (X, Δ) be a pair, and let $V_\bullet$ be a multigraded linear series containing an ample series and with bounded support. Let $\mathcal{F}$ be a filtration on $V_\bullet$. Then for any valuation v of linear growth on X and any subvariety $Z \subseteq X$, we have*

$$S(V_\bullet; v) = S(V_\bullet, \mathcal{F}; v), \ \ \delta(V_\bullet) = \delta(V_\bullet, \mathcal{F}), \ \ \textit{and} \ \ \delta_Z(V_\bullet) = \delta_Z(V_\bullet, \mathcal{F}).$$

Proof. It suffices to show that for any $m \in M(V_\bullet)$, we have

$$S_m(V_\bullet; v) = S_m(V_\bullet, \mathcal{F}; v), \ \ \delta_m(V_\bullet) = \delta_m(V_\bullet, \mathcal{F}), \ \text{ and } \ \delta_{Z,m}(V_\bullet) = \delta_{Z,m}(V_\bullet, \mathcal{F}),$$

the result then follows by taking the limit as $m \to \infty$. Let $\mathcal{F}_v$ be the filtration on $V_\bullet$ induced by v (see Example 2.2). It is clear from the definition that $S_m(V_\bullet; v) = v(D)$ for any m-basis type $\mathbb{Q}$-divisor D of $V_\bullet$ that is compatible with $\mathcal{F}_v$. In particular, if we choose an m-basis type $\mathbb{Q}$-divisor D of $V_\bullet$ that is compatible with both $\mathcal{F}_v$ and $\mathcal{F}$ (which exists by Lemma 3.1), then we see that $S_m(V_\bullet; v) = v(D) \leq S_m(V_\bullet, \mathcal{F}; v)$. But the reverse inequality $S_m(V_\bullet, \mathcal{F}; v) \leq S_m(V_\bullet; v)$ is trivial, and thus we prove the first equality $S_m(V_\bullet; v) = S_m(V_\bullet, \mathcal{F}; v)$. By definition, it is not hard to see that

$$\delta_{Z,m}(V_\bullet) = \inf_E \frac{A_{X,\Delta}(E)}{S_m(V_\bullet; E)} \quad \text{and} \quad \delta_{Z,m}(V_\bullet, \mathcal{F}) = \inf_E \frac{A_{X,\Delta}(E)}{S_m(V_\bullet, \mathcal{F}; E)},$$

where both infimums run over divisors E over X whose centres contain Z (here we use the fact that Z is irreducible), hence the equality $\delta_{Z,m}(V_\bullet) = \delta_{Z,m}(V_\bullet, \mathcal{F})$ follows. The proof of the equality $\delta_m(V_\bullet) = \delta_m(V_\bullet, \mathcal{F})$ is similar. □

Typically we will apply Proposition 3.1 to some Fano variety X and the complete linear series associated to $-rK_X$ for some sufficiently divisible integer $r > 0$. By choosing different filtrations $\mathcal{F}$ on $V_\bullet$, we get various consequences. Here we explore two of them corresponding to filtrations induced by primitive divisors or admissible flags. Throughout the remaining part of this section, we fix a klt pair (X, Δ), some Cartier divisors $L_1, \cdots, L_r$ on X and an $\mathbb{N}^r$-graded linear series $V_\bullet$ associated to the L_is such that $V_\bullet$ contains an ample series and has bounded support.

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3.1. Filtrations from primitive divisors

Let F be a primitive divisor over X with associated prime blowup $\pi : Y \to X$. Let $\mathcal{F}$ be the induced filtration on $V_{\bullet}$, and let

$$D = \frac{1}{mN_m} \sum_{\vec{a}} \sum_{i} \{s_{\vec{a},i} = 0\}$$

(where $N_m = h^0(V_{m,\bullet})$ and for each $\vec{a} \in \mathbb{N}^{r-1}$, $s_{\vec{a},i}$ $(1 \leq i \leq \dim(V_{m,\vec{a}}))$ form a basis of $V_{m,\vec{a}}$) be an m-basis type $\mathbb{Q}$-divisor of $V_{\bullet}$ that is compatible with $\mathcal{F}$. We may write

$$D = \frac{1}{mN_m} \sum_{\vec{a}} \sum_{j=0}^{\infty} D'_{\vec{a},j},$$

where

$$D'_{\vec{a},j} = \sum_{i, \mathrm{ord}_F(s_{\vec{a},i}) = j} \{s_{\vec{a},i} = 0\}.$$

Since D is compatible with $\mathcal{F}$, for each $\vec{a} \in \mathbb{N}^{r-1}$, the $s_{\vec{a},i}$s that appear in the expression of $D'_{\vec{a},j}$ restrict to form a basis of $\mathrm{Gr}^j_{\mathcal{F}} V_{m,\vec{a}}$. Now assume that F is either Cartier on Y or of plt type, and let $W_{\bullet}$ be the refinement of $V_{\bullet}$ by F (Example 2.6). Then after combining coefficients of F in $\pi^* D$, we see that

$$\pi^* D = S_m(V_{\bullet}; F) \cdot F + \frac{1}{mN_m} \sum_{\vec{a}} \sum_{j=0}^{\infty} D_{\vec{a},j} =: S_m(V_{\bullet}; F) \cdot F + \Gamma,$$

where each $D_{\vec{a},j}$ doesn't contain F in its support and $D_{\vec{a},j}|_F$ is a basis type divisor for $W_{m,\vec{a},j}$. In other words, $\Gamma|_F$ is an m-basis type $\mathbb{Q}$-divisor of $W_{\bullet}$ (notice that $h^0(W_{m,\bullet}) = h^0(V_{m,\bullet})$). Letting $m \to \infty$, we obtain

$$c_1(W_{\bullet}) = (\pi^* c_1(V_{\bullet}) - S(V_{\bullet}; F) \cdot F)|_F. \tag{3.1}$$

These observations also allow us to relate the stability thresholds of $V_{\bullet}$ and $W_{\bullet}$ via inversion of adjunction. In particular, we get the following consequence:

Theorem 3.2. *With the above notation and assumptions, let $Z \subseteq X$ be a subvariety, and let Z_0 be an irreducible component of $Z \cap C_X(F)$. Let Δ_Y be the strict transform of Δ on Y (but remove the component F as in Definition 2.6), and let $\Delta_F = \mathrm{Diff}_F(\Delta_Y)$ be the difference so that $(K_Y + \Delta_Y + F)|_F = K_F + \Delta_F$. Then we have*

$$\delta_Z(X, \Delta; V_{\bullet}) \geq \min\left\{ \frac{A_{X,\Delta}(F)}{S(V_{\bullet}; F)}, \inf_{Z'} \delta_{Z'}(F, \Delta_F; W_{\bullet}) \right\} \tag{3.2}$$

when $Z \subseteq C_X(F)$ and otherwise

$$\delta_Z(X, \Delta; V_{\bullet}) \geq \inf_{Z'} \delta_{Z'}(F, \Delta_F; W_{\bullet}), \tag{3.3}$$

where the infimums run over all subvarieties $Z' \subseteq Y$ such that $\pi(Z') = Z_0$. Moreover, if equality holds and $\delta_Z(V_{\bullet})$ is computed by some valuation v on X, then either $Z \subseteq C_X(F)$ and F computes $\delta_Z(V_{\bullet})$ or $C_Y(v) \nsubseteq F$; and for any irreducible component S of $C_Y(v) \cap F$ with $Z_0 \subseteq \pi(S)$, there exists some valuation v_0 on F with centre S computing $\delta_{Z'}(W_{\bullet}) = \delta_Z(V_{\bullet})$ for all subvarieties $Z' \subseteq S$ with $\pi(Z') = Z_0$.

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Loosely speaking, this means $\delta(V_\bullet)$ is either computed by the auxiliary divisor F or bounded from below by the stability threshold $\delta(W_\bullet)$ of the refinement by F, and in the latter case the inequality is usually strict.

Proof. We only prove equation (3.2), since the proof for equation (3.3) is almost identical. By Proposition 3.1, we have $\delta_Z(V_\bullet) = \delta_Z(V_\bullet, \mathcal{F})$ (where $\mathcal{F}$ is the filtration on $V_\bullet$ induced by F); thus it suffices to show that

$$\delta_{Z,m}(V_\bullet, \mathcal{F}) \geq \min\left\{\frac{A_{X,\Delta}(F)}{S_m(V_\bullet; F)}, \inf_{\pi(Z')=Z_0} \delta_{Z',m}(F, \Delta_F; W_\bullet)\right\} \tag{3.4}$$

for all $m \in M(V_\bullet)$. Letting $m \to \infty$, we obtain equation (3.2). Let D be an m-basis type $\mathbb{Q}$-divisor of $V_\bullet$ that's compatible with F. From the discussion before, we have

$$\pi^* D = S_m(V_\bullet; F) \cdot F + \Gamma, \tag{3.5}$$

where $\Gamma = D_Y$ is the strict transform of D on Y and $\Gamma|_F$ is an m-basis type $\mathbb{Q}$-divisor of $W_\bullet$. Let λ_m (respectively, λ) be the right-hand side of equation (3.4) (respectively, equation (3.2)). Then we have $\pi^*(K_X + \Delta + \lambda_m D) = K_Y + \Delta_Y + a_m F + \lambda_m \Gamma$, where $a_m = 1 - A_{X,\Delta}(F) + \lambda_m S_m(V_\bullet; F) \leq 1$. In addition, the non-lc centre of $(F, \Delta_F + \lambda_m \Gamma|_F)$ doesn't contain $Z_0 \subseteq Z$ in its image (under the morphism π) by the definition of stability thresholds, and hence by inversion of adjunction the same is true for $(Y, \Delta_Y + F + \lambda_m \Gamma)$. It follows that $(X, \Delta + \lambda_m D)$ is lc at the generic point of Z and indeed

$$A_{X,\Delta}(v) \geq \lambda_m v(D) + (1 - a_m) \cdot v(F)$$

for all valuations v on X whose centre contains Z (when $Z \nsubseteq C_X(F)$. The value of a_m doesn't matter to us since $v(F) = 0$; this is the main difference between the proof of equation (3.2) and equation (3.3)). Since D is arbitrary, we get $\delta_{Z,m}(V_\bullet, \mathcal{F}) \geq \lambda_m$, which proves equation (3.4), and

$$A_{X,\Delta}(v) \geq \lambda_m S_m(V_\bullet, \mathcal{F}; v) + (A_{X,\Delta}(F) - \lambda_m S_m(V_\bullet; F)) \cdot v(F).$$

As in the proof of Lemma 2.9, we have $\lim_{m\to\infty} \lambda_m = \lambda$. Thus, letting $m \to \infty$ and noting that $S(V_\bullet; v) = S(V_\bullet, \mathcal{F}; v)$ by Proposition 3.1, we obtain equation (3.2) as well as the following inequality:

$$A_{X,\Delta}(v) \geq \lambda \cdot S(V_\bullet; v) + (A_{X,\Delta}(F) - \lambda \cdot S(V_\bullet; F)) \cdot v(F). \tag{3.6}$$

Now assume that equality holds in equation (3.2) and $\delta_Z(V_\bullet)$ is computed by some valuation $v \in \mathrm{Val}_X^*$: that is, $Z \subseteq C_X(v)$ and $A_{X,\Delta}(v) = \lambda \cdot S(V_\bullet; v)$. By equation (3.6), we see that either $A_{X,\Delta}(F) = \lambda \cdot S(V_\bullet; F)$, in which case F computes $\delta_Z(V_\bullet)$ and we are done, or

$$\lambda = \inf_{\pi(Z')=Z_0} \delta_{Z'}(W_\bullet) < \frac{A_{X,\Delta}(F)}{S(V_\bullet; F)} \tag{3.7}$$

and $v(F) = 0$: that is, $C_Y(v) \nsubseteq F$. Now assume that we are in the latter case, and let S be an irreducible component of $C_Y(v) \cap F$ with $Z_0 \subseteq \pi(S)$. After rescaling the valuation v, we may also assume that $A_{Y,\Delta_Y}(v) = A_{X,\Delta}(v) = 1$. Let $\mathfrak{a}_\bullet(v) \subseteq \mathcal{O}_Y$ be the valuation ideals, and let $\mathfrak{b}_\bullet = \mathfrak{a}_\bullet(v)|_F$. Clearly $\mathrm{lct}_x(Y, \Delta_Y; \mathfrak{a}_\bullet(v)) \leq \frac{A_{Y,\Delta_Y}(v)}{v(\mathfrak{a}_\bullet(v))} \leq 1$ for any $x \in C_Y(v)$, hence by inversion of adjunction, we have $\mathrm{lct}(F, \Delta_F; \mathfrak{b}_\bullet) \leq 1$ at the generic point of S. By [27, Theorem A], there exists some valuation v_0 on F with centre S such that

$$\frac{A_{F,\Delta_F}(v_0)}{v_0(\mathfrak{b}_\bullet)} \leq 1. \tag{3.8}$$

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To finish the proof, it suffices to show that this valuation computes $\delta_{Z'}(W_\bullet)$ for any subvarieties $Z' \subseteq S$ with $\pi(Z') = Z_0$ and that $\delta_{Z'}(W_\bullet) = \lambda$. To see this, let D be an m-basis type $\mathbb{Q}$-divisor of $V_\bullet$ that's compatible with both F and v (which exists by Lemma 3.1), and let D_Y be its strict transform on Y. Then as before, we have $v(D_Y) = v(D) = S_m(V_\bullet; v)$ (here we use the fact that $C_Y(v) \not\subseteq F$) and $D_Y|_F$ is an m-basis type $\mathbb{Q}$-divisor of $W_\bullet$. Using equation (3.8), we further see that

$$A_{F,\Delta_F}(v_0) \leq \frac{v_0(D_Y|_F)}{v(D_Y)} = \frac{v_0(D_Y|_F)}{S_m(V_\bullet; v)},$$

hence $S_m(W_\bullet; v_0) \geq v_0(D_Y|_F) \geq S_m(V_\bullet; v) \cdot A_{F,\Delta_F}(v_0)$. Letting $m \to \infty$, we obtain

$$\delta_{Z'}(W_\bullet) \leq \frac{A_{F,\Delta_F}(v_0)}{S(W_\bullet; v_0)} \leq \frac{1}{S(V_\bullet; v)} = \frac{A_{X,\Delta}(v)}{S(V_\bullet; v)} = \lambda.$$

Combined with equation (3.7), this implies $\delta_{Z'}(W_\bullet) = \lambda$, and it's computed by v_0. □

Theorem 3.2 reduces the question of estimating stability thresholds to similar problems in lower dimensions. Certainly the lower bounds we get depend on the choice of the auxiliary divisor F. In general, if we want to calculate the precise value of the stability threshold, we should pick an 'optimal' F – that is, a divisor that computes $\delta(V_\bullet)$ – although the resulting refinement $W_\bullet$ can be quite complicated. On the other hand, if we are merely interested in an estimate, we can also choose some divisor F such that $W_\bullet$ is relatively simple. As a typical example, we have the following direct consequence of Theorem 3.2.

Corollary 3.3. *Let $(X, \Delta), L_i, V_\bullet, Z, F, Z_0, \pi\colon Y \to X, \Delta_F$ and $W_\bullet$ be as in Theorem 3.2. Assume that*

1. *$W_\bullet$ is almost complete (Definition 2.16);*
2. *$\delta_{Z'}(F, \Delta_F + \lambda F(W_\bullet); c_1(M_\bullet)) \geq \lambda$ for some $0 \leq \lambda \leq \frac{A_{X,\Delta}(F)}{S(V_\bullet;F)}$ and all subvarieties $Z' \subseteq Y$ with $\pi(Z') = Z_0$ (where $M_\bullet$ is the movable part of $W_\bullet$).*

Then $\delta_Z(X, \Delta; V_\bullet) \geq \lambda$. If equality holds and $\delta_Z(V_\bullet)$ is computed by some valuation v on X, then either $Z \subseteq C_X(F)$ and F computes $\delta_Z(V_\bullet)$, or $C_Y(v) \not\subseteq F$, and for any irreducible component S of $C_Y(v) \cap F$ with $Z_0 \subseteq \pi(S)$, there exists some valuation v_0 on F with centre Z computing

$$\delta_{Z'}(F, \Delta_F + \lambda F(W_\bullet); c_1(M_\bullet)) = \lambda$$

for all $Z' \subseteq S$ with $\pi(Z') = Z_0$.

Proof. This is immediate from Theorem 3.2 and Lemma 2.13. □

3.2. Filtrations from admissible flags

One can inductively apply Theorem 3.2 to refine the original graded linear series while lowering the dimension of the ambient variety. This is essentially equivalent to filtering the graded linear series via an admissible flag. For simplicity, consider the following situation. Let

$$Y_\bullet \quad : \quad X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_\ell$$

be an admissible flag of length ℓ *on* X. Assume that each Y_i in the flag is a Cartier divisor on Y_{i-1}. Then for each $1 \leq j \leq \ell$, we can define a boundary divisor Δ_j on Y_j inductively as follows: first set $\Delta_0 = \Delta$; for each Δ_i that's already defined, write $\Delta_i = a_i Y_{i+1} + \Gamma_i$, where Γ_i doesn't contain Y_{i+1} in its support and set $\Delta_{i+1} = \Gamma_i|_{Y_{i+1}}$. We also let $Y_\bullet^{(j)}$ be the flag given by $Y_0 \supseteq \cdots \supseteq Y_j$, and let $W_\bullet^{(j)}$ be the refinement of $V_\bullet$ by $Y_\bullet^{(j)}$ (Example 2.7): that is, it is the $\mathbb{N}^{r+j}$-graded linear series on Y_j given by

$$W^{(j)}_{\vec{a},b_1,\cdots,b_j} = V_{\vec{a}}(b_1, \cdots, b_j).$$

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Note that $W_\bullet^{(0)} = V_\bullet$. Also recall from Section 2.4 that the flag $Y_\bullet$ induces a filtration $\mathcal{F} = \mathcal{F}_{Y_\bullet}$ on each $W_\bullet^{(j)}$.

Theorem 3.4. *With the above notation and assumptions, we have*

$$\delta_Z(X,\Delta;V_\bullet) \geq \min\left\{\min_{0\leq i\leq j-1}\left\{\frac{A_{Y_i,\Delta_i}(Y_{i+1})}{S(W_\bullet^{(i)};Y_{i+1})}\right\}, \delta_{Z\cap Y_j}(Y_j,\Delta_j;W_\bullet^{(j)},\mathcal{F})\right\} \tag{3.9}$$

for any $1 \leq j \leq \ell$ *and any subvariety* $Z \subseteq X$ *that intersects* Y_j.

This will be a key ingredient in our proof of Theorem 1.1. Compared with Theorem 3.2, the main difference is that we allow (possibly) reducible centres $Z \cap Y_j$ when applying inversion of adjunction. In this case, we only have an inequality $\delta_{Z\cap Y_j}(W_\bullet^{(j)},\mathcal{F}) \geq \delta_{Z\cap Y_j}(W_\bullet^{(j)})$ (as opposed to the equality in Proposition 3.1). As such, we also need to keep track of the filtration $\mathcal{F}$ in the proof below.

Proof. By Proposition 3.1, we have $\delta_Z(V_\bullet) = \delta_Z(V_\bullet,\mathcal{F}) = \delta_{Z\cap Y_0}(Y_0,\Delta_0;W_\bullet^{(0)},\mathcal{F})$. Thus it suffices to prove that

$$\delta_{Z\cap Y_i}(Y_i,\Delta_i;W_\bullet^{(i)},\mathcal{F}) \geq \min\left\{\frac{A_{Y_i,\Delta_i}(Y_{i+1})}{S(W_\bullet^{(i)};Y_{i+1})}, \delta_{Z\cap Y_{i+1}}(Y_{i+1},\Delta_{i+1};W_\bullet^{(i+1)},\mathcal{F})\right\} \tag{3.10}$$

for all $0 \leq i \leq j-1$; equation (3.9) then follows by induction.

As in the proof of Theorem 3.2, let D be an m-basis type $\mathbb{Q}$-divisor of $W_\bullet^{(i)}$ that's compatible with $\mathcal{F}$. Then in particular it is compatible with Y_{i+1}, and we may write

$$D = S_m(W_\bullet^{(i)};Y_{i+1})\cdot Y_{i+1} + \Gamma,$$

where Γ doesn't contain Y_{i+1} in its support and $\Gamma|_{Y_{i+1}}$ is an m-basis type $\mathbb{Q}$-divisor of $W_\bullet^{(i+1)}$ (since this is the same as the refinement of $W_\bullet^{(i)}$ by Y_{i+1}) that is compatible with $\mathcal{F}$ (since the same is true for D and the filtration $\mathcal{F}$ on $W_\bullet^{(i)}$ is a refinement of the filtration induced by Y_{i+1}). Thus by inversion of adjunction as in the proof of Theorem 3.2, we get

$$\delta_{Z\cap Y_i,m}(Y_i,\Delta_i;W_\bullet^{(i)},\mathcal{F}) \geq \min\left\{\frac{A_{Y_i,\Delta_i}(Y_{i+1})}{S_m(W_\bullet^{(i)};Y_{i+1})}, \delta_{Z\cap Y_{i+1},m}(Y_{i+1},\Delta_{i+1};W_\bullet^{(i+1)},\mathcal{F})\right\}.$$

Letting $m \to \infty$, we obtain equation (3.10), and this finishes the proof. □

4. Applications

4.1. Tian's criterion and connection to birational superrigidity

As a first application of the general framework developed in Section 3, we give a new proof of Tian's criterion for K-stability [46] (see, for example, [24, 41] for other proofs).

Theorem 4.1 (Tian's criterion). *Let* (X,Δ) *be a log Fano pair of dimension* n. *Assume that* $(X,\Delta+\frac{n}{n+1}D)$ *is log canonical (respectively, klt) for any effective* $\mathbb{Q}$*-divisor* $D \sim_{\mathbb{Q}} -(K_X+\Delta)$. *Then* (X,Δ) *is K-semistable (K-stable).*

The proof is based on the following lemma, which is known to imply Tian's criterion (this is the strategy used in [24]). When v is a divisorial valuation, the statement is proved in [22, Proposition 2.1] and [2, Proposition 3.11]. Here we give a different proof using compatible divisors, which naturally generalises the statement to all valuations (see also [2, Remark 3.12]).

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Lemma 4.1. *Let X be a projective variety of dimension n, let L be an ample line bundle on X, and let v be a valuation of linear growth on X. Then*

$$S(L;v) \leq \frac{n}{n+1}T(L;v).$$

Proof. Let r be a sufficiently large integer such that rL is very ample, and let $H \in |rL|$ be a general member. Let $V_\bullet$ be the complete linear series associated to L, and let $\mathcal{F}$ be the filtration on $V_\bullet$ induced by H. By Proposition 3.1, we have $S(L;v) = S(V_\bullet;v) = S(V_\bullet,\mathcal{F};v)$. Let D be an m-basis type $\mathbb{Q}$-divisor of L that's compatible with $\mathcal{F}$. By the same discussion as at the beginning of Section 3.1, we have

$$D = S_m(L;H)\cdot H + \Gamma$$

for some effective $\mathbb{Q}$-divisor Γ whose support doesn't contain H. Since H is general, we have $C_X(v) \not\subseteq H$. Thus $v(D) = v(\Gamma) \leq T(L - S_m(L;H)\cdot H;v))$ and $S_m(V_\bullet,\mathcal{F};v) \leq T(L - S_m(L;H)\cdot H;v))$. Letting $m \to \infty$, we see that

$$S(L;v) \leq T(L - S(L;H)\cdot H;v)).$$

By direct calculation for any irreducible divisor $H \in |rL|$, we have

$$S(L;H) = \int_0^{1/r}(1-rx)^n \mathrm{d}x = \frac{1}{r(n+1)}; \tag{4.1}$$

putting it into the previous inequality we get $S(L;v) \leq \frac{n}{n+1}T(L;v)$ as desired. □

Proof of Theorem 4.1. We only prove the K-stability part since the argument for K-semistability is similar (and simpler). Let $r > 0$ be an integer such that $-r(K_X+\Delta)$ is Cartier. Following [23], we say that a divisor E over X is dreamy if the double graded algebra $\bigoplus_{k,j\in\mathbb{N}} H^0(Y, -kr\pi^*(K_X+\Delta) - jE)$ is finitely generated (where $\pi\colon Y \to X$ is a proper birational morphism such that the centre of E is a prime divisor on Y). For such E, there exists some effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X+\Delta)$ such that $T(-K_X-\Delta;E) = \mathrm{ord}_E(D)$. By assumption, $(X, \Delta + \frac{n}{n+1}D)$ is klt, hence $\frac{n}{n+1}T(-K_X-\Delta;E) < A_{X,\Delta}(E)$ and by Lemma 4.1, we have $\beta_{X,\Delta}(E) = A_{X,\Delta}(E) - S(-K_X-\Delta;E) > 0$. Since this holds for any dreamy divisor E over X, (X,Δ) is K-stable by [23, Theorem 1.6 and §6]. □

Using the same strategy, we can also give a new proof of the following statement, which implies the K-stability criterion from [44].

Theorem 4.2 [51, Theorem 1.5]. *Let (X,Δ) be a log Fano pair where X is $\mathbb{Q}$-factorial of Picard number 1 and dimension n. Assume that for every effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X+\Delta)$ and every movable boundary $M \sim_{\mathbb{Q}} -(K_X+\Delta)$, the pair $(X, \Delta + \frac{1}{n+1}D + \frac{n-1}{n+1}M)$ is log canonical (respectively, klt). Then X is K-semistable (respectively, K-stable).*

For the proof, we need some notation. Let X be a projective variety of dimension n, and let v be a valuation on X whose centre has codimension at least two on X. Let L be an ample line bundle on X. We define the movable threshold $\eta(L;v)$ (see [51, Definition 4.1]) as the supremum of all $\eta > 0$ such that the base locus of the linear system $|\mathcal{F}_v^{m\eta}H^0(X,mL)|$ has codimension at least 2 for some $m \in \mathbb{N}$. Analogous to Lemma 4.1, we have

Lemma 4.2 [51, Lemma 4.2]. *Notation as above. Assume that X is $\mathbb{Q}$-factorial and $\rho(X) = 1$. Then we have*

$$S(L;v) \leq \frac{1}{n+1}T(L;v) + \frac{n-1}{n+1}\eta(L;v).$$

Proof. We may assume that $T(L;v) > \eta(L;v)$, otherwise the statement follows from Lemma 4.1. We claim that there exists a unique *irreducible* $\mathbb{Q}$-divisor $G \sim_{\mathbb{Q}} L$ such that $v(G) > \eta$. The uniqueness

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simply follows from the definition of movable threshold. To see the existence, let $\widetilde{G} \sim_{\mathbb{Q}} L$ be an effective $\mathbb{Q}$-divisor on X such that $v(\widetilde{G}) > \eta$ (such $\widetilde{G}$ exists by the definition of pseudo-effective thresholds). Since X is $\mathbb{Q}$-factorial and has Picard number one, every divisor on X is $\mathbb{Q}$-linearly equivalent to a rational multiple of L. In particular, we may write $\widetilde{G} = \sum \lambda_i G_i$, where $\sum \lambda_i = 1$ and each $G_i \sim_{\mathbb{Q}} L$ is irreducible. As $v(\widetilde{G}) > \eta$, we have $v(G_i) > \eta$ for some i that proves the claim. Note that by the definition of pseudo-effective threshold, we then necessarily have $v(G) = T(L; v)$. Write $G = \lambda G_0$, where G_0 is a prime divisor on X.

As in the proof of Lemma 4.1, let $r \in \mathbb{Z}_+$ be such that rL is very ample, and let $H \in |rL|$ be a general member. Let $V_\bullet$ be the complete linear series associated to L, and let $\mathcal{F}$ be the filtration on $V_\bullet$ induced by H. By Proposition 3.1, we have $S(L; v) = S(V_\bullet, \mathcal{F}; v)$. Let D be an m-basis type $\mathbb{Q}$-divisor of L that's compatible with both $\mathcal{F}$ and $\mathcal{F}_v$ (which exists by Lemma 3.1). We have

$$D = S_m(L; H) \cdot H + \Gamma$$

for some effective $\mathbb{Q}$-divisor Γ whose support doesn't contain H. We further decompose $\Gamma = \mu G_0 + \Gamma_0$, where the support of Γ_0 doesn't contain G_0. Note that $v(\Gamma_0) \le \eta(\Gamma_0; v)$ by our choice of G_0. As H is general and D is of m-basis type, we have $\mu = \mathrm{ord}_{G_0}(\Gamma) = \mathrm{ord}_{G_0}(D) \le S_m(L; G_0)$; thus

$$\begin{aligned} S_m(V_\bullet, \mathcal{F}; v) = v(D) = v(\Gamma) &= \mu \cdot v(G_0) + v(\Gamma_0) \\ &\le S_m(L; G_0) \cdot v(G_0) + \eta(\Gamma - S_m(L; G_0) \cdot G_0; v) \\ &= T(S_m(L; G_0) \cdot G_0; v) + \eta(L - S_m(L; H) \cdot H - S_m(L; G_0) \cdot G_0; v). \end{aligned}$$

Since $\rho(X) = 1$, for any prime divisor F on X, we have $S(L; F) \cdot F \sim_{\mathbb{Q}} \frac{1}{n+1} L$ as in the proof of Lemma 4.1, hence letting $m \to \infty$ in the above inequality, we obtain

$$S(L; v) = \lim_{m \to \infty} S_m(V_\bullet, \mathcal{F}; v) \le \frac{1}{n+1} T(L; v) + \frac{n-1}{n+1} \eta(L; v)$$

as desired. □

Proof of Theorem 4.2. As in Theorem 4.1, we only prove the K-stability part. Let E be a dreamy divisor over X. If the centre of E is a prime divisor on X, then we have $-(K_X + \Delta) \sim_{\mathbb{Q}} \lambda E$ for some $\lambda > 0$ as X has Picard number one. By assumption $(X, \Delta + \frac{\lambda}{n+1} E)$ is klt, hence $\beta_{X,\Delta}(E) = A_{X,\Delta}(E) - S(-K_X - \Delta; E) = A_{X,\Delta}(E) - \frac{\lambda}{n+1} > 0$. If the centre of E has codimension at least two on X, then since E is dreamy there are effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X + \Delta)$ and movable boundary $M \sim_{\mathbb{Q}} -(K_X + \Delta)$ such that $\mathrm{ord}_E(D) = T(-K_X - \Delta; E)$ and $\mathrm{ord}_E(M) = \eta(-K_X - \Delta; E)$. By assumption $(X, \Delta + \frac{1}{n+1} D + \frac{n-1}{n+1} M)$ is klt, thus

$$\begin{aligned} A_{X,\Delta}(E) &> \frac{1}{n+1} \mathrm{ord}_E(D) + \frac{n-1}{n+1} \mathrm{ord}_E(M) \\ &= \frac{1}{n+1} T(-K_X - \Delta; E) + \frac{n-1}{n+1} \eta(-K_X - \Delta; E) \\ &\ge S(-K_X - \Delta; E), \end{aligned}$$

where the last inequality follows from Lemma 4.2. Therefore $\beta_{X,\Delta}(E) > 0$ for all dreamy divisors E over X, and (X, Δ) is K-stable by [23, Theorem 1.6 and §6]. □

Corollary 4.3 [44, Theorem 1.2]. *Let X be a birationally superrigid Fano variety. Assume that $(X, \frac{1}{2} D)$ is lc for all effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$. Then X is K-stable.*

Proof. By [12, Theorem 1.26], X is $\mathbb{Q}$-factorial of Picard number one and (X, M) has canonical singularities (in particular it is klt) for every movable boundary $M \sim_{\mathbb{Q}} -K_X$. Let $D \sim_{\mathbb{Q}} -K_X$ be an effective $\mathbb{Q}$-divisor. By assumption, $(X, \frac{1}{2} D)$ is lc. As $\frac{1}{n+1} D + \frac{n-1}{n+1} M = \frac{2}{n+1} \cdot \frac{1}{2} D + \frac{n-1}{n+1} M$ is a convex

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linear combination of $\frac{1}{2}D$ and M, we see that the conditions of Theorem 4.2 are satisfied and therefore X is K-stable. □

4.2. Fano manifolds of small degrees

As a second application of our general framework, we study K-stability of Fano manifolds of small degree using flags of complete intersection subvarieties. To do so, we first specialise Corollary 3.3 to the case when the auxiliary divisor is an ample Cartier divisor on the given variety.

Lemma 4.3. *Let X be a variety of dimension n, let L be an ample line bundle on X, and let $H \in |L|$. Assume that H is irreducible and reduced. Then*

$$\delta_x(L) \geq \min\left\{n+1, \frac{n+1}{n}\delta_x(L|_H)\right\}$$

at every $x \in H$. If equality holds, then either $\delta_x(L) = n+1$ and it is computed by H, or $\delta_x(L) = \frac{n+1}{n}\delta_x(L|_H)$ and $C_X(v) \not\subseteq H$ for any valuation v that computes $\delta_x(L)$. Moreover, in the latter case, for every irreducible component Z of $C_X(v) \cap H$ containing x, there exists a valuation v_0 on H with centre Z computing $\delta_x(L|_H)$.

Proof. Let $V_\bullet$ be the complete linear series associated to L, and let $W_\bullet$ be its refinement by H. By Example 2.12, $W_\bullet$ is almost complete and $F(W_\bullet) = 0$. By equation (4.1), we have $S(L;H) = \frac{1}{n+1}$. As discussed in Section 3, any m-basis type $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ that is compatible with H can be written as $D = S_m(L;H)\cdot H + \Gamma$, where $\Gamma|_H$ is an m-basis type $\mathbb{Q}$-divisor of $W_\bullet$; thus, letting $m \to \infty$, we see that

$$c_1(W_\bullet) \sim_{\mathbb{Q}} L|_H - S(L;H)\cdot H|_H \sim_{\mathbb{Q}} \frac{n}{n+1}L|_H$$

and $\delta_x(c_1(W_\bullet)) = \frac{n+1}{n}\delta_x(L|_H)$. The result now follows directly from Corollary 3.3 with $F = H$. □

Applying induction, we further deduce:

Lemma 4.4. *Let X be a variety of dimension n, and let L be an ample line bundle on X. Let $x \in X$ be a smooth point. Assume that*

(*) *there exists $H_1, \cdots, H_{n-1} \in |L|$ containing x such that $H_1 \cap \cdots \cap H_{n-1}$ is an integral curve that is smooth at x.*

Then $\delta_x(L) \geq \frac{n+1}{(L^n)}$. If equality holds, then either $(L^n) = 1$, or every valuation that computes $\delta_x(L)$ is divisorial and is induced by some prime divisor E on X.

Proof. First assume that $n = 1$: that is, X is a curve that is smooth at x (in this case, the statement should be well-known to experts). By direct calculation, we have $S(L;x) = \frac{1}{2}\deg L$. Hence $\delta_x(L) = \frac{2}{\deg L}$ as desired.

Assume now that the statement has been proved in smaller dimensions. Let $H \in |L|$ be a general divisor containing x. By (*), H is smooth at x and $L|_H$ also satisfies (*). By induction hypothesis, we have $\delta_x(L|_H) \geq \frac{n}{(L^{n-1}\cdot H)} = \frac{n}{(L^n)}$, hence by Lemma 4.3, we see that $\delta_x(L) \geq \frac{n+1}{(L^n)}$. Suppose that equality holds, $(L^n) > 1$, and let v be a valuation on X that computes $\delta_x(L)$. Then by Lemma 4.3, we see that the centre $C_X(v)$ of v is not contained in H, $\delta_x(L|_H) = \frac{n}{(L^n)}$, and it is computed by some valuation v_0 on H with centre $Z \subseteq C_X(v) \cap H$. But by induction hypothesis, v_0 is divisorial and its centre Z is a prime divisor on H, hence $C_X(v)$ has to be a divisor on X. It follows that v is divisorial as well and is induced by a divisor *on* X. □

We now restrict our attention to Fano manifolds of small degree:

Corollary 4.4. *Let X be a Fano manifold of dimension n. Assume that there exists an ample line bundle L on X such that*

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1. $-K_X \sim_{\mathbb{Q}} rL$ *for some* $r \in \mathbb{Q}$ *with* $(L^n) \leq \frac{n+1}{r}$*; and*
2. *for every* $x \in X$*, there exists* $H_1, \cdots, H_{n-1} \in |L|$ *containing* x *such that* $H_1 \cap \cdots \cap H_{n-1}$ *is an integral curve that is smooth at* x*.*

Then X *is K-semistable. If it is not uniformly K-stable, then* $(L^n) = \frac{n+1}{r}$*, and there exists some prime divisor* $E \subseteq X$ *such that* $\beta_X(E) = 0$*.*

Proof. By Lemma 4.4, we have $\delta_x(L) \geq \frac{n+1}{(L^n)}$ at every $x \in X$, hence $\delta(L) \geq \frac{n+1}{(L^n)}$. By (1), we then obtain $\delta(-K_X) \geq \frac{n+1}{r\cdot(L^n)} \geq 1$ and X is K-semistable. Assume that X is not uniformly K-stable: that is, $\delta(-K_X) = 1$. Then equality holds in (1) and $\delta(L) = \frac{n+1}{(L^n)}$. By Lemma 4.4, either $(L^n) = 1$ or $\delta(L)$ is computed by some prime divisor E on X. In the latter case, there is nothing left to prove. In the former case, we have $r = \frac{n+1}{(L^n)} = n+1$, hence $X \cong \mathbb{P}^n$ by [30] and $\beta_X(H) = 0$ for any hyperplane H on X. □

In particular, taking L to be the hyperplane class on $\mathbb{P}^n$, Corollary 4.4 gives a new algebraic proof of the K-semistability of $\mathbb{P}^n$ (see, for example, [34, 42] for other proofs). It also gives a unified treatment of the uniform K-stability of the following Fano manifolds.

Corollary 4.5. *The following Fano manifolds are all uniformly K-stable:*

1. [45] *del Pezzo surfaces of degree* ≤ 3*;*
2. [21] *hypersurfaces* $X \subseteq \mathbb{P}^{n+1}$ *of degree* $n+1$*;*
3. [16] *double covers of* $\mathbb{P}^n$ *branched over some smooth divisor* D *of degree* $d \geq n+1$*.*
4. *cyclic covers* $\pi : X \to Y$ *of degree* s *(where* $Y \subseteq \mathbb{P}^{n+1}$ *is a smooth hypersurface of degree* m*) branched along some smooth divisor* $D \in |dH|$ *(where* H *is the hyperplane class) with* $0 \leq n+2-m-(1-\frac{1}{s})d \leq \frac{n+1}{ms}$*.*
5. *del Pezzo threefolds of degree* 1*: that is, smooth weighted hypersurfaces* $X_6 \subseteq \mathbb{P}(1^3, 2, 3)$*.*

Proof. Note that (3) is a special case of (4) with $m = 1$. We will also treat (5) separately. In each remaining case, we will find an ample line bundle L on the Fano variety that satisfies the assumptions of Corollary 4.4. Indeed, for del Pezzo surfaces X of degree 2 or 3 (respectively, degree 1), we take $L = -K_X$ (respectively, $L = -2K_X$). We also set $L = -K_X$ for hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of degree $n+1$. In case (4), we choose $L = \pi^* H$. It is straightforward to verify that they all satisfy the assumptions of Corollary 4.4, hence by Corollary 4.4, all Fano manifolds X in (1)–(4) are K-semistable. Moreover, del Pezzo surface of degree 1 or 2 are uniformly K-stable since $(L^n) < \frac{n+1}{r}$ for our choice of L. It remains to check that there are no divisors E on X with $\beta_X(E) = 0$ in the other cases.

Let $\tau = T(-K_X; E)$ be the pseudo-effective threshold (Definition 2.2). If $\dim X \geq 3$ (so we are in case (2) or (4)), then X has Picard number one and $-K_X \sim_{\mathbb{Q}} \tau E$. A direct calculation gives $\beta_X(E) = 1 - \frac{\tau}{n+1}$. Since X is not isomorphic to $\mathbb{P}^n$, we have $\tau < n+1$ by [30] and thus $\beta_X(E) > 0$ in this case (compare to [20, Corollary 9.3]). If $\dim X = 2$, then X is a cubic surface. Clearly $S(E) < \tau$. Since $-K_X - \tau E$ is pseudo-effective, it has nonnegative intersection with $-K_X$ and thus $\tau \leq \frac{3}{(-K_X \cdot E)}$. It follows that if $\beta_X(E) = 1 - S(E) = 0$, then $\tau > S(E) = 1$ and $(-K_X \cdot E) \leq 2$: that is, E is a line or a conic. But in both cases, we have $\tau = 1$ and hence $S(E) < 1$: if E is a line, then $|-K_X - E|$ is base point free and defines a conic bundle $X \to \mathbb{P}^1$; if E is a conic and L_0 is the residue line (the other component of the hyperplane section that contains E), then $-K_X - E \sim L_0$ is a (-1)-curve. Thus $\beta_X(E) = 1 - S(E) > 0$ for all divisors E on the cubic surface X as well. We therefore conclude that all Fano manifolds in (1)–(4) are uniformly K-stable.

It remains to prove every Fano threefold X in (5) is uniformly K-stable. For such X, we have $-K_X = 2H$ for some ample line bundle H on X. We claim that for every $x \in X$ there exists a smooth member $S \in |H|$ that contains x. Indeed, it is not hard to check that $h^0(X, H \otimes \mathfrak{m}_x) \geq 2$. Let $S_1 \neq S_2 \in |H \otimes \mathfrak{m}_x|$, and let $\mathcal{M} \subseteq |H \otimes \mathfrak{m}_x|$ be the pencil they span. As H generates $\mathrm{Pic}(X)$, S_1 and S_2 doesn't have common component, and we have a well-defined 1-cycle $W = (S_1 \cdot S_2)$ on X. Since $(H \cdot W) = (H^3) = 1$, W is an integral curve. As W is also the complete intersection of any two members of $\mathcal{M}$, every $S \in \mathcal{M}$ is smooth at the smooth points of W. Let y be a singular point of W, and let S' be a general member

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of $|2H \otimes \mathfrak{m}_y|$. Then as $|2H|$ is base point free, $S_1 \cap S_2 \cap S$ is zero dimensional. If both S_1 and S_2 are singular at y, then we have $\mathrm{mult}_y S_i \geq 2$ $(i = 1, 2)$ and thus

$$2 = 2(H^3) = (S_1 \cdot S_2 \cdot S') \geq \mathrm{mult}_y S_1 \cdot \mathrm{mult}_y S_2 \cdot \mathrm{mult}_y S' \geq 4,$$

a contradiction. Hence a general member of $\mathcal{M}$ is smooth at y. Since there are only finitely many singular point of W and $\mathcal{M}$ is base point free outside W, we see that a general member of $\mathcal{M}$ is smooth, proving the claim.

Now let $x \in X$, and choose a smooth member $S \in |H|$ containing x. Note that S is a del Pezzo surface of degree 1. By Lemma 4.4 with $L = -2K_S$, we have $\delta(H|_S) = 2\delta(L) \geq \frac{3}{2}$ and if the equality is computed by some divisor E over S, then E is a divisor *on* S. By Lemma 4.3, it follows that $\delta_x(-K_X) = \frac{1}{2}\delta_x(H) \geq \frac{2}{3}\delta(H|_S) \geq 1$ for all $x \in X$, thus X is K-semistable. If it is not K-stable, then by another use of Lemma 4.3 and the same argument as in Corollary 4.4, we have $\beta_X(E) = 0$ for some divisor E *on* X. But since X has Picard number one and is not $\mathbb{P}^3$, this is a contradiction as before and therefore X is uniformly K-stable. □

4.3. Surface case

We next investigate the surface case where almost everything can be explicitly computed. Recall from [22, Theorem 1.5] that it is enough to test the K-stability of log Fano pairs using divisors of plt type. The nice feature in the surface case is that the corresponding refinements are always almost complete.

Lemma 4.5. *Let (S, Δ) be a surface pair, and let L be an big line bundle on S. Let E be a plt type divisor over S. Let $V_\bullet$ be the complete linear series associated to L, and let $W_\bullet$ be the refinement of $V_\bullet$ by E. Then $W_\bullet$ is almost complete.*

As in Example 2.12, the almost completeness of a refinement is related to the surjectivity of the natural restriction map on sections, hence the proof of Lemma 4.5 essentially boils down to the following vanishing-type result.

Lemma 4.6. *Let (S, Δ) be a surface pair. Then there exists some constant $A > 0$ such that $h^1(S, \mathcal{O}_S(D)) \leq A$ for all $\mathbb{Q}$-Cartier Weil divisor D on S such that $D - (K_S + \Delta)$ is nef and big.*

Proof. Let $f : T \to S$ be the minimal log resolution of (S, Δ), and let (T, Δ_T) be the crepant pullback of (S, Δ): that is, $K_T + \Delta_T = f^*(K_S + \Delta)$. Let E be the sum of all exceptional divisors. Since D has integer coefficients, $\{f^*D\}$ is exceptional over S, hence we have $\lfloor f^*D \rfloor + E \geq f^*D$ and $f_*\mathcal{O}_T(\lfloor f^*D \rfloor + E) = \mathcal{O}_S(D)$. Let $L = \lfloor f^*D \rfloor + E$, and let $\Delta' = \Delta_T + L - f^*D$. Then it is easy to check that $0 \leq \Delta' \leq \Delta_T + E$ and

$$L - (K_T + \Delta') \sim_{\mathbb{Q}} f^*(D - K_S - \Delta),$$

which is nef and big by assumption. By Lemma 4.7, we know that there exists some constant A depending only on the pair $(T, \Delta_T + E)$ such that $h^1(T, \mathcal{O}_T(L)) \leq A$. The lemma then follows as $h^1(S, \mathcal{O}_S(D)) = h^1(S, \pi_*\mathcal{O}_T(L)) \leq h^1(T, \mathcal{O}_T(L))$. □

The following result is used in the above proof.

Lemma 4.7. *Let S a smooth surface, and let Δ be an effective divisor on S with simple normal crossing support. Then there exists some constant A such that $h^1(T, \mathcal{O}_T(L)) \leq A$ for all Cartier divisor L such that $L - (K_T + \Delta')$ is nef and big for some $\mathbb{Q}$-divisor $0 \leq \Delta' \leq \Delta$.*

Proof. We prove by induction on the sum of all coefficients of $\lfloor \Delta \rfloor$. First note that if $\lfloor \Delta' \rfloor = 0$, then (T, Δ') is klt and $h^1(T, \mathcal{O}_T(L)) = 0$ by Kawamata-Viehweg vanishing. Thus it suffices to consider the case when $\lfloor \Delta' \rfloor \neq 0$. In particular, we may just take $A = 0$ when $\lfloor \Delta \rfloor = 0$. In general, let C be an irreducible component of $\lfloor \Delta' \rfloor \leq \lfloor \Delta \rfloor$. By assumption $(L - K_T - \Delta') \cdot C \geq 0$, which

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gives $\deg(L|_C - K_C) \geq (\Delta' - C)\cdot C$, thus by Serre duality $h^1(C, \mathcal{O}_C(L)) = h^0(C, \omega_C(-L)) \leq 1 + \deg(K_C - L|_C) \leq 1 + ((C - \Delta')\cdot C)$ is bounded by some constants A_1 that only depends on Δ. By induction hypothesis (applied to the pairs $(T, \Delta - C)$ for various components C of $\lfloor \Delta \rfloor$), we also have $h^1(T, \mathcal{O}_T(L - C)) \leq A_2$ for some constant A_2 that only depends on Δ, thus $h^1(T, \mathcal{O}_T(L)) \leq A_1 + A_2$ via the exact sequence $0 \to \mathcal{O}_T(L - C) \to \mathcal{O}_T(L) \to \mathcal{O}_C(L) \to 0$. □

Proof of Lemma 4.5. Let $T_1 \to S$ be the prime blowup associated to E. Note that E is a smooth curve on T_1 and T_1 is klt along E (as $(T_1, \Delta_{T_1} + E)$ is plt by assumption). Let $T \to T_1$ be the minimal resolution of T_1 over its non-klt locus, and let $\pi\colon T \to S$ be the induced morphism. Note that T is $\mathbb{Q}$-factorial. Let $I = \mathrm{Supp}(W_{\bullet}) \cap (\{1\} \times \mathbb{R})$, let $\gamma \in I^\circ \cap \mathbb{Q}$, and let $\pi^*L - \gamma E = P_\gamma + N_\gamma$ be the Zariski decomposition where P_γ (respectively, N_γ) is the nef (respectively, negative) part. We claim that there exists a divisor $G \subseteq T$ such that $\mathrm{Supp}(N_\gamma) \subseteq G$ for all γ. Indeed, for any $\gamma_1 < \gamma < \gamma_2$, since $\pi^*L - \gamma E$ is a convex linear combination of $\pi^*L - \gamma_1 E$ and $\pi^*L - \gamma_2 E$, we see that $\mathrm{Supp}(N_\gamma) \subseteq \mathrm{Supp}(N_{\gamma_1}) \cup \mathrm{Supp}(N_{\gamma_2})$. On the other hand, by [39, Proposition III.1.10], there are at most $\rho(T)$ irreducible components in each N_γ. It follows that $\cup_\gamma \mathrm{Supp}(N_\gamma)$ is a finite union of divisors in T, and we may simply take $G = \cup \mathrm{Supp}(N_\gamma)$. Note that $E \not\subseteq \mathrm{Supp}(G)$ as otherwise $E \subseteq \mathrm{Bs}(\pi^*L - \gamma E)$ for some γ, and thus $W_{m,m\gamma} = 0$ for all m.

Now fix $\gamma \in I^\circ \cap \mathbb{Q}$ and write P (respectively, N) for P_γ (respectively, N_γ). Then for sufficiently divisible m, we have $|m(\pi^*L - \gamma E)| = |mP| + mN$. It follows that $|M_{m,m\gamma}| \subseteq |D_m|$ for some divisor D with $0 < \deg D_m \leq m(P\cdot E)$ and in particular $(P\cdot E) > 0$. Since E is a curve, any divisors on E are numerically proportional. Thus $W_{\bullet}$ is almost complete (with respect to any line bundle of degree 1 on E) as long as

$$\lim_{m\to\infty} \frac{h^0(W_{m,m\gamma})}{m(P\cdot E)} = 1, \tag{4.2}$$

where the limit is taken over sufficiently divisible integers m. Indeed, if equation (4.2) holds, then as

$$h^0(W_{m,m\gamma}) = h^0(M_{m,m\gamma}) \leq h^0(E, D_m) \leq \deg D_m + 1 \leq m(P\cdot E) + 1,$$

we clearly have $\lim_{m\to\infty} \frac{h^0(W_{m,m\gamma})}{h^0(E,D_m)} = 1$, which verifies condition (2) in Definition 2.16. It also gives $\lim_{m\to\infty} \frac{\deg D_m}{m} = (P\cdot E)$, hence $\lim_{m\to\infty} \frac{F_{m,m\gamma}}{m} = N|_E$ for sufficiently divisible m (compare to the proof of Lemma 4.8 below). Since $\mathrm{Supp}(N) \subseteq G$, we see that $F(W_{\bullet})$ is supported on $G \cap E$, which verifies condition (1) in Definition 2.16.

It remains to prove equation (4.2). To see this, we note that P is big (since $\gamma \in I^\circ$) and hence $m_0P - E - K_T$ is effective for some divisible enough integer m_0. Let $Q \in |m_0P - E - K_T|$. Then by Lemma 4.6, there exists some constant A depending only on (T, Q) such that $h^1(T, \mathcal{O}_T(mP - E)) \leq A$ for all sufficiently divisible $m > m_0$ (as $mP - E - (K_T + Q) \sim (m - m_0)P$ is nef and big).

Using the exact sequence

$$0 \to \mathcal{O}_T(mP - E) \to \mathcal{O}_T(mP) \to \mathcal{O}_E(mP) \to 0$$

from Lemma 2.1, we obtain

$$\begin{aligned} h^0(W_{m,m\gamma}) &= \dim \mathrm{Im}(H^0(T, \mathcal{O}_T(mP)) \to H^0(E, \mathcal{O}_E(mP))) \\ &\geq h^0(E, \mathcal{O}_E(mP)) - h^1(T, \mathcal{O}_T(mP - E)) \\ &\geq h^0(E, \mathcal{O}_E(mP)) - A \\ &\geq m(P\cdot E) + 1 - g(E) - A, \end{aligned}$$

where the last inequality follows from Riemann-Roch. Letting $m \to \infty$, we get equation (4.2), and hence $W_{\bullet}$ is almost complete as desired. □

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For actual calculations, it would be convenient to have a formula for $F(W_\bullet^\cdot)$ before we apply Corollary 3.3 to the almost complete refinement $W_\bullet^\cdot$. This can be done using Zariski decomposition on surfaces.

Lemma 4.8. *In the setup of Lemma 4.5, assume that L is ample, and let $\pi\colon T \to S$ be the prime blowup associated to E. Then we have*

$$F(W_\bullet^\cdot) = \frac{2}{(L^2)} \int_0^\infty \left(\mathrm{vol}_{T|E}(\pi^*L - tE) \cdot N_\sigma(\pi^*L - tE)|_E\right) \mathrm{d}t, \tag{4.3}$$

where $\mathrm{vol}_{T|E}(\cdot)$ is the restricted volume function (see [19]) and $N_\sigma(\cdot)$ denotes the negative part in the Zariski decomposition of a (pseudo-effective) divisor.

Proof. Since L is ample, it is easy to see that $\mathrm{Supp}(W_\bullet^\cdot) \cap (\{1\} \times \mathbb{R}) = [0, T(L;E)]$. By Corollary 2.15, we then have

$$F(W_\bullet^\cdot) = \frac{2}{\mathrm{vol}(W_\bullet^\cdot)} \int_0^{T(L;E)} F(\gamma)\mathrm{vol}_{W_\bullet^\cdot}(\gamma)\mathrm{d}\gamma,$$

where $F(\gamma) = \lim_{m\to\infty} \frac{1}{m} F_{m,\lfloor m\gamma \rfloor}$. By construction, we have $\mathrm{vol}(W_\bullet^\cdot) = \mathrm{vol}(V_\bullet^\cdot) = \mathrm{vol}(L)$, $\mathrm{vol}_{W_\bullet^\cdot}(\gamma) = \mathrm{vol}_{T|E}(\pi^*L - \gamma E)$ and $\mathrm{vol}_{T|E}(\pi^*L - \gamma E) = 0$ when $\gamma > T(L;E)$. Thus it suffices to show that

$$F(\gamma) = N_\sigma(\pi^*L - \gamma E)|_E. \tag{4.4}$$

By continuity, it is enough to check equation (4.4) when $\gamma \in (0, T(L;E)) \cap \mathbb{Q}$. Let $\pi^*L - \gamma E = P + N$ be the Zariski decomposition as in the proof of Lemma 4.5, and let m be a sufficiently divisible integer. Since L is ample, E is not contained in the stable base locus $\mathrm{Bs}(\pi^*L)$ of π^*L. Since there always exists some $\gamma' \geq \gamma$ such that $E \nsubseteq \mathrm{Bs}(\pi^*L - \gamma' E)$ (e.g., we take $\gamma' = \mathrm{ord}_E(D)$ for any $D \sim_{\mathbb{Q}} L$ with $\mathrm{ord}_E(D) \geq \gamma$) and $\pi^*L - \gamma E$ is a convex linear combination of π^*L and $\pi^*L - \gamma' E$, we see that $E \nsubseteq \mathrm{Bs}(\pi^*L - \gamma E)$ as well. In particular, $E \nsubseteq \mathrm{Supp}(N)$. Then clearly $F_{m,m\gamma} \geq mN|_E$ and hence $F(\gamma) \geq N|_E$. From the proof of Lemma 4.5, we also see that there exists some constant A (depending only on (S, Δ) and E) such that the restricted linear series $|mP|_E$ has codimension at most A in $|\mathcal{O}_E(mP)|$, and thus the degree of $F_{m,m\gamma} - mN|_E$ is at most A. Letting $m \to \infty$, we obtain $\deg F(\gamma) = \deg(N|_E)$, which implies equation (4.4) as $F(\gamma) \geq N|_E$. □

As an illustration, we compute the δ-invariants of all smooth cubic surfaces. Some of these will be useful in our proof of the K-stability of cubic threefolds (Lemma 4.10).

Theorem 4.6. *Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface, and let $x \in X$ be a closed point. Let $C = T_x(X) \cap X$ be the tangent hyperplane section. Then*

$$\delta_x(X) = \begin{cases} 3/2 & \text{if } \mathrm{mult}_x C = 3, \\ 27/17 & \text{if } C \text{ has a tacnode at } x, \\ 5/3 & \text{if } C \text{ has a cusp at } x, \\ 18/11 & \text{if } C \text{ is the union of three lines and } \mathrm{mult}_x C = 2, \\ 12/7 & \text{if } C \text{ is irreducible and has a node at } x, \\ \frac{9}{25-8\sqrt{6}} & \text{if } C \text{ is the union of a line and a conic that intersects transversally.} \end{cases}$$

Moreover, in the first three cases, $\delta_x(X)$ is computed by the (unique) divisor that computes $\mathrm{lct}_x(X, C)$; in the next two cases, $\delta_x(X)$ is computed by the ordinary blowup of x; in the last case, $\delta_x(X)$ is computed by the quasi-monomial valuation over $x \in X$ with weights $1 + \sqrt{6}$ on the line and 2 on the conic, and if $0 < \varepsilon \ll 1$, then the log del Pezzo pair $(X, (1-\varepsilon)C)$ satisfies $\delta(X, (1-\varepsilon)C) = \frac{9}{25-8\sqrt{6}} \notin \mathbb{Q}$.

Proof. See Appendix A. □

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Corollary 4.7. *Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface. Then*

$$\delta(X) = \begin{cases} 3/2 & \text{if } X \text{ has an Eckardt point,} \\ 27/17 & \text{otherwise.} \end{cases}$$

It has been expected (see, for example, [37]) that given a klt Fano variety X with a $\mathbb{Q}$-complement Δ, the graded rings

$$\mathrm{gr}_v R := \oplus_{m,\lambda} \mathrm{Gr}^{\lambda}_{\mathcal{F}_v} H^0(X, -mrK_X)$$

are finitely generated for all lc places v of (X, Δ), where $r > 0$ is an integer such that $-rK_X$ is Cartier and $\mathcal{F}_v$ is the filtration induced by v. Unfortunately, this is not true in general, and we identify a counterexample through the calculations in Theorem 4.6.

Theorem 4.8. *Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface, and let $C \subseteq X$ be a hyperplane section such that $C = L \cup Q$ is the union of a line and a conic that intersects transversally. Then there exists an lc place v of (X, C) such that $\mathrm{gr}_v R$ is not finitely generated.*

Proof. This can be deduced from the fact that $\delta_x(X) \notin \mathbb{Q}$, where $x \in L \cap Q$. Here we give a more direct (and simpler) argument.

Let $x \in L \cap Q$, and let $a, b > 0$ be coprime integers. Let $\pi \colon Y = Y_{a,b} \to X$ be the weighted blowup at x with $\mathrm{wt}(L) = a$ and $\mathrm{wt}(Q) = b$. Let E be the exceptional divisor, and let $\widetilde{L}$ (respectively, $\widetilde{Q}$) be the strict transform of L (respectively, Q). Assume that $b < 2a$. We have $(\widetilde{L}^2) = -1 - \frac{a}{b}$, $(\widetilde{Q}^2) = -\frac{b}{a}$, $(\widetilde{L} \cdot \widetilde{Q}) = 1$ and in particular the intersection matrix of $\widetilde{L}$ and $\widetilde{Q}$ is negative definite. As $-\pi^*K_X - (a+b)E \sim \widetilde{L} + \widetilde{Q}$, it follows that $T(-K_X; E) = a + b$, the stable base locus of $-\pi^*K_X - tE$ is supported on $\widetilde{L} \cup \widetilde{Q}$ for all $0 \le t \le a + b$, and hence $N_\sigma(-\pi^*K_X - tE) = f(t)\widetilde{L} + g(t)\widetilde{Q}$ for some $f(t), g(t) \ge 0$. The coefficients $f(t)$ and $g(t)$ are computed as the smallest numbers such that $-\pi^*K_X - tE - f(t)\widetilde{L} - g(t)\widetilde{Q}$ is nef, and it is enough to check nefness against $\widetilde{L}$ and $\widetilde{Q}$. A straightforward computation then gives

$$N_\sigma(-\pi^*K_X - tE) = \begin{cases} 0 & \text{if } 0 \le t \le b, \\ \frac{t-b}{a+b}\widetilde{L} & \text{if } b < t \le \frac{a(2a+3b)}{2a+b}, \\ \frac{2t-2a-b}{b}\widetilde{L} + \frac{(2a+b)t - a(2a+3b)}{b^2}\widetilde{Q} & \text{if } \frac{a(2a+3b)}{2a+b} < t \le a+b. \end{cases} \tag{4.5}$$

The key point is that as a rational function, $\frac{a(2a+3b)}{(2a+b)}$ is not a linear combination of a and b. In particular, we may choose $a_0, b_0 \in \mathbb{R}_+$ such that $b_0 < 2a_0$ and $a_0, b_0, \frac{a_0(2a_0+3b_0)}{(2a_0+b_0)}$ are linearly independent over $\mathbb{Q}$ (thus $\frac{a_0}{b_0}$ is necessarily irrational). Let v_0 be the quasi-monomial valuation centred at x given by $\mathrm{wt}(L) = a_0$ and $\mathrm{wt}(Q) = b_0$. We claim that $\mathrm{gr}_{v_0} R$ is not finitely generated.

Suppose that $\mathrm{gr}_{v_0} R$ is finitely generated, and let f_i $(i = 1, 2, \cdots, s)$ be a finite set of homogeneous generators ($\mathrm{gr}_{v_0} R$ is naturally graded by $\mathbb{N} \times (\mathbb{N}a_0 + \mathbb{N}b_0)$). Let $\deg(f_i) = (m_i, \lambda_i = p_i a_0 + q_i b_0)$, where $m_i, p_i, q_i \in \mathbb{N}$. We may assume that $0 = \frac{\lambda_1}{m_1} \le \frac{\lambda_2}{m_2} \cdots \le \frac{\lambda_s}{m_s}$. Clearly $\frac{\lambda_s}{m_s} \ge a_0 + b_0 > \frac{a_0(2a_0+3b_0)}{(2a_0+b_0)}$ (otherwise $v_0(s) < a_0 + b_0$ for all $s \in H^0(X, -K_X)$; but $v_0(L + Q) = a_0 + b_0$). Since a_0, b_0 and $\frac{a_0(2a_0+3b_0)}{(2a_0+b_0)}$ are linearly independent over $\mathbb{Q}$, there exists $1 \le \ell < s$ such that

$$\frac{\lambda_\ell}{m_\ell} < \frac{a_0(2a_0 + 3b_0)}{(2a_0 + b_0)} < \frac{\lambda_{\ell+1}}{m_{\ell+1}}.$$

We may lift each f_i to $g_i \in R_{m_i} = H^0(X, -m_i K_X)$ such that $\mathrm{in}_{v_0}(g_i) = f_i$. Then for all $\alpha = (a, b) \in \mathbb{Q}^2$ with $|\alpha - (a_0, b_0)| \ll 1$, we have $\mu_i := v_\alpha(g_i) = p_i a + q_i b$ (where v_α is the quasi-monomial valuation with $\mathrm{wt}(L) = a$ and $\mathrm{wt}(Q) = b$); in particular, $v_\alpha(g_i) > m_i \cdot \frac{a(2a+2b)}{2a+b}$ when $i \ge \ell + 1$; thus by

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equation (4.5), g_i vanishes on Q for all $i \geq \ell + 1$. We may also assume that

$$0 = \frac{\mu_1}{m_1} \leq \cdots \leq \frac{\mu_\ell}{m_\ell} < \frac{a(2a+2b)}{2a+b} < \frac{\mu_{\ell+1}}{m_{\ell+1}} \leq \cdots \leq \frac{\mu_s}{m_s}. \tag{4.6}$$

By [35, Lemma 2.10], g_i restrict to a finite set of generators of $\mathrm{gr}_{v_\alpha} R$. It follows that for any $g \in R_m = H^0(X, -mK_X)$, we have

$$v_\alpha(g) = \max\{\mathrm{wt}(F) \mid F \in \mathbb{C}[x_1, \cdots, x_s] \text{ s.t. } F(g_1, \cdots, g_s) = g\}, \tag{4.7}$$

where we set $\mathrm{wt}(x_i) = \mu_i$ (clearly if $g = F(g_1, \cdots, g_s)$, then $v_\alpha(g) \geq \mathrm{wt}(F)$; conversely, as g_i generate $\mathrm{gr}_{v_\alpha} R$, there exists F such that $\mathrm{wt}(F) = v_\alpha(g)$ and $g = F(g_1, \cdots, g_s) \mathrm{mod}\ \mathcal{F}_{v_\alpha}^{>v_\alpha(g)} R_m$, one can then prove by induction on $v_\alpha(g)$ that $g = F(g_1, \cdots, g_s)$ for some $\mathrm{wt}(F) = v_\alpha(g)$). Now let $\lambda = \frac{a(2a+2b)}{2a+b}$. By equation (4.5), for sufficiently divisible integers $m > 0$, there exists $f \in H^0(X, -mK_X)$ such that $v_\alpha(f) = \lambda m$ and f does not vanish on Q. By equation (4.7), we have $f = F(g_1, \cdots, g_s)$ for some F with $\mathrm{wt}(F) = m\lambda$. However, by equation (4.6), we see that each monomial in F must contain some g_i with $i \geq \ell + 1$; it follows that $f = F(g_1, \cdots, g_s)$ vanishes along Q, a contradiction. Therefore, $\mathrm{gr}_{v_0} R$ is not finitely generated. □

4.4. Hypersurfaces with Eckardt points

Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$. Recall that $x \in X$ is called a generalised Eckardt point if the tangent hyperplane section $D = T_x X \cap X \subseteq X$ at x satisfies $\mathrm{mult}_x D = d$. In this case D is isomorphic to the cone over $F(X, x)$, the Hilbert scheme of lines in X passing through x, which is a hypersurface of degree d in $\mathbb{P}^{n-1}$. It is in fact smooth by the following easy lemma.

Lemma 4.9. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$, and let $x \in X$ be a generalised Eckardt point. Then $F(X, x)$ is smooth.*

Proof. We may assume that $x = [0 : \cdots : 0 : 1]$. Up to a change of coordinates, X is defined by an equation of the form $x_0 f(x_1, \cdots, x_{n+1}) + g(x_1, \cdots, x_n) = 0$, where $\deg f = d - 1$, $\deg g = d$ and f contains the monomial x_{n+1}^{d-1}. We then have $F(X, x) \cong (g = 0) \subseteq \mathbb{P}^{n-1}$. If $[a_1 : \cdots : a_n]$ is a singular point of $F(X, x)$, then for any a_{n+1} with $f(a_1, \cdots, a_{n+1}) = 0$ (such a_{n+1} exists since f contains the monomial x_{n+1}^{d-1}) it is not hard to check that X is singular at $[0 : a_1 : \cdots : a_{n+1}]$. This is a contradiction as X is smooth. Thus $F(X, x)$ is smooth. □

Theorem 4.9. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$, and let $x \in X$ be a generalised Eckardt point. Assume that $F(X, x)$ is K-semistable if $d \leq n - 1$ (i.e., when it's Fano). Then $\delta_x(H) = \frac{n(n+1)}{d+n-1}$ (where H is the hyperplane class on X), and it is computed by the ordinary blowup of x.*

Proof. Let $\pi : Y \to X$ be the blowup of x, and let E be the exceptional divisor. Let $V_\bullet$ be the complete linear series associated to H, and let $W_\bullet$ be its refinement by E. Since $x \in X$ is a generalised Eckardt point, the tangent hyperplane section $x \in D \subseteq X$ has $\mathrm{mult}_x D = d$. Let $\widetilde{D}$ be the strict transform of D on Y. Let $j, m \in \mathbb{N}$. Note that $|m\pi^* H - jE| \neq \emptyset$ if and only if $0 \leq j \leq dm$, and it is base point free when $0 \leq j \leq m$. We first show that

$$|m\pi^* H - jE| = \left| \left(m - \lceil \frac{j-m}{d-1} \rceil \right) \pi^* H - \left(j - d \cdot \lceil \frac{j-m}{d-1} \rceil \right) E \right| + \lceil \frac{j-m}{d-1} \rceil \widetilde{D} \tag{4.8}$$

is the decomposition into movable and fixed part when $m \leq j \leq dm$.

Suppose first that $n \geq 3$. Then D is irreducible. Let $D' \sim_{\mathbb{Q}} H$ be another effective $\mathbb{Q}$-divisor that doesn't contain D in its support. We have

$$d \cdot \mathrm{mult}_x D' \leq (D \cdot D' \cdot H_1 \cdot \cdots \cdot H_{n-2}) = d,$$

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where $H_1, \cdots, H_{n-2}$ are general hyperplane sections passing through x, hence $\mathrm{mult}_x D' \leq 1$. It follows that for any $G \in |m\pi^*H - jE|$, if we write $G = a\widetilde{D} + G'$, where $\widetilde{D} \not\subseteq \mathrm{Supp}(G')$, then $G' \in |(m-a)\pi^*H - (j-ad)E|$ and

$$j - ad \leq \mathrm{mult}_x \pi(G') \leq m - a.$$

In other words, $a \geq \lceil \frac{j-m}{d-1} \rceil$, which implies equation (4.8). If $n = 2$, then in the above same notation $\widetilde{D}$ is a disjoint union of d lines $L_1, \cdots, L_d$. If we take $G \in |m\pi^*H - jE|$ and write $G = \sum a_i L_i + G'$, where G' doesn't contain any L_i $(i = 1, \cdots, d)$ in its support, then as $(G' \cdot L_i) \geq 0$, we obtain $m - j = (G \cdot L_i) \geq a_i(L_i^2) = a_i(1-d)$; thus $a_i \geq \lceil \frac{j-m}{d-1} \rceil$ for all $1 \leq i \leq d$, and equation (4.8) still holds.

It is straightforward to check that for all $0 \leq j \leq m$, the natural restriction maps

$$H^0(Y, \mathcal{O}_Y(m\pi^*H - jE)) \to H^0(E, \mathcal{O}_E(j))$$

are surjective. It follows that

$$\mathrm{vol}(\pi^*H - tE) = \begin{cases} d - t^n & \text{if } 0 \leq t \leq 1, \\ \frac{(d-t)^n}{(d-1)^{n-1}} & \text{if } 1 < t \leq d, \end{cases}$$

and

$$W_{m,j} = \begin{cases} H^0(E, \mathcal{O}_E(j)) & \text{if } 0 \leq j \leq m, \\ \mathrm{Im}(H^0(E, \mathcal{O}_E(j - d\lceil \frac{j-m}{d-1} \rceil)) \xrightarrow{\cdot \lceil \frac{j-m}{d-1} \rceil D_0} H^0(E, \mathcal{O}_E(j))) & \text{if } m \leq j \leq dm \\ 0 & \text{otherwise,} \end{cases}$$

where $D_0 = \widetilde{D} \cap E \cong F(X, x)$. In particular, $W_{\bullet,\bullet}$ is almost complete, and through direct calculations we see that $S(H; E) = \frac{d+n-1}{n+1}$, $F(W_{\bullet,\bullet}) = \frac{1}{n+1}(1 - \frac{1}{d})D_0$ (by Corollary 2.15) and $c_1(W_{\bullet,\bullet}) \sim_{\mathbb{Q}} (\pi^*H - S(H; E) \cdot E)|_E \sim \frac{d+n-1}{n+1} H_0$ (see equation (3.1)), where H_0 is the hyperplane class on $E \cong \mathbb{P}^{n-1}$.

Clearly $\delta_x(H) \leq \lambda$, where $\lambda = \frac{n(n+1)}{d+n-1} = \frac{A_X(E)}{S(H;E)}$. It remains to prove $\delta_x(H) \geq \lambda$. Let $M_{\bullet,\bullet}$ be the movable part of $W_{\bullet,\bullet}$. By Corollary 3.3, it suffices to prove

$$\delta(E, \lambda F(W_{\bullet,\bullet}); c_1(M_{\bullet,\bullet})) \geq \lambda. \tag{4.9}$$

Note that by the above calculations, we have

$$\lambda c_1(M_{\bullet,\bullet}) + \lambda F(W_{\bullet,\bullet}) \sim_{\mathbb{Q}} \lambda c_1(W_{\bullet,\bullet}) \sim_{\mathbb{Q}} nH_0 \sim -K_E.$$

Thus equation (4.9) is equivalent to saying that the pair

$$(E, \lambda F(W_{\bullet,\bullet})) \cong (\mathbb{P}^{n-1}, \frac{n(d-1)}{d(d+n-1)} D_0)$$

is K-semistable. By [16, Lemma 2.6], this would be true if $(\mathbb{P}^{n-1}, \mu D_0)$ is K-semistable for some $\mu \geq \frac{n(d-1)}{d(d+n-1)}$ (as $\mathbb{P}^{n-1}$ is K-semistable). When $d \geq n$, $(\mathbb{P}^{n-1}, \frac{n}{d} D_0)$ is a log canonical log Calabi-Yau pair (note that D_0 is smooth) and therefore is K-semistable by [40, Corollary 1.1]; thus we may take $\mu = \frac{n}{d}$. When $d \leq n-1$, D_0 is Fano and K-semistable by assumption. We claim that $(\mathbb{P}^{n-1}, \mu D_0)$ is K-semistable, where $\mu = 1 - \frac{1}{d} + \frac{1}{n} > \frac{n(d-1)}{d(d+n-1)}$. Indeed, the divisor D_0 induces a special degeneration of $(\mathbb{P}^{n-1}, \mu D_0)$ to $(V, \mu V_\infty)$, where $V = C_p(D_0, N_{D_0/E})$ is the projective cone over D_0. By [36, Proposition 5.3], $(V, \mu V_\infty)$ is K-semistable, thus $(\mathbb{P}^{n-1}, \mu D_0)$ is also K-semistable by the openness of K-semistability [3, 49]. This proves the claim and also concludes the proof of the theorem. □

Restricting to Fano hypersurfaces, we have

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Corollary 4.10. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of degree d, and let $x \in X$ be a generalised Eckardt point. Assume that $F(X,x)$ is K-semistable if $d \leq n-1$. Then $\delta_x(X) = \frac{n(n+1)}{(n-1+d)(n+2-d)}$, and it is computed by the ordinary blowup of x.*

Proof. If $d = 1$, then $X \cong \mathbb{P}^n$ and $\delta_x(X) = 1 = \frac{n(n+1)}{(n-1+d)(n+2-d)}$ for all $x \in X$. If $d \geq 2$, then as $-K_X \sim (n+2-d)H$, where H is the hyperplane class, we have $\delta_x(X) = \frac{1}{n+2-d}\delta_x(H) = \frac{n(n+1)}{(n-1+d)(n+2-d)}$ by Theorem 4.9. □

Since every point of a smooth quadric hypersurface is a generalised Eckardt point, we obtain a new algebraic proof of the following well-known result.

Corollary 4.11. *Quadric hypersurfaces are K-semistable.*

Proof. Let $d = 2$ in Corollary 4.10. Since every $x \in X$ is a generalised Eckardt point and $F(X,x)$ is a smooth quadric hypersurface of smaller dimension, we get $\delta(X) = 1$ by induction on the dimension. □

4.5. Hypersurfaces of index two

The goal of this section is to prove the following result.

Theorem 4.12. *Let $X = X_n \subseteq \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of degree $n \geq 3$ (i.e., it has Fano index 2). Then X is uniformly K-stable.*

Note that when $n = 3$ – that is, X is a cubic threefold – the result is already known by [38]. Here we give a different proof using techniques developed in previous sections.

Lemma 4.10. *Let $X \subseteq \mathbb{P}^4$ be a smooth cubic threefold. Then X is uniformly K-stable.*

Proof. It suffices to show that $\delta_x(X) > 1$ for all $x \in X$. If x is a generalised Eckardt point, then $\delta_x(X) = \frac{1}{2}\delta_x(H) = \frac{6}{5} > 1$ by Theorem 4.9. If x is not a generalised Eckardt point, then there are only finitely many lines on X passing through x; thus if $Y \subseteq X$ is a general hyperplane section passing through x, then Y is a smooth cubic surface such that x is not contained in any lines on Y. By Theorem 4.6, we see that $\delta_x(Y) \geq \frac{5}{3}$. It then follows from Lemma 4.3 that $\delta_x(X) = \frac{1}{2}\delta_x(H) \geq \frac{2}{3}\delta_x(Y) \geq \frac{10}{9} > 1$. This completes the proof. □

In the remaining part of this section, we will henceforth assume that $n \geq 4$. As a key step toward the proof of Theorem 4.12, we observe the following K-stability criterion.

Lemma 4.11. *Let X be a Fano manifold of dimension n. Assume that*

1. *$\delta_Z(X) \geq \frac{n+1}{n}$ for any subvariety $Z \subseteq X$ of dimension ≥ 1;*
2. *$\beta_X(E_x) > 0$ for any $x \in X$, where E_x denotes the exceptional divisor of the ordinary blowup of x.*

Then X is uniformly K-stable.

Proof. We need to show that for any valuation $v \in \mathrm{Val}_X^*$ with $A_X(v) < \infty$, we have $\beta_X(v) > 0$. By our first assumption, this holds if the centre of v has dimension at least one. Thus we may assume that the centre of v is a closed point $x \in X$, and by our second assumption, we may assume that $v \neq c \cdot \mathrm{ord}_{E_x}$. Let r be a sufficiently large integer such that $-rK_X$ is very ample, and let $H \in |-rK_X|$ be a general member (in particular, $x \notin H$). By Proposition 3.1, we have $S(-K_X;v) = S(V_\bullet, \mathcal{F}; v)$, where $V_\bullet$ is the complete linear series associated to $-K_X$ and $\mathcal{F}$ is the filtration induced by H. Let $m \gg 0$, and let D be an m-basis type $\mathbb{Q}$-divisor of $-K_X$ that's compatible with $\mathcal{F}$. As in the proof of Lemma 4.1, we have $D = \mu_m \cdot H + \Gamma$, where $\mu_m = S_m(-K_X;H) \to S(-K_X;H) = \frac{1}{r(n+1)}$ $(m \to \infty)$ and $\Gamma \sim_{\mathbb{Q}} -(1 - r\mu_m)K_X$ is effective. By [2, Corollary 3.6], there exist constants $\epsilon_m \in (0,1)$ $(m \in \mathbb{N})$ depending only on X such that $\epsilon_m \to 1$ $(m \to \infty)$ and

$$S(-K_X;v) > \epsilon_m \cdot S_m(-K_X;v)$$

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for all valuations $v \in \mathrm{Val}_X^*$ with $A_X(v) < \infty$ and all $m \in \mathbb{N}$. Perturbing the ϵ_m, we will further assume that $\epsilon_m(1 - r\mu_m) < \frac{n}{n+1}$. Combining with our first assumption, we see that

$$\delta_{Z,m}(X) > \epsilon_m \cdot \delta_Z(X) \geq \frac{(n+1)\epsilon_m}{n}$$

for any subvariety $Z \subseteq X$ of dimension ≥ 1. It follows that $(X, \frac{(n+1)\epsilon_m}{n} D)$ is klt in a punctured neighbourhood of x and so does $(X, \frac{(n+1)\epsilon_m}{n}\Gamma)$. Note that $-(K_X + \frac{(n+1)\epsilon_m}{n}\Gamma) \sim_{\mathbb{Q}} -(1 - \frac{n+1}{n}\epsilon_m(1 - r\mu_m))K_X$ is ample; thus by the following Lemma 4.12, there exists some $\lambda = \frac{\mu}{\mu+1} \cdot \frac{n+1}{n} > 1$ (where $\mu = \frac{A_X(v)}{v(\mathfrak{m}_x)} > n$) such that

$$A_X(v) \geq \lambda\epsilon_m \cdot v(\Gamma) = \lambda\epsilon_m \cdot v(D),$$

where the last equality holds since $x \notin H$. Since D is arbitrary, we obtain $A_X(v) \geq \lambda\epsilon_m \cdot S_m(V_\bullet, \mathcal{F}; v)$; letting $m \to \infty$, we deduce $A_X(v) \geq \lambda S(-K_X; v) > S(-K_X; v)$. This completes the proof. □

The following result is used in the above proof.

Lemma 4.12. *Let $x \in X$ be a smooth point on a projective variety. Let D be an effective $\mathbb{Q}$-divisor on X such that (X, D) is klt in a punctured neighbourhood of x and $-(K_X + D)$ is ample. Let $v \in \mathrm{Val}_X^*$ be a valuation with $A_X(v) < \infty$ that's centred at x, and let $\mu = \frac{A_X(v)}{v(\mathfrak{m}_x)}$. Then*

1. *$\mu \geq \dim X$ and equality holds if and only if $v = c \cdot \mathrm{ord}_E$ for some $c > 0$, where E is the exceptional divisor of the blowup of x.*
2. *$A_X(v) \geq \frac{\mu}{\mu+1} \cdot v(D)$.*

Proof. Let $n = \dim X$. The first part follows from the fact that $(X, \mathfrak{m}_x^n)$ is lc, and the only lc place is the exceptional divisor coming from the blowup of x. The second part essentially follows from the proof of [51, Theorem 1.6], which we reproduce here for the reader's convenience. Let $\mathcal{J} = \mathcal{J}(X, D)$ be the multiplier ideal of (X, D). We may assume that (X, D) is not lc at x (otherwise $A_X(v) \geq v(D)$, and we are done), hence $\mathcal{J}_x \neq \mathcal{O}_{X,x}$. By assumption, we have $\mathcal{J} = \mathcal{O}_X$ in a punctured neighbourhood of x. Since $-(K_X + D)$ is ample, we have $H^1(X, \mathcal{J}) = 0$ by Nadel vanishing and hence a surjection $H^0(\mathcal{O}_X) \twoheadrightarrow H^0(\mathcal{O}_X/\mathcal{J}) \twoheadrightarrow H^0(\mathcal{O}_{X,x}/\mathcal{J}_x)$. Since $h^0(X, \mathcal{O}_X) = 1$, we see that $\mathcal{J}_x = \mathfrak{m}_x$ and thus $v(\mathcal{J}) = v(\mathfrak{m}_x) = \frac{A_X(v)}{\mu}$. Through the definition of multiplier ideals, we also have $v(\mathcal{J}) \geq v(D) - A_X(v)$. Combined with the previous equality it implies $A_X(v) \geq \frac{\mu}{\mu+1} \cdot v(D)$. □

To prove the K-stability of smooth hypersurfaces of Fano index two, it remains to verify the two conditions in Lemma 4.11. The following lemma takes care of the (easier) second condition.

Lemma 4.13. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of degree d. Let $r = n + 2 - d$ be its Fano index. Assume that $d \geq 3$ and $n + 1 \geq r^2$. Then $\beta_X(E_x) > 0$ for any $x \in X$, where E_x is the exceptional divisor of the ordinary blowup of x.*

Proof. Let H be the hyperplane class on X, and let $T = T(H; E_x)$ be the pseudo-effective threshold and $\eta = \eta(H; E_x)$ the movable threshold (see Lemma 4.2). Clearly $1 \leq \eta \leq T \leq d$. Let $\pi\colon Y \to X$ be the blowup of x. Then as $\pi^*H - E_x$ is nef, we see that

$$(\pi^*H - E_x)^{n-2} \cdot (\pi^*H - \eta E_x) \cdot (\pi^*H - TE_x) \geq 0,$$

and thus $\eta T \leq d$ and $\eta \leq \sqrt{d}$. By Lemma 4.2, we then have

$$S(H; E_x) \leq \frac{1}{n+1}T + \frac{n-1}{n+1}\eta \leq \frac{d}{(n+1)\eta} + \frac{n-1}{n+1}\eta.$$

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When $1 \le \eta \le \sqrt{d}$, the right-hand side of the above inequality achieves its maximum at either $\eta = 1$ or $\eta = \sqrt{d}$, hence as $3 \le d \le n+1$ and $n+1 \ge r^2$, we obtain

$$S(-K_X; E_x) = r \cdot S(H; E_x) \le \max\left\{\frac{(n+2-d)(n-1+d)}{n+1}, \frac{rn\sqrt{d}}{n+1}\right\} < n = A_X(E_x).$$

In other words, $\beta_X(E_x) > 0$. □

We now focus on checking the first condition of Lemma 4.11. The basic idea, similar to the proof of Lemma 4.4, is to apply Theorem 3.4 to an admissible flag of complete intersection subvarieties on the hypersurface X. At the end, we relate $\delta_Z(X)$ to the stability threshold of a divisor of degree close to 4 on a curve C (i.e., the 1-dimensional subvariety in the chosen flag). However, this only gives the naïve bound $\delta_Z(X) \ge \delta(-\frac{2}{n+1}K_X|_C) = \frac{n+1}{2n}$ (since $\delta(L) = \frac{2}{\deg L}$ for any ample line bundle L on a curve) and is not good enough for our purpose. To get a better estimate, we choose a flag such that C intersects Z in at least two points P, Q (which is possible since $\dim Z \ge 1$). We still have the freedom to choose another point $R \ne P, Q$ on C to put in our flag. The key observation is that (asymptotically) basis type $\mathbb{Q}$-divisors of degree 4 on C that are compatible with R have multiplicity 2 at the point R and therefore must be log canonical at one of P or Q for degree reason. In other words, the stability threshold along $Z \cap C$ is at least one, which is exactly what we need.

We work out the details in the next several lemmas. The first thing is to make sure that the admissible flag we want to use exists.

Lemma 4.14. *Let $Y \subseteq \mathbb{P}^{m+1}$ be a smooth hypersurface of dimension $m \ge 2$, and let $P \ne Q$ be two distinct points on Y. Let $H \subseteq Y$ be a general hyperplane section containing both P and Q. Then H is smooth unless $m = 2$ and Y contains the line joining P and Q.*

Proof. Let $\ell \subseteq \mathbb{P}^{m+1}$ be the line joining P and Q, and let $\mathcal{M} \subseteq |\mathcal{O}_Y(1)|$ be the linear system of hyperplane sections containing P, Q. If $\ell \not\subseteq Y$, then $\mathcal{M}$ only has isolated base points $\ell \cap Y$, and by Bertini's theorem, H is smooth away from these points. On the other hand, since H is general, it is different from the tangent hyperplane of any $x \in \ell \cap Y$; hence H is also smooth at any $x \in \ell \cap Y$. Thus we may assume that $\ell \subseteq X$. Again H is smooth away from ℓ by Bertini's theorem. The tangent hyperplanes of $x \in \ell$ give a 1-dimensional family of members of $\mathcal{M}$. Hence they are different from H as long as $\dim \mathcal{M} = m - 1 \ge 2$. It follows that H is smooth when $m \ge 3$. □

In the remaining part of this subsection, let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n \ge 4$ and $Z \subseteq X$ a subvariety of dimension at least one. We divide into two cases to show that $\delta_Z(X) \ge \frac{n+1}{n}$. First we treat the case when X doesn't contain the secant variety of Z.

Lemma 4.15. *In the above notation, assume that there exist closed points $P \ne Q \in Z$ such that the line joining P and Q is not contained in X. Then $\delta_Z(X) \ge \frac{n+1}{n}$.*

Proof. By Lemma 4.14, there exists a flag

$$Y_\bullet \quad : \quad X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

on X such that each Y_i $(1 \le i \le n-1)$ is a smooth hyperplane section of Y_{i-1} containing P, Q and Y_n is a smooth point on the curve Y_{n-1} that's different from P, Q. Let $V_\bullet$ be the complete linear series associated to $-K_X$, let $Y_\bullet^{(j)}$ be the truncated flag given by $Y_0 \supseteq \cdots \supseteq Y_j$, and let $W_\bullet^{(j)}$ be the refinement of $V_\bullet$ by $Y_\bullet^{(j)}$. It is equipped with a filtration $\mathcal{F}$ induced by $Y_\bullet$. By Example 2.12, $W_\bullet^{(j)}$ is almost complete, and it is clear that $F(W_\bullet^{(j)}) = 0$. Since any m-basis type $\mathbb{Q}$-divisor D of $W_\bullet^{(j)}$ that's compatible with $\mathcal{F}$ can be written as (see the discussion at the beginning of Section 3.1)

$$D = S_m(W_\bullet^{(j)}; Y_{j+1}) \cdot Y_{j+1} + \Gamma,$$

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where $\Gamma|_{Y_{j+1}}$ is an m-basis type $\mathbb{Q}$-divisor of $W_\bullet^{(j+1)}$, we have (see equation (3.1))

$$c_1(W_\bullet^{(j+1)}) = \left(c_1(W_\bullet^{(j)}) - S(W_\bullet^{(j)}; Y_{j+1}) \cdot Y_{j+1}\right)\Big|_{Y_{j+1}}. \tag{4.10}$$

Therefore, by Lemma 2.13 and induction on j, we have

$$c_1(W_\bullet^{(j)}) \sim_{\mathbb{Q}} -\left(1 - \frac{j}{n+1}\right) K_X|_{Y_j} = 2\left(1 - \frac{j}{n+1}\right) H$$

for all $0 \le j \le n-1$ and $S(W_\bullet^{(j)}; Y_{j+1}) = S(c_1(W_\bullet^{(j)}); Y_{j+1}) = \frac{2}{n+1}$ for $0 \le j \le n-2$. By Theorem 3.4 (applied to $j = n-1$), we see that to prove $\delta_Z(X) = \delta_Z(V_\bullet) \ge \frac{n+1}{n}$, it suffices to show that $\delta_{Z\cap Y_{n-1}}(Y_{n-1}; W_\bullet^{(n-1)}, \mathcal{F}) \ge \frac{n+1}{n}$. As $Z \cap Y_{n-1}$ contains at least two points P, Q and $\deg c_1(W_\bullet^{(n-1)}) = \frac{2}{n+1}(-K_X \cdot H^{n-1}) = \frac{4n}{n+1}$, this follows from the next lemma. □

Lemma 4.16. *Let C be a smooth curve, let $W_\bullet$ be a multigraded linear series with bounded support containing an ample series, let $P_1, \cdots, P_r, Q$ be distinct points on C, let $Z = P_1 \cup \cdots \cup P_r$, and let $\mathcal{F}$ be the filtration on $W_\bullet$ induced by Q. Assume that $W_\bullet$ is almost complete and $F(W_\bullet) = 0$. Then*

$$\delta_Z(C; W_\bullet, \mathcal{F}) \ge \frac{2r}{\deg c_1(W_\bullet)}.$$

Proof. Any m-basis type $\mathbb{Q}$-divisor D of $W_\bullet$ that's compatible with $\mathcal{F}$ has the form $D = S_m(W_\bullet; Q)\cdot Q + \Gamma$ for some effective $\mathbb{Q}$-divisor Γ. Since $Q \notin Z$, in order for Z to be contained in the non-lc centre of $(C, \lambda D)$, we need $\mathrm{mult}_{P_i}(\lambda\Gamma) > 1$ for all $i = 1, \cdots, r$. It follows that

$$\delta_{Z,m}(W_\bullet, \mathcal{F}) \ge \frac{r}{\deg \Gamma} = \frac{r}{\deg D - S_m(W_\bullet; Q)}.$$

Letting $m \to \infty$, we obtain

$$\delta_Z(W_\bullet, \mathcal{F}) \ge \frac{r}{\deg c_1(W_\bullet) - S(W_\bullet; Q)}.$$

The lemma then follows since $S(W_\bullet; Q) = S(c_1(W_\bullet); Q) = \frac{1}{2}\deg c_1(W_\bullet)$ by Lemma 2.13. □

The opposite case is when X contains the secant variety of Z.

Lemma 4.17. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n \ge 4$ and $Z \subseteq X$ a subvariety of dimension at least one. Assume that there exists closed points $P \ne Q \in Z$ such that the line joining P, Q is contained in X. Then $\delta_Z(X) \ge \frac{n+1}{n}$.*

Proof. The proof is similar to Lemma 4.15, except that we use a slightly different flag. Consider a flag

$$Y_\bullet \quad : \quad X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

on X such that each Y_i $(1 \le i \le n-2)$ is a smooth hyperplane section of Y_{i-1} containing P and Q, Y_{n-1} is the line joining P, Q and Y_n is a smooth point on Y_{n-1} that's different from P, Q. We use the same notation $W_\bullet^{(j)}$ and $\mathcal{F}$ as in Lemma 4.15. We claim that $W_\bullet^{(j)}$ is almost complete and $F(W_\bullet^{(j)}) = 0$ for all $0 \le j \le n-1$. Indeed this is evident when $0 \le j \le n-2$ by Example 2.12, so it remains to consider the case $j = n-1$. For ease of notation, let $S = Y_{n-2}$, $L = Y_{n-1}$, and let H be the hyperplane class. It is straightforward to check that on the surface S (which is a smooth surface of degree n in $\mathbb{P}^3$), we have

1. $(H \cdot L) = 1$, $(H^2) = n$, $(L^2) = 2 - n$ and $(H - L)^2 = 0$;
2. $H - L$ is nef.

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They together imply that $H^0(S, mH - jL) \neq 0$ $(m, j \in \mathbb{N})$ if and only if $0 \leq j \leq m$ and that

$$S(H; L) = \frac{1}{(H^2)} \int_0^1 (H - xL)^2 \mathrm{d}x = \frac{2}{3} - \frac{1}{3n}. \tag{4.11}$$

By Kodaira vanishing, we also have $H^1(S, \mathcal{O}_S(mH - jL)) = 0$ whenever $m - j > n - 4$, and thus the natural map $H^0(S, \mathcal{O}_S(mH - jL)) \to H^0(L, \mathcal{O}_L(mH - jL))$ is surjective when $m - (j + 1) > n - 4$. As $\mathcal{O}_L(mH - jL) \cong \mathcal{O}_L(m + j(n - 2))$, we see that the complete linear series $V_\bullet$ associated to H on S has almost complete refinement by L. Since $W_\bullet^{(n-2)}$ is almost complete (with respect to H), most graded pieces of $W_\bullet^{(n-2)}$ are a complete linear series $|mH|$ for some $m \in \mathbb{N}$ (see Example 2.12); hence its refinement $W_\bullet^{(n-1)}$ by L is also almost complete. As $L \cong \mathbb{P}^1$, the linear systems $|\mathcal{O}_L(mH - jL)|$ are all base point free; thus it is not hard to check that $F(W_\bullet^{(n-1)}) = 0$. By Lemma 2.13, equation (4.10) and equation (4.11), we find (as in the proof of Lemma 4.15)

$$S(W_\bullet^{(j)}; Y_{i+1}) = \frac{2}{n+1} < \frac{n}{n+1}$$

when $0 \leq j \leq n - 3$ and

$$S(W_\bullet^{(n-2)}; Y_{n-1}) = S(c_1(W_\bullet^{(n-2)}); Y_{n-1}) = S\left(\frac{6}{n+1}H; L\right) = \frac{6}{n+1}\left(\frac{2}{3} - \frac{1}{3n}\right) < \frac{n}{n+1},$$

where the last inequality uses $n \geq 4$. Using equation (4.10) one more time, we also obtain

$$\deg c_1(W_\bullet^{(n-1)}) = \frac{6}{n+1}\left(H - \left(\frac{2}{3} - \frac{1}{3n}\right) L \cdot L\right) = \frac{4}{n+1}\left(n - 1 + \frac{1}{n}\right) < \frac{4n}{n+1}.$$

Hence by Lemma 4.16, noting that $Z \cap Y_{n-1}$ contains at least two points P, Q, we deduce that $\delta_{Z \cap Y_{n-1}}(W_\bullet^{(n-1)}, \mathcal{F}) \geq \frac{n+1}{n}$, and therefore the lemma follows from Theorem 3.4 and the above computations. □

We are ready to prove the K-stability of index two hypersurfaces.

Proof of Theorem 4.12. By Lemma 4.10, we may assume that $n \geq 4$. It suffices to verify the two conditions of Lemma 4.11. The first condition follows from Lemmas 4.15 and 4.17 and the second condition follows from Lemma 4.13. □

A. Stability thresholds of cubic surfaces

In this appendix, we compute the δ-invariants of all smooth cubic surfaces and give the proof of Theorem 4.6. Throughout the section, we let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface, $x \in X$ a closed point and $C = T_x(X) \cap X$.

The proofs are similar between different cases. Note that the first case – that is, $\mathrm{mult}_x C = 3$ – is already treated by Theorem 4.9. We work out the details when C has a tacnode and sketch the argument in the remaining cases.

Lemma A.1. *Assume that C has a tacnode at x. Then $\delta_x(X) = \frac{27}{17}$, and it is computed by the (unique) divisor that computes* $\mathrm{lct}_x(X, C)$.

Proof. By assumption, $C = L \cup Q$, where L (respectively, Q) is a line (respectively, conic) and L is tangent to Q at x. We have $L = (u = 0)$ and $Q = (u - v^2 = 0)$ in some local coordinates u, v around x. Let $\pi\colon Y \to X$ be the weighted blowup at x with weights $\mathrm{wt}(u) = 2$, $\mathrm{wt}(v) = 1$, and let $E \subseteq Y$ be the exceptional divisor. Note that E is the unique divisor that computes $\mathrm{lct}_x(X, C)$. Let $\widetilde{L}$ (respectively, $\widetilde{Q}$) be the strict transform of L (respectively, Q). We have $(\widetilde{L}^2) = -3$, $(\widetilde{Q}^2) = -2$, $(\widetilde{L} \cdot \widetilde{Q}) = 0$ and

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$-\pi^*K_X - 4E \sim \widetilde{L} + \widetilde{Q}$. It follows that the stable base locus of $-\pi^*K_X - tE$ is contained in $\widetilde{L} \cup \widetilde{Q}$ for all $0 \leq t \leq 4$, and we have

$$N_\sigma(-\pi^*K_X - tE) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ \frac{t-1}{3}\widetilde{L} & \text{if } 1 < t \leq 2, \\ \frac{t-1}{3}\widetilde{L} + \frac{t-2}{2}\widetilde{Q} & \text{if } 2 < t \leq 4, \end{cases} \tag{A.1}$$

where $N_\sigma(L)$ (respectively, $P_\sigma(L)$) denotes the negative (respectively, positive) part in the Zariski decomposition of L. Therefore,

$$\begin{aligned} \mathrm{vol}(-\pi^*K_X - tE) &= ((P_\sigma(-\pi^*K_X - tE))^2 \\ &= \begin{cases} 3 - \frac{1}{2}t^2 & \text{if } 0 \leq t \leq 1, \\ 3 - \frac{1}{2}t^2 + \frac{1}{3}(t-1)^2 & \text{if } 1 < t \leq 2, \\ 3 - \frac{1}{2}t^2 + \frac{1}{3}(t-1)^2 + \frac{1}{2}(t-2)^2 & \text{if } 2 < t \leq 4 \end{cases} \end{aligned} \tag{A.2}$$

In particular, $T(-K_X; E) = 4$ (as $\mathrm{vol}(-\pi^*K_X - 4E) = 0$) and $S(-K_X; E) = \frac{17}{9}$. Let

$$\lambda = \frac{27}{17} = \frac{A_X(E)}{S(-K_X; E)}.$$

Clearly $\delta_x(X) \leq \lambda$ by definition, and it remains to show $\delta_x(X) \geq \lambda$. Let $V_\bullet$ be the complete linear series associated to $-K_X$, and let $W_\bullet$ be its refinement by E. By Theorem 3.2, it is enough to prove $\delta(E, \Delta_E; W_\bullet) \geq \lambda$, where $\Delta_E = \mathrm{Diff}_E(0) = \frac{1}{2}P_0$ and P_0 is the (unique) singular point of Y. Note that $P_0 \notin \widetilde{L} \cup \widetilde{Q}$.

By Lemma 4.5, $W_\bullet$ is almost complete. Therefore, by Corollary 3.3, it suffices to show

$$\delta(E, \Delta_E + \lambda F(W_\bullet); c_1(M_\bullet)) \geq \lambda. \tag{A.3}$$

As in the proof of Theorem 4.9, noting that

$$\begin{aligned} \lambda c_1(M_\bullet) + \lambda F(W_\bullet) &\sim_{\mathbb{Q}} \lambda c_1(W_\bullet) \sim_{\mathbb{Q}} \lambda(-\pi^*K_X - S(-K_X; E) \cdot E)|_E \\ &\sim_{\mathbb{Q}} -A_X(E) \cdot E|_E \sim_{\mathbb{Q}} -(K_Y + E)|_E \sim_{\mathbb{Q}} -(K_E + \Delta_E), \end{aligned}$$

we see that equation (A.3) is equivalent to saying the pair $(E, \Delta_E + \lambda F(W_\bullet))$ is K-semistable.

We apply Lemma 4.8 to compute $F(W_\bullet)$. Let $P_1 = \widetilde{L} \cap E$ and $P_2 = \widetilde{Q} \cap E$. Then $\mathrm{Supp}(F(W_\bullet)) \subseteq P_1 \cup P_2$. We have $\mathrm{vol}_{Y|E}(-\pi^*K_X - tE) = -\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(-\pi^*K_X - tE)$ by [33, Corollary C]. Combined with equation (A.1), equation (A.2) and Lemma 4.8, we deduce

$$\mathrm{mult}_{P_1} F(W_\bullet) = \frac{1}{(-K_X)^2}\left(\int_1^2 \frac{t-1}{3} \cdot \frac{t+2}{3}\mathrm{d}t + \int_2^4 \frac{t-1}{3} \cdot \frac{2(4-t)}{3}\mathrm{d}t\right) = \frac{17}{54},$$

$$\mathrm{mult}_{P_2} F(W_\bullet) = \frac{1}{(-K_X)^2}\int_2^4 \frac{t-2}{2} \cdot \frac{2(4-t)}{3}\mathrm{d}t = \frac{4}{27}.$$

Thus $(E, \Delta_E + \lambda F(W_\bullet)) \cong (\mathbb{P}^1, \frac{1}{2}P_0 + \frac{1}{2}P_1 + \frac{4}{17}P_2)$, which is K-semistable by Lemma A.2. This finishes the proof. □

The following result is used in the above proof.

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Lemma A.2. *Let $\Delta = a_1P_1+\cdots+a_mP_m$, where $P_1, \cdots, P_m$ are distinct points on $\mathbb{P}^1$ and $a_i \in (0,1)$ $(i = 1, \cdots, m)$ satisfy $a_1 + \cdots + a_m < 2$ (i.e., $(\mathbb{P}^1, \Delta)$ is log Fano). Then*

$$\delta(\mathbb{P}^1, \Delta) = \frac{1 - \max_{1\le i\le m}\{a_i\}}{1 - \frac{1}{2}(a_1 + \cdots + a_m)}.$$

In particular, $(\mathbb{P}^1, \Delta)$ is K-semistable if and only if $a_1 + \cdots + a_m \ge 2a_i$ for all $1 \le i \le m$.

Proof. We have $S(-K_{\mathbb{P}^1} - \Delta; P) = \frac{1}{2}\deg(-K_{\mathbb{P}^1} - \Delta) = 1 - \frac{1}{2}(a_1 + \cdots + a_m)$ and $A_{\mathbb{P}^1,\Delta}(P) = 1 - \mathrm{mult}_P(\Delta)$ for any $P \in \mathbb{P}^1$. The result then follows from the definition of stability thresholds. □

Lemma A.3. *Assume that C has a cusp at x. Then $\delta_x(X) = \frac{5}{3}$, and it is computed by the (unique) divisor that computes $\mathrm{lct}_x(X, C)$.*

Proof. The proof is very similar to that of Lemma A.1, so we only sketch the steps. In local coordinates, $C = (u^2 - v^3 = 0)$ around x. Let $\pi\colon Y \to X$ be the weighted blowup at x with $\mathrm{wt}(u) = 3$, $\mathrm{wt}(v) = 2$, and let E be the exceptional divisor. Let $\widetilde{C}$ be the strict transform of C. We have $T(-K_X; E) = 6$,

$$N_\sigma(-\pi^*K_X - tE) = \begin{cases} 0 & \text{if } 0 \le t \le 3, \\ \frac{t-3}{3}\widetilde{C} & \text{if } 3 < t \le 6 \end{cases}$$

and

$$\mathrm{vol}(-\pi^*K_X - tE) = \begin{cases} 3 - \frac{1}{6}t^2 & \text{if } 0 \le t \le 3, \\ \frac{1}{6}(6-t)^2 & \text{if } 3 < t \le 6. \end{cases}$$

Thus $S(-K_X; E) = 3$. Let $\lambda = \frac{5}{3} = \frac{A_X(E)}{S(-K_X;E)}$, and let $W_\bullet$ be the refinement by E of the complete linear series of $-K_X$. As in the proof of Lemma A.1, $W_\bullet$ is almost complete, and it suffices to show that $(E, \Delta_E + \lambda F(W_\bullet))$ is K-semistable. Note that $\Delta_E = \frac{1}{2}P_0 + \frac{2}{3}P_1$, where P_0, P_1 are the two singular points of Y. By Lemma 4.8, we find

$$\mathrm{mult}_{P_2}F(W_\bullet) = \frac{1}{(-K_X)^2}\int_3^6 \frac{t-3}{3}\cdot\frac{6-t}{3}\mathrm{d}t = \frac{1}{6},$$

where $P_2 = \widetilde{C} \cap E$. Thus $(E, \Delta_E + \lambda F(W_\bullet)) \cong (\mathbb{P}^1, \frac{1}{2}P_0 + \frac{2}{3}P_1 + \frac{5}{18}P_2)$, which is K-semistable by Lemma A.2. This concludes the proof. □

Lemma A.4. *Assume that C is irreducible and has a node at x. Then $\delta_x(X) = \frac{12}{7}$, and it is computed by the ordinary blowup of x.*

Proof. Again we only sketch the steps. Let $\pi\colon Y \to X$ be the ordinary blowup of x, and let E be the exceptional divisor. Let $\widetilde{C}$ be the strict transform of C. We have $T(-K_X; E) = 2$,

$$N_\sigma(-\pi^*K_X - tE) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{3}{2}, \\ (2t-3)\widetilde{C} & \text{if } \frac{3}{2} < t \le 2 \end{cases}$$

and

$$\mathrm{vol}(-\pi^*K_X - tE) = \begin{cases} 3 - t^2 & \text{if } 0 \le t \le \frac{3}{2}, \\ 3(2-t)^2 & \text{if } \frac{3}{2} < t \le 2. \end{cases}$$

Thus $S(-K_X; E) = \frac{7}{6}$. Let $\lambda = \frac{12}{7} = \frac{A_X(E)}{S(-K_X;E)}$, and let $W_\bullet$ be the refinement by E of the complete linear series of $-K_X$. Since $W_\bullet$ is almost complete by Lemma 4.5, it suffices to show that $(E, \lambda F(W_\bullet))$ is

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K-semistable as in the proof of Lemma A.1 (note that $\Delta_E = 0$ in this case). By Lemma 4.8, we see that $F(W_\bullet) = \mu(P_1 + P_2)$ for some $\mu > 0$, where $\{P_1, P_2\} = \widetilde{C} \cap E$. Thus $(E, \lambda F(W_\bullet))$ is K-semistable by Lemma A.2 (regardless of the value of μ). This proves the lemma. □

Lemma A.5. *Assume that C is a union of three lines and $\mathrm{mult}_x C = 2$. Then $\delta_x(X) = \frac{18}{11}$, and it is computed by the ordinary blowup of x.*

Proof. Write $C = L_1 + L_2 + L_3$, where $L_1 \cap L_2 = x$. Let $\pi\colon Y \to X$ be the ordinary blowup of x, and let E be the exceptional divisor. Let $\widetilde{L}_i$ be the strict transform of L_i $(i = 1, 2)$. We have $T(-K_X; E) = 2$,

$$N_\sigma(-\pi^* K_X - tE) = \begin{cases} 0 & \text{if } 0 \le t \le 1, \\ \frac{t-1}{2}(\widetilde{L}_1 + \widetilde{L}_2) & \text{if } 1 < t \le 2 \end{cases}$$

and

$$\mathrm{vol}(-\pi^* K_X - tE) = \begin{cases} 3 - t^2 & \text{if } 0 \le t \le 1, \\ 4 - 2t & \text{if } 1 < t \le 2. \end{cases}$$

Thus $S(-K_X; E) = \frac{11}{9}$. Let $\lambda = \frac{18}{11} = \frac{A_X(E)}{S(-K_X;E)}$, and let $W_\bullet$ be the refinement by E of the complete linear series of $-K_X$. As in previous cases, $W_\bullet$ is almost complete, and it suffices to show that $(E, \lambda F(W_\bullet))$ is K-semistable (note that $\Delta_E = 0$). By Lemma 4.8, we have $F(W_\bullet) = \mu(P_1 + P_2)$ for some $\mu > 0$, where $P_i = \widetilde{L}_i \cap E$. Thus $(E, \lambda F(W_\bullet))$ is K-semistable by Lemma A.2 (regardless of the value of μ). This proves the lemma. □

Lemma A.6. *Assume that $C = L \cup Q$, where L is a line, Q is a conic, and they intersect transversally at x. Then $\delta_x(X) = \frac{9}{25-8\sqrt{6}}$, and it is computed by the weighted blowup at x with $\mathrm{wt}(u) = 1 + \sqrt{6}$ and $\mathrm{wt}(v) = 2$ (where u, respectively v, is the local defining equation of L, respectively Q).*

Proof. For each $a, b > 0$, let $\nu_{a,b}$ be the quasi-monomial valuation over $x \in X$ defined by $\nu_{a,b}(u) = a$ and $\nu_{a,b}(v) = b$. We first identify the minimiser of $\frac{A_X(\nu_{a,b})}{S_X(\nu_{a,b})}$. For this choose coprime integers $a, b > 0$, and let $\pi\colon Y = Y_{a,b} \to X$ be the weighted blowup at x with $\mathrm{wt}(u) = a$ and $\mathrm{wt}(v) = b$. Let E be the exceptional divisor, and let $\widetilde{L}$ (respectively, $\widetilde{Q}$) be the strict transform of L (respectively, Q). Assume that $b < 2a$. Then similar to the calculations in previous cases, we have (compare to equation (4.5))

$$N_\sigma(-\pi^* K_X - tE) = \begin{cases} 0 & \text{if } 0 \le t \le b, \\ \frac{t-b}{a+b}\widetilde{L} & \text{if } b < t \le \frac{a(2a+3b)}{2a+b}, \\ \frac{2t-2a-b}{b}\widetilde{L} + \frac{(2a+b)t - a(2a+3b)}{b^2}\widetilde{Q} & \text{if } \frac{a(2a+3b)}{2a+b} < t \le a+b, \end{cases}$$

$$\mathrm{vol}_{Y|E}(-\pi^* K_X - tE) = \big(P_\sigma(-\pi^* K_X - tE) \cdot E\big) = -\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(-\pi^* K_X - tE)$$

$$= \begin{cases} \frac{t}{ab} & \text{if } 0 \le t \le b, \\ \frac{t+a}{a(a+b)} & \text{if } b < t \le \frac{a(2a+3b)}{2a+b}, \\ \frac{4(a+b-t)}{b^2} & \text{if } \frac{a(2a+3b)}{2a+b} < t \le a+b, \end{cases}$$

and

$$S(-K_X; E) = \frac{2}{(-K_X)^2}\int_0^{a+b} t \cdot \mathrm{vol}_{Y|E}(-\pi^* K_X - tE)\mathrm{d}t = \frac{10a^2 + 19ab + 3b^2}{9(2a+b)}.$$

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Note that $\frac{A_X(v_{a,b})}{S_X(v_{a,b})}$ only depends on the ratio $\frac{a}{b}$; thus by continuity [3, Proposition 2.4], we have

$$\frac{A_X(v_{a,b})}{S_X(v_{a,b})} = \frac{9(a+b)(2a+b)}{10a^2+19ab+3b^2}$$

for all $a, b \in \mathbb{R}_+$. It achieves its minimum $\lambda = \frac{9}{25-8\sqrt{6}}$ when $\frac{a}{b} = \mu := \frac{1+\sqrt{6}}{2} > \frac{1}{2}$. In particular, we have $\delta_x(X) \le \lambda$. It remains to show $\delta_x(X) \ge \lambda$.

Choose a sequence of coprime integers $a_m, b_m > 0$ $(m = 1, 2, \cdots)$ such that $\mu_m := \frac{a_m}{b_m} \to \mu$ $(m \to \infty)$. Let $\pi_m \colon Y_m = Y_{a_m,b_m} \to X$ be the corresponding weighted blowup, and let E_m be the exceptional divisor. Let $P_1^{(m)} = \widetilde{L} \cap E_m$, $P_2^{(m)} = \widetilde{Q} \cap E_m$, and let $W_\bullet^{(m)}$ be the refinement by E_m of the complete linear series associated to $-K_X$. As before, $W_\bullet^{(m)}$ is almost complete by Lemma 4.5. Using the above calculations and Lemma 4.8, we have

$$F(W_\bullet^{(m)}) = \frac{c_1^{(m)}}{b_m} P_1^{(m)} + \frac{c_2^{(m)}}{a_m} P_2^{(m)},$$

where

$$c_1^{(m)} = \frac{20\mu_m^3 - 8\mu_m^2 + \mu_m + 1}{9\mu_m(2\mu_m+1)^2}, \quad c_2^{(m)} = \frac{4}{9(2\mu_m+1)^2}.$$

Let $\Delta_m = \mathrm{Diff}_{E_m}(0) = (1 - \frac{1}{b_m})P_1^{(m)} + (1 - \frac{1}{a_m})P_2^{(m)}$, $\lambda_m = \frac{A_X(E_m)}{S(-K_X;E_m)}$, and let

$$r_m := \delta(E_m, \Delta_m + \lambda_m F(W_\bullet^{(m)}); \lambda_m c_1(M_\bullet^{(m)})) = \delta(E_m, \Delta_m + \lambda_m F(W_\bullet^{(m)})), \tag{A.4}$$

where the last equality follows from equation (3.1). Then by Corollary 3.3, we obtain

$$\delta_x(X) \ge \min\{\lambda_m, r_m\lambda_m\} \tag{A.5}$$

for all m. Note that $\lambda_m \to \lambda$ as $m \to \infty$.

We claim that $r_m \to 1$ as $m \to \infty$. Since $E_m \cong \mathbb{P}^1$, by Lemma A.2 this is equivalent to

$$\frac{1 - \mathrm{mult}_{P_1^{(m)}}(\Delta_m + \lambda_m F(W_\bullet^{(m)}))}{1 - \mathrm{mult}_{P_2^{(m)}}(\Delta_m + \lambda_m F(W_\bullet^{(m)}))} \to 1$$

when $m \to \infty$. It is straightforward (though a bit tedious) to check that

$$\mathrm{LHS} = \mu_m \cdot \frac{1 - \lambda_m c_1^{(m)}}{1 - \lambda_m c_2^{(m)}} \to \mu \cdot \frac{9\mu(2\mu+1)^2 - \lambda(20\mu^3 - 8\mu^2 + \mu + 1)}{9\mu(2\mu+1)^2 - 4\lambda\mu} = 1.$$

This proves the claim. Letting $m \to \infty$ in equation (A.5), we obtain $\delta_x(X) \ge \lambda$ as desired. □

Corollary A.1. *In the situation of Lemma A.6, let $0 < \varepsilon \ll 1$ be a rational number. Then the pair $(X, (1-\varepsilon)C)$ is log Fano and $\delta(X, (1-\varepsilon)C) = \frac{9}{25-8\sqrt{6}}$.*

Proof. We continue to use the notation from Lemma A.6. Since (X, C) is lc, it is clear that $(X, (1-\varepsilon)C)$ is log Fano. Let $L \cap Q = \{x, y\}$, let v be the quasi-monomial valuation that computes $\delta_x(X)$ in Lemma A.6, and let $\lambda = \frac{9}{25-8\sqrt{6}}$. Note that v is an lc place of (X, C): that is, $A_X(v) = v(C)$. Then by Lemma A.6, we get $A_{X,(1-\varepsilon)C)}(v) = \varepsilon A_X(v) = \varepsilon\lambda S_X(v) = \lambda S_{X,(1-\varepsilon)C}(v)$ and hence $\delta(X, (1-\varepsilon)C) \le \lambda$. To get

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the reverse inequality, we shall prove

$$\delta_z(X,(1-\varepsilon)C) \geq \lambda \tag{A.6}$$

for any closed point $z \in X$. In any case, we have $A_X(v) \geq v(C)$, and hence $A_{X,(1-\varepsilon)C}(v) \geq \varepsilon A_X(v) \geq \delta_z(X) \cdot \varepsilon S_X(v) = \delta_z(X) \cdot S_{X,(1-\varepsilon)C}(v)$ for any divisorial valuation v whose centre contains z. It follows that $\delta_z(X,(1-\varepsilon)C) \geq \delta_z(X)$, and hence by Lemma A.6, equation (A.6) holds when $z \in \{x,y\}$. If $z \notin \mathrm{Supp}(C)$, then $v(C) = 0$ for any divisorial valuation v whose centre contains z, hence by the definition of stability thresholds we get $\delta_z(X,(1-\varepsilon)C) = \frac{\delta_z(X)}{\varepsilon} \geq \lambda$ when $0 < \varepsilon \ll 1$. Thus it remains to consider the case when $z \in \mathrm{Supp}(C) \setminus \{x,y\}$. For simplicity, we assume $z \in Q$ (the other case $z \in L$ is similar). Consider the refinement (denote it by $W_\bullet$) by Q of the complete linear series associated to $-K_X$. Note that $\delta(W_\bullet) > 0$. Since $-(K_X + (1-\varepsilon)C) \sim_{\mathbb{Q}} -\varepsilon K_X$, by Theorem 3.2, we have

$$\delta_z(X,(1-\varepsilon)C) = \varepsilon^{-1}\delta_z(X,(1-\varepsilon)C;-K_X) \geq \min\left\{\frac{A_{X,(1-\varepsilon)C}(Q)}{S_{X,(1-\varepsilon)C}(Q)}, \varepsilon^{-1}\delta_z(W_\bullet)\right\} \geq \lambda$$

when $0 < \varepsilon \ll 1$. Hence equation (A.6) also holds in this case. The proof is now complete. □

Proof of Theorem 4.6. This follows from the combination of Theorem 4.9, Lemmas A.1, A.3, A.4, A.5, A.6 and Corollary A.1. □

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Conflicts of Interest. None.

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